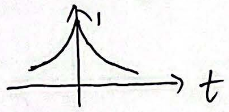
几个不太常见的傅里叶变换对

① 双边指数信号

$$x(t)=e^{-a|t|}\xrightarrow{F}\frac{2a}{a^2+w^2}\qquad(a>0)$$



推导: $X(jw)=\int_{-\infty}^{+\infty}e^{-a|t|}e^{-jwt}dt$

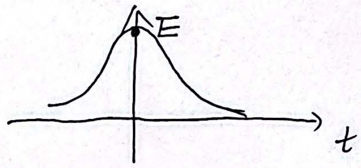
$$=\int_{-\infty}^{0}e^{at}e^{-jwt}dt+\int_{0}^{+\infty}e^{-at}e^{-jwt}dt$$

$$=\frac{1}{a-jw}e^{(a-jw)t}\Big|_{-\infty}^{0}+\left(-\frac{1}{a+jw}\right)e^{-(a+jw)t}\Big|_{0}^{+\infty}$$

$$=\frac{1}{a-jw}+\frac{1}{a+jw}=\frac{2a}{a^2+w^2}$$

③ 高斯脉冲信号

$$x(t)=Ee^{-\left(\frac{t}{\tau}\right)^2}\xrightarrow{F}\sqrt{\pi}E\tau e^{-\left(\frac{w\tau}{2}\right)^2}$$



先证明 $\int_{-\infty}^{+\infty}e^{-x^2}dx=\sqrt{\pi}$

设 $A=\int_{-\infty}^{+\infty}e^{-x^2}dx$

则 $X^2=\left[\int_{-\infty}^{+\infty}e^{-x^2}dx\right]\left[\int_{-\infty}^{+\infty}e^{-y^2}dy\right]$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{-(x^2+y^2)}dxdy$$

设 $x=r\cos\theta$, $y=r\sin\theta$

Jacobi行列式 $=\begin{vmatrix}\frac{\partial x}{\partial r}&\frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r}&\frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\cos\theta&-r\sin\theta\\ \sin\theta&r\cos\theta\end{vmatrix}=r$

则有 $A^2=\int_0^{2\pi}d\theta\int_0^{+\infty}re^{-r^2}dr$

$$=\int_0^{2\pi}\frac{1}{2}\int_0^{+\infty}e^{-r^2}d(r^2)$$

$$=\frac{1}{2}\times 2\pi\cdot\times 1=\pi$$

所以 $A=\sqrt{\pi}$

$$X(jw)=E\int_{-\infty}^{+\infty}e^{-\frac{t^2}{\tau^2}}e^{-jwt}dt$$

$$=E\int_{-\infty}^{+\infty}e^{-\frac{\left(t+\frac{jw\tau^2}{2}\right)^2}{\tau^2}}dt\cdot e^{-\frac{w^2\tau^4}{4\tau^2}}$$

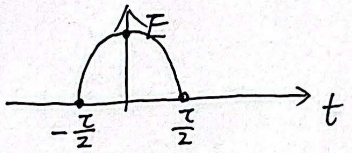
设 $t'=\dfrac{t+\frac{jw\tau^2}{2}}{\tau}$，则有

$$X(jw)=E\tau\int_{-\infty}^{+\infty}e^{-t'^2}dt'\cdot e^{-\left(\frac{w\tau}{2}\right)^2}$$

$$=\sqrt{\pi}E\tau e^{-\left(\frac{w\tau}{2}\right)^2}$$

③ 半波余弦信号

$$x(t)=\begin{cases}E\cos\left(\frac{\pi t}{\tau}\right) & |t|<\frac{\tau}{2}\\ 0 & |t|>\frac{\tau}{2}\end{cases}\xrightarrow{F}\frac{2E\tau}{\pi}\frac{\cos\left(\frac{w\tau}{2}\right)}{1-\left(\frac{w\tau}{\pi}\right)^2}$$



推导: $X(jw)=\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}E\cos\left(\frac{\pi t}{\tau}\right)e^{-jwt}dt$

$$=\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}E\cos\left(\frac{\pi t}{\tau}\right)\cos(wt)dt-\underbrace{j\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}E\,\underbrace{\cos\left(\frac{\pi t}{\tau}\right)\sin(wt)}_{t\text{的奇函数}}dt}_{=0}$$

$$=\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}E\cos\left(\frac{\pi t}{\tau}\right)\cos(wt)dt$$

$$= 2E\int_0^{\frac{\tau}{2}} \cos\left(\frac{\pi}{\tau}t\right)\cos(wt)\,dt$$

$$= E\int_0^{\frac{\tau}{2}} \cos\left[\left(\frac{\pi}{\tau}+w\right)t\right] + \cos\left[\left(\frac{\pi}{\tau}-w\right)t\right]dt$$

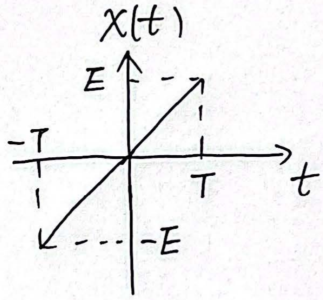
$$= E\left[\frac{1}{\frac{\pi}{\tau}+w}\sin\left[\left(\frac{\pi}{\tau}+w\right)t\right]\Big|_0^{\frac{\tau}{2}} + \frac{1}{\frac{\pi}{\tau}-w}\sin\left[\left(\frac{\pi}{\tau}-w\right)t\right]\Big|_0^{\frac{\tau}{2}}\right]$$

$$= E\left[\frac{1}{\frac{\pi}{\tau}+w}\sin\left(\frac{\pi}{2}+\frac{\tau}{2}w\right) + \frac{1}{\frac{\pi}{\tau}-w}\sin\left(\frac{\pi}{2}-\frac{\tau}{2}w\right)\right]$$

$$= E\left[\frac{1}{\frac{\pi}{\tau}+w}\cos\left(\frac{\tau}{2}w\right) + \frac{1}{\frac{\pi}{\tau}-w}\cos\left(\frac{\tau}{2}w\right)\right]$$

$$= \frac{2E\tau}{\pi}\,\frac{\cos\left(\frac{w\tau}{2}\right)}{1-\left(\frac{w\tau}{\pi}\right)^2}$$

④ 奇对称斜线



$$\xrightarrow{F} j\frac{2E}{wT}\left[T\cos(wt) - \frac{1}{w}\sin(wT)\right]$$

推导: $$X(jw) = \int_{-T}^{T}\frac{E}{T}t\,e^{-jwt}\,dt$$

$$= \frac{E}{T}\left[\underbrace{\int_{-T}^{T}\underbrace{t\cos(wt)}_{\text{为奇函数}}dt}_{=0} - j\int_{-T}^{T}t\sin(wt)\,dt\right]$$

$$= -j\frac{2E}{T}\int_0^T t\sin(wt)\,dt$$

$$= j\frac{2E}{wT}\int_0^T t\,d[\cos(wt)]$$

$$= j\frac{2E}{wT}\left[t\cos(wt)\Big|_0^T - \int_0^T\cos(wt)\,dt\right]$$

$$= j\frac{2E}{wT}\left[T\cos(wt) - \frac{1}{w}\sin(wT)\right]$$