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Can Stability Analysis be really simplified? (Revisiting Lyapunov, Barbalat, LaSalle and all that)

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Abstract.
(Presented at 2016 ICNPAA, La Rochelle, France) Even though Lyapunov approach is the most commonly used method for stability analysis, its use has been hindered by the realization that in most applications the so-called Lyapunov derivative is at most negative semidefinite and not negative definite as desired. Many different approaches have been used in an attempt to overcome these difficulties. Until recently, the most widely accepted stability analysis has been based on Barbalat's Lemma which seems to require uniform continuity of practically all signals involved. Recently, stability analysis methods for nonautonomous nonlinear systems have been revisited. Even though new developments based on unknown works of LaSalle attempted to mitigate these continuity conditions, counterexamples are suggested to contradict these results. New analysis shows that these counterexamples, which are making use of well-known mathematical expressions, are actually using them beyond their domain of validity. Therefore, the restrictive condition of uniform continuity required by Barbalat's Lemma and even the milder conditions required by LaSalle's extension of the Invariance Principle to nonautonomous systems can be further mitigated. A new Invariance Principle only required that bounded trajectories cannot pass an infinite distance in finite time. Finally, a new Theorem of Stability, which is formulated as a direct extension and a generalization of Lyapunov's Theorem, not only simplifies the stability analysis of nonlinear systems, but also leads to conclusive results about the system under analysis.

INTRODUCTION

Unless mentioned otherwise, this section deals with stability of nonlinear systems of the form

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, t). \tag{1}$$

How can one decide about stability of this nonlinear system? The first and direct idea is to just solve the differential equation and check the resulting trajectories. Not only would this approach require explicit solutions for the nonlinear system (which may not be available for many other than simple class-room examples), but would also require analysing its trajectories for all possible input commands and all initial conditions, as various trajectories of same system could behave differently. Even simulating and numerically solving the differential equation seems like mission impossible for same reasons. Because of the extreme importance of proper stability analysis of the nonlinear systems, we present here a few old and new results.

For convenience of the following presentation, we first try to define the aim of the stability analysis. First one would want to establish that all trajectories are bounded, or, in more general situations, which trajectories are bounded. Second, with regard to bounded trajectories, one would want to know where those trajectories may end. This leads to the next concept of limit point or point of accumulation of the trajectory. As colleagues commented, the very definition of limit point and limit set can be confusing in nonlinear non-autonomous systems. Therefore, before we proceed with stability analysis of trajectories, it is useful to define those limit points (or points of accumulation).

A limit point is defined as such a point about which any *arbitrarily small* neighborhood contains an infinite number of points of the trajectory. This definitions might be confusing, because every point on a continuous trajectory seems to conform to this definition. Therefore, we define an infinite sequence of discrete times $t_0, t_1, t_2, ..., t_i, ...$ and the corresponding points on the trajectory $\mathbf{x}(t_0), \mathbf{x}(t_1), \mathbf{x}(t_2), ..., \mathbf{x}(t_i), ...$ A limit point is thus defined as such a (discrete) point about which any *arbitrarily small* neighborhood contains an infinite number of (discrete) points $\mathbf{x}(t_i)$ of the trajectory.

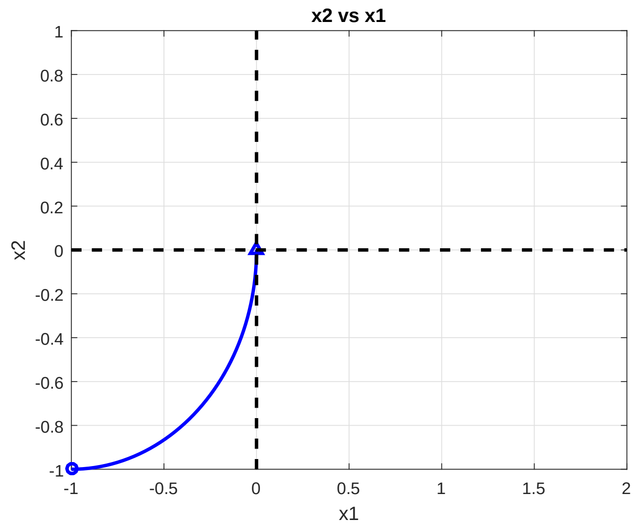

FIGURE 1. Equilibrium-point-type limit point

We will later define the various types of limit points and limit sets, yet here we start with a simple example of limit point, namely, a stable equilibrium point that is asymptotically approached by the trajectory as time goes to infinity. Thus, for any neighborhood, arbitrarily small, around the equilibrium point there exists some finite time t_0 such that *all* points on the corresponding trajectory belong to this neighborhood for all times larger than t_0.

1. Lyapunov direct method The so-called Lyapunov direct method is the methodology that allowed analysis of nonlinear systems stability, without requiring to actually solve the nonlinear differential equations for all initial conditions. Lyapunov proposed to associate with the system an appropriate function, say positive definite and radially unbounded. Such functions have later been called "Lyapunov functions" $V(\mathbf{x}(t))$, where $\mathbf{x}(t)$ is the entire state vector. The Lyapunov theorem states that if the time-derivative of $V(\mathbf{x}(t))$ *along the trajectories of the system* is negative definite, then the system is globally asymptotically stable.

In other words, this result of Lyapunov approach would describe trajectories as shown in Figure 1.

Although Lyapunov works have been published at the start of 20th century [1], it took more than another half a century before it became the basic tool upon which modern stability analysis is based (for good presentations of Lyapunov direct method, see, for example the excellent books [2] and [3]).

Nevertheless, in spite of the initial enthusiasm, pretty soon developers were forced to realize that, in real world applications, it is not easy to find an appropriate Lyapunov function with a negative definite derivative. Besides the difficulty of finding the appropriate Lyapunov function, even when in principle it existed, real-world applications could be more complex than just globally asymptotically stable. Some systems may have more than one equilibrium point, some trajectories may end in a limit cycle or may even contain what we called rosette-type limit points, where the trajectory may come, leave and then keep coming and leaving an infinite number of times (to further discuss below), some could be bounded only within some restricted domain and unbounded elsewhere, etc.

Still, the main difficulty that limited the applicability of the direct Lyapunov approach was due to the fact that, in most applications, the derivative of the Lyapunov function usually was *at most* negative semidefinite. Here, things started looking pretty complex.

2. Krasovskii-LaSalle Invariance Principle First extensions to Lyapunov-style approach for the case when the derivative of the Lyapunov function is only negative semidefinite were first attributed in the West to LaSalle [4], yet now are also attributed to Krasovskii and Barbashin [5]. Their result has become known as the Krasovskii-LaSalle Invariance Principle, that was only covering strictly *autonomous* functions of the form $\dot{\mathbf{x}}(t) = f(\mathbf{x})$ (where the time does not explicitly show). In this case, they show that bounded trajectories still end within the domain that is defined by $\dot{V}(t) = 0$. We notice that, unlike the strictly negative definite case, this result does not necessarily imply asymptotic stability as its implications depend on the meaning of $\dot{V}(t) = 0$.

Moreover, most non-trivial nonlinear systems are *nonautonomous* of the form (1), where the time variable my appear either through input command signals or a time-varying parameters, and, thus, one may usually read in the professional literature "unfortunately, this system is nonautonomous and so, the invariance principle cannot be applied."

Therefore, because control systems are nonautonomous, new extensions to the basic Lyapunov stability theory have been sought.

3. Barbalat's lemma One of these extensions was provided by Barbalat's lemma that we state here as it appears in [6]:

Lemma 1 (Barbalat's Lemma): If the differentiable function $V(t)$ has a finite limit as $t \to \infty$ and if $\dot{V}(t)$ is *uniformly continuous* (or $\ddot{V}(t)$ is bounded), then $\dot{V}(t) \to 0$ as $t \to \infty$.

Barbalat's Lemma is very simple and thus, very attractive. Furthermore, under some conditions, it allowed to finally show that the *function* $\dot{V}(t)$ ultimately vanishes and in many cases even it allowed reaching the desirable asymptotic stability or asymptotically perfect tracking conclusion. Nevertheless, it also leaves the burdensome impression that any input command, distortion or disturbance that may affect the uniform continuity of Lyapunov derivative may also affect the proof and thus, the very guarantee of stability of nonlinear systems. However, as we show below, it is only because Barbalat's lemma deals with the *functions* and not with the *systems* that it imposes those strict conditions on continuity of functions and even of their derivative. These conditions *may* happen to hold in various systems, yet if they are not satisfied under less than ideal conditions, it is not necessarily a result of some lack of stability.

In any case, most publications work very hard to guarantee uniform continuity of practically all signals involved and end claiming that "according to Barbalat's Lemma, the system ends with $\dot{V}(t) = 0$." However, this claim is not necessarily true, as it seems to ignore that Barbalat's Lemma requires *prior* knowledge that the Lyapunov function $V(t)$ itself has a finite limit. As a mathematical result, the lemma is perfectly correct, yet as a stability tool, why would the Lyapunov function reach a finite limit *unless* its derivative tends to zero, but then why would the derivative tend to zero *unless* the Lyapunov function reaches a finite limit?

4. (The real) LaSalle's Invariance Principle As a matter of fact, extensions of LaSalle's Invariance Principle to nonautonomous systems have been available at least since 1976 [7]-[11] and have been immediately adopted and used since 1980 for such special problems as adaptive control of large space structures and other applications (see for example [12] for a proof and a brief presentation of the theory along with some early examples). Nonetheless, as classical books in nonlinear systems [6, 13, 14, 15, 16, 17] and even recent publications [18, 19] seem to show, these extensions of the Invariance Principle have remained largely unknown or, at least, misunderstood. Actually, one may still see the still pretty common opinion among researchers that "LaSalle's Invariance Principle only holds for time-invariant (or, more exactly, *autonomous*) systems."

Therefore, it is important to first emphasize LaSalle's simple and ingenious idea. Instead of dealing with the properties of some general *function*, LaSalle's Invariance Principle goes back to establishing some milder conditions on the *system*. Using the notation $\|f\| = \sqrt{f_1^2 + f_2^2 + ...f_n^2}$, satisfaction of one of the following assumption is *checked along the trajectories* of a system:

Assumption 1: $\int_\alpha^\beta \|f(\mathbf{x}, \tau)\| d\tau = \mu(\beta - \alpha)$.

We note that in LaSalle's formulation, the function $\mu(\tau)$ is a "modulus of continuity," to be discussed in continuation. (For convenience, we note that, if $\|f(\mathbf{x}(t), t)\|$ happens to be uniformly bounded for any *bounded* $\mathbf{x}$, the assumption is satisfied.)

Furthermore, as LaSalle observed, nonlinear systems can be more complex than just asymptotically stable or unstable. Some systems could be stable in one domain and unstable in another. For example, for a simple second order system, a Positive Definite Lyapunov function could end with a derivative of the form $\dot{V} = 10^6 - x^2 - y^2$, which, in spite of the first disappointment, guarantees that all trajectories are bounded. In other nonlinear systems, as mentioned, some trajectories could be bounded, yet not necessarily all trajectories. On the other hand, one Lyapunov function may vaguely guarantee that all trajectories are bounded, yet one would still want to know the ultimate behavior of the bounded trajectories. Therefore, LaSalle's Invariance Principle deals with the more general case when not all trajectories are necessarily bounded and so, it only attempts to locate the limit points of those trajectories that are bounded. Thus, its Lyapunov function is only required to be bounded from below and is not necessarily required to be Positive Definite.

Under these assumptions, we can present LaSalle's Invariance Principle for nonautonomous systems:

Theorem 1 (LaSalle's Invariance Principle): Consider the nonlinear non-autonomous system (1). Assume that there exists a Lyapunov function $V(\mathbf{x})$, which is bounded from below, and that its derivative $\dot{V}(x,t)$ along the trajectories of (1) is Negative Semi-Definite and satisfies a relation of the form $\dot{V}(\mathbf{x},t) \leq W(\mathbf{x}) \leq 0$. Now, define the domain $\Omega_w = \{\mathbf{x}|W(\mathbf{x}) = 0\}$. Then, if Assumptions 1 holds, the entire state vector $x(t)$ ultimately reaches the domain Ω_w [7], [12].

Of course, whenever the case is and the appropriate Lyapunov function is found, one may still show that *all* trajectories are bounded. Besides, LaSalle's formulation is that bounded trajectories end within "an invariant set within the domain Ω_w." The actual meaning of this formulation may look pretty obscure and it will be explain later below.

As we mentioned, the ultimate limit points could be either equilibrium points, limit cycles, or rosette-type limit points (the kind that most preoccupied LaSalle and others).

Actually, the finite-distance-in-finite-time condition is first of all needed to guarantee that any rosette-type limit point $\mathbf{p}$ must be located such that $\dot{V}(\mathbf{p},t) = 0$. Otherwise, the rosette-type limit point $\mathbf{p}$ could be located at $\dot{V}(\mathbf{p},t) < 0$ and the trajectory could leave $\mathbf{p}$ towards a neighborhood where $\dot{V}(\mathbf{p},t) = 0$ and then come back to $\mathbf{p}$ an infinite number of times in finite time. Therefore, if the trajectory could pass an infinite distance in finite time, the integral of $\dot{V}(\mathbf{p},t) < 0$ in the neighborhood of $\mathbf{p}$ would remain finite. The finite-distance-in-finite-time condition makes this situation impossible, such that *any* limit point must belong to $\dot{V}(\mathbf{p},t) = 0$.

Although they require different conditions for stability, *all* these methodologies above, either for autonomous or nonautonomous systems, end with the conclusion that all limit points belong to the domain defined by $\dot{V}(\mathbf{x},t) = 0$. If at first look this may appear as a satisfactory conclusion, a second look shows, however, that this result is less conclusive that one may think. Assume for example that the system contains the variables $x_1, x_2, x_3, ...$ and that the Lyapunov derivative is $\dot{V}(\mathbf{p},t) = -x_2^2 \leq 0$. Although at first look the conclusion that limit points belong to the domain defined by $x_2 = 0$ may look satisfactory, it not only tells nothing about all the other variables, but does not even tell if:

a) the result implies the desired asymptotic stability, i.e., the trajectories end at the origin, like in Figure 1,

b) the trajectories end at another point on the axis $x_2 = 0$ like in Figure 2,

c) the trajectories just reach a given point on the axis $x_2 = 0$ and then keep moving back and forth along this axis like in Figure 3,

In all three cases above, the trajectories that reach the domain defined by $\dot{V}(\mathbf{p},t) = 0$ cannot *stay* there, *unless* higher order derivatives of $\dot{V}(\mathbf{p},t)$ also end at zero. Otherwise, one has to check other eventual alternatives, such as:

d) the trajectories end with a limit cycle like in Figure 4,

e) the trajectories end with a rosette-type limit point like in Figure 5,

f) the trajectories may even just cross the axis $x_2 = 0$ at some occasional point with *no stability significance* whatsoever and then may keep leaving and coming back to cross the axis at various *other* points for an infinite number of times.

Nevertheless, it is difficult to shake the impression that such a result like $x_2 = 0$ implies the desired result that everything stops there. Therefore, for a better illustration, assume that the stability analysis results in the Lyapunov derivative $\dot{V}(\mathbf{p},t) = -(x_1 - x_2)^2$. The derivative is negative semidefinite and the conclusion is that the trajectories end et $x_1 - x_2 = 0$. Again, any of the conclusions above is valid and possible (Figs 6-10).

This conclusion is strongly emphasized by Matrosov's methodology [20]. Because the common result $\dot{V}(\mathbf{x}) = 0$ of all stability analysis approaches could not be considered to be satisfactory, the methodology initiated by Matrosov has been trying to maybe invent and use a few Lyapunov functions and their derivatives in an attempt to reach more satisfactory conditions by combining their individual partial results (see for example the recent [21]).

For a better illustration, consider any of the various adaptive model following methodologies. In all schemes, the model following state-error $\mathbf{e}_x(t)$ is forced to vanish using the adaptive control gain vector $\mathbf{K}(t)$. Because both the state-error and the adaptive gains are dynamical values, the total system state-vector is $\{\mathbf{e}_x(t), \mathbf{K}(t)\}$ and the positive definite Lyapunov function, which must contains them both, has the form

$$V(\mathbf{e}_x, \mathbf{K}) = \mathbf{e}_x^T \mathbf{P} \mathbf{e}_x + tr\left[\left(\mathbf{K}(t) - \widetilde{\mathbf{K}}\right)\Gamma^{-1}\left(\mathbf{K}(t) - \widetilde{\mathbf{K}}\right)^T\right]. \tag{2}$$

One now computes the Lyapunov derivative and all adaptive control methodologies end with a Lyapunov derivative of the form

$$\dot{V}(\mathbf{e}_x, \mathbf{K}) = -\mathbf{e}_x^T \mathbf{Q} \mathbf{e}_x, \tag{3}$$

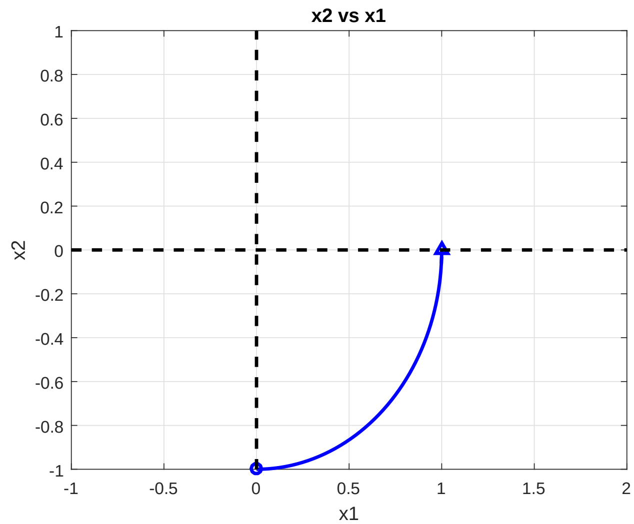

FIGURE 2. Equilibrium-type limit point

which is only negative semidefinite, as it does not contain the adaptive gain. The negativity of the Lyapunov derivative (with respect to $\mathbf{e}_x$) does guarantees that both the errors and the adaptive gains are bounded. Although one cannot say anything about the ultimate adaptive gain behavior, it is attractive to conclude (as all proofs of stability of adaptive control systems actually do) that the state-error vanish and this result may still seem satisfactory.

In reality, this mere result, which only says that "the error tends to zero as time tends to infinity," does not say much, as one cannot know whether the following error indeed ends at zero or its ultimate behavior follows any one of the six a)-f) possible situations described above.

In this context, it would seem that no customary adaptive control stability analysis ends with any *real* proof of stability, *unless* one follows the Matrosov methodology [20] and at least tries to invent supplementary Lyapunov functions and uses the *combined* results of their derivatives in order to reach any sort of real conclusion.

Instead, as we show below, even for the case of *one* Lyapunov function with a negative *semidefinite* derivative *along the bounded trajectories* of the system, new developments not only allow the conditions for stability to be further mitigated, but actually the stability analysis also ends with conclusive results on *all* system variables. As just a first introductory remark, we observe that, in order to guarantee results such as shown in Figure 1, Figure 2, or even Figure 3, then along with $\dot{V}(\mathbf{p}, t) = 0$ one must, at least, also get $\ddot{V}(\mathbf{p}, t) = 0$.

5. The new Invariance Principle As recently observed [23], even the milder condition of LaSalle's original formulation still seems to impose are both difficult to satisfy in realistic applications and, more important, are not necessarily needed. Therefore, we present a new Invariance Principle for nonautonomous systems in a form that is appropriate for the problems we discuss and that further relaxes even the milder conditions of LaSalle. Here, we only require satisfaction of the following assumptions along trajectories of a system:

Assumption A: $\int_\alpha^\beta \|f(x(\tau), \tau)\| d\tau$ is bounded along any *bounded* trajectory $\mathbf{x}(t)$ and for any *finite* time interval $p = \beta - \alpha$.

Note that our presentation of Assumption A above is different from other presentations of LaSalle's original presentations, where Assumption 1 is written in the form $\int_\alpha^\beta \|f(x, \tau)\| d\tau = \mu(\beta - \alpha)$. In LaSalle's formulation, the function $\mu(\tau)$ is a "modulus of continuity," from which one can imply that the trajectory must be a continuous function of time. Here, we replaced the modulus of continuity by a simple bound, because, although continuity is desirable, it cannot be guaranteed in practical environments and it is not necessarily needed for stability. For a simple illustration, one may compare the "decent" uniformly continuous converging function $x_1(t) = e^{-t}$ with the equally converging, though not continuous, ladder function described by $x_2(t) = e^{-(k+1)}$ for $k < t \leq k + 1$, k=0,1,2,... Nor is continuity actually needed for stability under the Invariance Principle approach, because, as the proof of stability shows, the only condition that is needed is that the trajectory, continuous or not continuous, *cannot* pass an infinite distance in finite

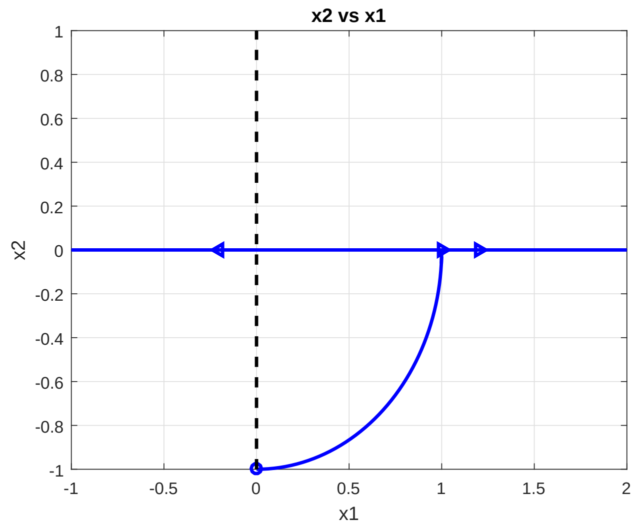

FIGURE 3. Limit-cycle-type limit points

time.

Note: The following development will show how helpful the satisfaction of the simple Assumption A is for the stability analysis. Nevertheless, this is only an *assumption* that must be *checked*, because it may or may not necessarily hold. However, we will show below that it holds in most cases of interest. Furthermore, it is important to emphasize that, instead of dealing with properties of the general function $\mathbf{f}(\mathbf{x}, t)$, the Invariance Principle approach only tests its properties "along the trajectories of the system." For example, it is easy to see that $f(x, t) = x + sint$ satisfies Assumption A "along bounded trajectories," while $f(x, t) = x/(1 - x) + sint$ or $f(x, t) = x + e^t$ may not. On the other hand, depending on the system and on its trajectories, all three examples may or may not satisfy Assumption A. As the many examples of references [22, 23, 24] show, Assumption A is satisfied in most situations, even when the velocity $\mathbf{f}(\mathbf{x}, t)$ is not bounded and, instead, may contain impulses, diverging exponentials, etc. Finally, it is useful to note that $\mathbf{f}(\mathbf{x}, t)$ and its integral are *not required to be continuous* and that stability theory based on the Invariance Principle approach only needs the guarantee that x(t) *cannot* pass an *infinite* distance in *finite* time.

As readers commented, at first glance Assumption A could seem to imply that $\mathbf{f}(\mathbf{x}, t)$ is required to be a $L_1[\alpha, \beta]$ function. However, similarly to LaSalle's Assumption 1, the new Assumption A is also a property of the function $\mathbf{f}(\mathbf{x}, t)$ only along the system *trajectories* and therefore it may or may not hold for *same* function.

For better clarification, it is helpful to think in what situations a bounded trajectory could pass an infinite distance in *finite* time. After some thought, one may see that this is possible *only if the bounded trajectory would perform an infinite number of (bounded, of course) jumps in finite time*. In other words, even though we do allow $\mathbf{f}(\mathbf{x}, t)$ to contain impulse functions and even an infinite sequence of impulse functions, we only assume that there cannot be an infinite number of impulse functions within any finite interval. Therefore, for any practical purposes, Assumption A is not a limiting condition at all.

On another point, LaSalle's stability analysis could only deal with those Lyapunov derivatives that satisfy a relation of the form $\dot{V}(\mathbf{x}, t) \le W(\mathbf{x}) \le 0$. However, although in many cases this relation could be sufficient, it may still *unnecessarily* restrict the applicability of stability theory. For example, while for $\dot{V}_1(x, t) = -x^2(2 + sint)$ one can define $W(x) = -x^2$ and then write $\dot{V}_1(x, t) \le W(x) \le 0$, this is not possible for $\dot{V}_2(x, t) = -x^2(1 + sint)$. Nevertheless, it is clear that $\dot{V}_2(x, t) \le 0$ or, in other words, that $\dot{V}_2(x, t)$ is *uniformly* negative semi-definite. In a more general case, such as $\dot{V}_3(\mathbf{x}, t) = -x_1^2(1 + sint) - x_2^2(1 + cost)$ it is still clear that $\dot{V}_3(\mathbf{x}, t)$ is *uniformly* negative semi-definite although it cannot be written in any one of the (more convenient) previous forms. Therefore, whenever needed, we will show that one can directly deal with *uniformly* positive and negative semidefinite *explicit* functions of time.

Theorem 2 (The new Invariance Principle): Consider the nonlinear non-autonomous system (1). Assume that there exists a Lyapunov function $V(\mathbf{x})$ which is bounded from below and that its derivative $\dot{V}(\mathbf{x}, t)$ along the trajectories of (1) is Negative Semi-Definite, i.e., satisfies $\dot{V}(\mathbf{x}, t) \le 0$. Now, define the domain Ω_e where the Lyapunov derivative equals zero, $\Omega_e = \{\mathbf{x}|\dot{V}(\mathbf{x}, t) = 0)\}$. Also, define the domain Ω_i where the Lyapunov derivative is *identically* zero (i.e.,

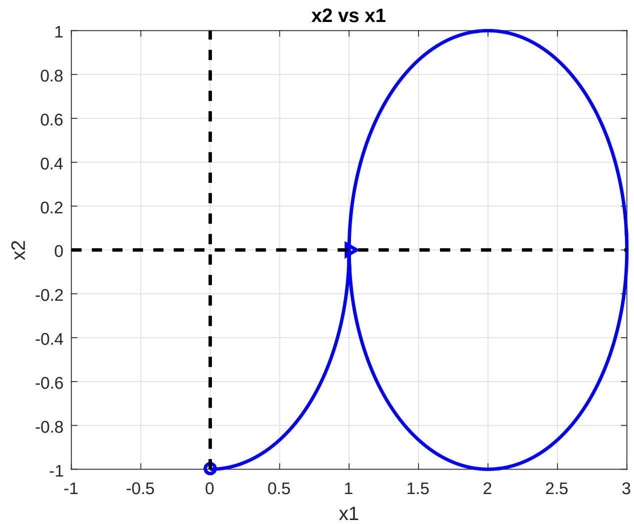

FIGURE 4. Limit-cycle-type limit points

not just *equal* zero), $\Omega_i = \{\mathbf{x}|\dot{V}(\mathbf{x}, t) \equiv 0)\}$. Then, if Assumptions A holds, the limit points of bounded trajectories $\mathbf{x}(t)$ belong to the domain Ω_e. In particular, limit points of the type equilibrium point and limit cycles belong to the domain Ω_i [23, 24].

Note: In order to prevent eventual difficulties in accepting Assumption A, we again emphasize that it is only an assumption that must be checked for any specific problem at task. Therefore, it is better to follow the theorem in the following order: In the first part of the theorem, the positive definite Lyapunov function and its negative semidefinite derivative guarantee that all trajectories are bounded. Because in many common cases $f(\mathbf{x}, t)$ contains "decent" functions of $\mathbf{x}$ and t, in general it is easy to see whether $f(\mathbf{x}, t)$, which is the velocity $\dot{\mathbf{x}}(t)$, is bounded. However, sometimes $f(\mathbf{x}, t)$ may contain unbounded functions, such as impulses, diverging exponentials, etc., so boundedness of $f(\mathbf{x}, t)$ cannot be guaranteed. Nevertheless, this is not a problem that may affect stability, because even in this case, the first part of Theorem still guarantees that the state $\mathbf{x}$(t) is bounded. As a result, the *integral* of $f(\mathbf{x}, t)$, which is a measure of a segment of a bounded trajectory that the system passes in a finite amount of time, *cannot* be unbounded (unless, as we said, the system equation $f(\mathbf{x}, t)$ would contain an infinite number of impulse functions). Therefore, although at first look Assumption A may seem to be difficult to accept, it actually holds in most cases (including all examples we could find).

Note also that this work emphasizes the *identity* relation $\Omega_i = \{\mathbf{x}|\dot{V}(\mathbf{x}, t) \equiv 0\}$ instead of simple *equality* relation $\Omega_e = \{\mathbf{x}|\dot{V}(\mathbf{x}, t) = 0\}$. Examples illustrate the practicality and usefulness of defining the limit set by the identity relation instead of simple equality relation, because it results in sharper conclusions.

As we mentioned above, many real-world examples [23] show that in many practical situations one cannot define Ω_w that LaSalle required (as no appropriate $W(\mathbf{x})$ may exist), yet nonetheless one can define $\Omega_e = \{\mathbf{x}|\dot{V}(\mathbf{x}, t) = 0\}$ and $\Omega_i = \{\mathbf{x}|\dot{V}(\mathbf{x}, t) \equiv 0\}$. As one can prove [23], the new definition is legitimate and is covered by the new Invariance Principle and allows extending the stability analysis to large classes of systems that were not covered before.

In other words, under fairly mild conditions, the Invariance Principle extension to nonautonomous systems guarantees that all trajectories ultimately reach the domain Ω_e. However, as examples illustrate [23, 24], its significance and efficiency and sometimes even its mere existence seems to have remained unknown to a large section of potential users.

6. Towards a new Theorem of Stability Note that stability theory based on the new Invariance Principle approach eliminates the previous requirement for *uniform* continuity of the Lyapunov derivative and actually any other requirement, except for the guarantee that any bounded trajectory $\mathbf{x}$(t) *cannot* pass an *infinite* distance in *finite* time.

The many examples of [23] showed that the use of Ω_i was very efficient in locating equilibrium points and limit cycles. Also, as far as equilibrium points and limit cycles are concerned, no one of the prior assumptions above was actually needed, because the way the trajectory reaches the isolated equilibrium point or the limit cycle (i.e., in finite

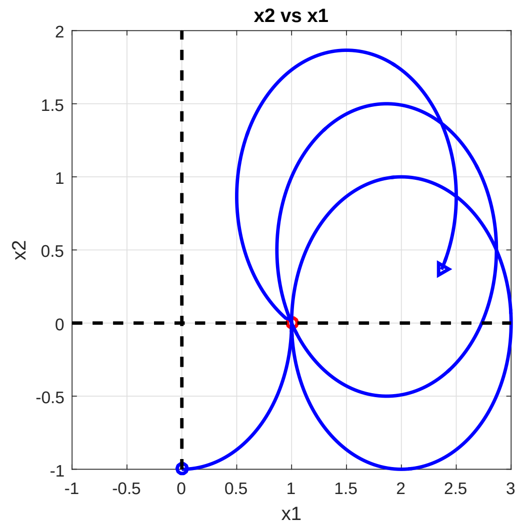

FIGURE 5. Rosette-type limit point

or infinite time) is immaterial. Assumption A was needed only in order to show that limit points of type *rosette*, i.e., those isolated rosette-type limit points, that the trajectories might reach, leave, and then come back to them an infinite number of times, must belong to $\Omega_e = \{\mathbf{x}|\dot{V}(\mathbf{x}, t) = 0\}$. However, already in [23] the usefulness of the equality relation was already considered to be very doubtful, because without supplementary knowledge about a particular system, it would be hard to differentiate between rosette-type limit points and all other points of the trajectory, with no special meaning whatsoever, where at one time or other the Lyapunov derivative just occasionally happens to be zero.

For a very simple illustration, assume that the Lyapunov function for a system in R_{20} is $V(\mathbf{x}) = V(x_1, x_2, ..., x_{20}) = x_1^2 + x_2^2 + ... + x_{20}^2$, while the derivative *along the trajectories* of the system is $\dot{V}(\mathbf{x}) = -x_1^2$. All limit points must satisfy $\dot{V}(\mathbf{x}) = -x_1^2 = 0$, which in turn results in $x_1 = 0$. However, the converse is not necessarily relevant. Although at first look the "solution" $x_1 = 0$ may look satisfactory, a second look shows that it does not have much relevance in terms of system stability. The result $x_1 = 0$ may allow finding some eventual rosette time limit points, yet this is only in principle, as this result not only contains all system trajectories that keep moving within R_{19} but also all points and *all* trajectories of the *entire* space R_{20} which, at this time or other, may occasionally cross the axis $x_1 = 0$.

Here, it could be useful to recall that in the case of the original Lyapunov Stability Theorem, where the Lyapunov derivative is negative *definite*, the conclusion $\dot{V}(\mathbf{x}) = 0$ is *equivalent* to $\mathbf{x} = 0$. Furthermore, because the vector $\mathbf{x}$ contains the *entire* dynamics of the system, there was no need to even mention that the result $\mathbf{x} = 0$ actually is *equivalent* to $\mathbf{x} \equiv 0$. However, although it may be difficult to accept it and even though some points that satisfy the simple equality $\dot{V}(\mathbf{x}) = 0$ in the *semi*definite case $\dot{V}(\mathbf{x}) \leq 0$ could also have relevance with respect to stability, most have no relevance at all and it is almost impossible to separate those that may have any relevance. In this context, although the result "trajectories ultimately end within the domain defined by $\dot{V}(\mathbf{x}) = 0$" seemed as a good result, it should have been clear that no "end of motion" is guaranteed by $\dot{V}(\mathbf{x}) = 0$ *unless* its next derivative is also zero, $\ddot{V}(\mathbf{x}) = 0$, and then next derivative and so on, or in other words unless one requires that at least the dynamics of the Lyapunov derivative vanish, or in other words that $\dot{V}(\mathbf{x}) \equiv 0$.

Nevertheless, although the great effort to guarantee the mere $\dot{V}(\mathbf{x}) = 0$ was motivated by the fear of missing those "special" rosette-type limit points, now one can see [24] that, although most probably a necessary step in the development of a complex idea, the special treatment that rosette-type limit points have received might have been exaggerated and that the eventual use of the assumptions and of $\Omega_e = \{\mathbf{x}|\dot{V}(\mathbf{x}, t) = 0\}$ could be redundant.

In retrospect, it is amazing that, while so much thought and effort had been invested to investigate what must happen *at* those particular rosette-type point locations, not much room or thought was left for all those segments that a trajectory must pass *after* leaving the limit point and *before* coming back to it. If the trajectory happens to only pass a rosette-type point and then come back to it a *finite* number of times, then this rosette-type point is not a limit point at all. Only if the trajectory revisits the point an *infinite* number of times would the rosette-type point become a limit point. In this case, however, except for those moments when the trajectory coincides with the rosette-

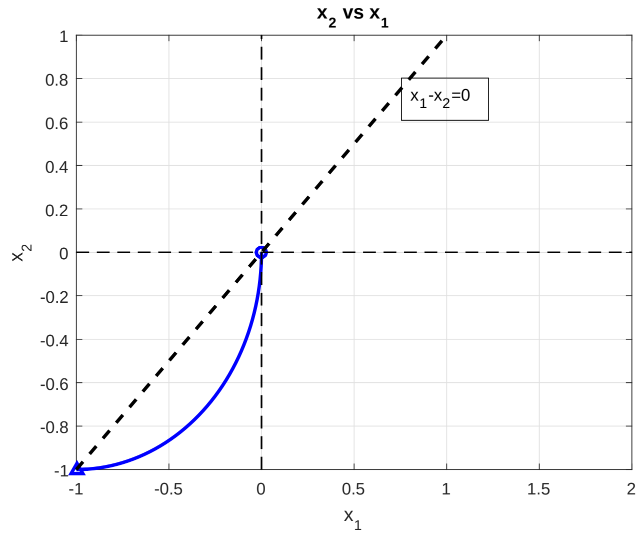

FIGURE 6. Equilibrium-point-type limit point

type point and when indeed $\dot{V}(\mathbf{x}, t) = 0$, at other times and along most sections of the trajectory we are supposed to have $\dot{V}(\mathbf{x}, t) < 0$ and this *strict* negativity of the derivative situation is repeated again and again, for all times up to infinity. In such a case, assuming that the trajectory first reaches the rosette-type limit point at time $t = t_1$, then $V(\mathbf{x}(t), t) = V(\mathbf{x}(t_1), t_1) + \int_{t_1}^{t} \dot{V}(\mathbf{x}(\tau), \tau) d\tau$ and therefore $\lim_{t\to\infty} (V(\mathbf{x}(t), t))$ would tend to $-\infty$ *unless* for any ϵ positive and arbitrarily small there exists some *finite* time t_2 such that $|\dot{V}(\mathbf{x}, t)| \leq \epsilon$ for any $t \geq t_2$. Thus, ultimately, $\dot{V}(\mathbf{x}, t)$ tends to zero *all along* the trajectory or, in other words, even what might have started looking as a rosette-type limit point *ultimately* must also belong to Ω_i, as part of a limit cycle or even as an equilibrium point.

Another point to emphasize again before going on to the theorem of next section is the very definition of limit points. A limit point, or point of accumulation, of a trajectory is such a point that any neighborhood, arbitrarily small, around it contains an infinite number of points of the trajectory. When one first hears this definition, one could be confused, because it seems pretty clear that any point of a continuous curve satisfies this condition. Therefore, as we mentioned, when one deals with trajectories, one defines a *discrete* time sequence $\{t_k\}$ and, correspondingly, discrete-time *points* on the trajectory $\{\mathbf{x}(t_k)\}$. In this context, a limit point is that point of the *discrete* sequence $\{\mathbf{x}(t_k)\}$ that any neighborhood, arbitrarily small, around it contains an infinite number of *discrete* points of the trajectory. Next, any *bounded* trajectory that leads to the creation of such an infinite number of discrete points, *must* contain at least one such accumulation. Therefore, as we only want to know where these discrete-time limit points are located, even though the ever diminishing distances between points around a limit point may sound similar to the definition of continuity, they have nothing in common with, neither do they need to even mention any continuity.

7. The new Theorem of Stability As explained above, *all* limit points of any bounded trajectory must ultimately either become equilibrium points or belong to a limit cycle. In order to formulate the new Theorem of Stability in its most general form, we only need the domain Ω_i that we define as

$$\Omega_i = \left\{ \mathbf{x} | \lim_{t\to\infty} \left(\dot{V}(\mathbf{x}, t) \right) \equiv 0 \right\}. \tag{4}$$

Now we can write the new Theorem of Stability in the following simple formulation:

Theorem 3 (*The new Theorem of Stability*) Consider the nonlinear non-autonomous system (1). Let $V(\mathbf{x})$ be a one-to-one differentiable function bounded from below. (Note that V($\mathbf{x}$) is not required to be Positive Definite.) Assume that its derivative $\dot{V}(\mathbf{x}, t)$ *along the trajectories* of (1) is Negative Semidefinite, i.e., satisfies $\dot{V}(\mathbf{x}, t) \leq 0$. Then, all limit points of any bounded trajectory $\mathbf{x}$(t) belong to the domain Ω_i [24].

Because limit points of trajectories are those points that the trajectory reaches as time approaches infinity, it is important to emphasize that *it is sufficient* if the identity relation that defines Ω_i is also only satisfied as time approaches

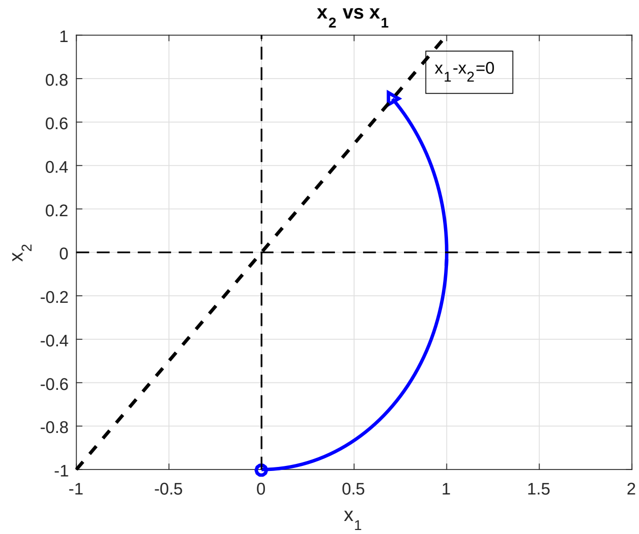

FIGURE 7. Equilibrium-type limit point

infinity. In special cases, though, as examples of [23] and [24] show, the identity could be satisfied after some finite time and even for any time, implying that for some limit points, if the trajectory starts there or reaches there at some finite time (in particular, in *autonomous* systems), it stays there thereafter. Note also that, for convenience, because $V(\mathbf{x})$ is a selected function, we assume that both $V(\mathbf{x})$ and $\dot{V}(\mathbf{x}, t)$ are continuous functions of $\mathbf{x}$. However, as shown in [23] and [24], their differentiability with respect to t implies the Dini derivatives (see for example [25, 26]). In other words, while it is nice to have continuous functions that are also differentiable in the classical sense, stability is not affected if eventual discontinuity of $\mathbf{x}(t)$ leads to discontinuity of $V(\mathbf{x}(t))$. In this context, a piece-wise continuous function may still be differentiable everywhere, even if its derivatives at the points of discontinuity are δ-functions.

In its most general form, the new Theorem of Stability does not require that $V(\mathbf{x})$ be positive definite and thus, it does not guarantee that *all* trajectories are bounded. Of course, when special selections, such as positive definite functions, functions of class K, etc. [2, 3], are available, boundedness of either some trajectories or of *all* trajectories *is* guaranteed.

It is also important to again explain the use of the identity relation $\dot{V}(\mathbf{x}, t) \equiv 0$. Because it implies that higher order derivatives must also be zero, which in turn seems to imply that the function must be infinitely differentiable, it is easy to think about counterexamples. Nevertheless, before thinking of counterexamples in the general context of mathematical functions, the reader is encouraged to check the various examples of [23] and [24] where, again *in the context of systems of equations* that are entirely defined by the first derivative $\mathbf{f}(\mathbf{x}, t)$ of the n-dimensional state-vector and, because only differentiation *along the trajectories* is concerned and only as time tends to infinity, when the Lyapunov derivative reaches zero and comes to rest there, the conditions are satisfied in most relevant cases and allow solving situations that would seem unsolvable otherwise.

Examples and Counterexamples

The initial intention of these recent works was only to continue along the lines of LaSalle and to show that stability can be obtained even without requiring the customary continuity conditions. However, because they seem to change concepts that have become well-established and commonly accepted, some common counterexamples were proposed which were meant to still show the opposite. Instead, they show how careful one must be with even basic definitions, which otherwise, may lead to false conclusions that may confuse the best of minds, if not carefully treated.

Example 1

As an attentive reader observed, if the functions $\mathbf{f}(\mathbf{x}, t)$ and $\dot{V}(\mathbf{x}, t)$ are allowed to just be general functions and are not required to be continuous, they could also contain other local discontinuities. For example, one may get

$$\dot{x}(t) = f(x, t) = f_1(x, t) + g(t_k) = -x^3(t) + g(t_k). \tag{5}$$

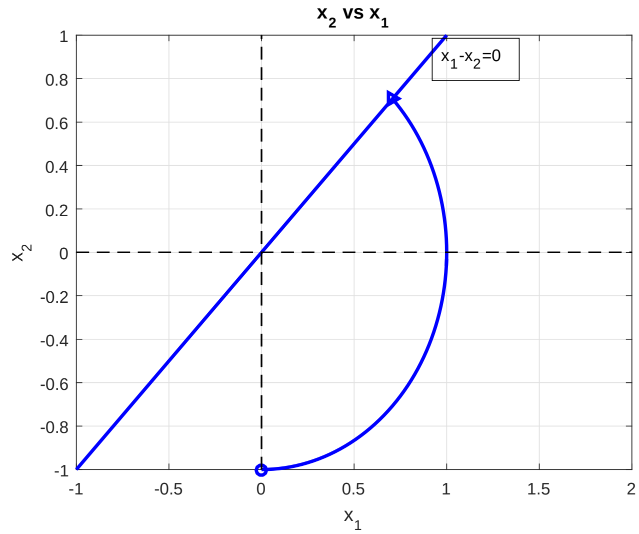

FIGURE 8. Limit-cycle-type limit points

Here, t_k represents an infinite sequence of discrete times $t_k = \{t_1, t_2, ...\}$ and $g(t_k)$ is a corresponding sequence of local spikes of finite or even infinite values, that may also show in $\dot{V}(\mathbf{x}, t)$. To avoid any possible confusion, it is worth emphasising that because, as conceived, the spikes (unlike δ-functions, etc.) do not affect the integration, the claim is that, although it is clear that the trajectory $x(t)$ converges, the Lyapunov derivative cannot reach zero, as the spikes in the derivative continue for ever.

So, in general, if this is the case for

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, t) = \mathbf{f}_1(\mathbf{x}, t) + \mathbf{g}(t_k), \tag{6}$$

apparently one cannot claim that $\mathbf{f}(\mathbf{x}, t) \to 0$ as $t \to \infty$, and therefore, one cannot claim that $\dot{V}(\mathbf{x}, t) = \frac{\partial V(\mathbf{X})}{\partial \mathbf{X}}\dot{\mathbf{x}} = \frac{\partial V(\mathbf{X})}{\partial \mathbf{X}}\mathbf{f}(\mathbf{x}, t)$ ends being zero, because these discontinuities may continue for ever.

This example was presented because, on the face of it, it seems to contradict the validity of any stability conclusions of this paper and, actually, of all stability results of the latest 40 years, which started with LaSalle's late works.

However, at a second thought, this example only shows how careful one must be with fine mathematical concepts. The formal solution of (5) is

$$x(t) = \int_0^t f(x(\tau), \tau)d\tau = \int_0^t (-x^3(\tau) + g(t_k))d\tau = -\int_0^t x^3(\tau)d\tau, \tag{7}$$

because the integral of $g(t_k)$ is zero. Therefore, because in this particular case, this very specific discontinuity does *not* affect the trajectory, then, contrary to its intention and before even mentioning stability *analysis*, this "counterexample" actually points to the fact that system stability does not necessarily have to be conditioned by the uniform continuity of some derivative. Moreover, as the derivative of a function is the result of actual differentiation, then, as peculiar as it may first look, we get

$$\frac{d(x(t))}{dt} = -\frac{d(\int_0^t x^3(\tau)d\tau)}{dt} = -x^3(t), \tag{8}$$

because the integral $x(t)$ contains no hint on the presence of the additional terms in the integrant.

In general, if in (6), $\mathbf{f}_1(\mathbf{x}, t)$ is a piece-wise uniformly continuous function that may also contain δ-functions, and $\mathbf{g}(t_k)$ is a locally-defined function that does *not* contribute to the integral, then

$$\mathbf{x}(t) = \int_0^t \mathbf{f}(\mathbf{x}(\tau), \tau)d\tau = \int_0^t \mathbf{f}_1(\mathbf{x}(\tau), \tau)d\tau + \int_0^t \mathbf{g}(t_k)d\tau = \int_0^t \mathbf{f}_1(\mathbf{x}(\tau), \tau)d\tau. \tag{9}$$

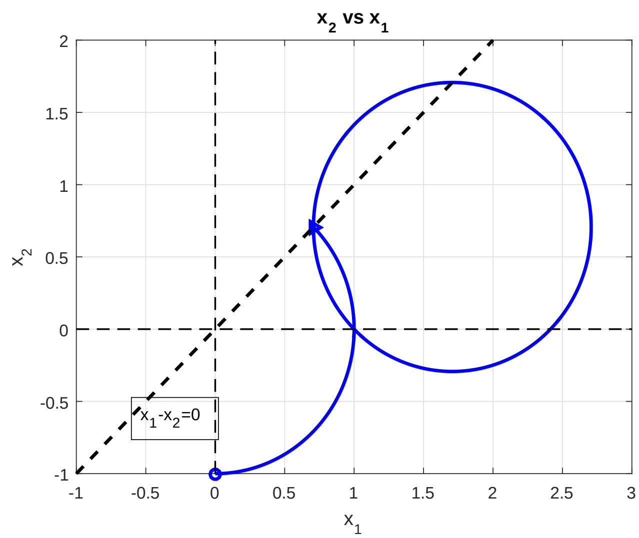

FIGURE 9. Limit-cycle-type limit points

From here, the derivative $\frac{d\mathbf{x}}{dt}$ is obtained by differentiation

$$\frac{d\mathbf{x}}{dt} = \frac{d\left(\int_0^t \mathbf{f}_1(\mathbf{x}(\tau), \tau)d\tau\right)}{dt} = \mathbf{f}_1(\mathbf{x}, t) \tag{10}$$

and those terms that do not contribute to the integral do not appear in the derivative.

Therefore, even if at fist look this may look bizarre,

$$\dot{V}(\mathbf{x}, t) = \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt} = \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\mathbf{f}_1(\mathbf{x}, t)$$

and *not* $\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}, t)$ as one might have thought.

This example only illustrates that, in spite of our usual and well-established notation of differential equations, its right term is only an *integrant*, which cannot be called "*derivative*" *unless* it can be obtained by *actually differentiating* the integral function.

Because, as mentioned, the recent publications on stability analysis only intended to relax the strict uniform continuity conditions of Barbalat and allow such discontinuities as pulses, square-waves or, at most, δ-functions, first thought was to maybe restrict the scope of stability analysis and to exclude such strictly locally defined functions. However, because the integral of those local discontinuities is zero, they do not affect either $\mathbf{x}(t)$, the integral of $\mathbf{f}(\mathbf{x}, t)$, or $V(\mathbf{x}, t)$, the integral of $\dot{V}(\mathbf{x}, t)$. Therefore, as they do *not* affect the stability, one may not see any need to restrict the scope of the theorems to functions that cannot contain such sequences.

In retrospect, this last (assumed) counterexample was very important, as it forced a thorough revision [27] of all those customary counterexamples which are used to motivate the need for Barbalat's Lemma and uniform continuity when dealing with stability of nonlinear systems.They lead to such authoritative claims as "of course a function can reach a constant limit while its derivative may keep going up-and-down" or the opposite "of course the derivative of a function can reach a constant limit while the function itself may keep going up-and-down." A thorough analysis of those counterexamples [27] shows that they all use similarly, *apparently* well-established mathematical formulas, yet automatically used where they do not hold and, as a result, they are all (surprisingly) wrong.

Example 2

The example

$$\begin{aligned} \dot{e}(t) &= -e(t) + \theta w(t) \\ \dot{\theta}(t) &= -e(t)w(t) \end{aligned} \tag{11}$$

of a very simple adaptive control system was used in [6] (and often repeated in many other publications) for an illustration of Barbalat's Lemma application to stability analysis. Here, $e(t)$ is the tracking error, while $\theta(t)$ is the

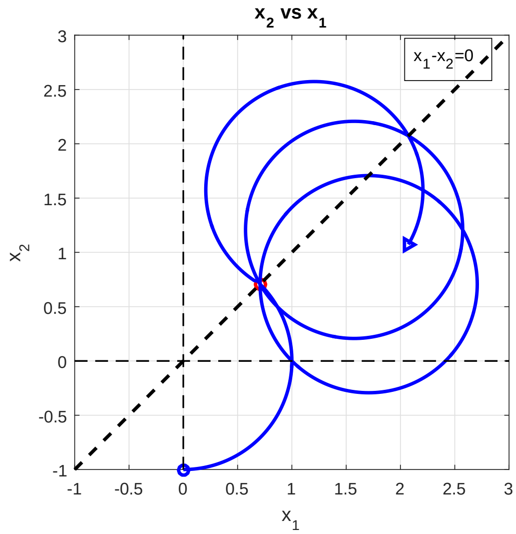

FIGURE 10. Rosette-type limit point

adaptive gain. Selecting the Lyapunov function $V(t) = e^2(t)+\theta^2(t)$ results in the derivative $\dot{V}(t) = -2e^2(t) \leq 0$. Because, as stated in [6], "one cannot use the Invariance Principle to conclude the convergence of e(t), because the dynamics is nonautonomous," [6] imposes conditions on the input command w(t) that would guarantee *uniform continuity* of Lyapunov derivative and therefore, application of Barbalat's Lemma indicates that $e \to 0$ as $t \to \infty$. Because the system satisfies (the fairly mild) Assumption 1 for any bounded w(t), LaSalle's Invariance Principle directly tells us that the state-vector $\{e, \theta\}$ tends towards the domain defined by $W(e(t)) = -2e^2(t) = 0$ which immediately results in same result $e \to 0$ as $t \to \infty$ as above, without requiring that $W(e(t))$ necessarily be uniformly continuous. Still, as discussed above, this result is more restricted than it may look at first sight. Not only there is no clear conclusion that can be drawn with respect to θ, but even $e \to 0$ is not a clear result, because it does not say whether the trajectory ends at a given point on the $e(t) = 0$ axis, just some points that the trajectories pass when they occasionally cross the axis, or if ultimately the entire motion occurs only along this axis.

Instead, a satisfactory result *is* provided by the new Theorem of Stability. The clear satisfaction of Assumption A for any bounded w(t) directly tells us that the state-vector $\{e, \theta\}$ ends within the domain defined by $W(e(t)) = -2e^2(t) \equiv 0$ which immediately results in same result $e(t) = 0$ as above. However, because the conclusion of the new Theorem of Stability is $W(e(t)) \equiv 0$, this in turn implies that the derivatives of $e(t)$ are also zero, or $\dot{e}(t) = 0$. For a system of two equation, this first of all is sufficient to imply that trajectories *indeed* reach $e(t) = 0$ in order to stay there. Moreover, substituting in both equations leads to the new results $\dot{\theta}(t) = 0$ and $\theta w(t) = 0$. The fairly straightforward result for this *nonautonomous* system is that the tracking error ends at zero and *stays* at zero, while the adaptive gain $\theta(t)$ ends at some constant value. If the input command $w(t)$ ends at a nonzero value, then the gain $\theta(t)$ ends at zero, yet even if $w(t)$ ends at a zero, the gain $\theta(t)$ still ends at some constant value, without requiring any of the "customary" persistent excitation conditions [24].

Similar results have been obtained for the Simple Adaptive Control (SAC) methodology in general, where results of the form (3) not only imply that the following error indeed ultimately vanishes, but also that the adaptive gains ultimately reach constant values [22, 28, 29, 30].

Example 3

As mentioned above, the authors of [21] use examples where the results of the analysis in the semidefinite case is not satisfactory. To help reaching better conclusions, they use a few Lyapunov functions for same system. In combination with the 'lim inf' results, those Lyapunov functions allow better stability analysis results.

Still, at least in the case when the one Lyapunov function ends with the negative semidefinite Lyapunov derivative, the recent developments *do* allow obtaining conclusive results without requiring any other Lyapunov function and any other mathematical tools.

For illustration, we use this example from [21], which is the 2-dimensional representation describing the Duffing

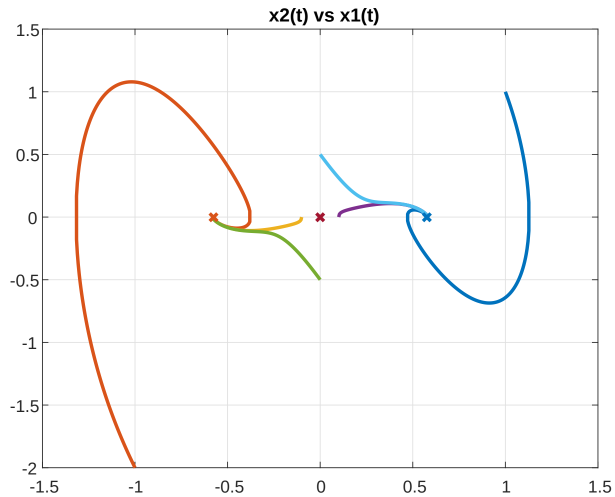

FIGURE 11. Example 3 – Various trajectories

oscillator

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha x_1 - \beta x_2 - \gamma x_1^3 \end{aligned} \tag{12}$$

with equilibrium points $\{x_1, x_2\} = \{0, 0\}$ and $\{x_1, x_2\} = \{\pm\sqrt{\alpha/\gamma}, 0\}$.

In [21], they first use the Lyapunov function

$$V = \frac{1}{\beta}\left(\gamma\frac{x_1^4}{4} - \alpha\frac{x_1^2}{2} + \frac{x_2^2}{2}\right) \tag{13}$$

The derivative of the Lyapunov function is $\dot{V} = -x_2^2$. As this leads to the customary result $x_2 \to 0$ as $t \to \infty$, which the authors of [21] correctly consider to not be satisfactory, they follow [20] and select a second Lyapunov function. Then, developing 'lim inf' results and after *some* algebra, they ultimately manage to get more conclusive results, namely, that trajectories converge to at least one equilibrium point.

Instead, the new Theorem of Stability needs just one single Lyapunov function to provide a complete response, if it only results in a negative semidefinite derivative. As it actually ends with $x_2 \equiv 0$, it first of all results in the same $x_2 = 0$, but it also implies $\dot{x}_2 = 0$. Together, they not only guarantee that x_2 reaches zero *in order to stay* there, but they also lead to $\dot{x}_1 = 0$ from first equation and then, from second equation, they end with $x_1 = 0$ or $x_1 = \pm\sqrt{\alpha/\gamma}$.

Moreover, because the Lyapunov derivative is negative at any other point besides these three limit points, all trajectories converge to at least one equilibrium point. Still, the nonlinear world can be complex and, unlike the case of a single equilibrium point, the direct result of this analysis does not tell to which equilibrium the various trajectories arrive. As simulations with $\alpha = 1, \beta = 2, \gamma = 3$ (Figure 11) and with $\alpha = 5, \beta = 2, \gamma = 1$ (Figure 12) show, the origin is unstable, while the other two are stable equilibrium points.

One can check the (in)stability of the origin by linearizing the equation in the strict neighborhood of the origin. This simply results in the linear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha x_1 - \beta x_2 \end{aligned} \tag{14}$$

and it is easy to see that, for any positive values of α and β, it indeed has a RHP pole.

More interesting is the case when some discontinuous function seems to violate all necessary conditions for all (other) customary stability analysis. For illustration, we define s(a,b,T) the square-wave of lower value a, upper value b and period T and test stability of the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha x_1 - \beta x_2 s(0, 1, 2) - \gamma x_1^3 \end{aligned} \tag{15}$$

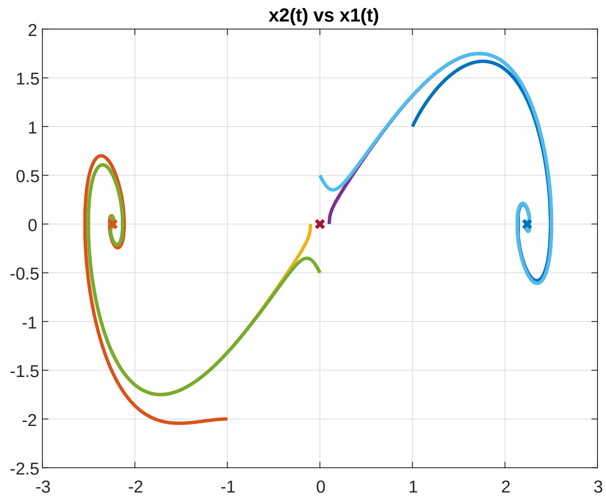

FIGURE 12. Example 3 – Various trajectories

which, because of the time-varying function, has become nonautonomous. The derivative of the same Lyapunov function is now $\dot{V} = -x_2^2 s(0, 1, 2)$ and this seems to violate any assumably necessary conditions and to end any stability analysis. However, as it satisfies the assumption of the new Theorem of Stability, it ends with same result $x_2 \equiv 0$, which in turn leads to same conclusion as above about the limit points of the system trajectories.

Example 4

In spite of the many examples of [23, 24, 27], new examples are continuously devised to deny the validity of the entire approach presented here and in the relevant references. In reality, they actually shows how confusing things can be and how careful one must be when dealing with basic things.

The example is $\dot{x} = -e^{-t}$ with the obvious solution $x(t) = x_o + e^{-t}$. So, it is clear that any finite x is a limit point of some trajectory.

However, the claim is that, if we consider the Lyapunov function $V(x) = e^x$, then we have $\dot{V}(t, x) = -e^{x-t} \leq 0$. This result seemed sufficient to conclude that no x can make $\dot{V}(t, x) = 0$ and therefore, the example may seem to justify the (apparently obvious) claim that $\Omega_i = \emptyset$ and therefore, that, even starting with LaSalle's 1976-1980 works, the new stability theory does not reveal any limit point and, therefore, cannot be trusted and could be totally wrong, as it does not hold even in this simple case.

Although similar examples have already been treated in the references, this case is still worth our attention, because it again shows how confusing these simple things, such the very definition of limit points in nonautonomous systems, can be for all of us. Writing again $\dot{V}(t, x) = -e^{x-t} = -e^{-t}e^x$, this indeed at first look may seem to show that no x can make the derivative equal zero, yet this is only true at finite times. Although in autonomous systems, where the time-variable does not explicitly appear, and even in some nonautonomous systems, trajectories may reach equilibrium points at finite times and remain there, the minimal and important property of equilibrium points is that the trajectories reaches them as time tends to infinity. As time tends to infinity, the coefficient e^{-t} vanishes and therefore the stability analysis conclusion is that Ω_i contains *all* finite x, as indeed it should be.

Example 5

For a more elaborate example where the results of the analysis in the semidefinite case is not satisfactory, the authors of [21] use the example

$$\begin{aligned}
\dot{x}_1 &= \eta x_1(1 - x_1^2 - x_2^2)x_3^r + k_1 x_2 \left[\Psi(x_{1+}) + x_3^p\right] \\
\dot{x}_2 &= \eta x_2(1 - x_1^2 - x_2^2)x_3^r - k_1 x_1 \left[\Psi(x_{1+}) + x_3^p\right] \\
\dot{x}_3 &= -k_2 x_3^q
\end{aligned} \tag{16}$$

where $(x_1, x_2, x_3) \in R3$, $x_{1+} = max\{x_1, 0\}$, Ψ is a positive definite function, p and $r \geq p$ are positive even integers, q is a positive odd integer, k_1, k_2 are positive and $\eta \geq 0$. The set of equilibrium points is given by $\{(x_1, x_2, x_3) : x_{1+} = x_3 = 0\}$.

We also assume that $q > r$.

In [21], they first use the Lyapunov function

$$V = x_1^2 + x_2^2 + x_3^2 \tag{17}$$

The derivative of the Lyapunov function is

$$\dot{V} = -\eta(x_1^2 + x_2^2)(x_1^2 x_2^2 - 1)x_3^r - k_2 x_3^{q+1}. \tag{18}$$

At this point the authors of [21] only conclude that all solutions are bounded and then move to devising and using more Lyapunov functions and their derivatives, in order to get more conclusive results.

Instead, the new Theorem of Stability directly implies that al trajectories end at $\dot{V} \equiv 0$, which first of all implies that $\dot{V}$ tends to zero *in order to stay* there. As we show below, a complex nonlinear system requires fine attention. Indeed, a first look at

$$\eta(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1)x_3^r + k_2 x_3^{q+1} = 0 \tag{19}$$

or

$$[\eta(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1) + k_2 x_3^{q-r+1}]x_3^r = 0 \tag{20}$$

seems to imply that we can have either $x_3 = 0$, or, if $x_3 \neq 0$, then $(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1) + k_2 x_3^{q-r+1} = 0$.

However, a second look shows that $x_3 = 0$ also results in $\dot{x}_3 = 0$. In other words, when trajectories reach $x_3 = 0$, they remain at $x_3 = 0$. Because all limit points then satisfy $x_3 = 0$, the first term, $(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1)$, is allowed to get any real value. Let us denote this value w and this results in

$$(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1) = (x_1^2 + x_2^2)^2 + (x_1^2 + x_2^2) = w \tag{21}$$

or

$$(x_1^2 + x_2^2)^2 + (x_1^2 + x_2^2) - w = 0. \tag{22}$$

From here,

$$x_1^2 + x_2^2 = \frac{-1 \pm \sqrt{1 + 4w}}{2} \tag{23}$$

Because $x_1^2 + x_2^2$ must be real and nonnegative, w must also be nonnegative and the final conclusion is that the ultimate limit sets of the various trajectories are circles with the center at the origin in the plane $x_3 = 0$ and their radii depend on the initial conditions. All that is the direct result of using just *one* Lyapunov function with a negative semidefinite derivative.

Example 6

For another elaborate example, where the results of the analysis in the semidefinite case is not even negative semidefinite, we select the system

$$\begin{aligned} \dot{x} &= e^{-t}x - \left(x^3 + xy^2\right) \\ \dot{y} &= e^{-t}y - \left(x^2 y + y^3\right) \end{aligned} \tag{24}$$

We try the "natural" selection

$$V(x) = \left(x^2 + y^2\right)/2 \tag{25}$$

and get

$$\dot{V}(x, y, t) = x\dot{x} + y\dot{y} = x\left(e^{-t}x - \left(x^3 + xy^2\right)\sin^2 t\right) + y\left(e^{-t}y - \left(x^y y + y^3\right)\sin^2 t\right)$$

$$\dot{V}(x, y, t) = e^{-t}x^2 - \left(x^4 + x^2y^2\right)\sin^2 t + e^{-t}y^2 - \left(x^2y^2 + y^4\right)\sin^2 t$$

$$\dot{V}(x, y, t) = e^{-t}\left(x^2 + y^2\right) - \left(x^4 + 2x^2y^2 + y^4\right)\sin^2 t$$

$$\dot{V}(x, y, t) = e^{-t}\left(x^2 + y^2\right) - \left(x^2 + y^2\right)^2 \sin^2 t$$

This result seems to be a disappointment, because the Lyapunov derivative is not Negative Definite or even Negative Semidefinite, as we got

$$\dot{V}(x, y, t) = W_1 + W_2$$

$$W_1 = -\left(x^2 + y^2\right)^2 \sin^2 t \le 0$$

$$W_2 = e^{-t}\left(x^2 + y^2\right)$$

Because the Lyapunov derivative contains the positive term W_2, it seems to end any stability analysis. However, the Lyapunov derivative satisfies the conditions of the new Theorem of Stability, where the first term, W_1, is negative semidefinite, while the non-negative second term, W_2, is bounded along all bounded trajectories and vanishes in time. Therefore, all *bounded* trajectories of the system end within the domain Ω_i defined by

$$\Omega_1 = \{\{x, y\} | W_1 \equiv 0\} = \left\{\{x, y\} | \left(x^2 + y^2\right)^2 \sin^2 t \equiv 0\right\}$$

At first sight, except for its pure mathematical significance, this result seems to have no practical relevance, because it does not show what trajectories, if any, are bounded.

However, now we may try to use Gronwall-Bellman Lemma. For

$$\dot{V}(t) \le f(t)V(t)$$

Gronwall-Bellman Lemma ends with

$$V(t) \le V(0) exp\left(\int_0^t f(s)ds\right).$$

In our specific case, we can ignore the negative term and get

$$\dot{V}(x, y, t) \le e^{-t}\left(x^2 + y^2\right)$$

which in our case ends with

$$V(t) \le V(0) exp\left(\int_0^t e^{-s} ds\right) = V(0) \exp\left(e^0 - e^{-t}\right) \le V(0)$$

and shows that the Lyapunov function V is bounded and so, that all trajectories are bounded.

Now, we go back to see the significance of domain Ω_i defined by

$$\Omega_1 = \{\{x, y\} | W_1 \equiv 0\} = \left\{\{x, y\} | \left(x^2 + y^2\right)^2 \sin^2 t \equiv 0\right\}$$

It requires

$$-\left(x^2 + y^2\right)^2 \sin^2 t \equiv 0,$$

which is satisfied by $x^2 + y^2 \equiv 0$, which in turn is equivalent to $x = 0$, $y = 0$. In other words, in spite of the apparently complex and inconclusive first impression, the system *is globally asymptotically stable*.

Conclusion

This paper revisits the Lyapunov style stability analysis for the customary situations when the Lyapunov derivative is at most negative semidefinite. After asking whether stability analysis can be simplified, this paper shows that, following along the lines of some of the late results of LaSalle, new developments manage not only to simplify nonlinear systems stability analysis, but also to reach satisfactory conclusions regarding the stability and the ultimate behavior of trajectories of the system under analysis.

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