12-A1

优先级队列

概述：需求与动机

I cannot choose the best.

The best chooses me.

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循优先级访问

❖ 应用举例

- 离散事件模拟
- 操作系统：任务调度/中断处理/MRU/...
- 输入法：词频调整

❖ 极值元素：须反复地、快速地定位

集合组成：可动态变化

元素优先级：可动态变化

❖ 作为底层数据结构所支持的高效操作

是很多高效算法的基础

- 内部、外部、在线排序
- 贪心算法：Huffman编码、Kruskal
- 平面扫描算法中的事件队列
- ...

优先级队列

```
template <typename T> struct PQ { //priority queue
    virtual void insert( T ) = 0;
    virtual T getMax() = 0;
    virtual T delMax() = 0;
}; //作为ADT的PQ有多种实现方式，各自的效率及适用场合也不尽相同
```

- Stack和Queue，都是PQ的特例——优先级完全取决于元素的**插入次序**
- Steap和Queap，也是PQ的特例——插入和删除的**位置受限**

12-A2

优先级队列

概述：基本实现

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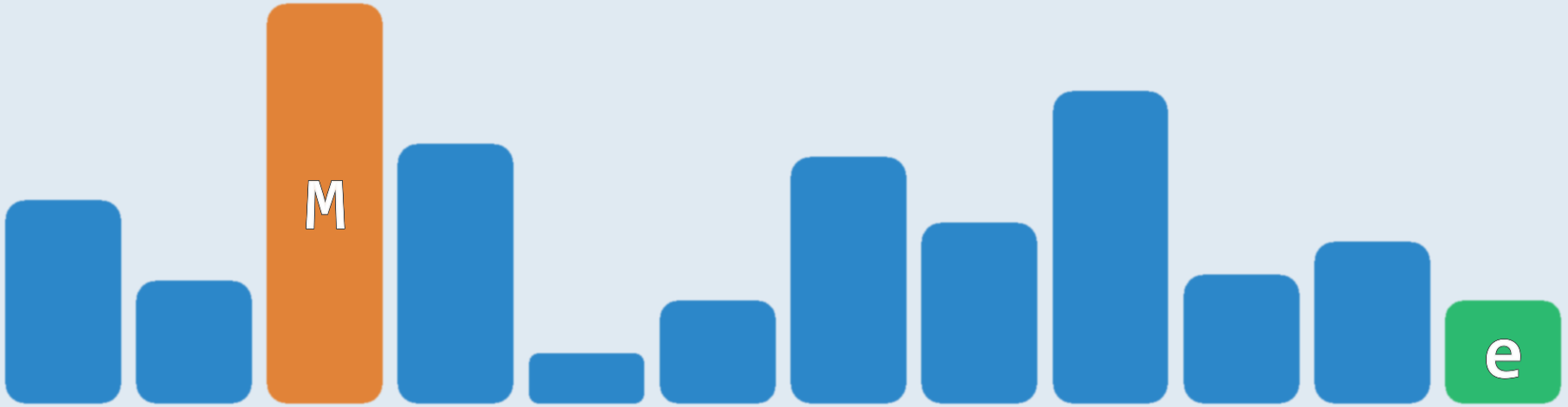
大兒孔文舉，小兒楊德祖。與子碌碌，莫足數也

Vector



getMax()	delMax()	insert()
traverse() $\Theta(n)$	remove(traverse()) $\Theta(n) + \mathcal{O}(n) = \Theta(n)$	insertAsLast(e) $\mathcal{O}(1)$

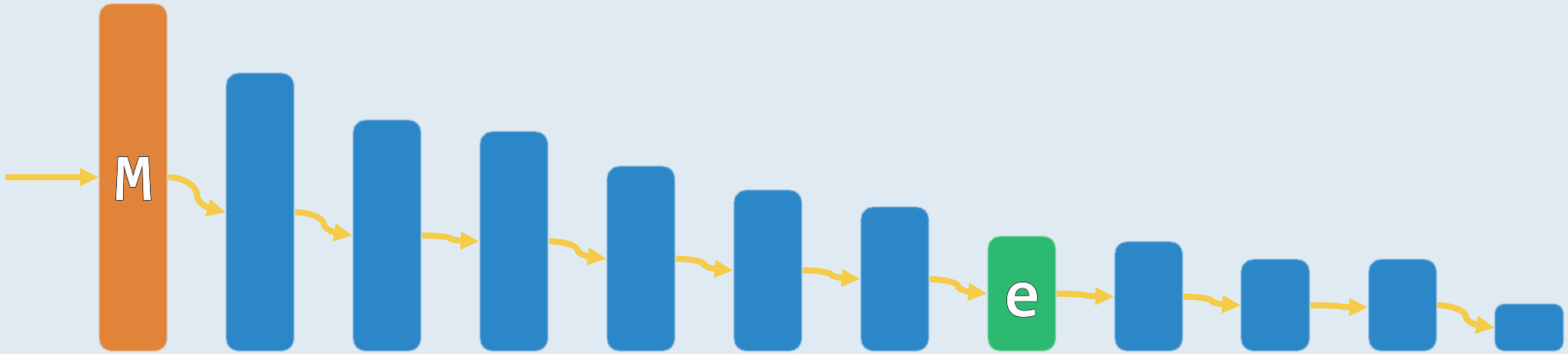
Sorted Vector

getMax()	delMax()	insert()
[n – 1] $O(1)$	remove(n – 1) $O(1)$	insert(1 + search(e), e) $O(\log n) + O(n) = O(n)$

List

getMax()	delMax()	insert()
traverse() Θ(n)	remove(traverse()) Θ(n) + *O*(1) = Θ(n)	insertAsFirst(e) *O*(1)

Sorted List



getMax()	delMax()	insert()
first() $O(1)$	remove(first()) $O(1)$	insertA(search(e), e) $O(n) + O(1) = O(n)$

BBST

- AVL、Splay、Red-black：三个接口均只需$O(\log n)$时间

  但是，BBST的功能远远超出了PQ的需求...

- PQ = 1 × insert() + 0.5 × search() + 0.5 × remove()

- 若只需查找极值元，则不必维护所有元素之间的全序关系，偏序足矣

- 因此有理由相信，存在某种更为简单、维护成本更低的实现方式

  使得各功能接口的时间复杂度依然为$O(\log n)$，而且实际效率更高

- 当然，就最坏情况而言，这类实现方式已属最优——为什么？

统一测试

```
template <typename PQ, typename T> void testHeap( int n ) {
   T* A = new T[ 2 * n / 3 ]; //创建容量为2n/3的数组，并
   for ( int i = 0; i < 2 * n / 3; i++ ) A[i] = dice( (T) 3 * n ); //随机化
   PQ heap( A + n / 6, n / 3 ); delete [] A; //Robert Floyd
   while ( heap.size() < n ) //随机测试
      if ( dice( 100 ) < 70 ) heap.insert( dice( (T) 3 * n ) ); //70%概率插入
      else if ( ! heap.empty() ) heap.delMax(); //30%概率删除
   while ( ! heap.empty() ) heap.delMax(); //清空
}
```

12-B1

优先级队列

完全二叉堆：结构

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逊问曰：“何人将乱石作堆？如何乱石堆中有杀气冲起？”

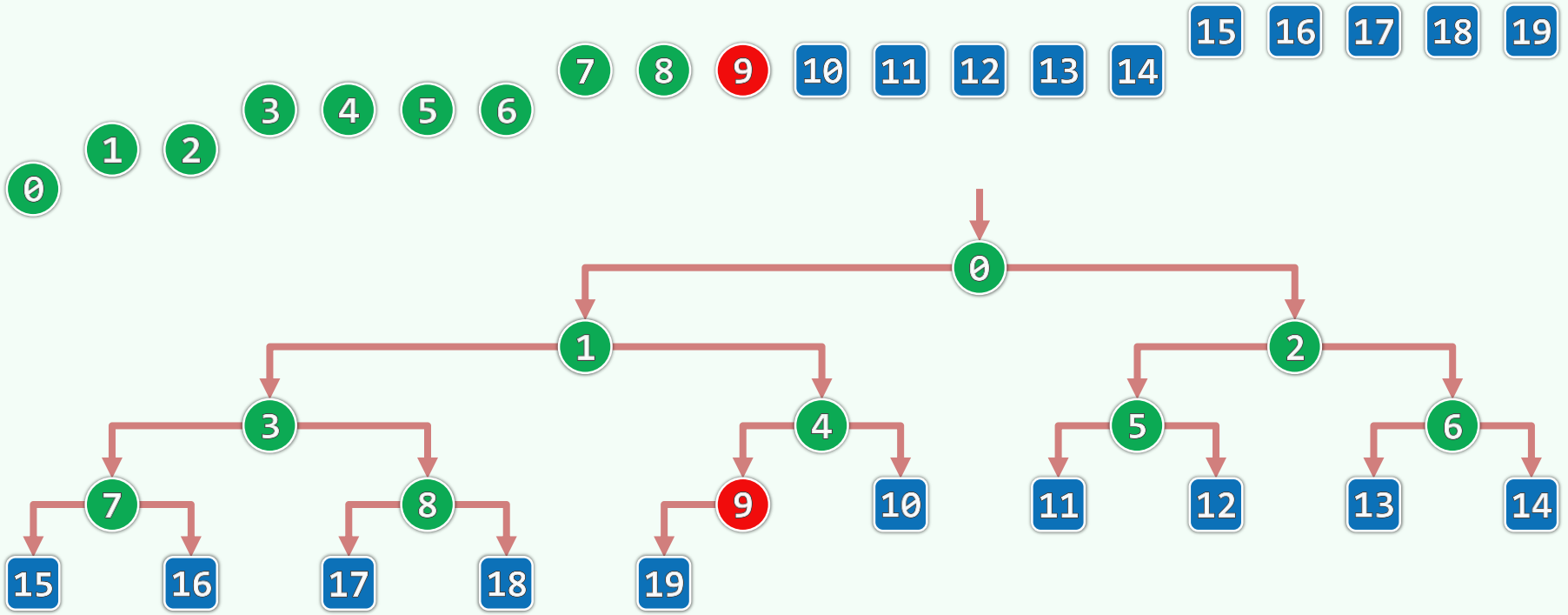
结构性：逻辑元素、物理节点依层次遍历次序彼此对应

```
#define  Parent(i)  ( ((i) - 1) >> 1 )
#define  LChild(i)  ( 1 + ((i) << 1) )
#define  RChild(i)  ( (1 + (i)) << 1 )
```

- 逻辑上，等同于完全二叉树

  物理上，直接借助向量实现

- 内部节点的最大秩 $= \lfloor \frac{n-2}{2} \rfloor = \lceil \frac{n-3}{2} \rceil$



PQ_ComplHeap = PQ + Vector

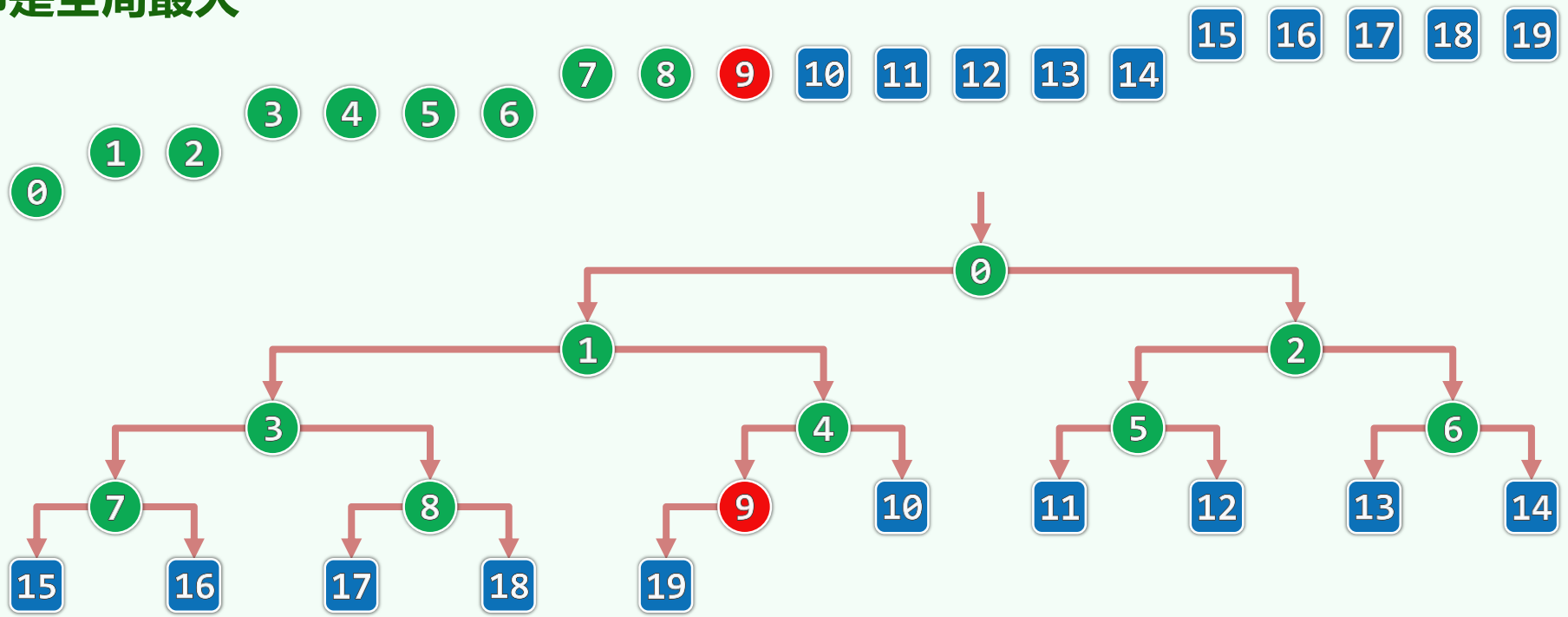
```
❖ template <typename T> struct PQ_ComplHeap : public PQ<T>, public Vector<T> {
      PQ_ComplHeap( T* A, Rank n ) { copyFrom( A, 0, n ); heapify( _elem, n ); }
      void insert( T ); T getMax(); T delMax();
  };
❖ template <typename T> Rank percolateDown( T* A, Rank n, Rank i ); //下滤
❖ template <typename T> Rank percolateUp( T* A, Rank i ); //上滤
❖ template <typename T> void heapify( T* A, Rank n); //Floyd建堆算法
```

堆序性

❖ `template <typename T> T PQ_ComplHeap<T>::getMax() { return _elem[0]; }`

❖ 只要 0<i，必满足 H[i] ≤ H[Parent(i)]

故H[0]即是全局最大



12-B2

优先级队列

完全二叉堆：插入

时迁看见土地庙后一株大柏树，便把两只腿夹定，一节节爬将上去树头顶，骑马儿坐在枝柯上。

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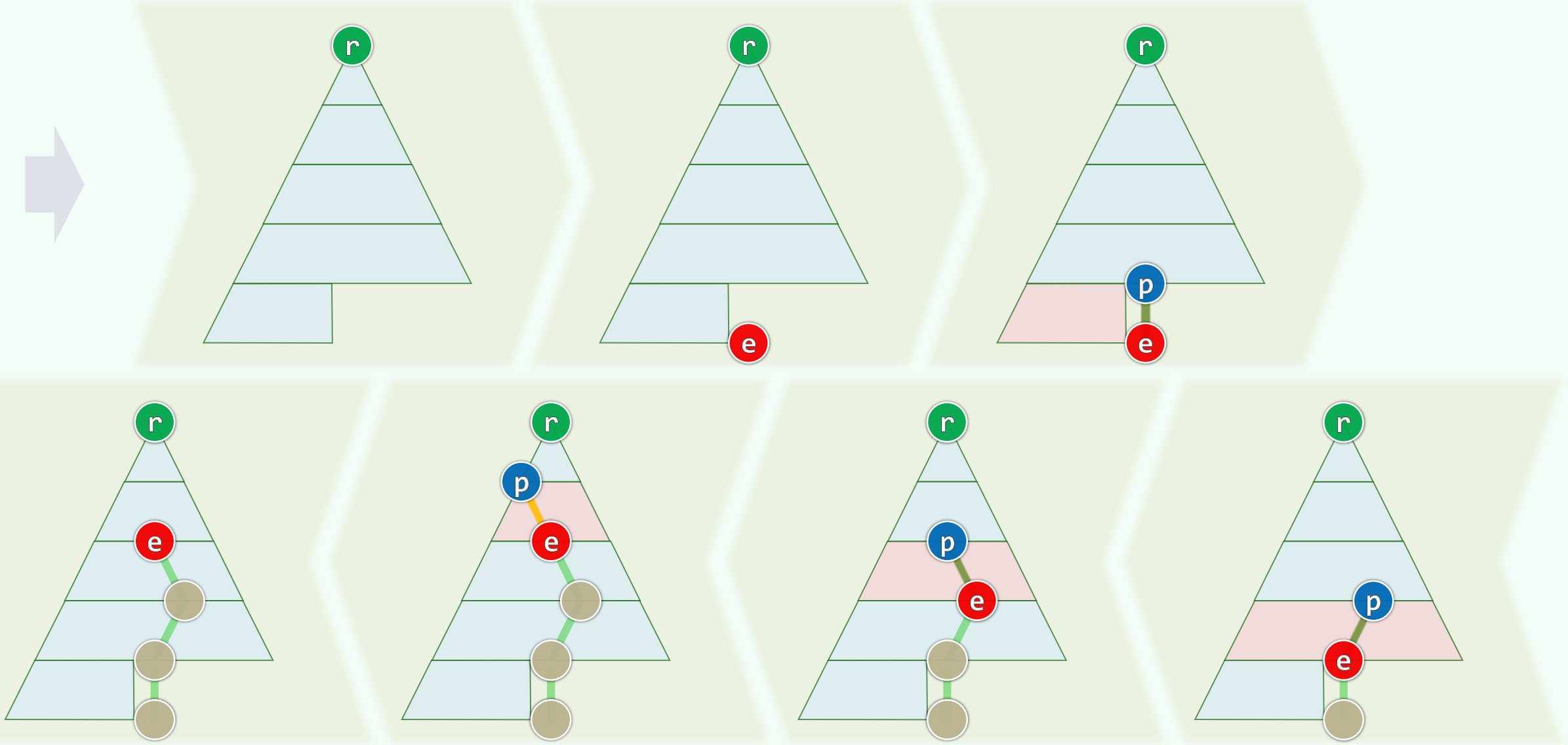
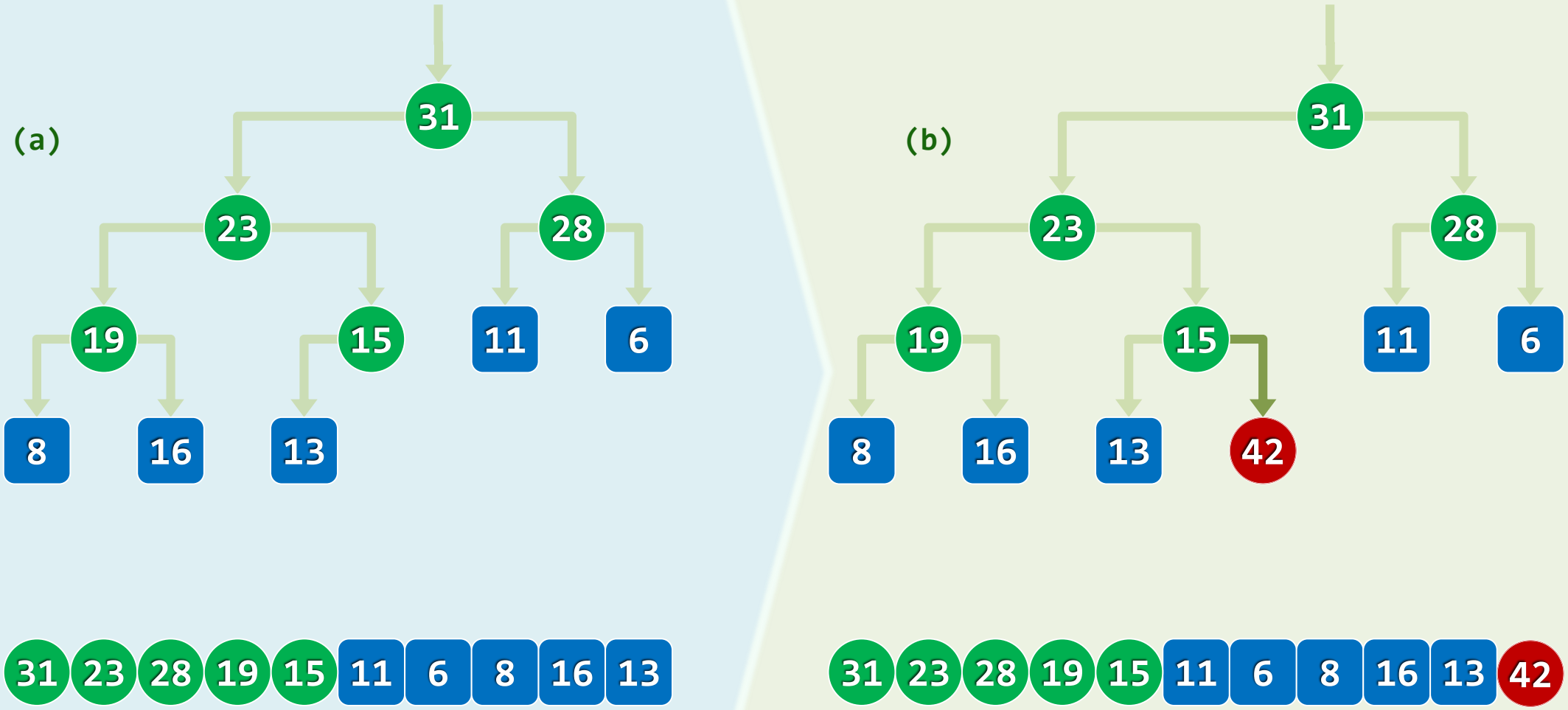
算法：逐层上滤



实例（1/5）


(a)
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(b)
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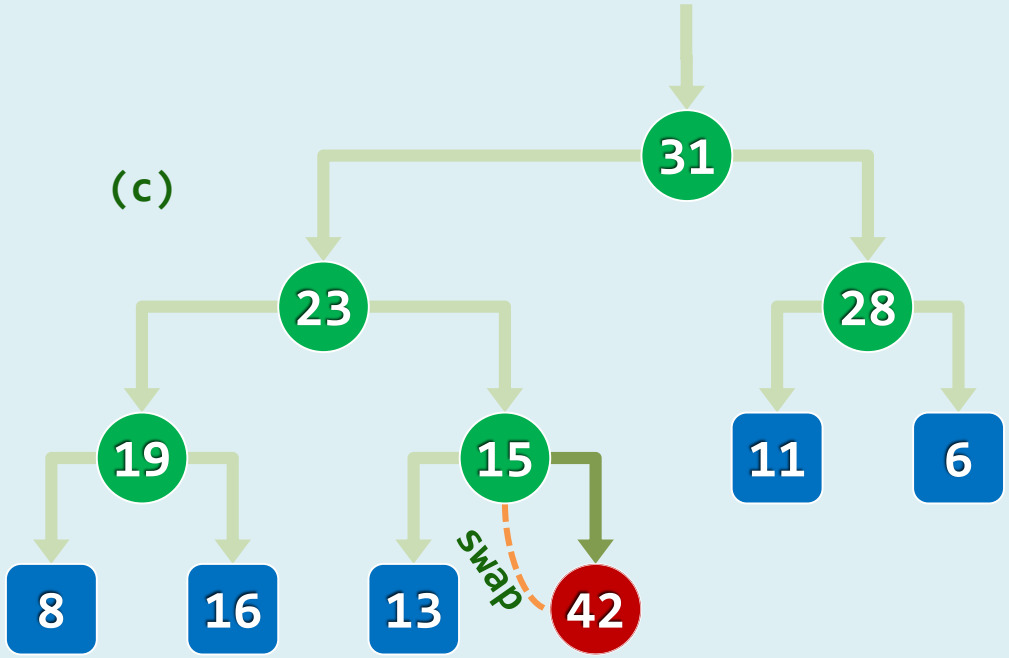
31 23 28 19 15 11 6 8 16 13 42

实例（2/5）


(c)
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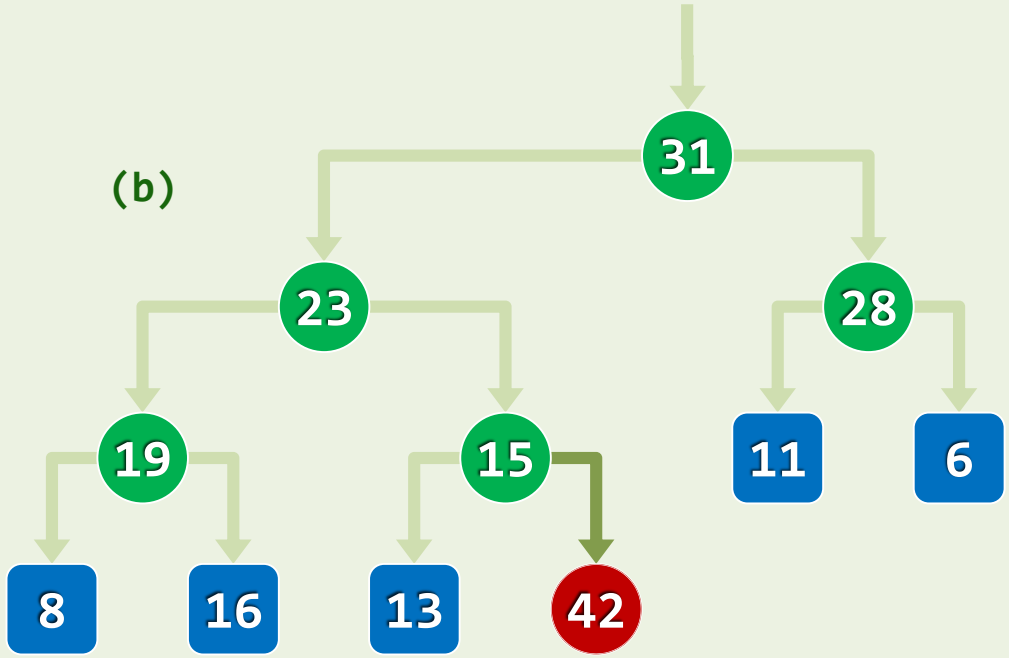
16
13
swap
42


(b)
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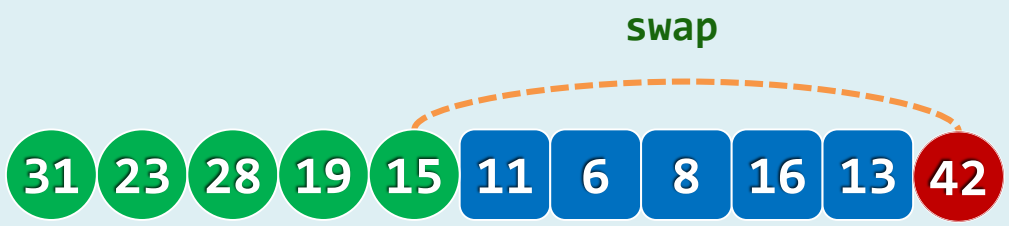
42


swap
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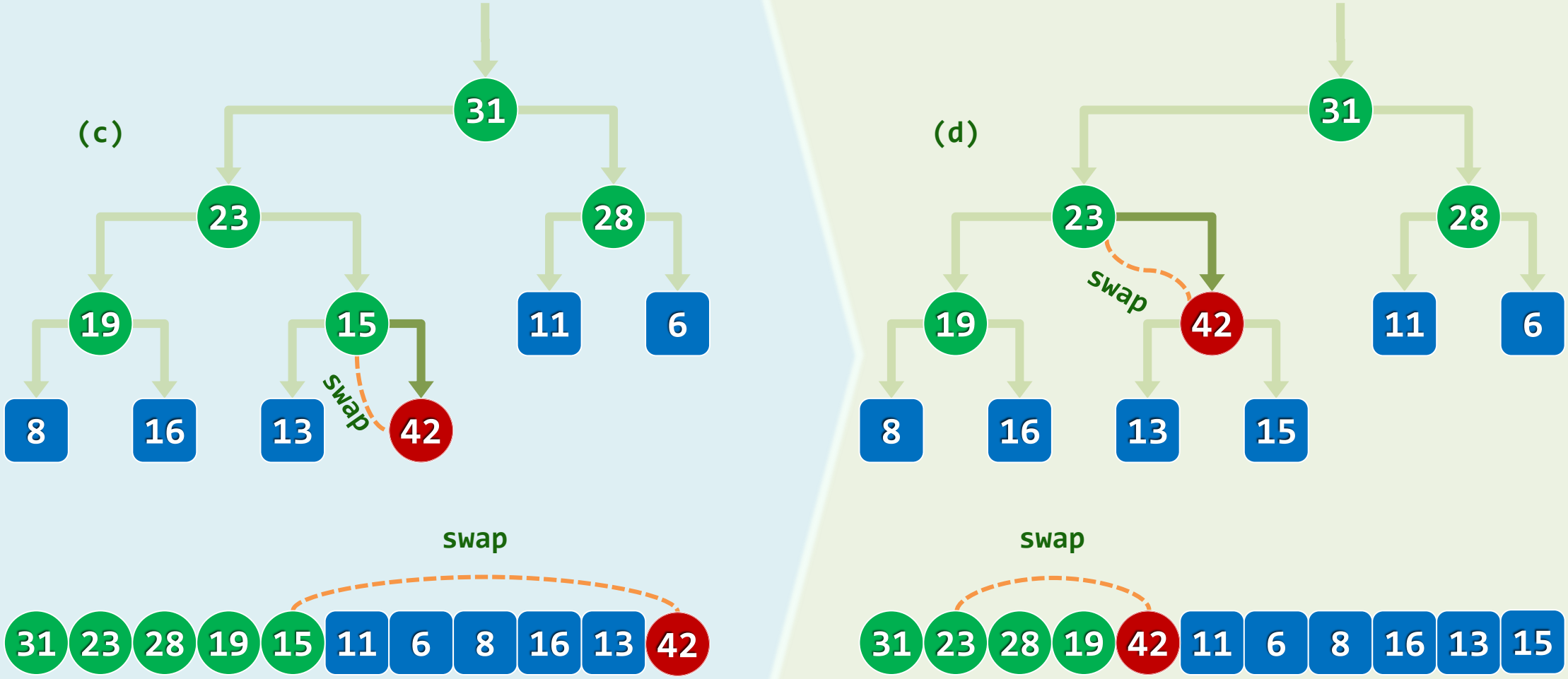
42

实例（3/5）


(c)
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11
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8
16
13
swap
42
swap
31 23 28 19 15 11 6 8 16 13 42
(d)
31
23
28
swap
19
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11
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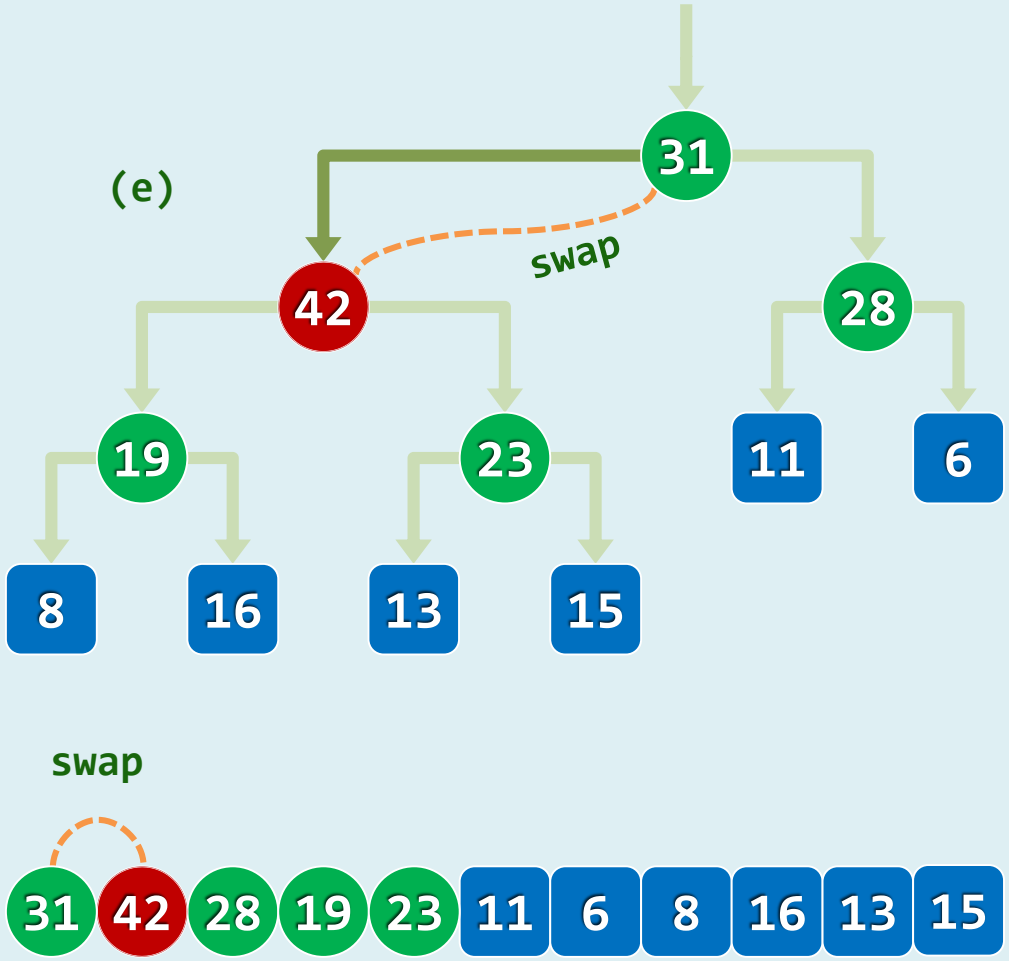
8
16
13
15
swap
31 23 28 19 42 11 6 8 16 13 15

实例（4/5）


(e)
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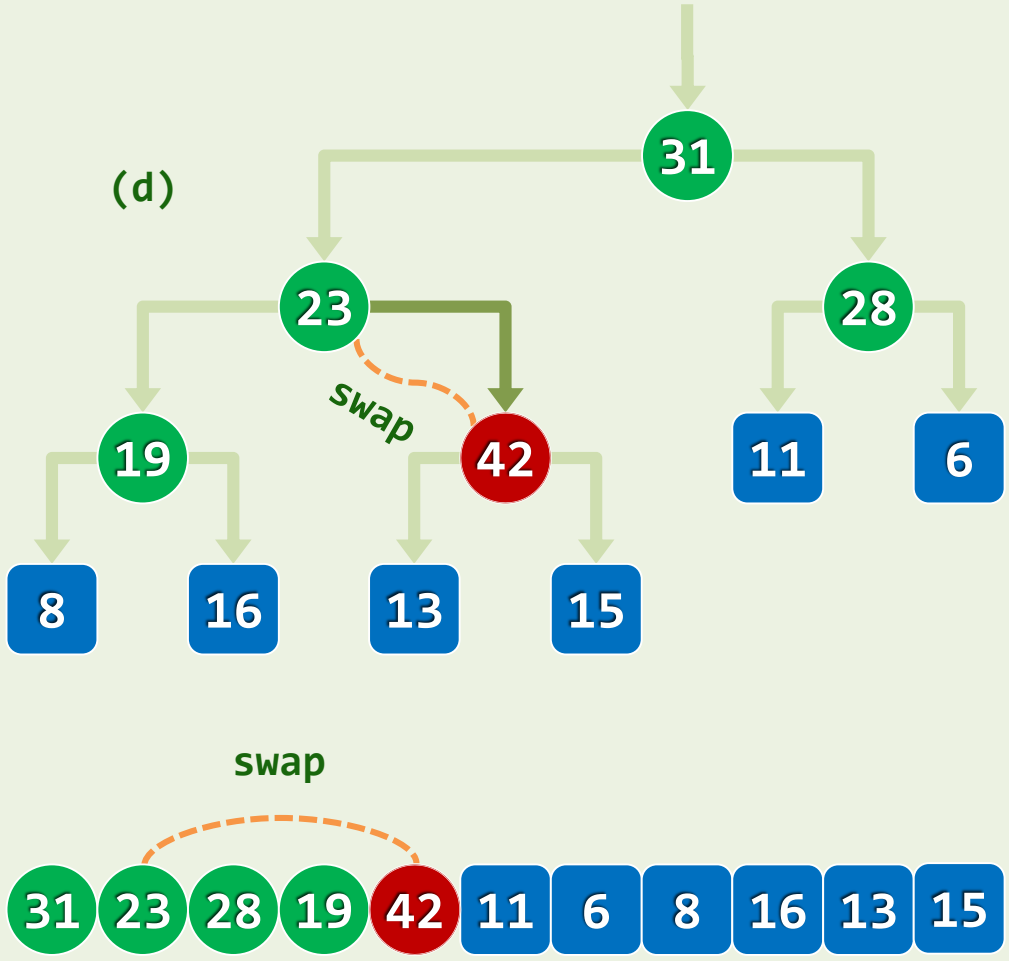
(d)
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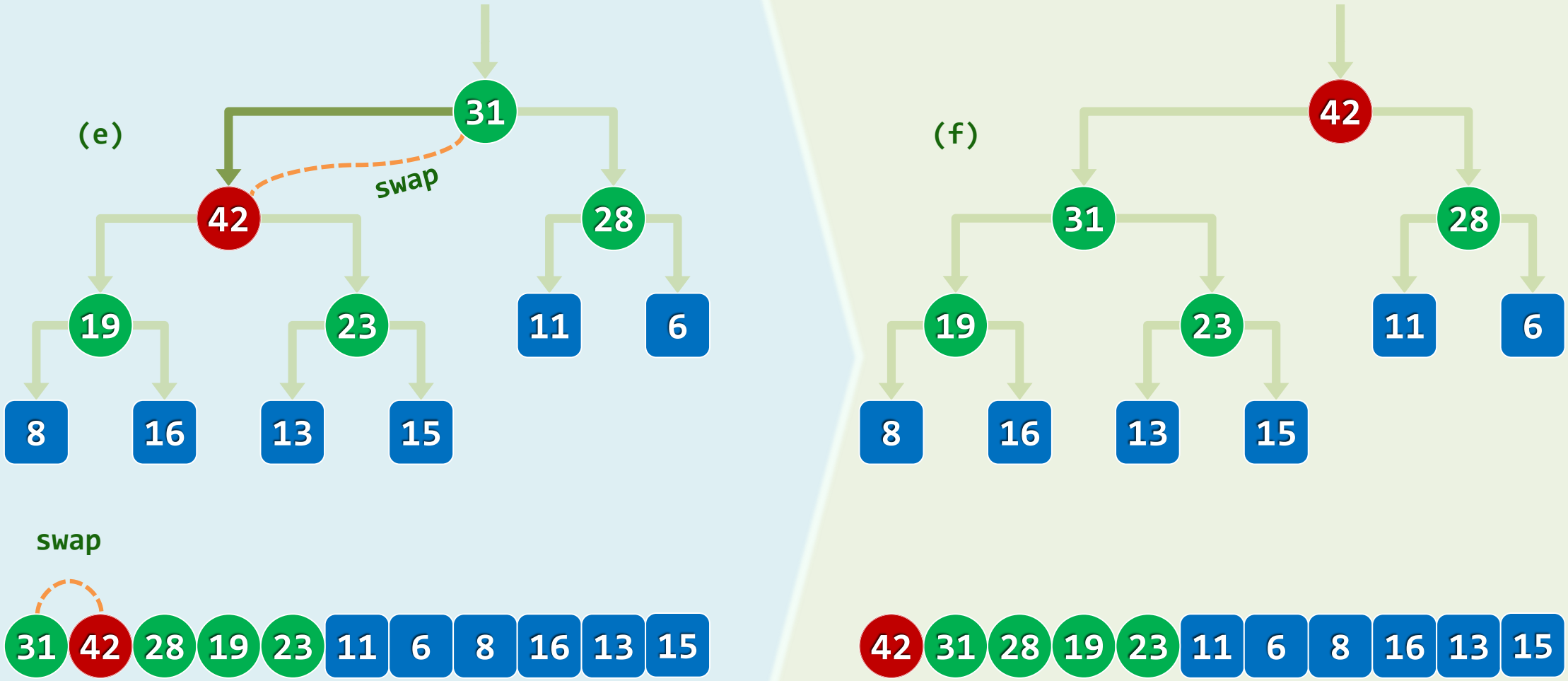
实例（5/5）


(e)
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swap
31 42 28 19 23 11 6 8 16 13 15
(f)
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31
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19
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11
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8
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13
15
42 31 28 19 23 11 6 8 16 13 15

实现

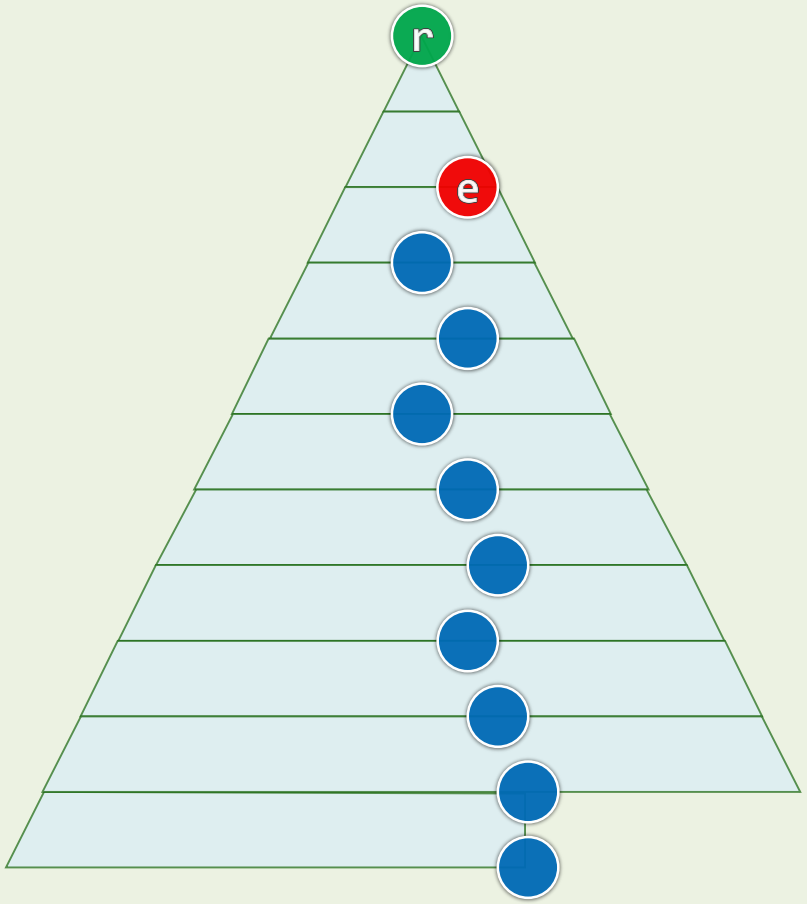
```
❖ template <typename T> void PQ_ComplHeap<T>::insert( T e ) //插入
    { Vector<T>::insert( e ); percolateUp( _elem, _size - 1 ); } //先接入，再上滤
```

```
❖ template <typename T> Rank percolateUp( T* A, Rank i ) { //0 <= i < _size
    while ( 0 < i ) { //在抵达堆顶之前，反复地
       Rank j = Parent( i ); //考查[i]之父亲[j]
       if ( lt( A[i], A[j] ) ) break; //一旦父子顺序，上滤旋即完成；否则
       swap( A[i], A[j] ); i = j; //父子换位，并继续考查上一层
    } //while
    return i; //返回上滤最终抵达的位置
  }
```

效率

- e在上滤过程中，只可能与祖先们交换
- 完全树必平衡，e的祖先不超过$O(\log n)$个
- 故知插入操作可在$O(\log n)$时间内完成
- 然而就数学期望而言

  实际效率往往远远更高...



优先级队列

完全二叉堆：删除

12-B3

I have scaled the peak and

found no shelter in

fame's bleak and barren height.

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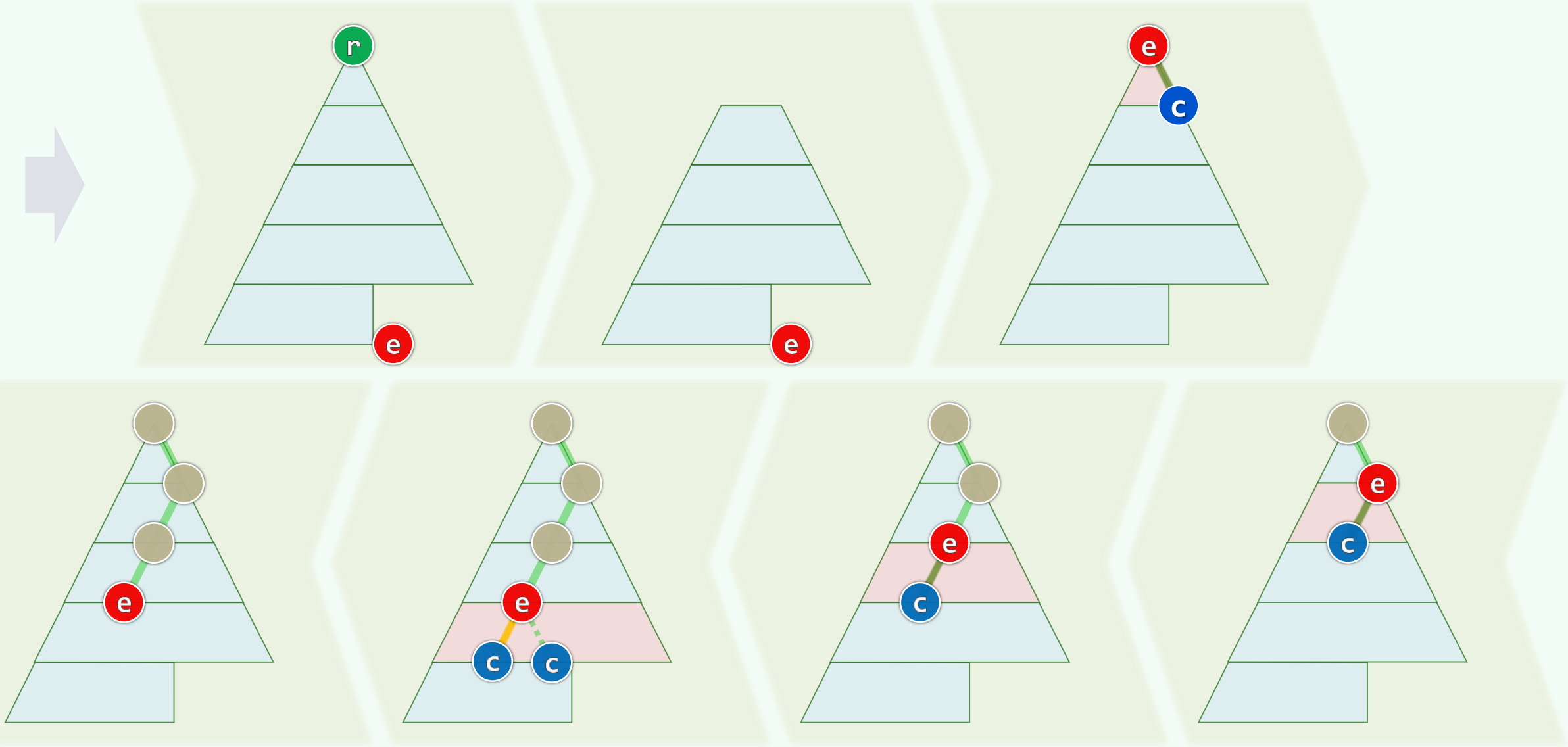
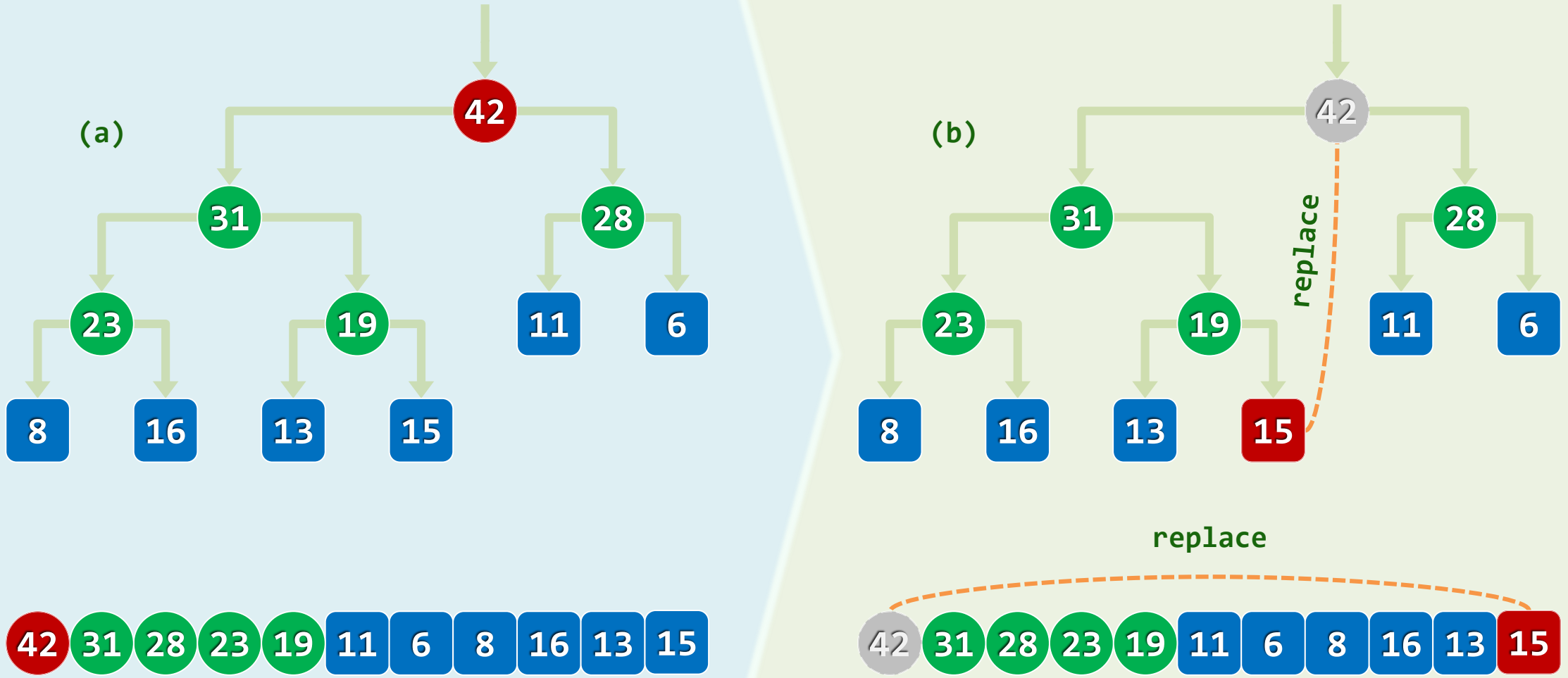
算法：割肉补疮 + 逐层下滤



实例（1/5）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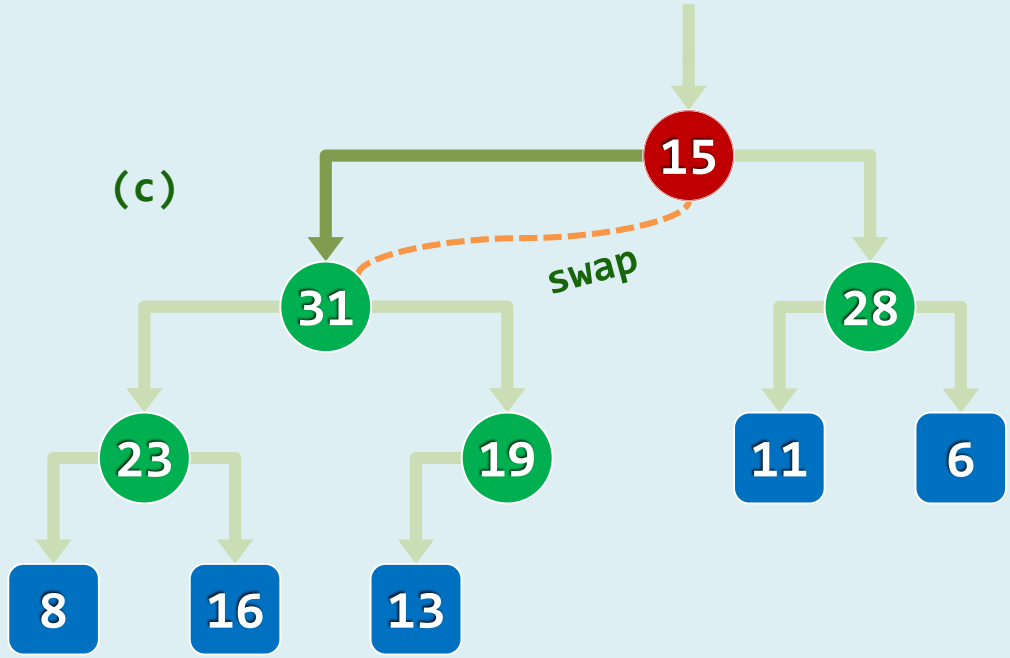
实例（2/5）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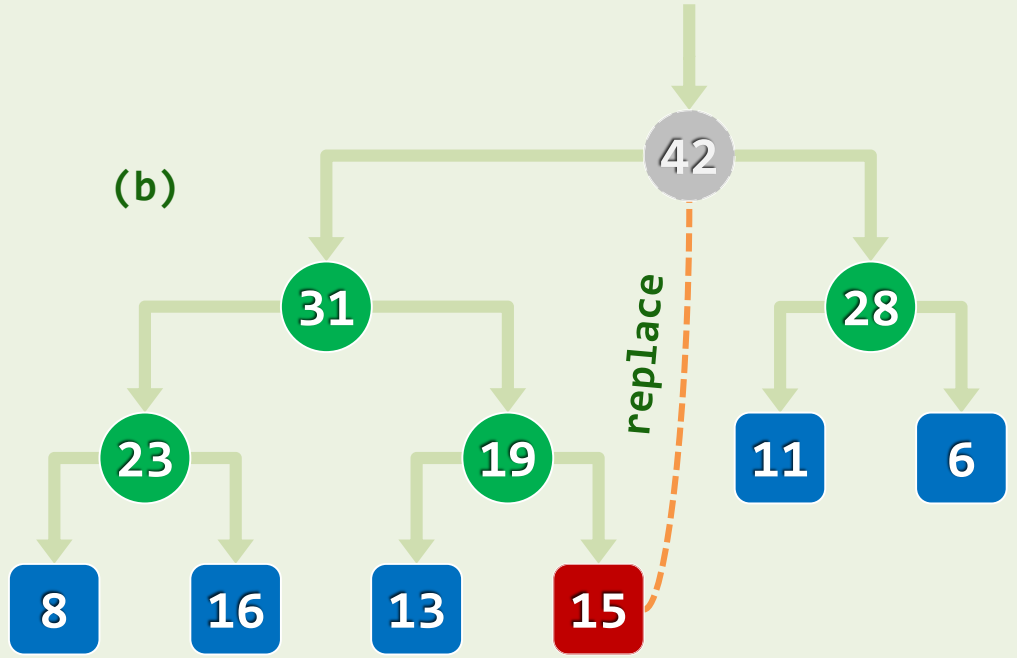
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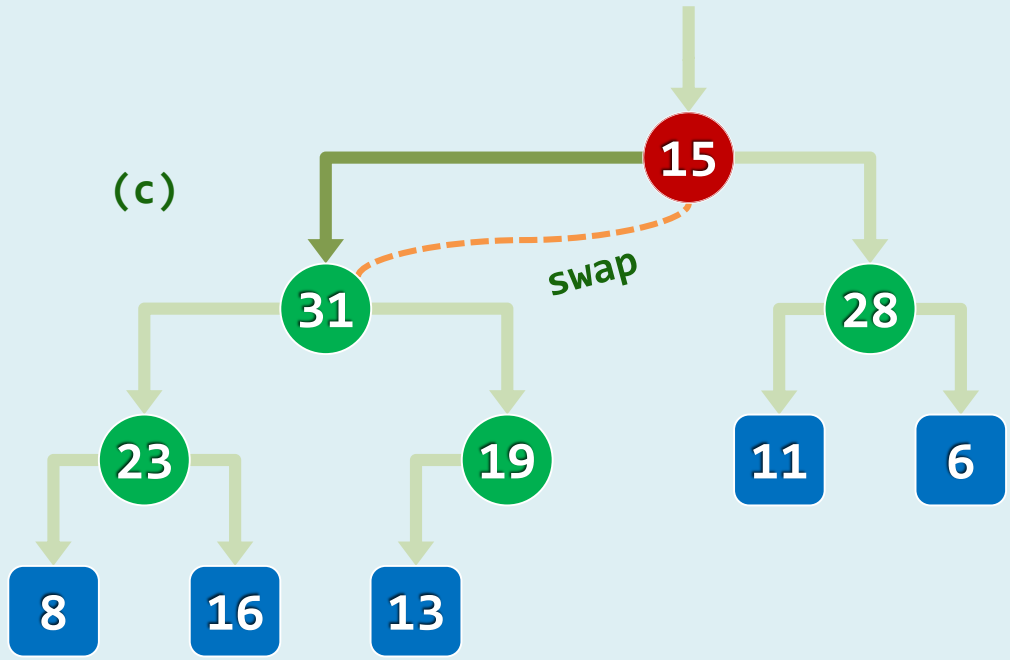
实例（3/5）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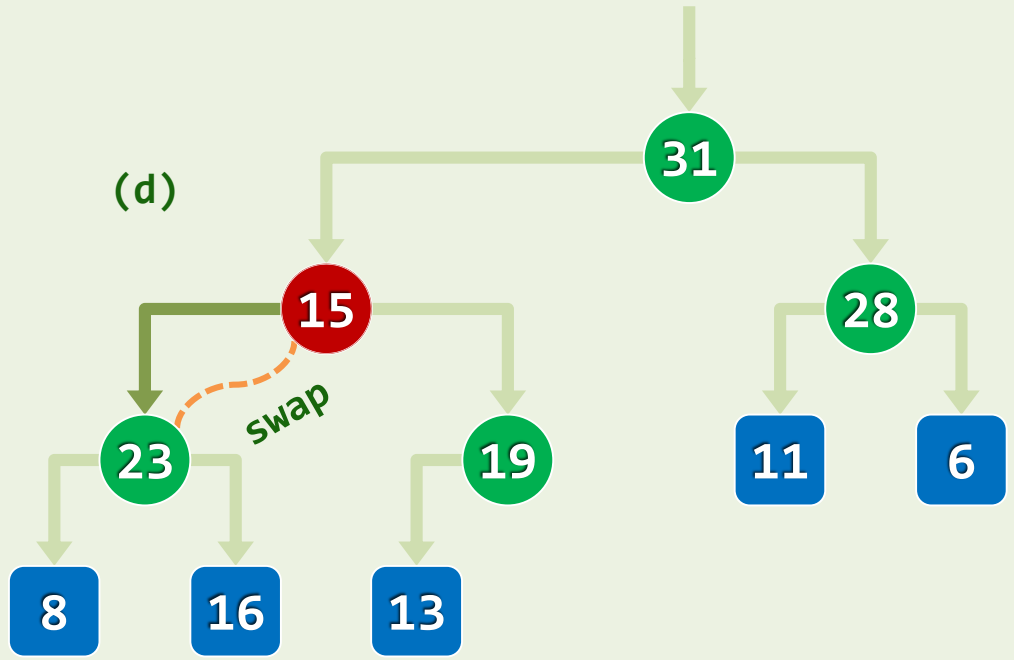
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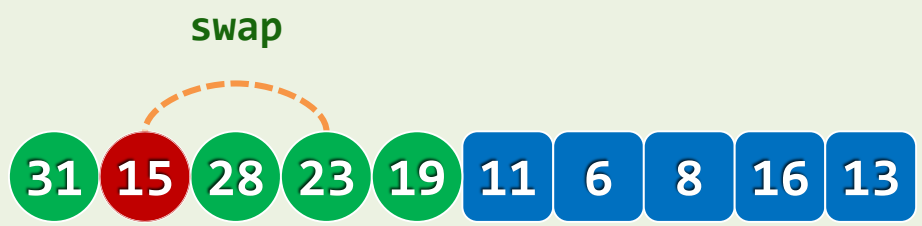
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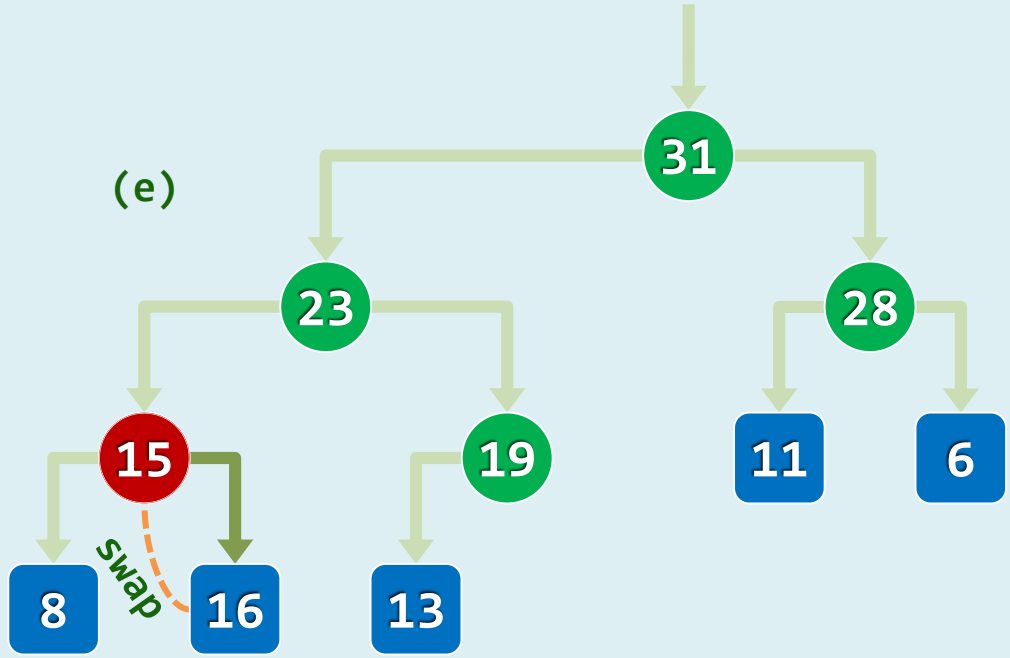
实例（4/5）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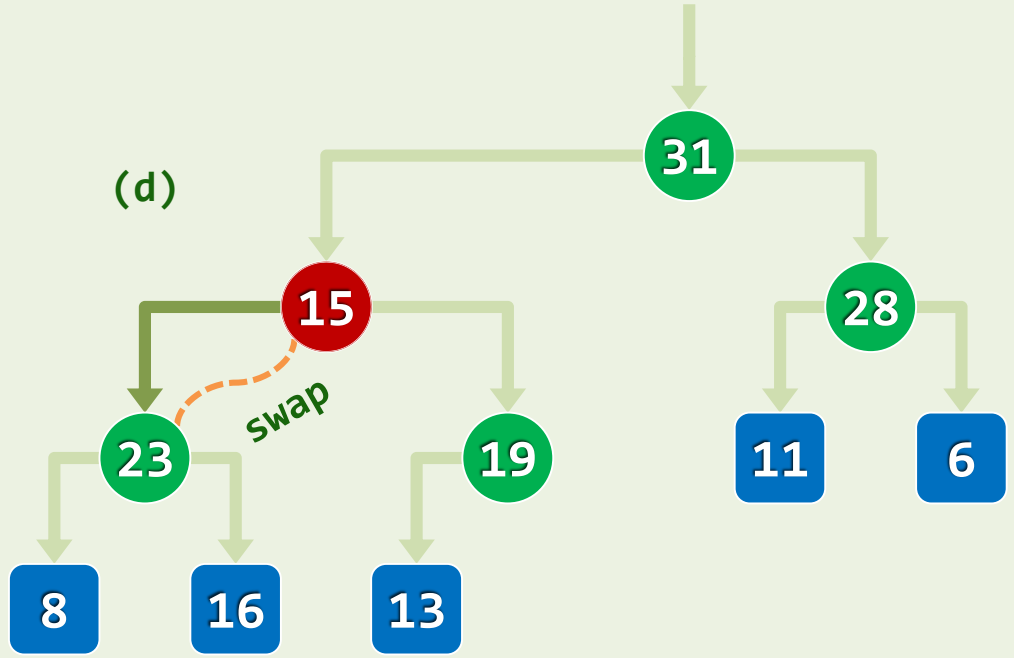
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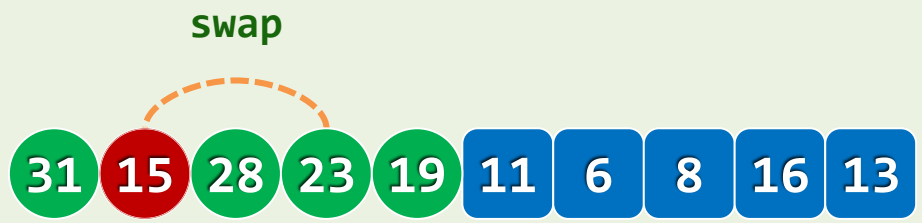
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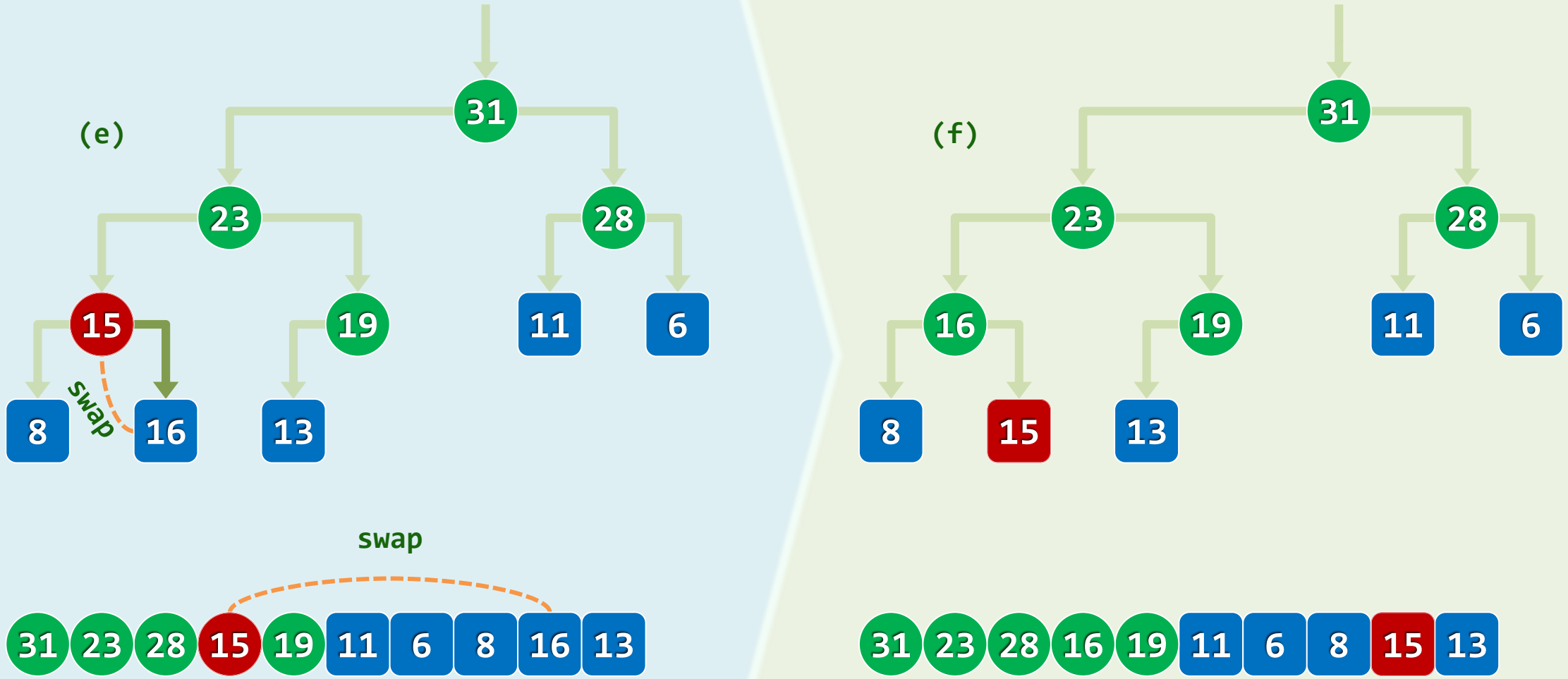
实例（5/5）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31 23 28 15 19 11 6 8 16 13
(f)
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31 23 28 16 19 11 6 8 15 13

实现

```
❖ template <typename T> T PQ_ComplHeap<T>::delMax() { //取出最大词条
    swap( _elem[0], _elem[ --_size ] ); //堆顶、堆尾互换
    percolateDown( _elem, _size, 0 ); //新堆顶下滤
    return _elem[_size]; //返回原堆顶
  }
❖ template <typename T> Rank percolateDown( T* A, Rank n, Rank i ) { //0 <= i < n
    Rank j; //i及其（至多两个）孩子中，堪为父者
    while ( i != ( j = ProperParent( A, n, i ) ) ) //只要i非j，则
       swap( A[i], A[j] ), i = j; //换位，并继续考察i
    return i; //返回下滤抵达的位置（亦i亦j）
  }
```

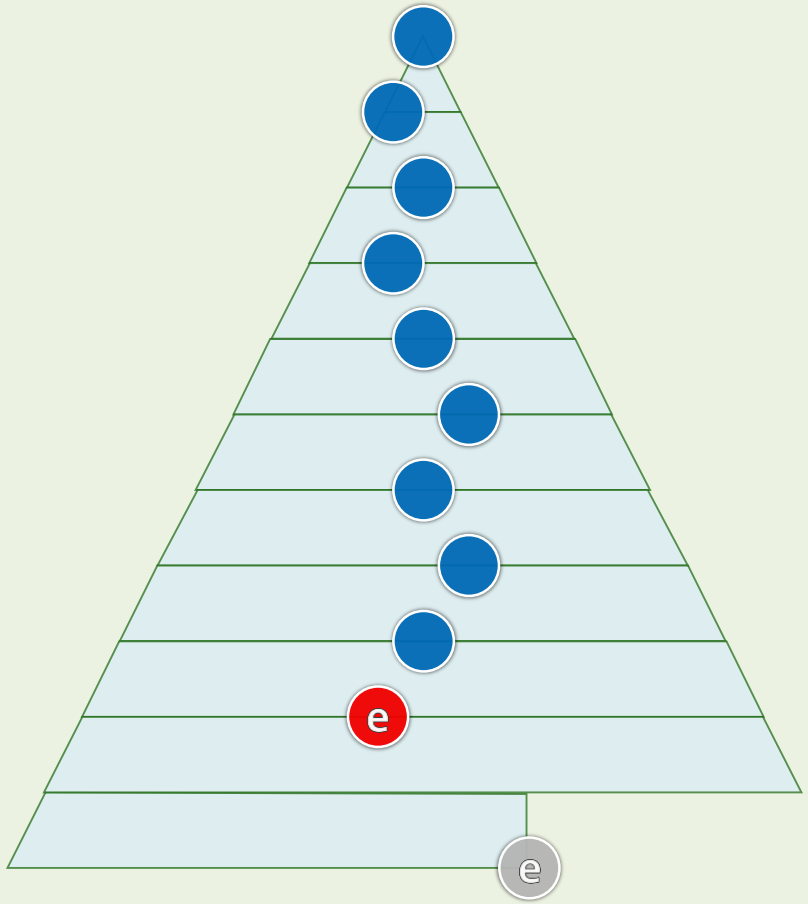
效率

- e在每一高度至多交换一次

  累计交换不超过$O(\log n)$次

- 通过下滤，可在$O(\log n)$时间内
  - 删除堆顶节点，并
  - 整体重新调整为堆

- 数学期望呢?



优先级队列

完全二叉堆：批量建堆

12-B4

今世号通人者，务为艰深之文，陈过高之义，以为士大夫劝，而独不为彼什伯千万倍里巷乡闾之子计，则是智益智，愚益愚，智日少，愚日多也。

现在，培训富人应对贫穷要比教育穷人获得财富更切合实际，因为从人数比例上来说，富人破产的越来越多，而穷人变富的越来越少。

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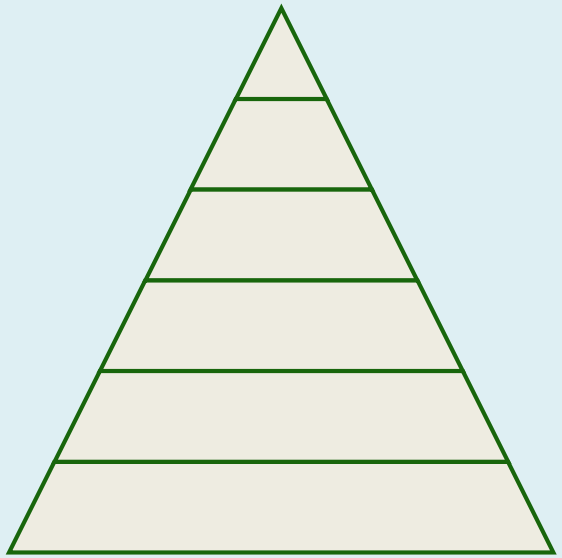
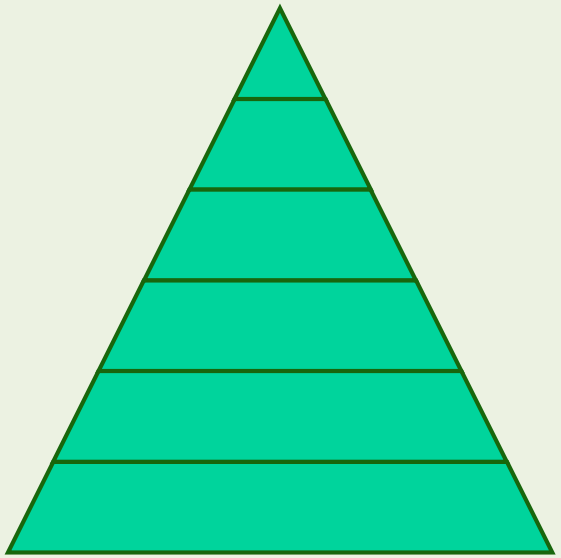
自上而下的上滤（1/2）

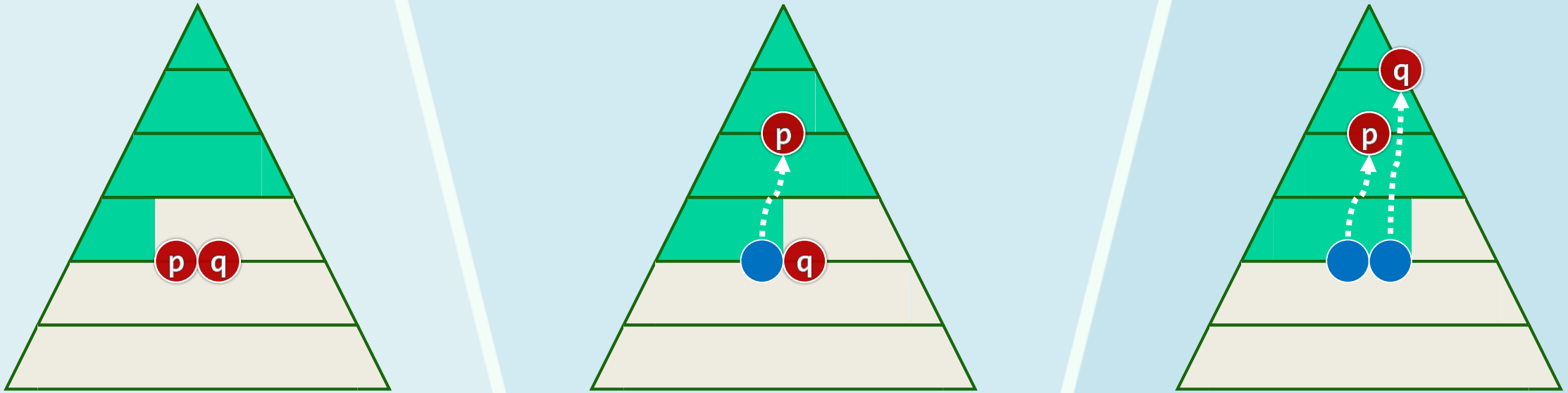
```
PQ_ComplHeap( T* A, Rank n )

   { copyFrom( A, 0, n ); heapify( _elem, n ); }
```





自上而下的上滤（2/2）

```
template <typename T> void heapify( T* A, const Rank n ) { //蛮力
   for ( Rank i = 1; i < n; i++ ) //按照逐层遍历次序逐一
      percolateUp( A, i ); //经上滤插入各节点
}
```

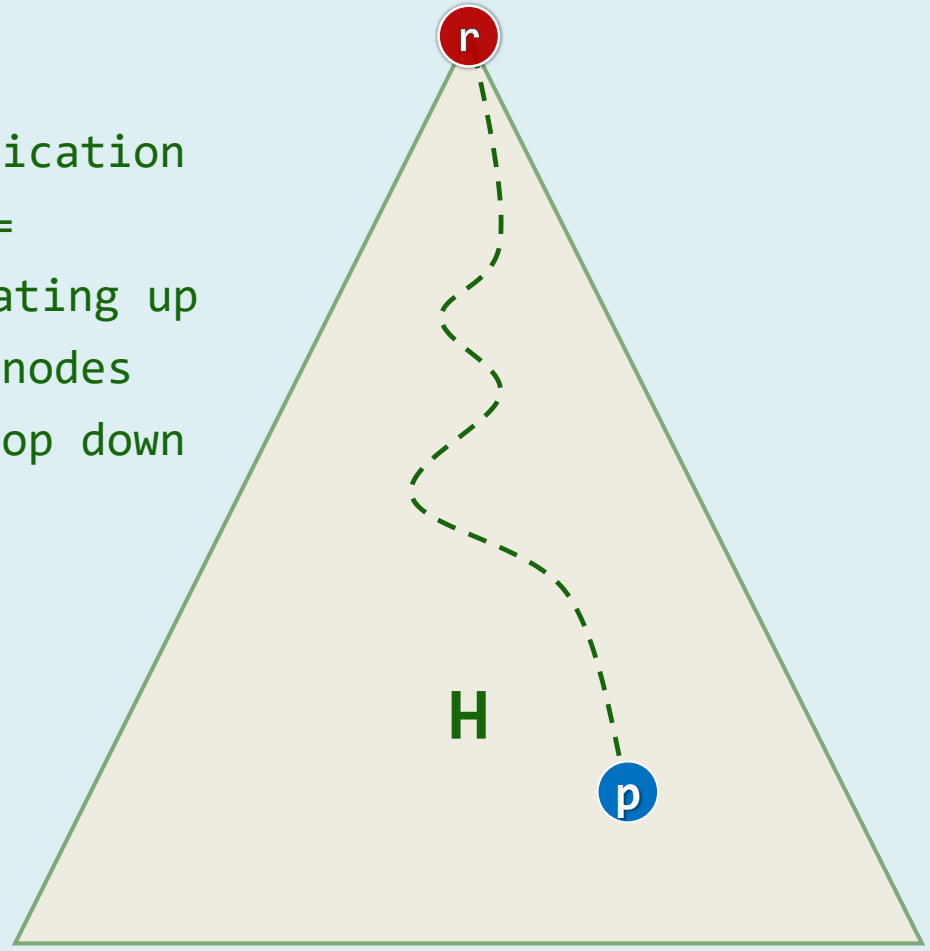


效率

- 最坏情况下
  - 每个节点都需上滤至根
  - 所需成本线性正比于其深度
- 即便只考虑底层
  - $n/2$ 个叶节点，深度均为 $\mathcal{O}(\log n)$
  - 亦累计耗时 $\mathcal{O}(n \log n)$
- 这样长的时间，本足以全排序！

  所以...应该能够...更快的...

heapification
=
percolating up
all nodes
from top down

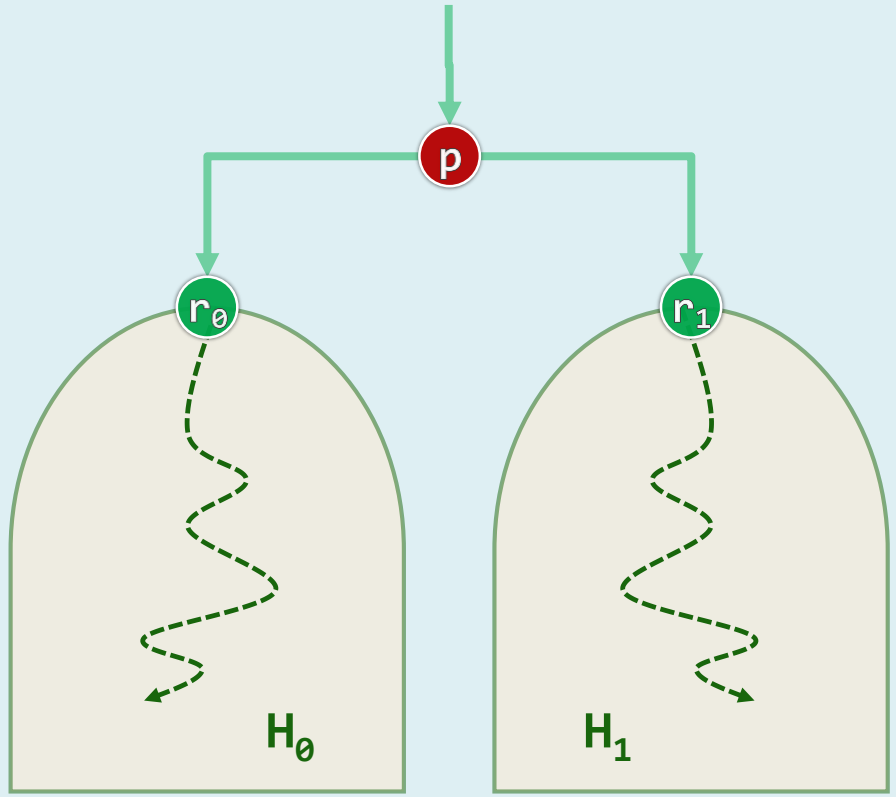


自下而上的下滤

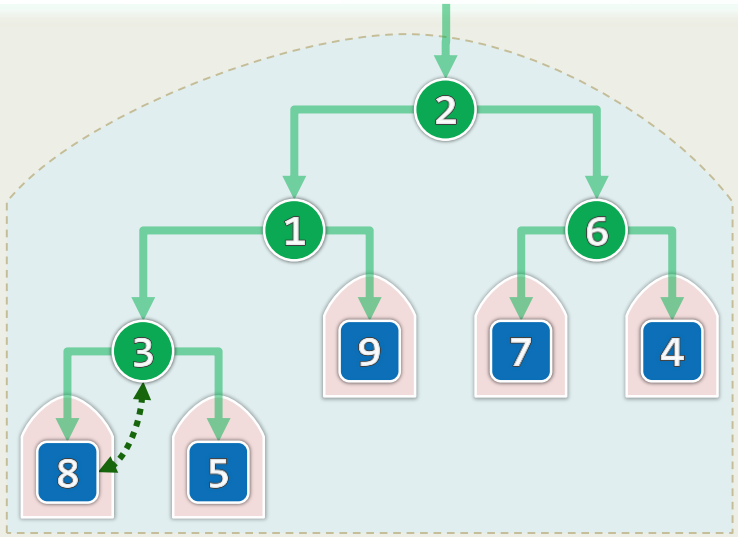
- 任意给定堆 $\mathcal{H}_0$ 和 $\mathcal{H}_1$，以及节点 p
- 为得到堆 $\mathcal{H}_0 \cup \{p\} \cup \mathcal{H}_1$，只需

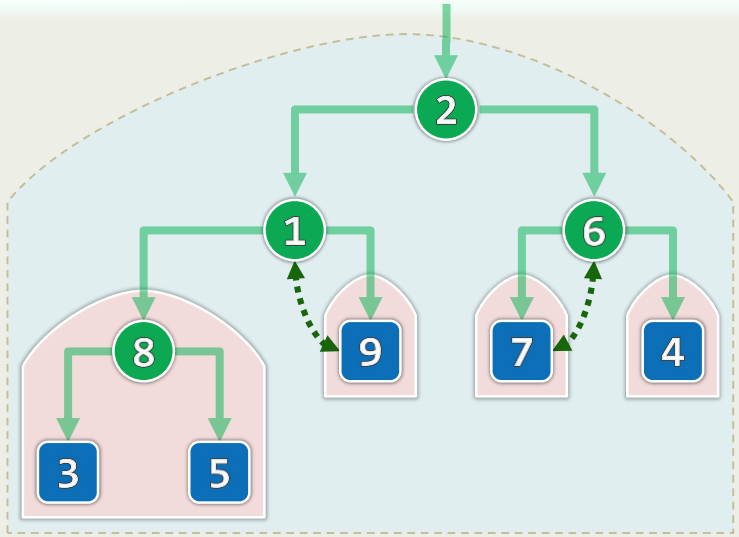
  将 r_0 和 r_1 当作 p 的孩子，再对 p 下滤

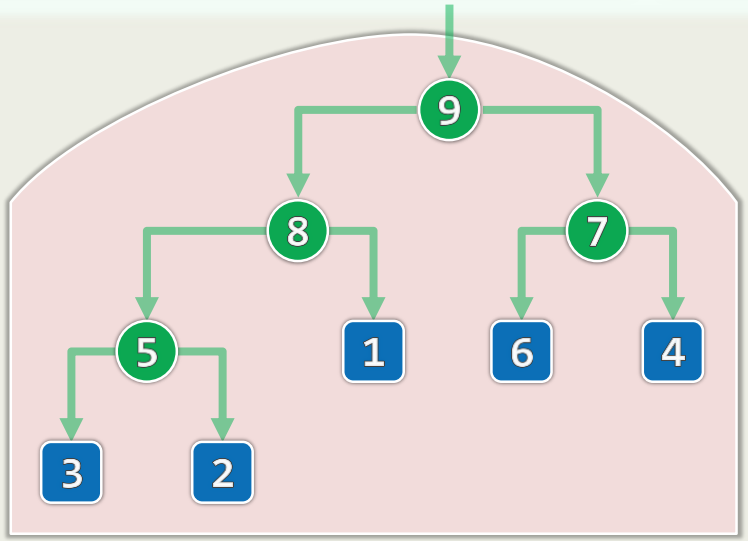
```
template <typename T> //Robert Floyd, 1964
void heapify( T* A, Rank n ) { //自下而上
   for ( Rank i = n/2 - 1; -1 != i; i-- ) //依次
      percolateDown( A, n, i ); //经下滤合并子堆
} //可理解为子堆的逐层合并，堆序性最终必然在全局恢复
```

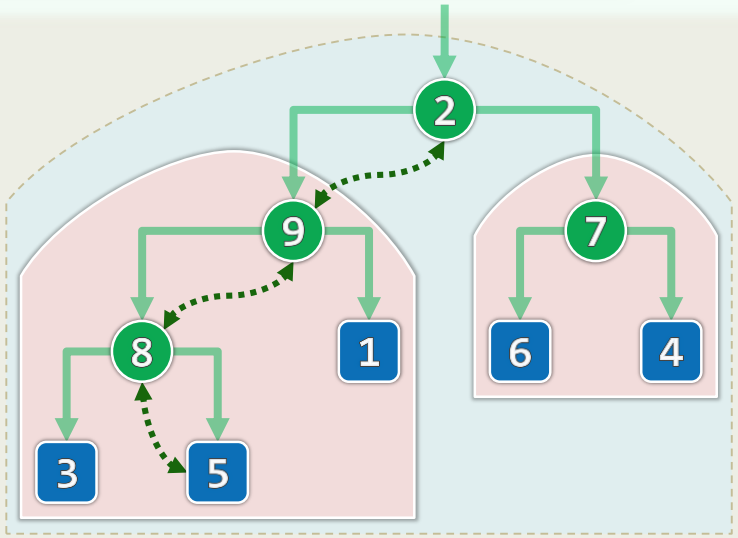


实例









效率

❖ 每个内部节点所需的调整时间，正比于其高度而非深度

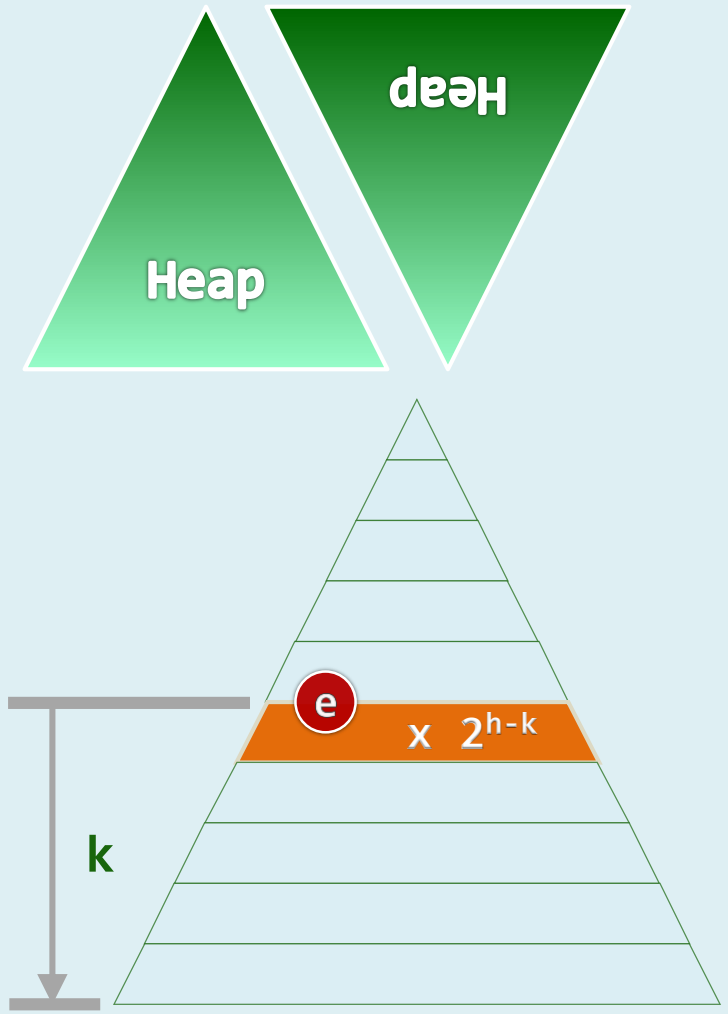
❖ 不失一般性，考查满树：$n = 2^{h+1} - 1$

❖ 所有节点的高度总和

$$S(n) = \sum_{k=1}^{h} k \cdot 2^{h-k} = \sum_{k=1}^{h} \sum_{i=1}^{k} 2^{h-k} = \sum_{i=1}^{h} \sum_{k=i}^{h} 2^{h-k}$$

$$= \sum_{i=1}^{h} \sum_{k=0}^{h-i} 2^{k} = \sum_{i=1}^{h} \{2^{h-i+1} - 1\} = \sum_{i=1}^{h} 2^{h-i+1} - h$$

$$= \sum_{i=1}^{h} 2^{i} - h = 2^{h+1} - 2 - h = \mathcal{O}(n)$$



课后

- insert()：最坏情况下效率为O(logn)，平均情况呢？
- heapify()：构造次序颠倒后，为什么复杂度会有实质降低？

  这一算法在哪些场合不适用？
- 扩充接口：
```
decrease( i, delta ) //任一元素_elem[i]的数值减小delta
increase( i, delta ) //任一元素_elem[i]的数值增加delta
remove( i ) //删除任一元素_elem[i]
```
- 借助完全堆，在$\mathcal{O}(n \log n)$时间内构造Huffman树
- 大顶堆的delMin()操作，能否也在$\mathcal{O}(\log n)$时间内完成？

  难道，为此需要同时维护一个小顶堆？

12-C

优先级队列

堆排序

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选取

- 在selectionSort()中

  将U替换为H...

- J. Williams, 1964

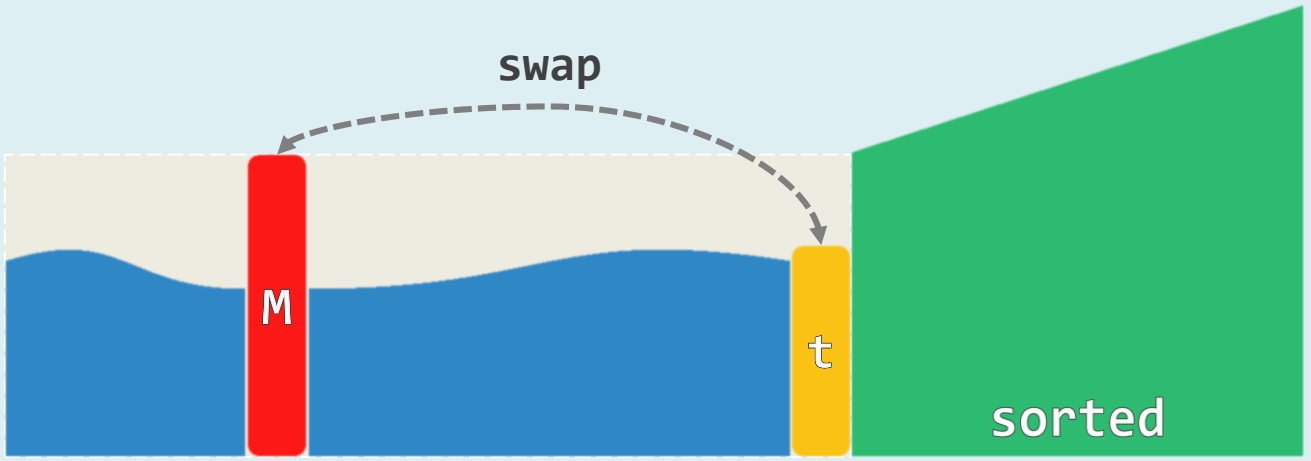
  初始化 : heapify(), $O(n)$

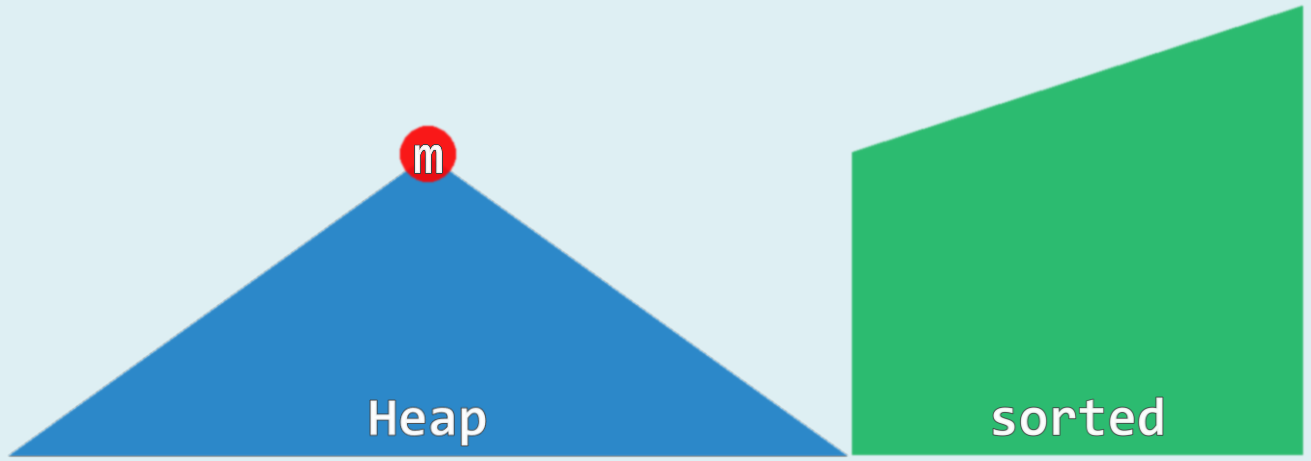
  迭代 : delMax(), $O(\log n)$

  不变性 : $H \le S$

- $O(n) + n \times O(\log n)$

  $= O(n \log n)$





就地

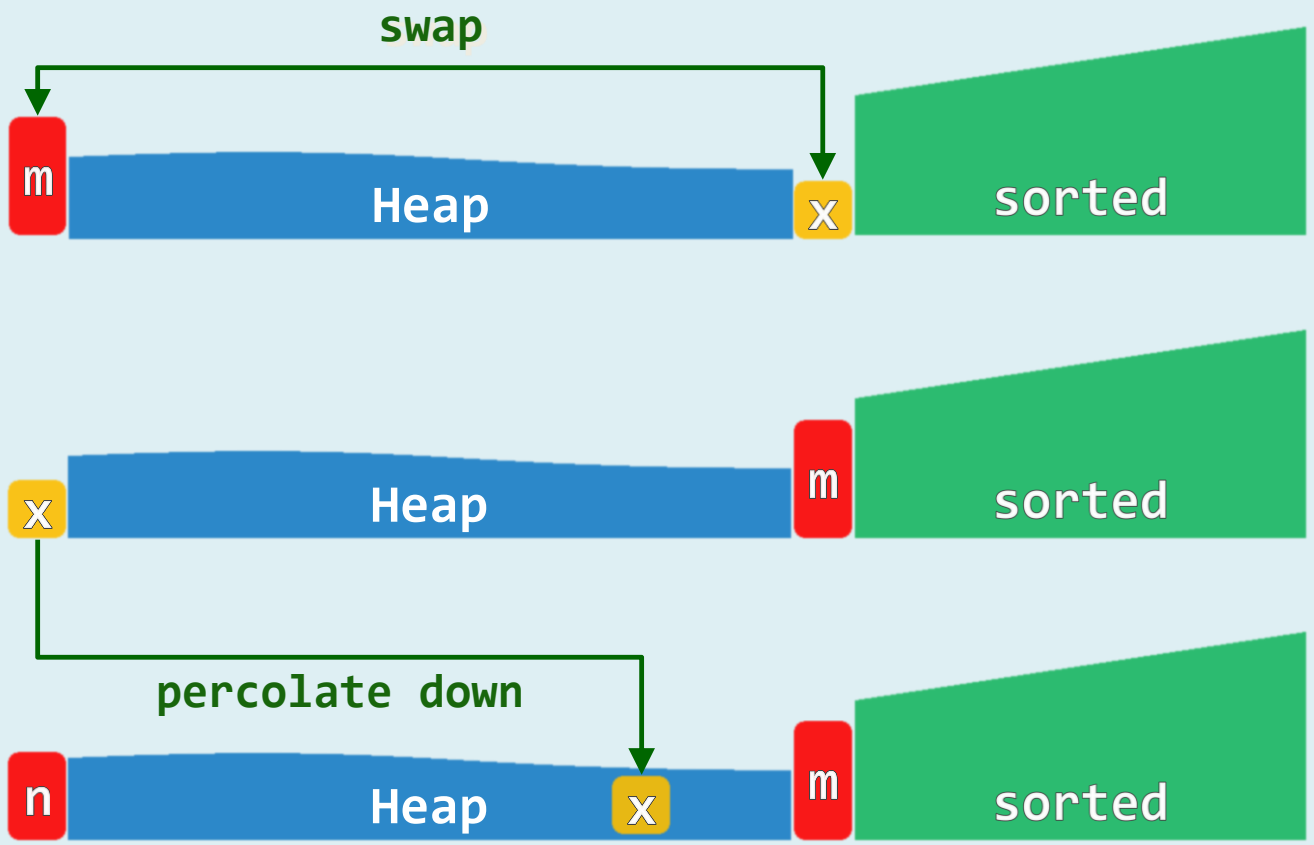
❖ 在物理上

完全二叉堆即是向量

❖ 既然此前有：

- m = H[0]
- x = H[n - 1]

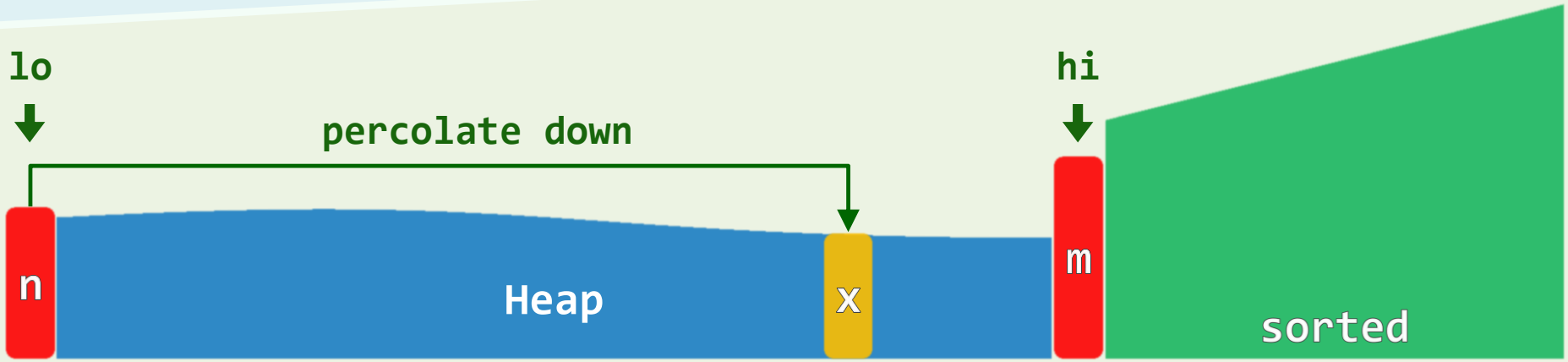
不妨随即就：



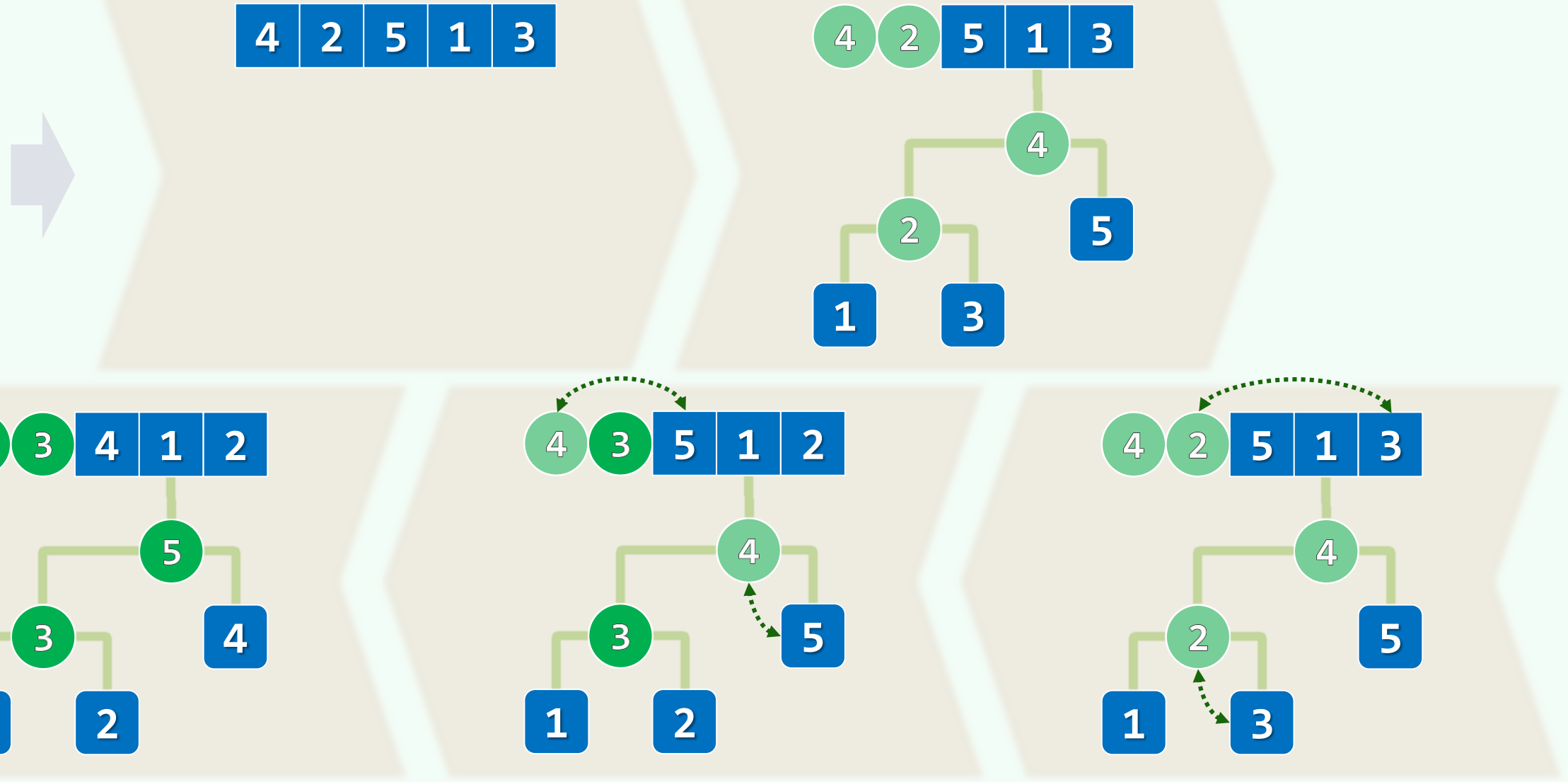
- swap(m , x) = H.insert(x) + S.insert(m)

实现

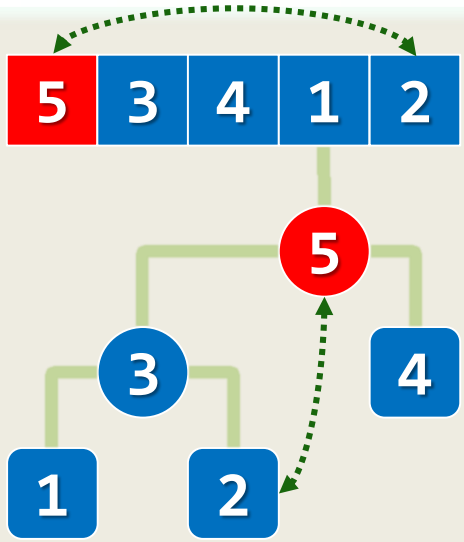
```
template <typename T> void Vector<T>::heapSort( Rank lo, Rank hi ) { //就地堆排序
   T* A = _elem + lo; Rank n = hi - lo; heapify( A , n ); //待排序区间建堆，O(n)
   while ( 0 < --n ) //反复地摘除最大元并归入已排序的后缀，直至堆空
      { swap( A[0], A[n] ); percolateDown( A, n, 0 ); } //堆顶与末元素对换后下滤
}
```

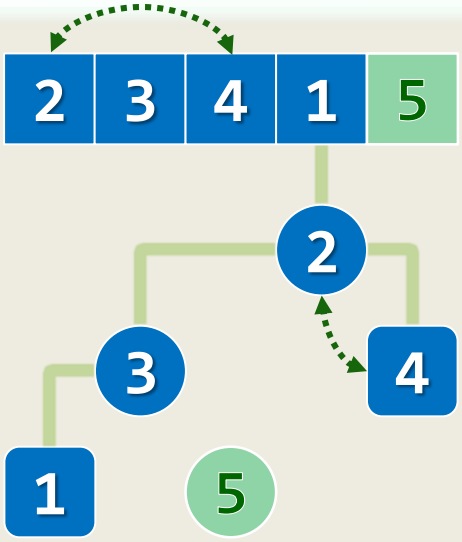


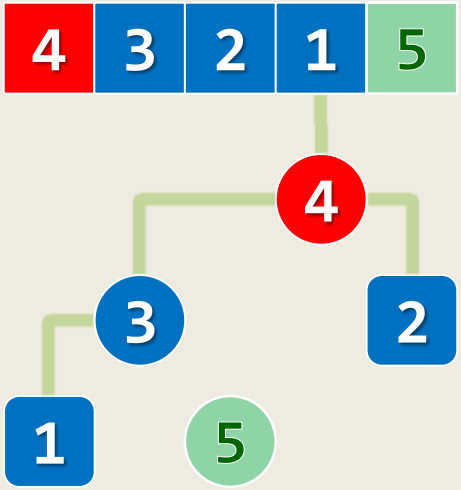
实例：建堆

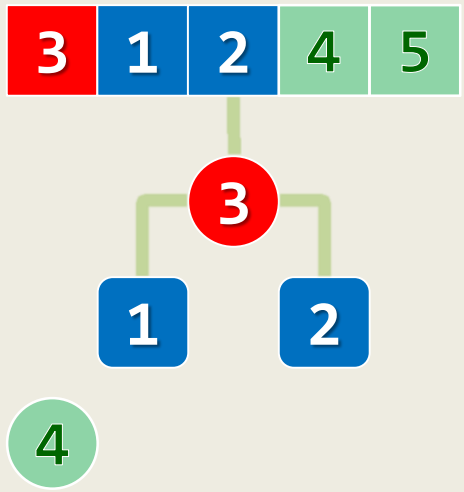


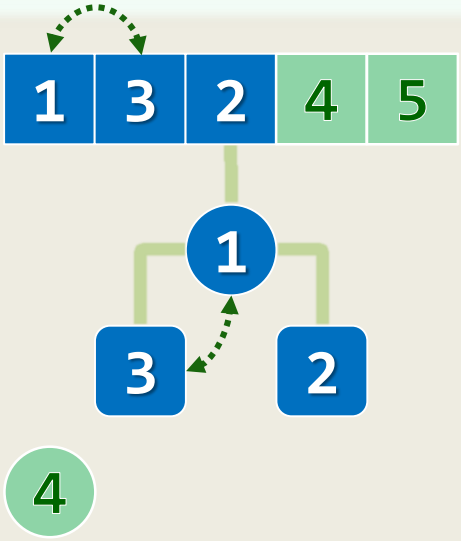
实例：选取 + 调整（1/2）

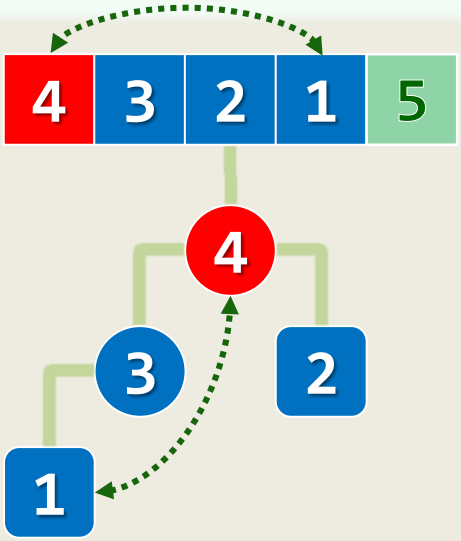

5 3 4 1 2
5
3 4
1 2


2 3 4 1 5
2
3 4
1 5

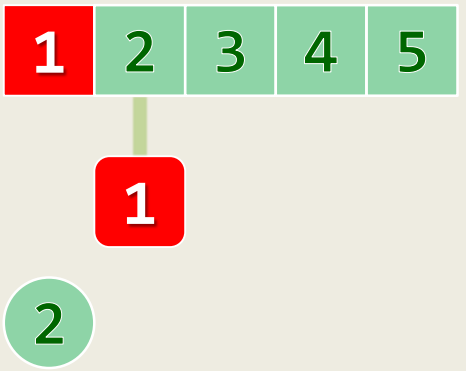

4 3 2 1 5
4
3 2
1 5

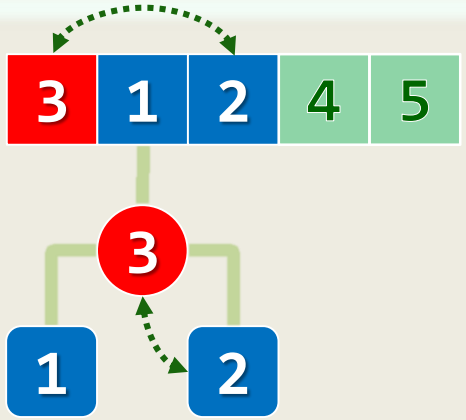

3 1 2 4 5
3
1 2
4

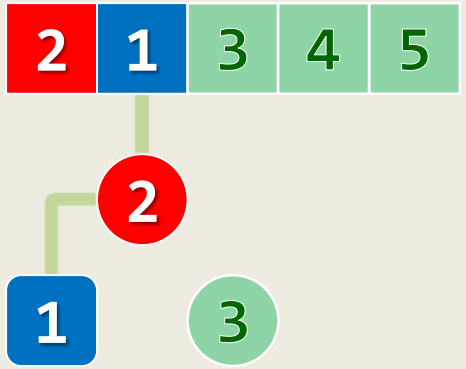

1 3 2 4 5
1
3 2
4

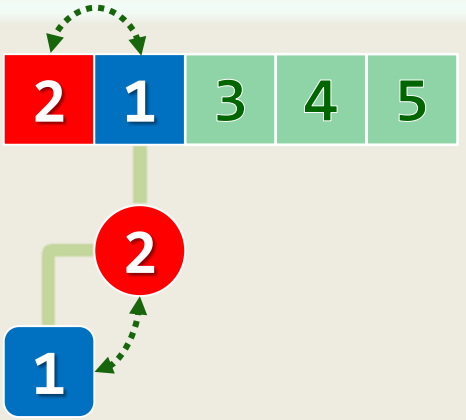

4 3 2 1 5
4
3 2
1

实例：选取 + 调整（2/2）









12-D1

优先级队列

锦标赛树：胜者树

老妖道：“怎么叫做分瓣梅花计？”

小妖道：“如今把洞中大小群妖，点将起来，千中选百，百中选十，十中只选三个...”

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锦标赛树/Tournament Tree

❖ **完全二叉树**

- 叶节点：　参赛选手/团队
- 内部节点：孩子对决后之**胜者**

　　或有重复（连胜）

❖ `create()` `//O(n)`

`remove()` `//O(logn)`

`insert()` `//O(logn)`

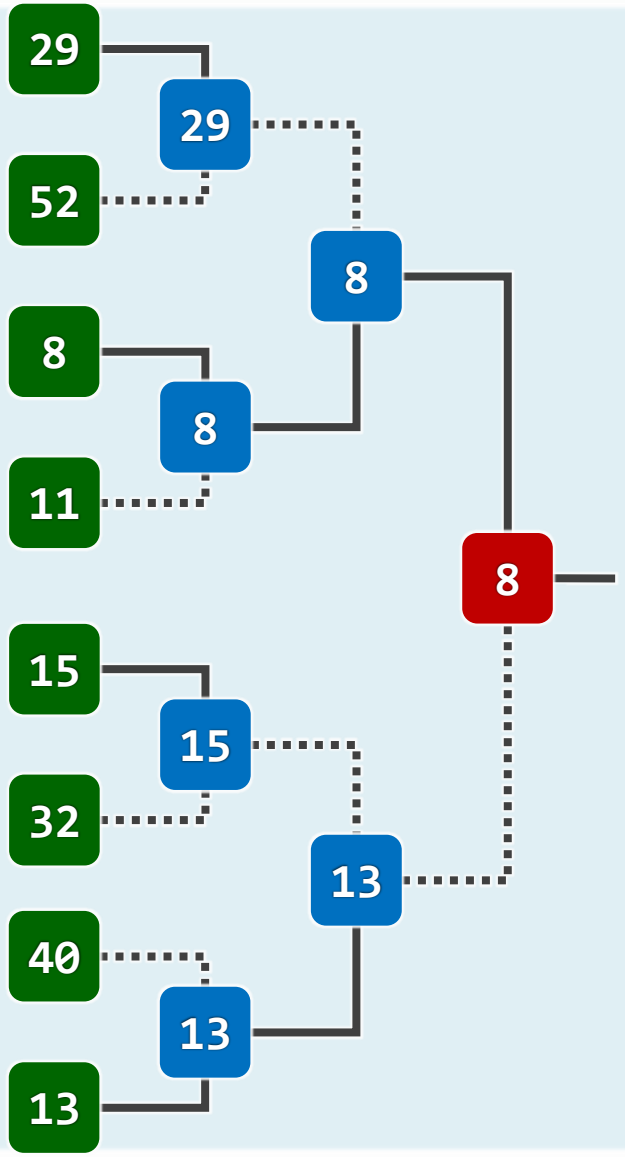
❖ **树根总是全局冠军：类似于堆顶**

Tournamentsort()

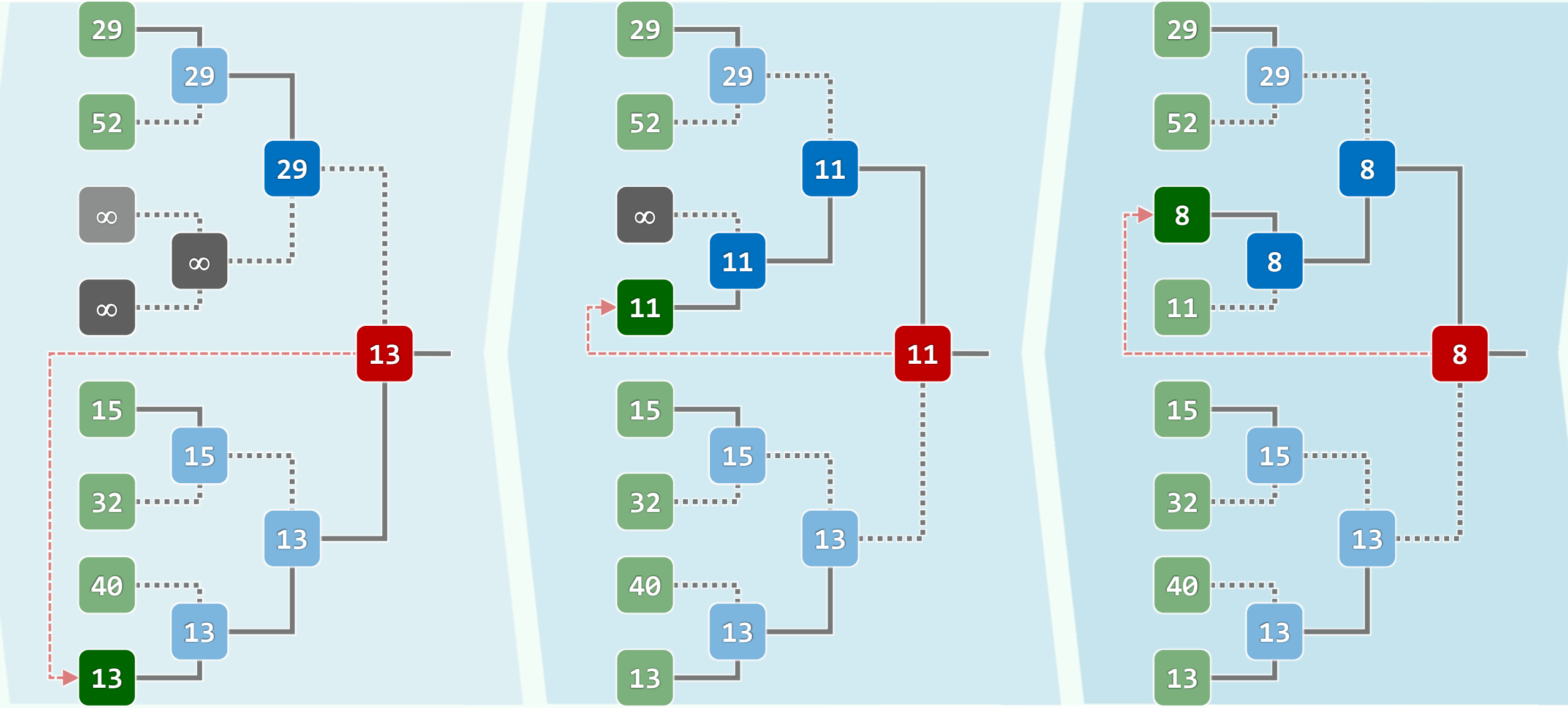
CREATE a tournament tree for the input list

while there are active leaves

- REMOVE the root
- RETRACE the root down to its leaf
- DEACTIVATE the leaf
- REPLAY along the path back to the root



实例（1/2）


29
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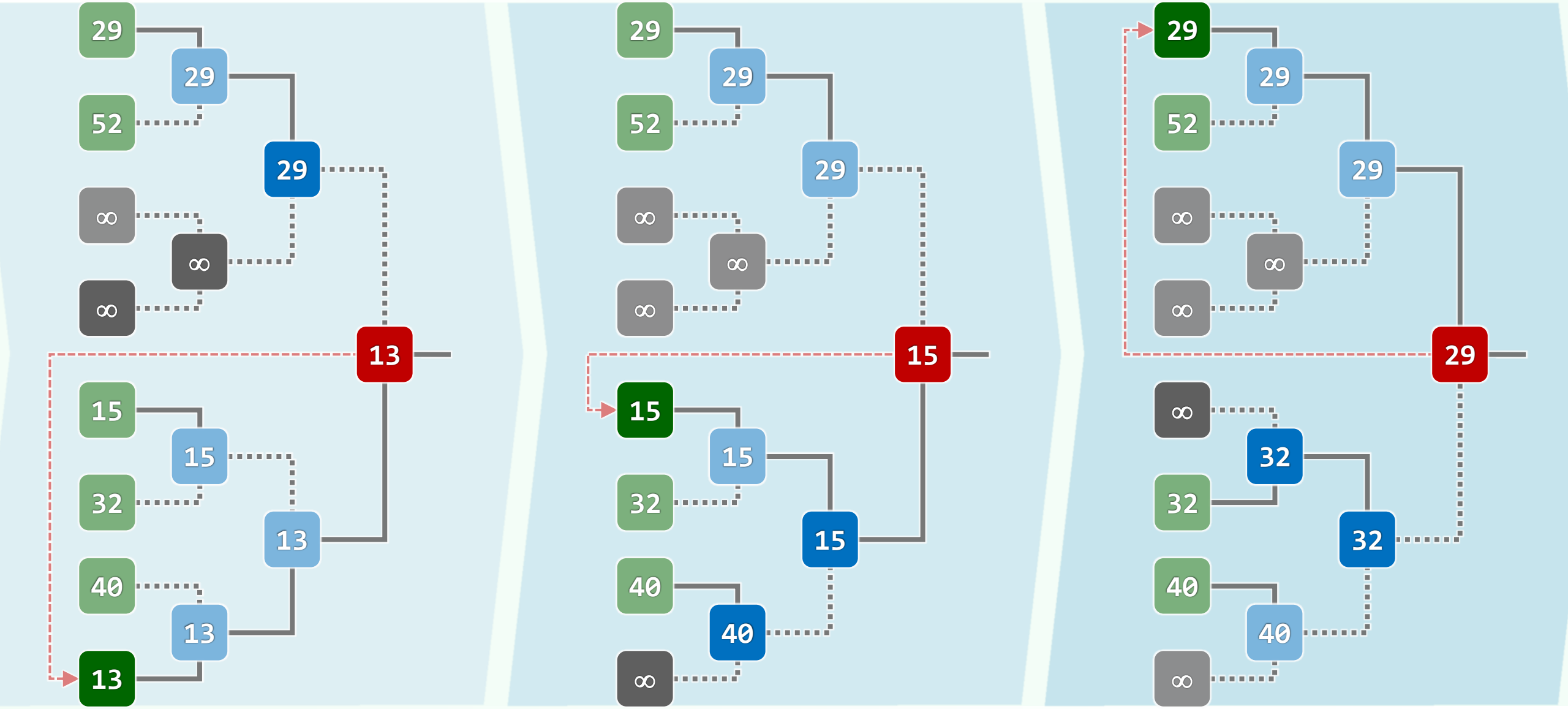
实例（2/2）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32
32
32
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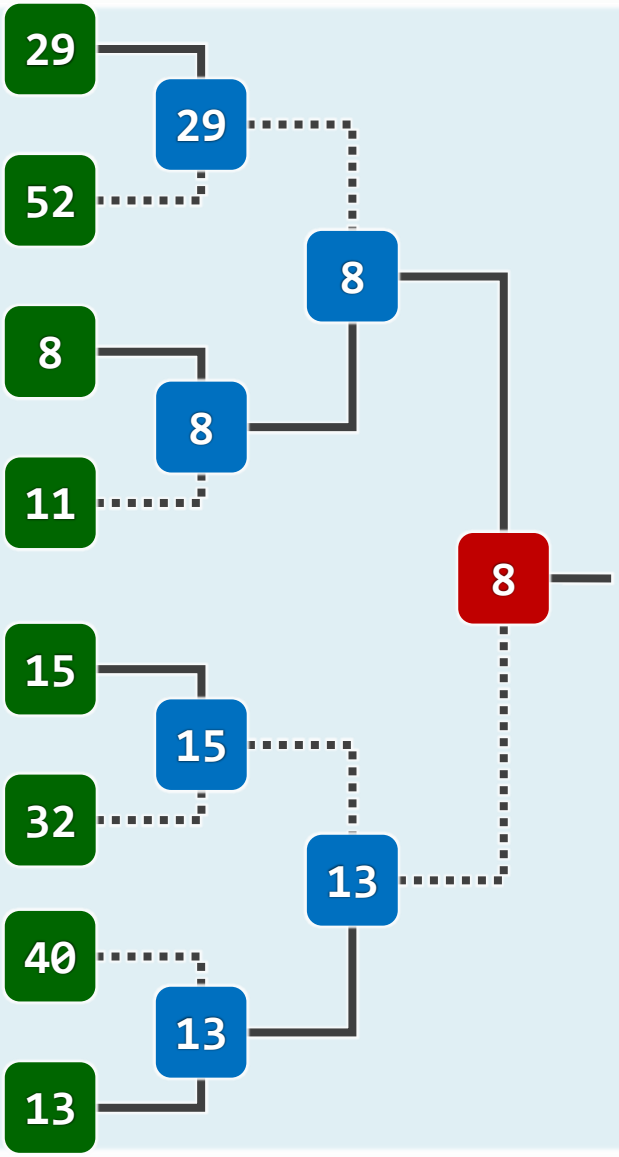
∞

效率

- ❖ 空间：　O(节点数) = O(叶节点数) = O(n)
- ❖ 构造：　仅需O(n)时间
- ❖ 更新：　每次都须全体重赛`replay`？

  　　唯上一优胜者的祖先，才有必要！
- ❖ 为此　　只需从其所在叶节点出发，逐层上溯直到树根

  　如此　　为确定各轮优胜者，总共所需时间仅O(logn)
- ❖ 时间：　n轮 × O(logn) = O(nlogn)，达到下界



选取

- 借助锦标赛树，从n个元素中找出最小的k个，k << n
  - 初始化：　$O(n)$　//n-1次比较
  - 迭代k步：　$O(k*\log n)$　//每步$\log n$次比较

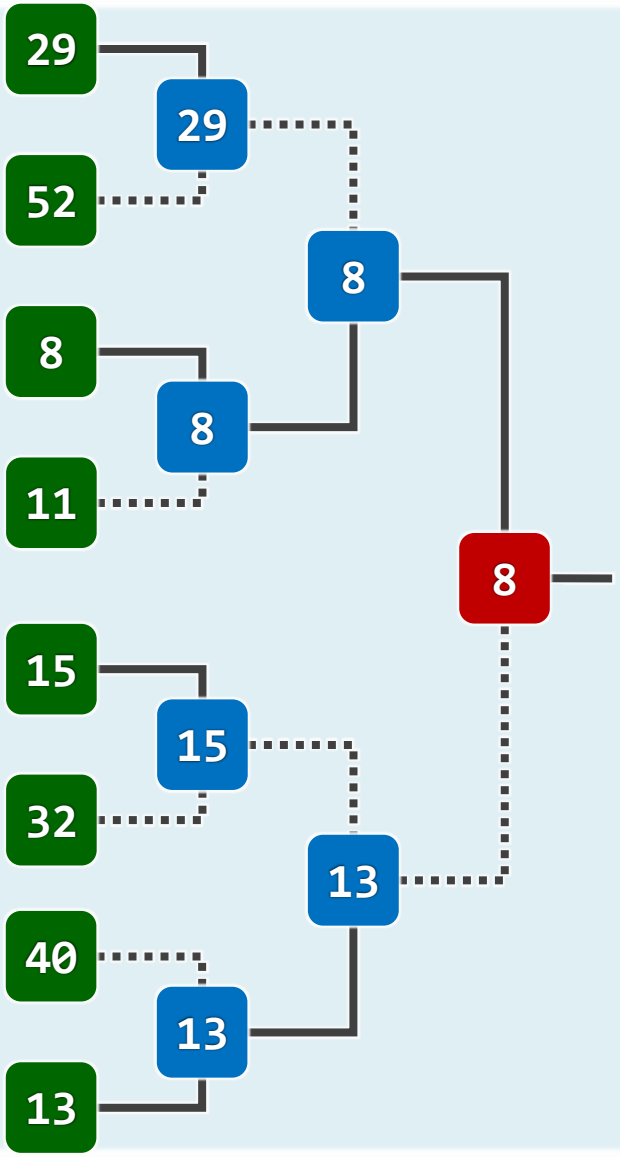
  与小顶堆旗鼓相当?

- 渐近意义上，的确如此

  但就常系数而言，区别不小...

- Floyd算法、delMax()中的percolateDown()

  在每一层需做2次比较，累计大致$2*\log n$次



优先级队列

锦标赛树：败者树

12-D2

We are the champions.

No time for losers

'Cause we are the champions of the world.

善胜者不阵，善阵者不战，善战者不败，善败者终胜

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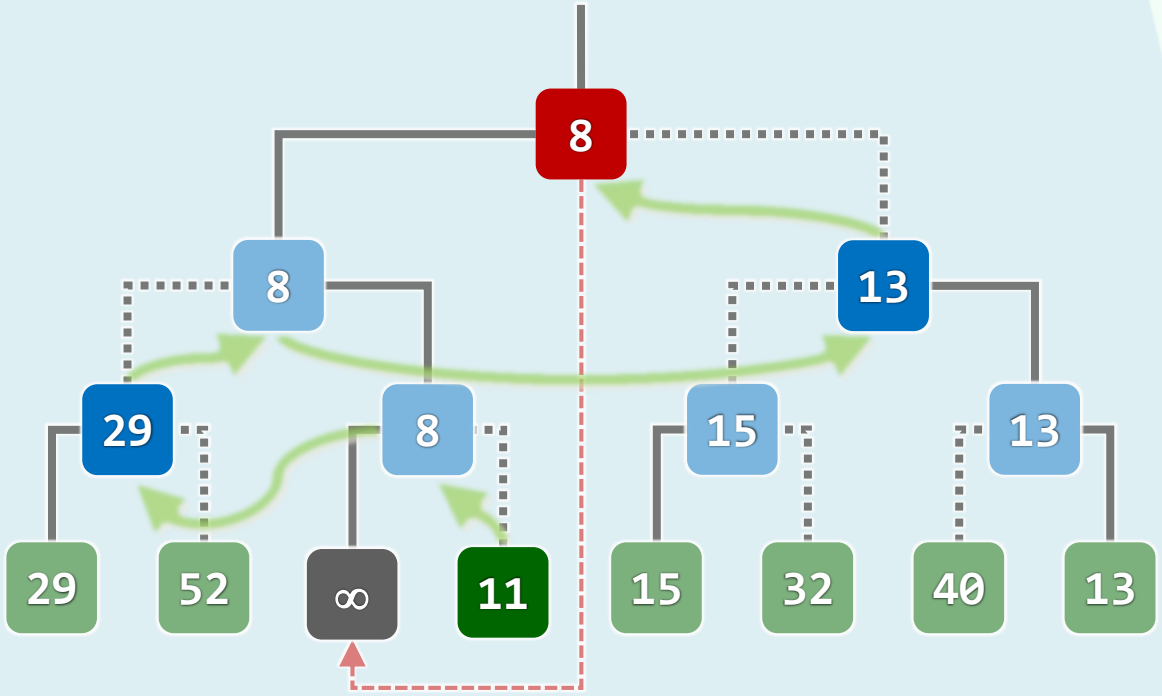
结构 + 重赛算法

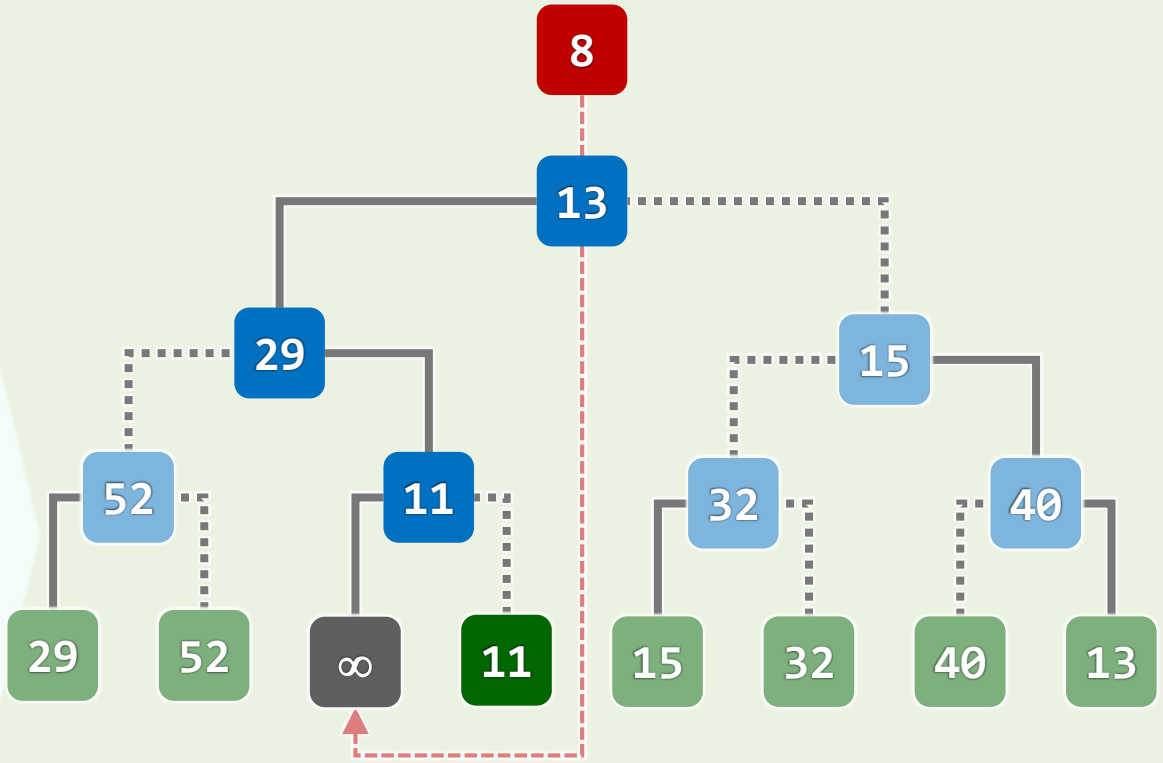
❖ 重赛过程中，须交替访问沿途节点及其兄弟

如何避免这类迂回?

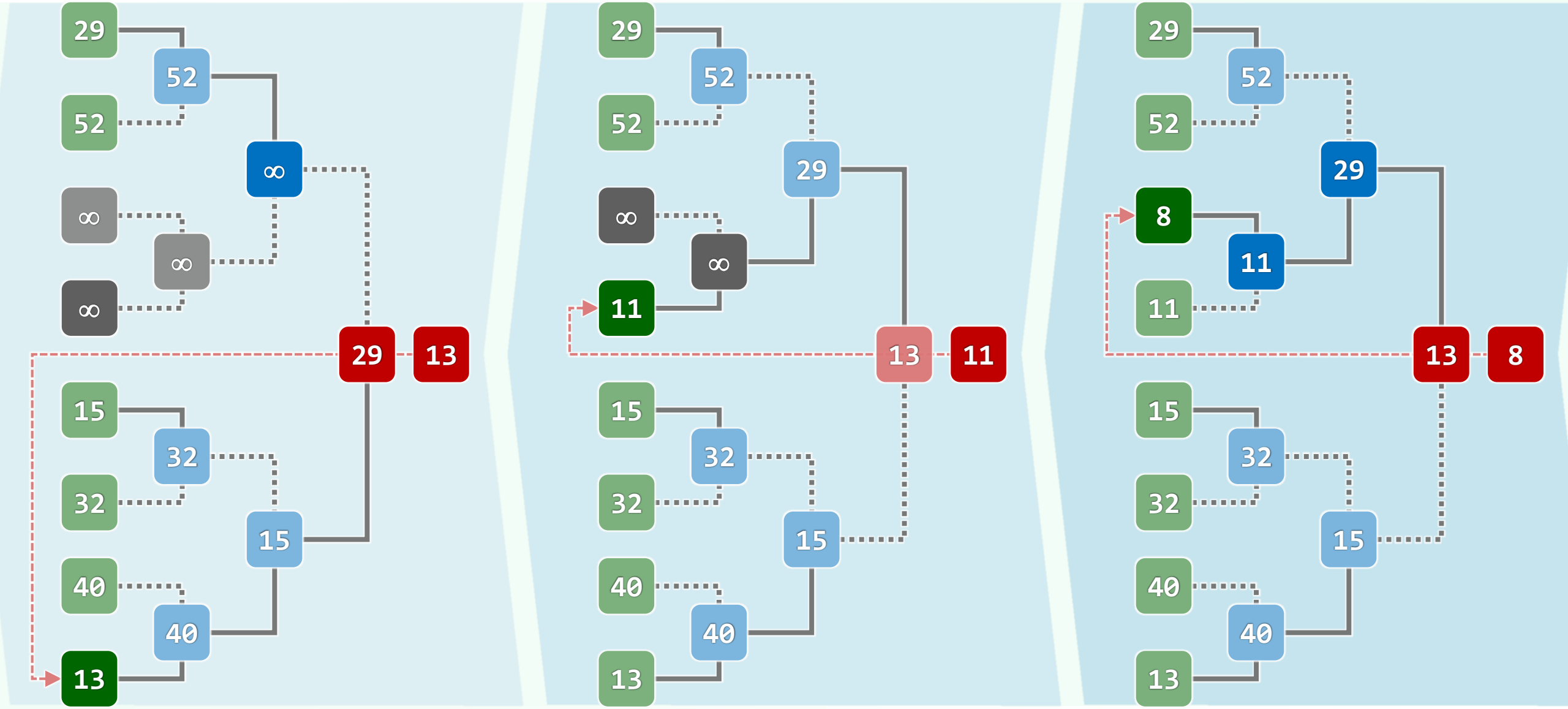


❖ 内部节点，记录对应比赛的败者

增设根的“父节点”，记录冠军



实例（1/2）


29
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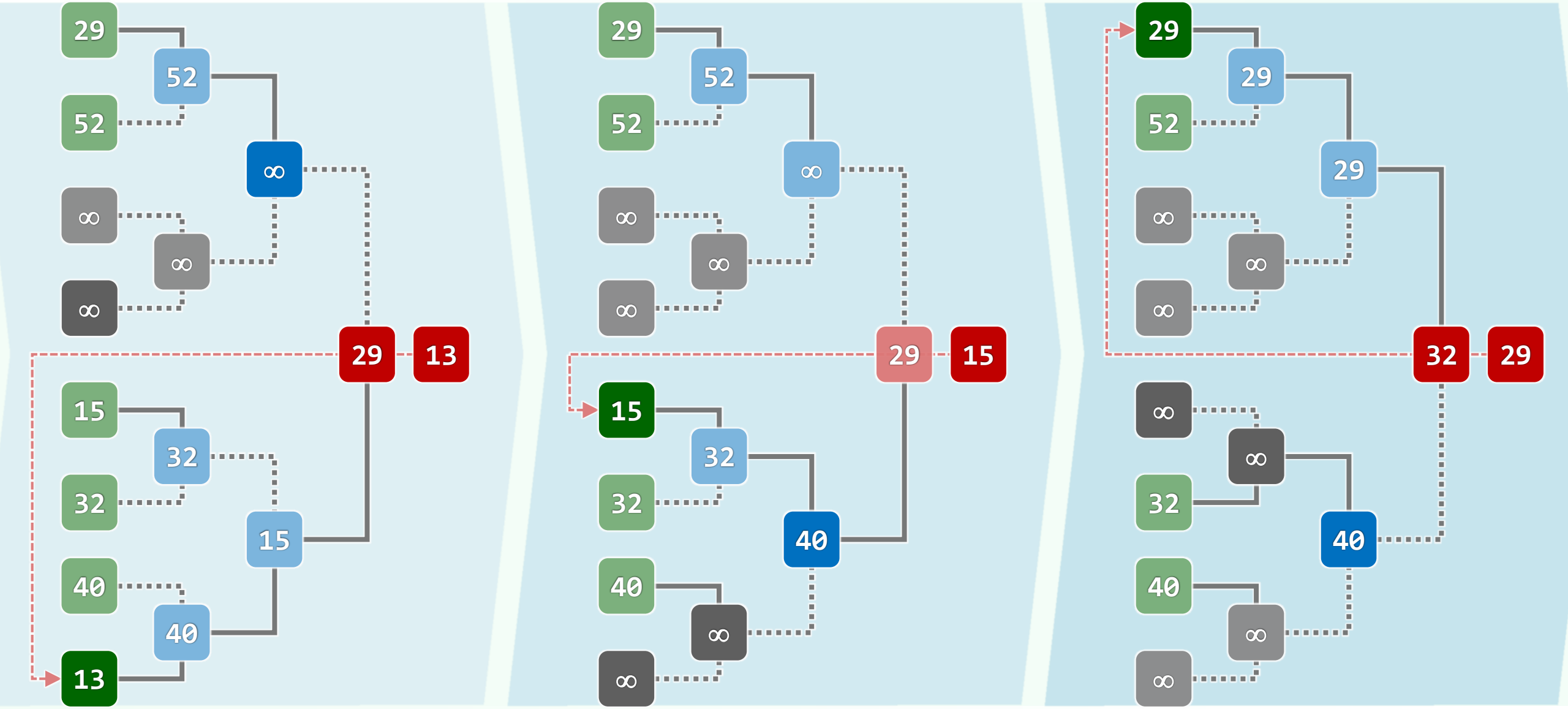
实例（2/2）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12-E

优先级队列

多叉堆

他说到一件事，时常萦绕我的心头，就是每个人若能窥清其他人的心意，那么愿意下来的人会多于愿意高升的人

匏有苦叶，濟有深涉

深則厲，淺則揭

邓俊辉

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优先级搜索

- 回顾图的PFS以及统一框架：g->pfs()...
- 无论何种算法，差异仅在于所采用的优先级更新器prioUpdater()
  - Prim算法：g->pfs(0, PrimPU());
  - Dijkstra算法：g->pfs(0, DijkPU());
- 每一节点引入遍历树后，都需要
  - 更新树外顶点的优先级（数），并
  - 选出新的优先级最高者
- 若采用邻接表，两类操作的累计时间，分别为$O(n+e)$和$O(n^2)$
- 能否更快呢?

优先级队列

❖ 自然地，PFS中的各顶点可组织为优先级队列

❖ 为此需要使用PQ接口

heapify():	由n个顶点创建初始PQ	总计$O(n)$
delMax():	取优先级最高（极短）跨边(u, w)	总计$O(\ n\ *\ \log n\ )$
increase():	更新所有关联顶点到U的距离，提高优先级	总计$O(\ e\ *\ \log n\ )$

❖ 总体运行时间 = $O(\ (n+e)*\log n\ )$

- 对于稀疏图，处理效率很高
- 对于稠密图，反而不如常规实现的版本

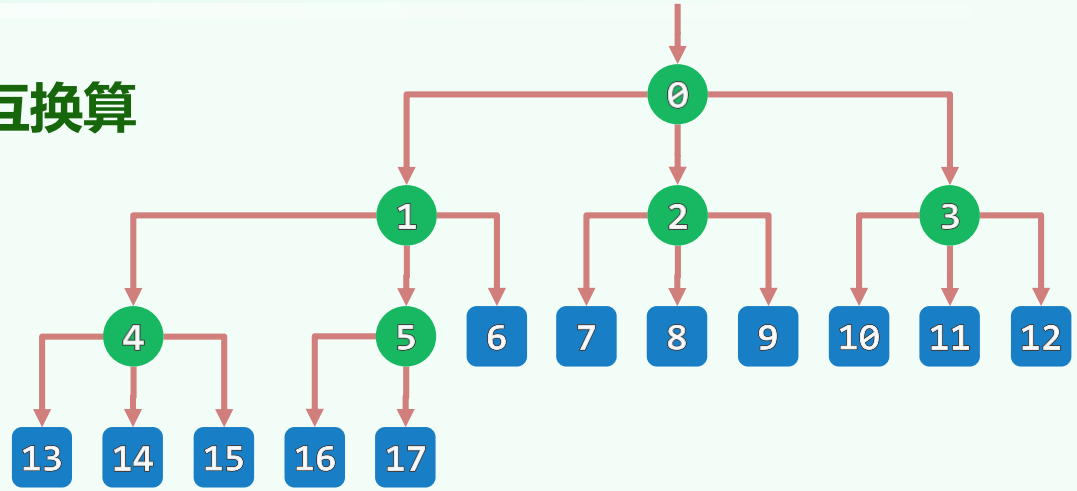
❖ 有无更好的办法?

多叉堆

❖ 仍可基于向量实现，且父、子节点的秩可简明地相互换算

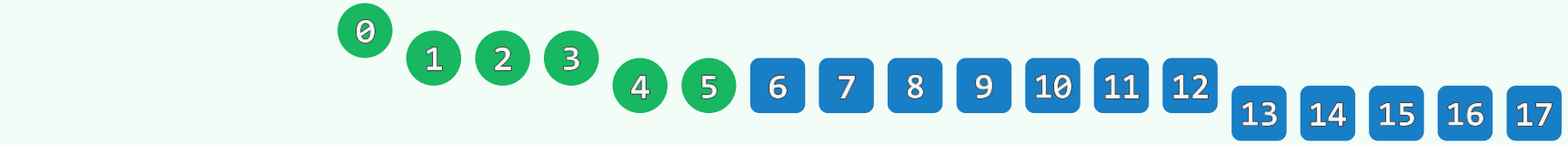
- $parent(k) = \lfloor (k-1)/d \rfloor$
- $child(k,i) = k \cdot d + i,\ 0 < i \leq d$

//d不是2的幂时，不能借助移位加速秩的换算



❖ heapify(): O(n)

//不可能再快了



❖ delMax(): O(logn)

//实质就是percolateDown()，已是极限了——为什么？

❖ increase(): O(logn)

//实质就是percolateUp() —— 似乎仍有改进空间...

上山容易下山难

❖ 若将二叉堆改成多叉堆（d-heap）

则堆高降至

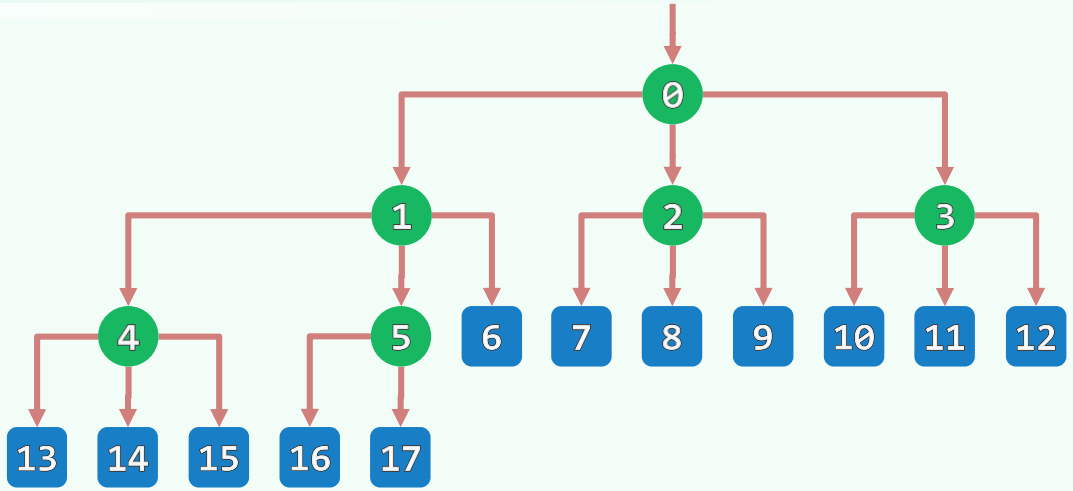
$$\mathcal{O}(\log_d n)$$

❖ 相应地，上滤成本降至

$$\log_d n$$

但（只要d>4）下滤成本却增至

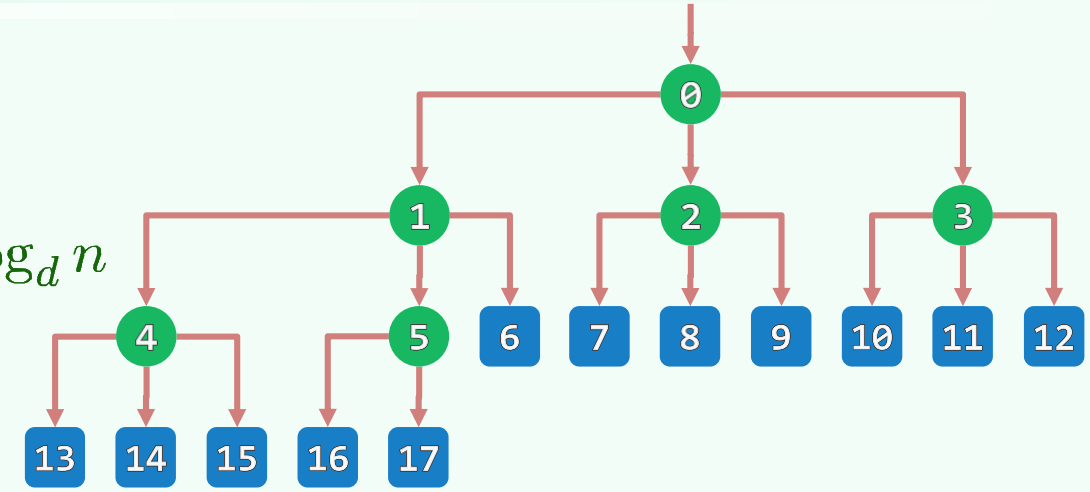
$$d \cdot \log_d n = \frac{d}{\ln d} \cdot \ln n$$



PFS

❖ 如此，PFS的运行时间将是：

$$n \cdot d \cdot \log_d n \ + \ e \cdot \log_d n \ = \ (n \cdot d \ + \ e) \cdot \log_d n$$



❖ 取 $d \approx e/n + 2$ 时

总体性能达到最优：$\mathcal{O}(e \cdot \log_{(e/n+2)} n)$

❖ 对于稀疏图保持高效：$e \cdot \log_{(e/n+2)} n \ \approx \ n \cdot \log_{(n/n+2)} n \ = \ \mathcal{O}(n \log n)$

对于稠密图改进极大：$e \cdot \log_{(e/n+2)} n \ \approx \ n^2 \cdot \log_{(n^2/n+2)} n \ \approx \ n^2 \ = \ \mathcal{O}(e)$

对于一般的图，会自适应地实现最优

12-F1

优先级队列

左式堆：沿藤合并

God's right hand is gentle

But terrible is his left hand

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堆合并

- H = merge(A, B): 将堆A和B合二为一

  //不妨设|A| = n ≥ m = |B|

- 方法一: A.insert(B.delMax())

  $O(m * (\log m + \log(n + m)))$

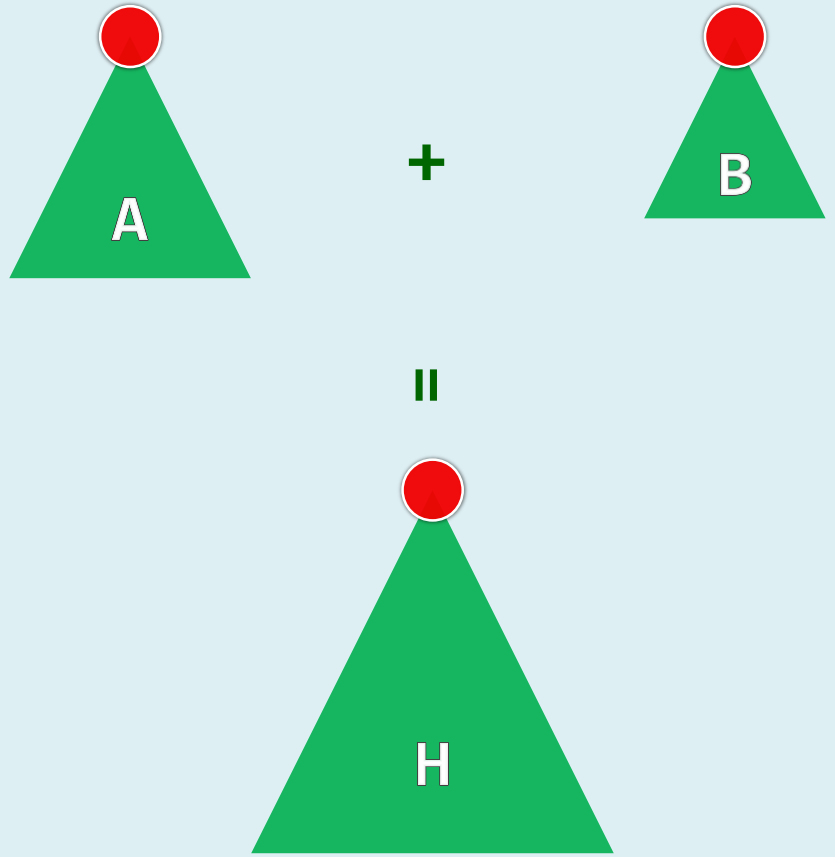
  $= O(m * \log(n + m))$

- 方法二: union(A, B).heapify(n + m)

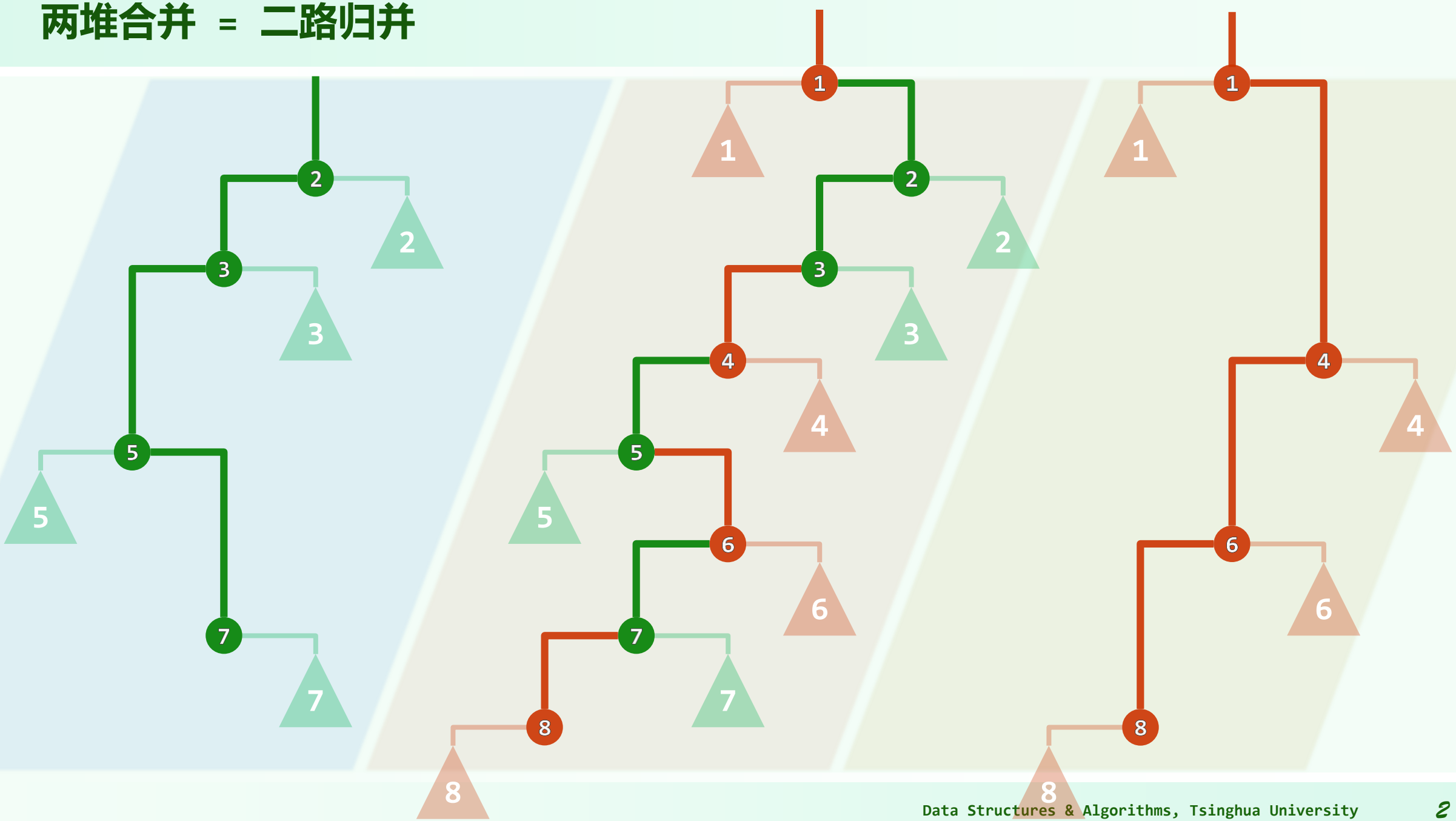
  $O(m + n)$

- 有没有更好的办法? 比如...

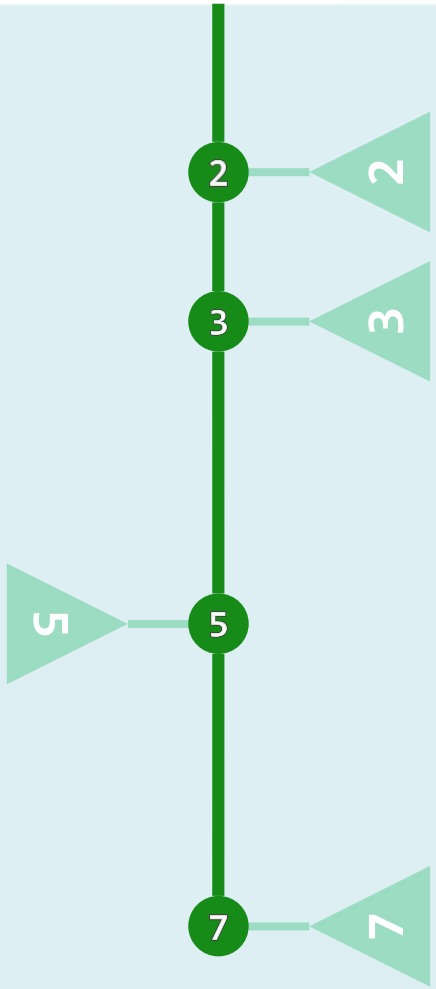
- 可否奢望在...$O(\log n)$...时间内实现merge()?

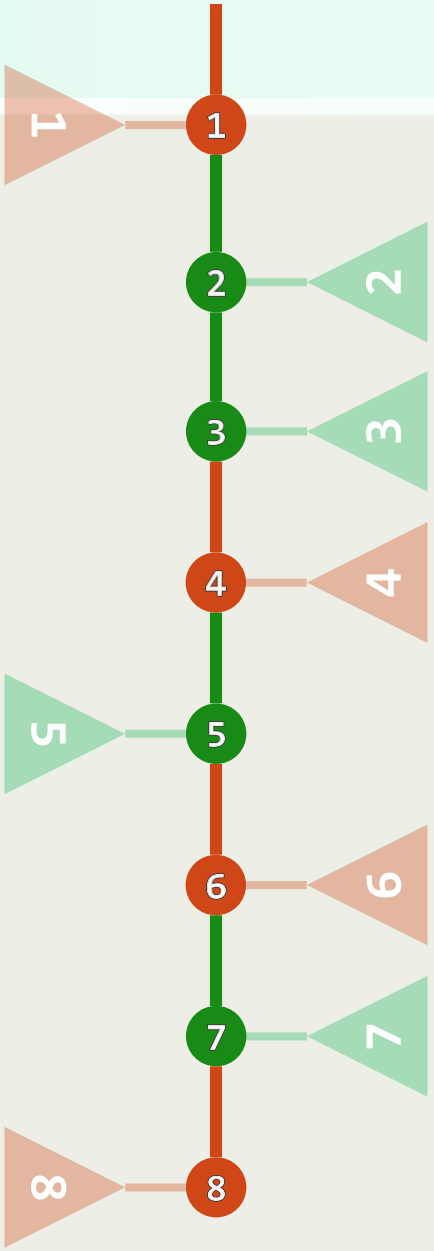


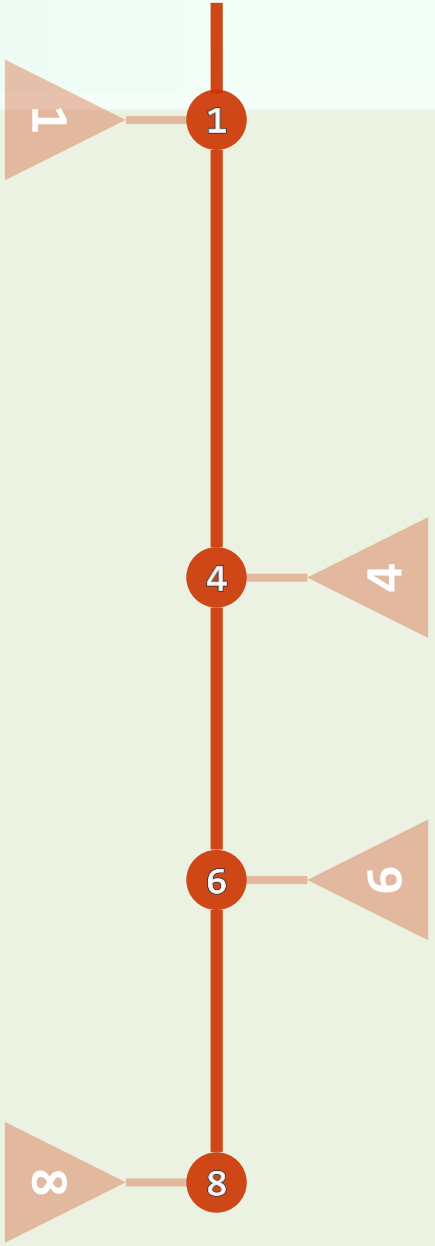
两堆合并 = 二路归并



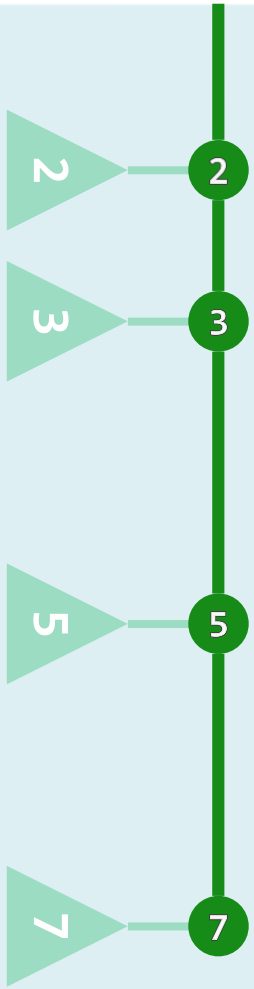
两堆合并 = 二路归并

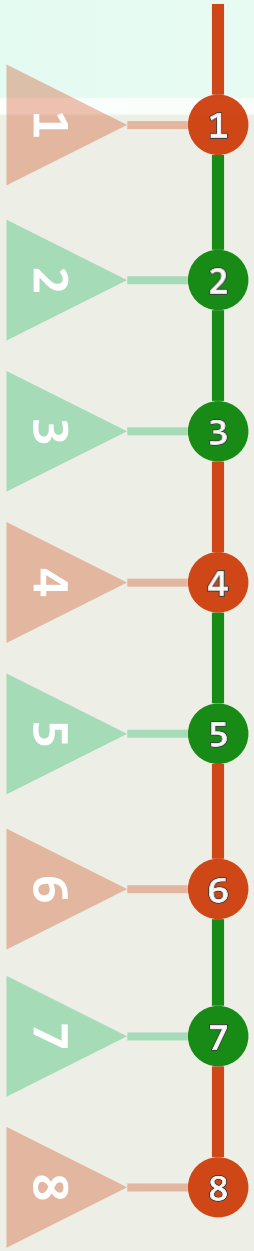


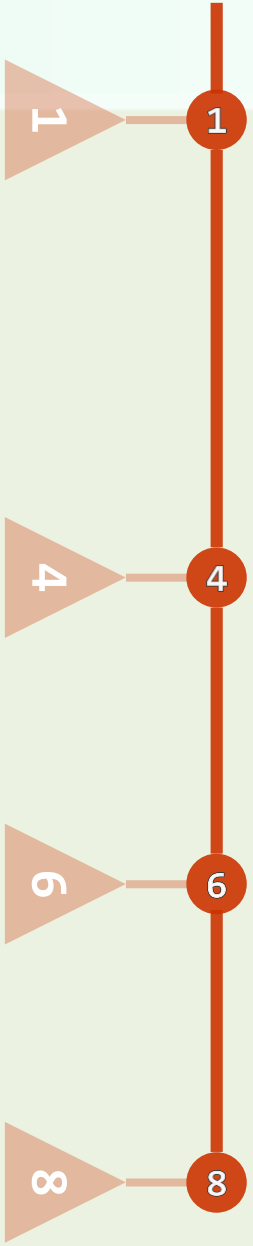




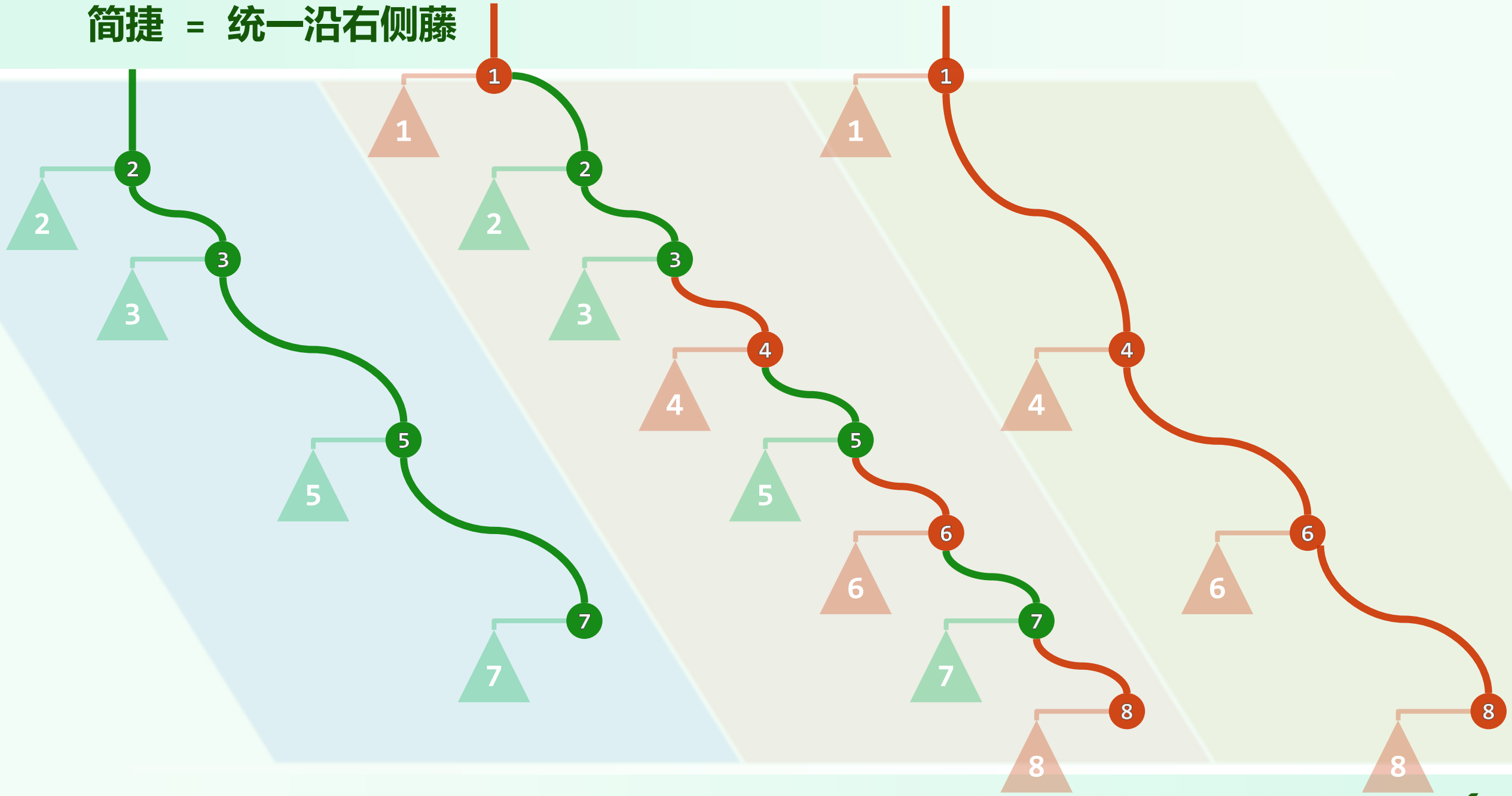
简捷 = 统一沿右侧藤







简捷 = 统一沿右侧藤



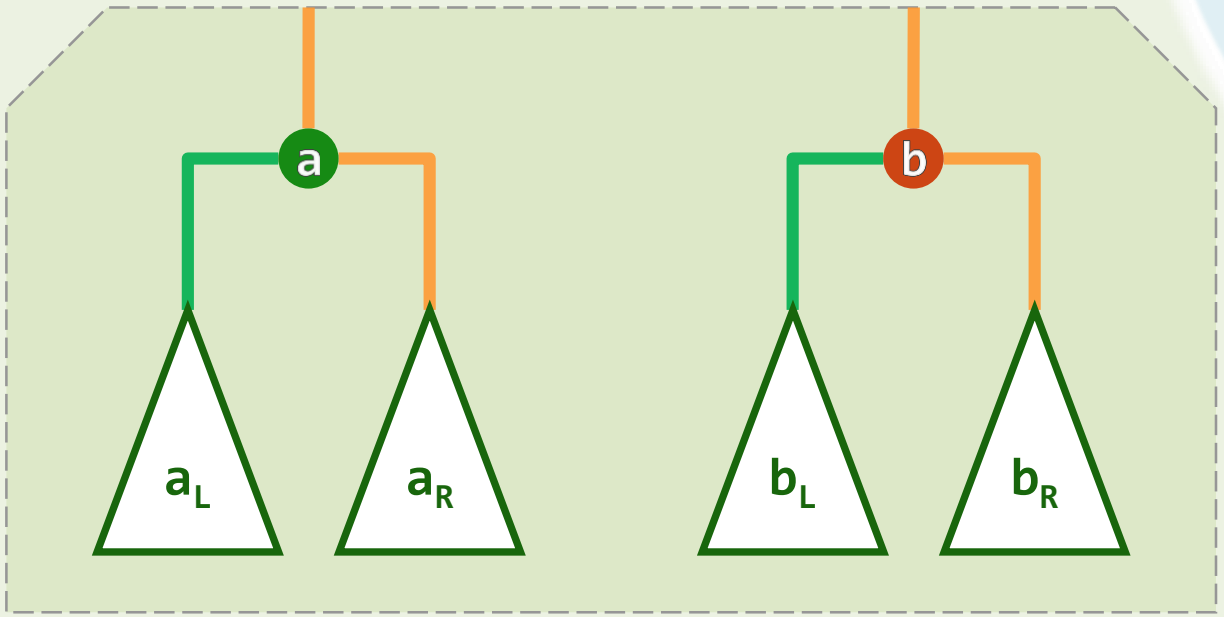
LeftHeap

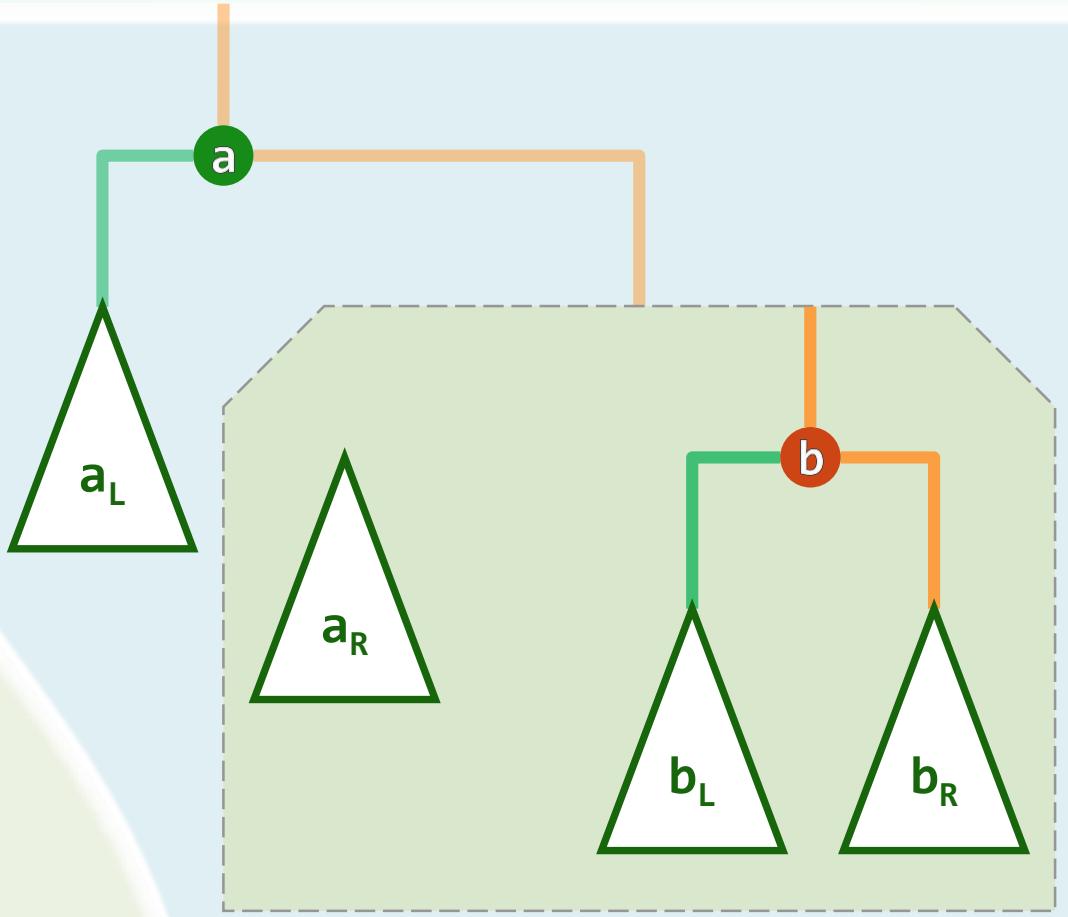
```
template <typename T> //基于二叉树，以左式堆形式实现的优先级队列
class PQ_LeftHeap : public PQ<T>, public BinTree<T> {
public:  T getMax() { return _root->data; }
         void insert(T); T delMax(); //均基于统一的合并操作实现...
         PQ_LeftHeap( PQ_LeftHeap & A, PQ_LeftHeap & B ) {
            _root = merge(A._root, B._root); _size = A._size + B._size;
            A._root = B._root = NULL; A._size = B._size = 0;
         }
};

template <typename T> BinNodePosi<T> merge(BinNodePosi<T>, BinNodePosi<T>);
```

递归实现

所需时间 ∝ 右侧藤总长

$\cancel{\mathcal{O}(n)}$ $\mathcal{O}(\log n)$





如何...控制藤长以...持续合并?

12-F2

优先级队列

左式堆：NPL与控制藤长

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君子居则贵左，用兵则贵右

可持续 = 单侧倾斜

- C. A. Crane, 1972:

  保持堆序性，附加新条件，使得

  在堆合并过程中，只涉及少量节点：$O(\log n)$

- 新条件 = 单侧倾斜：

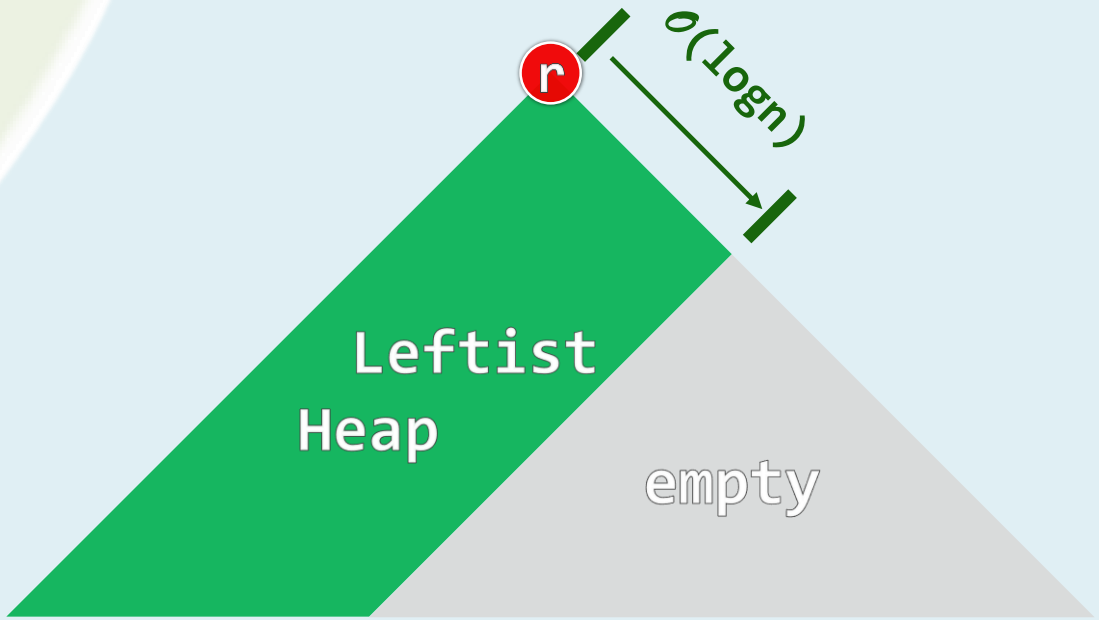
  节点分布偏向于左侧

  合并操作只涉及右侧

- 可是，果真如此，则拓扑上...

  不见得是完全二叉树，结构性无法保证！？

- 是的，实际上，结构性并非堆结构的本质要求



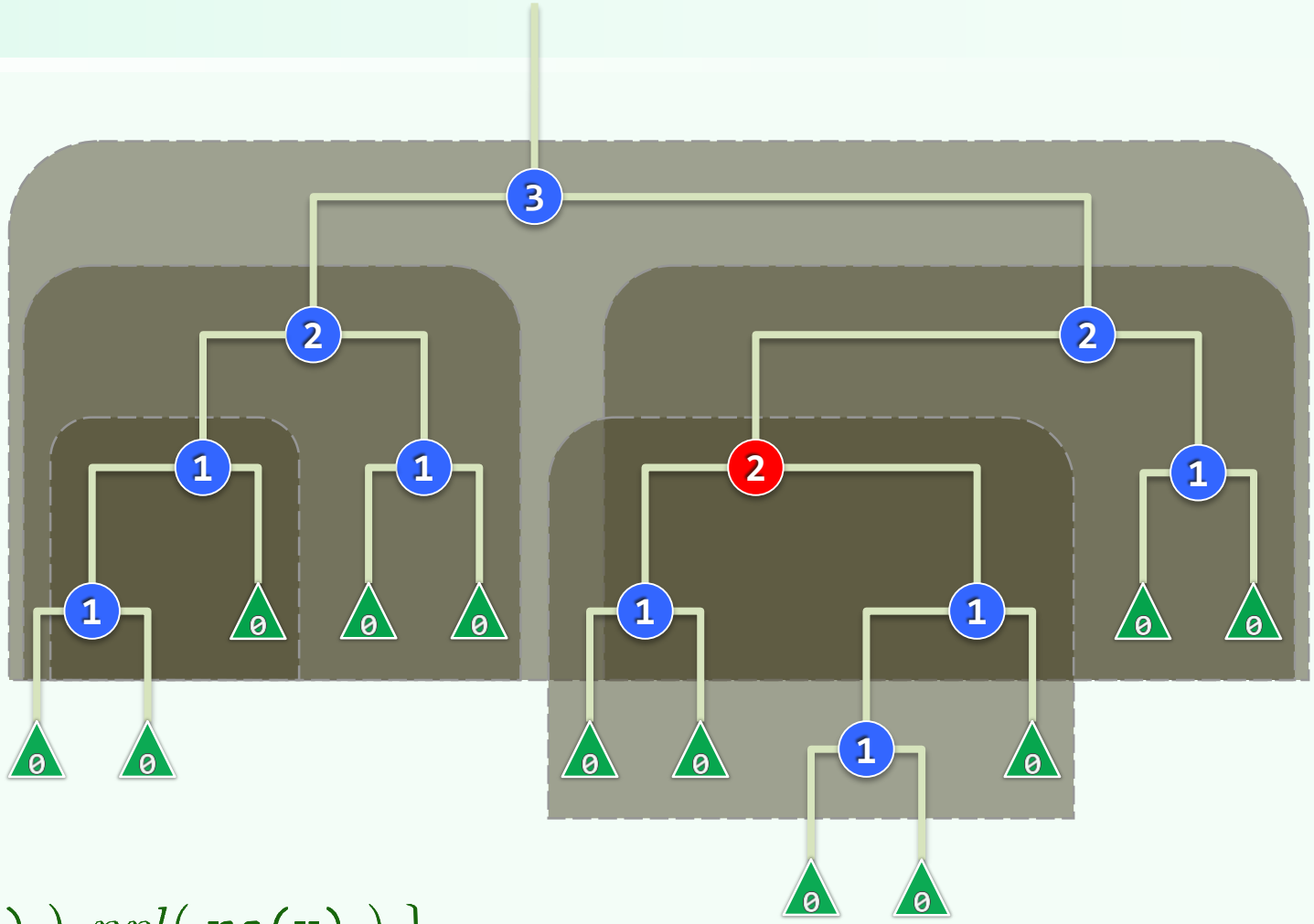
空节点路径长度

❖ 引入所有的外部节点

- 消除一度节点
- 转为真二叉树

❖ Null Path Length

- $npl(\texttt{NULL}) = 0$
- $npl(x) = 1 + \min\{\, npl(\texttt{lc(x)}), npl(\texttt{rc(x)}) \,\}$



❖ 验证： npl(x) = x到外部节点的最近距离 = 以x为根的最大满子树的高度

左式堆 = 处处左倾

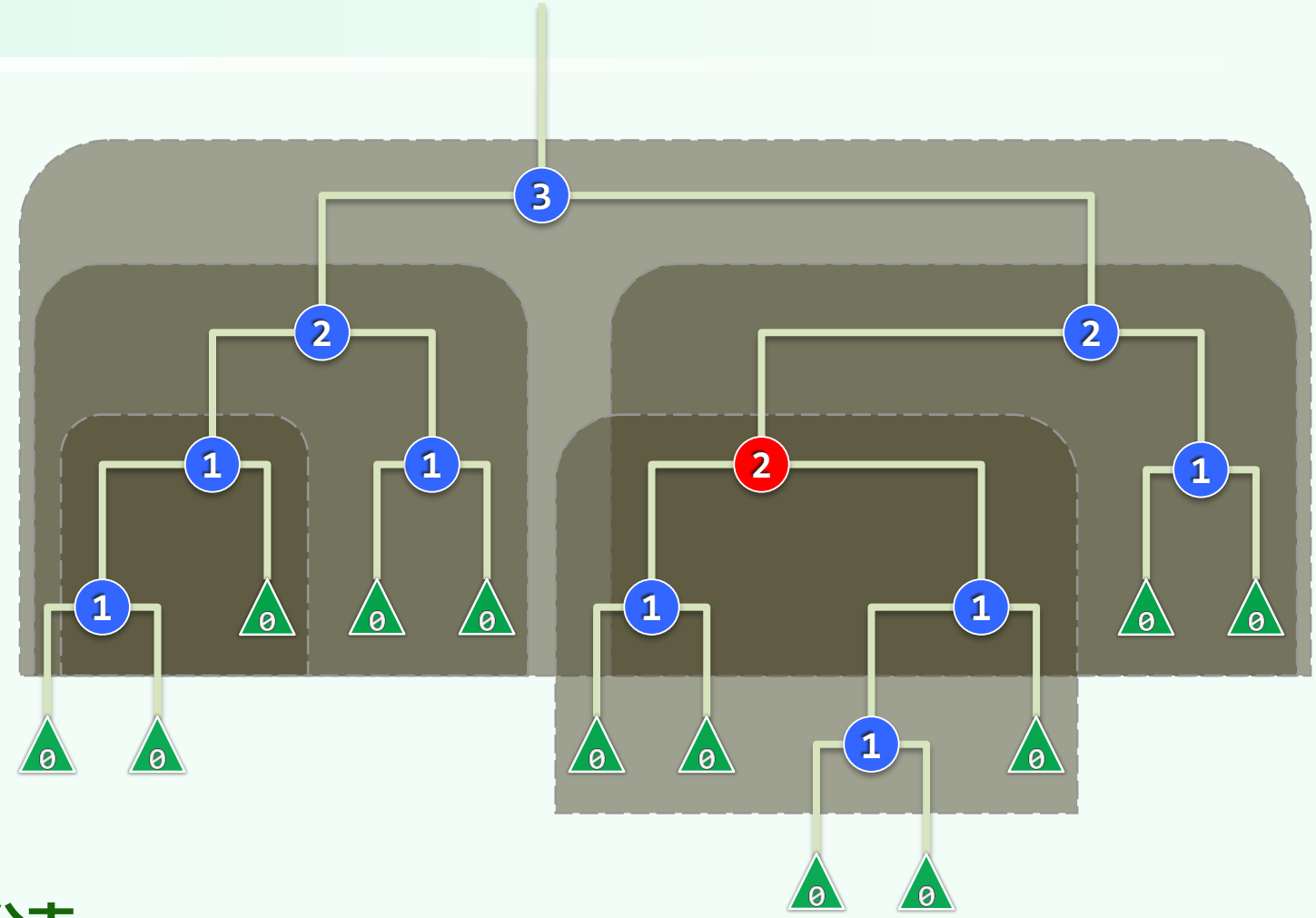
- 对任何内节点x，都有：

$$npl(\texttt{lc(x)}) \geq npl(\texttt{rc(x)})$$

- 推论：

$$npl(\texttt{x}) = 1 + npl(\texttt{rc(x)})$$

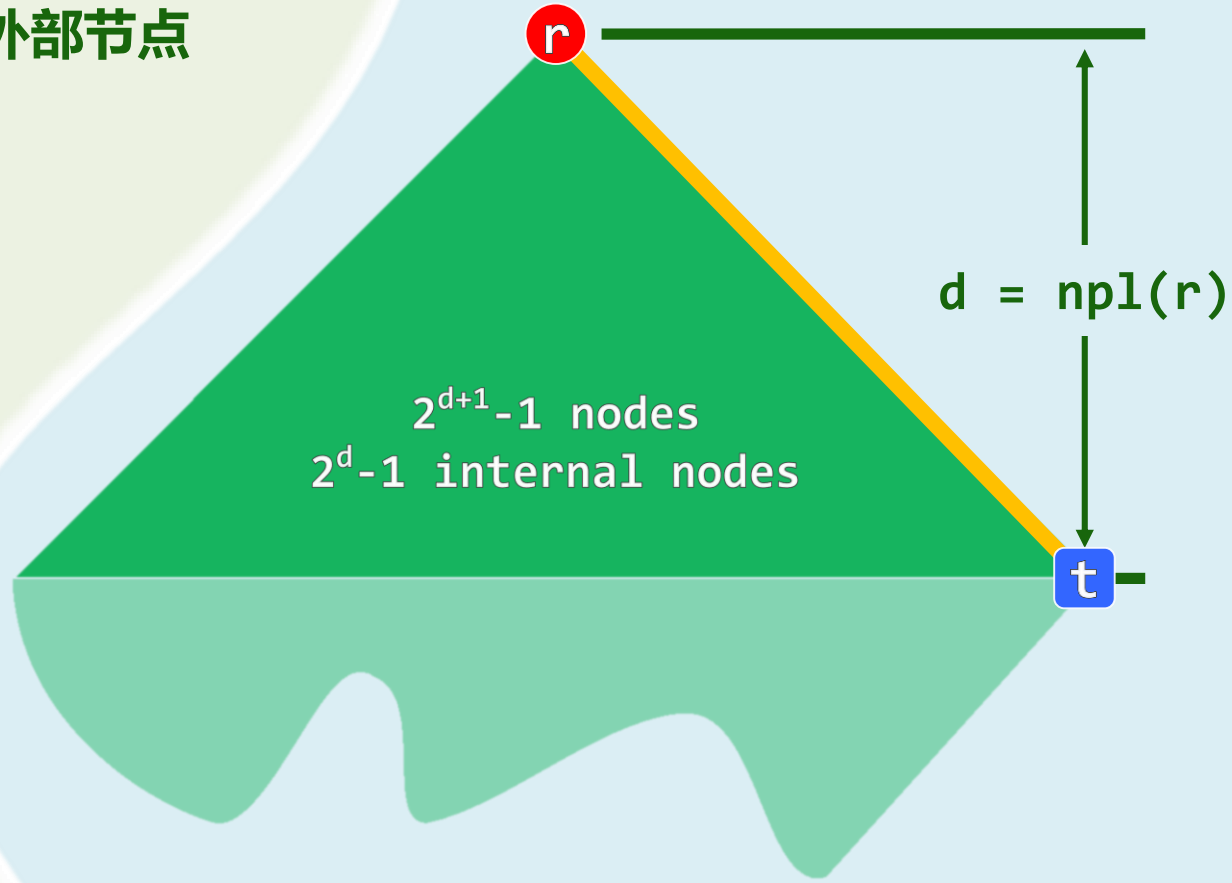
- 左倾性与堆序性，相容而不矛盾
- 左式堆的子堆，必是左式堆
- 左式堆倾向于更多节点分布于左侧分支
- 这是否意味着，左子堆的规模和高度必然大于右子堆?



右侧链

- rChain(x)：从节点x出发，一直沿右分支前进
- 特别地，rChain(r)的终点，即全堆中最浅的外部节点
  - $npl(r) \equiv |rChain(r)| = d$
  - 存在一棵以r为根、高度为d的满子树
- 右侧链长为d的左式堆，至少包含
  - $2^d - 1$ 个内部节点
  - $2^{d+1} - 1$ 个节点
- 反之，包含n个节点的左式堆，右侧链长度

$$d \leq \lfloor \log_2 (n+1) \rfloor - 1 = \mathcal{O}(\log n)$$



12-F3

优先级队列

左式堆：合并算法

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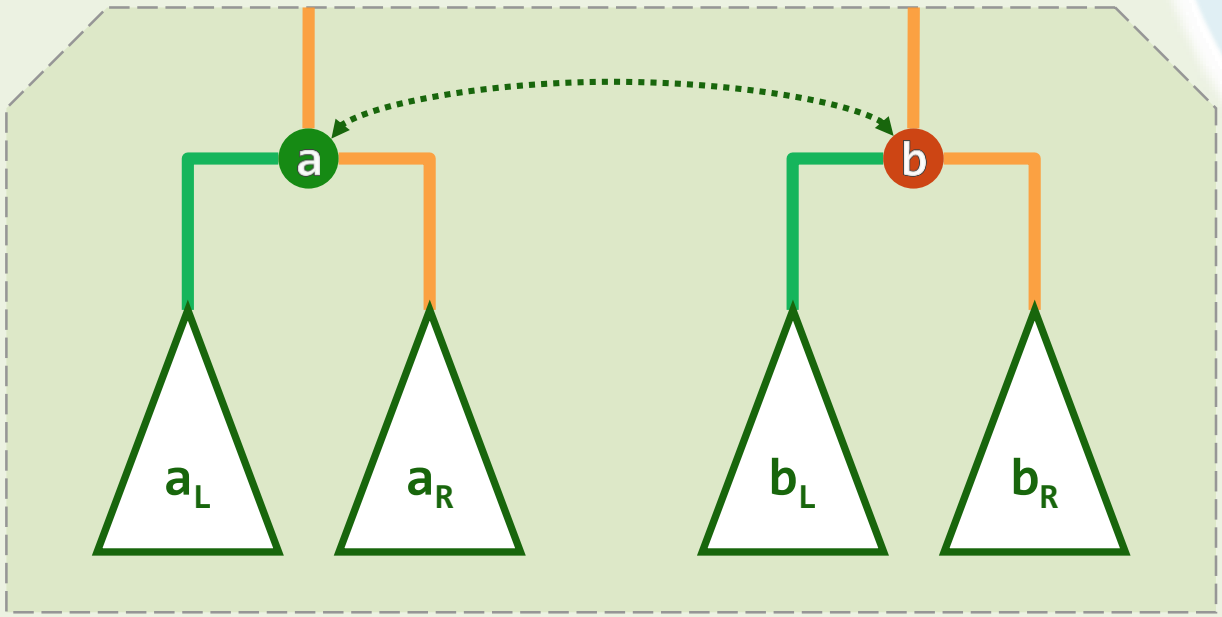
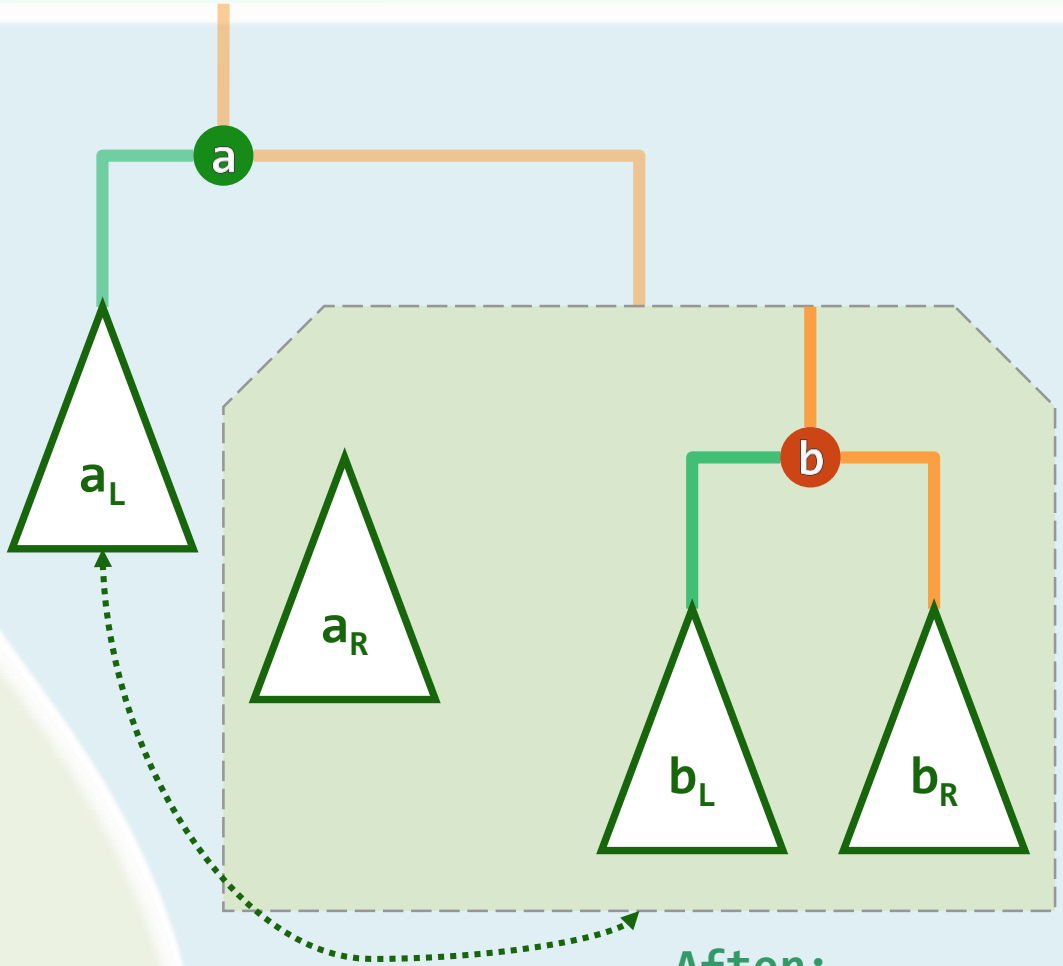
左之左之，君子宜之；右之右之，君子有之

递归：前处理 + 后处理

Before:
compare priority &
swap if necessary



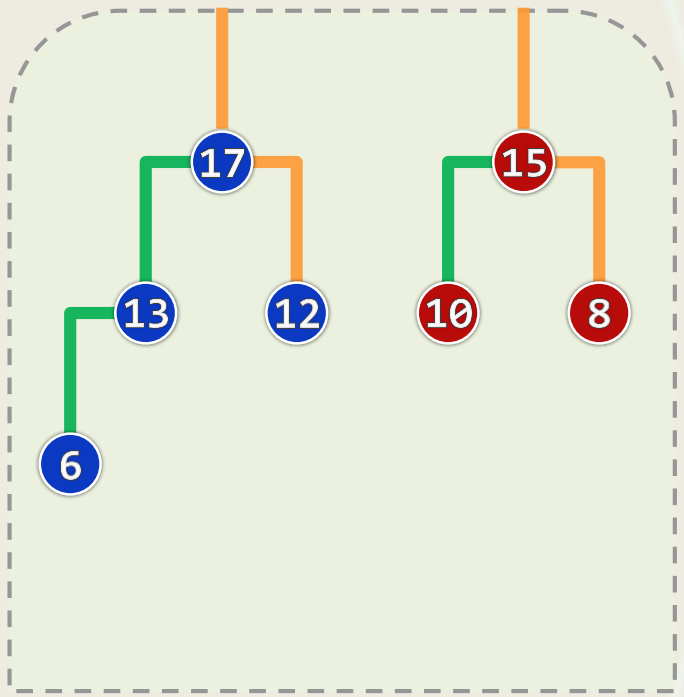


After:
compare NPL &
flip if necessary

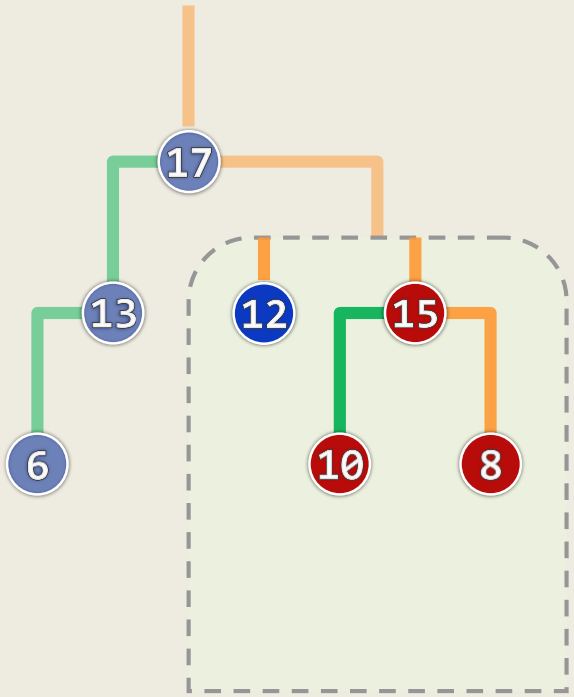
递归实现

```
template <typename T> BinNodePosi<T> merge( BinNodePosi<T> a, BinNodePosi<T> b ) {
   if ( !a ) return b; if ( !b ) return a; //递归基
   if ( lt( a->data, b->data ) ) swap( a, b ); //确保a>=b
   ( a->rc = merge( a->rc, b ) )->parent = a; //将a的右子堆，与b合并
   if ( !a->lc || (a->lc->npl < a->rc->npl) ) //若有必要
      swap( a->lc, a->rc ); //交换a的左、右子堆，以确保左子堆的npl不小
   a->npl = a->rc ? 1 + a->rc->npl : 1; //更新a的npl
   return a; //返回合并后的堆顶
}
```

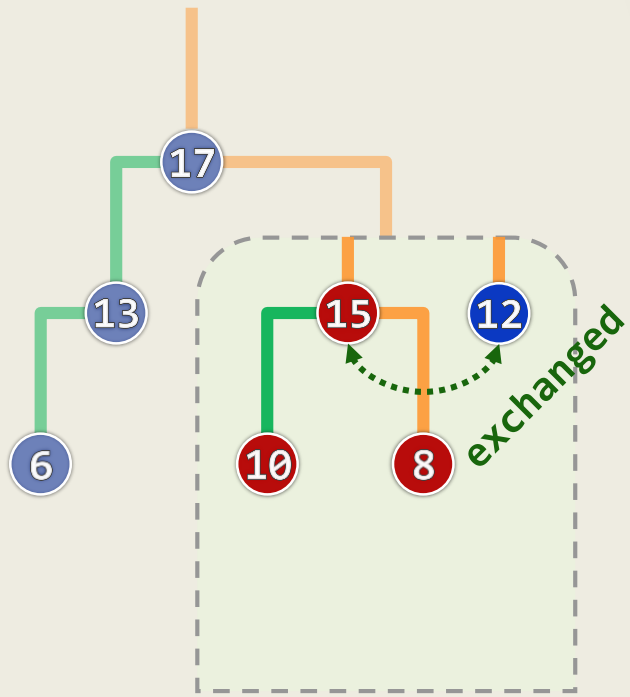
实例（1/5）



(a)

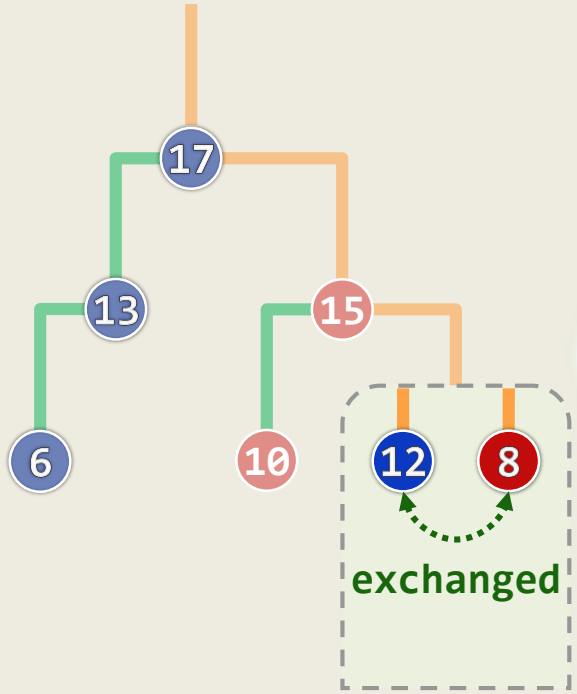


(b)



(c)

实例（2/5）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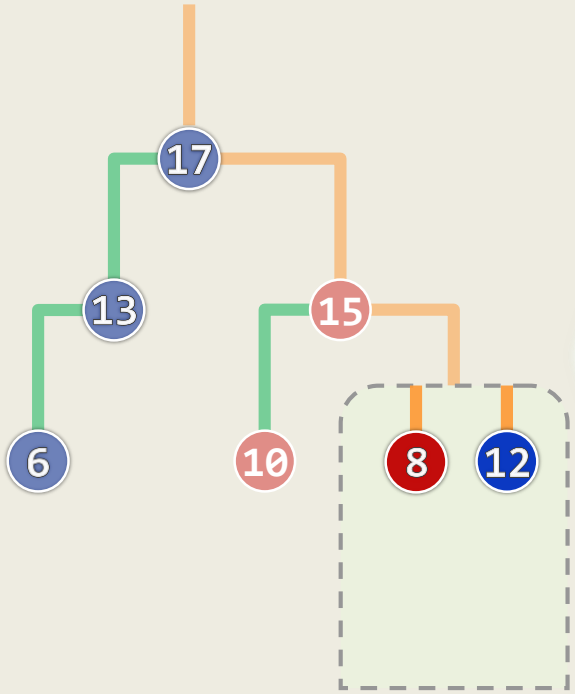
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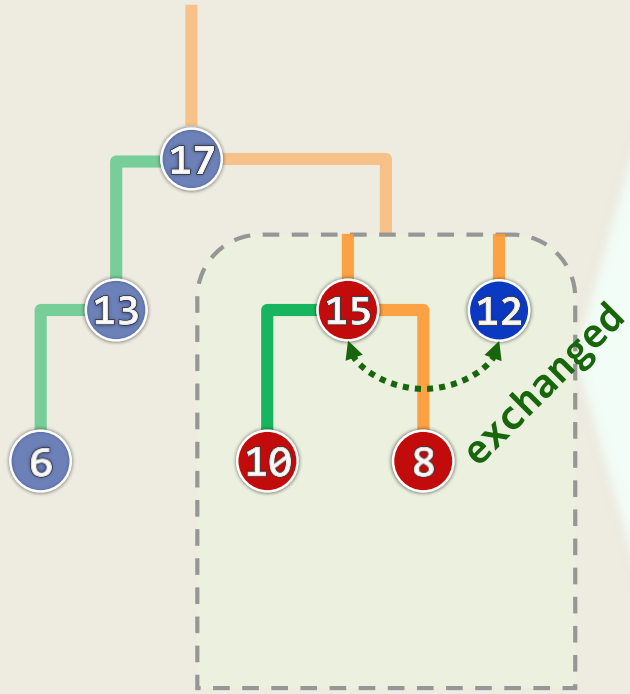
(d)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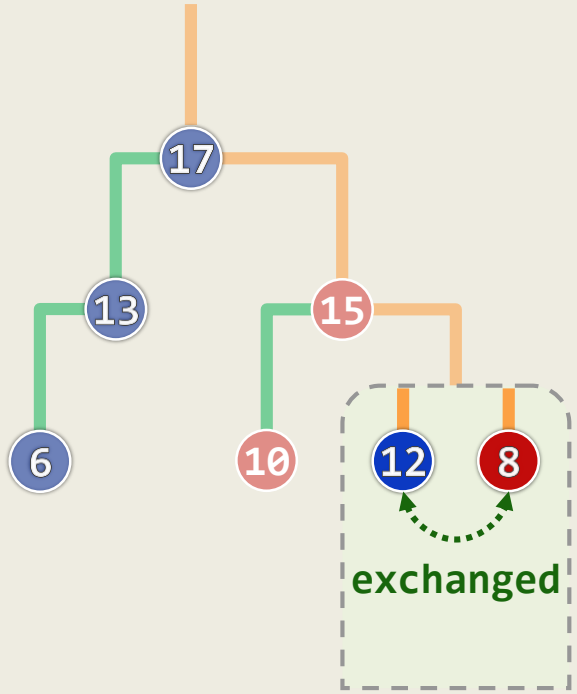
(c)

实例（3/5）



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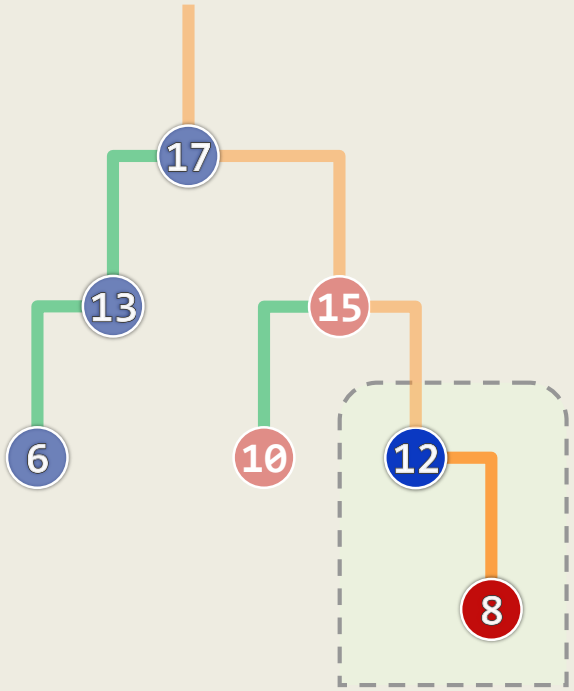
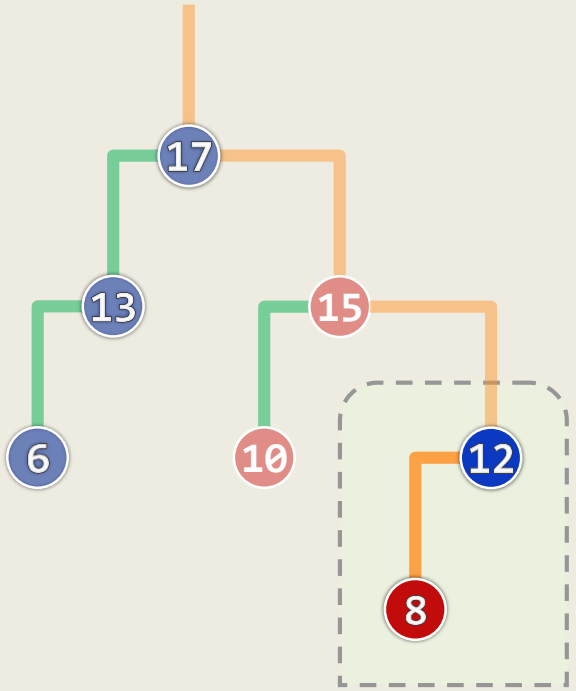
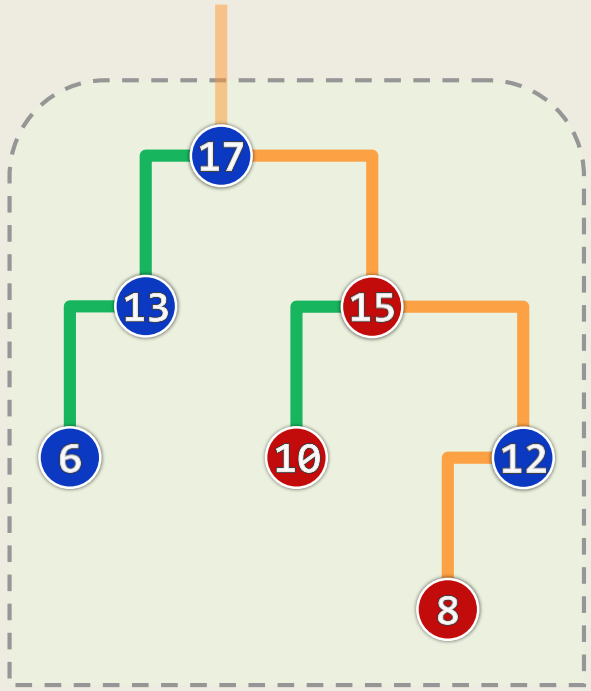


(f)

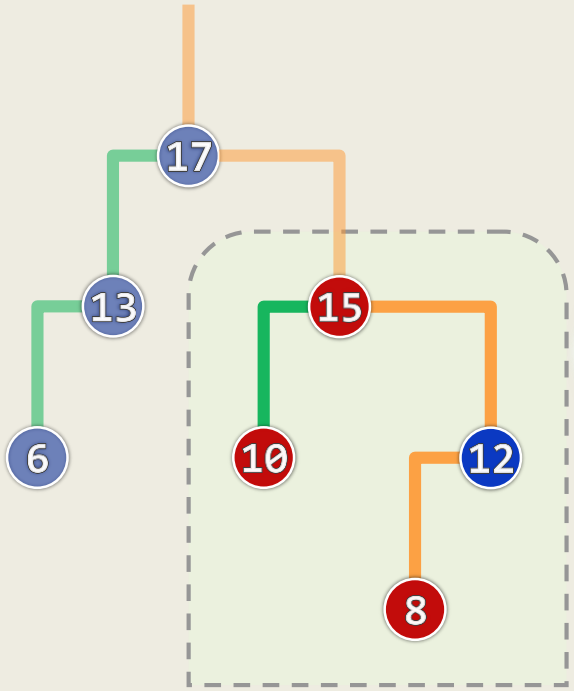


(g)

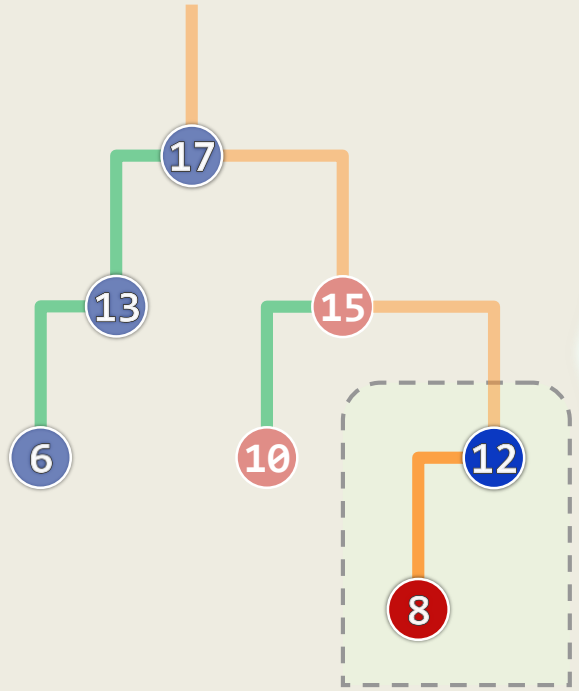
实例（4/5）



(i)

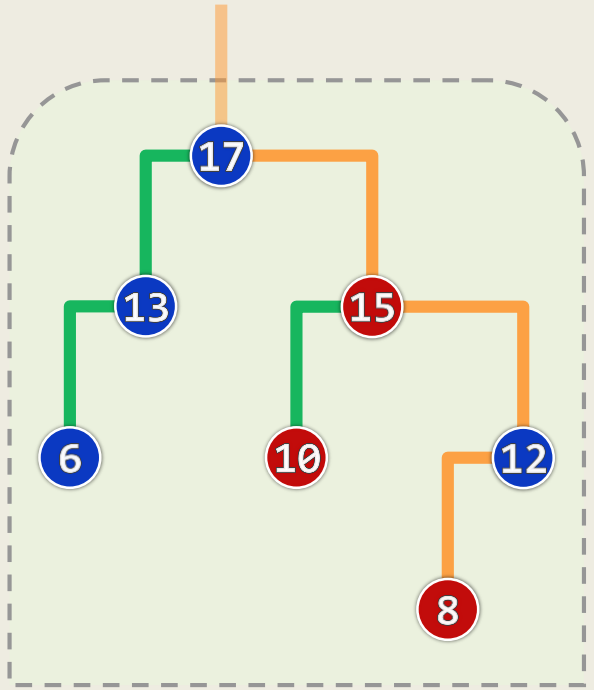


(h)

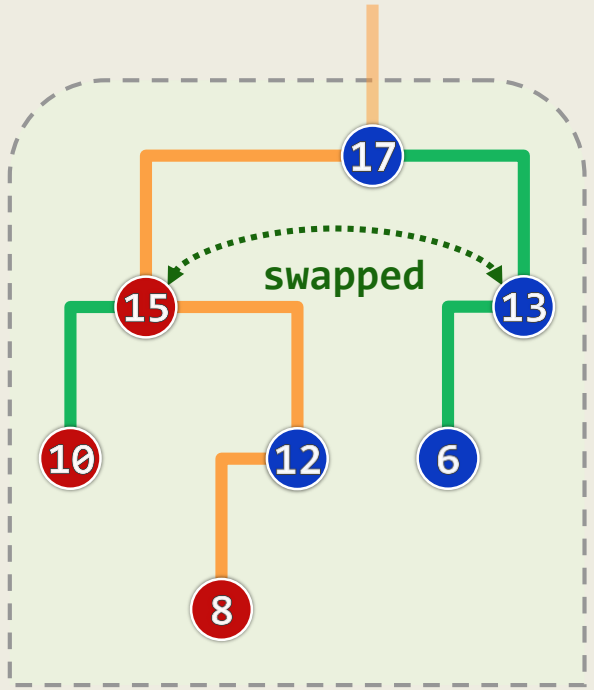


(g)

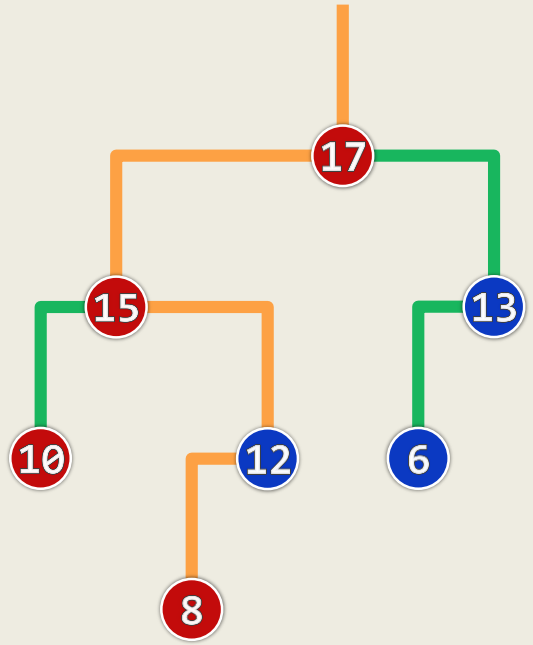
实例（5/5）



(i)



(j)



(k)

迭代实现

```
template <typename T> BinNodePosi<T> merge( BinNodePosi<T> a, BinNodePosi<T> b ) {
   if ( !a ) return b; if ( !b ) return a; //退化情况
   if ( lt( a->data, b->data ) ) swap( a, b ); //确保a>=b
   for ( ; a->rc; a = a->rc ) //沿右侧链做二路归并，直至堆a->rc先于b变空
      if ( lt( a->data, b->data ) ) { b->parent = a; swap( a->rc, b); } //接入b
   (a->rc = b)->parent = a; //直接接入b的残余部分（必然非空）
   for ( ; a; b = a, a = a->parent ) { //从a出发沿右侧链逐层回溯（b == a->rc）
      if ( !a->lc || (a->lc->npl < a->rc->npl) ) swap( a->lc, a->rc ); //确保npl合法
      a->npl = a->rc ? a->rc->npl + 1 : 1; //更新npl
   }
   return b; //返回合并后的堆顶
} //merge()
```

12-F4

优先级队列

左式堆：插入与删除

一招鲜，吃遍天

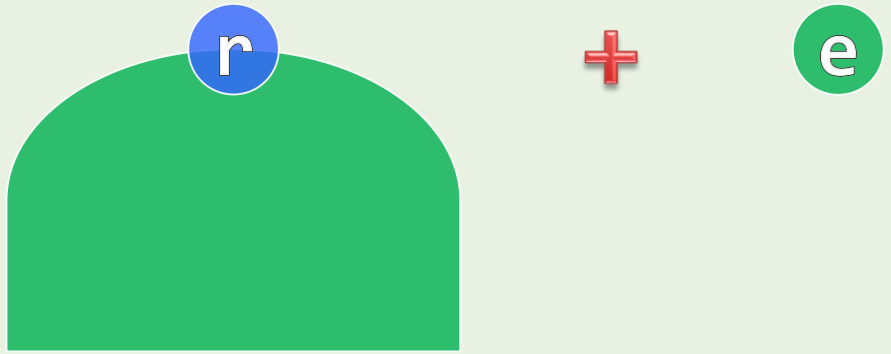
良田千顷，不如薄技在身

邓俊辉

deng@tsinghua.edu.cn

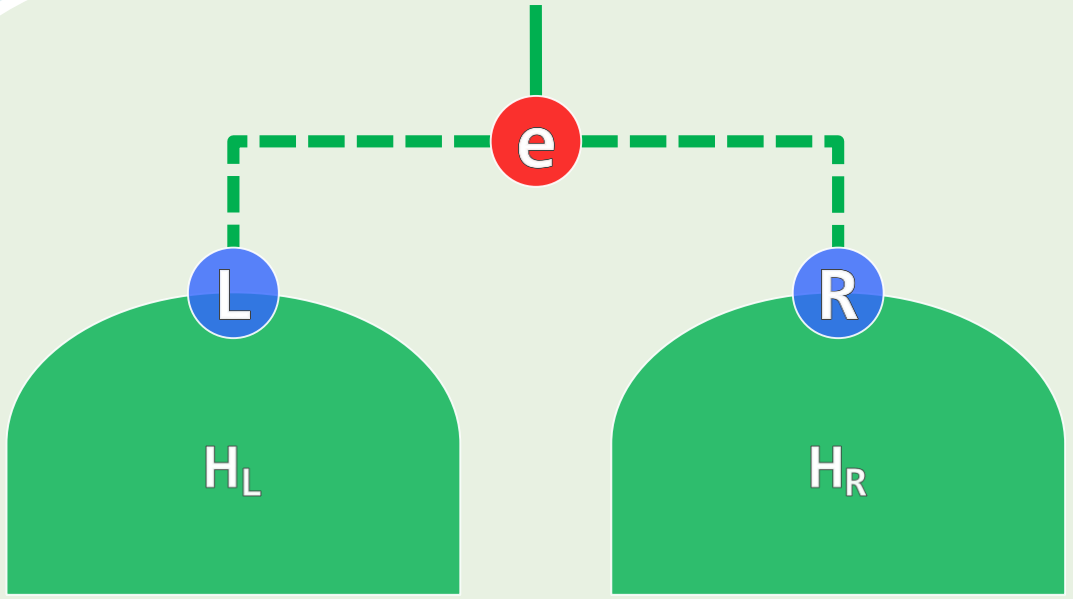
插入

```
template <typename T> void PQ_LeftHeap<T>::insert( T e ) { //O(logn)

   _root = merge( _root, new BinNode<T>( e, NULL ) );

   _size++;

}
```



删除

```
template <typename T> T PQ_LeftHeap<T>::delMax() { //O(logn)
   BinNodePosi<T> lHeap = _root->lc; if (lHeap) lHeap->parent = NULL;
   BinNodePosi<T> rHeap = _root->rc; if (rHeap) rHeap->parent = NULL;
   T e = _root->data;
   delete _root; _size--;
   _root = merge( lHeap, rHeap );
   return e;
}
```



12-XA

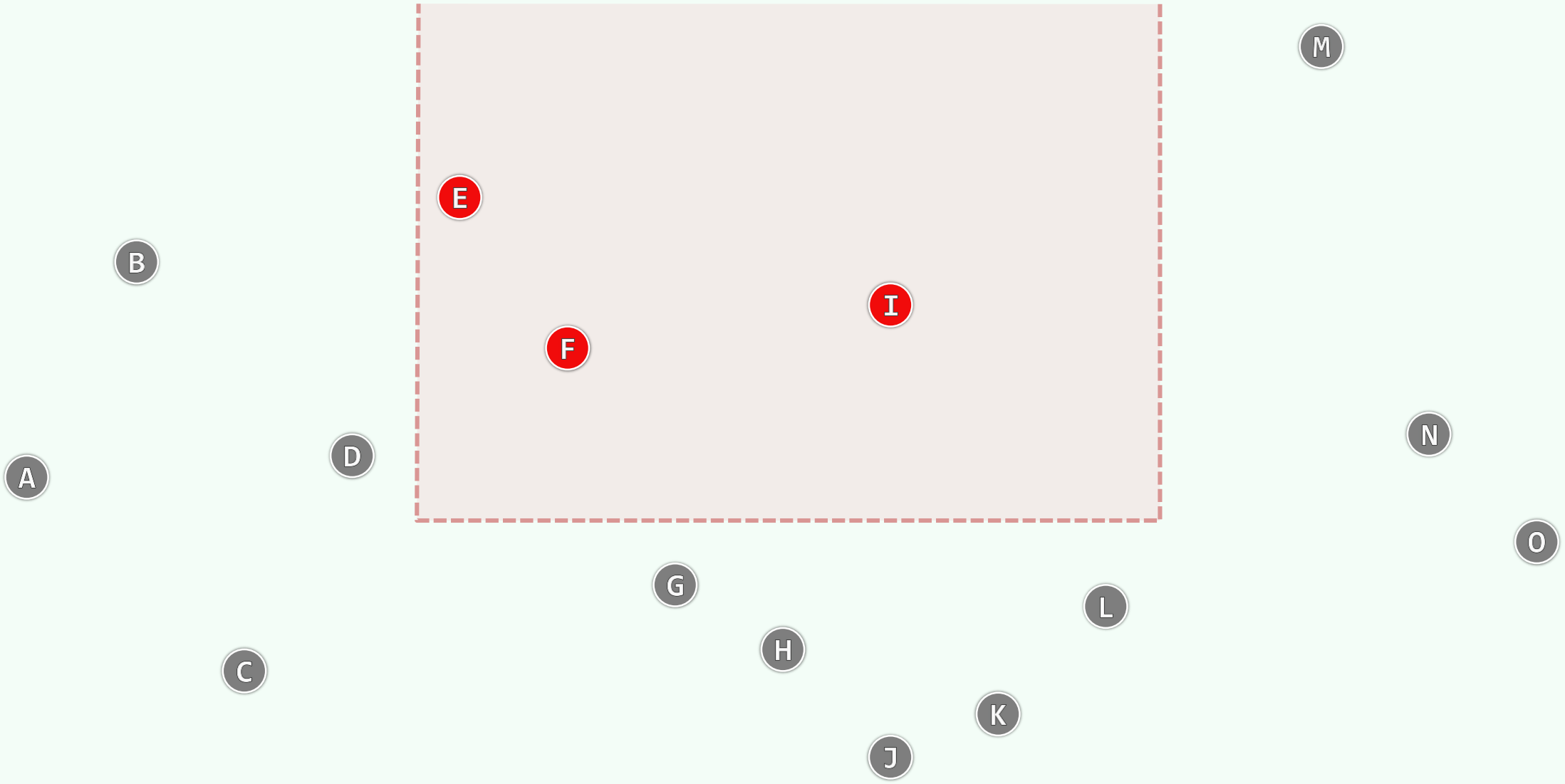
优先级队列

优先级搜索树

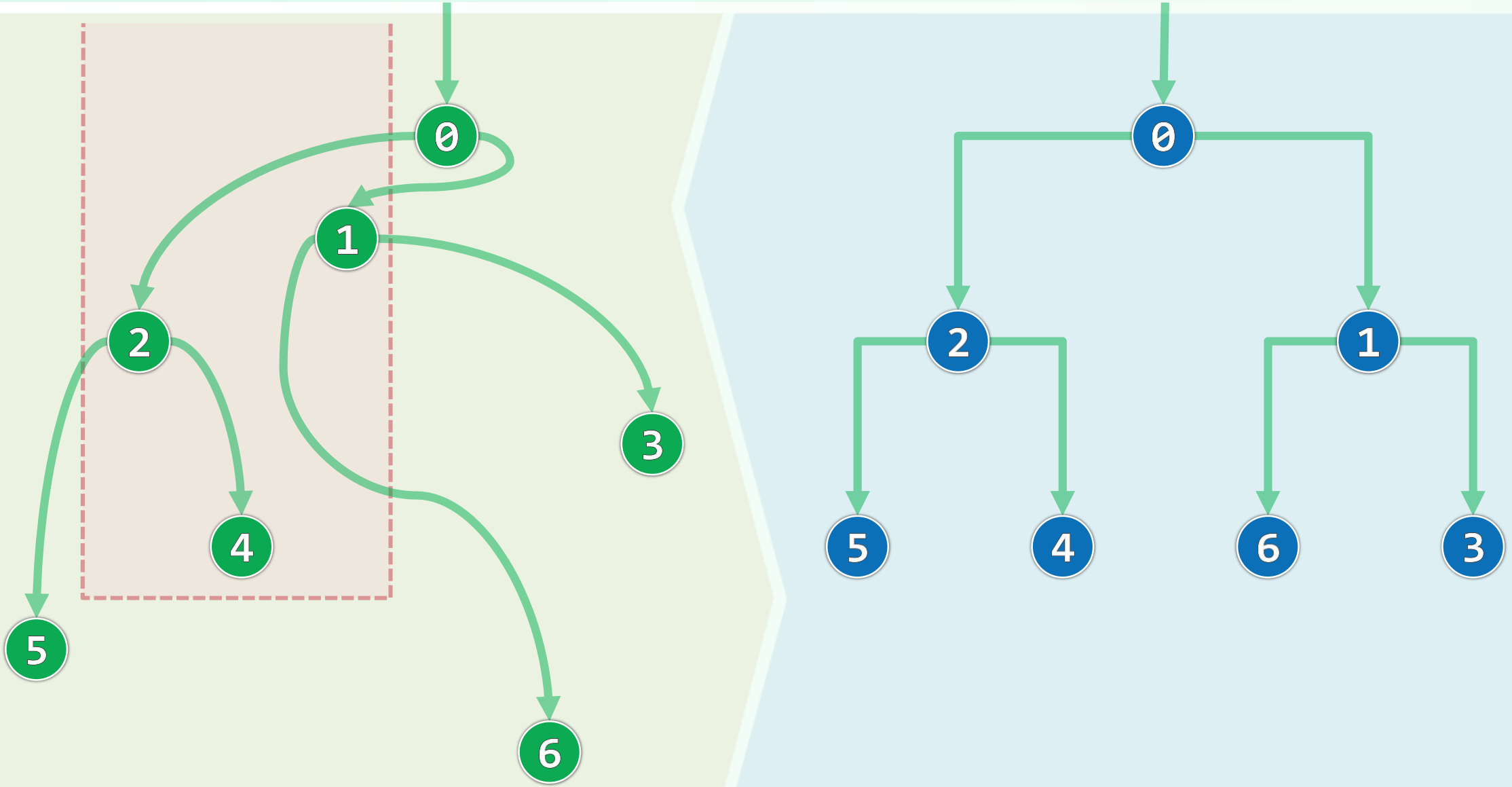
邓俊辉

deng@tsinghua.edu.cn

Grounded Range Query

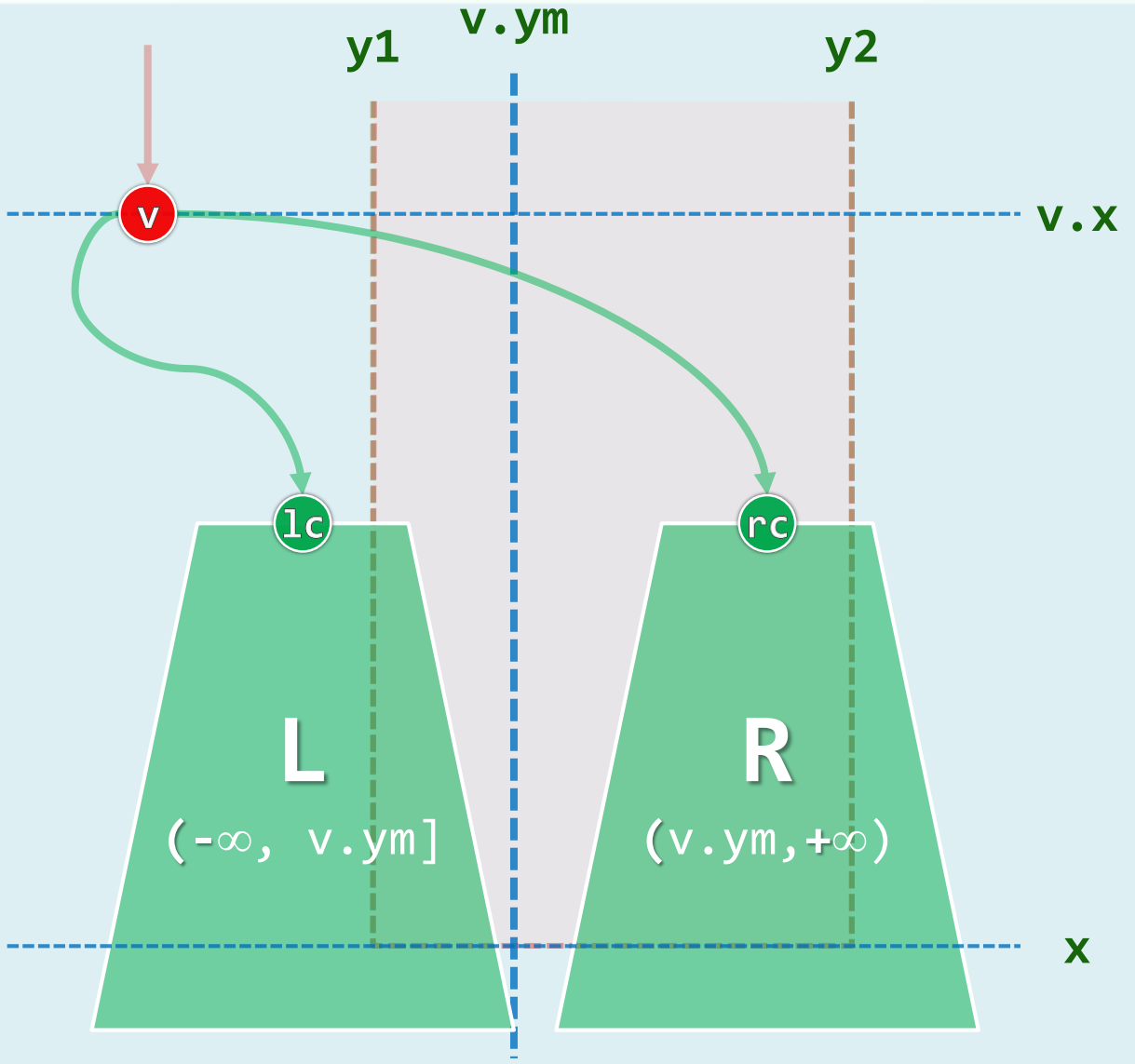


Priority Search Tree = BST + PQ

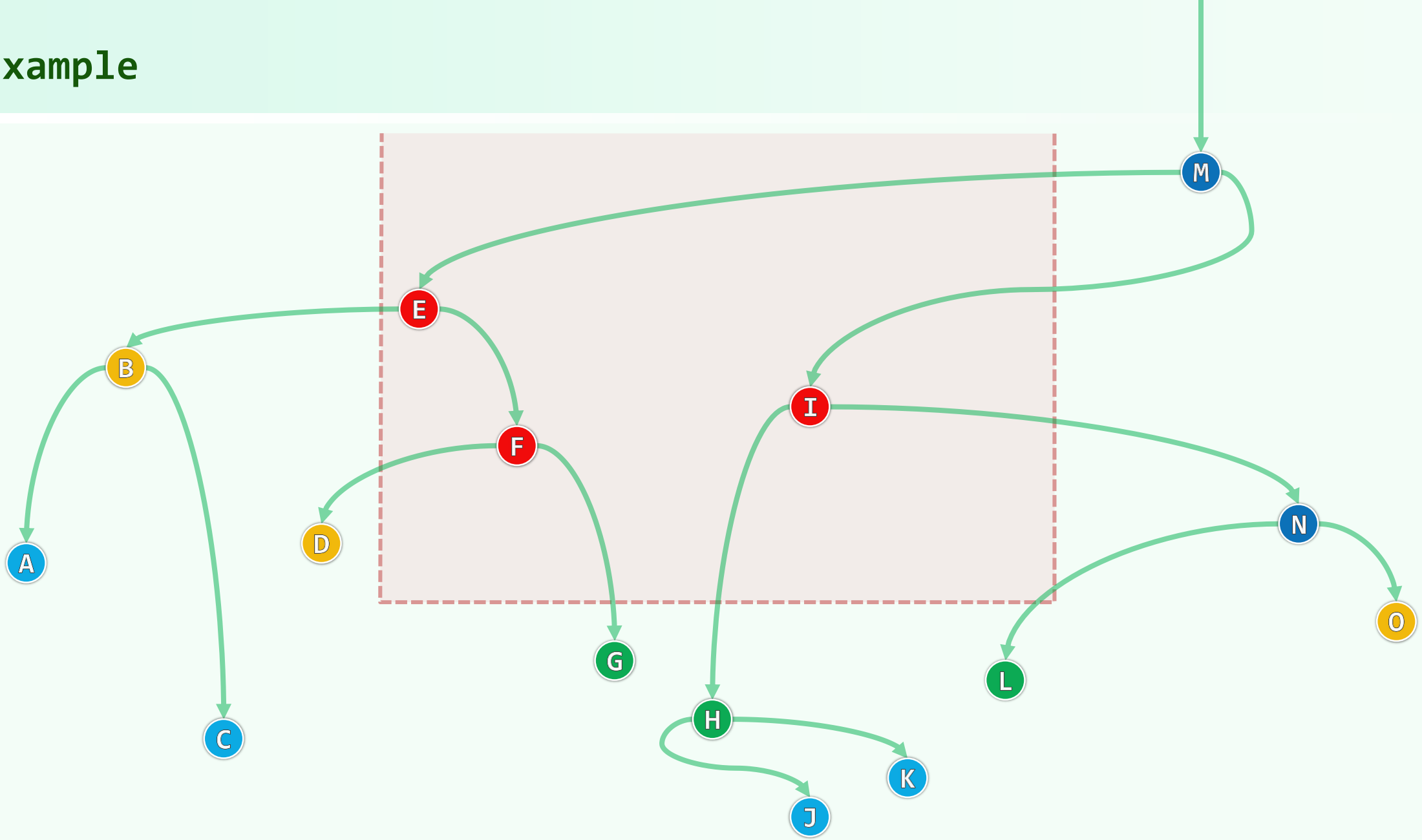


queryPST(PSTNode v, int x, int y1, int y2)

```
if ( !v || x < v.x )
    return //pruned for bad X
if ( ( y1 < v.y ) && ( v.y < y2 ) )
    output(v) //accepted
//else rejected for bad Y
if ( y1 ≤ v.ym )
    queryPST( v.lc, x, y1, y2 )
//else pruned for bad Y
if ( v.ym < y2 )
    queryPST( v.rc, x, y1, y2 )
//else pruned for bad Y
```



Example



Query Time

- P: Pruned with descendants due to bad Y
  - no more time cost
- A: Visited and accepted
  - exactly r = output size
- BY: Visited but rejected due to bad Y
  - no more than 2 for each level
  - altogether $O(\log n)$
- BX: Visited but rejected due to bad X
  - having an A or BY parent
  - no more than $O(r + \log n)$