07-A1

BST Application

Range Query: 1D

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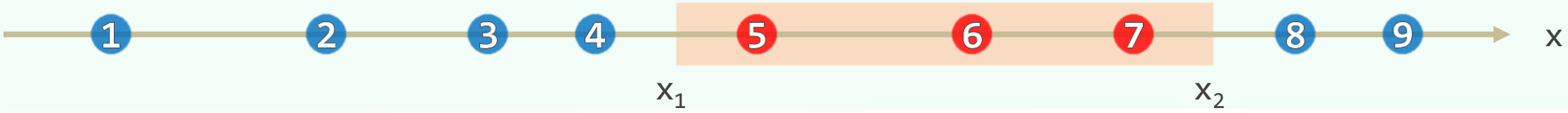
你这个人太敏感了。这个社会什么都需要，唯独不需要敏感。

1D Range Query

- Let $P = \{ p_1, p_2, p_3, \ldots, p_n \}$ be a set of n points on the x-axis
- For any given interval $I = (x_1, x_2]$
  - COUNTING: how many points of P lies in the interval?
  - REPORTING: enumerate all points in $I \cap P$ (if not empty)
- [Online] P is fixed while I is randomly and repeatedly given
- How to PREPROCESS P into a certain data structure s.t.

  the queries can be answered efficiently?

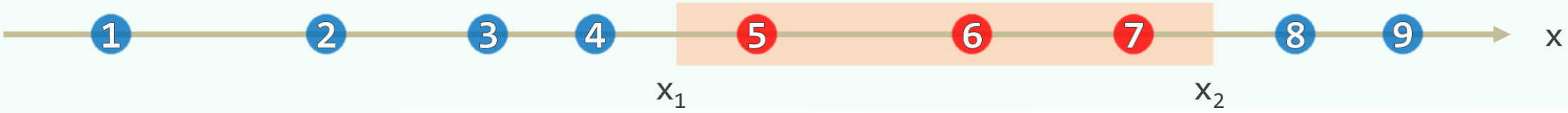


Brute-Force

- For each point p of P, test if $p \in (x_1, x_2]$
- Thus each query can be answered in LINEAR time
- Can we do it faster? It seems we can't, for ...
- In the worst case,

  the interval contains up to $O(n)$ points, which need $O(n)$ time to enumerate
- However, how if we

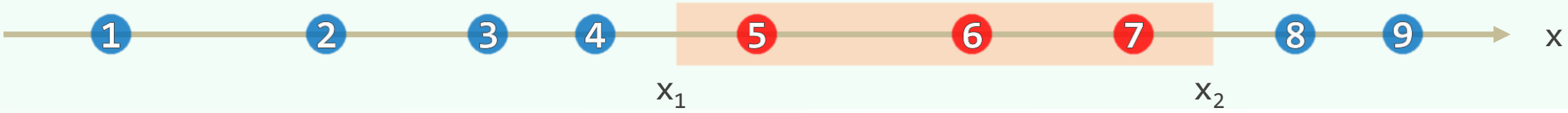
  ignore the time for enumerating and count only the searching time?



Binary Search

- Sort all points into a sorted vector and add an extra sentinel p[0] = -∞
- For any interval $I = (x_1, x_2]$
  - Find t = search(x_2) = max{ i | p[i] ≤ x_2 } //$O(\log n)$
  - Traverse the vector BACKWARD from p[t] and report each point //$O(r)$

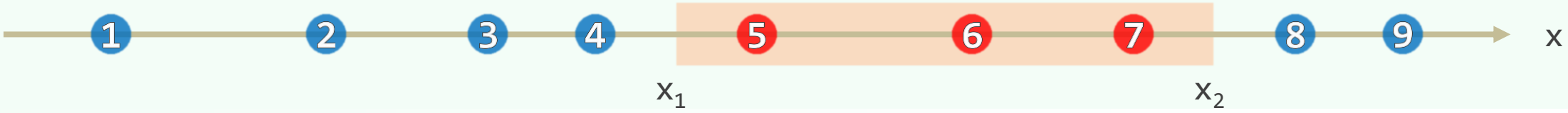
    until escaping from I at point p[s]
  - return r = t - s //output size



Output-Sensitivity

- An enumerating query can be answered in $O(r + \log n)$ time
- p[s] can also be found by binary search in $O(\log n)$ time
- Hence for COUNTING query, $O(\log n)$ time is enough //independent to r
- Can this simple strategy be extended to PLANAR range query?

  TTBOMK, unfortunately, no!



07-A2

BST Application

Range Query: 2D

昔者明王必尽知天下良士之名；既知其名，又知其数；
既知其数，又知其所在。

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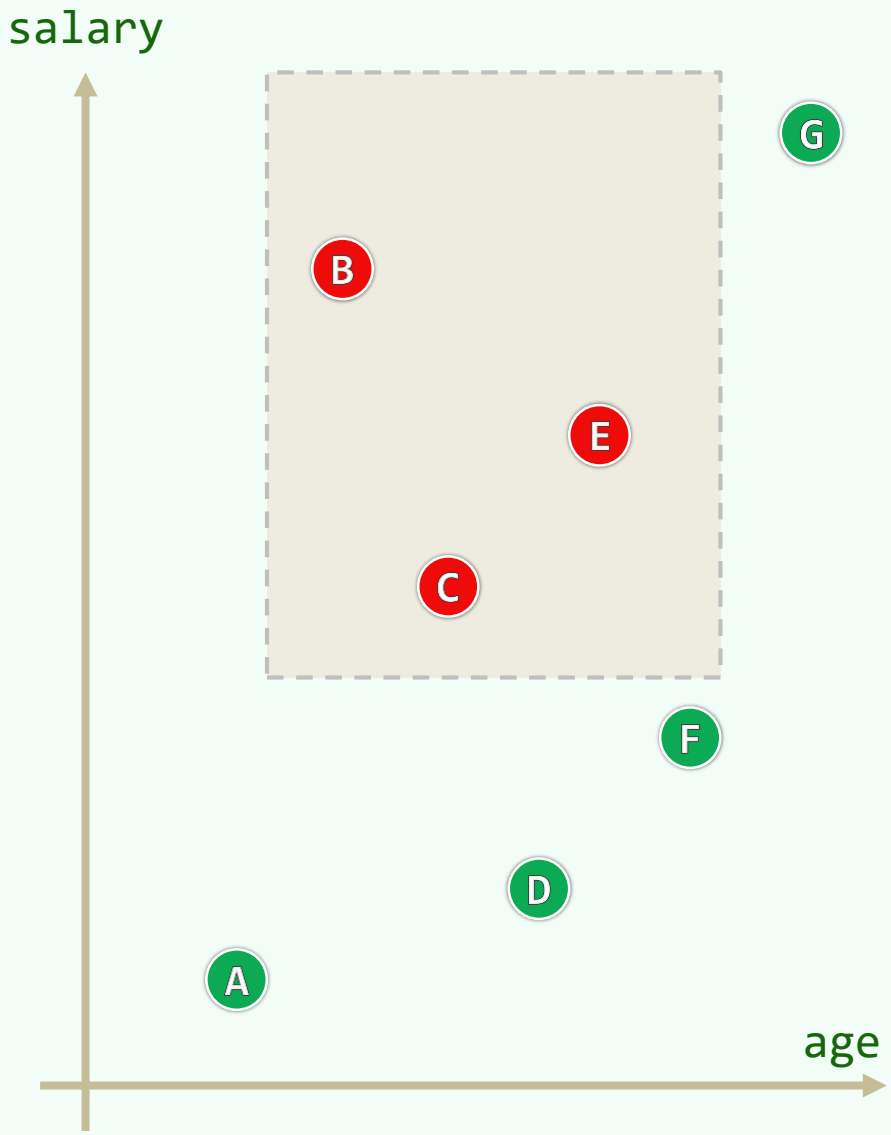
Planar Range Query

- Let $P = \{ p_1, p_2, p_3, \ldots, p_n \}$ be a planar set
- Given $R = (x_1, x_2] \times (y_1, y_2]$
  - COUNTING: $|R \cap P| = ?$
  - REPORTING: $R \cap P = ?$
- Binary search

  doesn't help this kind of query
- You might consider to

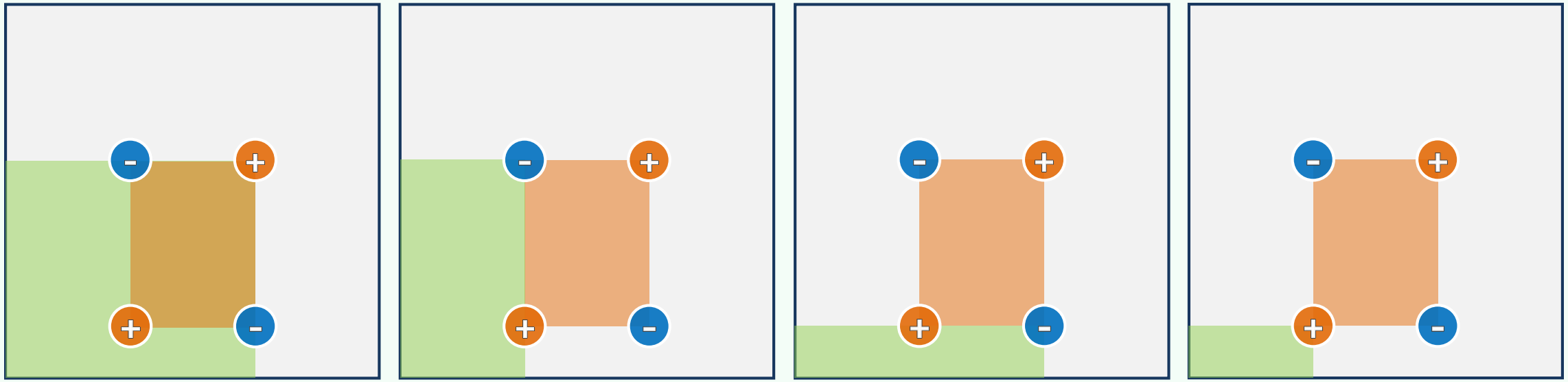
  expand the counting method using

  the Inclusion-Exclusion Principle



Preprocessing

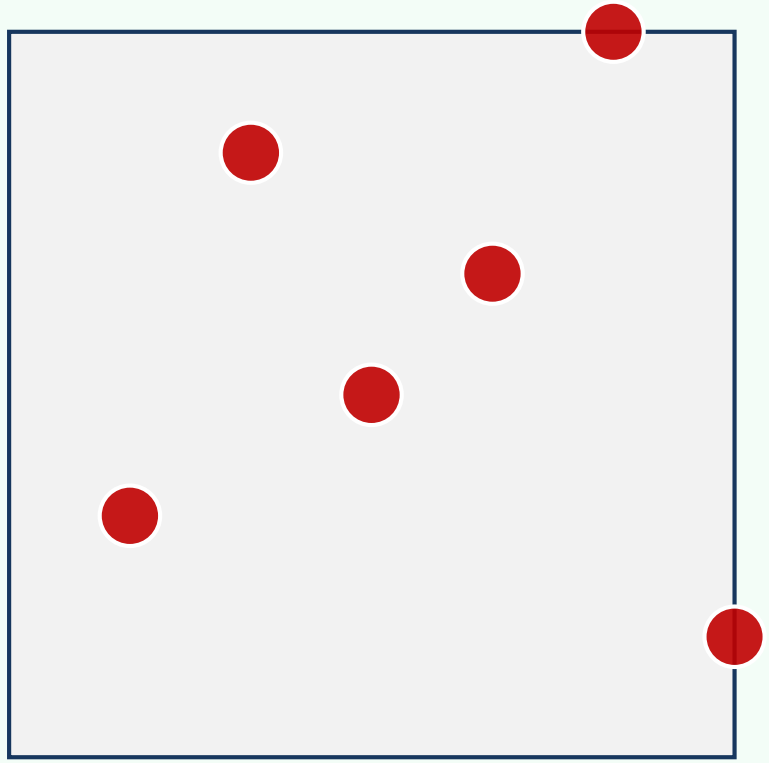
- $\forall$ point (x, y), let $n(x, y) = \left|((0, x] \times (0, y]) \cap P\right|$
- **This requires $O(n^2)$ time/space**

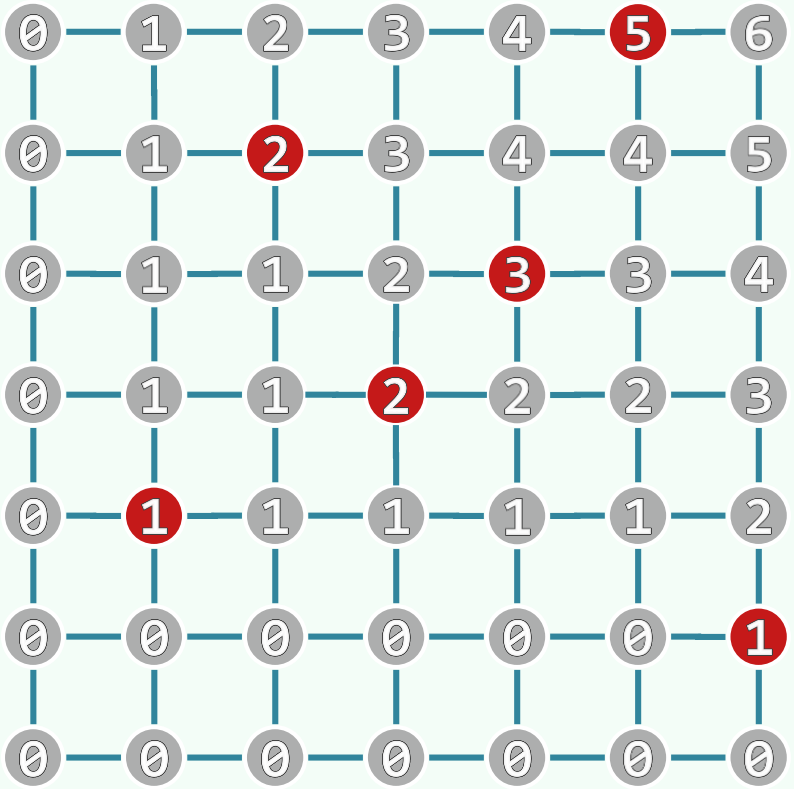


Domination

❖ A point (u, v) is called to be **DOMINATED** by point (x, y) if

$u \le x$ and $v \le y$

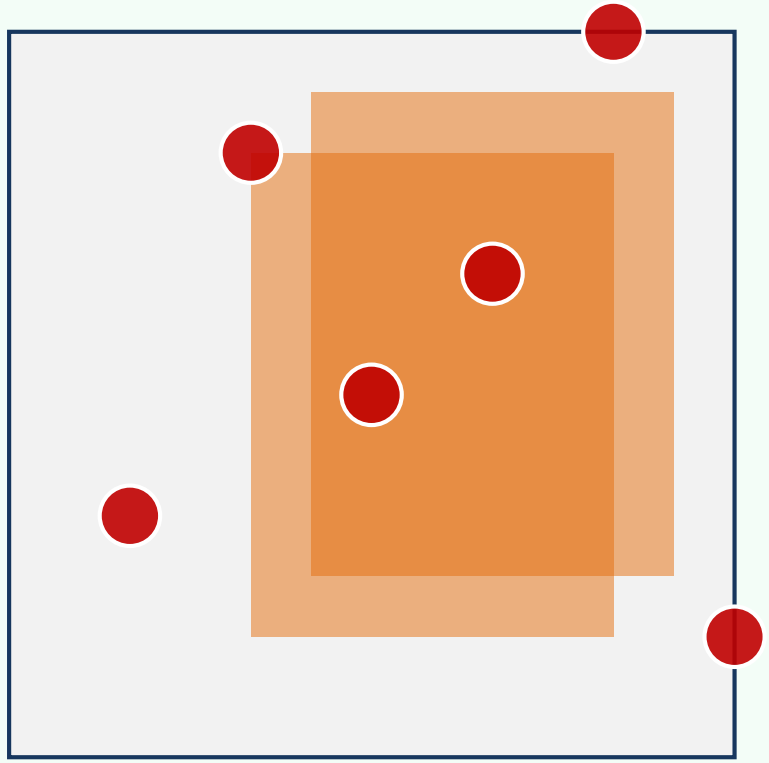


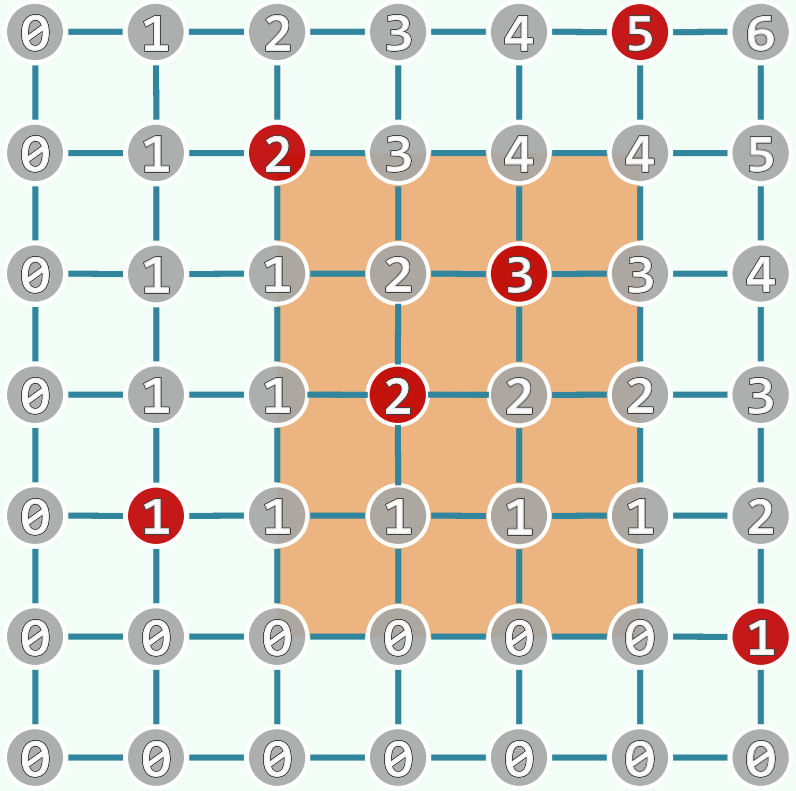


Inclusion-Exclusion Principle

❖ Then for any rectangular range $\mathcal{R} = (x_1, x_2] \times (y_1, y_2]$, we have

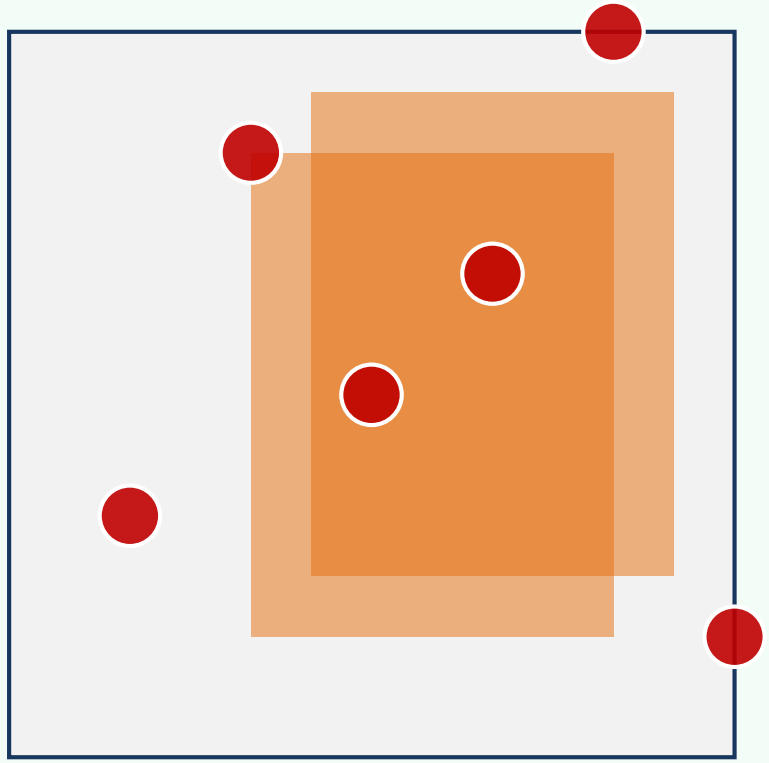
$$\left|\mathcal{R} \cap \mathcal{P}\right| = n(x_1, y_1) + n(x_2, y_2) - n(x_1, y_2) - n(x_2, y_1)$$

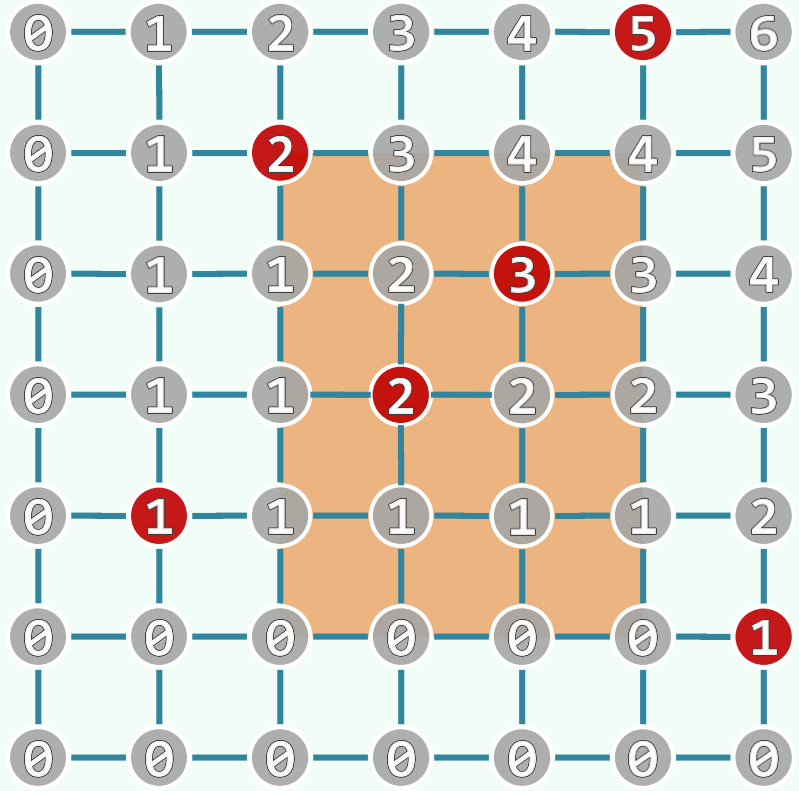




Performance

- Each query needs only $\mathcal{O}(\log n)$ time
- Uses $\Theta(n^2)$ storage and even more for higher dimensions
- To find a better solution, let's go back to the 1D case ...





07-B1

BST Application

kd-Tree: 1D

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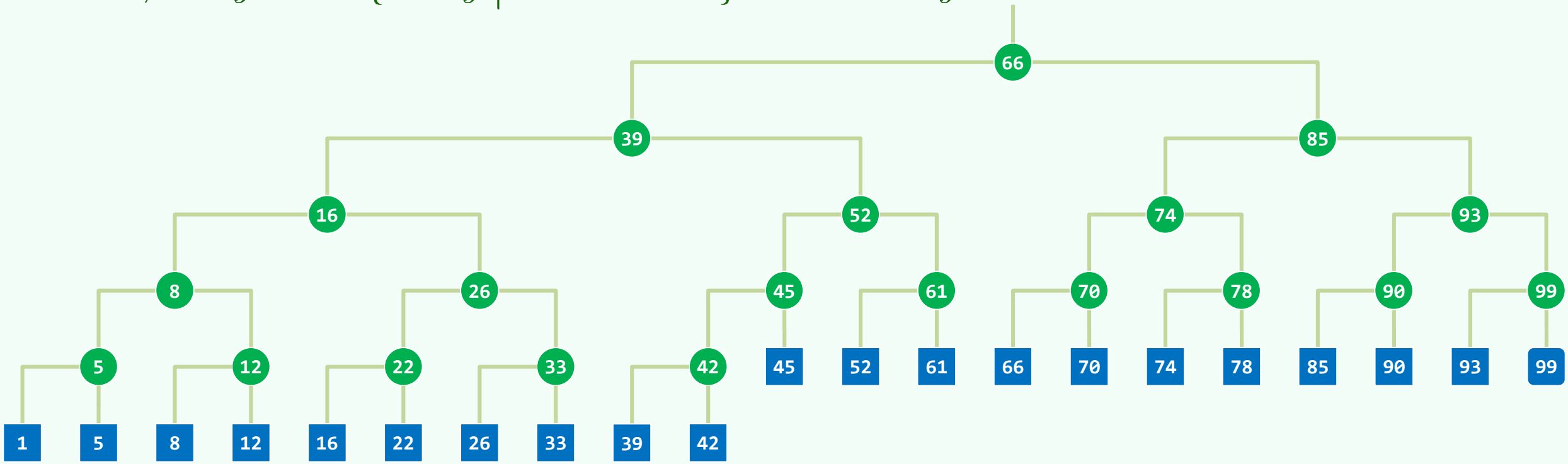
Structure: A Complete (Balanced) BST

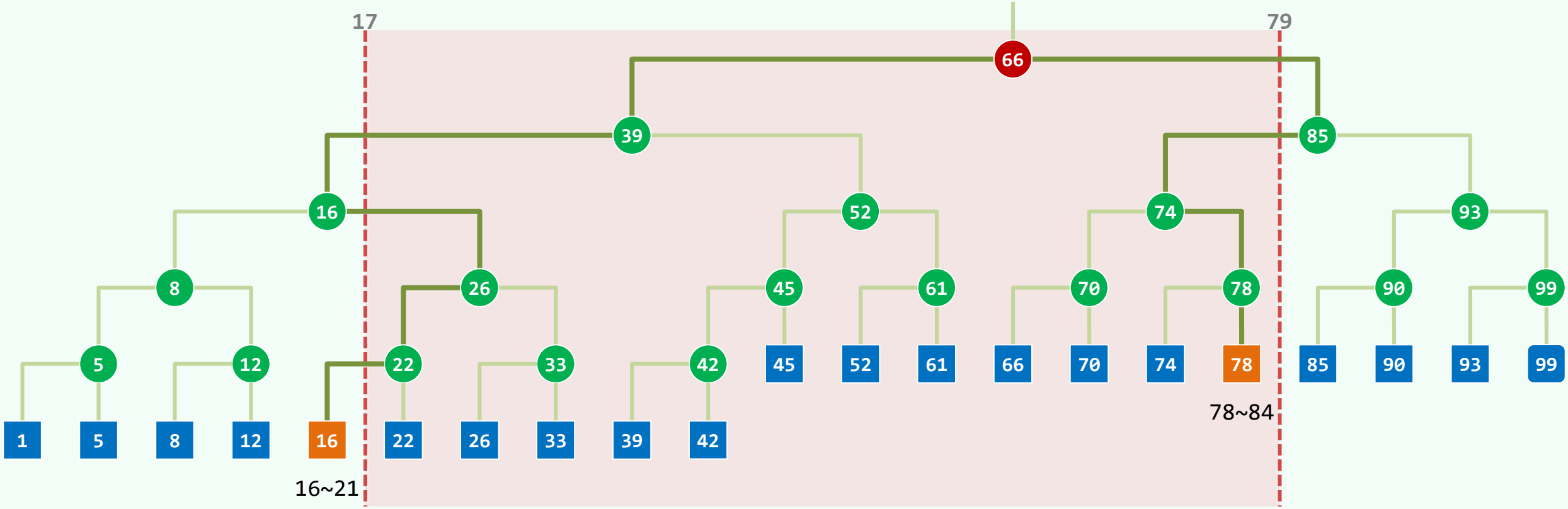
- $\forall v,\ v.key = \min\{\, u.key \mid u \in v.rTree \,\} = v.succ.key$



- $\forall u \in v.lTree,\ u.key < v.key$ $\qquad$ $\forall u \in v.rTree,\ u.key \geq v.key$
- **search(x) : returns the MAXIMUM key not greater than x**

Lowest Common Ancestor

- Consider, as an example, the query for [17, 79] ...

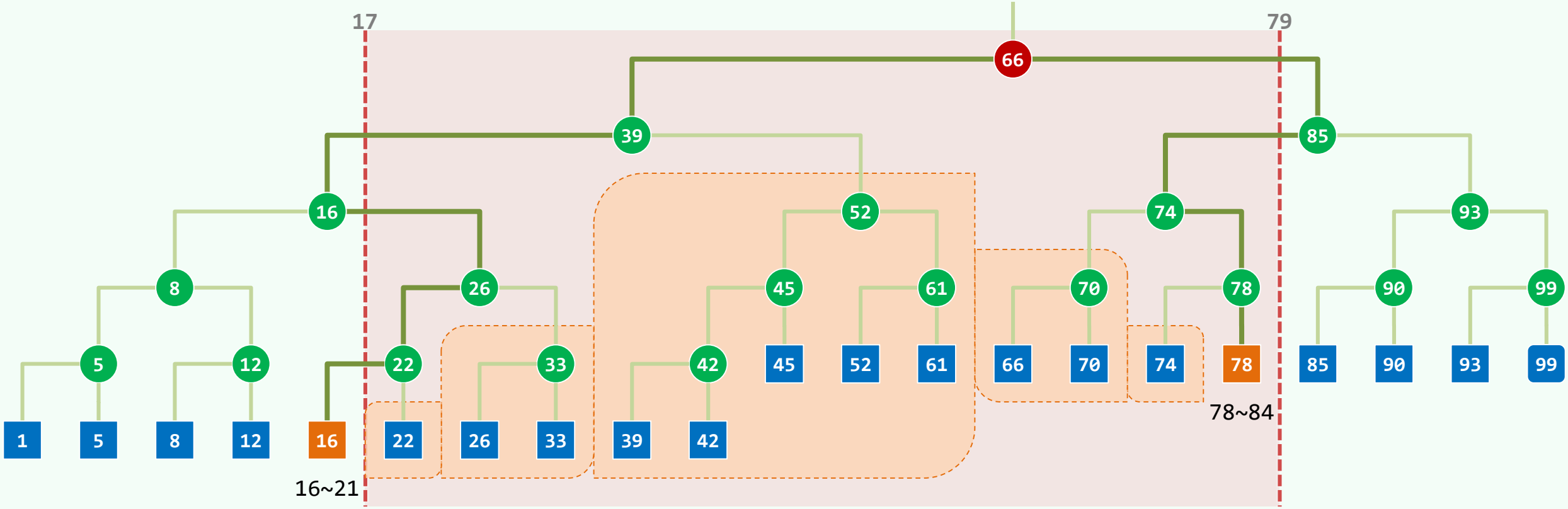


- Do search(17) = 16 (might rejected) and search(79) = 78 (must accepted)

- Consider LCA(16, 78) = 66 ...

Union of $\mathcal{O}$(logn) Disjoint Subtrees

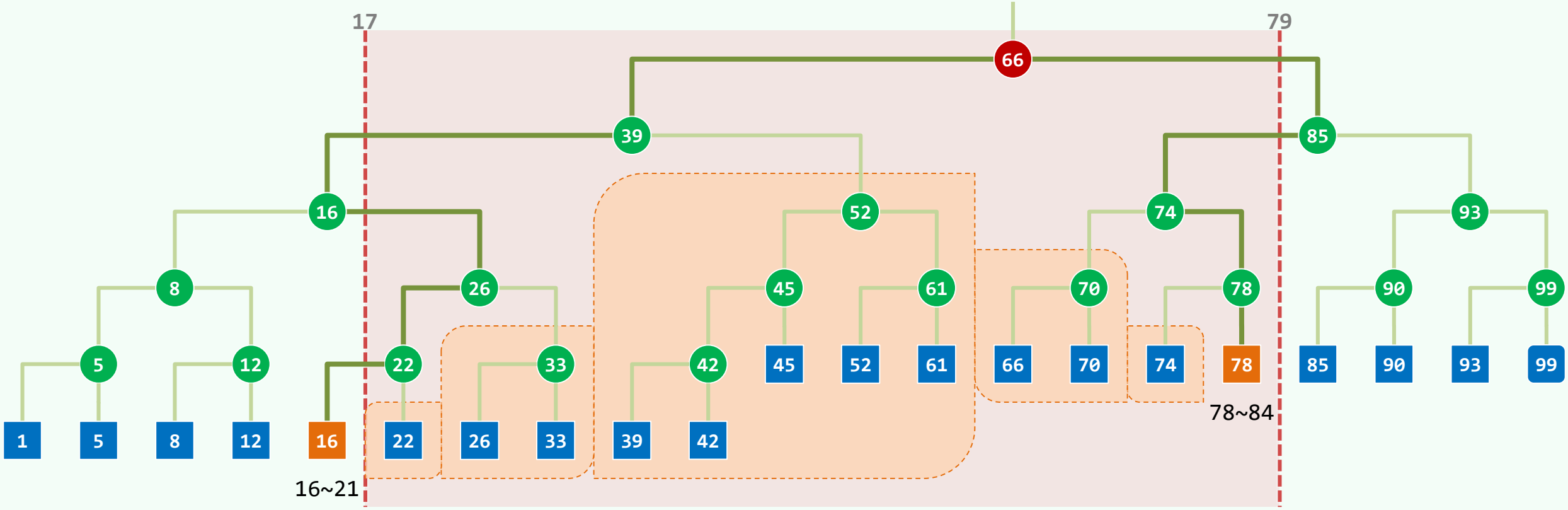
❖ Starting from the LCA, traverse path(16) and path(78) once more resp.



- All R/L-turns along path(16)/path(78) are ignored and

- the R/L subtree at each L/R-turn is reported

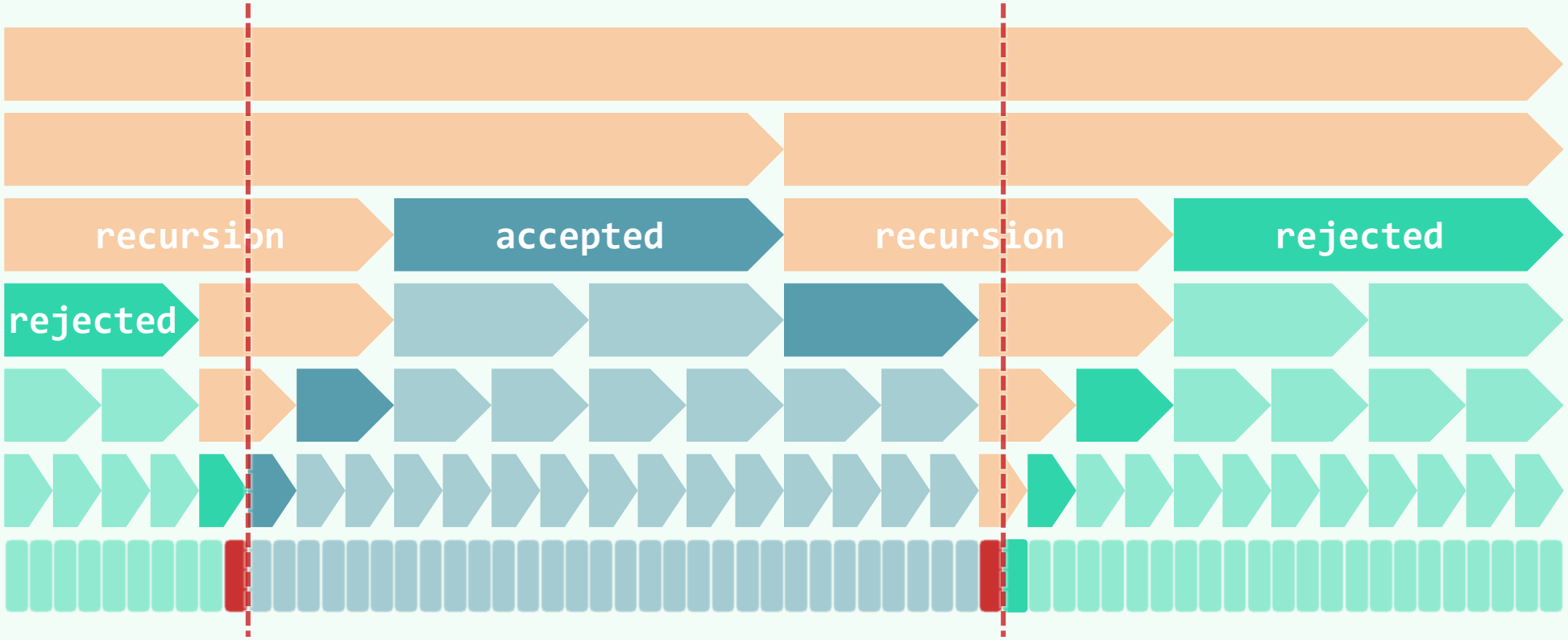
Complexity



Query: $\mathcal{O}(\log n)$ **Preprocessing:** $\mathcal{O}(n \log n)$ **Storage:** $\mathcal{O}(n)$

Hot Knives Through A Chocolate Cake of Height $O(\log n)$

- Region(u) is enclosed by Region(v) iff u is a descendant of v in the 1d-tree
- Region(u) and Region(v) are disjoint iff neither is the ancestor of the other



- All nodes are partitioned into 3 types: accepted + rejected + recursion

07-B2

BST Application

kd-Tree: 2D

凡见字数，如停匀，即平分一半为上卦，一半为下卦。如字数不均，即少一字为上卦，取天轻清之义，以多一字为下卦，取地重浊之义

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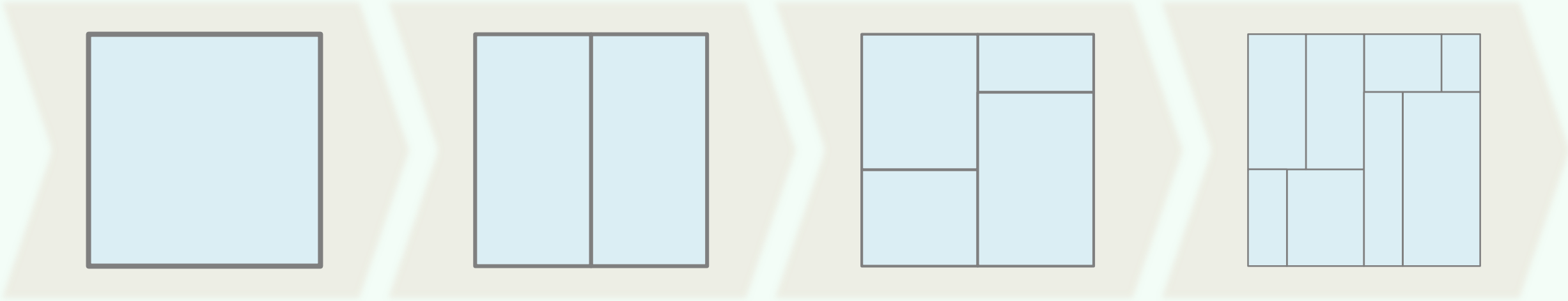
Divide-And-Conquer

- To extend the BBST method to planar GRS, we
  - divide the plane recursively and
  - arrange the regions into a kd-tree
- Start with a single region (the entire plane)

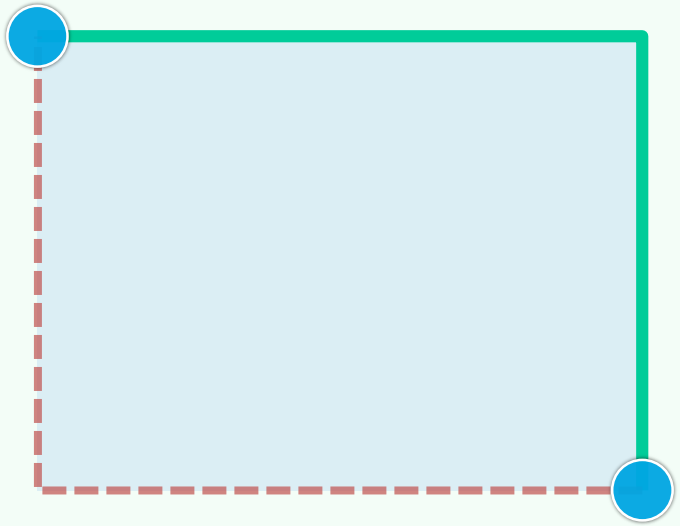
  Partition the region vertically/horizontally on each even/odd level

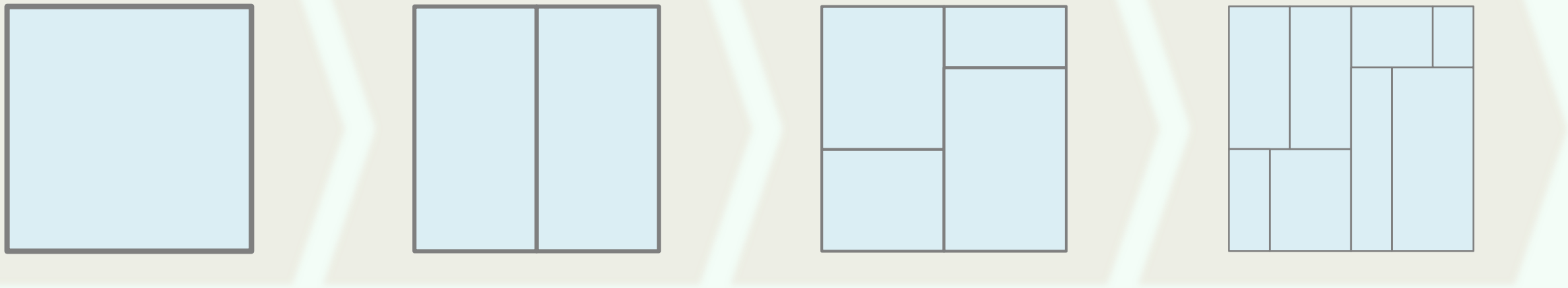
  Partition the sub-regions recursively



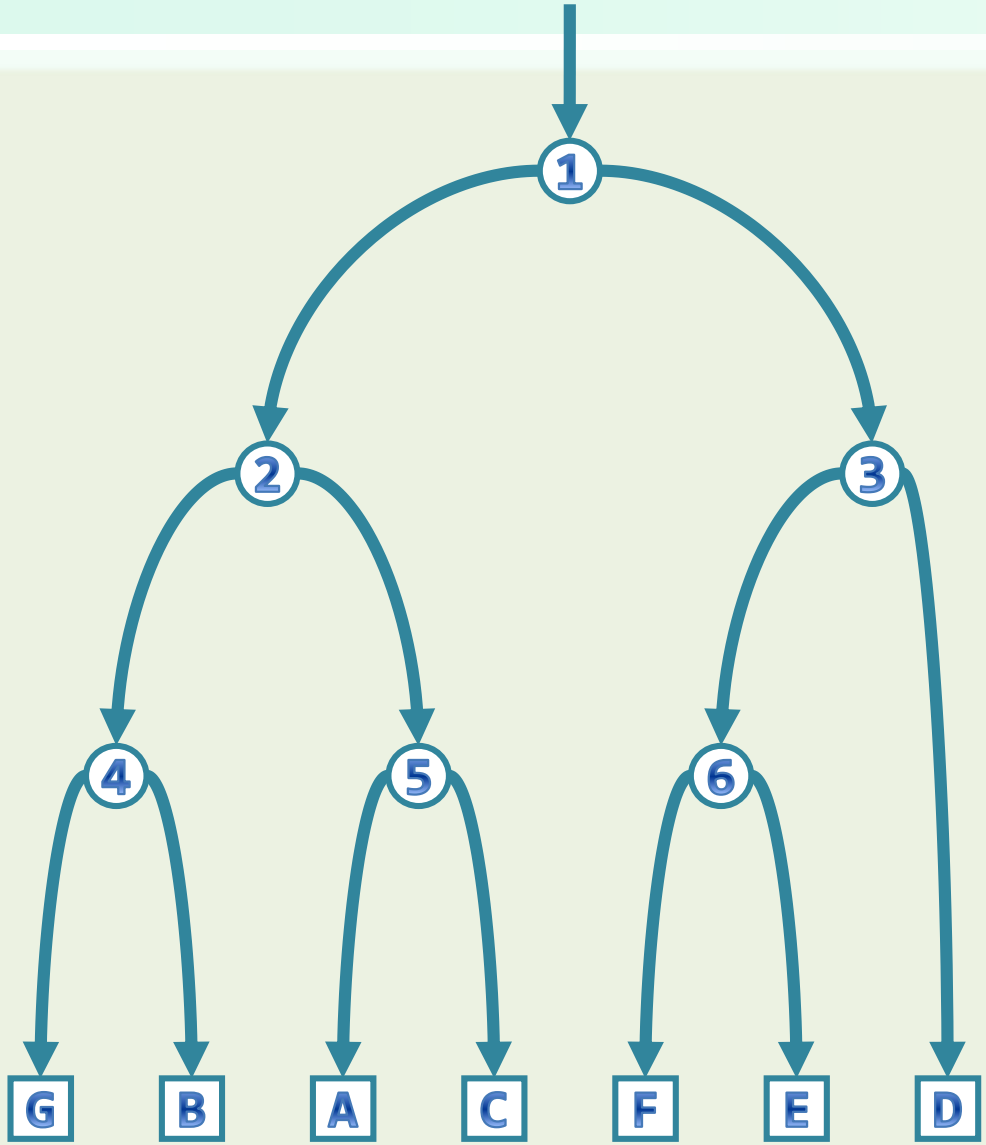
More Details

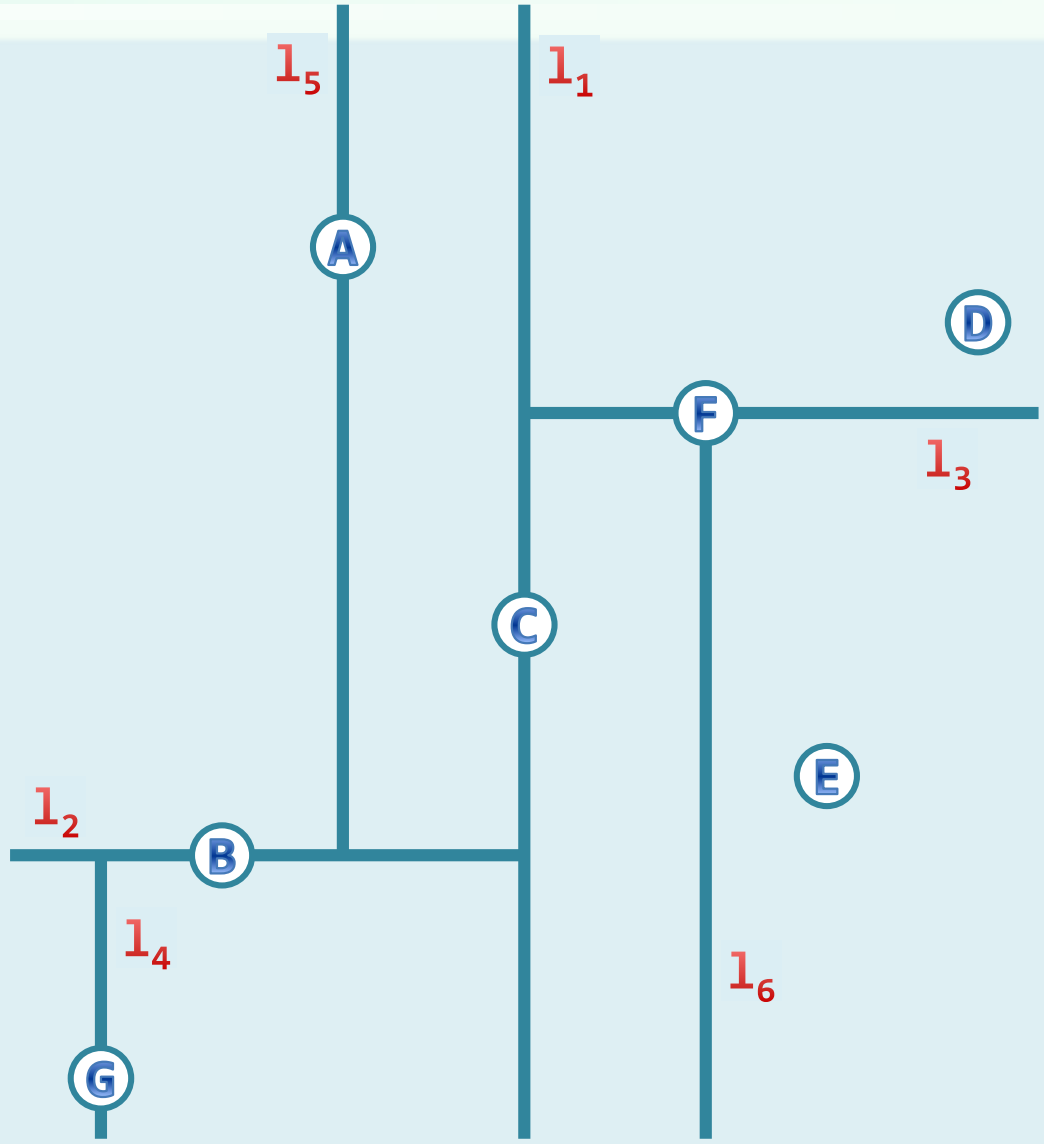
- To make it work,
  - each partition should be done as evenly as possible (at median)
  - each region is defined to be open/closed on the left-lower/right-upper sides



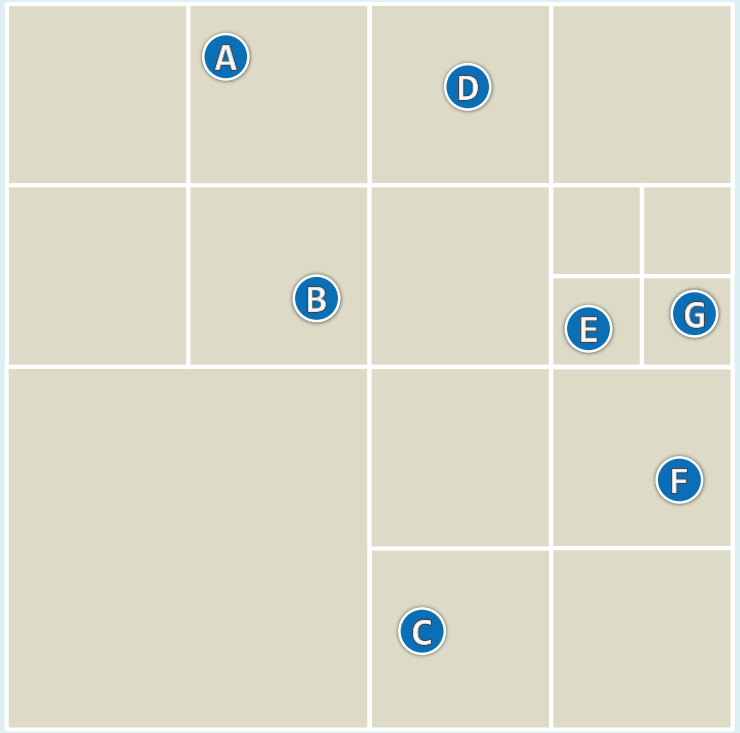


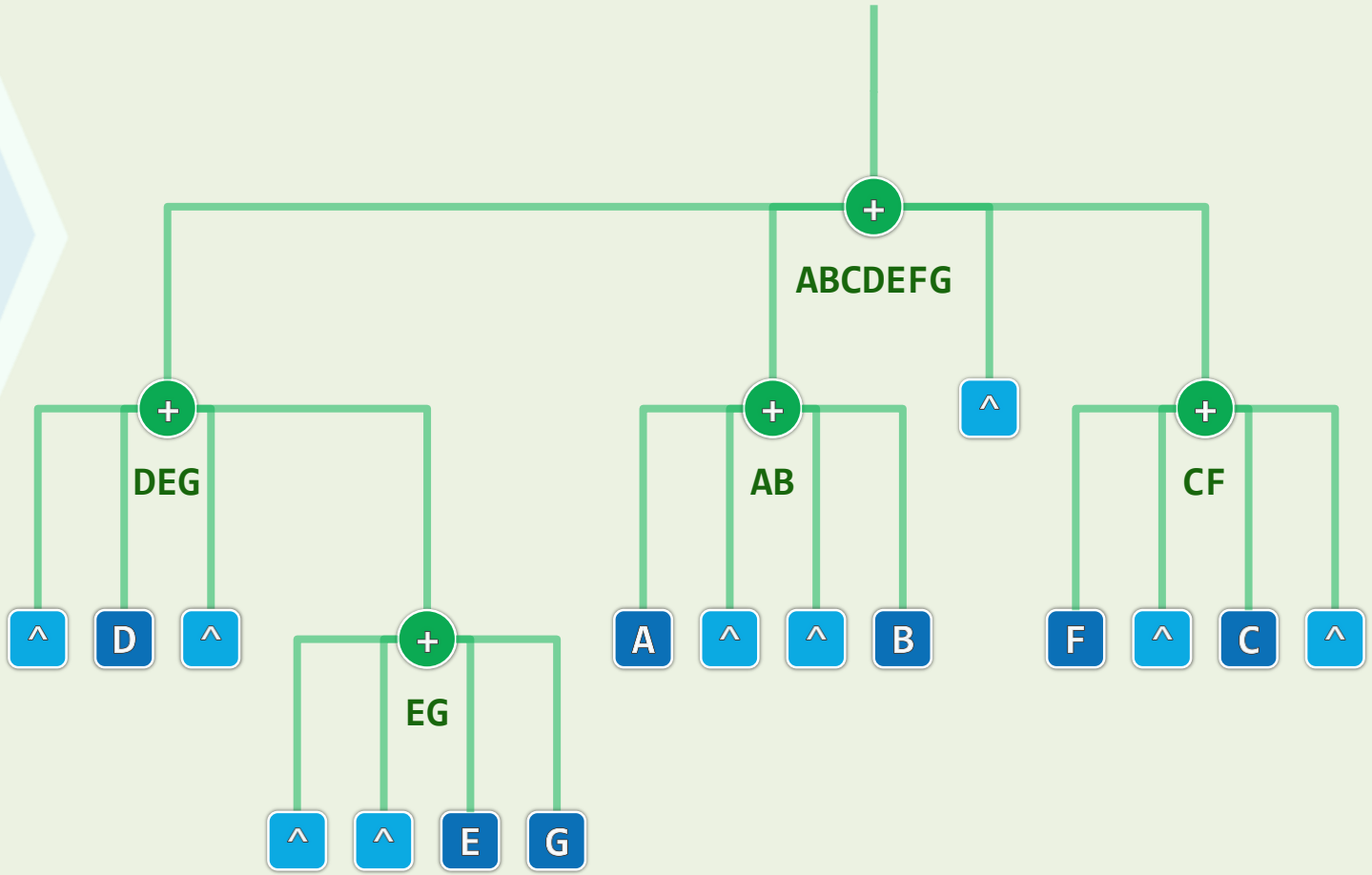
Example





Quadtree





07-B3

BST Application

kd-Tree: Construction

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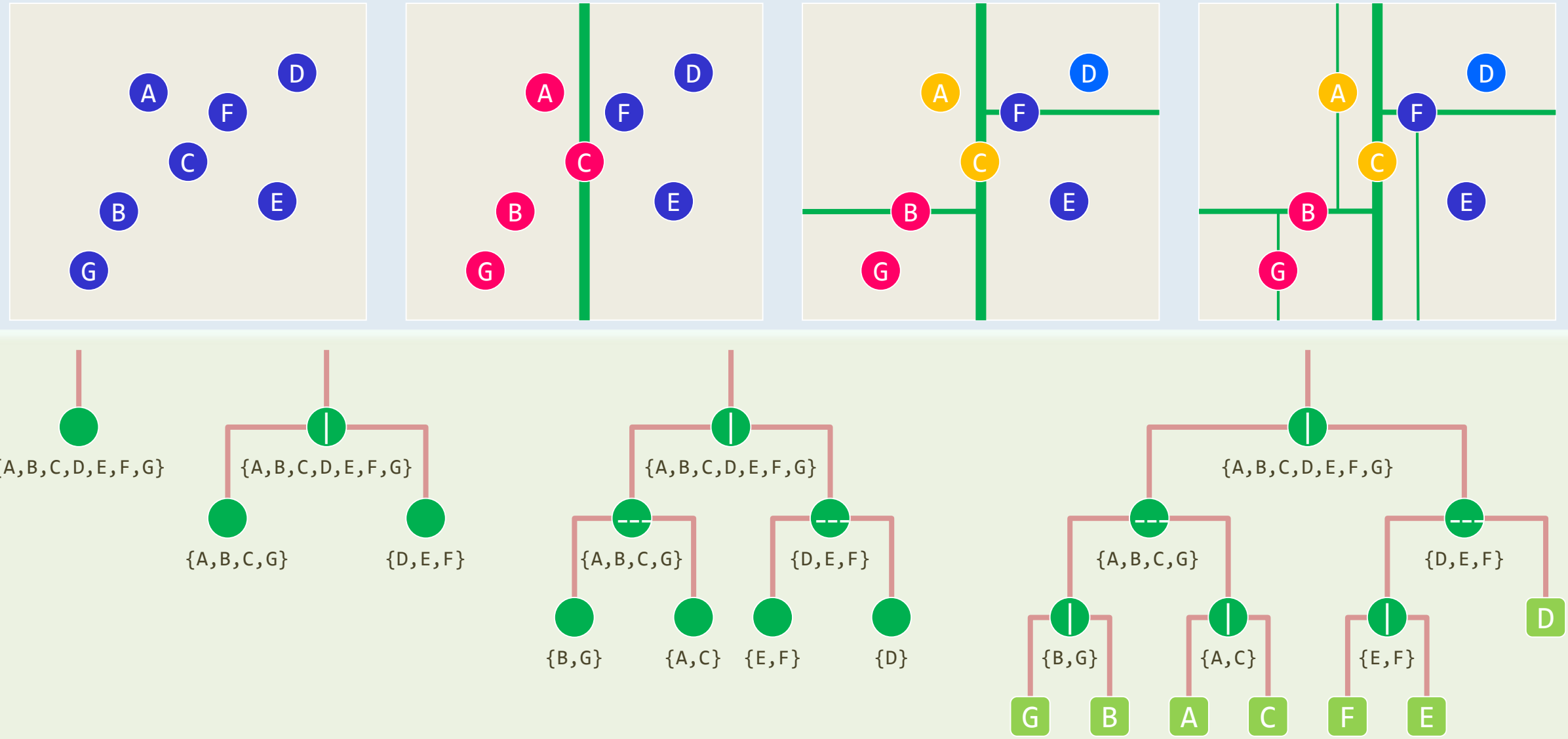
God found himself by creating.

buildKdTree(P,d)

```
// construct a 2d-(sub)tree for point (sub)set P at depth d
   if ( P == {p} ) return CreateLeaf( p )  //base
   root = CreateKdNode()
   root->splitDirection = Even(d) ? VERTICAL : HORIZONTAL
   root->splitLine = FindMedian( root->splitDirection, P )  //O(n)!
   ( P1, P2 ) = Divide( P, root->splitDirection, root->splitLine )  //DAC
   root->lc = buildKdTree( P1, d + 1 )  //recurse
   root->rc = buildKdTree( P2, d + 1 )  //recurse
   return( root )
```

Example



07-B4

BST Application

kd-Tree: Canonical Subsets

韦小宝跟著她走到桌边，只见桌上大白布上钉满了几千枚绣花针，几千块碎片已拼成一幅完整无缺的大地图，难得的是几千片碎皮拼在一起，既没多出一片，也没少了一片。

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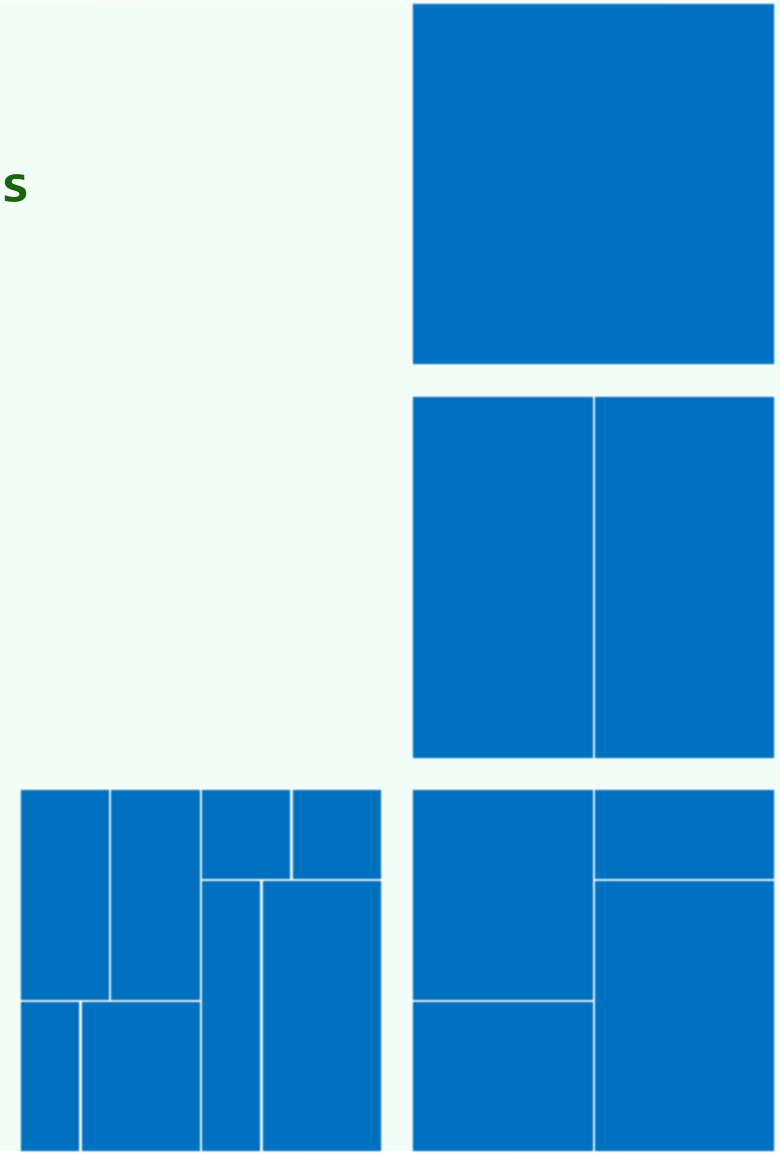
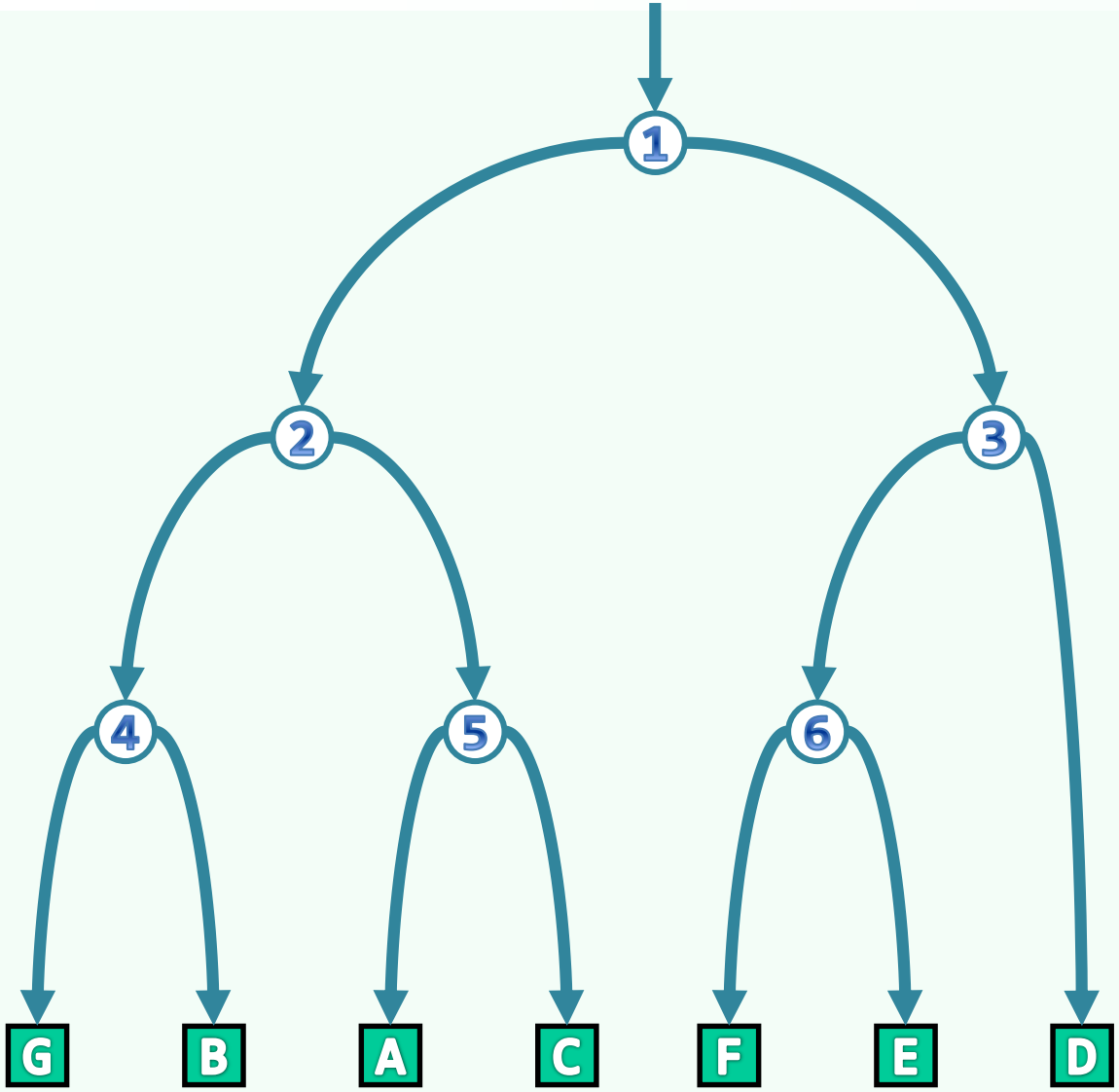
Canonical Subset

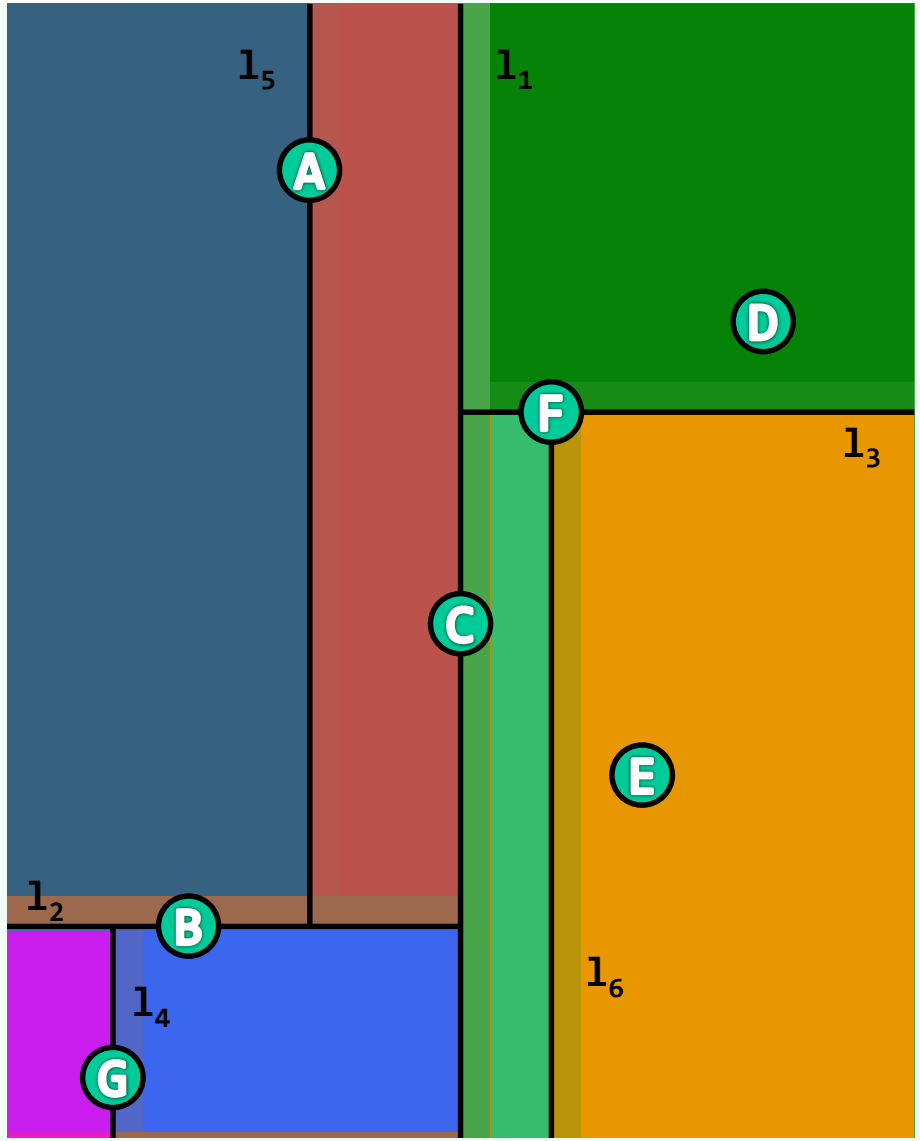
- Each node corresponds to
  - a rectangular **sub-region** of the plane, as well as
  - the **subset** of points contained in the sub-region
- Each of these subsets is called a **canonical subset**
- For each internal node X with children L and R,

  region(X) = region(L) ∪ region(R)
- Sub-regions of nodes at a same depth
  - never **intersect** with each other, and
  - their **union** covers the entire plane
- We will see soon that each 2D GRS can be answered by the **union** of a number of CS's



Example





07-B5

BST Application

kd-Tree: Query

我们竟为这无用的找寻浪费了这么多天!

我想找寻的心上人绝对不会在这里出现。

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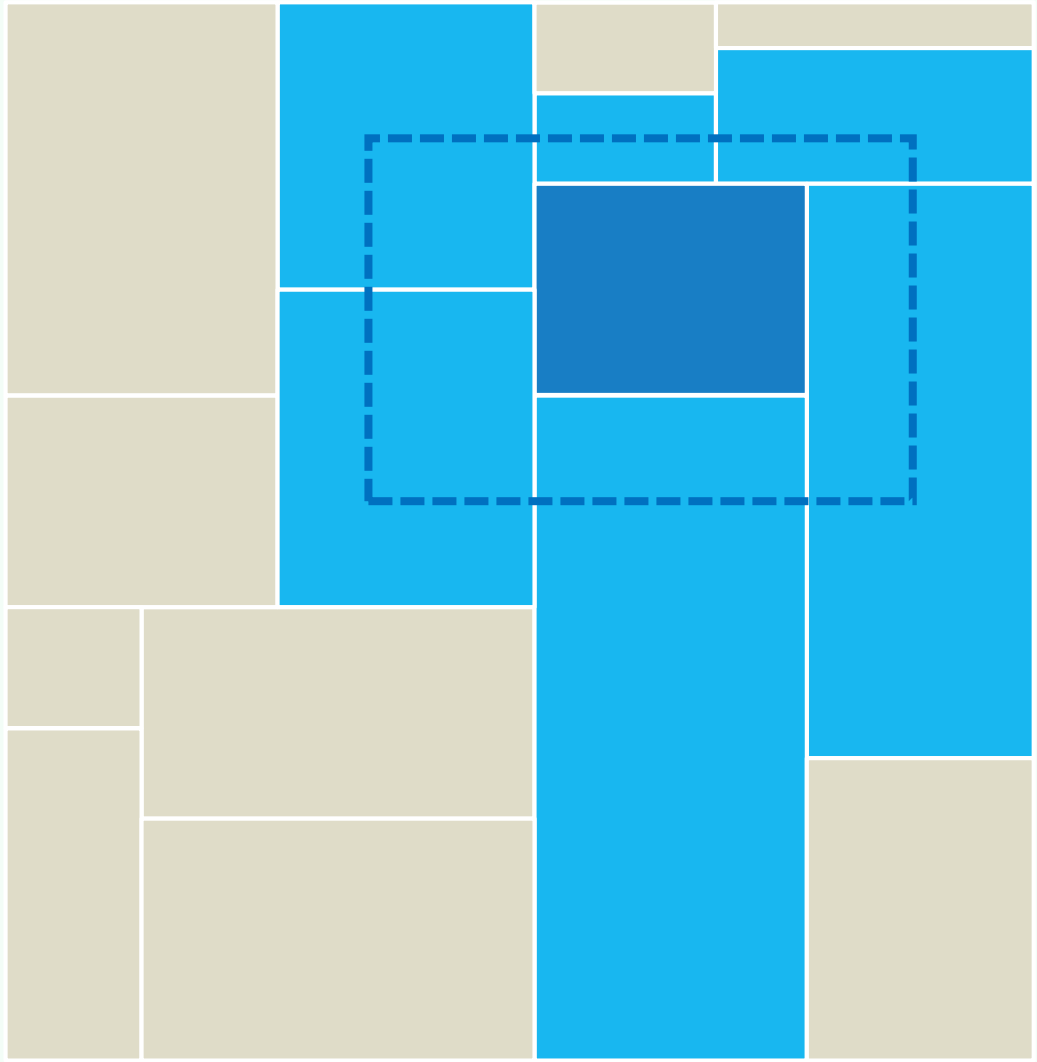
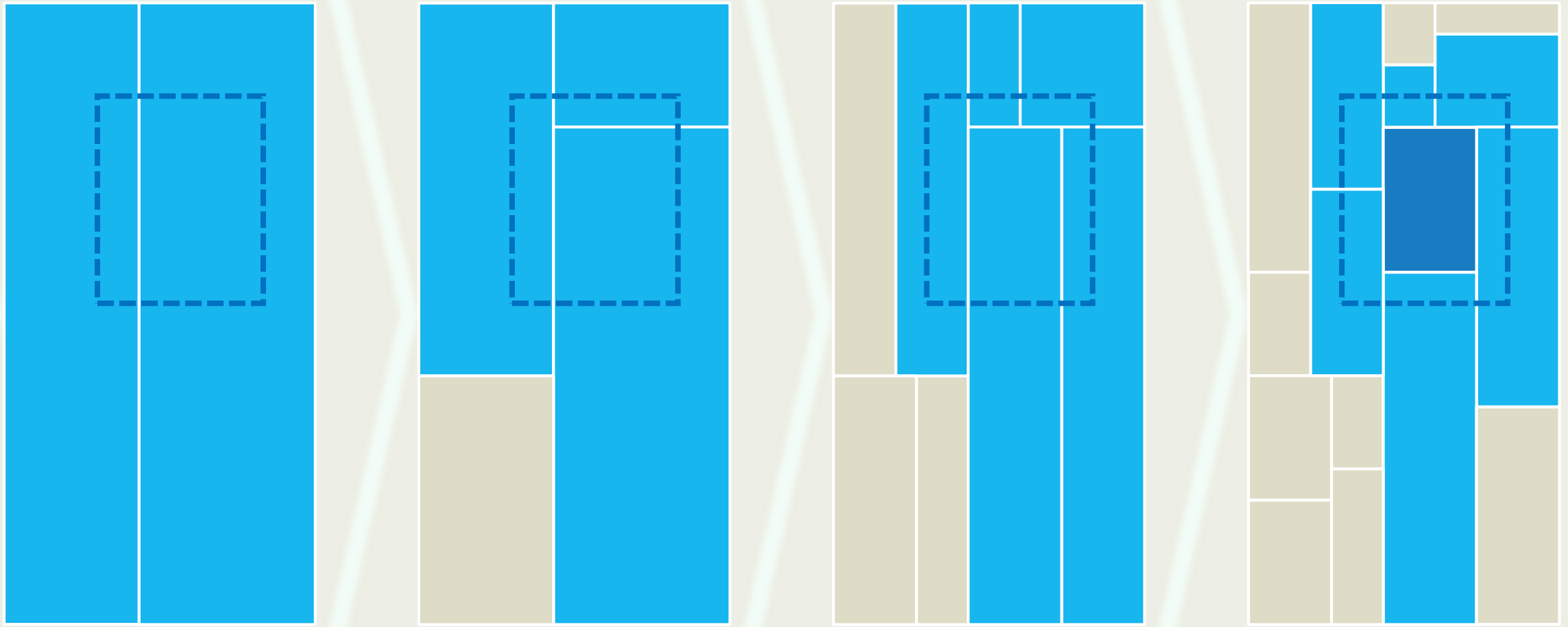
kdSearch(v,R)：热刀来切干(logn)层巧克力

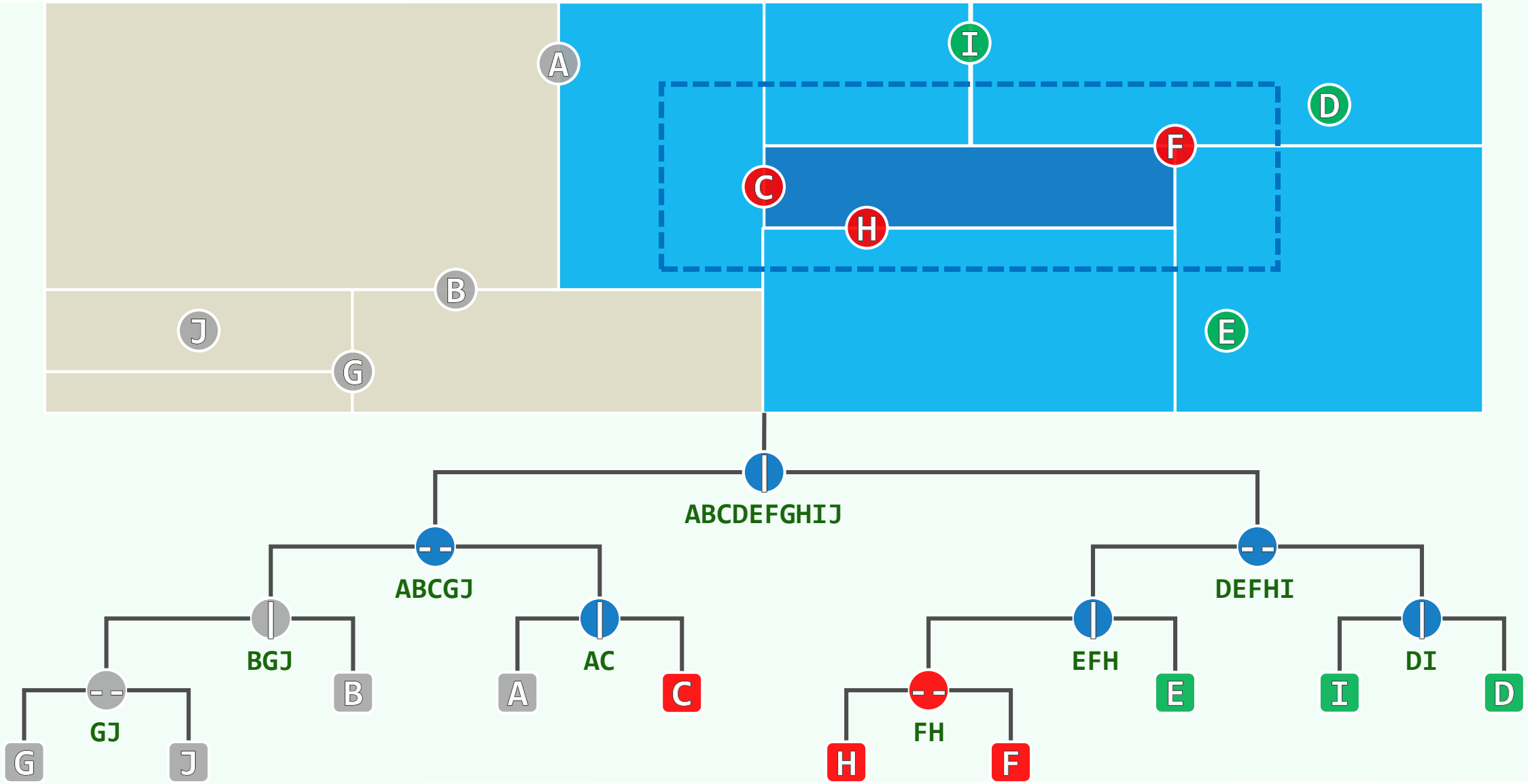
```
❖ if ( isLeaf( v ) )
      if ( inside( v, R ) ) report(v)
      return
❖ if ( region( v->lc ) ⊆ R )
      reportSubtree( v->lc )
   else if ( region( v->lc ) ∩ R ≠ ∅ )
      kdSearch( v->lc, R )
❖ if ( region( v->rc ) ⊆ R )
      reportSubtree( v->rc )
   else if ( region( v->rc ) ∩ R ≠ ∅ )
      kdSearch( v->rc, R )
```



Example



Example


A
I
D
F
C
H
B
J
G
E
ABCDEFGHIJ
ABCGJ
DEFHI
BGJ
AC
EFH
DI
GJ
FH
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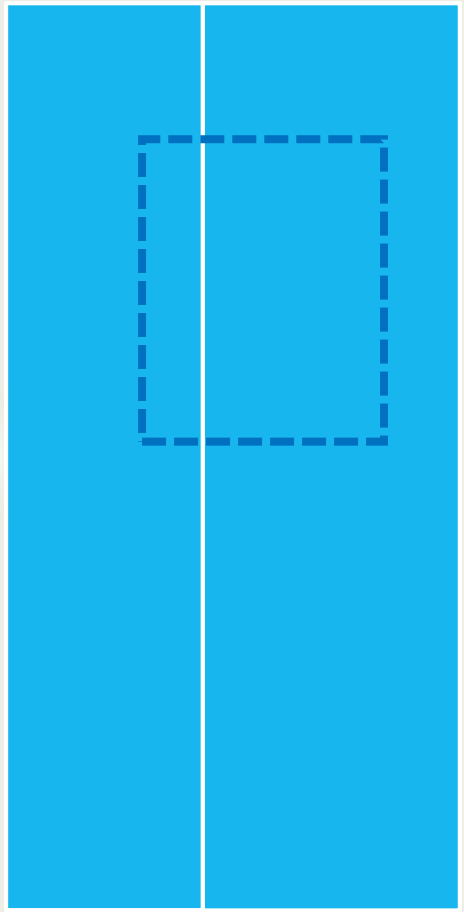
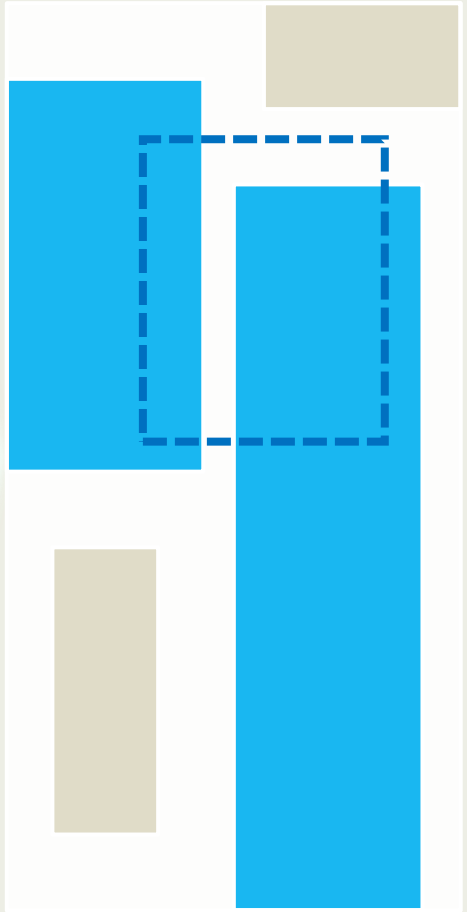
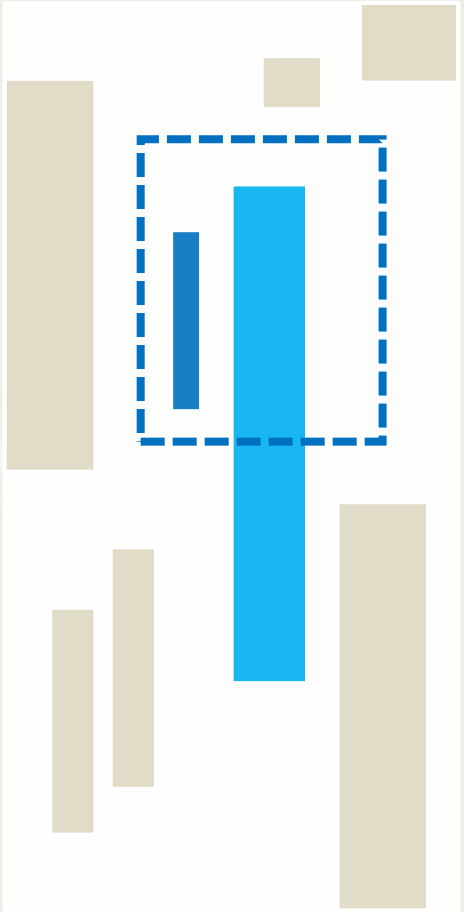
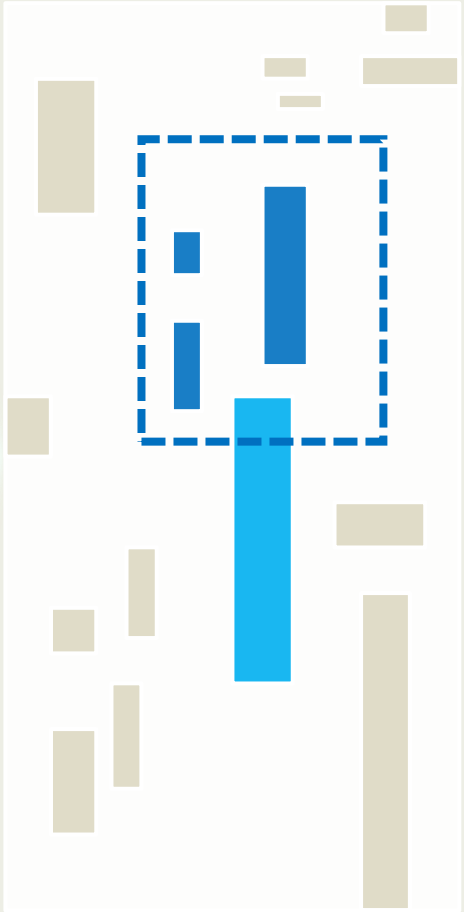
Bounding Box









07-B6

BST Application

kd-Tree: Complexity

肉眼看不清细节，但他们都知道那是木星所在的位置，这颗太阳系最大的行星已经坠落到二维平面上了。

有人嘲笑这种体系说：为了能发现这个比例中项并组成政府共同体，按照我的办法，只消求出人口数字的平方根就行了。

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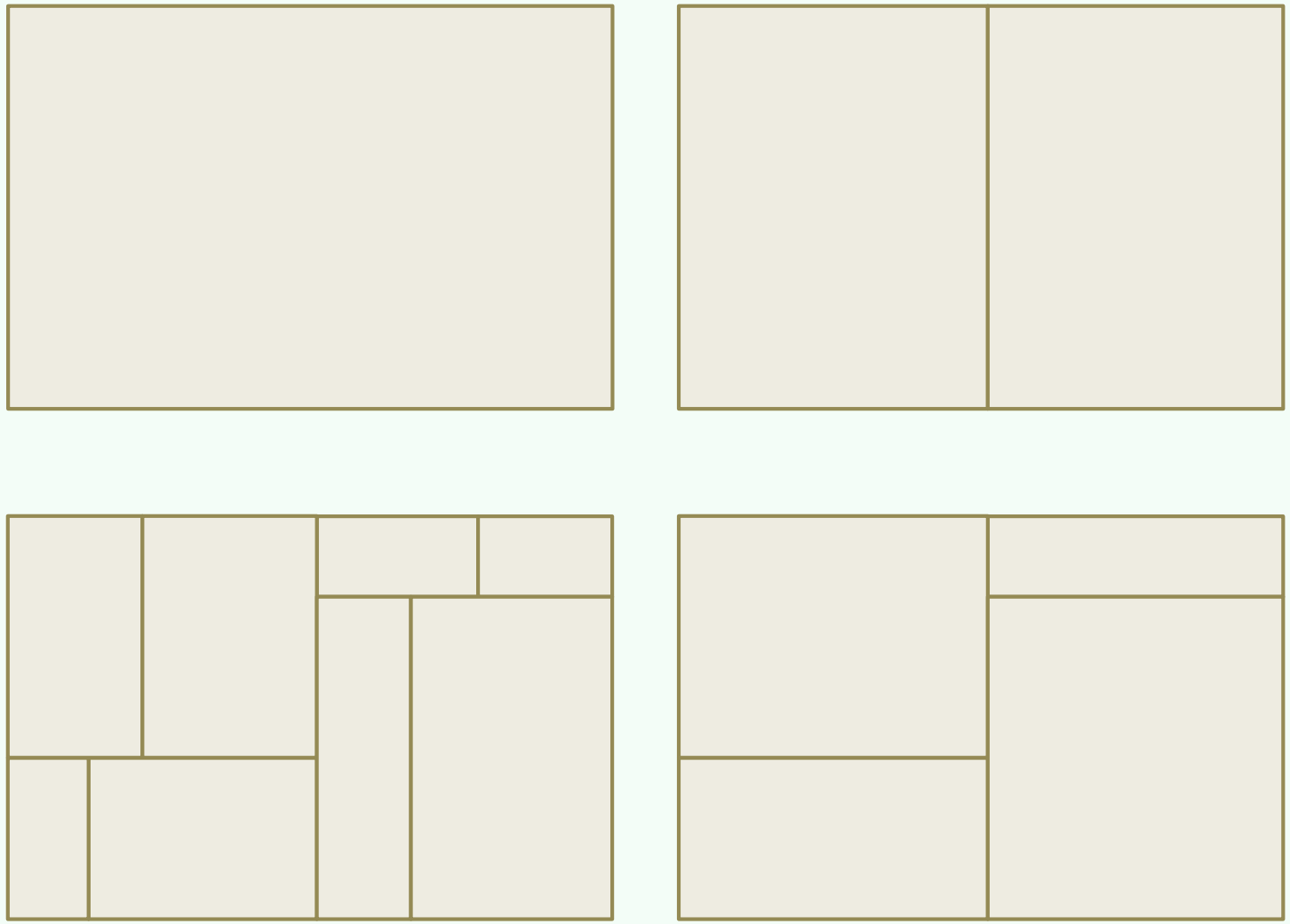
Preprocessing

- T(n)

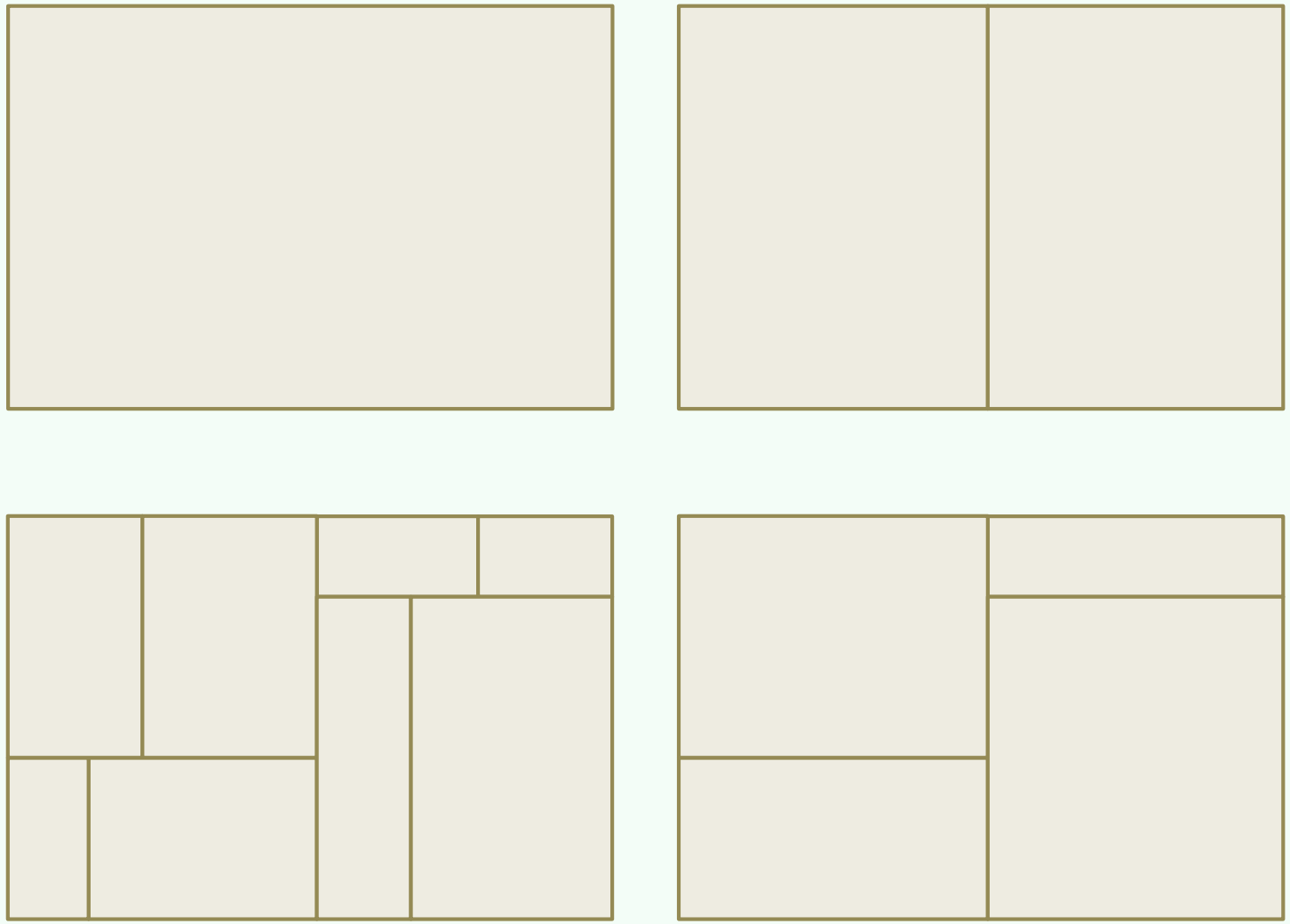
  = 2*T(n/2) + $\mathcal{O}$(n)

  = $\mathcal{O}$(nlogn)



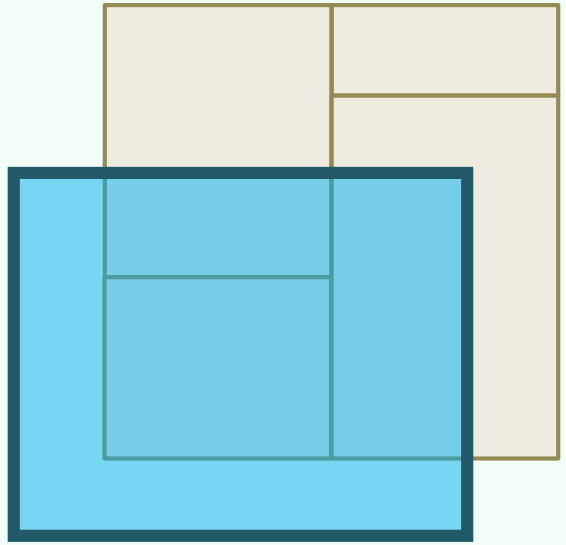
Storage

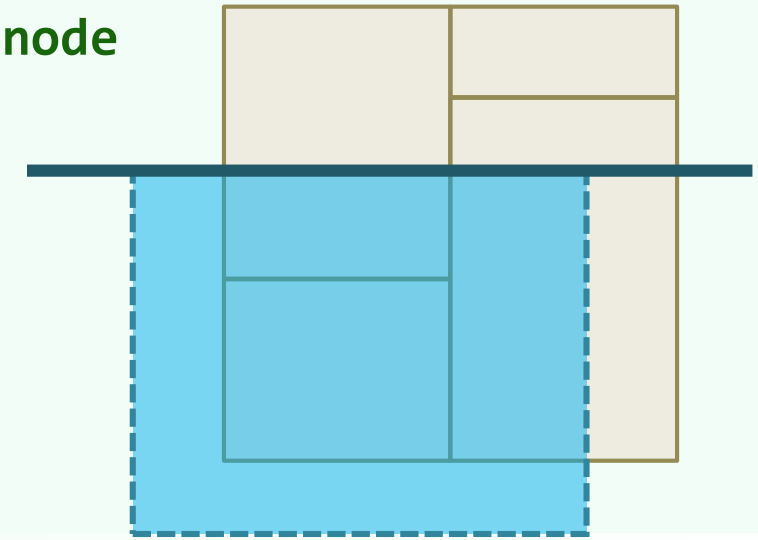
- The tree has a height of $O(\log n)$
- $1 + 2 + 4 + \ldots + O(2^{\log n}) = O(n)$

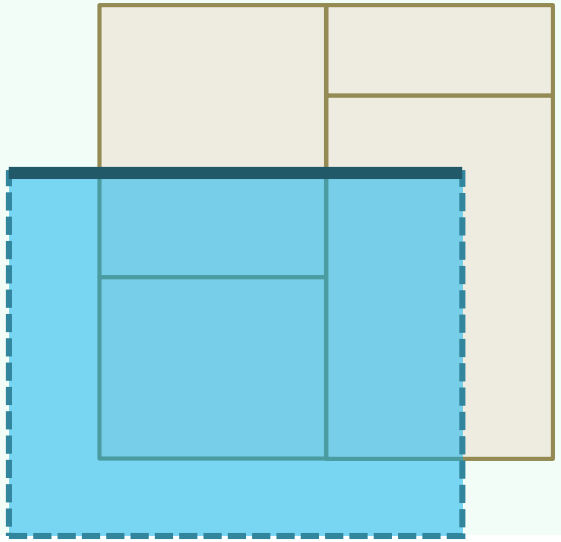


Query Time

- Claim: Report + Search = $\mathcal{O}(r + \sqrt{n})$
- The searching time depends on Q(n), the number of
  - recursive calls, or
  - sub-regions intersecting with R (at all levels)
- No more than 2 of the 4 grandchildren of each node will recurse
  - $Q(1) = \mathcal{O}(1)$
  - $Q(n) = 2 \cdot Q(n/4) + \mathcal{O}(1)$
- Solve to $Q(n) = \mathcal{O}(\sqrt{n})$







Beyond 2D

- Can 2d-tree be extended to kd-tree and help HIGHER dimensional GRS?

  If yes, how efficiently can it help?

- A kd-tree in k-dimensional space is constructed by

  recursively divide $\mathcal{E}^d$ along the 1st, 2nd, ..., kth dimensions

- An orthogonal range query on a set of n points in $\mathcal{E}^d$
  - can be answered in $\mathcal{O}(r + n^{1-1/d})$ time,
  - using a kd-tree of size $\mathcal{O}(n)$, which
  - can be constructed in $\mathcal{O}(n \log n)$ time

07-C

BST Application

Multi-Level Search Tree

几株不知名的树，已脱下了黄叶

只有那两三片，多么可怜在枝上抖怯

它们感到秋来到，要与世间离别

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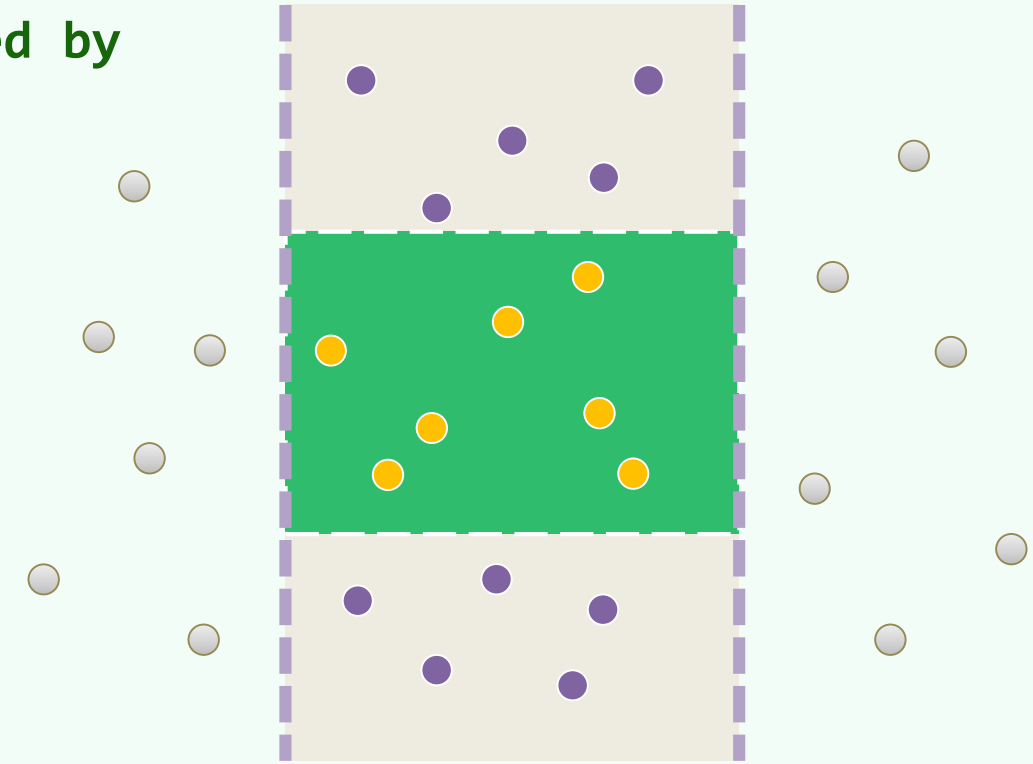
2D Range Query = x-Query + y-Query

- Is there any structure which answers range query **FASTER** than kd-trees?
- An m-D orthogonal range query can be answered by the **INTERSECTION** of m 1D queries
- For example, a 2D range query can be divided into two 1D range queries:
  - find all points in $[x_1, x_2]$; and then
  - find in these candidates those lying in $[y_1, y_2]$



Worst Cases

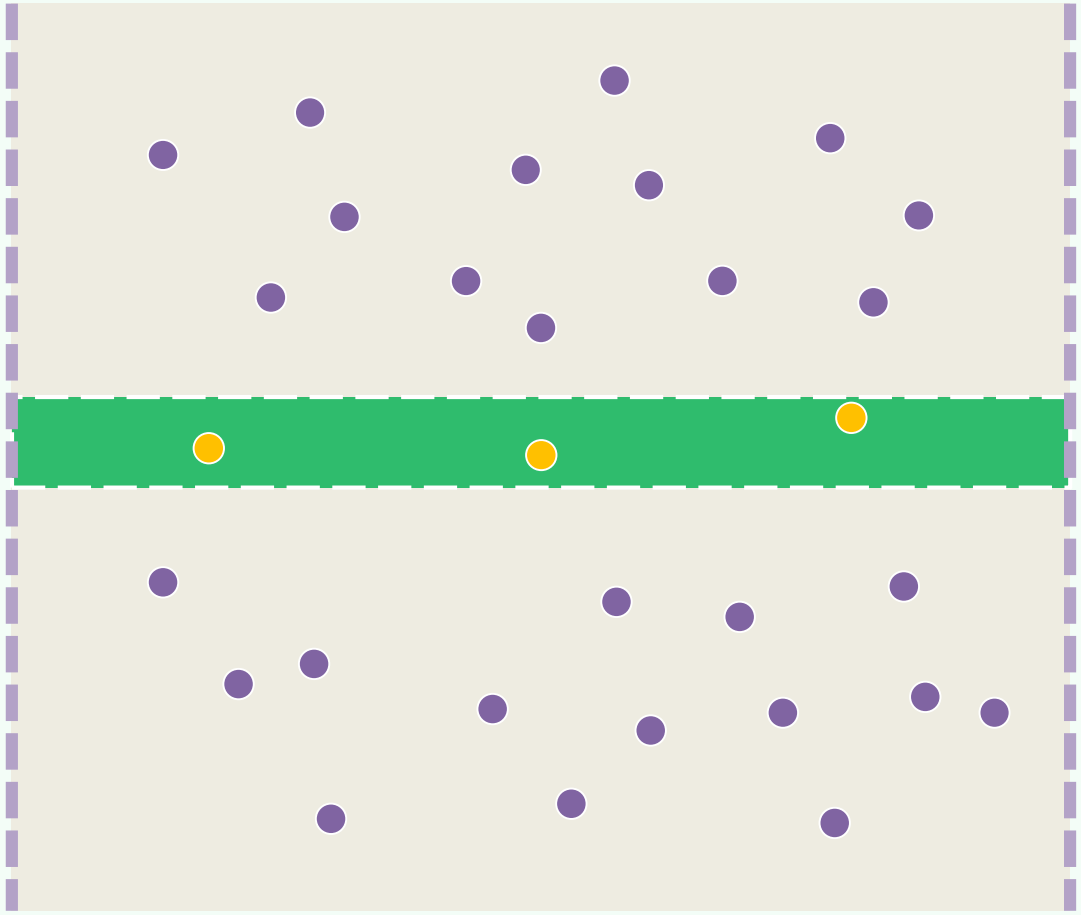
- Using kd-trees needs $\mathcal{O}(1+\sqrt{n})$ time. But here ...
- The x-query returns

  (almost) all points whereas

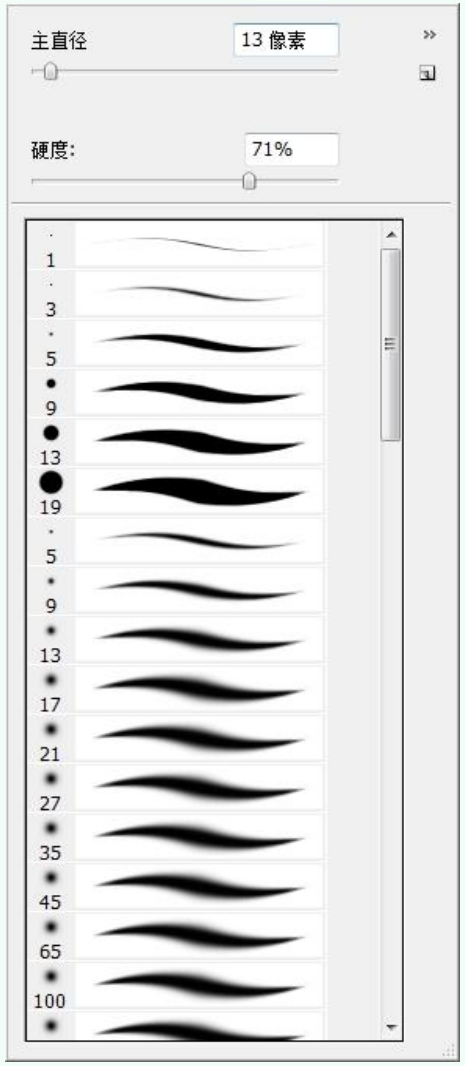
  the y-query rejects

  (almost) all
- We spent $\Omega(n)$ time

  before getting r = 0 points

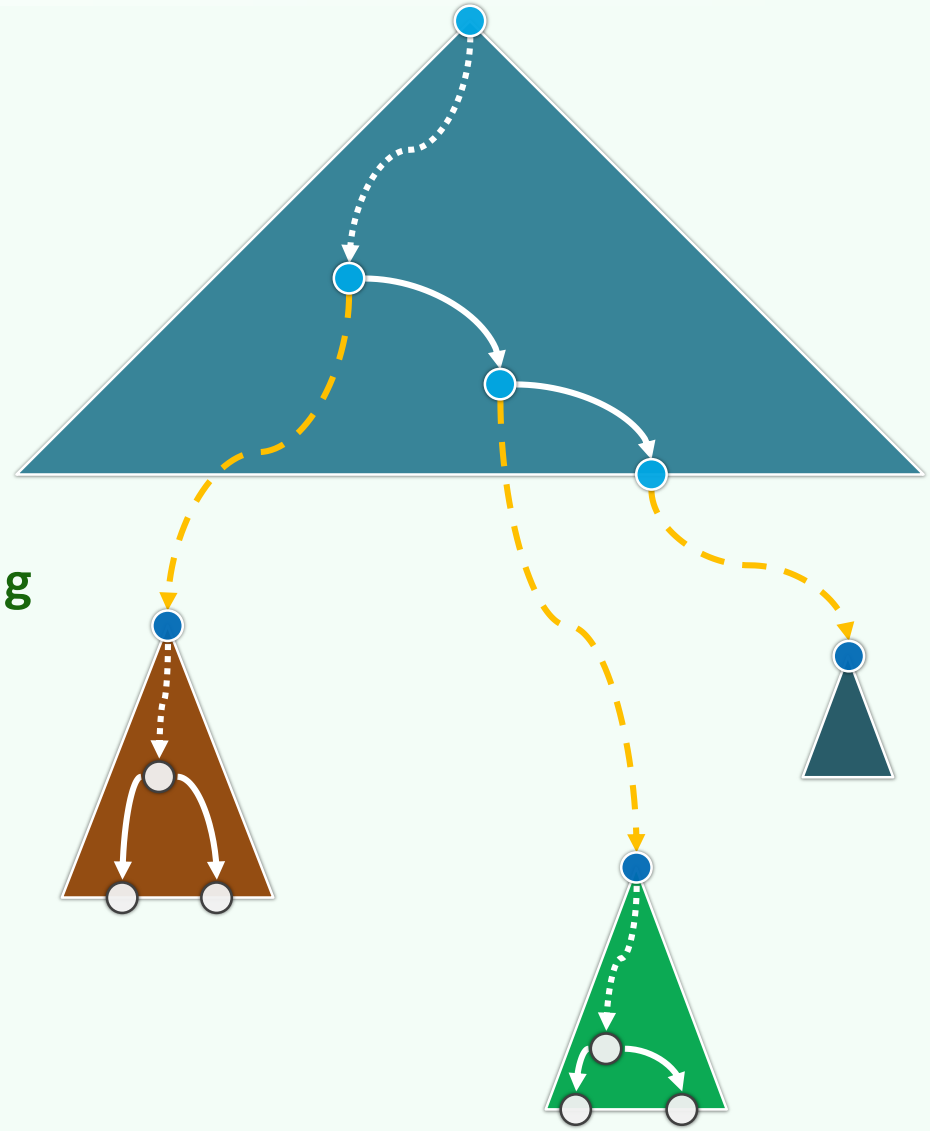


Painters' Strategy



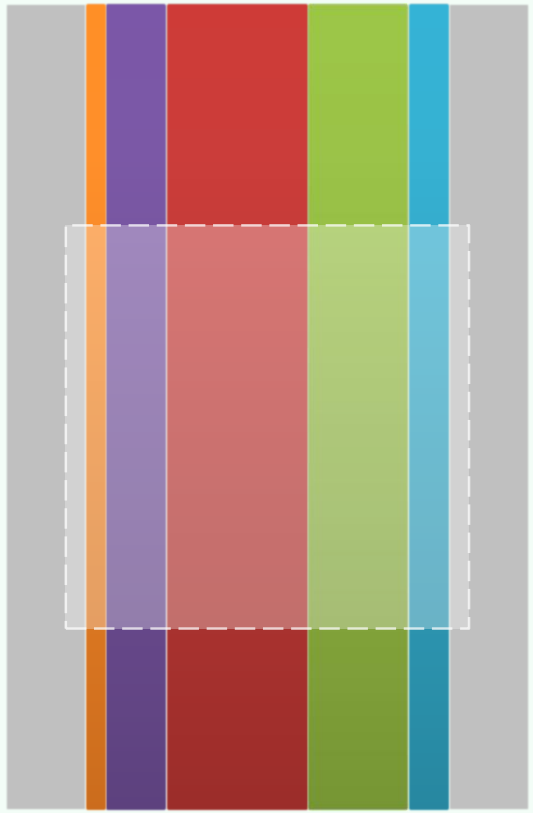
2D Range Query = x-Query * y-Query

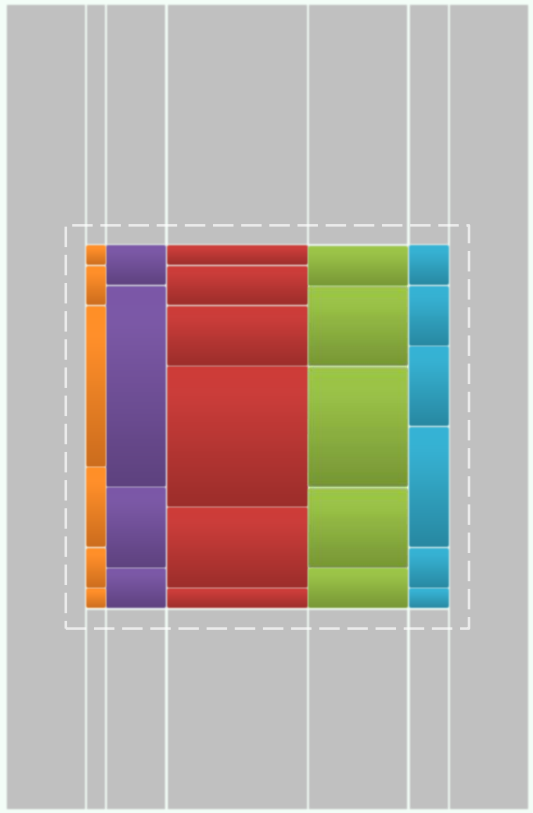
- Tree of trees
  - build a 1D BBST (called x-tree)
    for the first range query (x-query);
  - for each node v in the x-range tree,
    build a y-coordinate BBST (y-tree), containing
    the canonical subset associate with v
- It's an x-tree of (a number of) y-trees,
  called a Multi-Level Search Tree
- How to answer range queries with such an MLST?

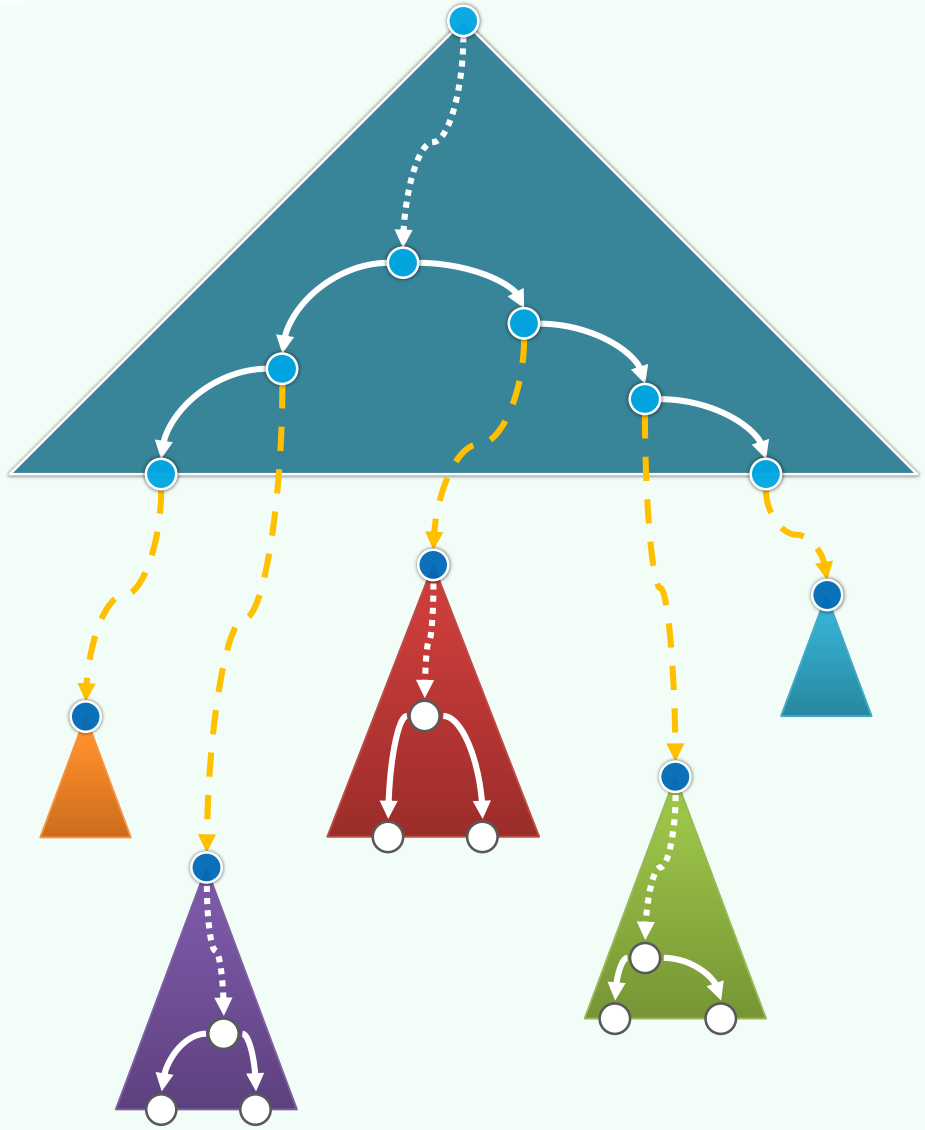


2D Range Query = x-Query * y-Queries

❖ Query Time = $\mathcal{O}(r + \log^2 n)$ ~ $\mathcal{O}(r + \log n)$







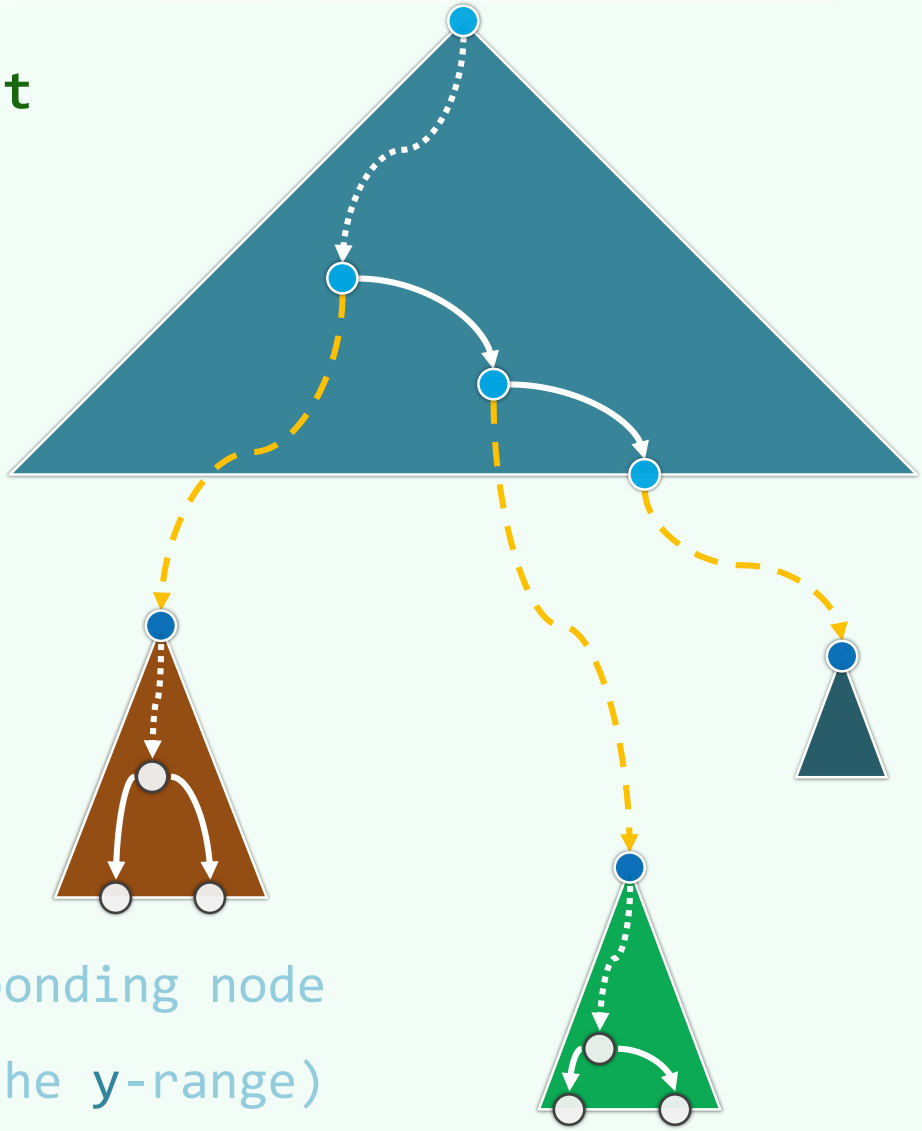
Query Algorithm

1. Determine the canonical subsets of points that satisfy the first query

// there will be $O(\log n)$ such canonical sets,
// each of which is just represented as
// a node in the x-tree

2. Find out from each canonical subset which points lie within the y-range

// To do this,
// for each canonical subset,
// we access the y-tree for the corresponding node
// this will be again a 1D range search (on the y-range)



Complexity: Preprocessing Time + Storage

- A 2-level search tree

  for n points in the plane

  can be built

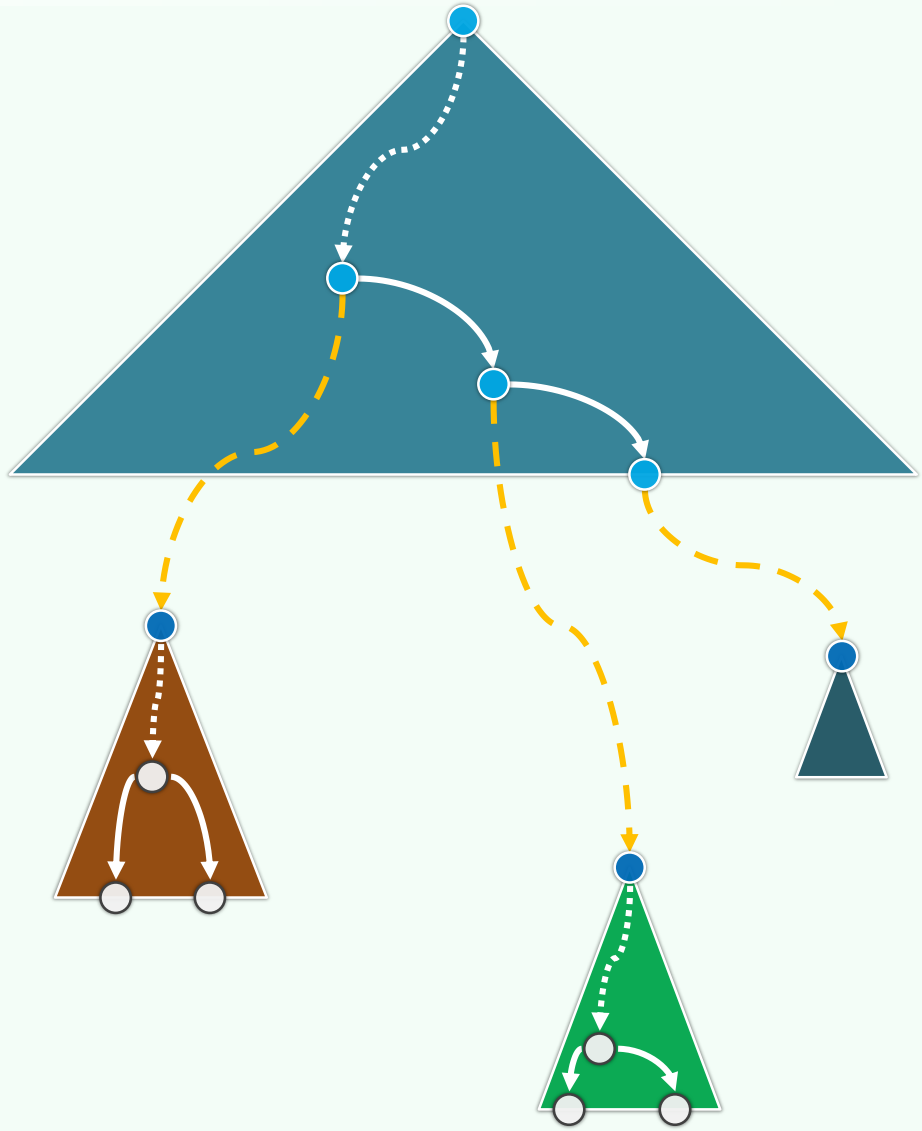
  in $\mathcal{O}(n \log n)$ time

- Each input point is stored in $\mathcal{O}(\log n)$ y-trees

- A 2-level search tree

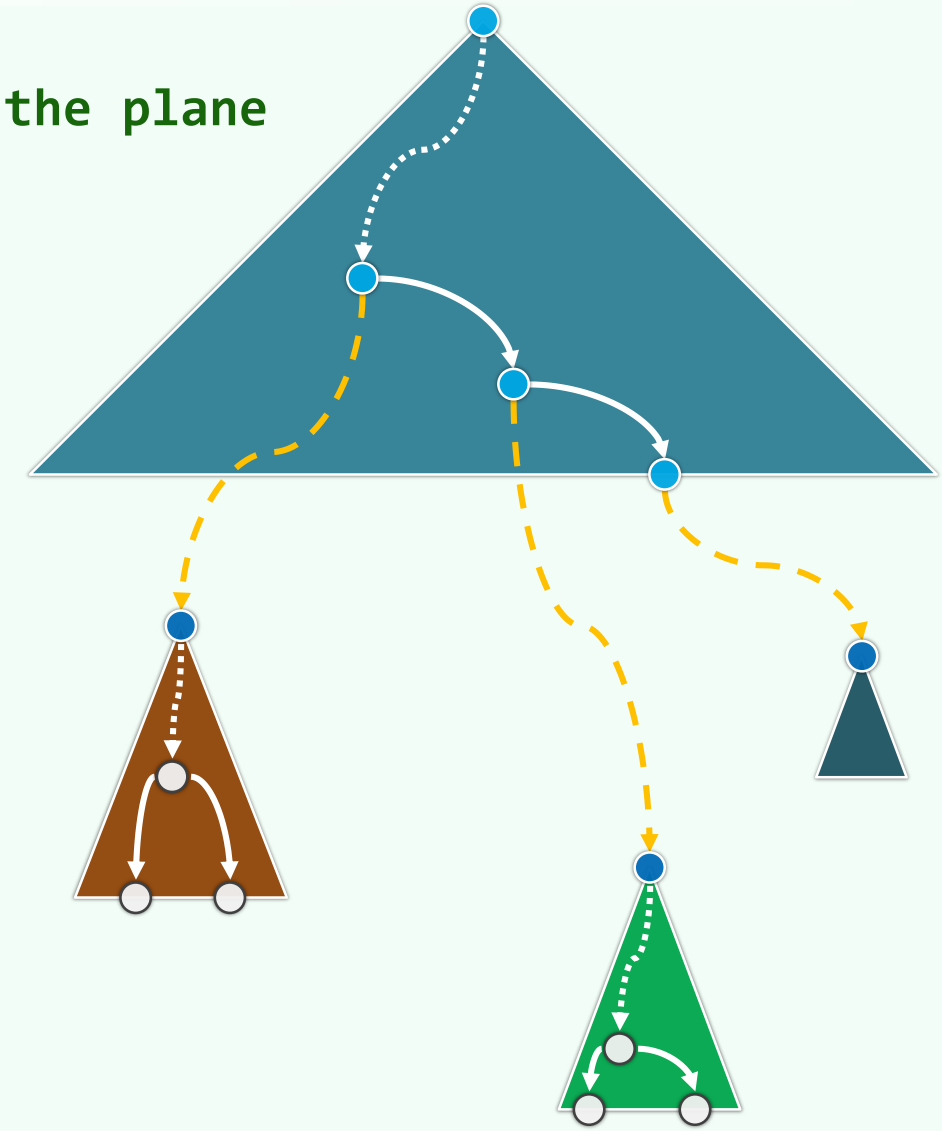
  for n points in the plane

  needs $\mathcal{O}(n \log n)$ space



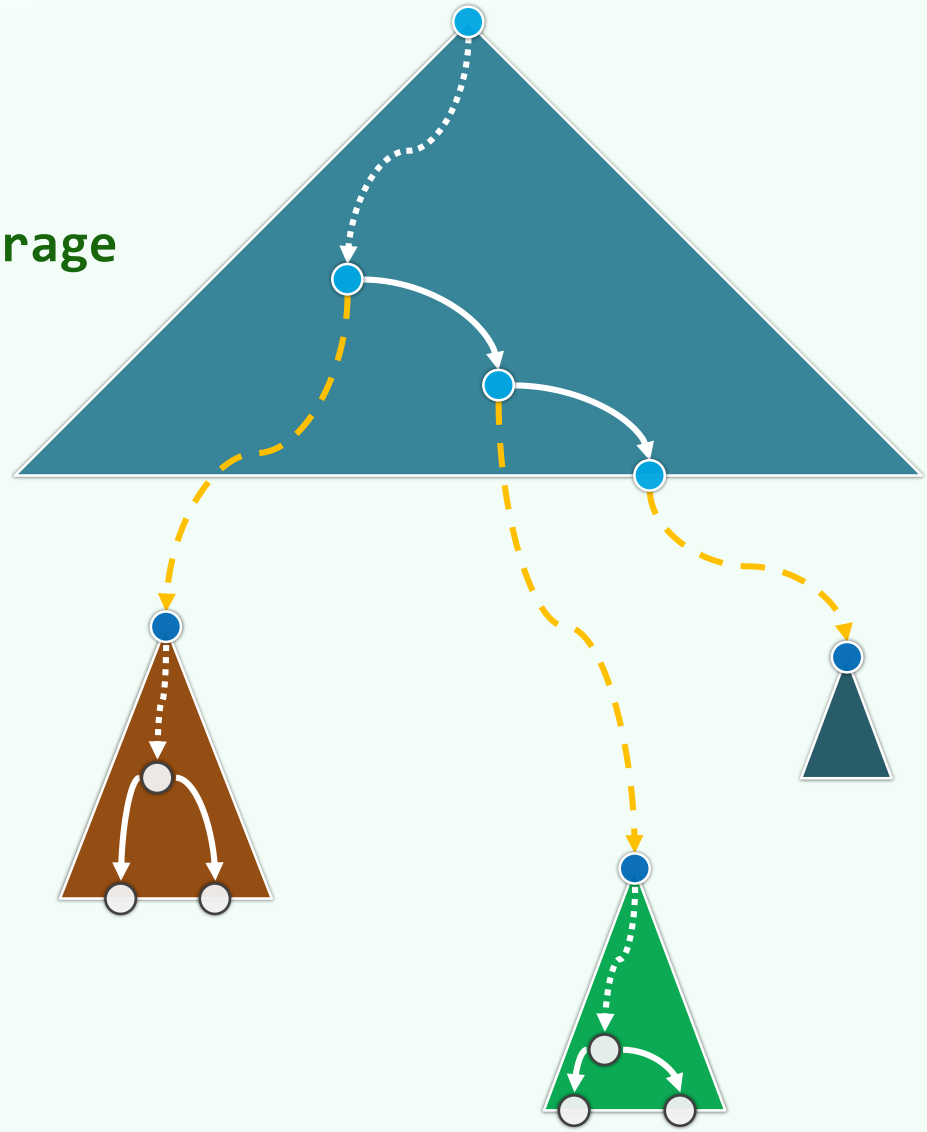
Complexity: Query Time

- Claim: A 2-level search tree for n points in the plane
  answers each planar range query
  in $\mathcal{O}(r + \log^2 n)$ time
- The x-range query needs $\mathcal{O}(\log n)$ time
  to locate the $\mathcal{O}(\log n)$ nodes
  representing the canonical subsets
- Then for each of these nodes,
  a y-range search is invoked,
  which needs $\mathcal{O}(\log n)$ time



Beyond 2D

- Let S be a set of n points in $\mathcal{E}^d$, d ≥ 2
  - A d-level tree for S uses $\mathcal{O}(n \cdot \log^{d-1} n)$ storage
  - Such a tree can be constructed in $\mathcal{O}(n \cdot \log^{d-1} n)$ time
  - Each orthogonal range query of S can be answered in $\mathcal{O}(r + \log^{d} n)$ time
- For planar case, can the query time be improved to, say, $\mathcal{O}(\log n)$?



07-D

BST Application

Range Tree

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Coherence

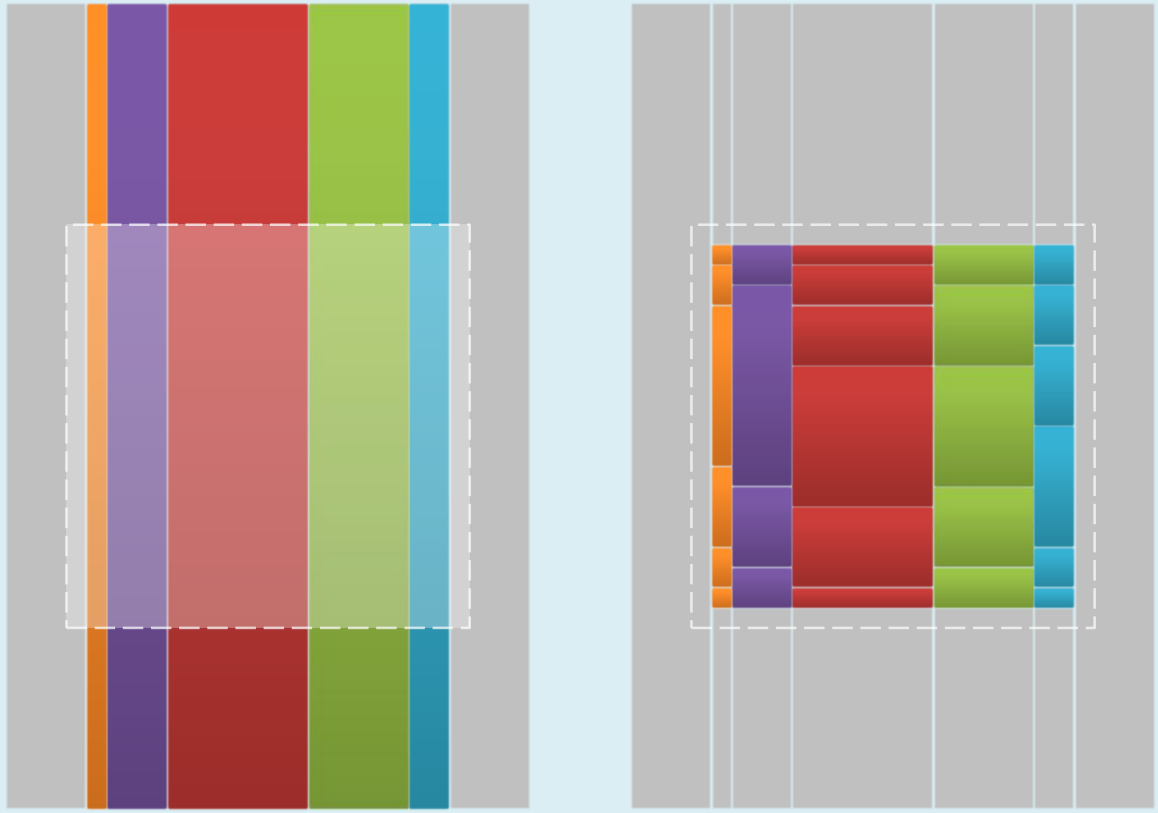
❖ For each query, we

- need to repeatedly search

DIFFERENT y-lists,

- but always with

the SAME key



BBST<BBST<T>> --> BBST<List<T>>

- Each y-search is just

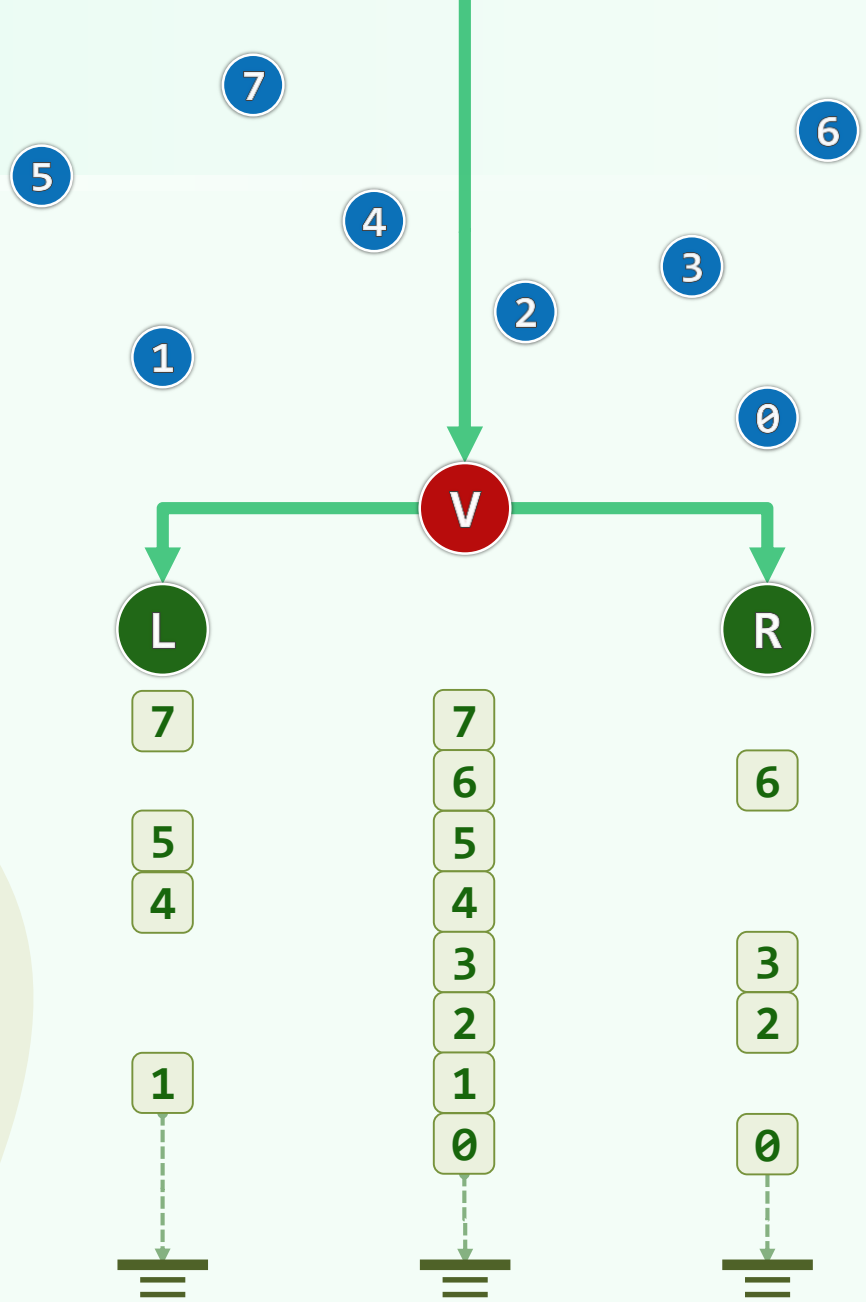
  a 1D query without further recursions

- So it not necessary

  to store each canonical subset

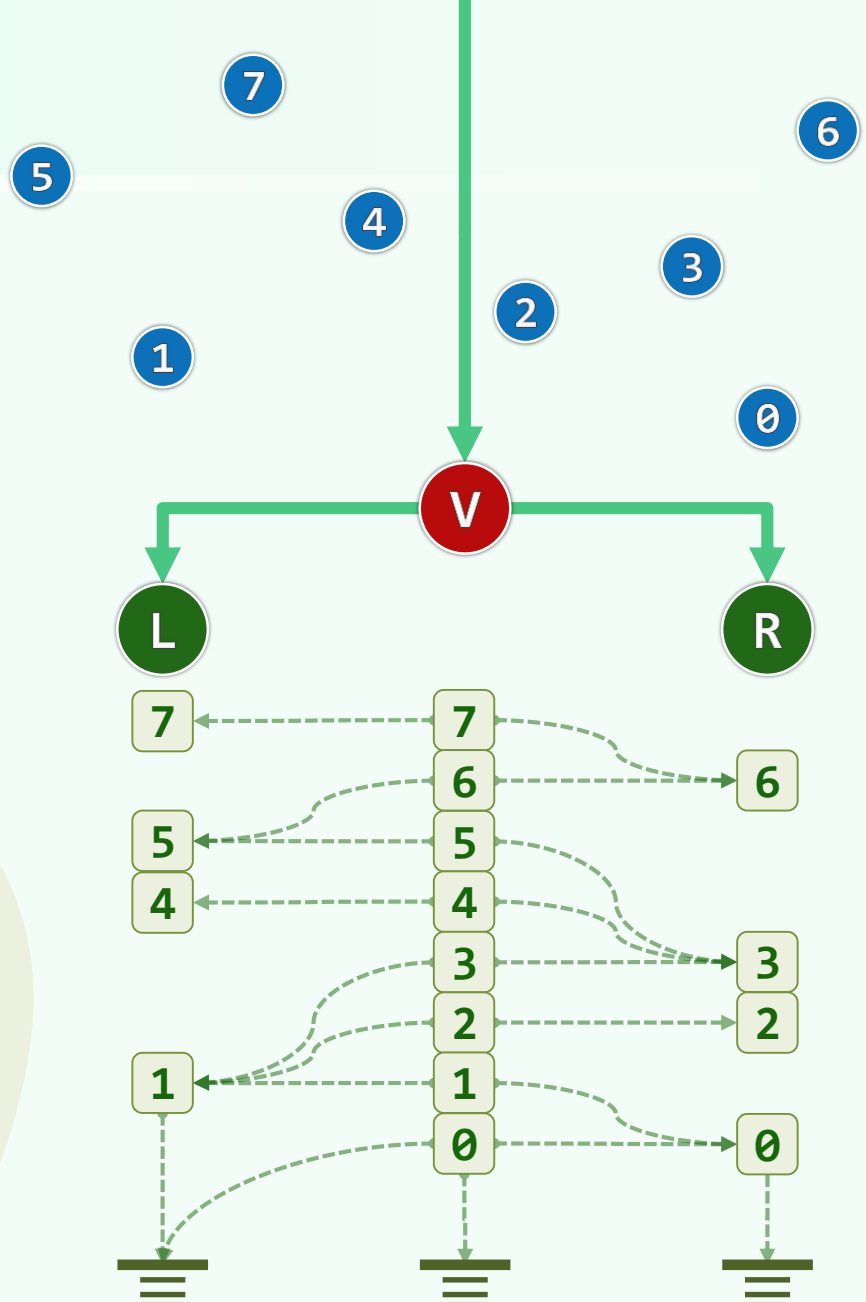
  as a BBST

- Instead, a sorted y-list simply works



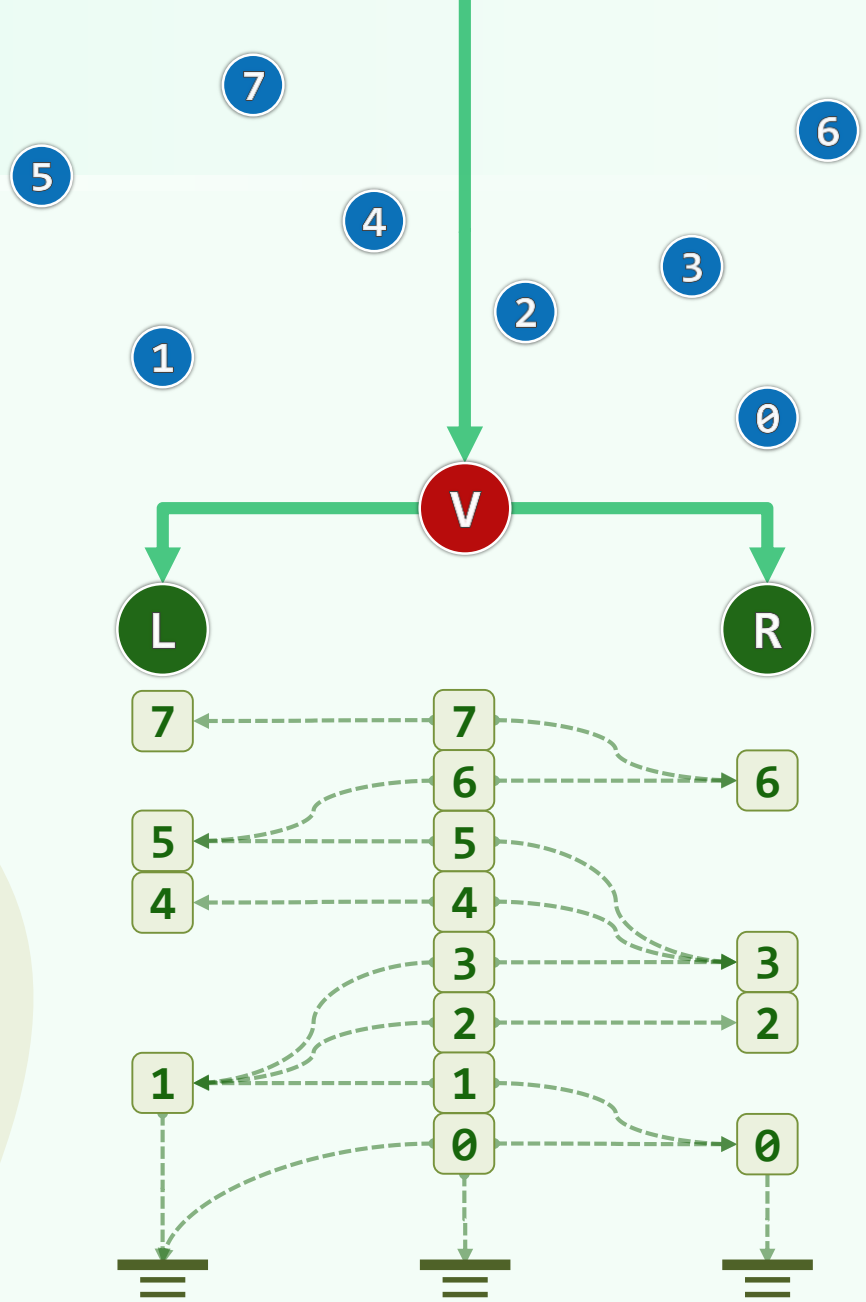
Links Between Lists

- We can **CONNECT** all the different lists into a **SINGLE** massive list
- Thus, once a parent y-list is searched, we can get, in *O*(1) time, the entry for child y-list by following the link between them



Using Coherence

- To answer a 2D range query, we will
  - do an $O(\log n)$ search
    - on the y-list for the LCA
- Thereafter, while descending the x-tree, we can
  - keep track of the position of y-range
    - in each successive list in $O(1)$ time
- This technique is called ......



Fractional Cascading

- For each item in A_v,

  we store two pointers to

  the item of NLT value

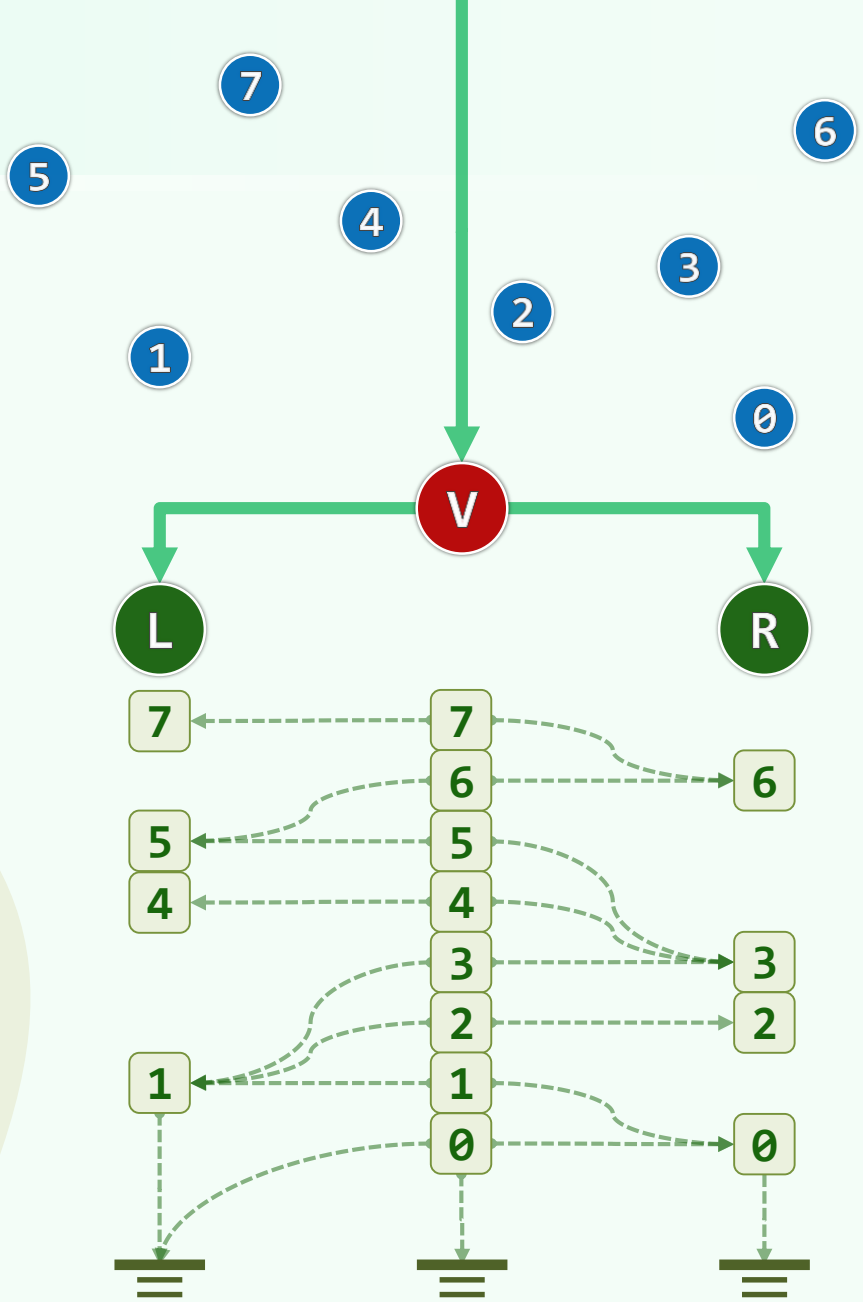
  in A_L and A_R resp.

- Hence for any y-query with q_y,

  once we know its entry in A_v, we can

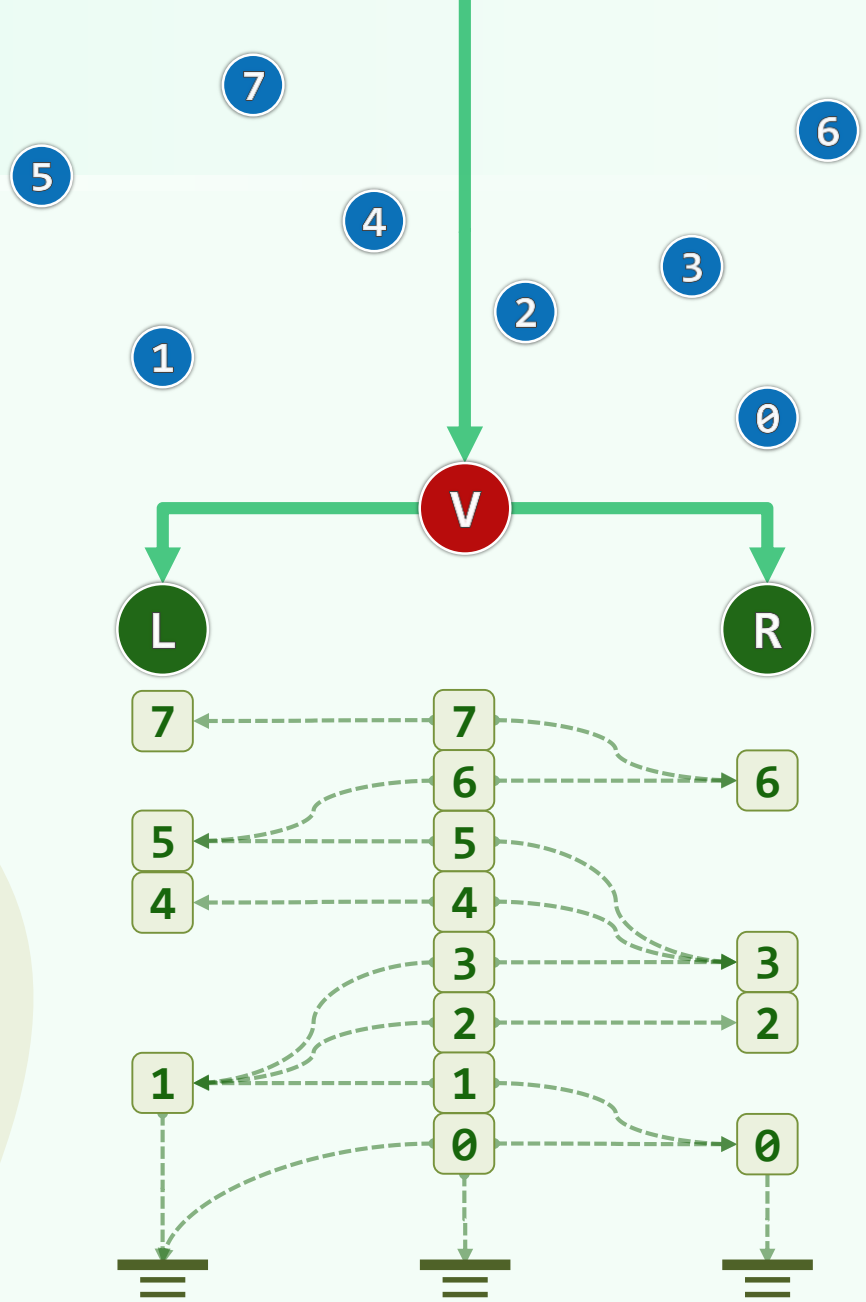
  determine its entry in either A_L or A_R

  in $O(1)$ additional time



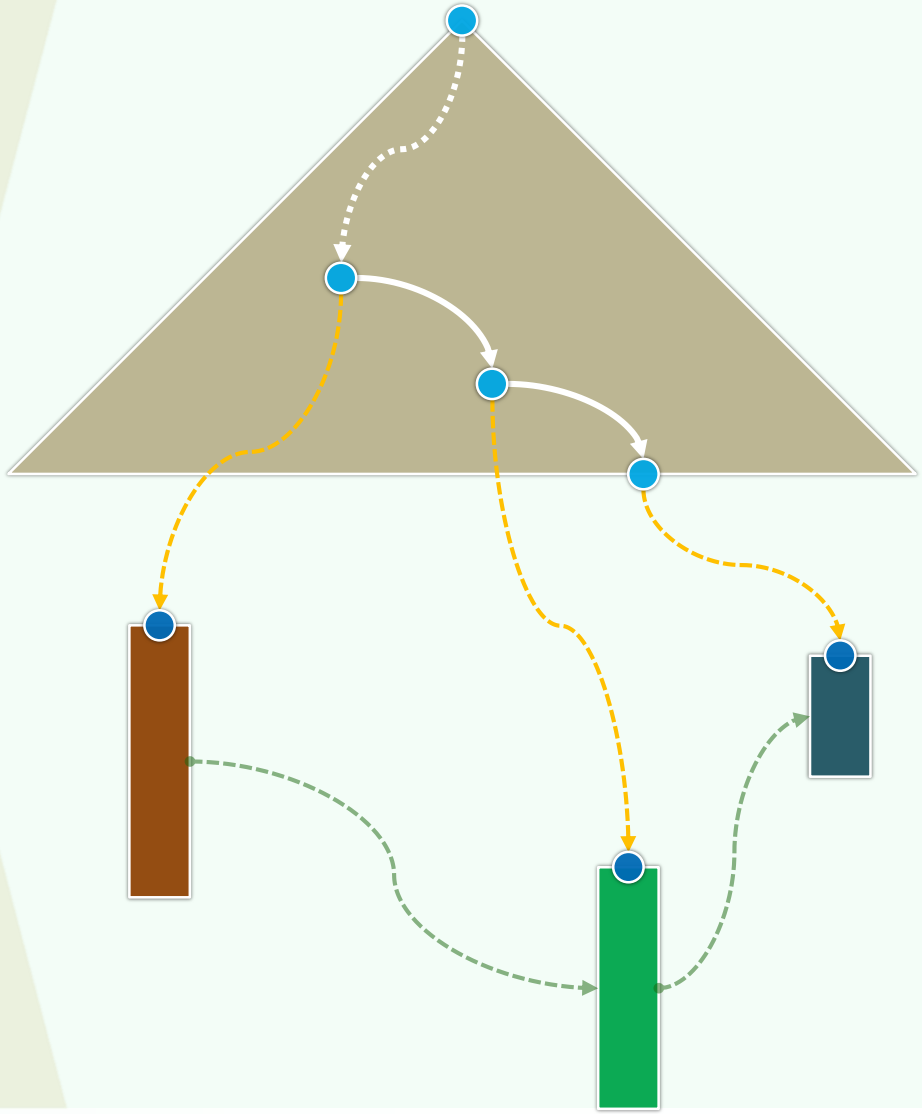
Construction By 2-Way Merging

- Let V be an internal node in the x-tree with L/R its left/right child resp.
- Let A_V be the y-list for v and A_L/A_R be the y-lists for its children
- Assuming no duplicate y-coordinates, we have
  - A_V is the disjoint union of A_L and A_R, and hence
  - A_V can be obtained by merging A_L and A_R (in linear time)



Complexity

- An MLST with fractional cascading is called a **range tree**
- A y-search for root is done in $\mathcal{O}(\log n)$ time
- To drop down to each next level, we can get, in $\mathcal{O}(1)$ time, the current y-interval from that of the prior level
- Hence, each 2D orthogonal range query
  - can be answered in $\mathcal{O}(r + \log n)$ time
  - from a data structure of size $\mathcal{O}(n \cdot \log n)$,
  - which can be constructed in $\mathcal{O}(n \cdot \log n)$ time



Beyond 2D

- Unfortunately, the trick of fractional cascading

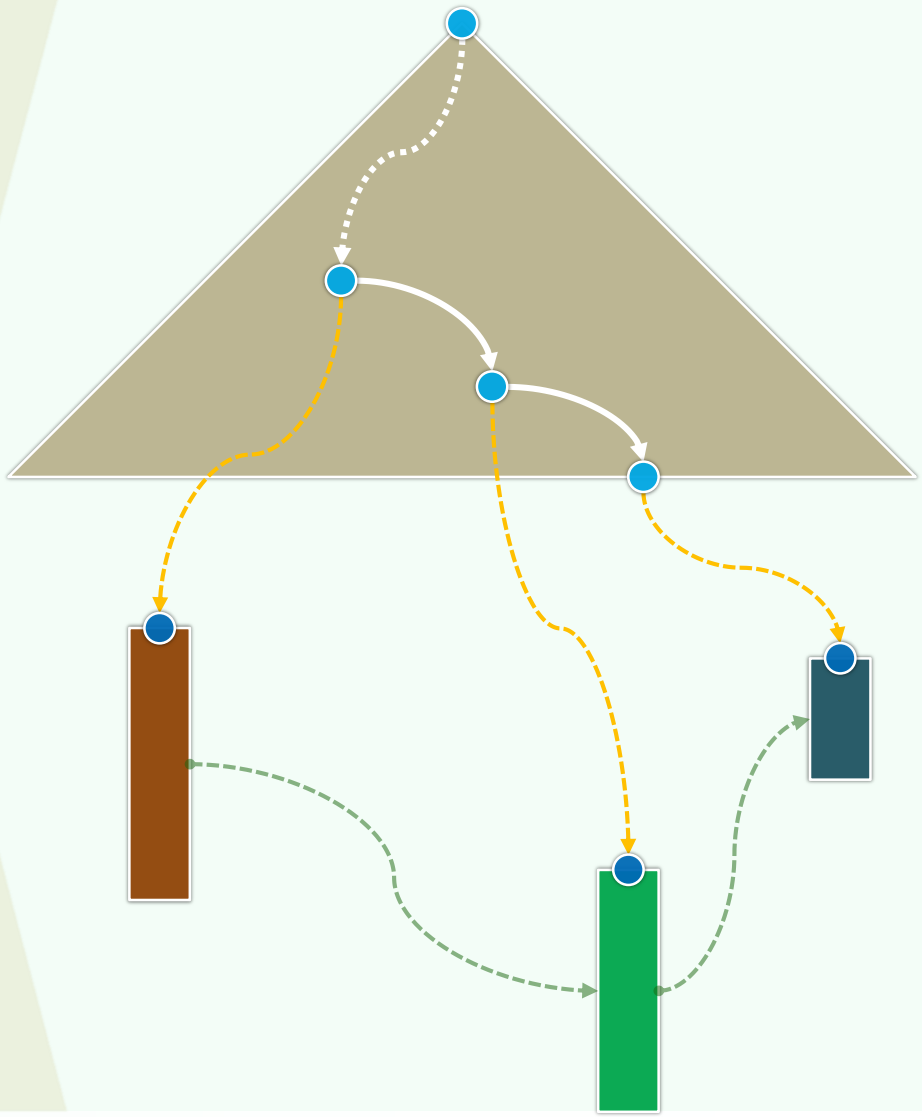
  can **ONLY** be applied to

  the **LAST** level the search structure

- Given a set of n points in $\mathcal{E}^d$,

  an orthogonal range query

  - can be answered in $\mathcal{O}(r + \log^{d-1} n)$ time
  - from a data structure of size $\mathcal{O}(n \cdot \log^{d-1} n)$,
  - which can be constructed in $\mathcal{O}(n \cdot \log^{d-1} n)$ time



07-E

BST Application

Interval Tree

Your instinct, rather than precision stabbing, is more about just random bludgeoning.

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Stabbing Query

- Given a set of intervals in general position

  on the x-axis:

  $$S = \{s_i = [x_i, x_i'] \mid 1 \le i \le n\}$$

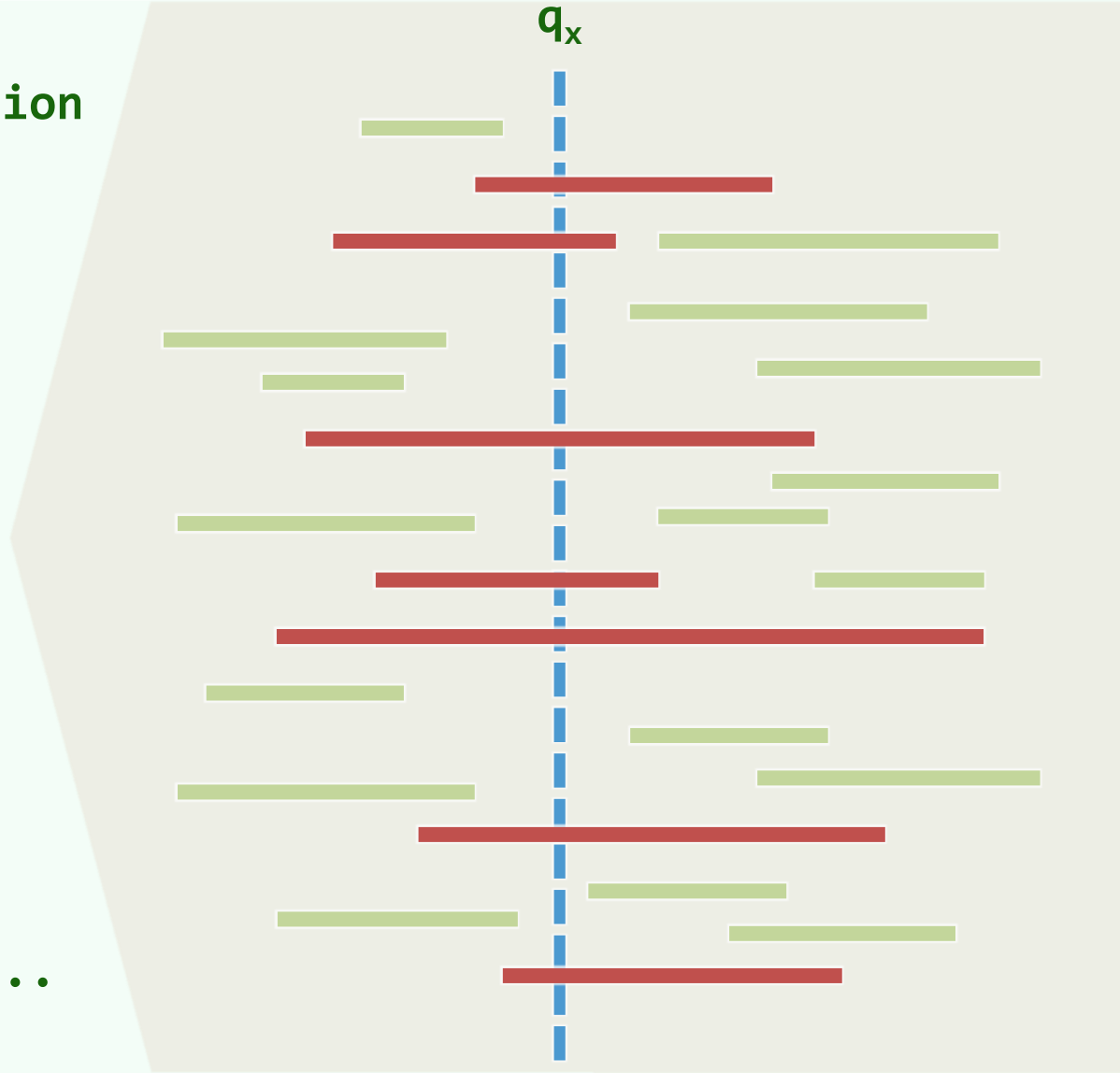
  and a query point q_x

- Find all intervals that contain q_x

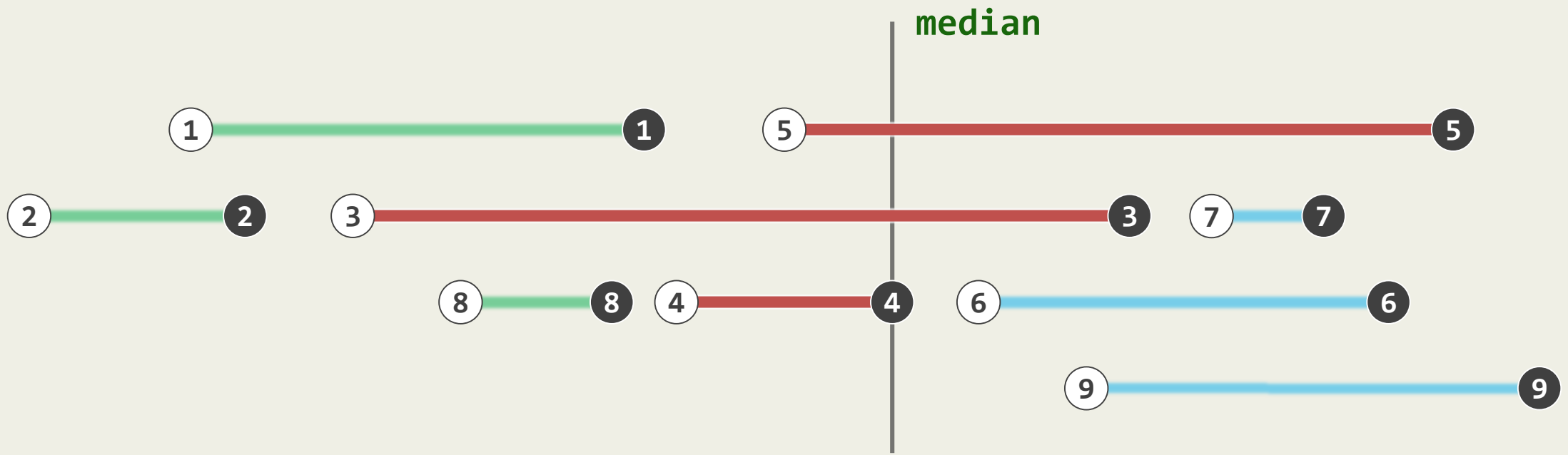
  $$\{s_i = [x_i, x_i'] \mid x_i \le q_x \le x_i'\}$$

- To solve this query,

  we will use the so-called interval tree ...



Median

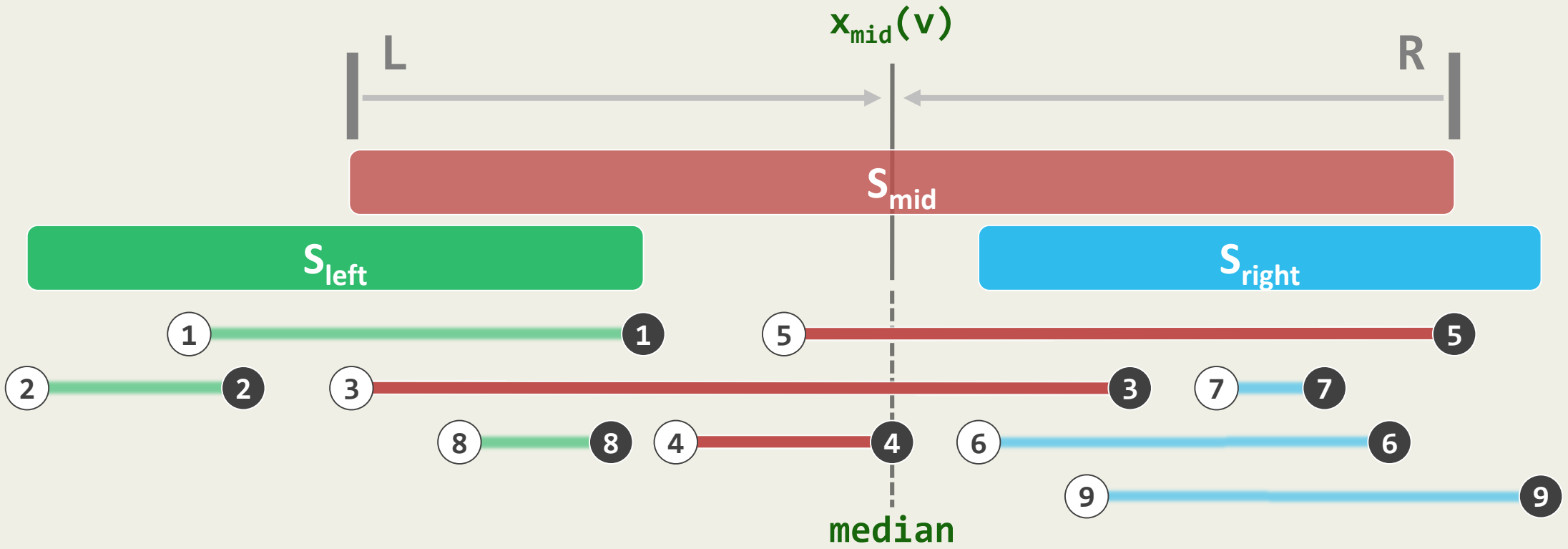


- Let $P = \partial S$ be the set of all endpoints

  (By general position assumption, $|P| = 2n$)

- Let $x_{mid} = median(P)$ be the median of P

Partitioning



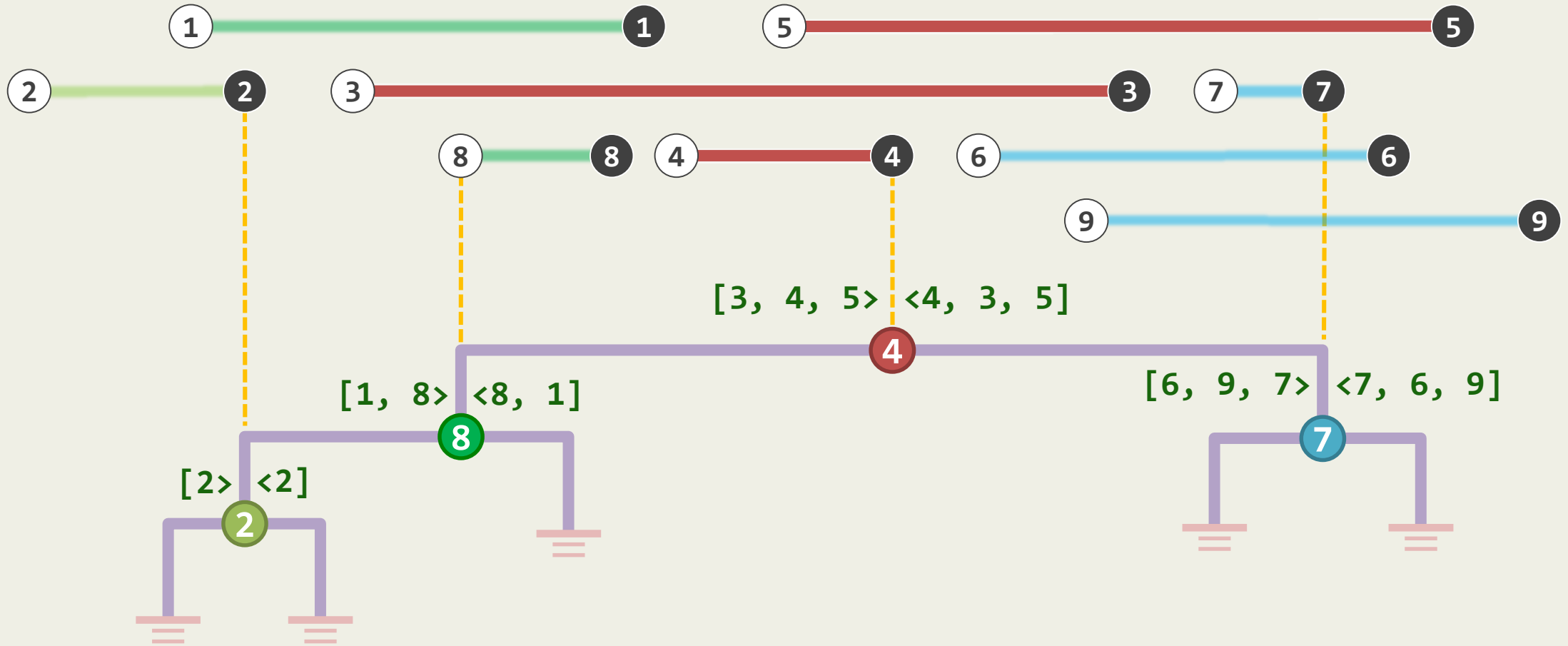
- All intervals can be then categorized into 3 subsets:

$$S_{left} = \{ S_i \mid x_i' < x_{mid} \} \quad S_{mid} = \{ S_i \mid x_i \le x_{mid} \le x_i' \} \quad S_{right} = \{ S_i \mid x_{mid} < x_i \}$$

- $S_{left/right}$ will be recursively partitioned until they are empty (leaves)

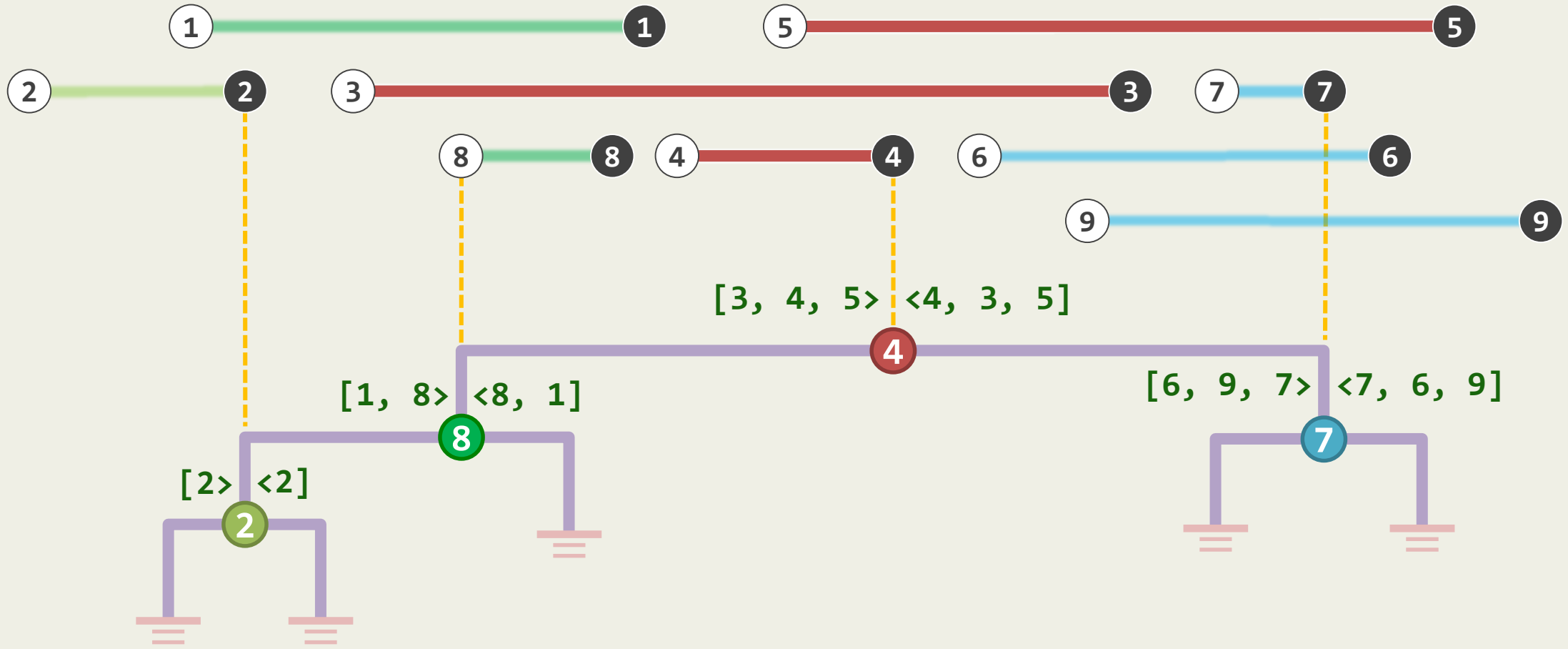
Balance & $\mathcal{O}(\log n)$ Depth

$\max\{\ |S_{left}|, |S_{right}|\ \} \leq n/2$ **Best case:** $|S_{mid}| = n$ **Worst case:** $|S_{mid}| = 1$



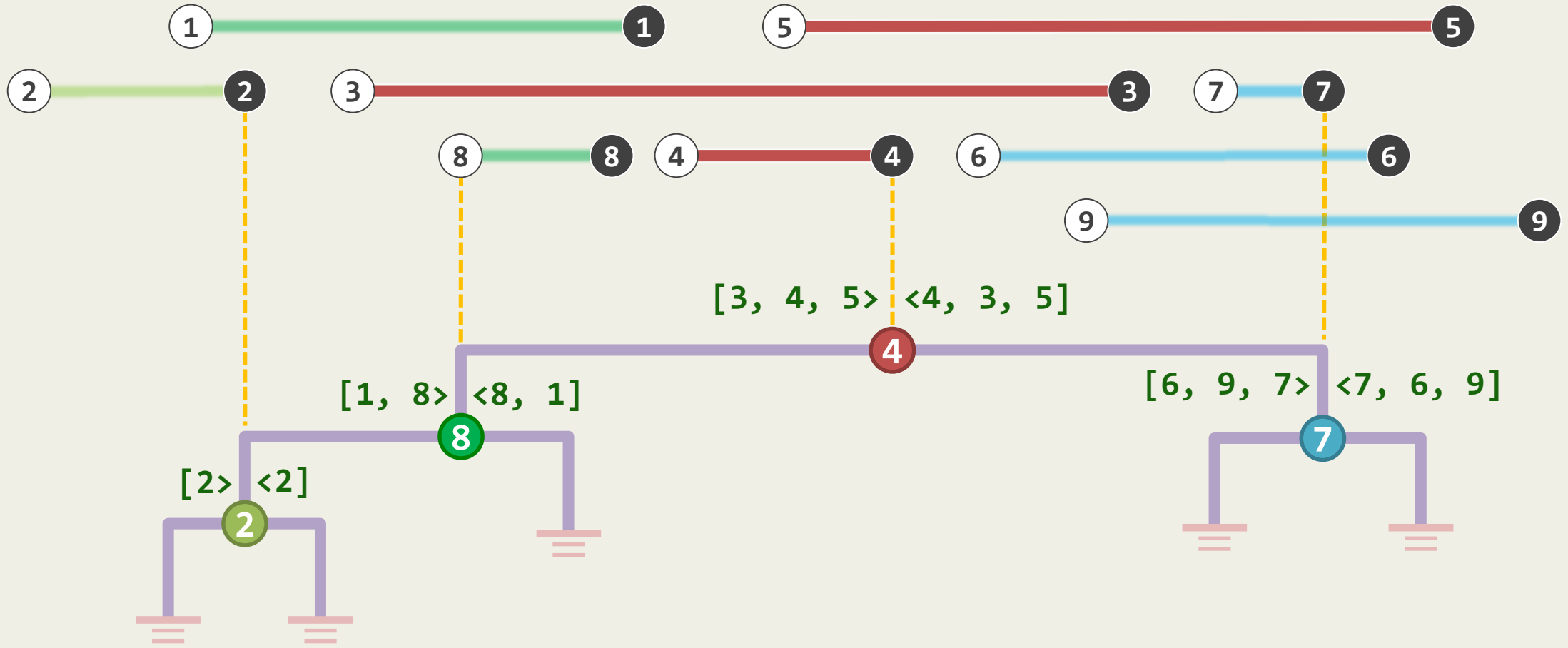
Associative Lists

❖ $L_{left/right}$ = all intervals of S_{mid} sorted by the left/right endpoints



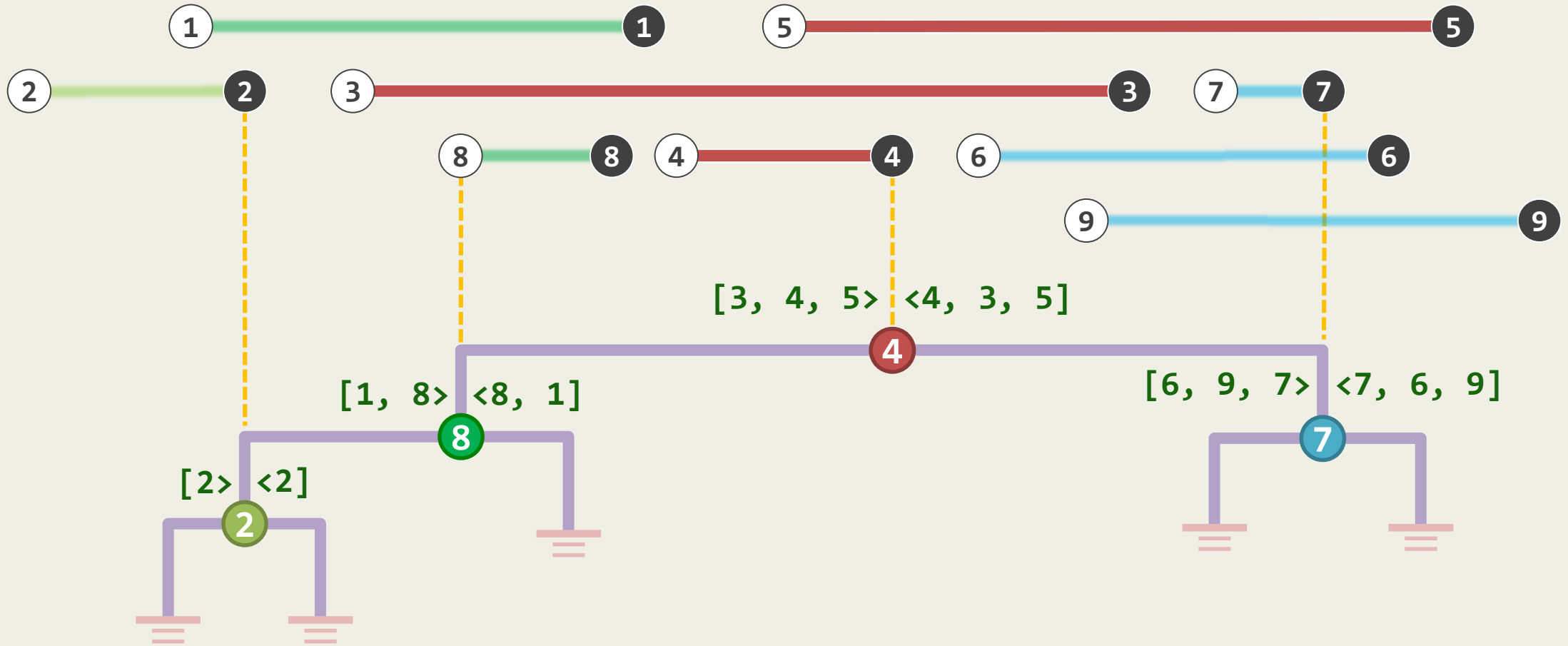
$O(n)$ Size

❖ Each segment appears twice (one in each list)



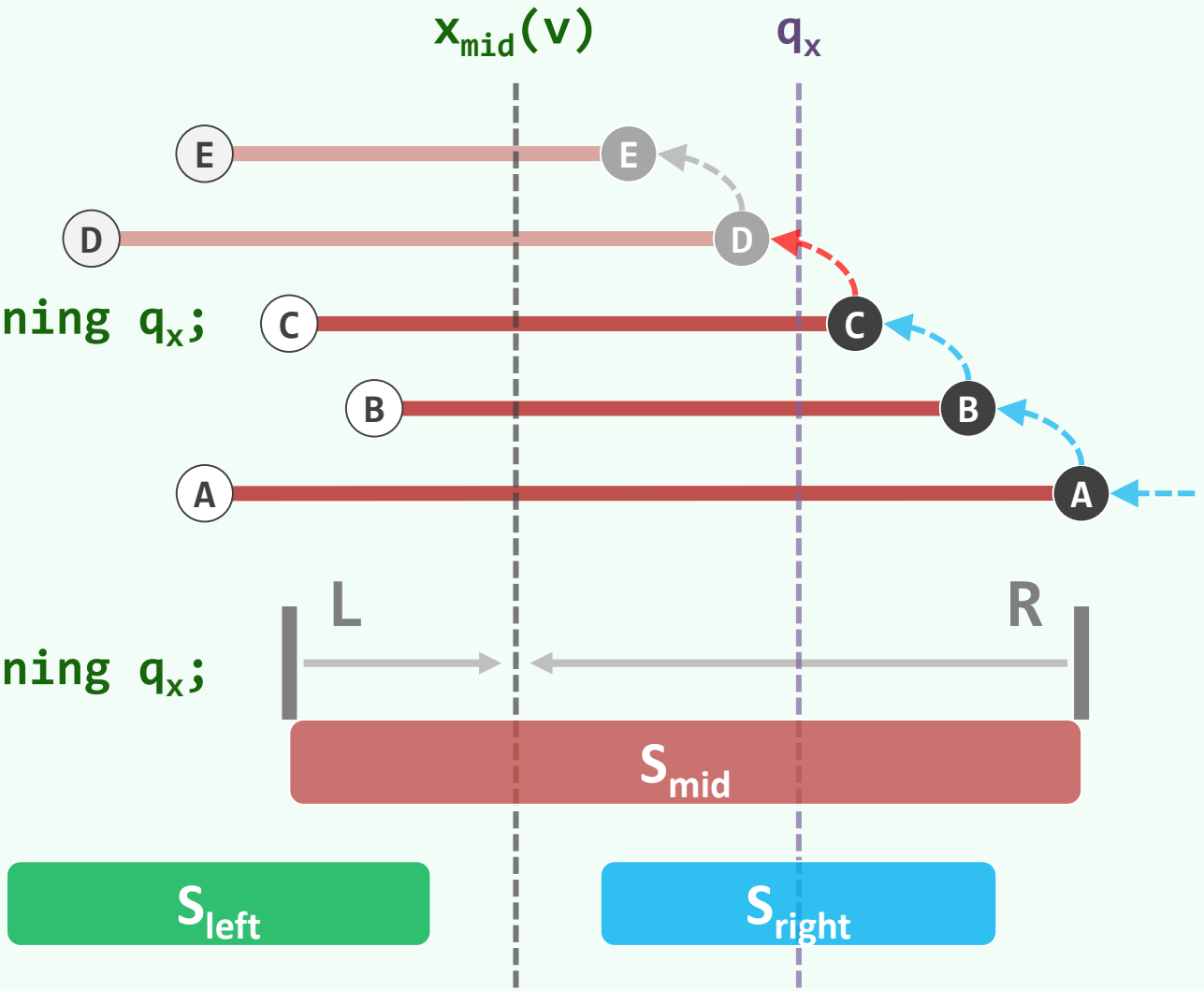
O(nlogn) Construction Time

❖ Hint: avoid repeatedly sorting



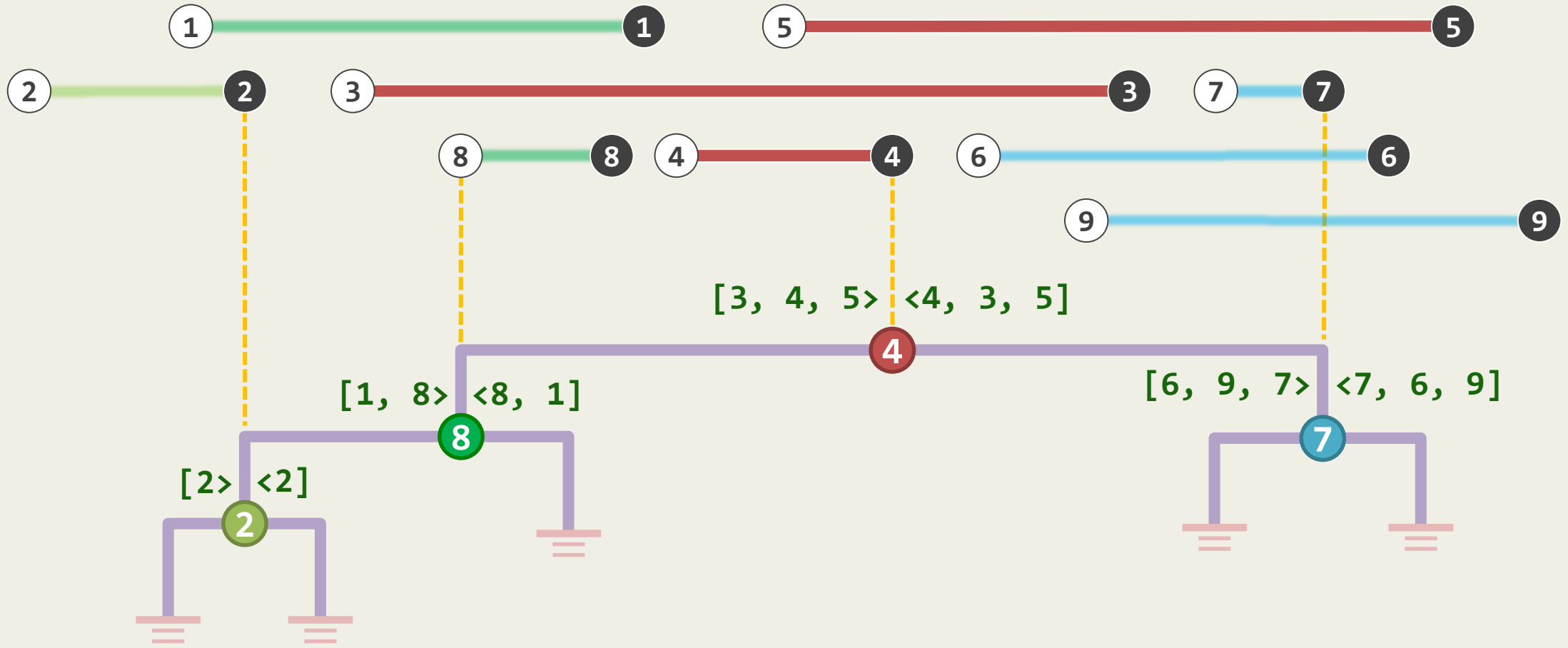
queryIntervalTree(v, q_x)

```
if ( ! v ) return;   //base
if ( q_x < x_mid(v) )
   report all segments of S_mid(v) containing q_x;
   queryIntervalTree( lc(v), q_x );
else if ( x_mid(v) < q_x )
   report all segments of S_mid(v) containing q_x;
   queryIntervalTree( rc(v), q_x );
else //with a probability ≈ 0
   report all segments of S_mid( v ); //both rc(v) & lc(v) can be ignored
```



$O(r + \log n)$ Query Time

❖ Each query visits $O(\log n)$ nodes //LINEAR recursion



07-F

BST Application

Segment Tree

把一条线分割成不相等的两段，再把这两段按照同样的比例再分成两个部分。假设第一次分出来的两段中，一段代表可见世界，另一段代表理智世界。然后再看第二次分成的两段，他们分别代表清楚与不清楚的程度，你便会发现，可见世界那一段的第一部分是它的影像。

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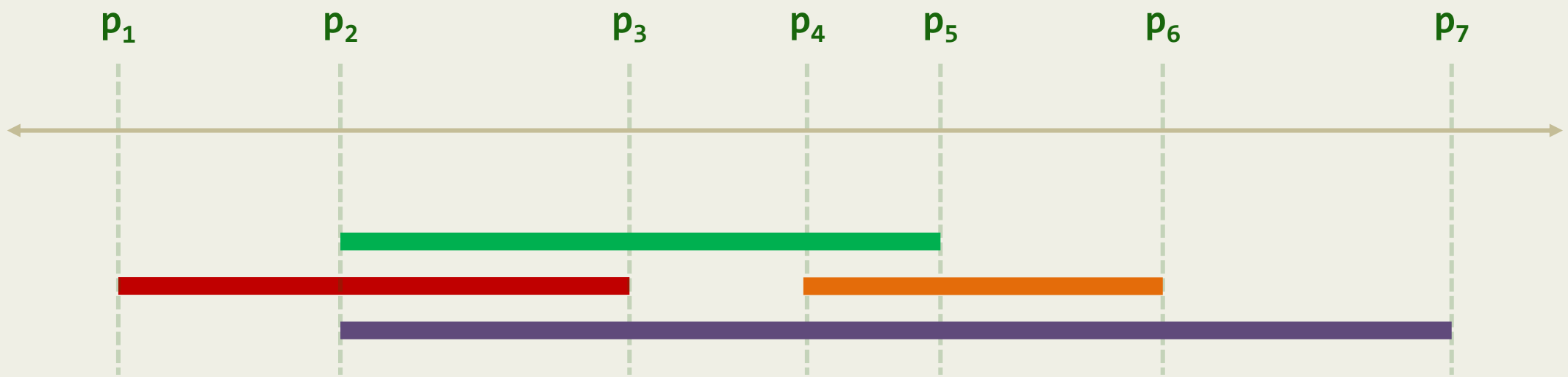
Elementary Intervals

- Let $I = \{ [x_i, x_i'] \mid i = 1, 2, 3, \ldots, n \}$ be n intervals on the x-axis
- Sort all the endpoints into $\{ p_1, p_2, p_3, \ldots, p_m \}$, $m \leq 2n$



- m+1 elementary intervals are hence defined as:

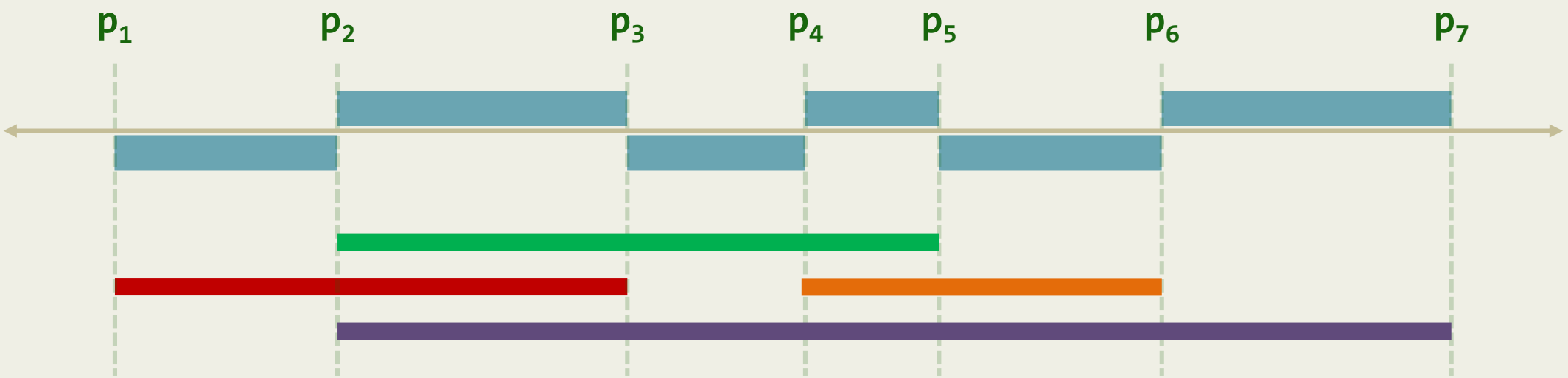
$$(-\infty, p_1],\ (p_1, p_2],\ (p_2, p_3],\ \ldots,\ (p_{m-1}, p_m],\ (p_m, +\infty]$$

Discretization

- Within each EI, all stabbing queries share a same output

∴ If we sort all EI's into a vector and

store the corresponding output with each EI, then ...

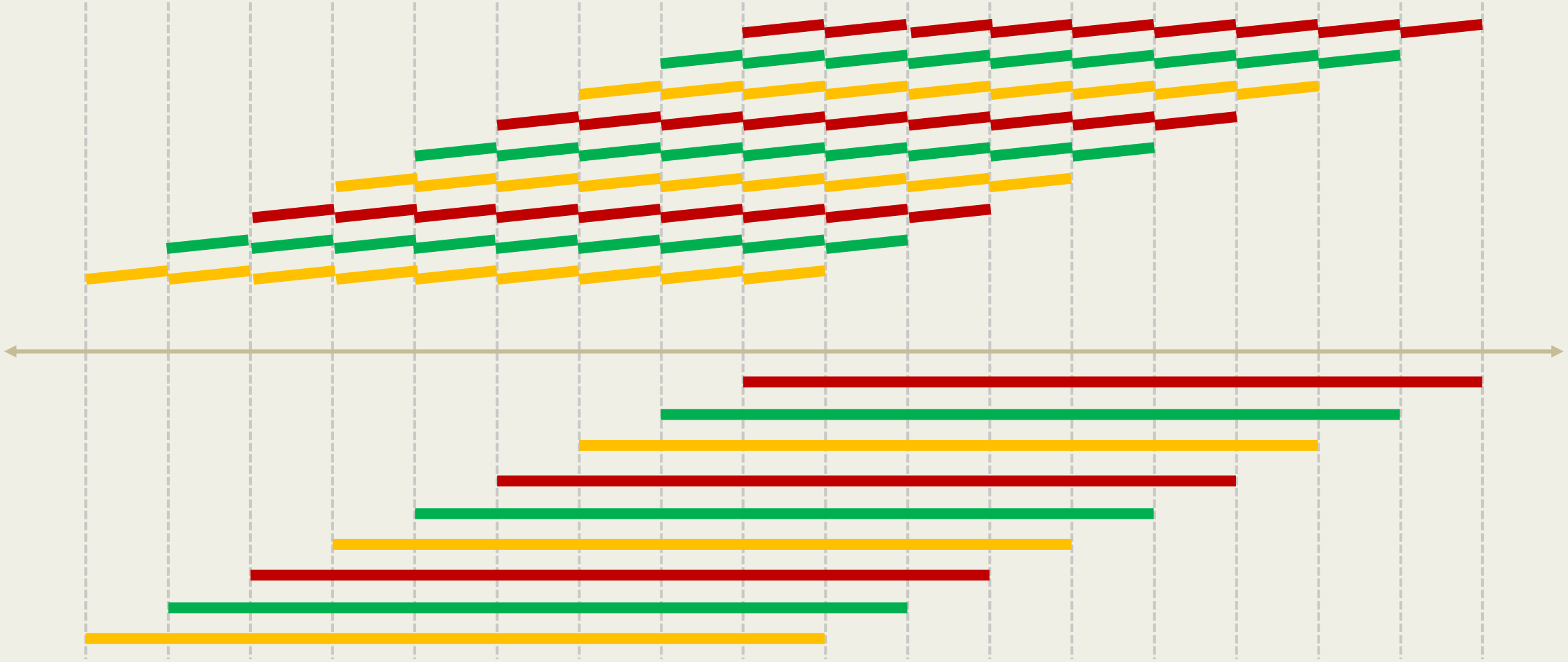


∴ Once a query position is determined, //by an $O(\log n)$ time binary search

the output can then be returned directly //$O(r)$

Worst Case

- Every interval spans $\Omega(n)$ EI's and a total space of $\Omega(n^2)$ is required

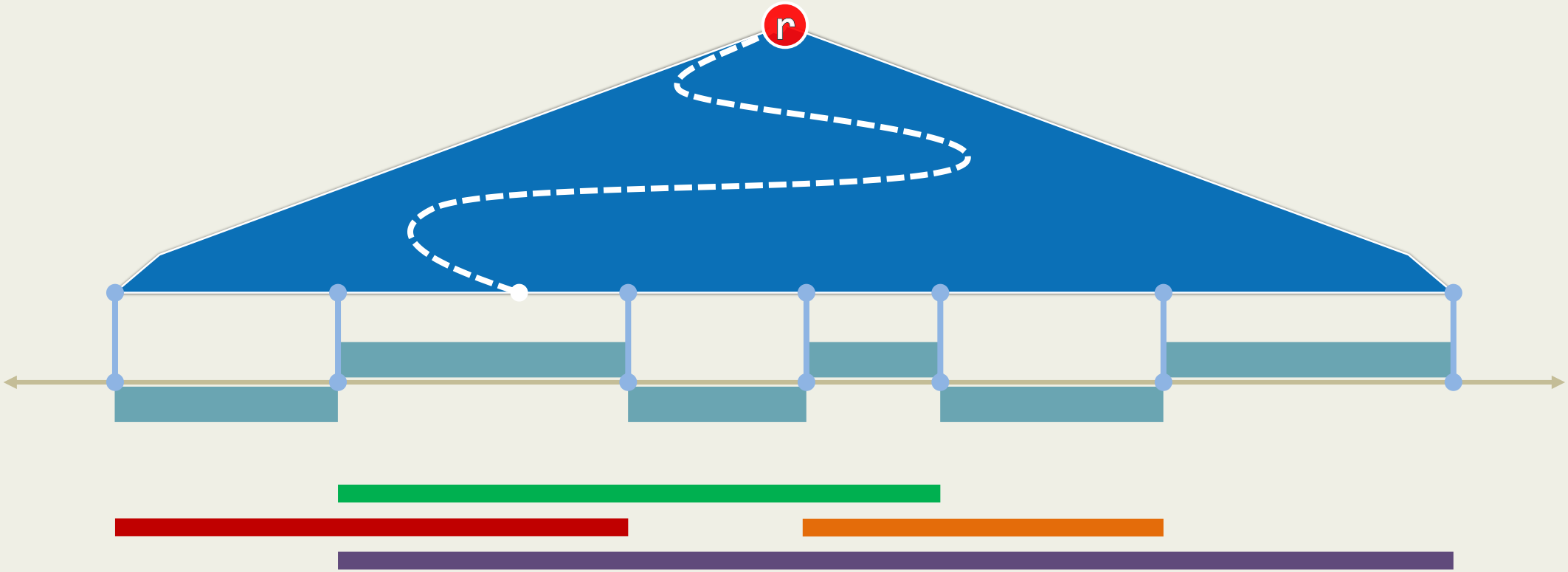


Sorted Vector --> BBST

- For each leaf v,

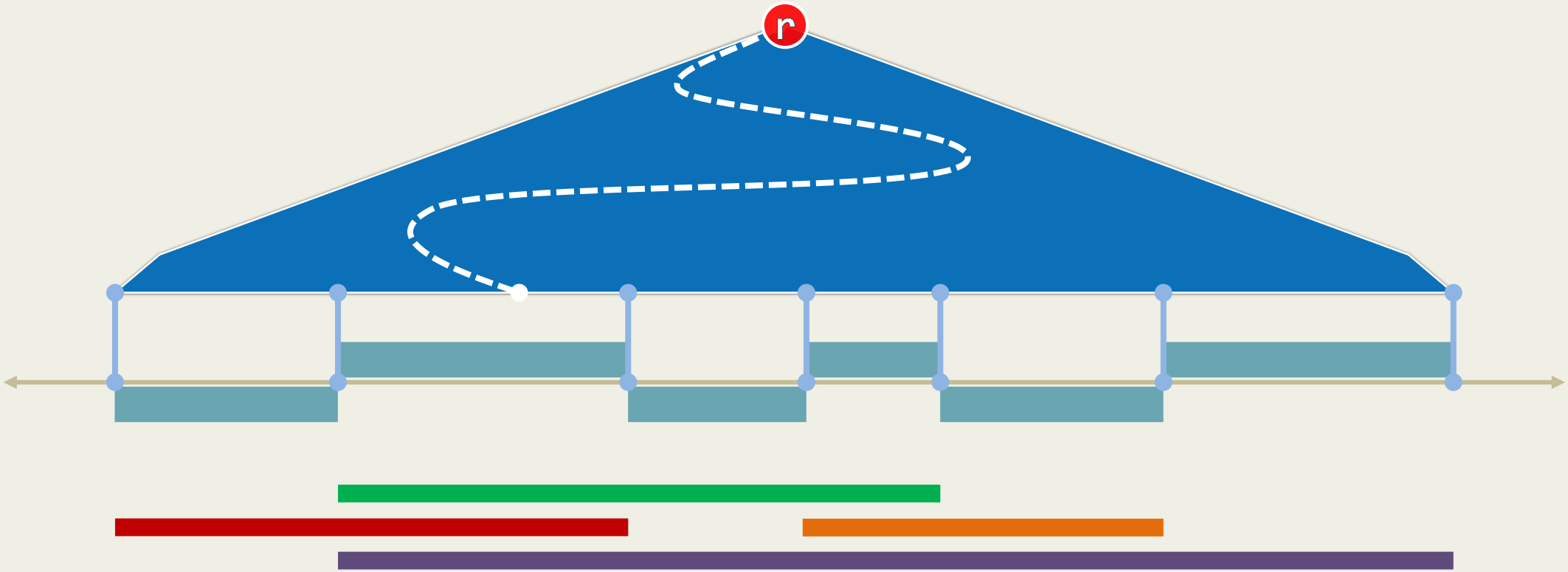
  denote the corresponding elementary interval as R(v), //range of domination

  denote the subset of intervals spanning R(v) as Int(v) and store Int(v) at v

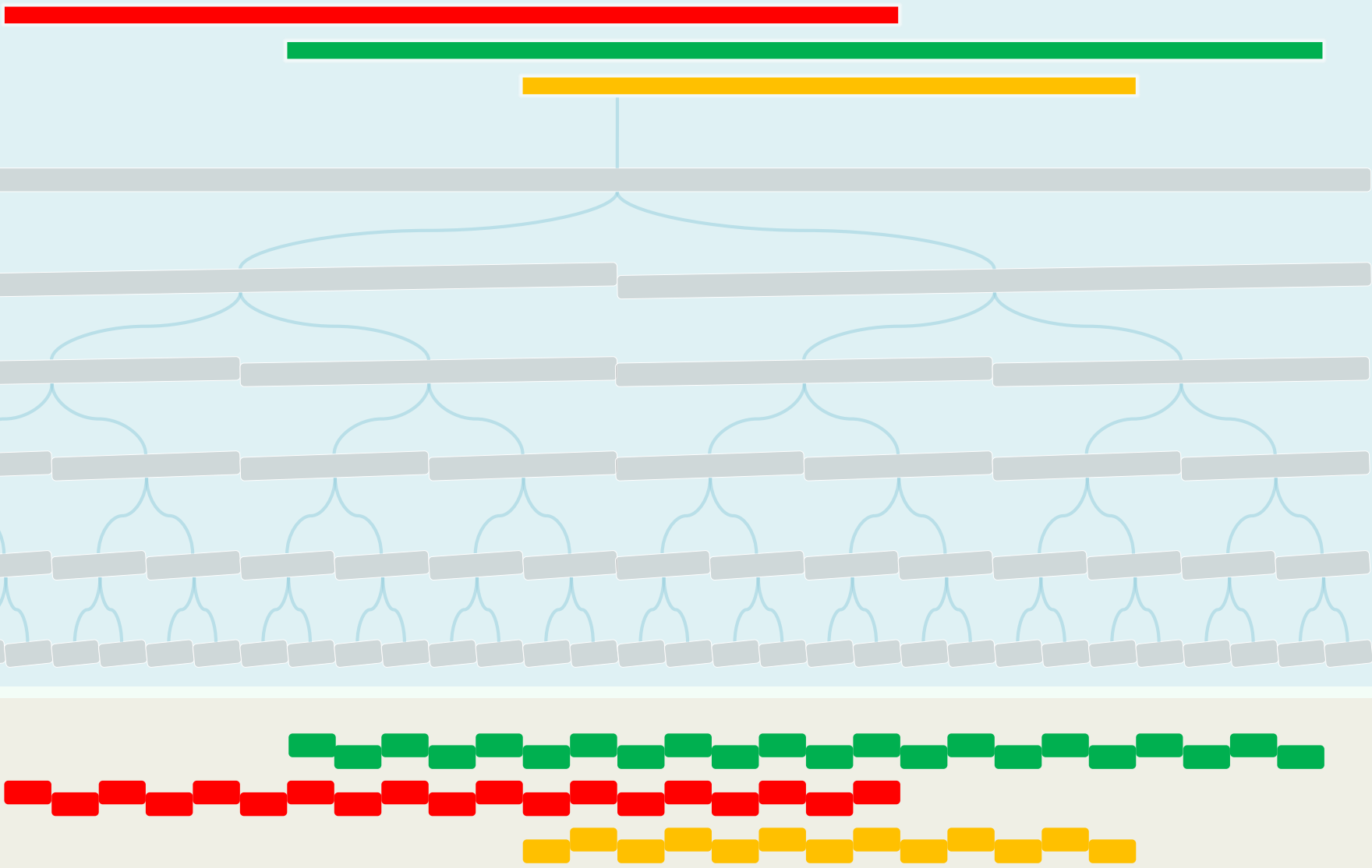


1D Stabbing Query with BBST

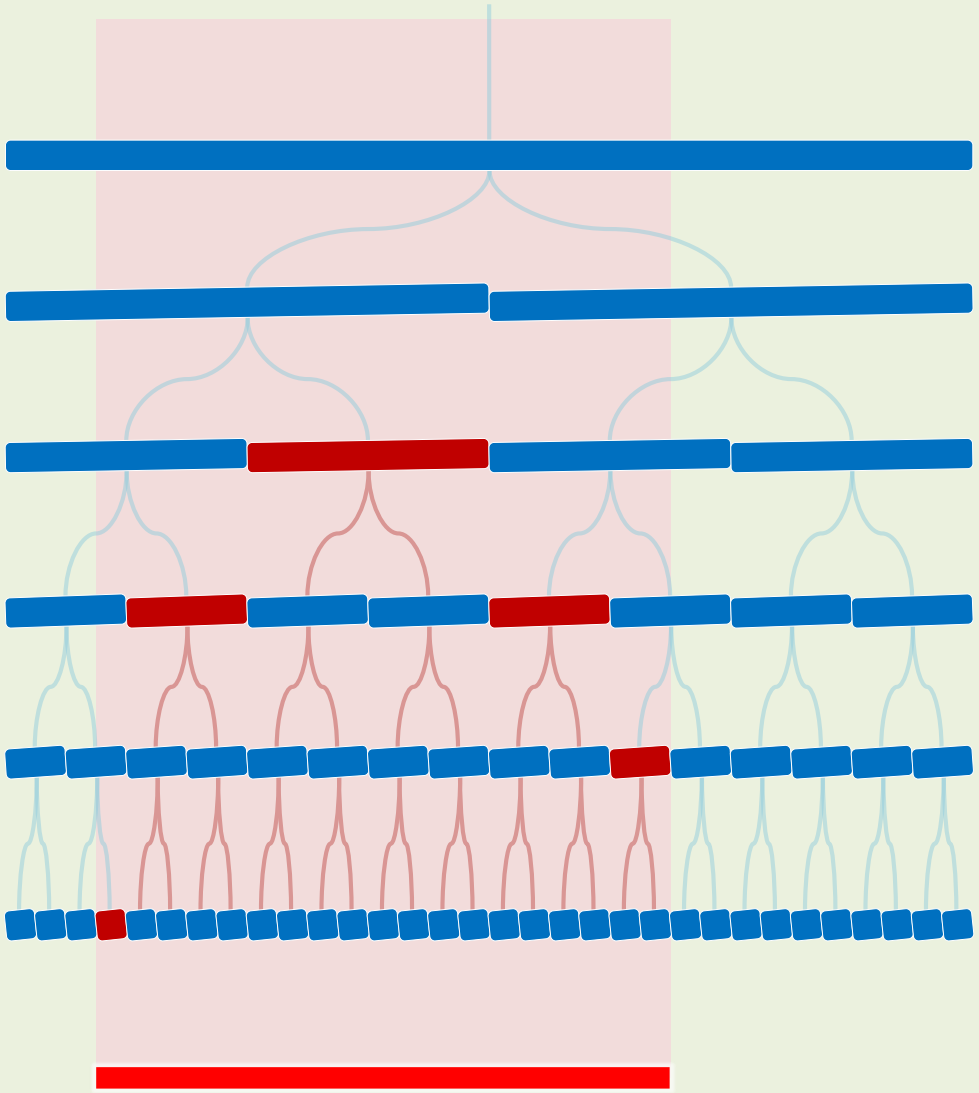
- To find all intervals containing q_x, we can
  - find the leaf v whose R(v) contains q_x //$O(\log n)$ time for a BBST
  - and then report Int(v) //$O(1 + r)$ time

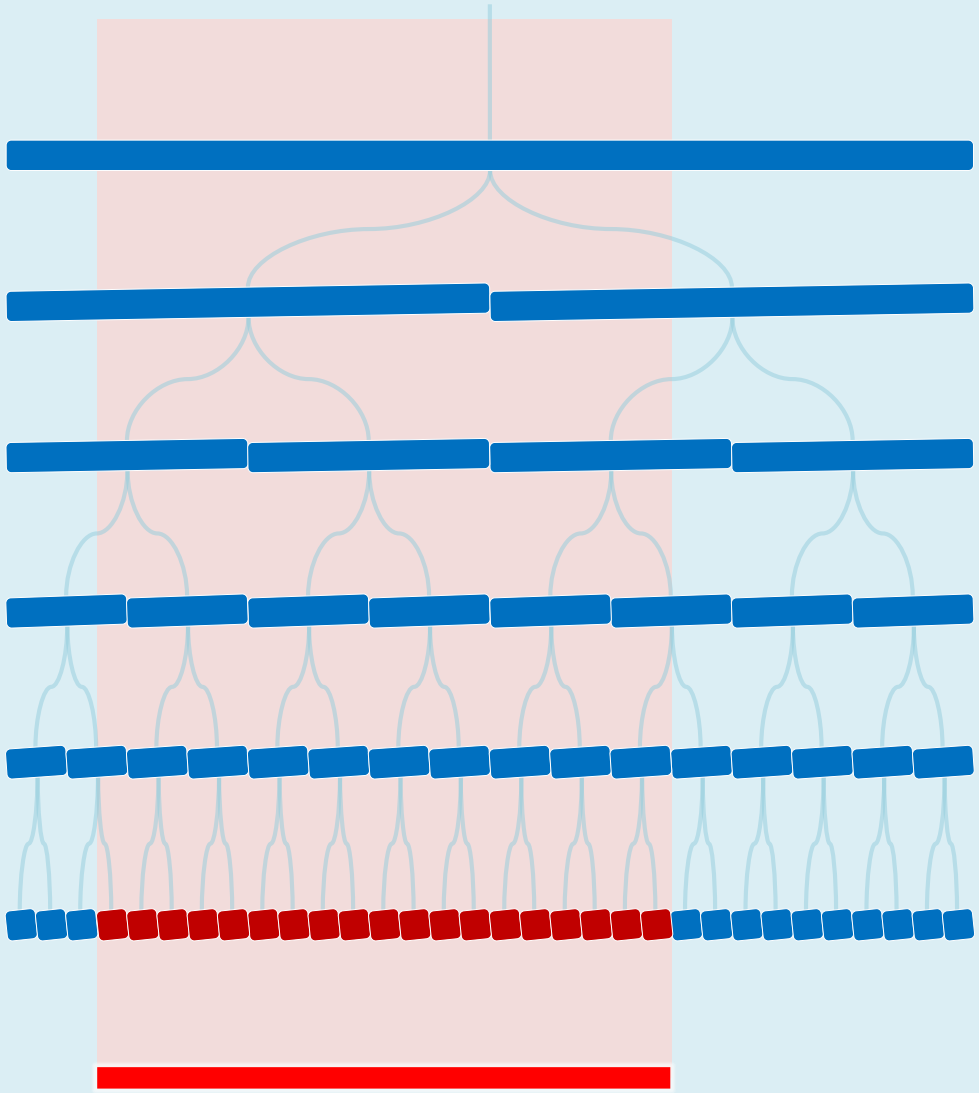


$\Omega(n^2)$ Total Space In The Worst Cases



Store each interval at $\mathcal{O}(\log n)$ common ancestors by greedy merging

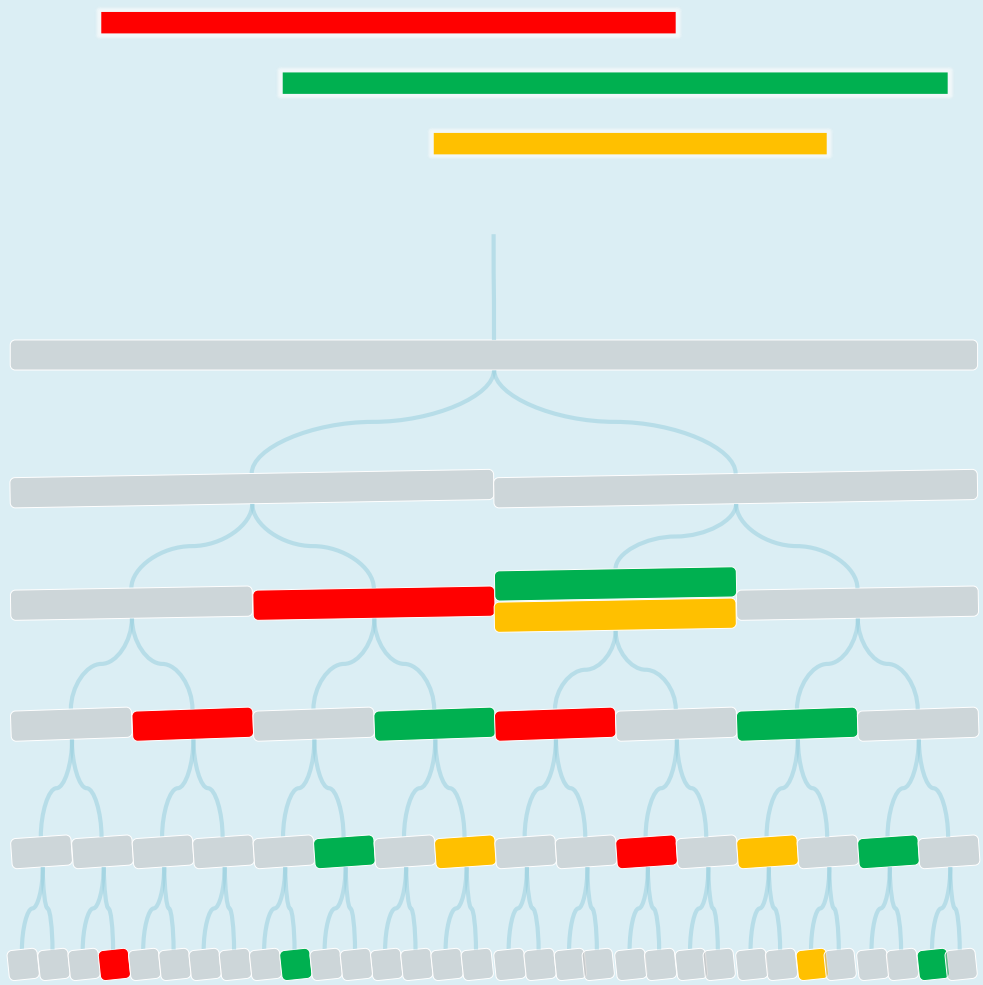




Canonical Subsets with $O(n \log n)$ Space

BuildSegmentTree(I)

```
Sort all endpoints in I before
    determining all the EI's //O(nlogn)
Create T a BBST on all the EI's //O(n)
Determine R(v) for each node v
    //O(n) if done in a bottom-up manner
For each s of I
    InsertSegment( T.root, s )
```



InsertSegment(v , s)

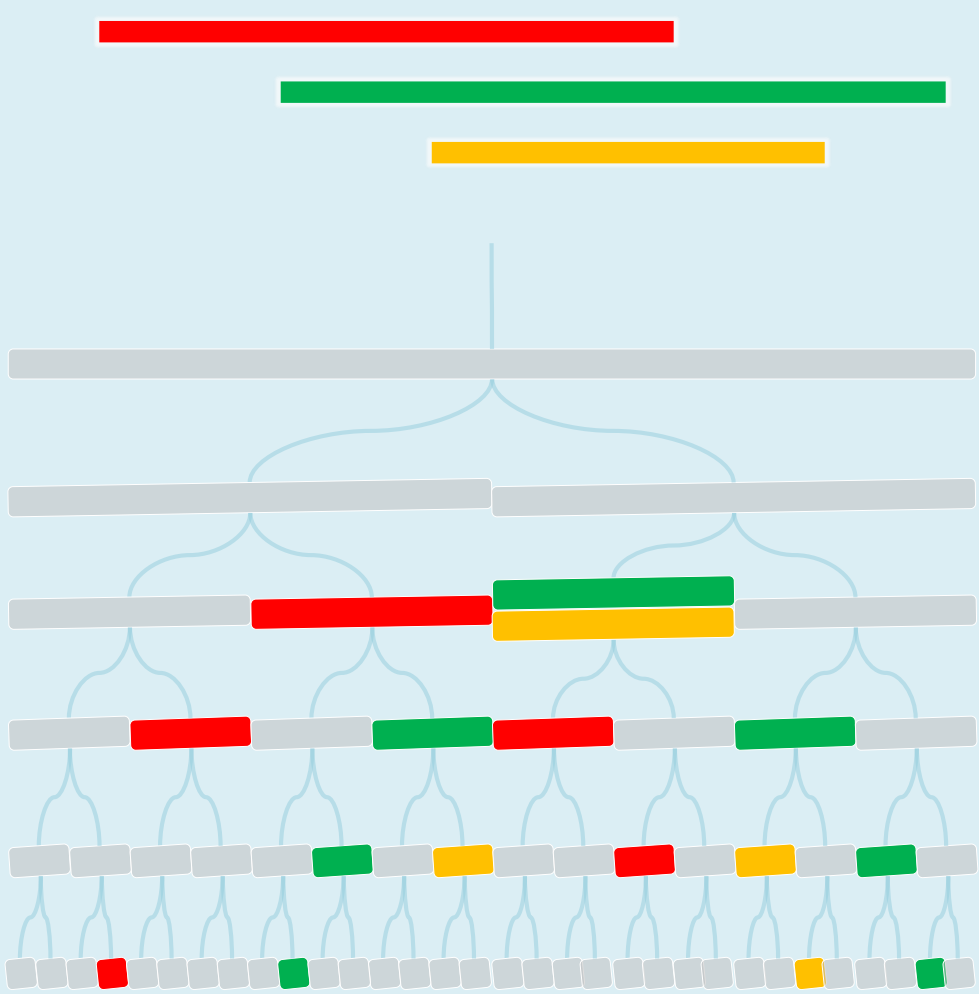
```
if ( R(v) ⊆ s ) //greedy by top-down
   store s at v and return;
if ( R( lc(v) ) ∩ s ≠ ∅ ) //recurse
   InsertSegment( lc(v), s );
if ( R( rc(v) ) ∩ s ≠ ∅ ) //recurse
   InsertSegment( rc(v), s );
```

👁 At each level,

≤ 4 nodes are visited

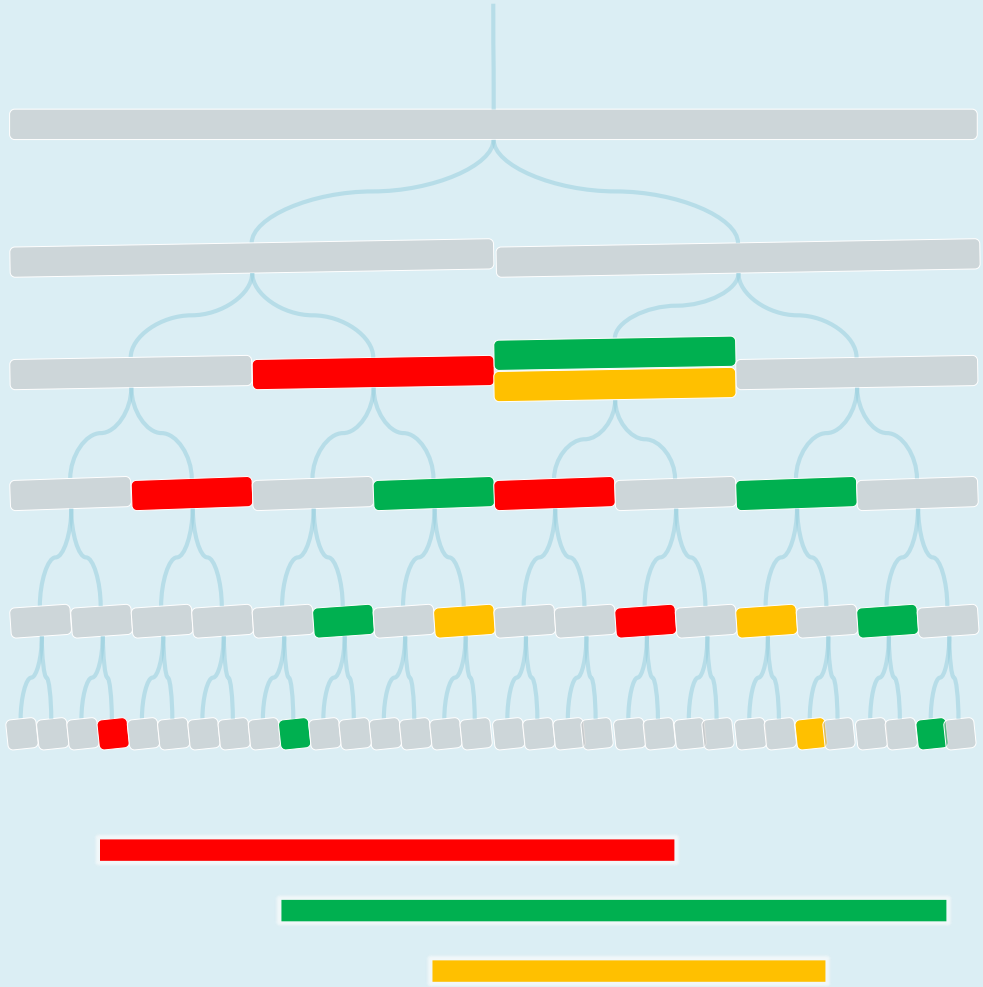
(2 stores + 2 recursions)

∴ $O(\log n)$ time



Query(v , q_x)

```
report all the intervals in Int(v)

if ( v is a leaf )

   return

if ( q_x ∈ R( lc(v) ) )

   Query( lc(v), q_x )

else //q_x ∈ R( rc(v) )

   Query( rc(v), q_x )
```



$O(r + \log n)$

- Only one node is visited per level,

  altogether $O(\log n)$ nodes

- At each node v

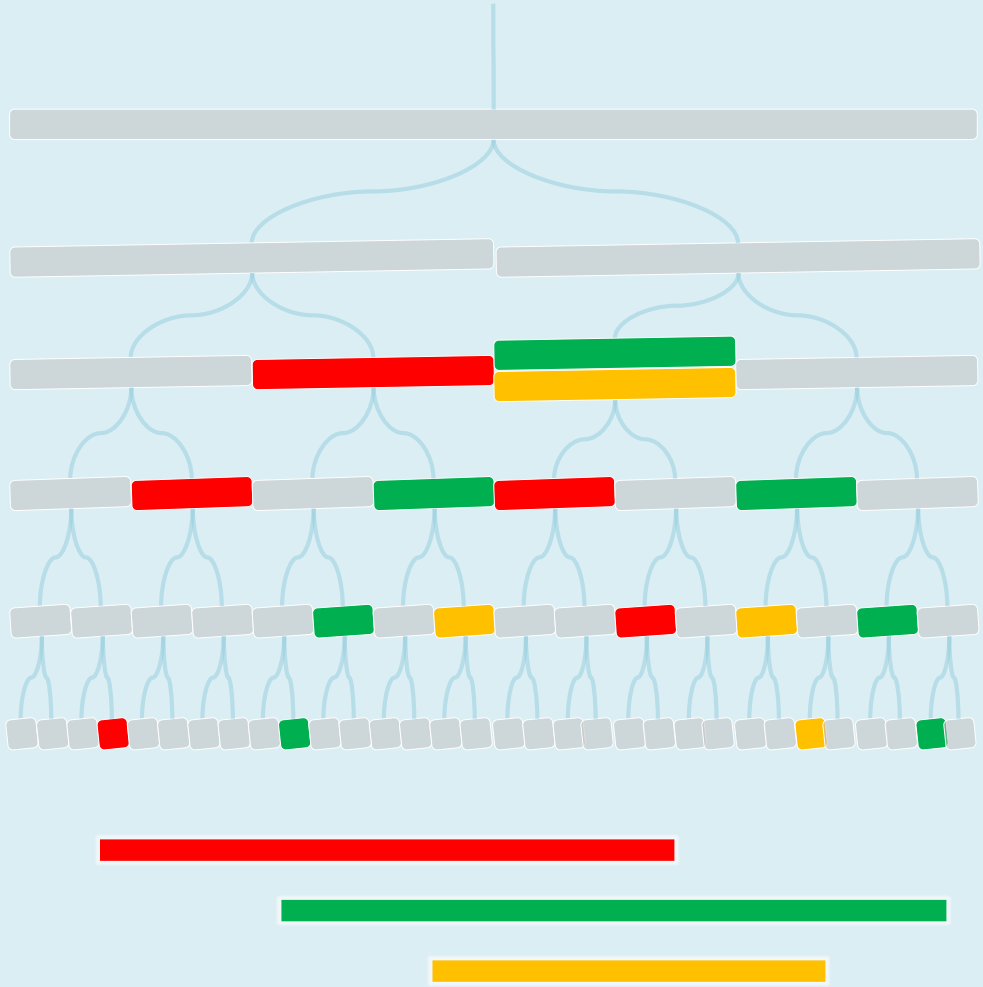
  - the CS Int(v) is reported

  - in time

    $$1 + |\text{Int}(v)| = O(1 + r_v)$$

∴ Reporting all the intervals

  costs $O(r)$ time



Conclusion

❖ For a set of n intervals,

- a segment tree of size $O(n \log n)$

- can be built in $O(n \log n)$ time

- which reports all intervals

containing a query point

in $O(r + \log n)$ time