08-A4

高级搜索树

伸展树：分摊分析

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所谓物价，其实就是我称之为生命的那部分，必须在交换时支付：要么立即支付，要么以后支付

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S的势能

❖ （任何时刻的）任何一棵伸展树S，都可以**假想地**被认为具有势能：

$$\Phi(S) = \log\left(\prod_{v \in S} size(v)\right) = \sum_{v \in S} \log\left(size(v)\right) = \sum_{v \in S} rank(v) = \sum_{v \in S} \log V$$

❖ 直觉：越**平衡**/**倾侧**的树，势能越**小**/**大**

- 单链：$\Phi(S) = \log n! = \mathcal{O}(n \log n)$

- 满树：$\Phi(S) = \log \prod_{d=0}^{h} (2^{h-d+1} - 1)^{2^d} \leq \log \prod_{d=0}^{h} (2^{h-d+1})^{2^d}$

$$= \log \prod_{d=0}^{h} 2^{(h-d+1) \cdot 2^d} = \sum_{d=0}^{h} (h-d+1) \cdot 2^d = (h+1) \cdot \sum_{d=0}^{h} 2^d - \sum_{d=0}^{h} d \cdot 2^d$$

$$= (h+1) \cdot (2^{h+1} - 1) - [(h-1) \cdot 2^{h+1} + 2] = 2^{h+2} - h - 3 = \mathcal{O}(n)$$

T的上界

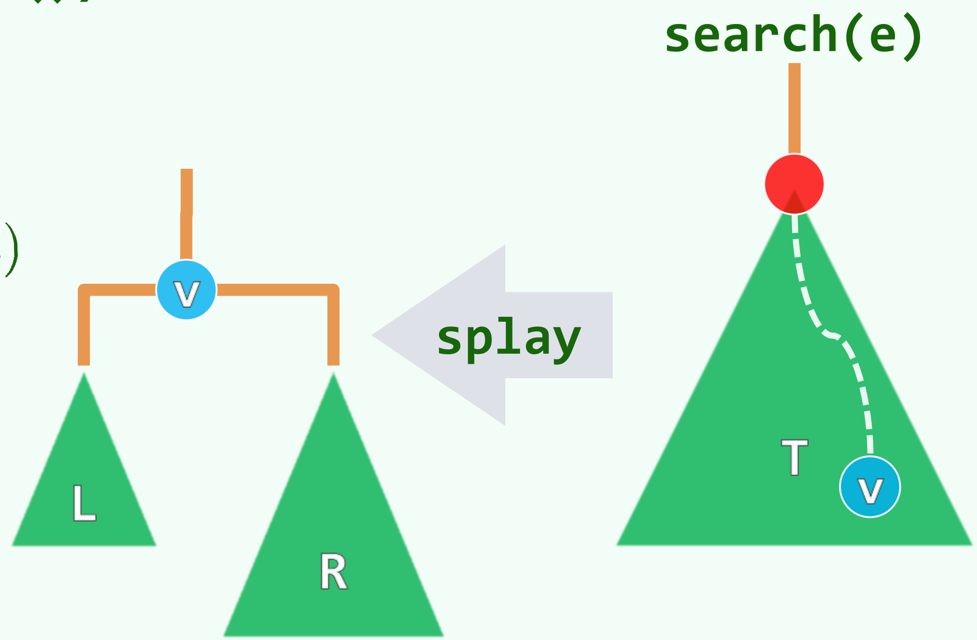
❖ 考查对伸展树 S 的 $m \gg n$ 次连续访问（**不妨**仅考查search()）

❖ 若记：$A^{(k)} = T^{(k)} + \Delta\Phi^{(k)}$, $k = 0, 1, 2, \ldots, m$

则有：$A - \mathcal{O}(n \log n) \leq \underline{T = A - \Delta\Phi} \leq A + \mathcal{O}(n \log n)$

❖ 故若能证明：$A = \mathcal{O}(m \log n)$

则必有：$T = \mathcal{O}(m \log n)$

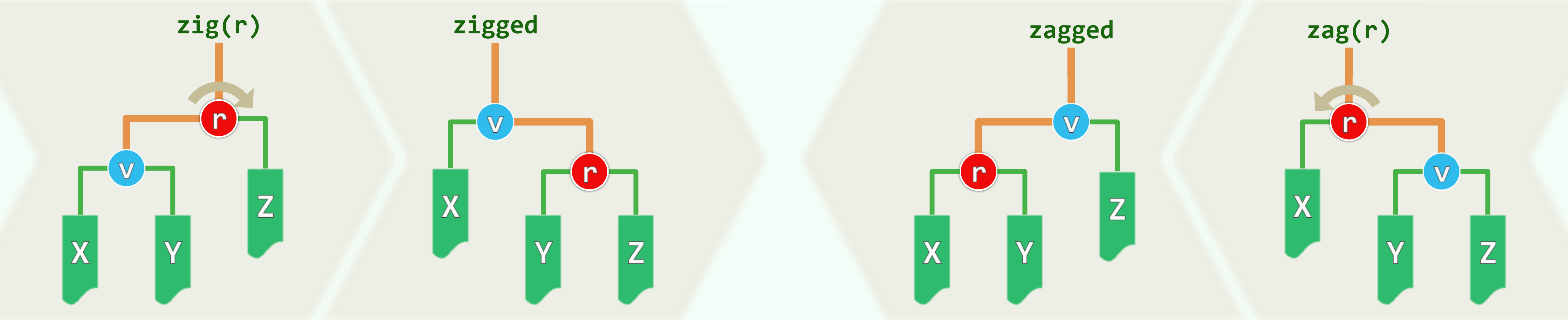


❖ 好消息是，尽管 $T^{(k)}$ 的变化幅度可能很大，我们却能证明：

$A^{(k)}$ 都不致超过节点v的势能变化量，即：$\mathcal{O}(\ rank^{(k)}(v) - rank^{(k-1)}(v)\ ) = \mathcal{O}(\log n)$

❖ 事实上，$A^{(k)}$ 不过是v的若干次连续伸展操作（时间成本）的累积，这些操作无非**三种**情况...

Zig / Zag

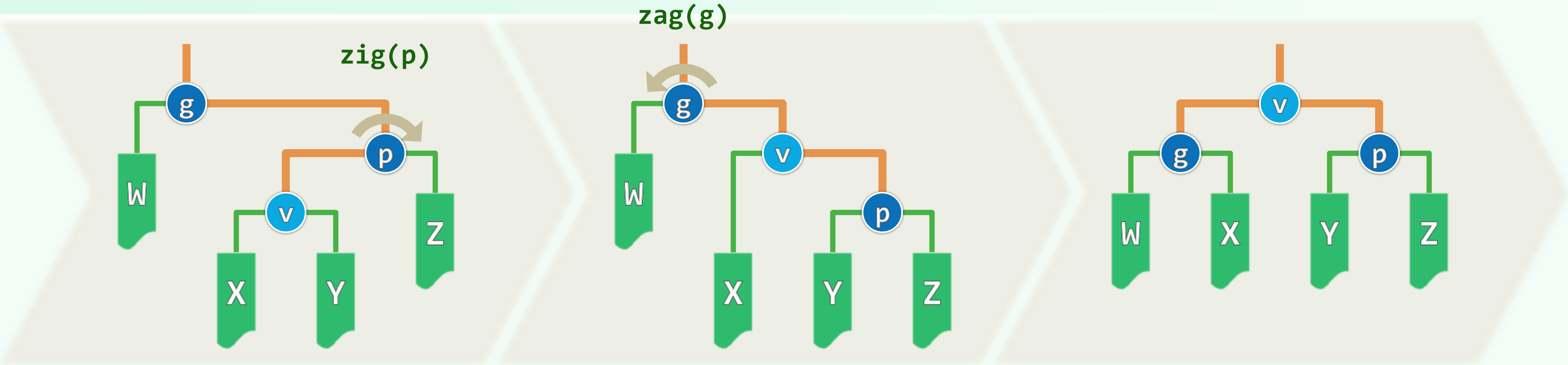


$$A_i^{(k)} = T_i^{(k)} + \Delta\Phi(S_i^{(k)}) = 1 + \Delta rank_i(v) + \Delta rank_i(r)$$

$$= 1 + [rank_i(v) - rank_{i-1}(v)] + [\underline{rank_i(r) - rank_{i-1}(r)}]$$

$$< {\color{red}1} + [rank_i(v) - rank_{i-1}(v)]$$

zig-zag / zag-zig



$$A_i^{(k)} = T_i^{(k)} + \Delta\Phi(S_i^{(k)}) = 2 + \Delta rank_i(g) + \Delta rank_i(p) + \Delta rank_i(v)$$

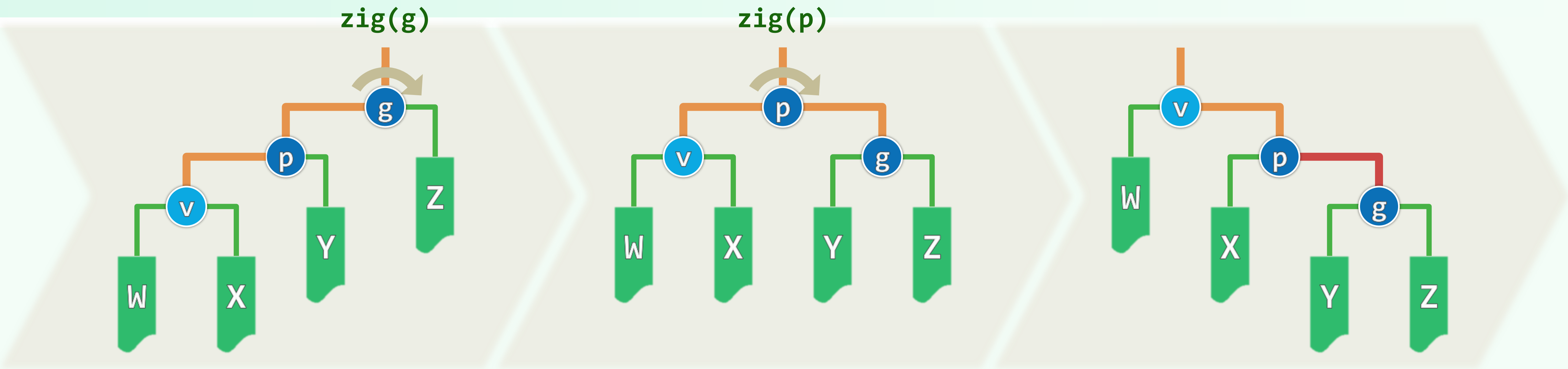
$$= 2 + [rank_i(g) - \cancel{rank_{i-1}(g)}] + [rank_i(p) - \underline{rank_{i-1}(p)}] + [\cancel{rank_i(v)} - \underline{rank_{i-1}(v)}]$$

$$< 2 + \underline{rank_i(g) + rank_i(p)} - 2 \cdot rank_{i-1}(v) \qquad (\because\ rank_{i-1}(p) > rank_{i-1}(v))$$

$$< \cancel{2} + \underline{2 \cdot rank_i(v) - \cancel{2}} - 2 \cdot rank_{i-1}(v) \qquad (\because\ \frac{\log G_i + \log P_i}{2} \leq \log \frac{G_i + P_i}{2} < \log \frac{V_i}{2})$$

$$= 2 \cdot (rank_i(v) - rank_{i-1}(v))$$

zig-zig / zag-zag



$$A_i^{(k)} = T_i^{(k)} + \Delta\Phi(S_i^{(k)}) = 2 + \Delta rank_i(g) + \Delta rank_i(p) + \Delta rank_i(v)$$

$$= 2 + [rank_i(g) - \cancel{rank_{i-1}(g)}] + [rank_i(p) - \underline{rank_{i-1}(p)}] + [\cancel{rank_i(v)} - \underline{rank_{i-1}(v)}]$$

$$< 2 + rank_i(g) + \underline{rank_i(p)} - 2 \cdot rank_{i-1}(v) \qquad (\because\ rank_{i-1}(p) > rank_{i-1}(v))$$

$$< 2 + rank_i(g) + \underline{rank_i(v)} - 2 \cdot rank_{i-1}(v) \qquad (\because\ rank_i(p) < rank_i(v))$$

$$< 3 \cdot (rank_i(v) - rank_{i-1}(v)) \qquad (\because\ \frac{\log G_i + \log V_{i-1}}{2} \le \log\frac{G_i + V_{i-1}}{2} < \log\frac{V_i}{2})$$