04-A

栈与队列

栈接口与实现

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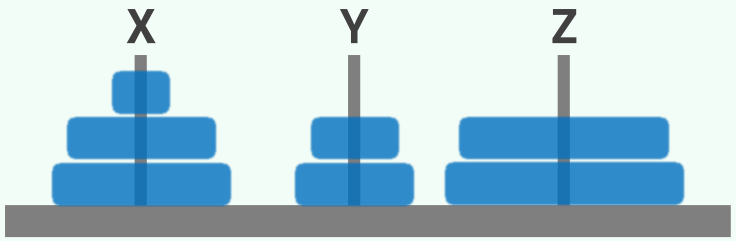
陛下用群臣，如积薪耳，后来者居上

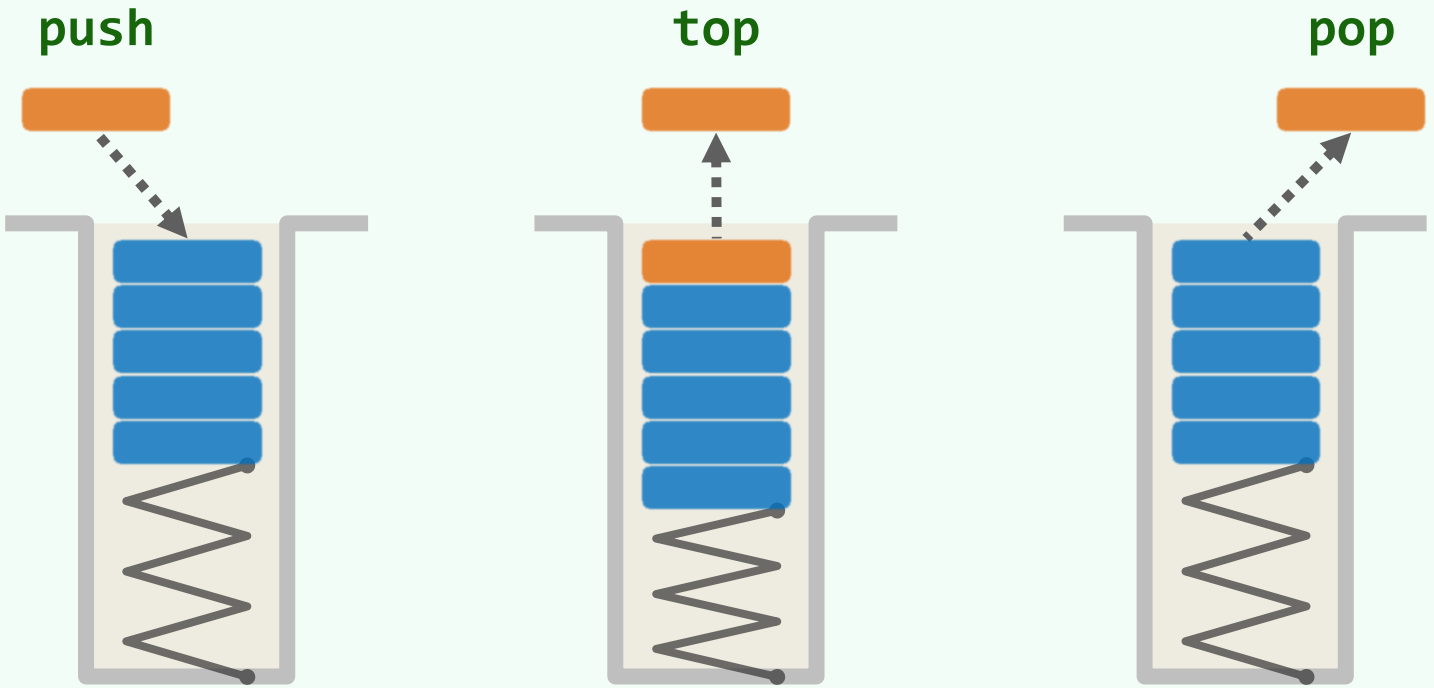
操作与接口

- 栈（stack）是受限的序列
  - 只能在栈顶（top）插入和删除
  - 栈底（bottom）为盲端



- 基本接口
  - size() / empty()
  - push() 入栈
  - pop() 出栈
  - top() 查顶
- 后进先出（LIFO）

  先进后出（FILO）
- 扩展接口：getMax()...



实例

操作	输出	栈（左侧栈顶）
Stack()		
empty()	true	
push(5)		5
push(3)		3 5
pop()	3	5
push(7)		7 5
push(3)		3 7 5
top()	3	3 7 5
empty()	false	3 7 5

操作	输出	栈（左侧栈顶）
push(11)		11 3 7 5
size()	4	11 3 7 5
push(6)		6 11 3 7 5
empty()	false	6 11 3 7 5
push(7)		7 6 11 3 7 5
pop()	7	6 11 3 7 5
pop()	6	11 3 7 5
top()	11	11 3 7 5
size()	4	11 3 7 5

实现：既然属于序列的特例，故可直接基于向量或列表派生

```
template <typename T> class Stack: public Vector<T> {
public: //原有接口一概沿用
  void push( T const & e ) { insert( e ); } //入栈
  T pop() { return remove( size() - 1 ); } //出栈
  T & top() { return (*this)[ size() - 1 ]; } //取顶
}; //以向量首/末端为栈底/顶——颠倒过来呢?
```

- 确认：如此实现的栈各接口，均只需$O(1)$时间
- 课后：基于列表，派生定义栈模板类；你所实现的栈接口，效率如何?

04-B1

栈与队列

调用栈：原理与算法

命运把我们的大脑当作一个容器，不停地把各种见解装进去、取出来，但总是现在的和最后的那个见解是可靠没错的

Yessiree. We do not doubt his word, an stack ourselfs into the bus like flapjacks.

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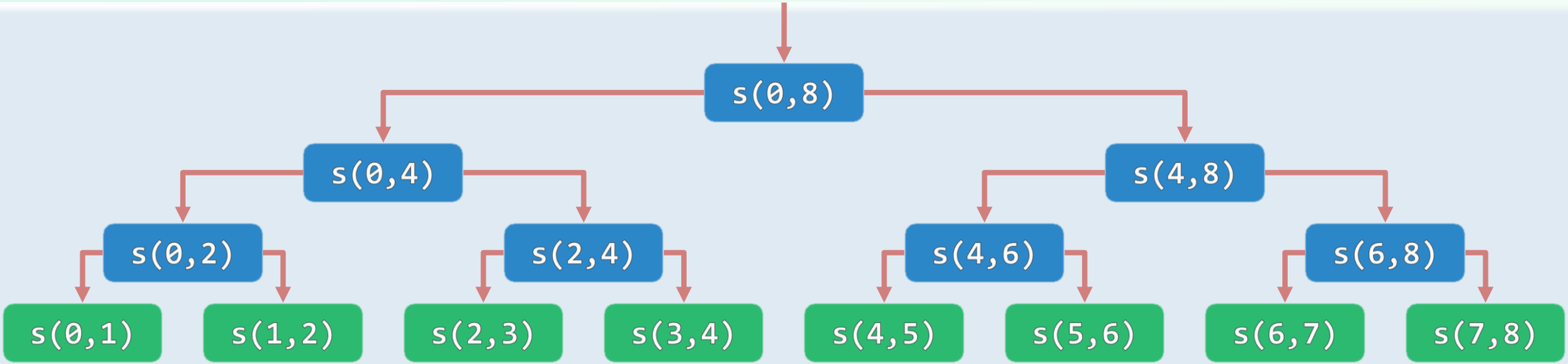
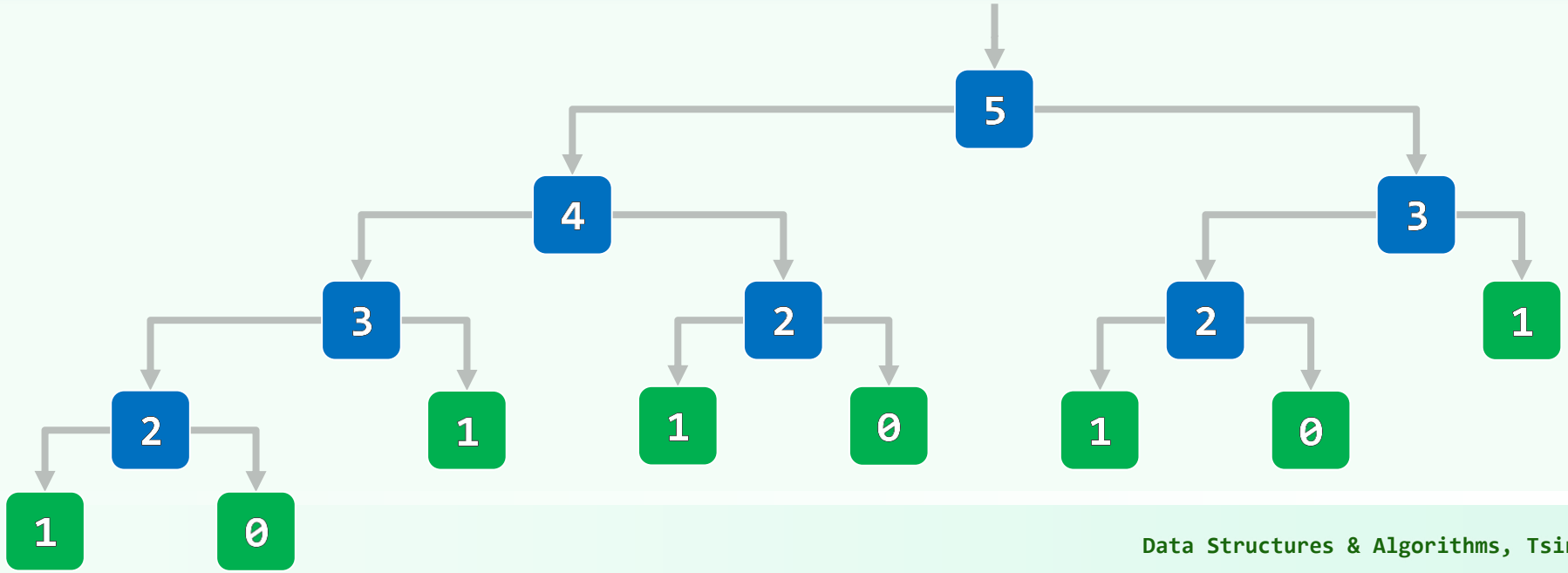
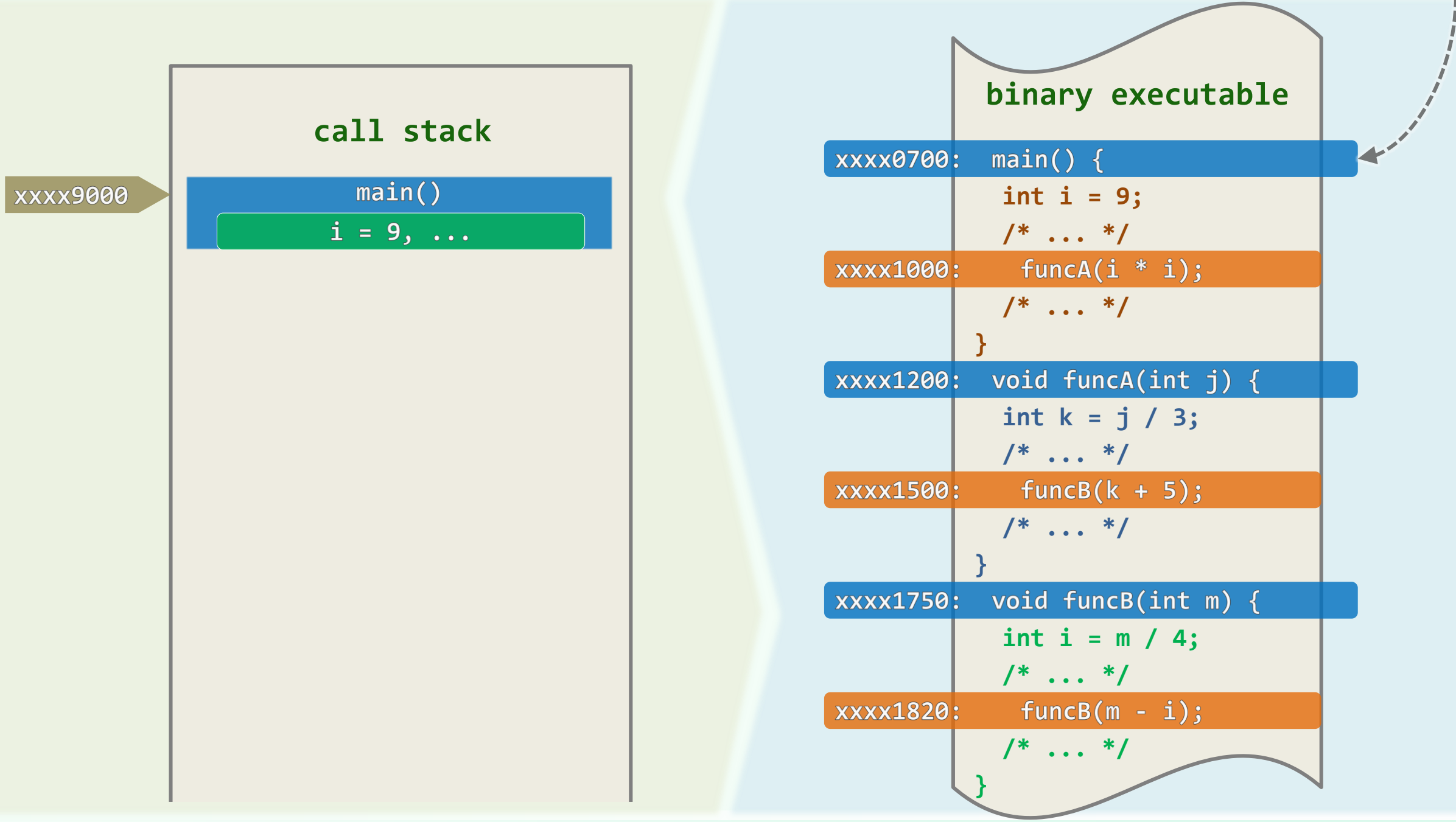
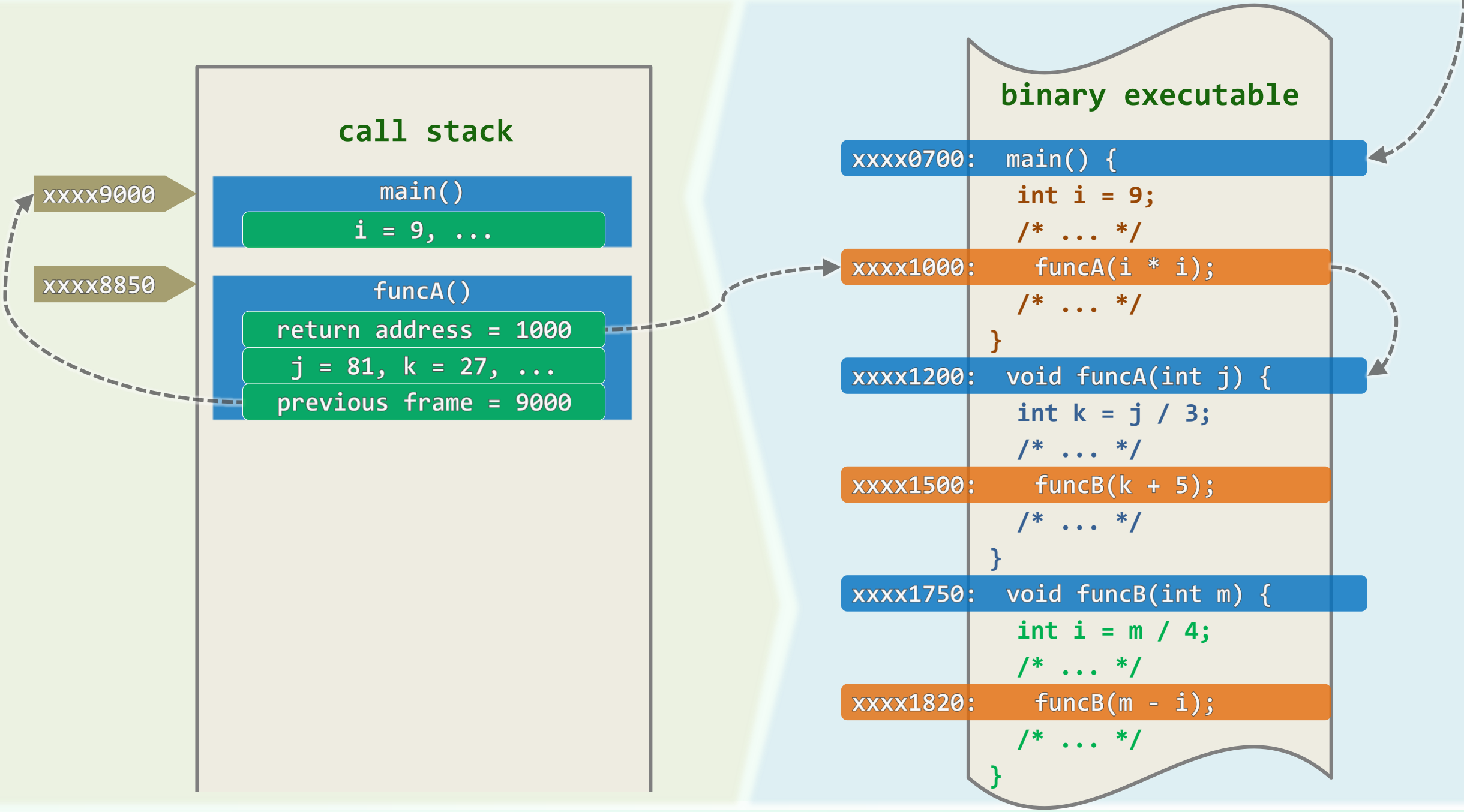
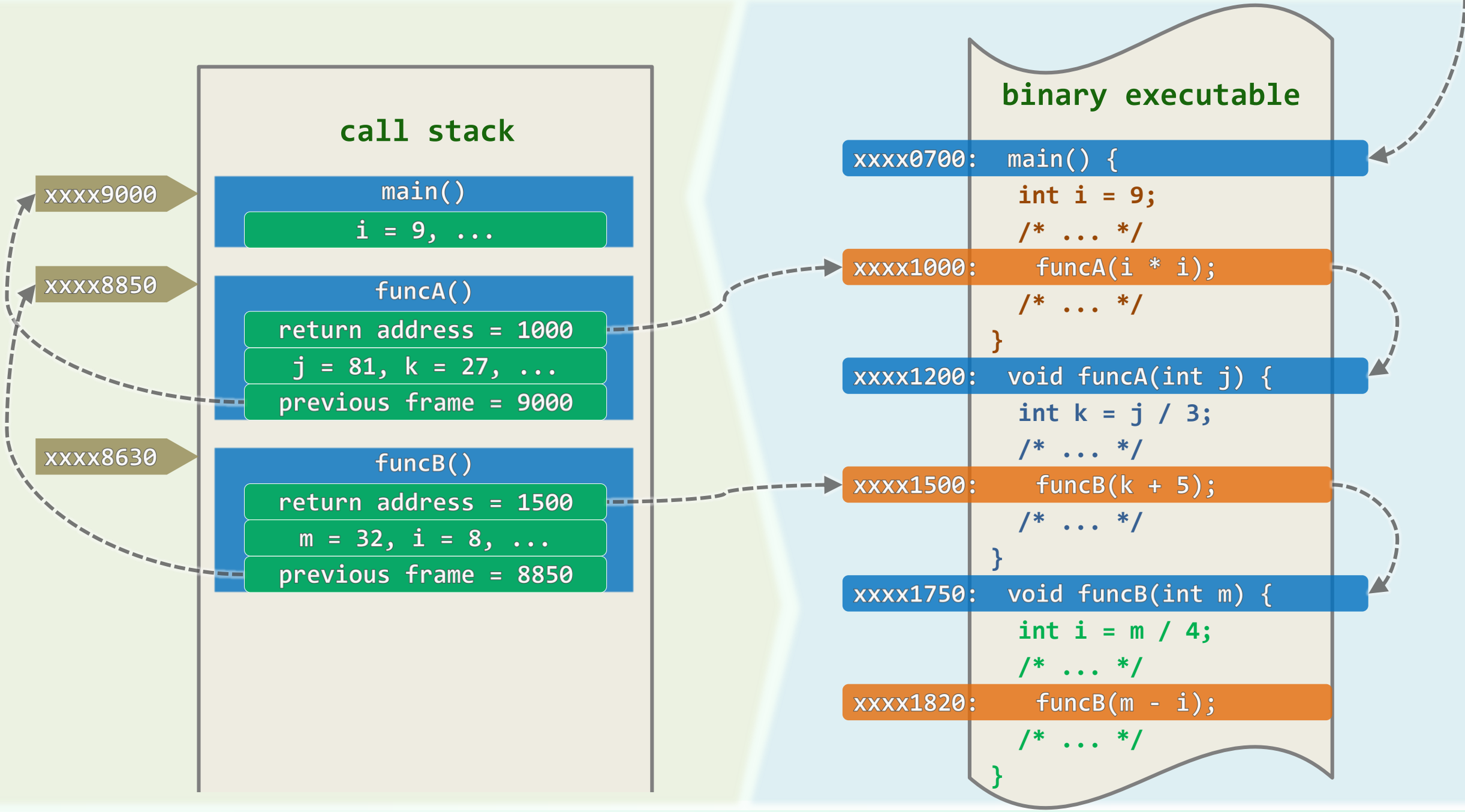
函数调用树：如何实现？Theseus的线团 + 粉笔

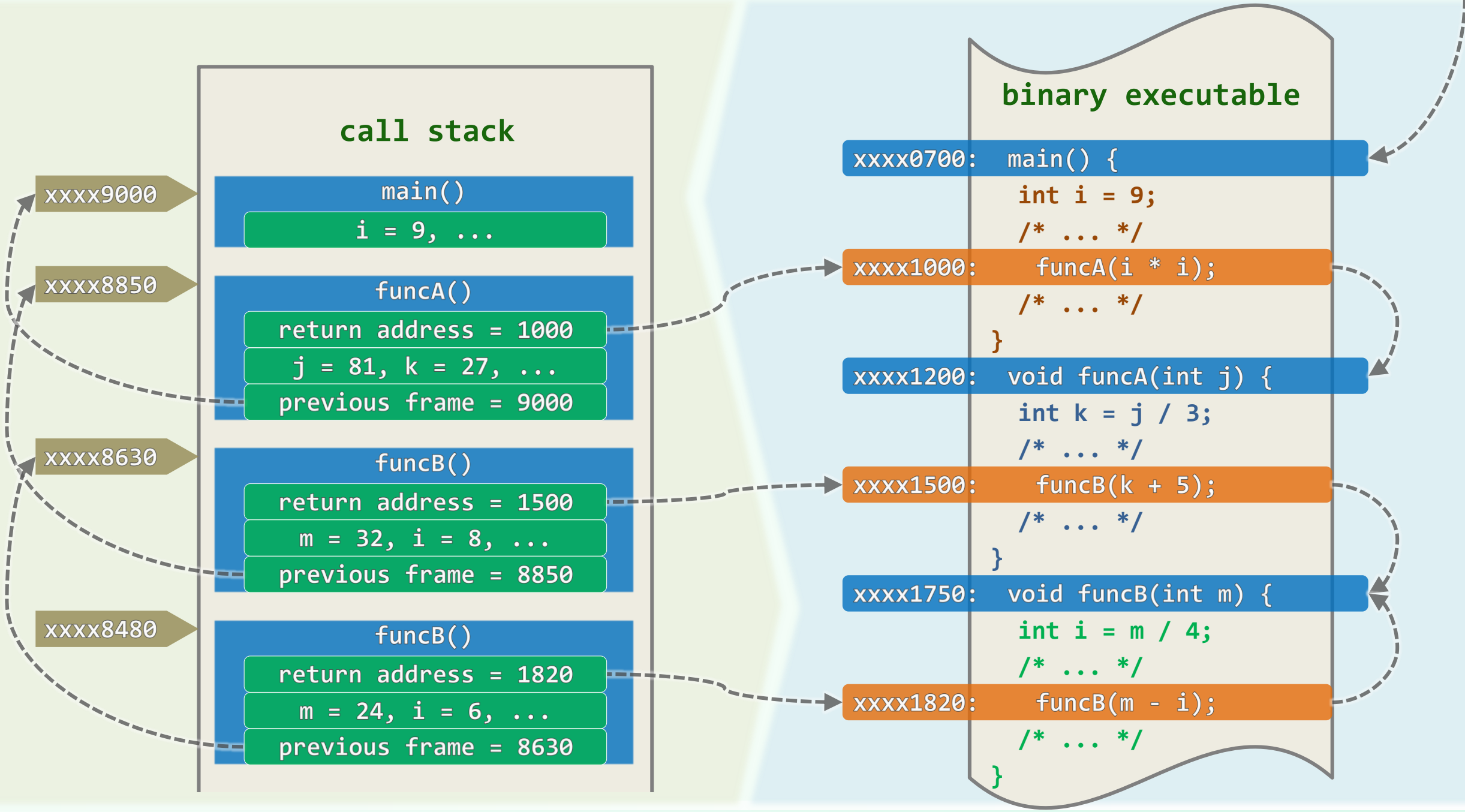





call stack
xxxx9000
main()
i = 9, ...
binary executable
xxxx0700: main() {
int i = 9;
/* ... */
xxxx1000: funcA(i * i);
/* ... */
}
xxxx1200: void funcA(int j) {
int k = j / 3;
/* ... */
xxxx1500: funcB(k + 5);
/* ... */
}
xxxx1750: void funcB(int m) {
int i = m / 4;
/* ... */
xxxx1820: funcB(m - i);
/* ... */
}


call stack
xxxx9000
main()
i = 9, ...
xxxx8850
funcA()
return address = 1000
j = 81, k = 27, ...
previous frame = 9000
binary executable
xxxx0700: main() {
int i = 9;
/* ... */
xxxx1000: funcA(i * i);
/* ... */
}
xxxx1200: void funcA(int j) {
int k = j / 3;
/* ... */
xxxx1500: funcB(k + 5);
/* ... */
}
xxxx1750: void funcB(int m) {
int i = m / 4;
/* ... */
xxxx1820: funcB(m - i);
/* ... */
}


call stack
xxxx9000
main()
i = 9, ...
xxxx8850
funcA()
return address = 1000
j = 81, k = 27, ...
previous frame = 9000
xxxx8630
funcB()
return address = 1500
m = 32, i = 8, ...
previous frame = 8850
binary executable
xxxx0700: main() {
int i = 9;
/* ... */
xxxx1000: funcA(i * i);
/* ... */
}
xxxx1200: void funcA(int j) {
int k = j / 3;
/* ... */
xxxx1500: funcB(k + 5);
/* ... */
}
xxxx1750: void funcB(int m) {
int i = m / 4;
/* ... */
xxxx1820: funcB(m - i);
/* ... */
}


call stack
xxxx9000
main()
i = 9, ...
xxxx8850
funcA()
return address = 1000
j = 81, k = 27, ...
previous frame = 9000
xxxx8630
funcB()
return address = 1500
m = 32, i = 8, ...
previous frame = 8850
xxxx8480
funcB()
return address = 1820
m = 24, i = 6, ...
previous frame = 8630
binary executable
xxxx0700: main() {
int i = 9;
/* ... */
xxxx1000: funcA(i * i);
/* ... */
}
xxxx1200: void funcA(int j) {
int k = j / 3;
/* ... */
xxxx1500: funcB(k + 5);
/* ... */
}
xxxx1750: void funcB(int m) {
int i = m / 4;
/* ... */
xxxx1820: funcB(m - i);
/* ... */
}

04-B2

栈与队列

调用栈：深度与空间

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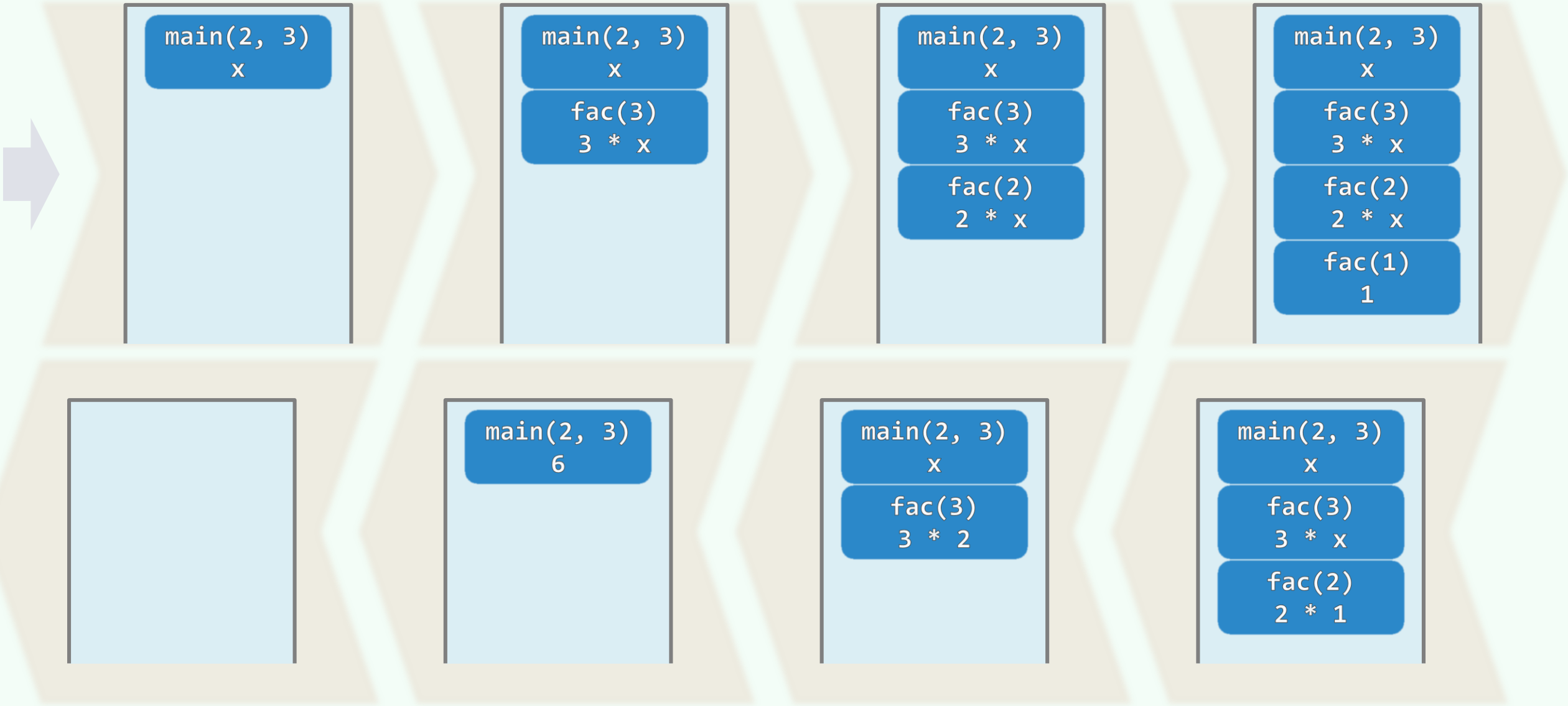
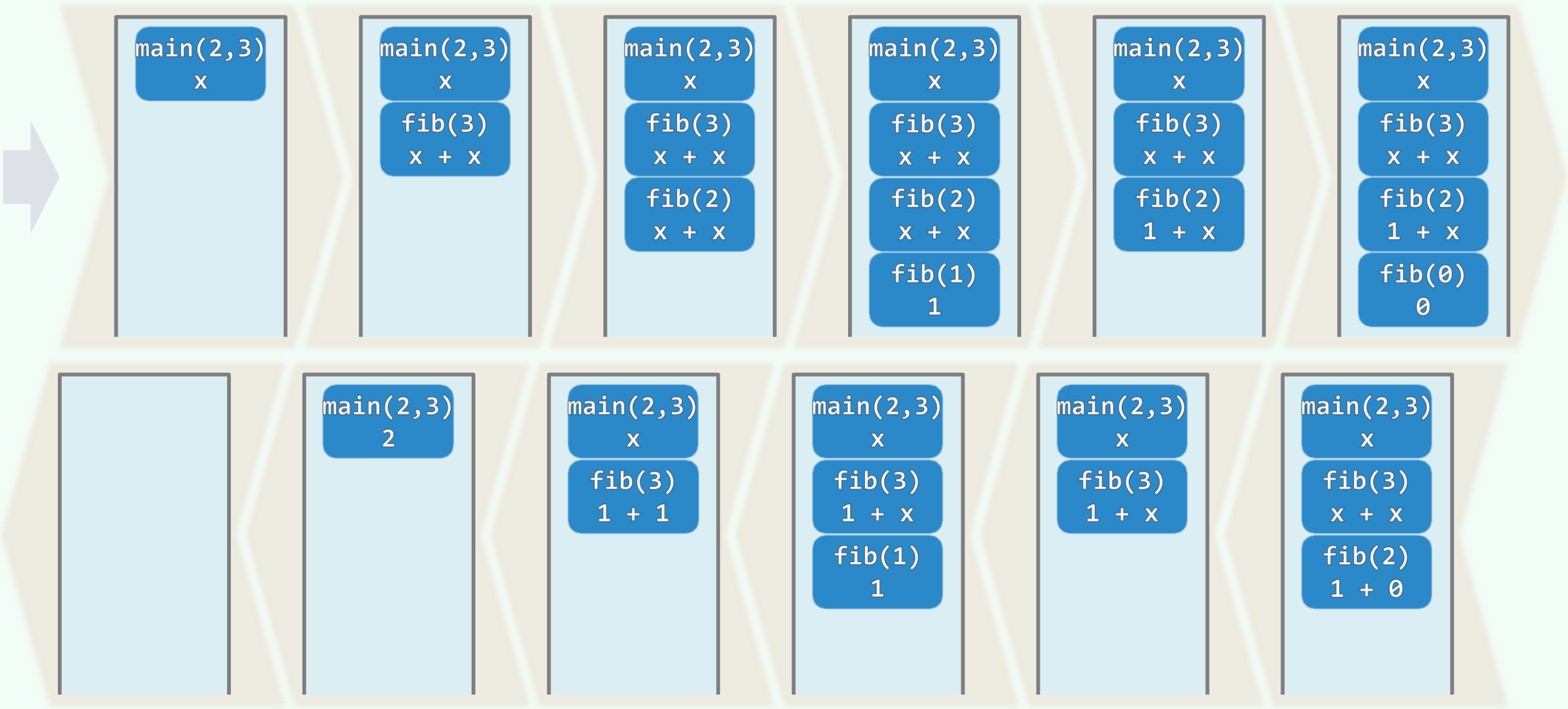
```
int fac(int n) { return (n < 2) ? 1 : n * fac(n - 1); }
```



```
int fib( int n ) { return (n < 2) ? n : fib(n - 1) + fib(n - 2); }
```



空间复杂度

```
❖ hailstone(int n) {
      if ( 1 < n )
         n % 2  ?  odd( n ) : even( n );
  }
❖ even( int n ) { hailstone( n / 2 ); }
  odd( int n ) { hailstone( 3*n + 1 ); }
❖ main( int argc, char* argv[] )
  { hailstone( atoi( argv[1] ) ); }
```

❖ 可见，递归算法所需的空间

主要取决于递归深度，而非递归实例总数

04-B3

栈与队列

调用栈：消除递归

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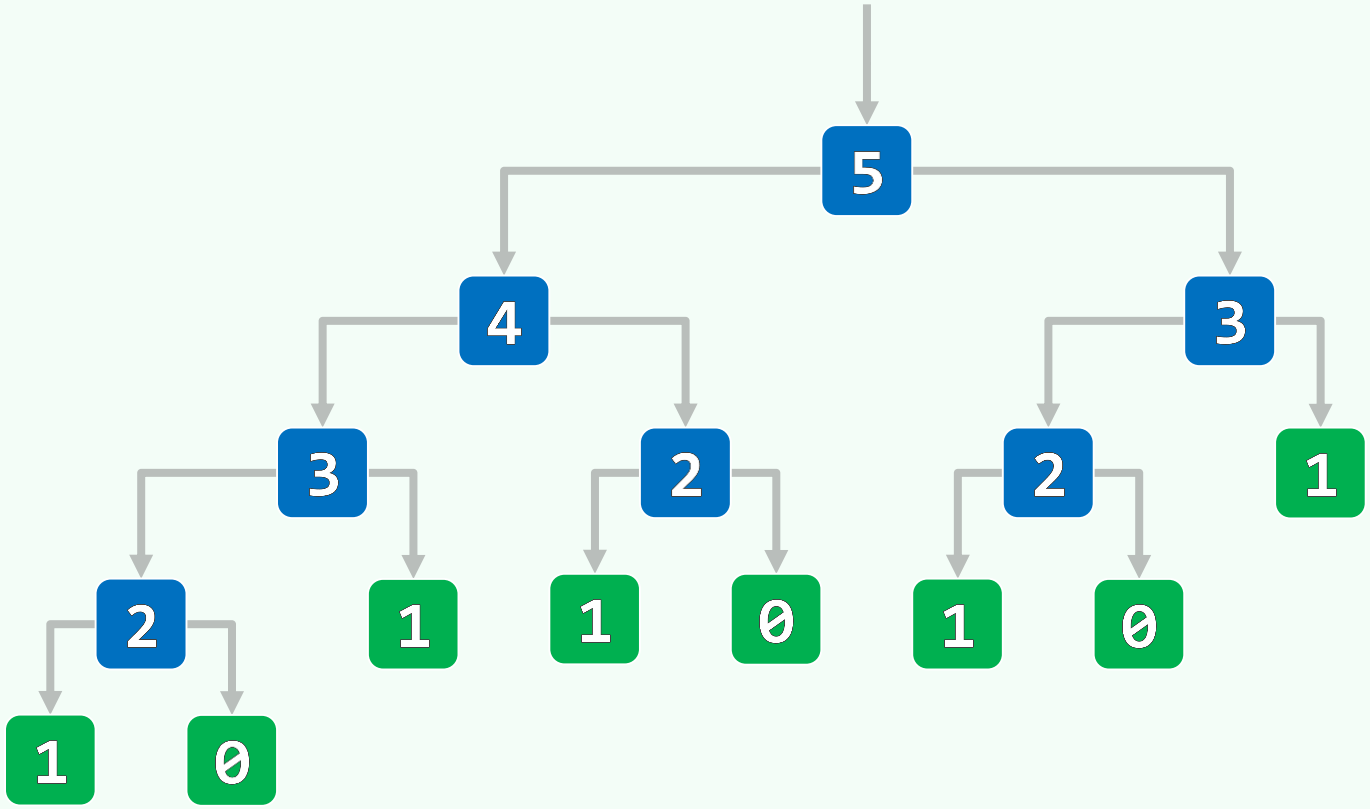
动机 + 方法

- ❖ 递归函数的空间复杂度
  - 主要取决于最大递归深度
  - 而非递归实例总数
- ❖ 为隐式地维护调用栈

  需花费额外的时间、空间
- ❖ 为节省空间，可
  - 显式地维护调用栈
  - 将递归算法改写为迭代版本...



实例

- 通常，消除递归只是在常数意义上优化空间
- 但也可能有实质改进

```
int fac( int n ) {
   int f = 1; //O(1)空间
   while ( n > 1 )
      f *= n--;
   return f;
}
```

```
void hailstone( int n ) { //O(1)空间
   while ( 1 < n )
      n = n % 2 ? 3*n + 1 : n/2;
}
```

```
int fib( int n ) { //O(1)空间
   int f = 0, g = 1;
   while ( 0 < n-- )
      { g += f; f = g - f; }
   return f;
}
```

栈与队列

调用栈：尾递归

04-B4

《星期评论》问我“女子解放从那里做起？”我的答案是：
“女子解放当从女子解放做起。此外更无别法。”

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定义

- 在递归实例中，作为最后一步的递归调用
- fac(n) {

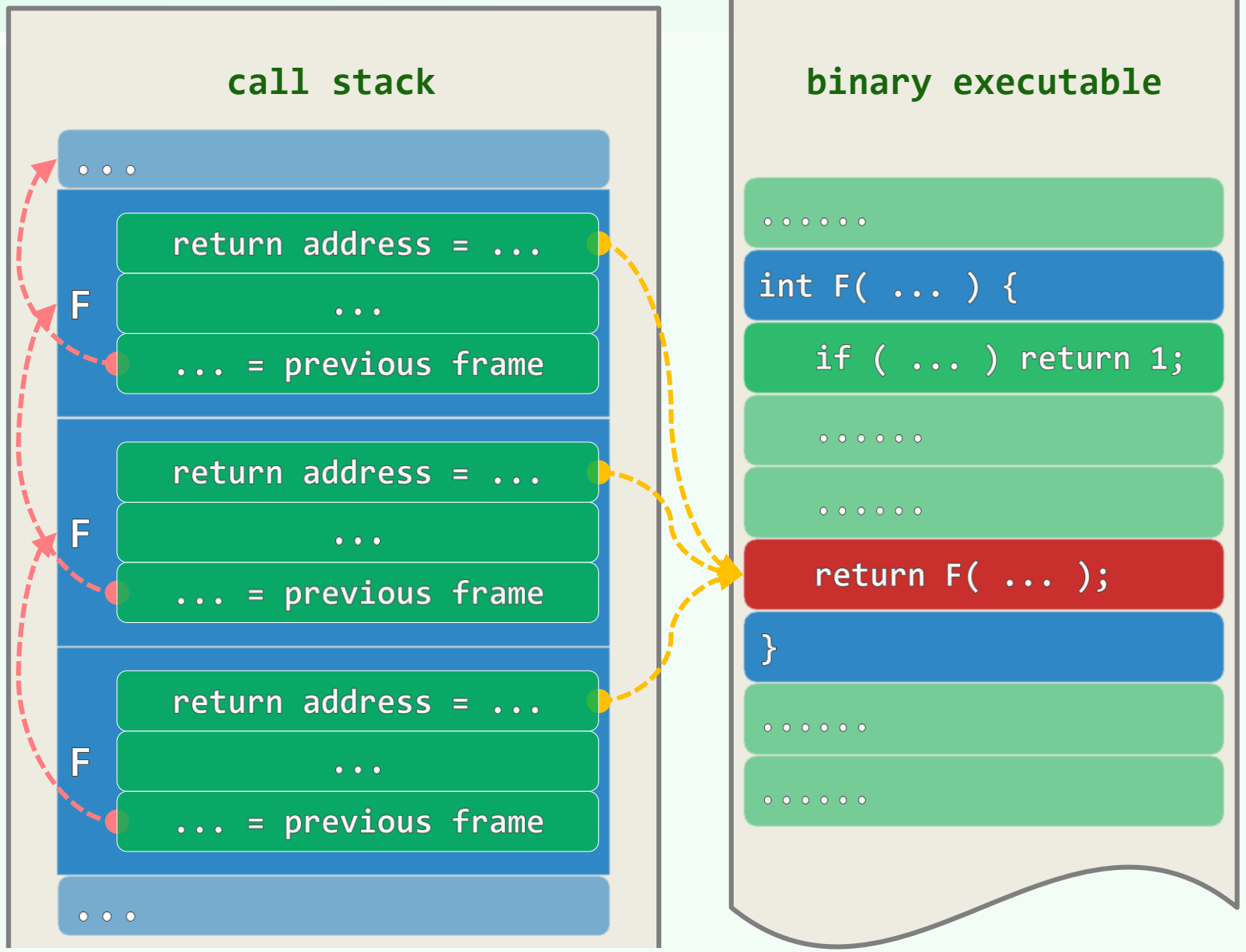
```
    if (1 > n) return 1; //base
    return n * fac( n-1 ); //tail recursion
}
```

- 严格地讲，上例在递归返回前的最后一步是乘法

  但这不是本质问题

性质

- 系最简单的递归模式
- 一旦抵达递归基，便会
  - 引发一连串的return
    （且返回地址相同）
  - 调用栈相应地连续pop
- 故不难改写为迭代形式
- 越来越多的编译器可以
  自动识别并代为改写
- 时间复杂度有常系数改进
  空间复杂度或有渐近改进



消除

```
fac(n) { //尾递归
   if (1 > n) return 1;
   return n * fac( n-1 );
} //O(n)时间 + O(n)空间
```

```
fac(n) { //统一转换为迭代
   int f = 1; //记录子问题的解
next: //转向标志，模拟递归调用
   if (1 > n) return f;
   f *= n--;
   goto next; //模拟递归返回
} //O(n)时间 + O(1)空间
```

```
fac(n) { //简捷
   int f = 1;
   while (1 < n)
      f *= n--;
   return f;
} //O(n)时间 + O(1)空间
```

栈与队列

进制转换

04-C

Hickory, Dickory, Dock

The mouse ran up the clock

wan, twan, tetbera, metbera, pimp,

setbera, letbera, bovera, dovera, dick,

wanadick, twanadick, tetberadick, metberadick, pimpdick,

setberadick, letberadick, boveradick, doveradick, bumfit,

wanabumfit, ...

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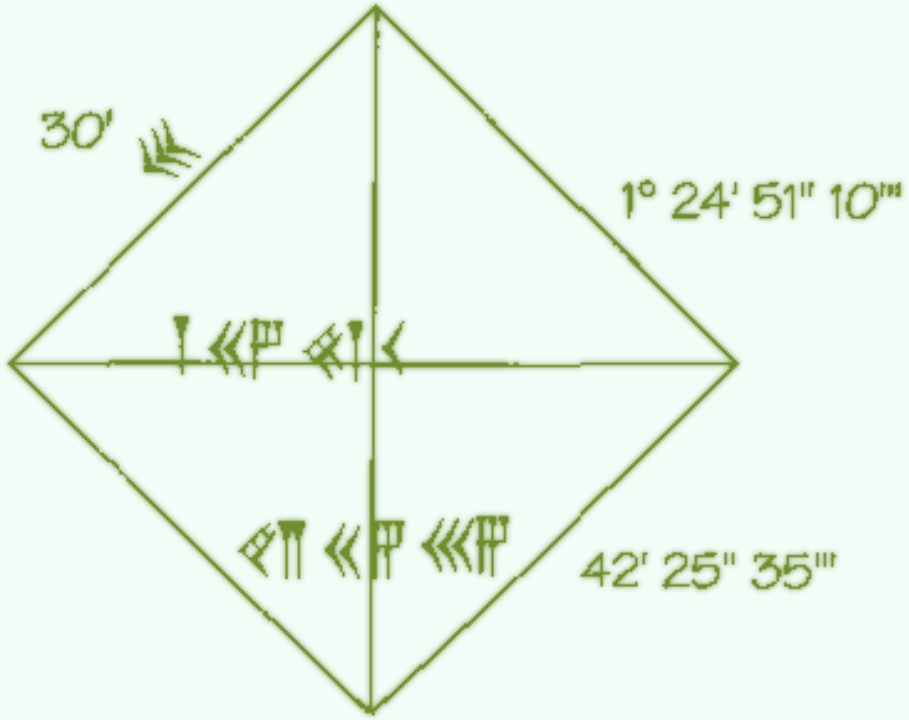
进制转换

- 给定任一10进制非负整数，将其转换为λ进制表示形式
  - $12345_{(10)} = 30071_{(8)}$
  - printf("%d | %I64d | %b | %o | %x" , n);
- 巴比伦楔形文字 (Babylonic cuneiform) 中的60进制...
- 1°24'51''10'''
  = 1 + 24/60 + 51/60^2 + 10/60^3
  = 1.41421296296296... //正方形的对角线
- 误差 ~ |1°24'51''10''' - sqrt(2)|
  < 0.000,000,6 = 0.6 x 10^{-6} //即便边长为1km，误差亦不足1mm

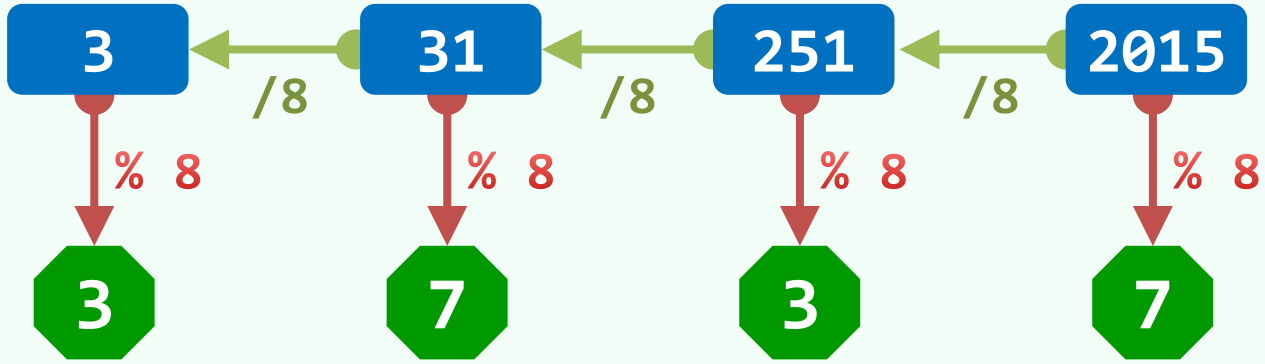


短除法：整商 + 余数

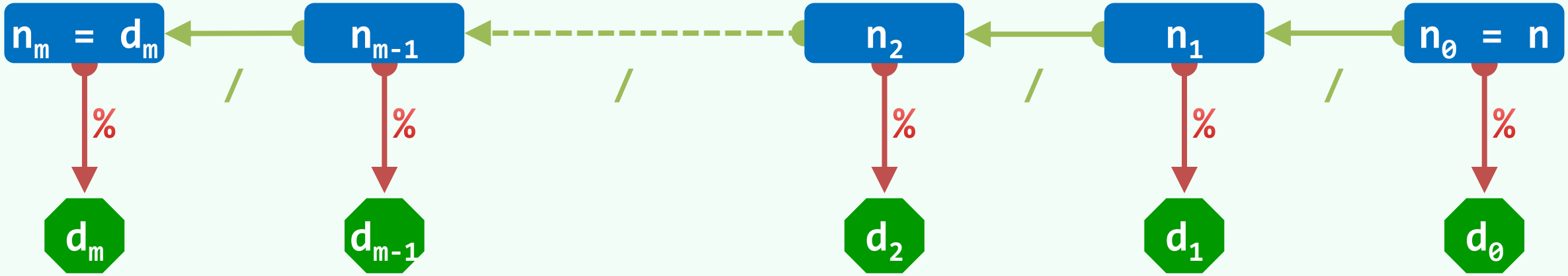
❖ 设：$n = (d_m \dots d_2 d_1 d_0)_\lambda = d_m \cdot \lambda^m + \cdots + d_2 \cdot \lambda^2 + d_1 \cdot \lambda^1 + d_0 \cdot \lambda^0$

令：$n_i = (d_m \dots d_{i+2} d_{i+1} d_i)_\lambda$

❖ 则有：$n_{i+1} = n_i \,/\, \lambda$ 和 $d_i = n_i \,\%\, \lambda$



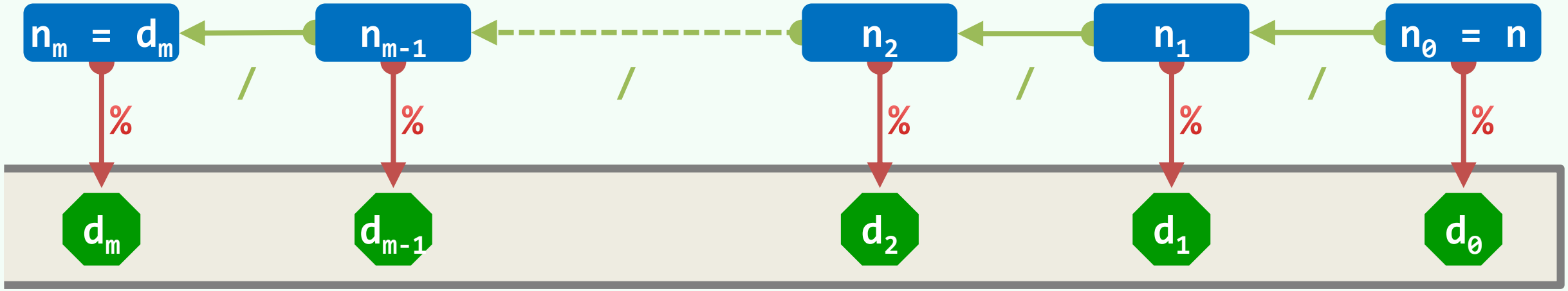
❖ 如此对λ反复整除、留余，即可自低而高得出λ进制的各位



难点 + 算法

❖ 位数m并不确定，如何正确记录并输出转换结果？具体地

- 如何支持足够大的m，同时空间也不浪费？
- 自低而高得到的数位，如何自高而低输出？



❖ 若使用向量，则扩容策略必须得当；若使用列表，则多数接口均被闲置

❖ 使用栈，既可满足以上要求，亦可有效控制计算成本

实现

```
void convert( Stack<char> & S, __int64 n, int base ) {
   char digit[] = "0123456789ABCDEF"; //数位符号，如有必要可相应扩充
   while ( n > 0 ) //由低到高，逐一计算出新进制下的各数位
      {  S.push( digit[ n % base ] ); n /= base;  } //余数入栈，n更新为除商
} //新进制下由高到低的各数位，自顶而下保存于栈S中
main() {
  Stack<char> S; convert( S, n, base ); //用栈记录转换得到的各数位
  while ( ! S.empty() ) printf( "%c", S.pop() ); //逆序输出
}
```

04-D

栈与队列

括号匹配

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实例

❖ `( a [ i - 1 ] [ j + 1 ] ) + a [ i + 1 ] [ j - 1 ] ) * 2` //失配

`( a [ i - 1 ] [ j + 1 ] + a [ i + 1 ] [ j - 1 ] ) * 2` //匹配

❖ 观察：除了各种括号，其余符号均可暂时忽略

`( [ ] [ ] ) [ ] [ ] )` //失配

`( [ ] [ ] [ ] [ ] )` //匹配

❖ 从简单入手，先来考查只有一种括号的情况...

尝试：由外而内

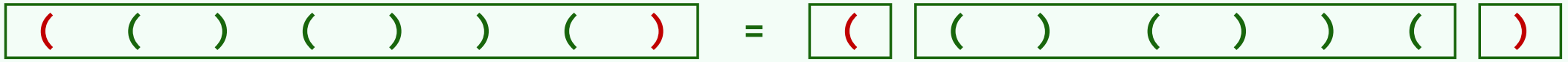
0）平凡：　无括号的表达式是匹配的

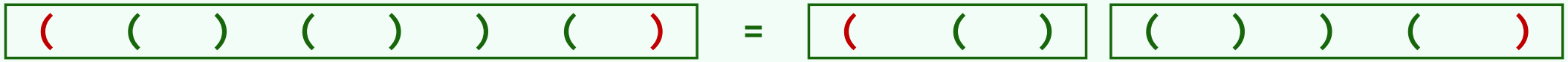
1）减治？　E 匹配，仅当 (E) 匹配

2）分治？　E 和 F 均匹配，仅当 E F 匹配

❖ 然而，根据以上性质，却不易直接应用已知的策略

❖ 究其根源在于，1）和2）均为**充分性**，比如反例：

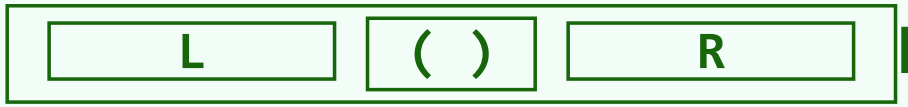




❖ 而为使问题有效简化，必须发现并借助**必要性**

构思：由内而外

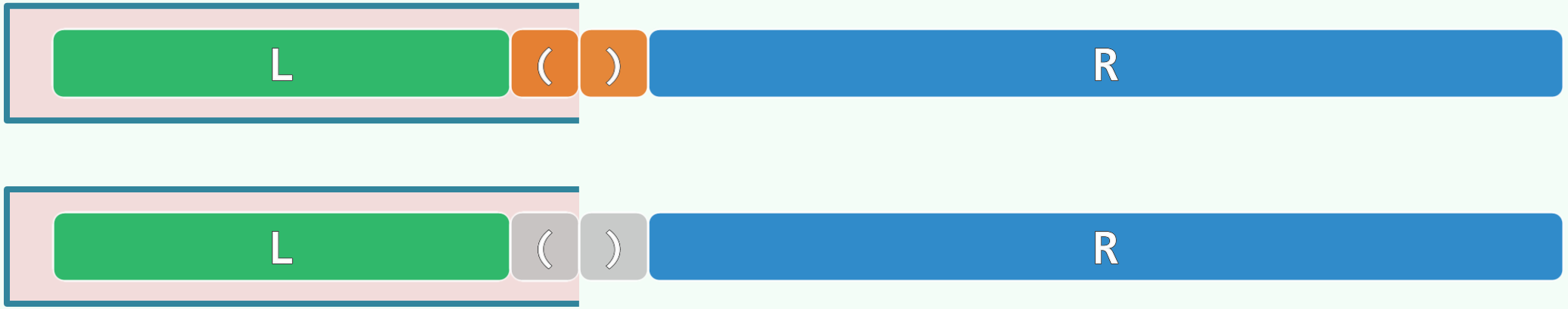
❖ 颠倒以上思路：消去一对紧邻的左右括号，不影响全局的匹配判断

亦即： 

匹配，仅当 

匹配

❖ 那么，如何找到这对括号？再者，如何使问题的这种简化得以持续进行？



❖ 顺序扫描表达式，用栈记录已扫描的部分　　//实际上只需记录左括号

反复迭代：凡遇"("，则进栈；凡遇")"，则出栈

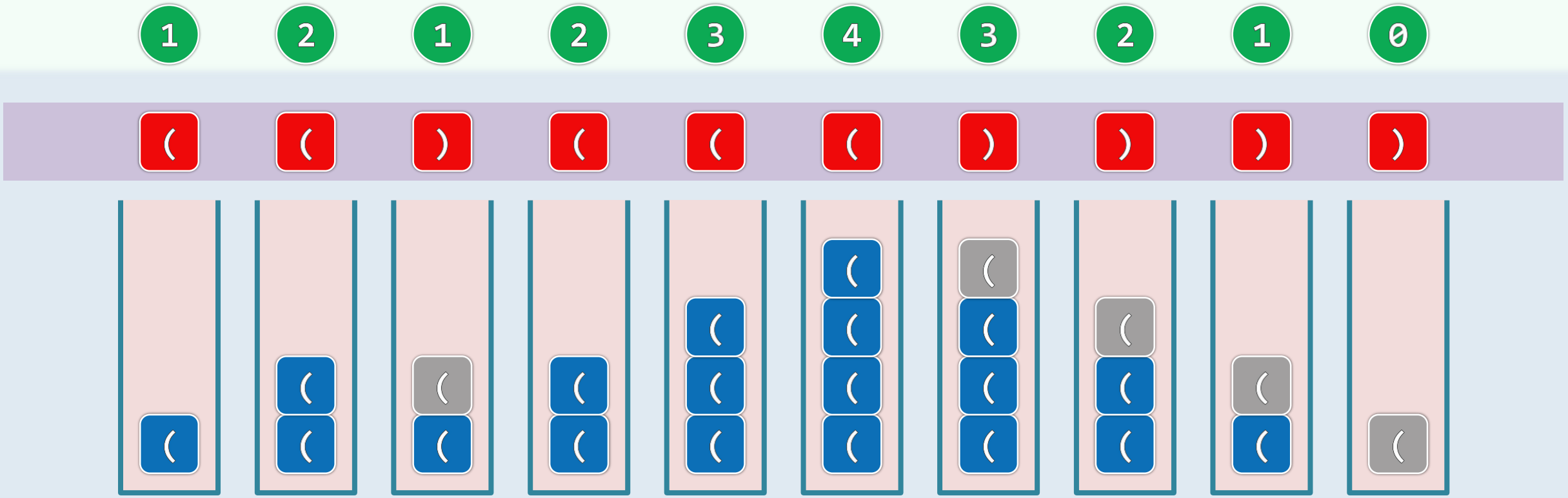
实现

```
bool paren( const char exp[], Rank lo, Rank hi ) { //exp[lo, hi)
   Stack<char> S; //使用栈记录已发现但尚未匹配的左括号
   for ( Rank i = lo; i < hi; i++ ) //逐一检查当前字符
      if ( '(' == exp[i] ) S.push( exp[i] ); //遇左括号：则进栈
      else if ( ! S.empty() ) S.pop(); //遇右括号：若栈非空，则弹出对应的左括号
      else return false; //否则（遇右括号时栈已空），必不匹配
   return S.empty(); //最终栈空，当且仅当匹配
}
```

实例：一种括号

❖ 实际上，若仅考虑一种括号，只需一个计数器足矣：S.size()

❖ 一旦转负，则为失配（右括号多余）；最后归零，即为匹配（否则左括号多余）

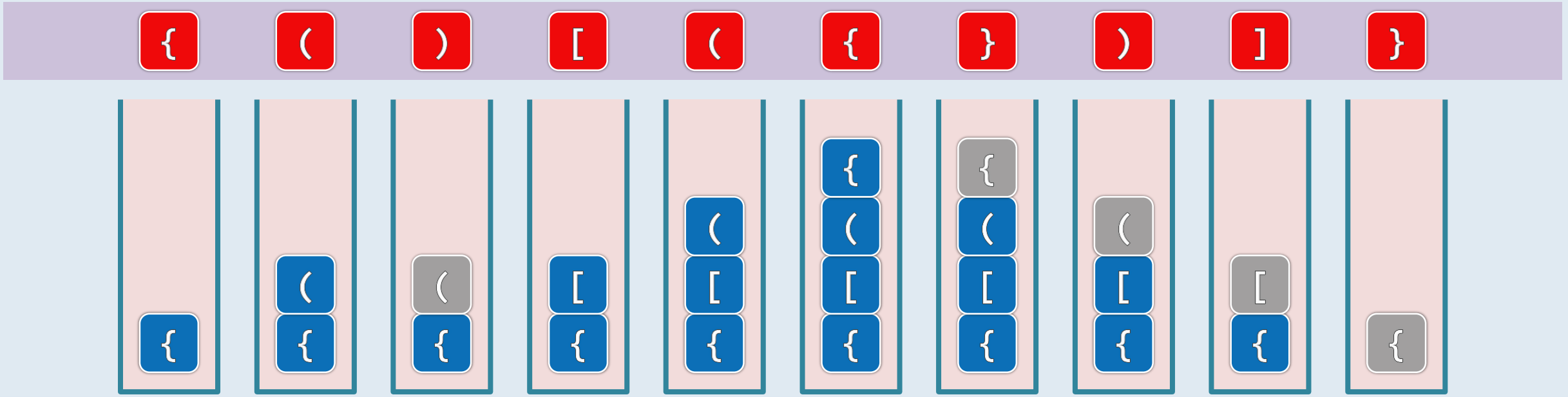


拓展：多类括号

❖ 以上思路及算法，可否推广至多种括号并存的情况？反例：[( ])

❖ 甚至，只需约定“括号”的通用格式，而不必事先固定括号的类型与数目

比如：<body>|</body>，<h1>|</h1>，|， | ， |，...



04-E

栈与队列

栈混洗

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Stack Permutation

❖ 考查栈 $\mathcal{A} = \langle\, a_1,\ a_2,\ a_3,\ \ldots,\ a_n \,]$

$\mathcal{B} = \mathcal{S} = \varnothing$

❖ 只允许

- 将A的顶元素弹出并压入S，或
- 将S的顶元素弹出并压入B

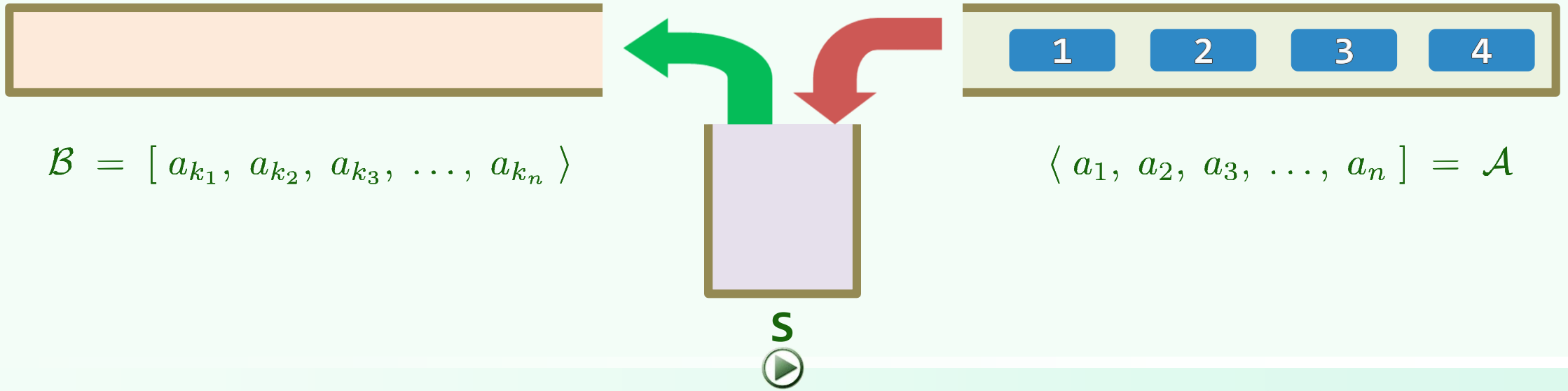
❖ 亦即 $\mathcal{S}.\text{push}(\mathcal{A}.\text{pop}())$

$\mathcal{B}.\text{push}(\mathcal{S}.\text{pop}())$

❖ 若经一系列以上操作后，A中元素全部转入B中

$\mathcal{B} = [\, a_{k_1},\ a_{k_2},\ a_{k_3},\ \ldots,\ a_{k_n} \,\rangle$

则称为A的一个**栈混洗**



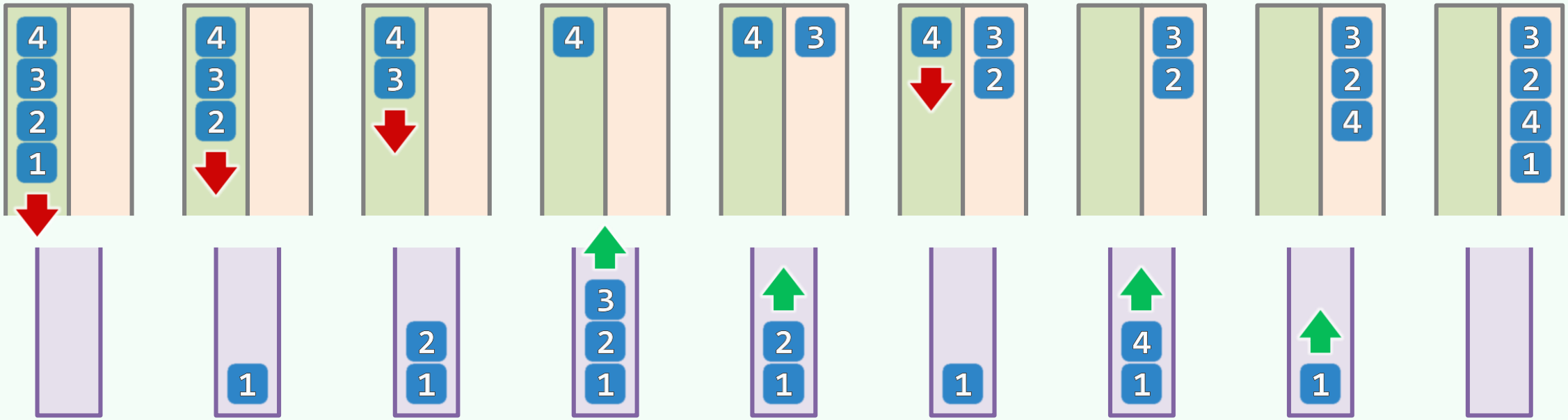
$\mathcal{B} = [\, a_{k_1},\ a_{k_2},\ a_{k_3},\ \ldots,\ a_{k_n} \,\rangle$

$\langle\, a_1,\ a_2,\ a_3,\ \ldots,\ a_n \,] = \mathcal{A}$

计数：SP(n)

❖ 同一输入序列，可有多种栈混洗：[1，2，3，4 >，[4，3，2，1 >，[3，2，4，1 > ...

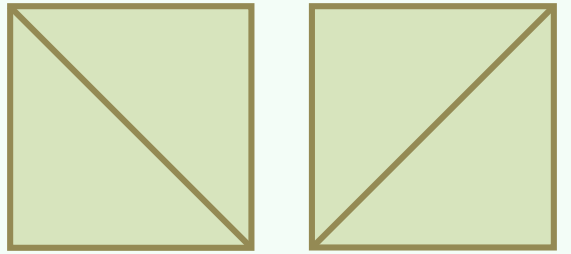
❖ 一般地，对于长度为n的序列，混洗总数SP(n) = ？



❖ 显然，SP(n) <= n!；更准确地呢?

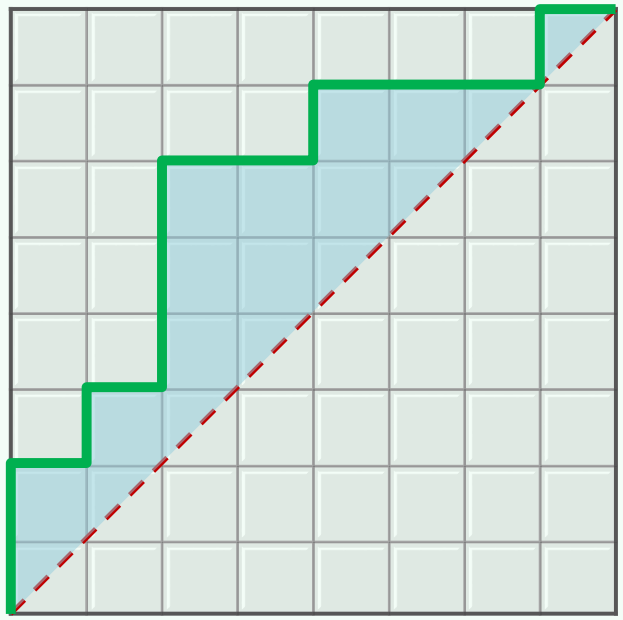
计数：catalan(n)

- $SP(1) = 1$
- **考查S再度变空（A首元素从S中弹出）的时刻，无非n种情况：**



$$SP(n) = \sum_{k=1}^{n} SP(k-1) \cdot SP(n-k)$$

$$= catalan(n) = \frac{(2n)!}{(n+1)! \cdot n!}$$

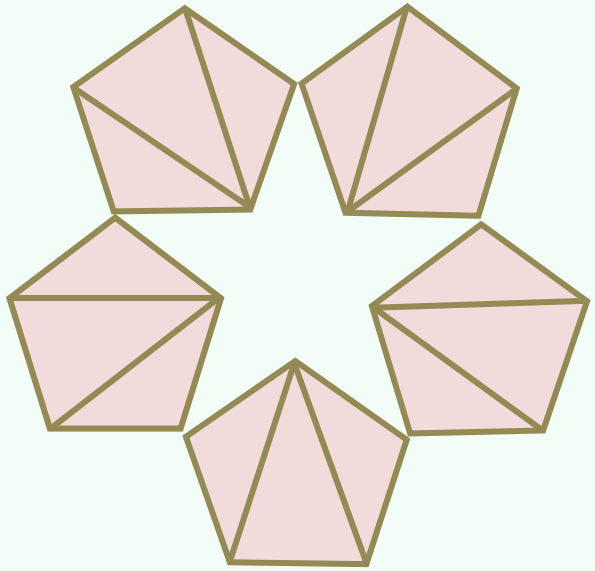


$SP(2) = 4!/3!/2! = 2$

$SP(3) = 6!/4!/3! = 5$

... ...

$SP(6) = 12!/7!/6! = 132$



计数：catalan(n)

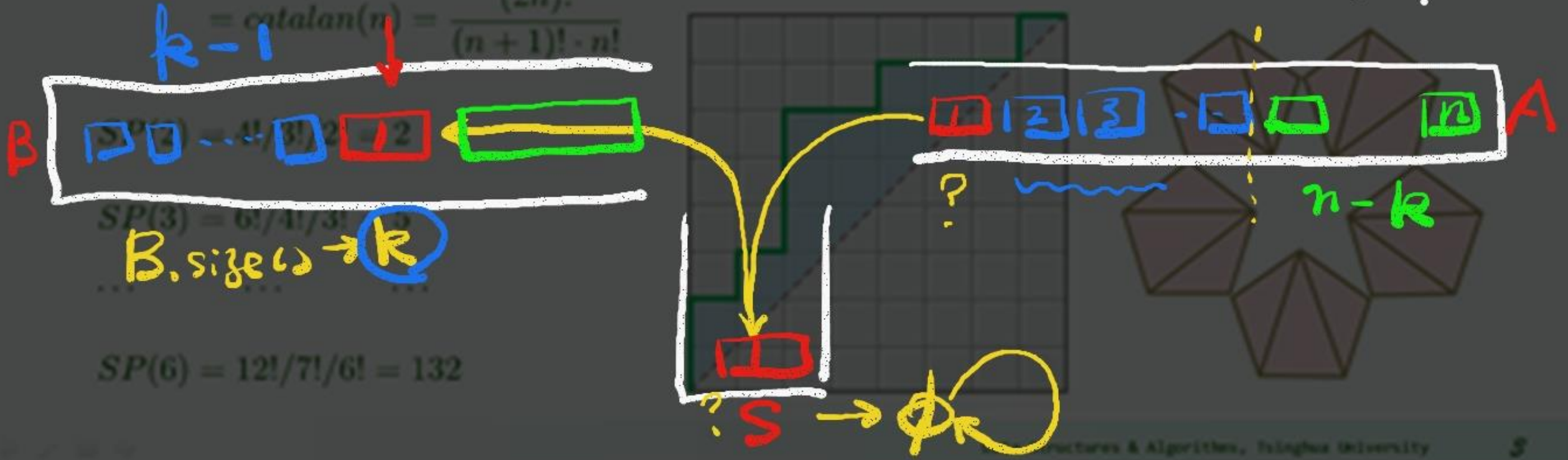
$$SP(n) = \sum_{k=1}^{n} SP(k-1) \cdot SP(n-k)$$

- $SP(1) = 1$
- 考查S再度变空（A首元素从S中弹出）的时刻，无非n种情况：

$$SP(n) = \sum_{k=1}^{n} SP(k-1) \cdot SP(n-k) = catalan(n) = \frac{(2n)!}{(n+1)! \cdot n!}$$

$$SP(3) = 6!/4!/3! = 30$$

$$SP(6) = 12!/7!/6! = 132$$



甄别：检测禁形

- 输入序列< 1, 2, 3, ..., n >的任一排列[p_1, p_2, p_3, ..., p_n >是否为栈混洗?
- 先考查简单情况：n = 3, A = < 1, 2, 3 >
  - 栈混洗共 6! / 4! / 3! = 5 种；全排列共 3! = 6 种 //少了一种...
- [3, 1, 2 > //为什么是它?
- 观察：任意三个元素能否按某相对次序出现于混洗中，与其它元素无关 //故可推而广之...
- 禁形：对任何1 ≤ i < j < k ≤ n, [..., k, ..., i, ..., j, ... > 必非栈混洗
- 反过来，不存在“312”模式的序列，一定是栈混洗吗?

甄别：直接模拟

❖ 充要性：　　A permutation is a stack permutation iff

(Knuth, 1968)　　it does NOT involve the permutation 312　　//习题[4-3]

❖ 如此，可得一个$O(n^3)$的甄别算法　　//进一步地...

❖ [p_1, p_2, p_3, ..., p_n]是[1, 2, 3, ..., n]的栈混洗，当且仅当

对于任意i < j，不含模式[..., j+1, ..., i, ..., j, ...]

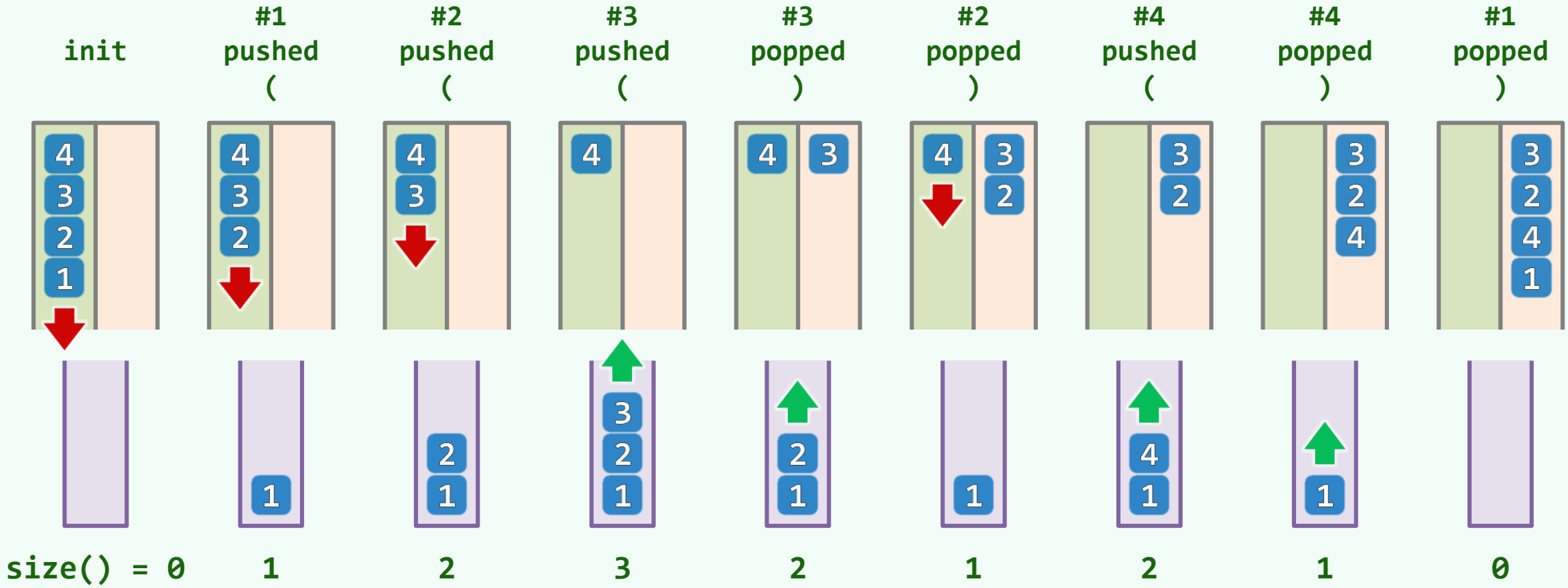
❖ 如此，可得一个$O(n^2)$的甄别算法　　//再进一步地...

❖ $O(n)$算法：　直接借助栈A、B和S，模拟混洗过程　　//为何可行?

每次S.pop()之前，检测S是否已空；或需弹出的元素在S中，却非顶元素

括号匹配

❖ 观察：每一栈混洗，都对应于栈S的n次push与n次pop操作构成的某一序列；反之亦然



❖ n个元素的栈混洗，等价于n对括号的匹配；二者的组合数，也自然相等

04-F1

栈与队列

中缀表达式求值：问题与构思

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知实而不知名，知名而不知实，皆不知也

应用

❖ 给定语法正确的算术表达式S，计算与之对应的数值

❖ UNIX: $ echo $((0 + (1 + 23) / 4 * 5 * 67 - 8 + 9))

❖ DOS: \> set /a (!0 ^<^< (1 - 2 + 3 * 4)) - 5 * (6 ^| 7) / (8 ^^ 9)

❖ PS: GS> 0 1 23 add 4 div 5 mul 67 mul add 8 sub 9 add =

❖ Excel: = COS(0) + 1 - (2 - POWER((FACT(3) - 4), 5)) * 67 - 8 + 9

❖ Word: = NOT(0) + 12 + 34 * 56 + 7 + 89

❖ calc: 0 ! + 12 + 34 * 56 + 7 + 89 =

❖ calc: 0 ! + 1 - (2 - (3 ! - 4) y 5) * 67 - 8 + 9 =

减而治之

❖ 优先级高的局部执行计算，并被代以其数值

运算符渐少，直至得到最终结果

❖ str(v)：数值v对应的字符串（名）

val(S)：符号串S对应的数值（实）

❖ 设表达式：$S = S_L + S_0 + S_R$

- S_0可优先计算，且
- $val(S_0) = v_0$

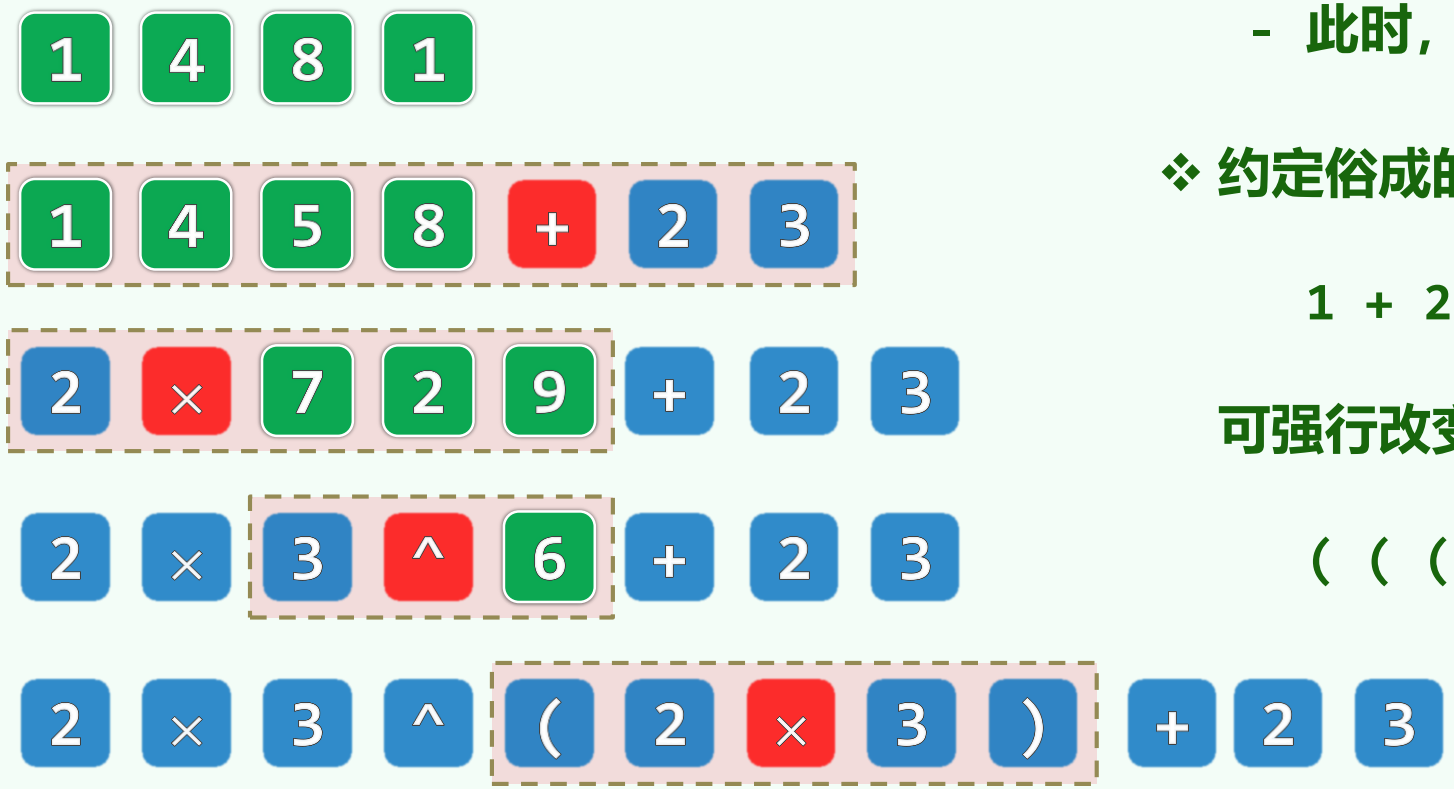
❖ 则有递推化简关系

$$val(S) = val(S_L + str(v_0) + S_R)$$

优先级

❖ 难点：如何高效地找到可优先计算的s_0

（亦即，其对应的运算符）？



❖ 与括号匹配迭代版类似，但亦不尽相同

- 不能简单地按“左先右后”次序处理各运算符
- 此时，需要考虑更多因素...

❖ 约定俗成的优先级：

1 + 2 * 3 ^ 4 !

可强行改变次序的括号：

(((1 + 2) * 3) ^ 4) !

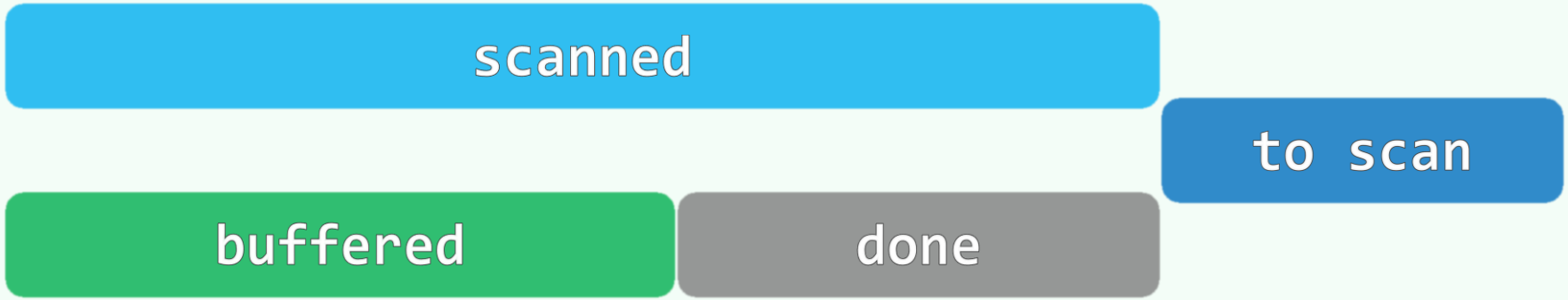
延迟缓冲

❖ 仅根据表达式的前缀，不足以确定各运算符的计算次序

只有获得足够的后续信息，才能确定其中哪些运算符可以执行

❖ 体现在求值算法的流程上

为处理某一前缀，必须提前预读并分析更长的前缀



❖ 为此，需借助某种支持延迟缓冲的机制...

求值算法 = 栈 + 线性扫描

- 自左向右扫描表达式

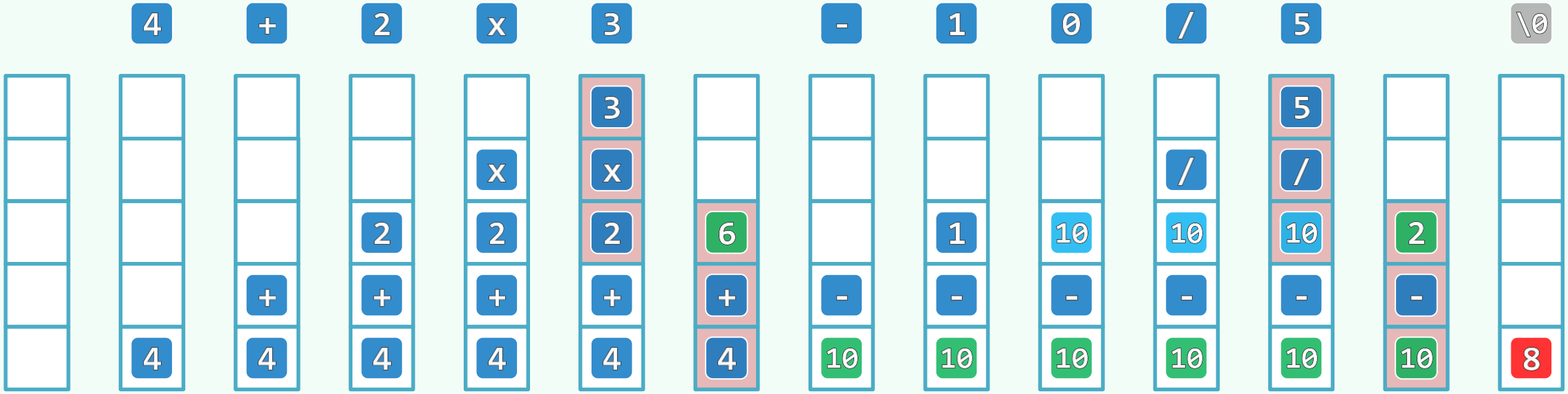
  用栈记录已扫描的部分，以及中间结果

- 栈内最终所剩的那个元素，即表达式之值

- If（栈的顶部存在可优先计算的子表达式）

  Then 令其退栈并计算；计算结果进栈

  Else 当前字符进栈，转入下一字符



栈与队列

中缀表达式求值：算法

04-F2

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主算法

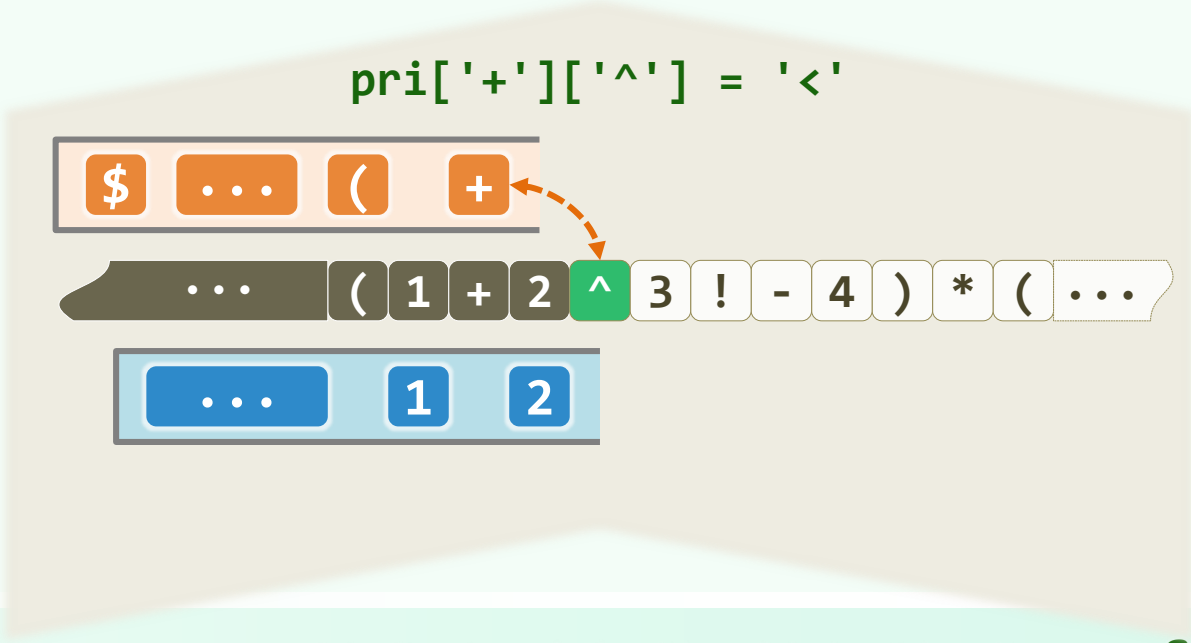
```
double evaluate( char* S, char* RPN ) { //S保证语法正确
   Stack<double> opnd; Stack<char> optr; //运算数栈、运算符栈
   optr.push('\0'); //哨兵
   while ( ! optr.empty() ) { //逐个处理各字符，直至运算符栈空
      if ( isdigit( *S ) ) //若为操作数（可能多位、小数），则
         readNumber( S, opnd ); //读入
      else //若为运算符，则视其与栈顶运算符之间优先级的高低
         switch( priority( optr.top(), *S ) ) { /* 分别处理 */ }
   } //while
   return opnd.pop(); //弹出并返回最后的计算结果
}
```

优先级表

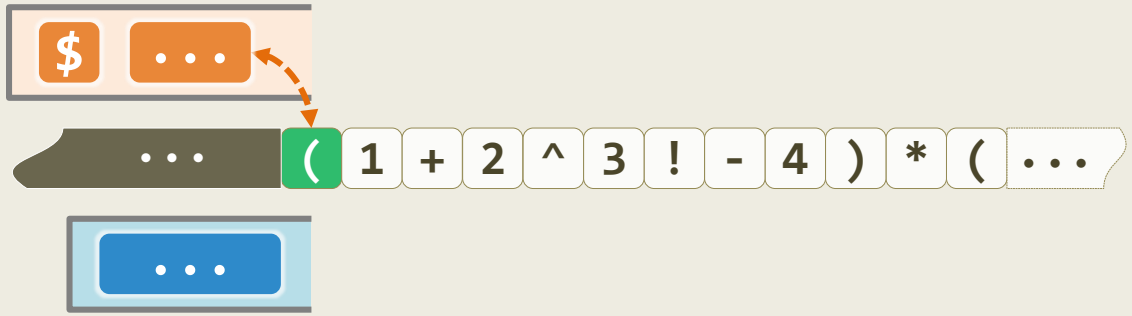
```
const char pri[N_OPTR][N_OPTR] = { //运算符优先等级 [栈顶][当前]
   /* --  + */  '>', '>', '<', '<', '<', '<', '<', '>', '>',
   /*  |  - */  '>', '>', '<', '<', '<', '<', '<', '>', '>',
   /* 栈  * */  '>', '>', '>', '>', '<', '<', '<', '>', '>',
   /* 顶  / */  '>', '>', '>', '>', '<', '<', '<', '>', '>',
   /* 运  ^ */  '>', '>', '>', '>', '>', '<', '<', '>', '>',
   /* 算  ! */  '>', '>', '>', '>', '>', '>', ' ', '>', '>',
   /* 符  ( */  '<', '<', '<', '<', '<', '<', '<', '=', ' ',
   /*  |  ) */  ' ', ' ', ' ', ' ', ' ', ' ', ' ', ' ', ' ',
   /* -- \0 */  '<', '<', '<', '<', '<', '<', '<', ' ', '='
   //           +    -    *    /    ^    !    (    )    \0
   //           |-------------- 当前运算符 --------------|
};
```

'<': 静待时机: 算法

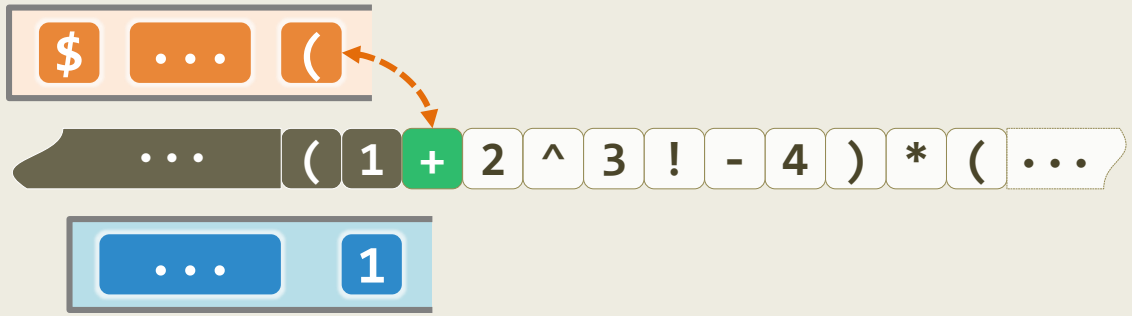
```
switch( priority( optr.top(), *S ) ) {
   case '<': //栈顶运算符优先级更低
      optr.push( *S ); S++; break; //计算推迟，当前运算符进栈
   case '=':
      /* ..... */
   case '>': {
      /* ..... */
      break;
   } //case '>'
} //switch
```



'<'：静待时机：实例

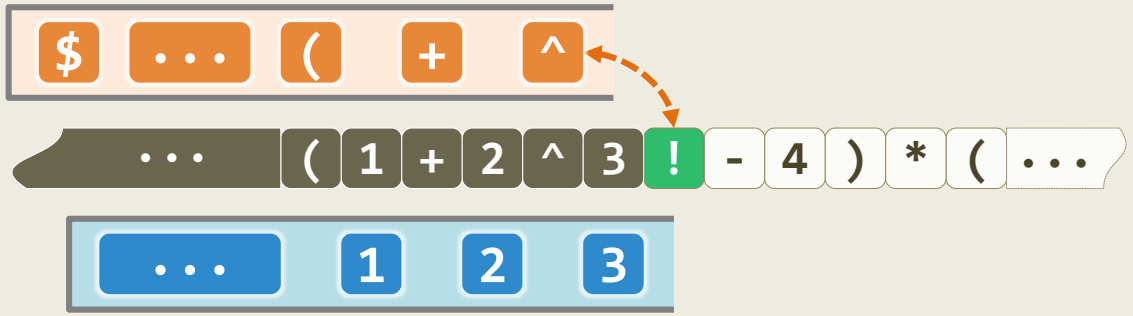


pri[ ]['('] = '<'

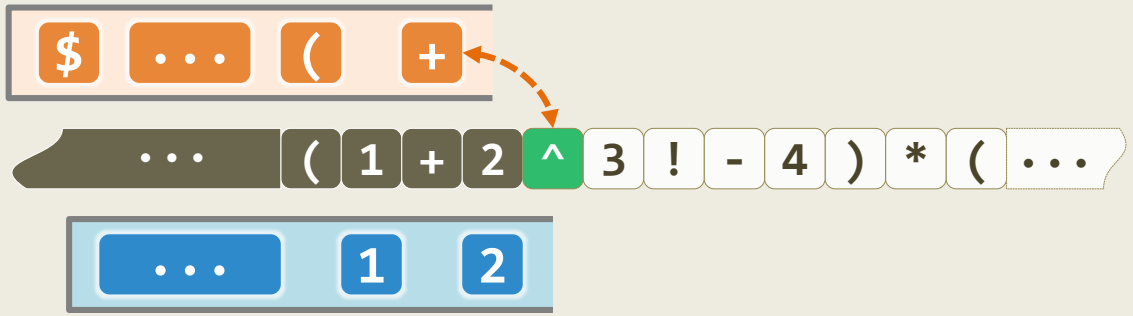


pri['('][+'] = '<'

pri['^']['!'] = '<'

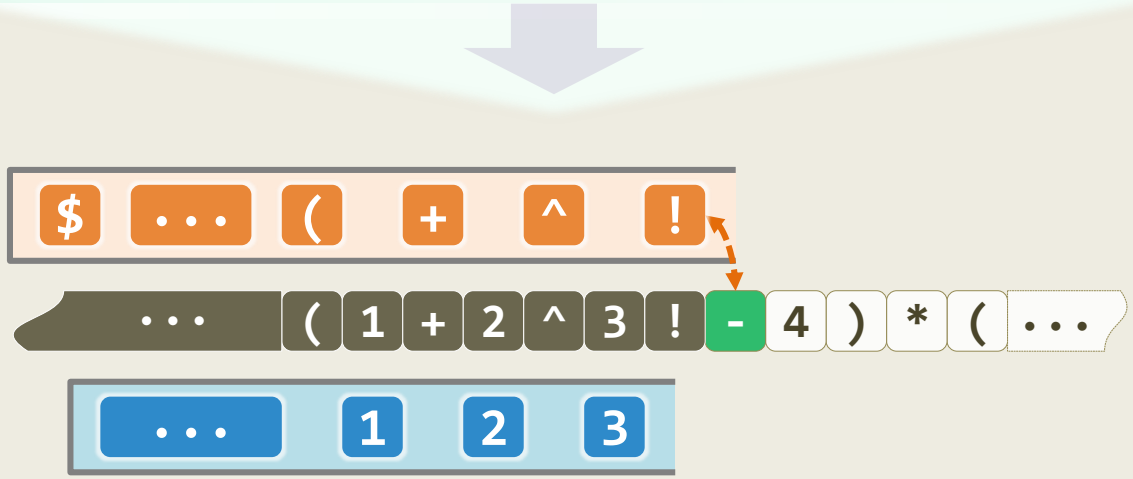


pri['+']['^'] = '<'



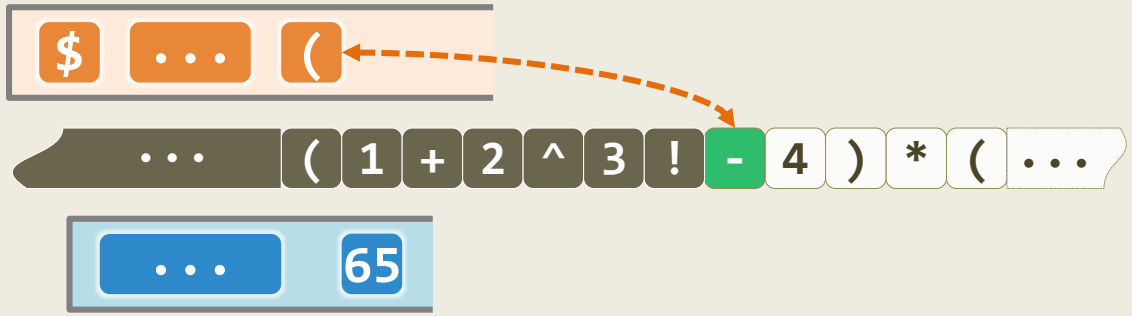
'>': 时机已到：算法

```
switch( priority( optr.top(), *S ) ) {
   /* ..... */
   case '>': {
      char op = optr.pop();
      if ( '!' == op ) opnd.push( calcu( op, opnd.pop() ) ); //一元运算符
      else { double opnd2 = opnd.pop(), opnd1 = opnd.pop(); //二元运算符
             opnd.push( calcu( opnd1, op, opnd2 ) ); //实施计算，结果入栈
      } //为何不直接：opnd.push( calcu( opnd.pop(), op, opnd.pop() ) )?
      break;
   } //case '>'
} //switch
```

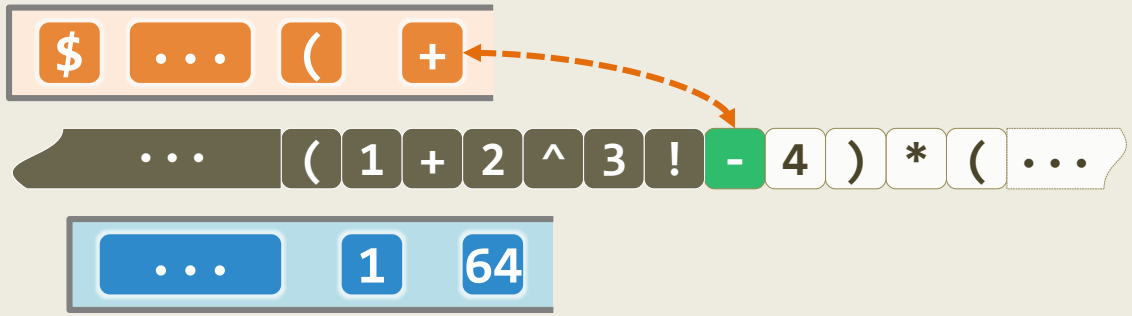


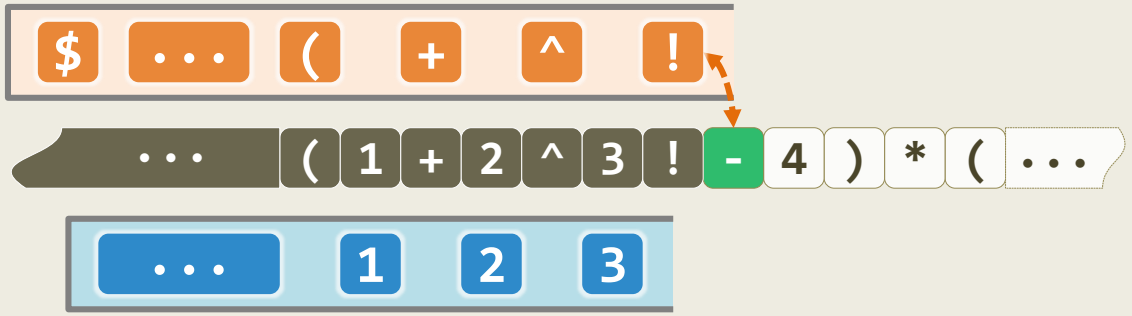
'>': 时机已到：实例

pri['('['-'] = '<'

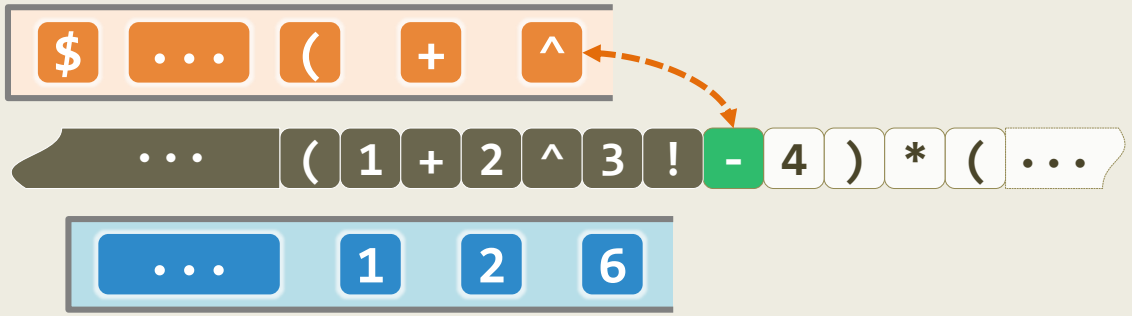


pri['+']['-'] = '>'





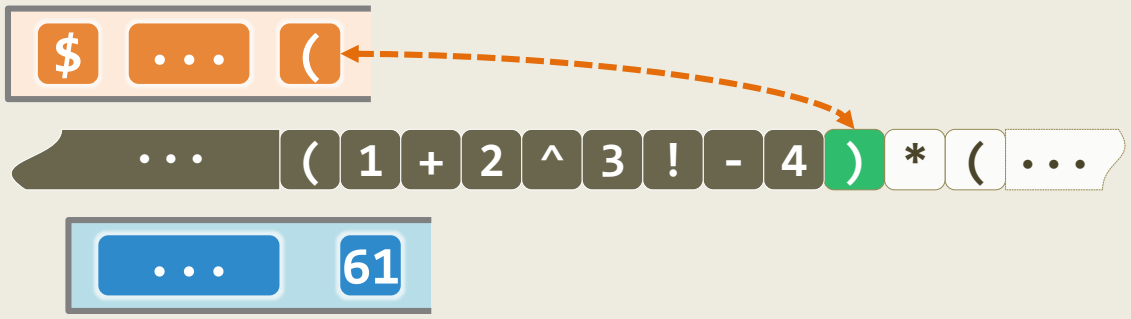
pri['!']['-'] = '>'



pri['^']['-'] = '>'

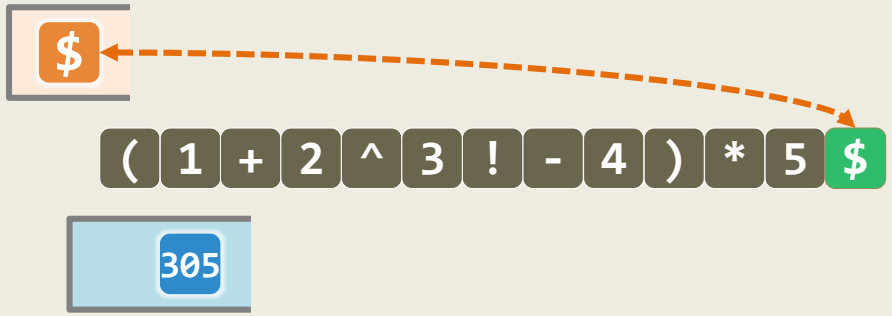
'=': 终须了断：算法

```
switch( priority( optr.top(), *S ) ) {
   case '<':
      /* ..... */
   case '=': //优先级相等（当前运算符为右括号，或尾部哨兵'\0'）
      optr.pop(); S++; break; //脱括号并接收下一个字符
   case '>': {
      /* ..... */
      break;
   } //case '>'
} //switch
```

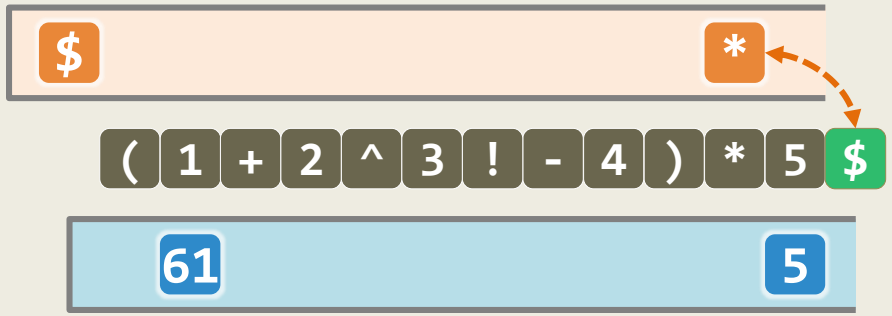


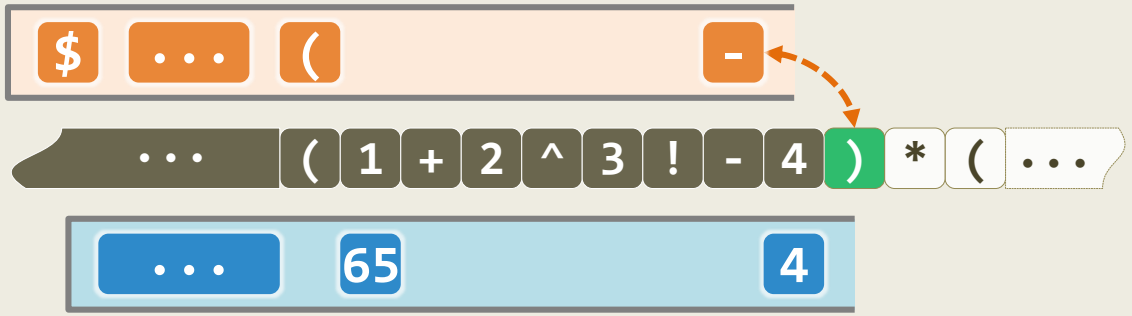
'=': 终须了断: 实例

pri['$']['$'] = '='

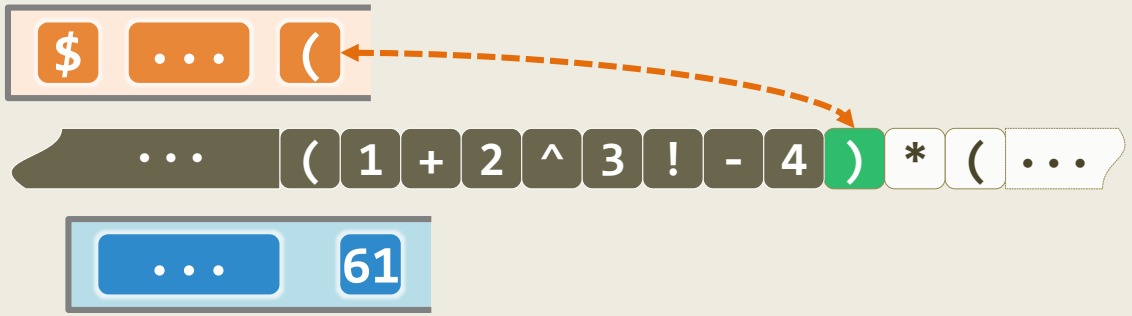


pri['*']['$'] = '>'





pri['-'][') '] = '>'



pri['('][')'] = '='

+ - * / !: 芸芸众生

```
const char pri[N_OPTR][N_OPTR] = { //运算符优先等级 [栈顶][当前]
   /* --  + */  '>', '>', '<', '<', '<', '<', '<', '>', '>',
   /*  |  - */  '>', '>', '<', '<', '<', '<', '<', '>', '>',
   /* 栈  * */  '>', '>', '>', '>', '<', '<', '<', '>', '>',
   /* 顶  / */  '>', '>', '>', '>', '<', '<', '<', '>', '>',
   /* 运  ^ */  '>', '>', '>', '>', '>', '<', '<', '>', '>',
   /* 算  ! */  '>', '>', '>', '>', '>', '>', ' ', '>', '>',
   /* 符  ( */  '<', '<', '<', '<', '<', '<', '<', '=', ' ',
   /*  |  ) */  ' ', ' ', ' ', ' ', ' ', ' ', ' ', ' ', ' ',
   /* -- \0 */  '<', '<', '<', '<', '<', '<', '<', ' ', '='
   //            +    -    *    /    ^    !    (    )    \0
   //              |-------------- 当前运算符 --------------|
};
```

'(': 我不下地狱，谁下地狱

```
const char pri[N_OPTR][N_OPTR] = { //运算符优先等级 [栈顶][当前]
   /* --  + */  '>', '>', '<', '<', '<', '<', '<', '>', '>',
   /*  |  - */  '>', '>', '<', '<', '<', '<', '<', '>', '>',
   /* 栈  * */  '>', '>', '>', '>', '<', '<', '<', '>', '>',
   /* 顶  / */  '>', '>', '>', '>', '<', '<', '<', '>', '>',
   /* 运  ^ */  '>', '>', '>', '>', '>', '<', '<', '>', '>',
   /* 算  ! */  '>', '>', '>', '>', '>', '>', ' ', '>', '>',
   /* 符  ( */  '<', '<', '<', '<', '<', '<', '<', '=', ' ',
   /*  |  ) */  ' ', ' ', ' ', ' ', ' ', ' ', ' ', ' ', ' ',
   /* -- \0 */  '<', '<', '<', '<', '<', '<', '<', ' ', '='
   //           +    -    *    /    ^    !    (    )    \0
   //           |------------- 当前运算符 --------------|
};
```

')': 死线已至（然后满血复活）

```
const char pri[N_OPTR][N_OPTR] = { //运算符优先等级 [栈顶][当前]
   /* --  + */  '>', '>', '<', '<', '<', '<', '<', '>', '>',
   /*  |  - */  '>', '>', '<', '<', '<', '<', '<', '>', '>',
   /* 栈  * */  '>', '>', '>', '>', '<', '<', '<', '>', '>',
   /* 顶  / */  '>', '>', '>', '>', '<', '<', '<', '>', '>',
   /* 运  ^ */  '>', '>', '>', '>', '>', '<', '<', '>', '>',
   /* 算  ! */  '>', '>', '>', '>', '>', '>', ' ', '>', '>',
   /* 符  ( */  '<', '<', '<', '<', '<', '<', '<', '=', ' ',
   /*  |  ) */  ' ', ' ', ' ', ' ', ' ', ' ', ' ', ' ', ' ',
   /* -- \0 */  '<', '<', '<', '<', '<', '<', '<', ' ', '='
   //            +    -    *    /    ^    !    (    )    \0
   //            |-------------- 当前运算符 --------------|
};
```

'\0': 从创世纪，到世界末日

```
const char pri[N_OPTR][N_OPTR] = { //运算符优先等级 [栈顶][当前]
   /* --  + */ '>', '>', '<', '<', '<', '<', '<', '>', '>',
   /*  |  - */ '>', '>', '<', '<', '<', '<', '<', '>', '>',
   /* 栈  * */ '>', '>', '>', '>', '<', '<', '<', '>', '>',
   /* 顶  / */ '>', '>', '>', '>', '<', '<', '<', '>', '>',
   /* 运  ^ */ '>', '>', '>', '>', '>', '<', '<', '>', '>',
   /* 算  ! */ '>', '>', '>', '>', '>', '>', ' ', '>', '>',
   /* 符  ( */ '<', '<', '<', '<', '<', '<', '<', '=', ' ',
   /*  |  ) */ ' ', ' ', ' ', ' ', ' ', ' ', ' ', ' ', ' ',
   /* -- \0 */ '<', '<', '<', '<', '<', '<', '<', ' ', '='
   //            +    -    *    /    ^    !    (    )    \0
   //            |-------------- 当前运算符 --------------|
};
```

04-F3

栈与队列

中缀表达式求值：实例

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表达式	运算符栈	操作数栈	注解
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$		表达式起始标识入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ (		左括号入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ (	0	操作数0入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ (!	0	运算符'!'入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ (	1	运算符'!'出栈执行
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ (+	1	运算符'+'入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ (+	1 1	操作数1入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ (	2	运算符'+'出栈执行
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$	2	左括号出栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ *	2	运算符'*'入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ *	2 2	操作数2入栈

表达式	运算符栈	操作数栈	注解
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ * ^	2 2	运算符'^'入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ * ^ (	2 2	左括号入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ * ^ (	2 2 3	操作数3入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ * ^ (!	2 2 3	运算符'!'入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ * ^ (	2 2 6	运算符'!'出栈执行
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ * ^ (+	2 2 6	运算符'+'入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ * ^ (+	2 2 6 4	操作数4入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ * ^ (	2 2 10	运算符'+'出栈执行
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ * ^	2 2 10	左括号出栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ *	2 1024	运算符'^'出栈执行
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$	2048	运算符'*'出栈执行

表达式	运算符栈	操作数栈	注解
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ -	2048	运算符'-'入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ - (	2048	左括号入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ - (	2048 5	操作数5入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ - (!	2048 5	运算符'!'入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ - (	2048 120	运算符'!'出栈执行
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ - (-	2048 120	运算符'-'入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ - (-	2048 120 67	操作数67入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ - (	2048 53	运算符'-'出栈执行
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ - (-	2048 53	运算符'-'入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ - (- (	2048 53	左括号入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ - (- (	2048 53 8	操作数8入栈

表达式	运算符栈	操作数栈	注解
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ - (- (+	2048 53 8	运算符'+'入栈
(0!+1)*2^(3!+4)-(5!-67-(8+**9**))$	$ - (- (+	2048 53 8 9	操作数9入栈
(0!+1)*2^(3!+4)-(5!-67-(8+9**)**)$	$ - (- (	2048 53 17	运算符'+'出栈执行
(0!+1)*2^(3!+4)-(5!-67-(8+9**)**)$	$ - (-	2048 53 17	左括号出栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ - (	2048 36	运算符'-'出栈执行
(0!+1)*2^(3!+4)-(5!-67-(8+9))$	$ -	2048 36	左括号出栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))**$**	$	2012	运算符'-'出栈执行
(0!+1)*2^(3!+4)-(5!-67-(8+9))**$**		2012	表达式起始标识出栈
(0!+1)*2^(3!+4)-(5!-67-(8+9))$			返回唯一的元素2012

栈与队列

逆波兰表达式：定义与求值

04-G1

日月逝矣，岁不我与

碰见我爱人的面，要替我说到

说我在这儿，依旧在为她颠倒

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Reverse Polish Notation

- 逆波兰表达式：J. Lukasiewicz (1878 ~ 1956)
- 在由运算符（operator）和操作数（operand）组成的表达式中

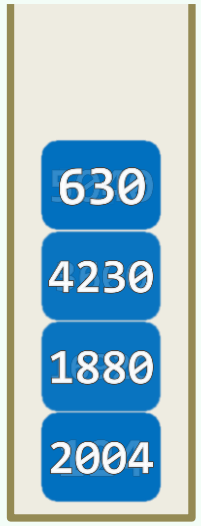
  不使用括号（parenthesis-free），即可表示带优先级的运算关系
- 例如：0 ! + 123 + 4 * (5 * 6 ! + 7 ! / 8) / 9

  0 ! 123 + 4 5 6 ! * 7 ! 8 / + * 9 / +
- 又如：(0 ! + 1) ^ (2 * 3 ! + 4 - 5) - 6 ! / (7 + 8 + 9)

  0 ! 1 + 2 3 ! * 4 + 5 - ^ 6 ! 7 8 + 9 + / -
- 相对于日常使用的中缀式（infix），RPN亦称作后缀式（postfix）
- 作为补偿，须额外引入一个起分隔作用的元字符（比如空格） //较之原表达式，未必更短

栈式求值

0	!	123	+	4	5	6	!	*	7	!	8	/	+	*	9	/	+



- 引入栈S //存放操作数

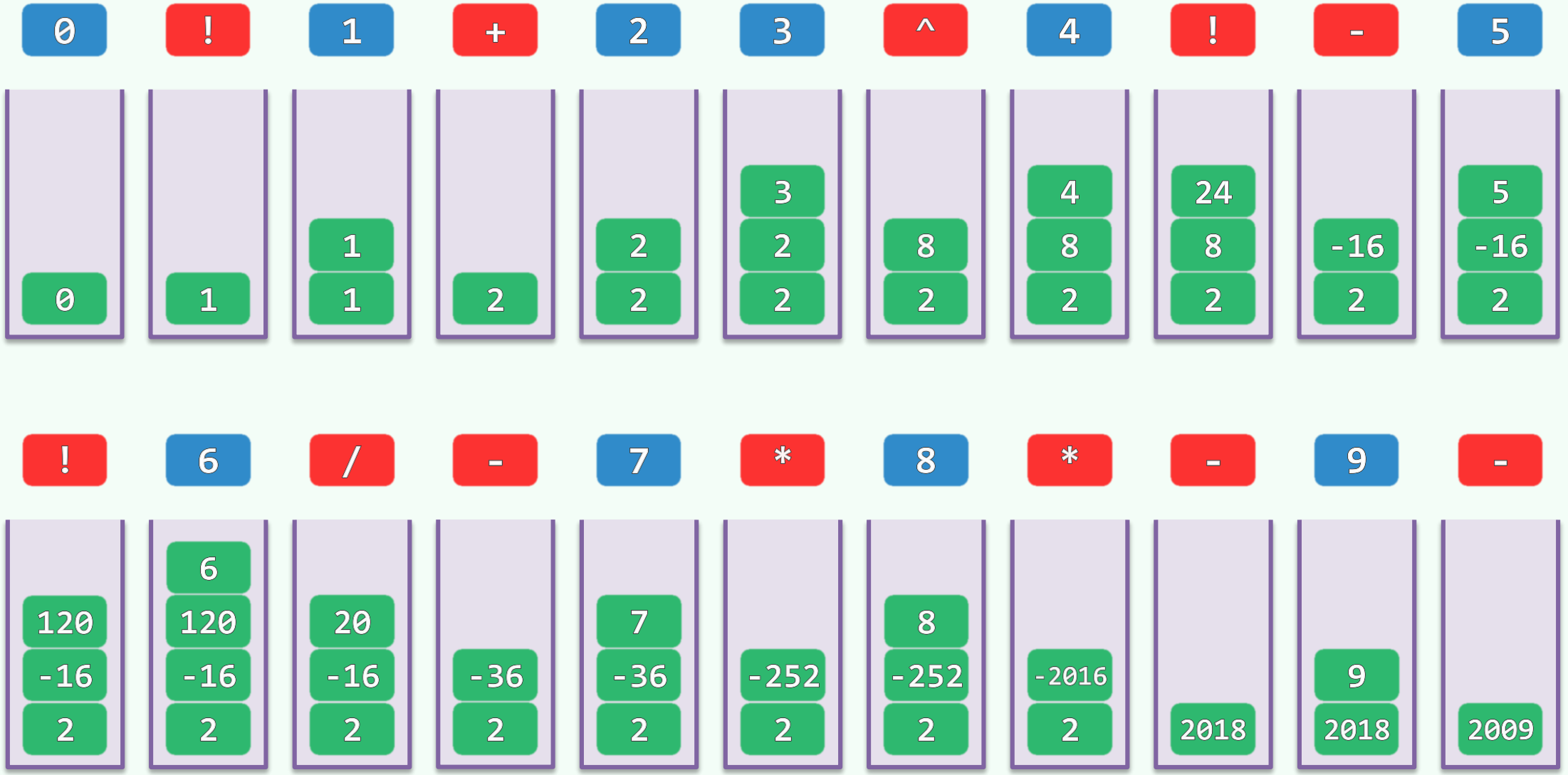
  逐个处理下一元素x

  ```
  if ( x是操作数 ) 将x压入S
  else //运算符无需缓冲
      从S中弹出x所需数目的操作数
      执行相应的计算，结果压入S //无需顾及优先级！
  ```

  返回栈顶

- 只要输入的RPN语法正确，此时的栈顶亦是栈底，对应于最终的计算结果

0 ! 1 + 2 3 ^ 4 ! - 5 ! 6 / - 7 * 8 * - 9 -


0 ! 1 + 2 3 ^ 4 ! - 5
0 1 1 2 2 2 3 2 8 2 4 8 2 24 8 2 -16 2 5 -16 2
! 6 / - 7 * 8 * - 9 -
120 -16 2 6 120 -16 2 20 -16 2 -36 2 7 -36 2 -252 2 8 -252 2 -2016 2 2018 9 2018 2009

04-G2

栈与队列

逆波兰表达式：转换

知与之为取，政之宝也

巨大的需求源于巨大的财富，而且要得到我们想要的，最好的办法往往是舍弃我们已拥有的。

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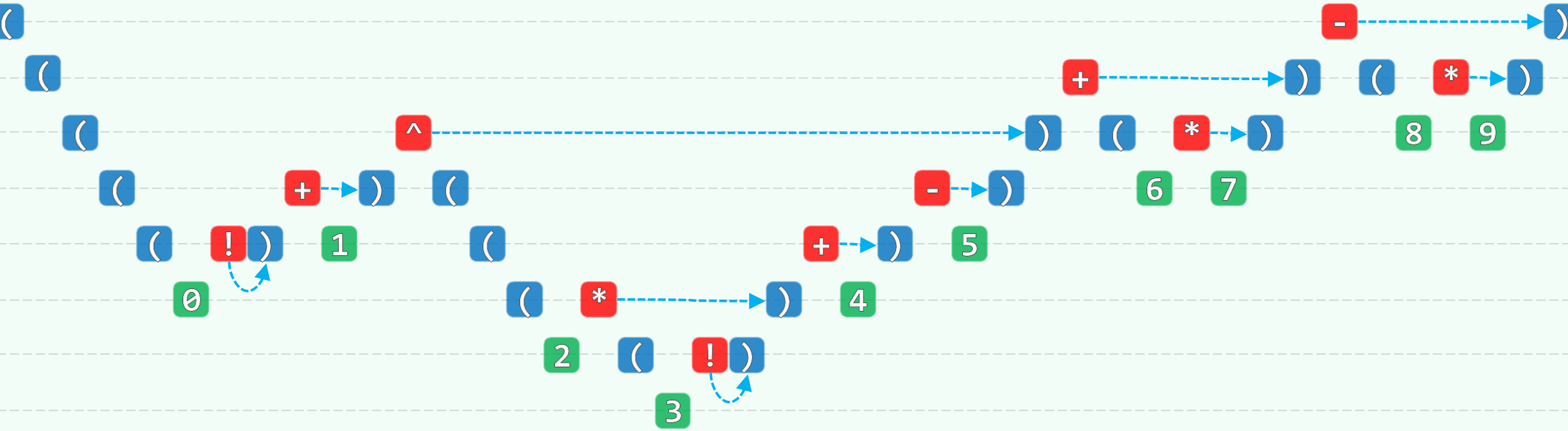
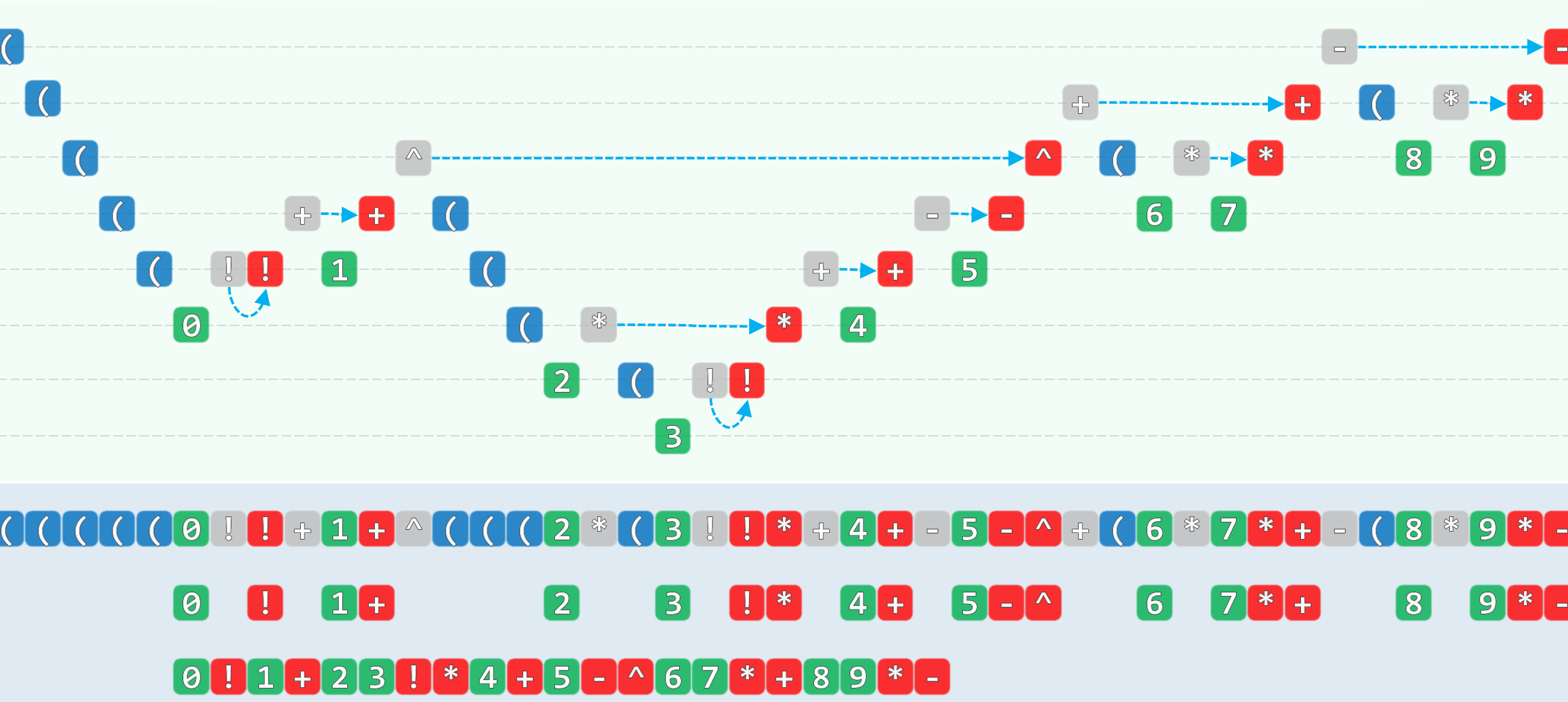
手工转换：添加括号



(0 ! + 1) ^ (2 * 3 ! + 4 - 5) + 6 * 7 - 8 * 9

手工转换：以运算符替换右括号，清除左括号



自动转换

```
double evaluate( char* S, char* RPN ) { //RPN转换
   /* ..................................... */
   while ( ! optr.empty() ) { //逐个处理各字符，直至运算符栈空
      if ( isdigit( * S ) ) //若当前字符为操作数，则直接
         { readNumber( S, opnd ); append( RPN, opnd.top() ); } //将其接入RPN
      else //若当前字符为运算符
         switch( priority( optr.top(), *S ) ) {
            /* ..................................... */
            case '>': { //且可立即执行，则在执行相应计算的同时
               char op = optr.pop(); append( RPN, op ); //将其接入RPN
               /* ..................................... */
            } //case '>'
         /* ..................................... */
```

04-G3

栈与队列

逆波兰表达式：PostScript

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Stack x 5

❖ PostScript诞生于1985

支持设备独立的图形描述

❖ （ 1个解释器 + 5个栈 ）× RPN语法

❖ operand stack：存放操作数及运算结果

❖ 提供基础且强大的图形功能

支持数据类型、变量、函数/宏 ...

❖ 一旦遇到操作符，则

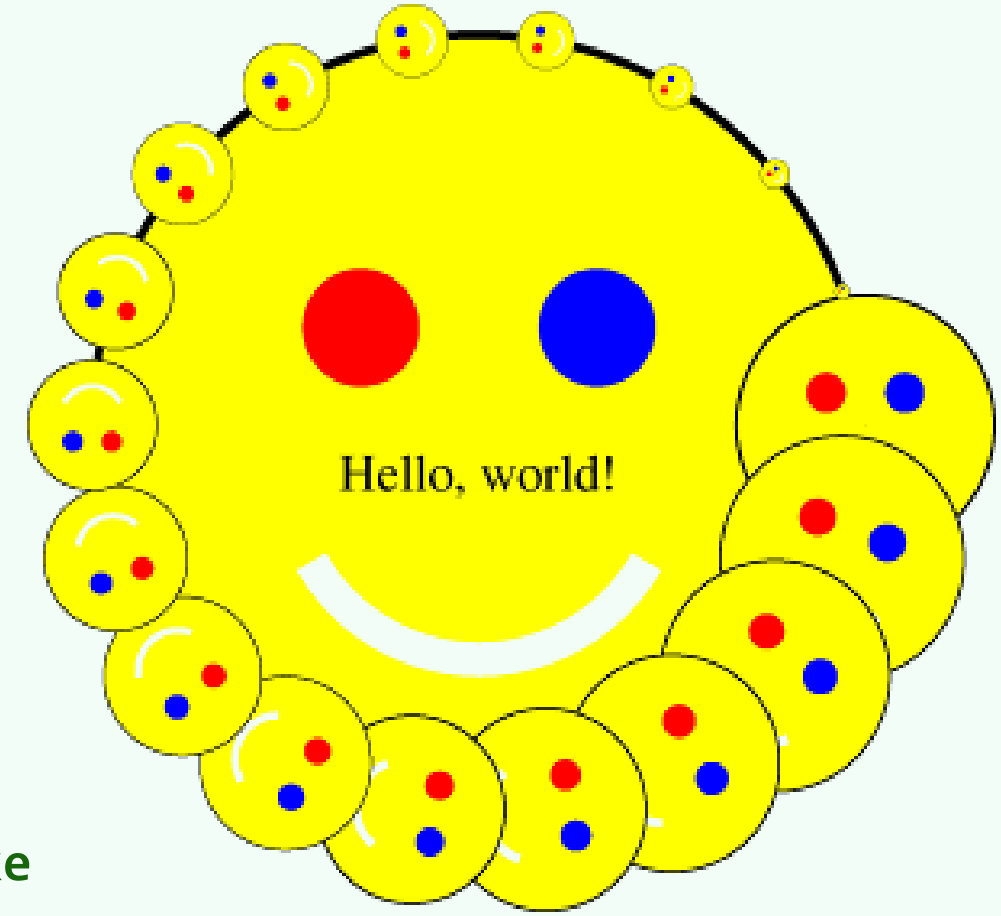
- 弹出相应数目的元素
- 实施计算，并
- 将（可能多个、一个或零个）结果入栈

❖ 实例

```
4 4 mul 5 5 mul add 7 mul 7 mul
```

实例

```
/smile {
    newpath
    gsave
    rotate
    0 translate
    180 div dup scale
    yellow   0 0 180 0 360 arc fill
    red      -55 45 27 0 360 arc fill
    blue     55 45 27 0 360 arc fill
    fatline  white 0 -18 90 210 330 arc stroke
    thinline black 0 0 180 0 360 arc stroke
    grestore
} def
```



04-H

栈与队列

队列接口与实现

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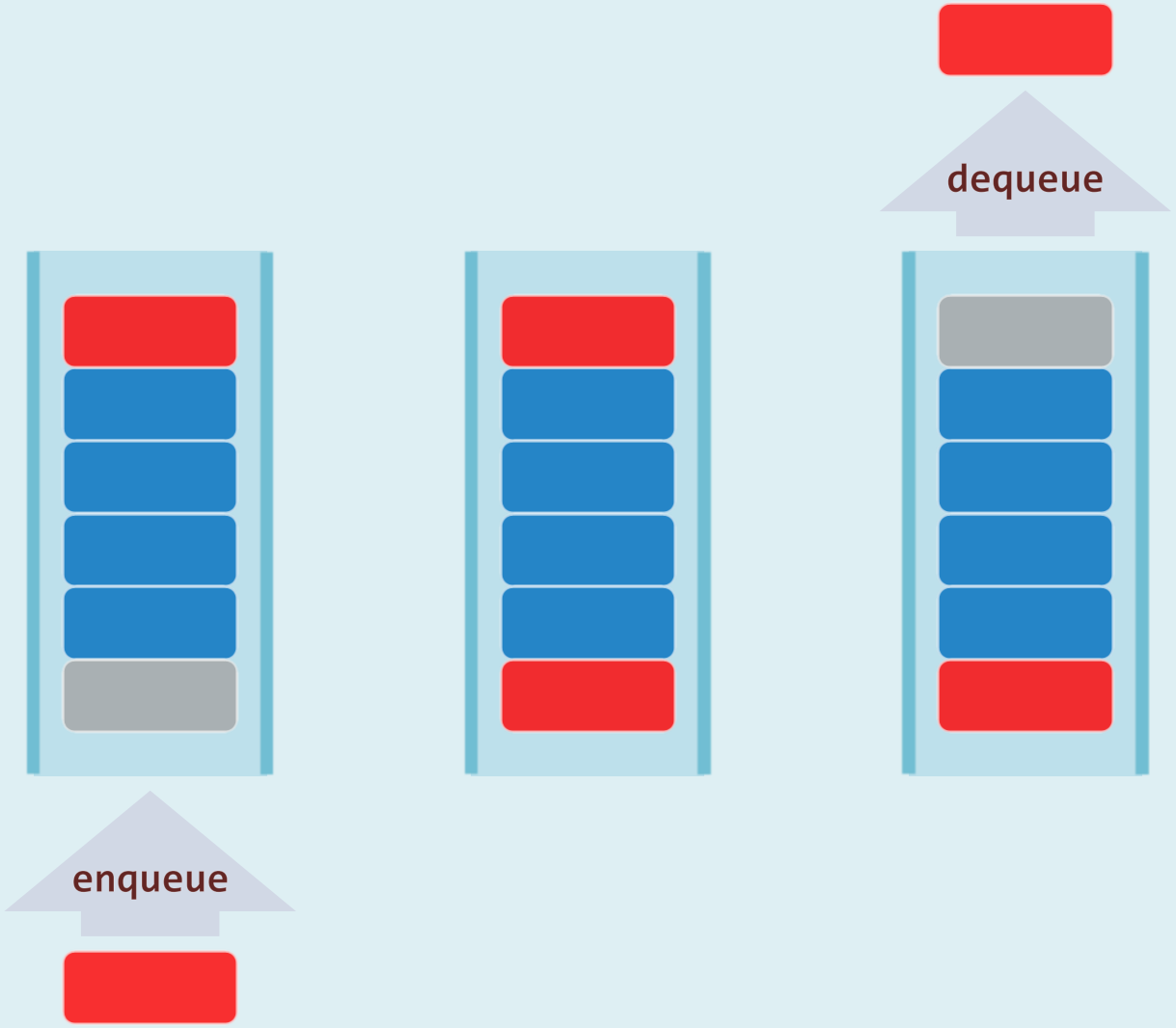
操作与接口

- 队列（queue）也是受限的序列
- 只能在队尾插入（查询）：
  - enqueue() / rear()
- 只能在队头删除（查询）：
  - dequeue() / front()
- 先进先出（FIFO）

  后进后出（LILO）
- 扩展接口：getMax()...



实例

操作	输出	队列（右侧为队头）			
Queue()					
empty()	true				
enqueue(5)		5			
enqueue(3)		3	5		
dequeue()	5	3			
enqueue(7)		7	3		
enqueue(3)		3	7	3	
front()	3	3	7	3	
empty()	false	3	7	3	

操作	输出	队列（右侧为队头）					
enqueue(11)		11	3	7	3		
size()	4	11	3	7	3		
enqueue(6)		6	11	3	7	3	
empty()	false	6	11	3	7	3	
enqueue(7)		7	6	11	3	7	3
dequeue()	3	7	6	11	3	7	
dequeue()	7	7	6	11	3		
front()	3	7	6	11	3		
size()	4	7	6	11	3		

实现

❖ 队列既然属于序列的特例，故亦可直接基于向量或列表派生

❖
```
template <typename T> class Queue: public List<T> {
  public: //原有接口一概沿用
     void enqueue( T const & e ) { insertAsLast( e ); } //入队
     T dequeue() { return remove( first() ); } //出队
     T & front() { return first()->data; } //队首
  }; //以列表首/末端为队列头/尾——颠倒过来呢?
```

❖ 确认：如此实现的队列接口，均只需$O(1)$时间

❖ 课后：基于向量派生实现队列模板类，并就其效率做一评估

04-1

栈与队列

队列应用

墙上一溜挂着五个烟斗。张大哥不等旧的已经不能再用才买新的，
而是使到半路就买个新的来；新旧替换着用，能多用些日子。

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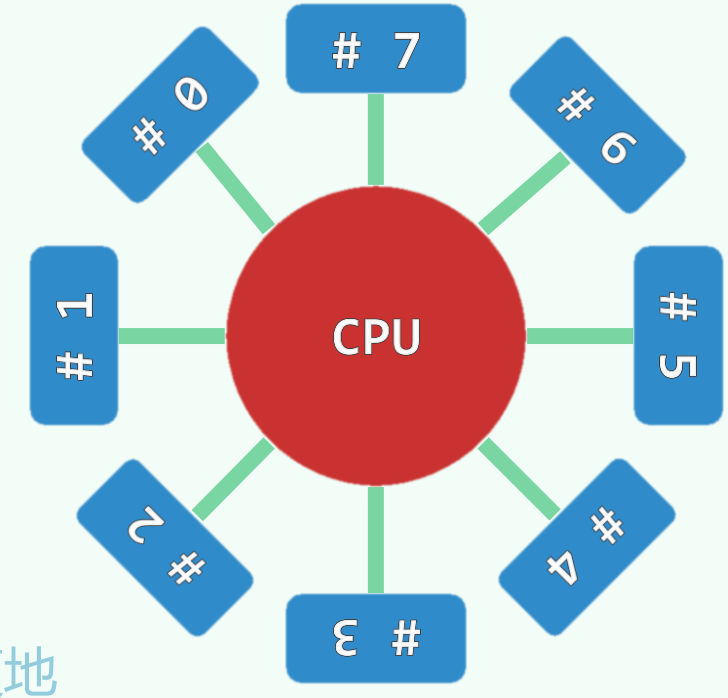
资源循环分配

❖ 一组客户（client）共享同一资源时，如何兼顾公平与效率？

比如，多个应用程序共享CPU，实验室成员共享打印机，...



❖ RoundRobin //循环分配器

```
Queue Q( clients ); //共享资源的所有客户组成队列
while ( ! ServiceClosed() ) //在服务关闭之前，反复地
    e = Q.dequeue(); //令队首的客户出队，并
    serve( e ); Q.enqueue( e ); //接受服务，然后重新入队
```

银行服务模拟：模型

- 提供n个服务窗口
  - 任一时刻，每个窗口至多接待一位顾客

    其他顾客排队等候
  - 顾客到达后，自动地

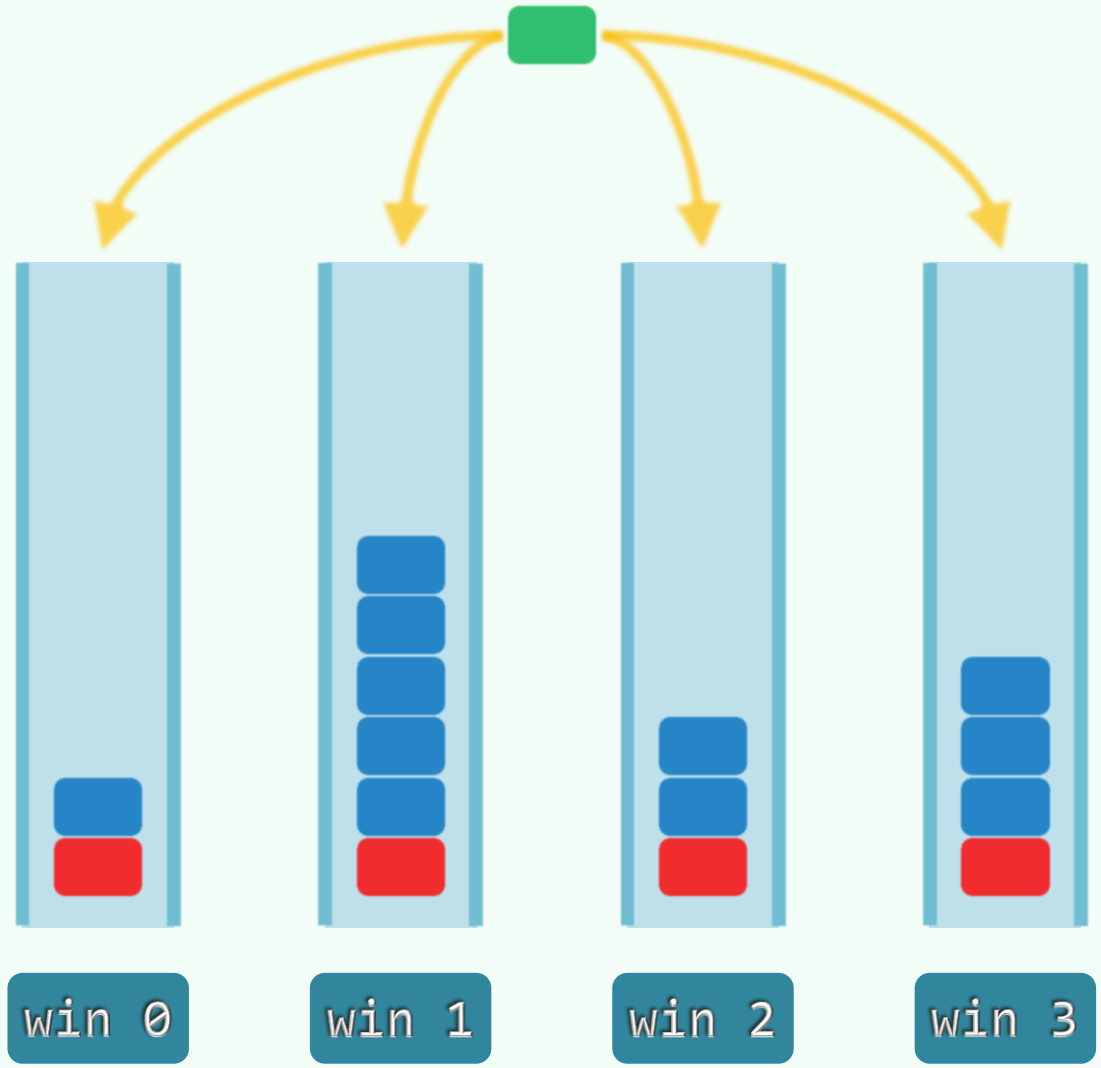
    选择和加入最短队列（的末尾）
- 参数： nWin //窗口（队列）数目

        servTime //营业时长
- struct Customer { //顾客类

    int window; //所属窗口（队列）

    unsigned int time; //服务时长

  };



银行服务模拟：实现

```
void simulate( int nWin, int servTime ) {
   Queue<Customer> * windows = new Queue<Customer>[ nWin ];
   for ( int now = 0; now < servTime; now++ ) { //在下班之前，每隔单位时间
      Customer c ; c.time = 1 + rand() % 50; //一位新顾客到达，其服务时长随机指定
      c.window = bestWindow( windows, nWin ); //找出最佳（最短）服务窗口
      windows[ c.window ].enqueue( c ); //新顾客加入对应的队列
      for ( int i = 0; i < nWin; i++ ) //分别检查
         if ( ! windows[ i ].empty() ) //各非空队列
            if ( -- windows[ i ].front().time <= 0 ) //队首顾客接受服务
               windows[ i ].dequeue(); //服务完毕则出列，由后继顾客接替
   } //for
   delete [] windows; //释放所有队列
}
```

04-J

栈与队列

直方图内最大矩形

就这么着，我有了一所严丝密缝、涂抹灰泥的木板房子，七英尺宽，十五英尺长，立柱有八英尺高...

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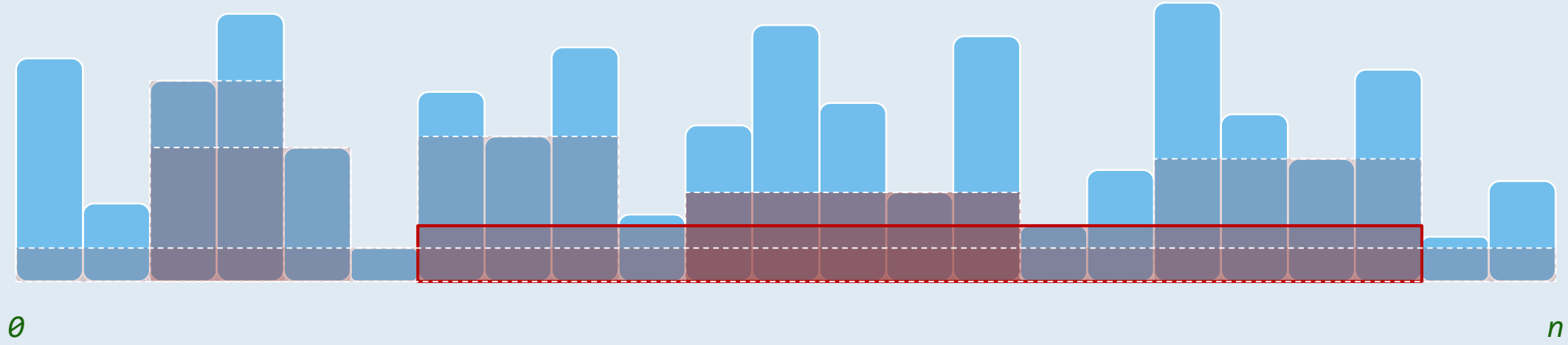
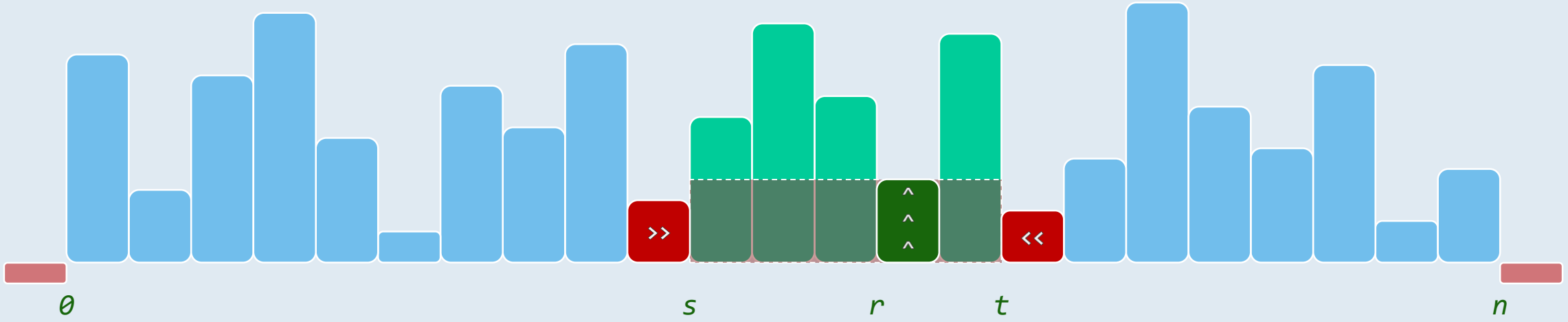
Maximum Rectangle



- Let H[0,n) be a histogram of non-negative integers
- How to find the largest orthogonal rectangle in H[]?
- To eliminate possible ambiguity

  we can, for example, choose the **LEFTMOST** one

Maximal Rectangles



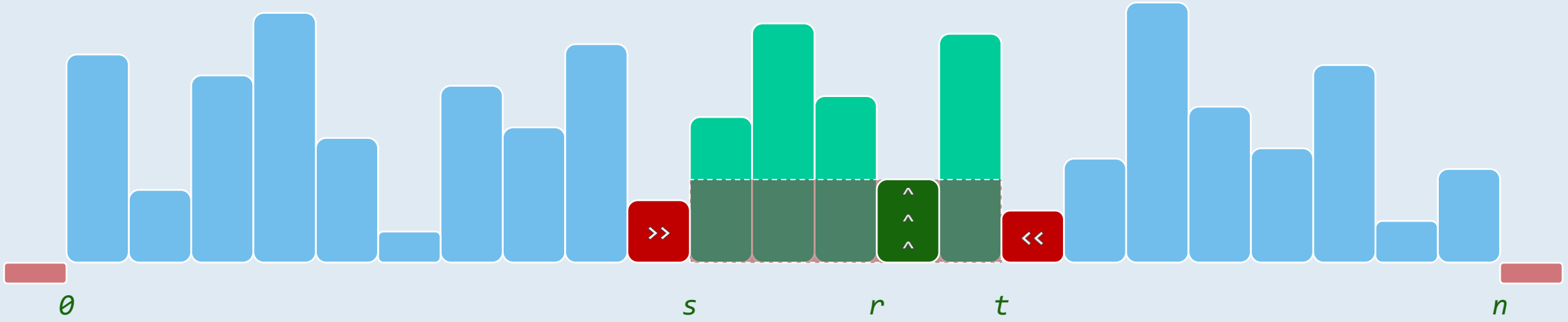
❖ **Maximal rectangle supported by H[r]:** $maxRect(r) = H[r] \cdot (t(r) - s(r))$

where

$$s(r) = \max\{\, k \mid 0 \le k \le r \text{ and } H[k-1] < H[r] \,\}$$

$$t(r) = \min\{\, k \mid r < k \le n \text{ and } H[r] > H[k] \,\}$$

Brute-force



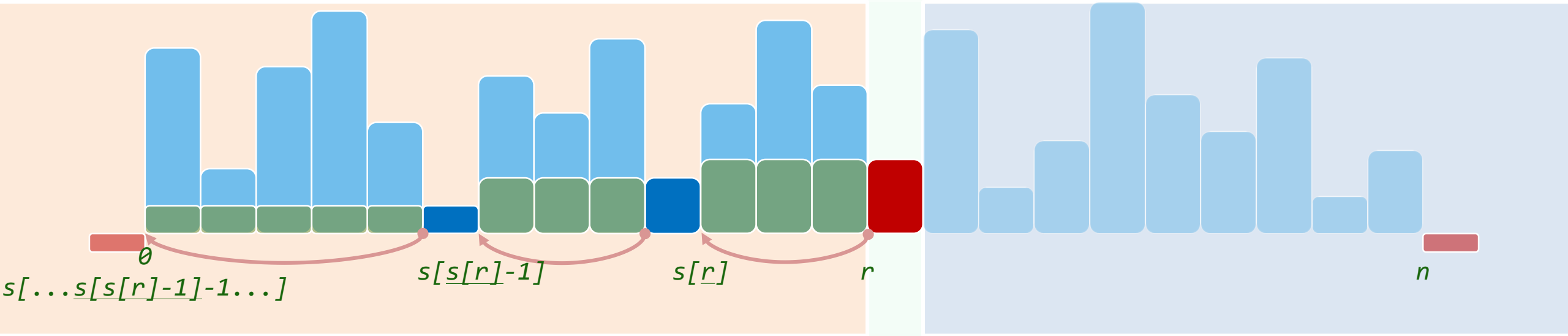
- Determining s(r) and t(r) for all r's requires $\mathcal{O}(n^2)$ time

$$s(r) = \max\{\ k \mid 0 \le k \le r \text{ and } H[k-1] < H[r]\ \}$$

$$t(r) = \min\{\ k \mid r < k \le n \text{ and } H[r] > H[k]\ \}$$

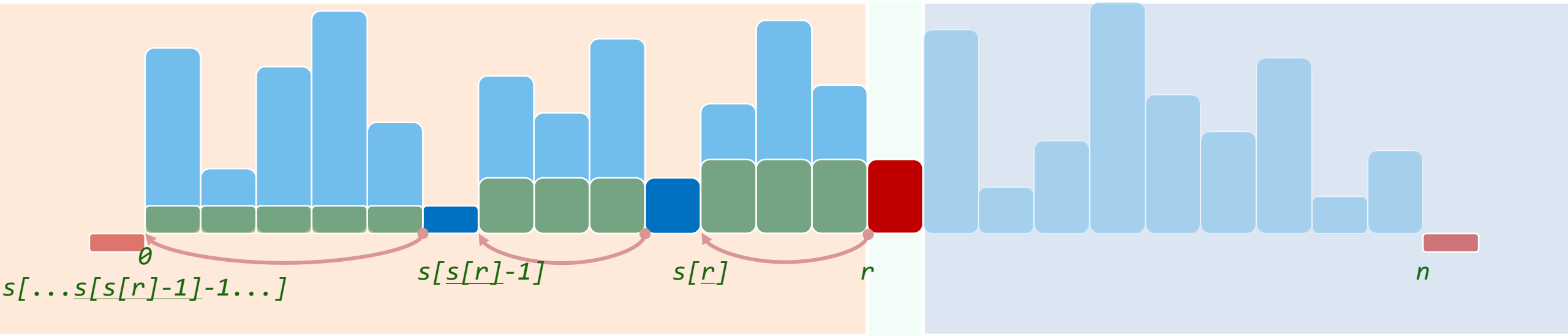
- Actually, all s(r)'s can be determined by a LINEAR scan of the histogram ...

Using Stack: Algorithm




```
Rank* s = new Rank[n]; Stack<Rank> S;
for ( Rank r = 0; r < n; r++ ) //try using SENTINEL for simplicity by yourself
   while ( !S.empty() && ( H[S.top()] >= H[r] ) ) S.pop(); //until H[top] < H[r]
   s[r] = S.empty() ? 0 : 1 + S.top();   S.push(r); //S is always ASCENDING
while( !S.empty() ) S.pop();
```

Using Stack: Loop Invariant & Correctness



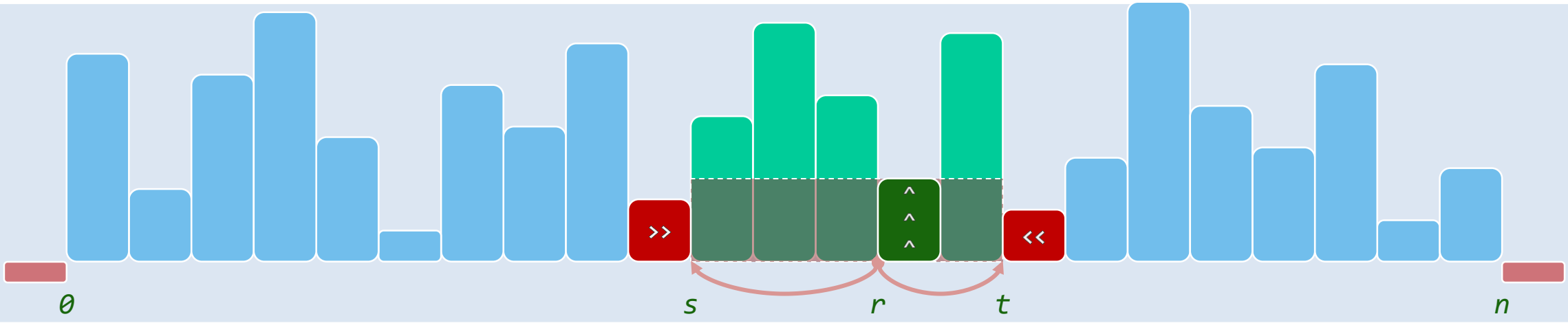
- All bars are scanned and pushed into S in turn
- After each iteration of the outer loop, S stores a chain in terms of s[]

$$S[S.size()-1] = S.top() = r \quad \textbf{and} \quad \forall\, 0 \le k < S.size(),\ \ S[k-1]+1 = s[S[k]]$$

- Every bar is popped when it will be of no use thereafter, i.e.

$$\forall\, 0 \le k < r \ \ but \ \ k \notin S, \quad \nexists\, r \le t \ \ s.t. \ \ k+1 = s(t)$$

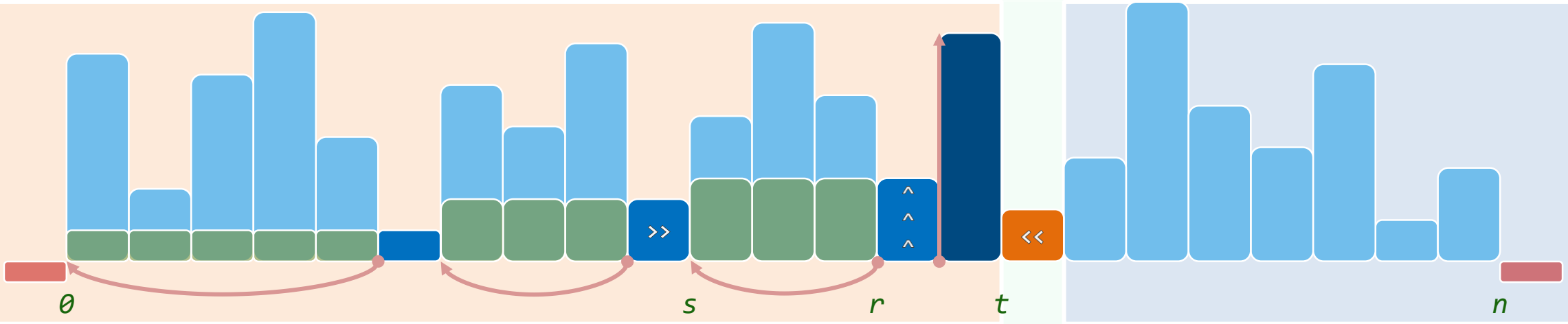
Using Stack: Complexity



- And t(r)'s can be determined by another scan in the REVERSED direction
- Hence all maximal rectangles can be computed in $O(n)$ time and using $O(n)$ space
- However, what if the histogram is given in an IN-PLACE and ON-LINE manner?

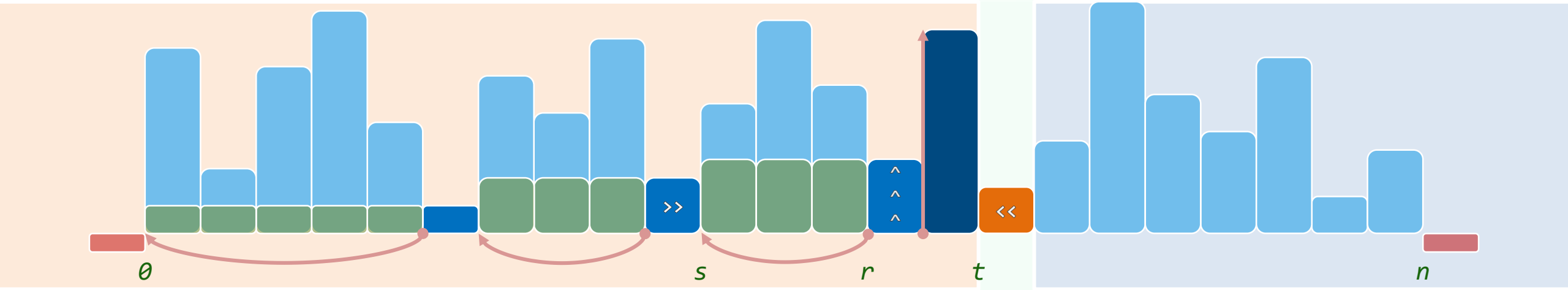
  Note that, the t(r)'s CAN'T be determined until the ENTIRE input is ready
- Is it possible to compute BOTH s(r)'s and t(r)'s by a SINGLE scan? //on-fly

One-Pass Scan: Algorithm




```
Stack<Rank> SR;  __int64 maxRect = 0; //SR.2ndTop() == s(r)-1 & SR.top() == r
for ( Rank t = 0; t <= n; t++ ) //amortized-O(n)
   while ( !SR.empty() && ( t == n || H[SR.top()] > H[t] ) )
      Rank r = SR.pop(), s = SR.empty() ? 0 : SR.top() + 1;
      maxRect = max( maxRect, H[r] * ( t - s ) );
   if ( t < n ) SR.push( t );
return maxRect;
```

One-Pass Scan: Loop Invariant & Correctness



❖ Again, at each iteration of the outer loop, we always have

$$\forall\, 0 \le k < SR.size(),\ \ SR[k-1] + 1 = s[SR[k]]$$

❖ For each bar r popped in the inner loop, we have

$$t(r) = t \quad \textbf{and} \quad s[r] = SR.top() + 1$$

04-K

栈与队列

Steap + Queap

A hero is born among a hundred, a wise man is found among a thousand, but an accomplished one might not be found even among a hundred thousand men.

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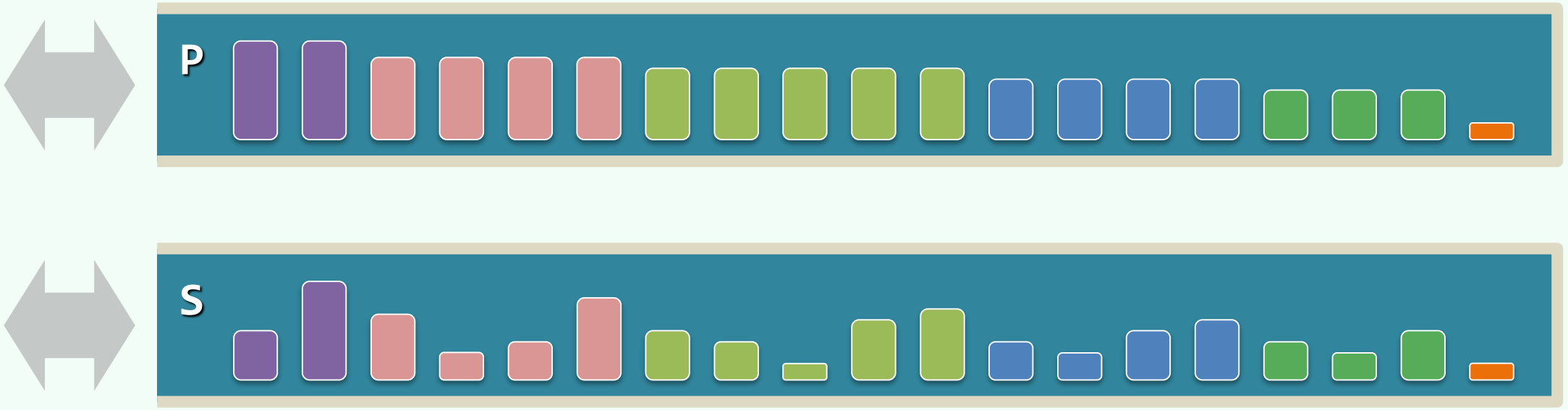
Steap = Stack + Heap = push + pop + getMax

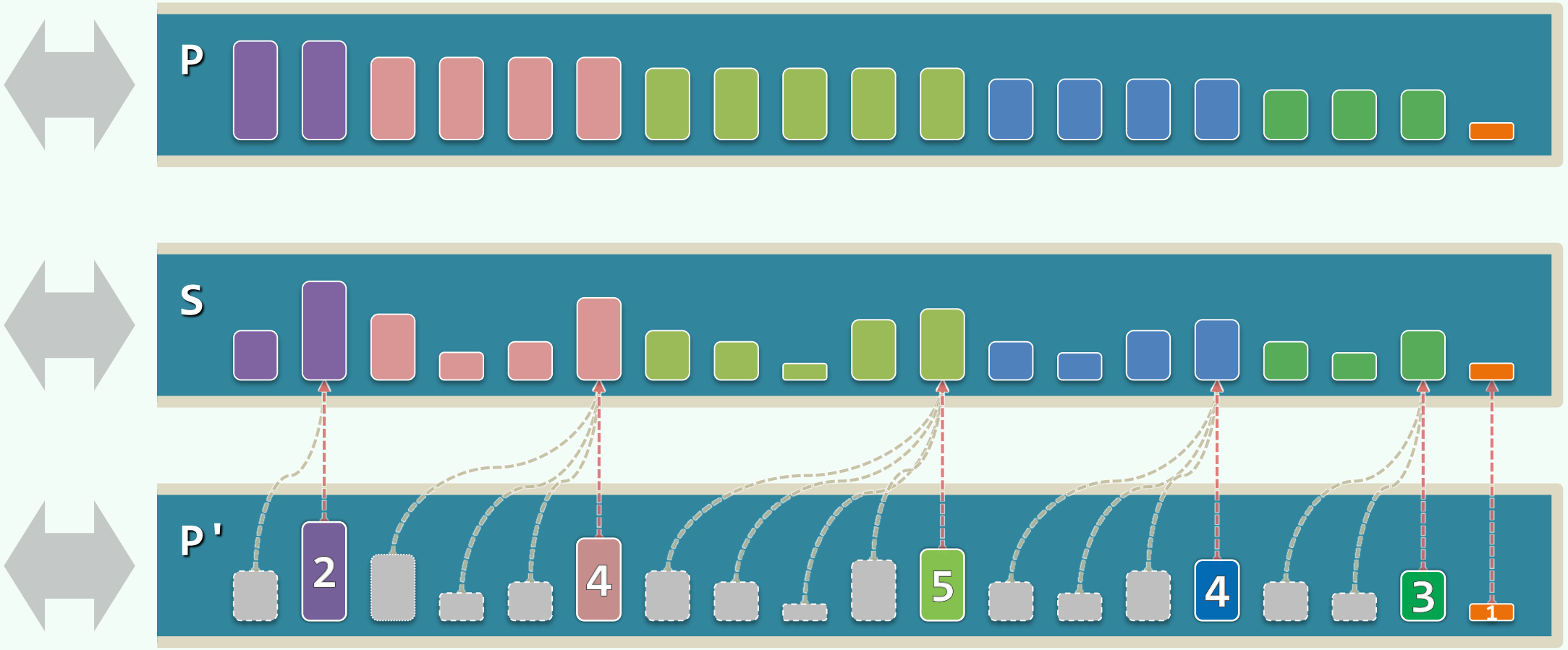
❖ P中每个元素，都是S中对应后缀里的最大者：Steap::getMax() { return P.top(); }



❖ Steap::pop() { P.pop(); return S.pop(); } //O(1)

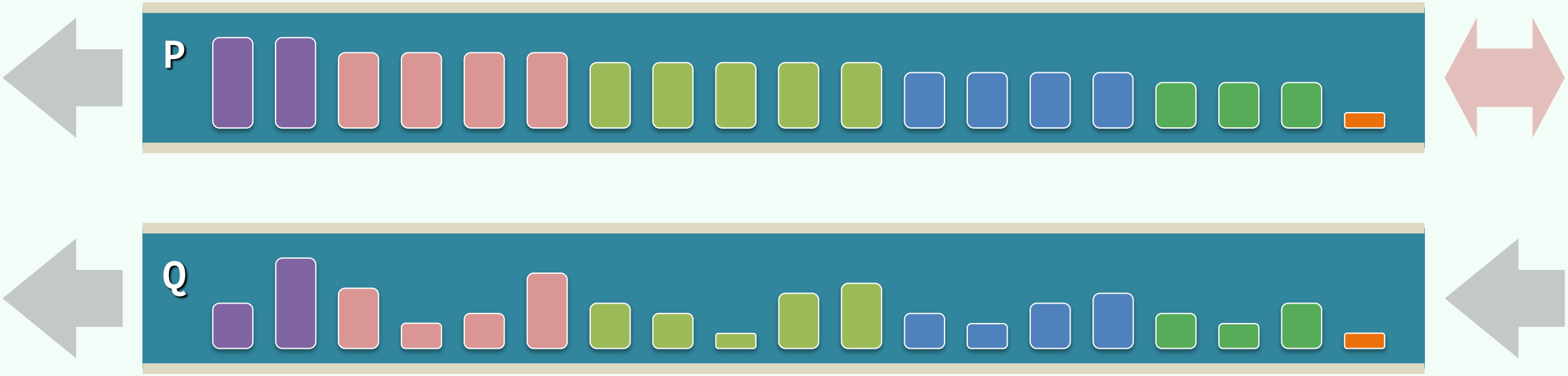
❖ Steap::push(e) { P.push(max(e, P.top())); S.push(e); } //O(1)

Steap = Stack + Heap = push + pop + getMax



Queap = Queue + Heap = enqueue + dequeue + getMax

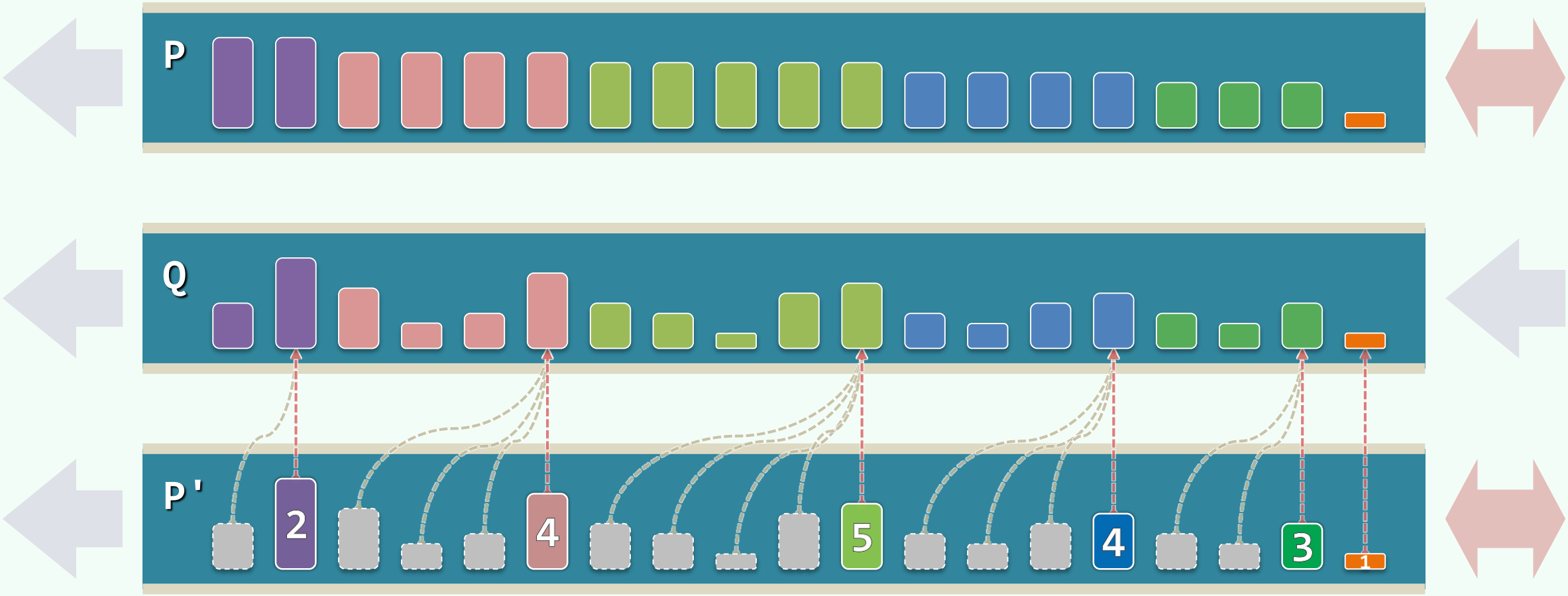
❖ Queap::dequeue() { P.dequeue(); return Q.dequeue(); } //O(1)



❖ Queap::enqueue(e) {

```
    Q.enqueue(e); P.enqueue(e);
    for ( x = P.rear(); x && (x->key <= e); x = x->pred ) //最坏情况O(n)
        x->key = e;
}
```

Queap = Queue + Heap = enqueue + dequeue + getMax



04-L

栈与队列

双栈当队

不恒其德，或承之羞，贞吝

我们也不用当场付款，要了什么东西都由店家记在一个小账本上，每两星期结一次账。

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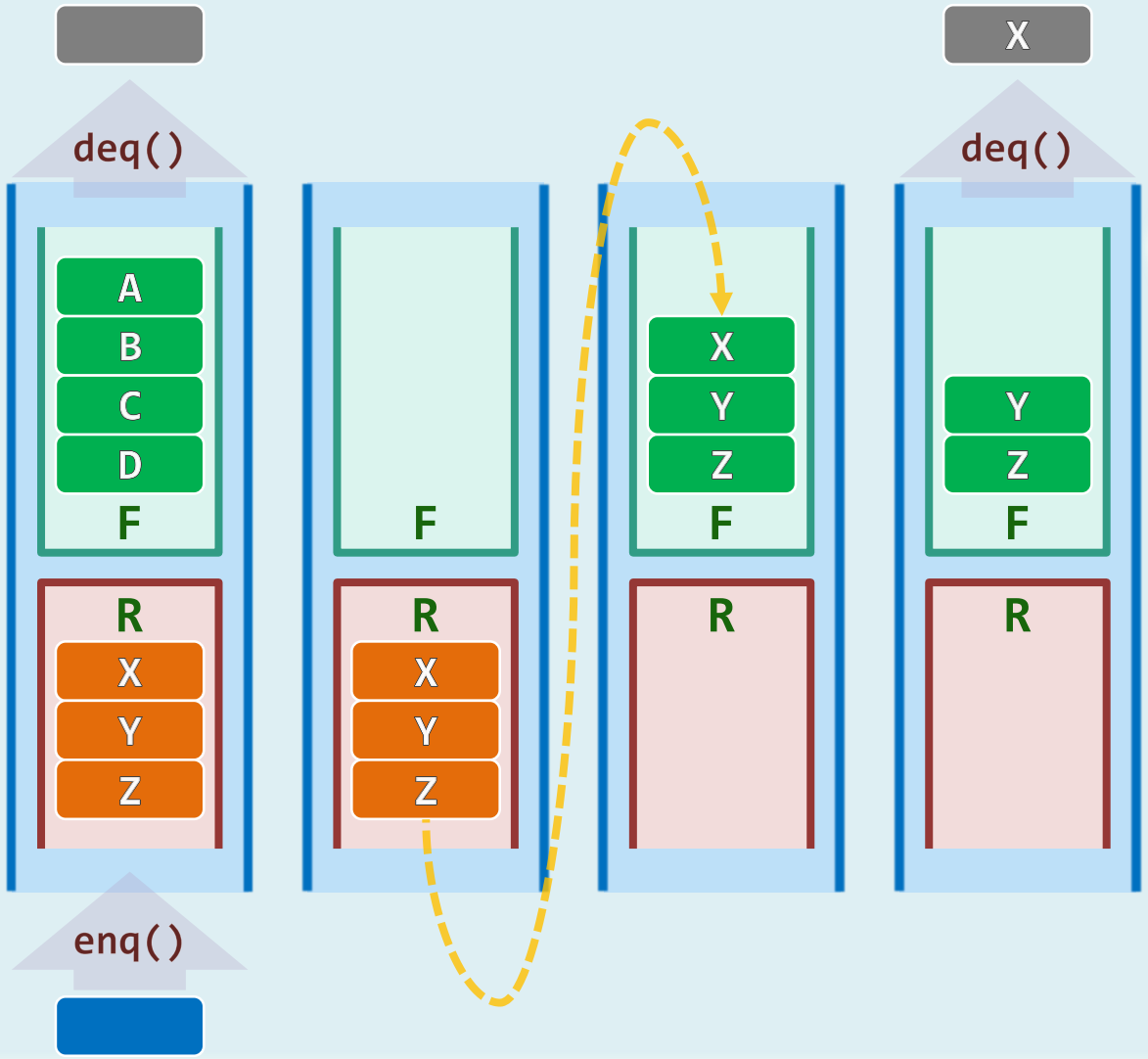
Queue = Stack x 2



- `def Q.enqueue(e)`

```
    R.push(e);
```

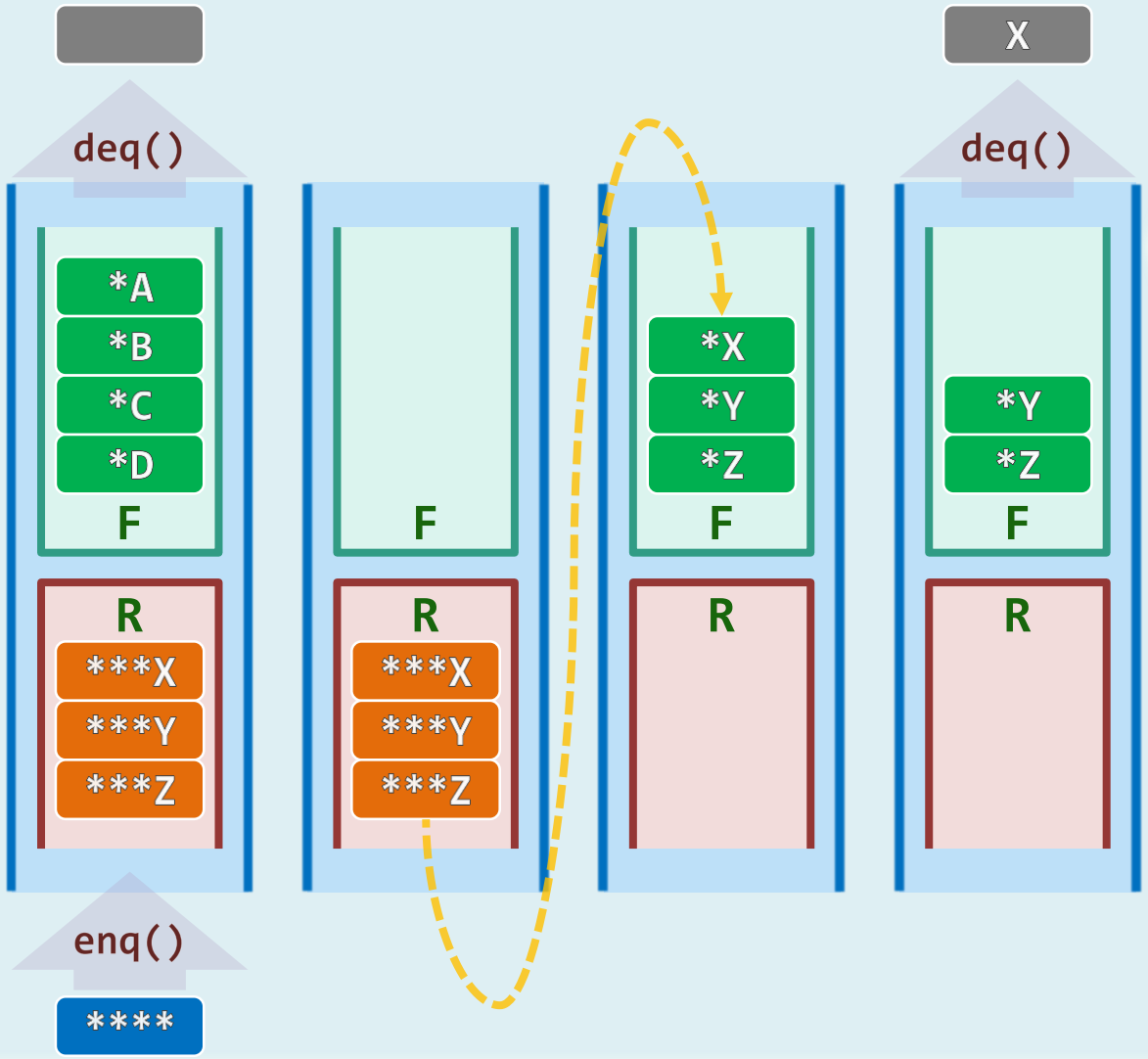
- `def Q.dequeue() # 0 < Q.size()`

```
    if ( F.empty() )
        while ( !R.empty() )
            F.push( R.pop() );
    return F.pop();
```

- Best/worst case: $O(1)/O(n)$

  Average? Amortization!

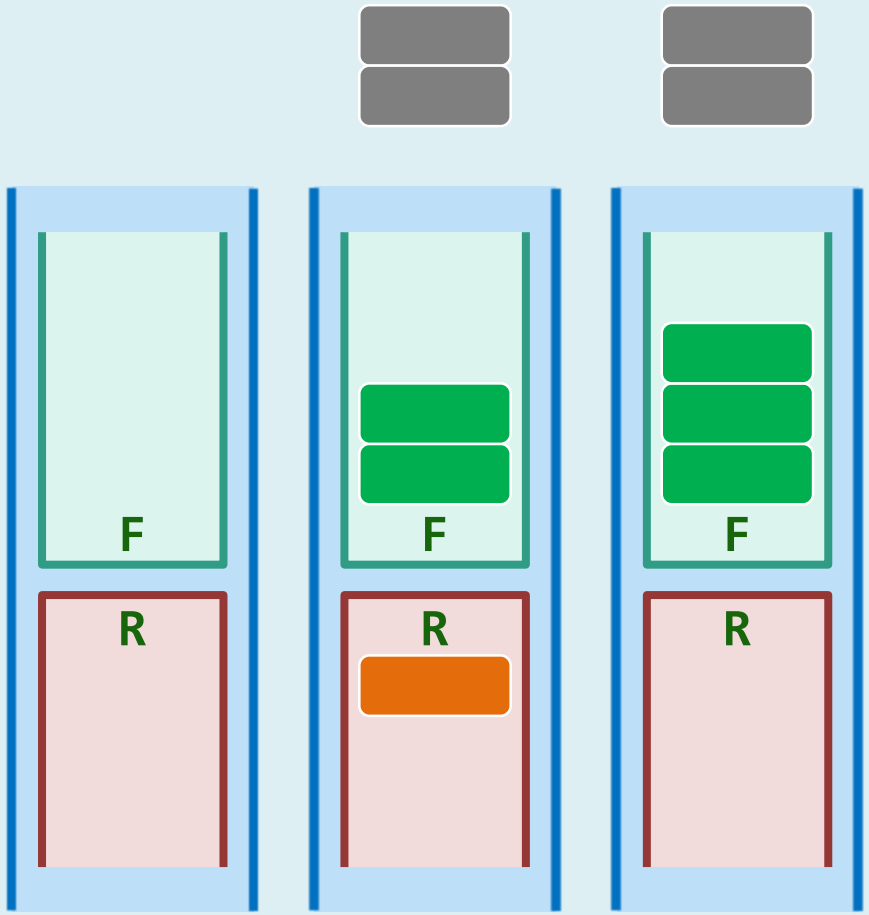
Amortization By Accounting



- Assign each new element with 4 coins //deposit
  - 1 for its enqueue()
  - 2 for transfer, and
  - the last 1 for dequeue()
- Hence every operation is pre-paid and ...
- The structure will never run out of credit
- Amortized cost of any operation sequence involving n ITEMs is

  $4n = O(n)$

Amortization By Aggregate



- Consider the moment when

  d dequeue()'s and **e** enqueue()'s

  have been done //d <= e

- The time cost

  for **ALL** the operations is

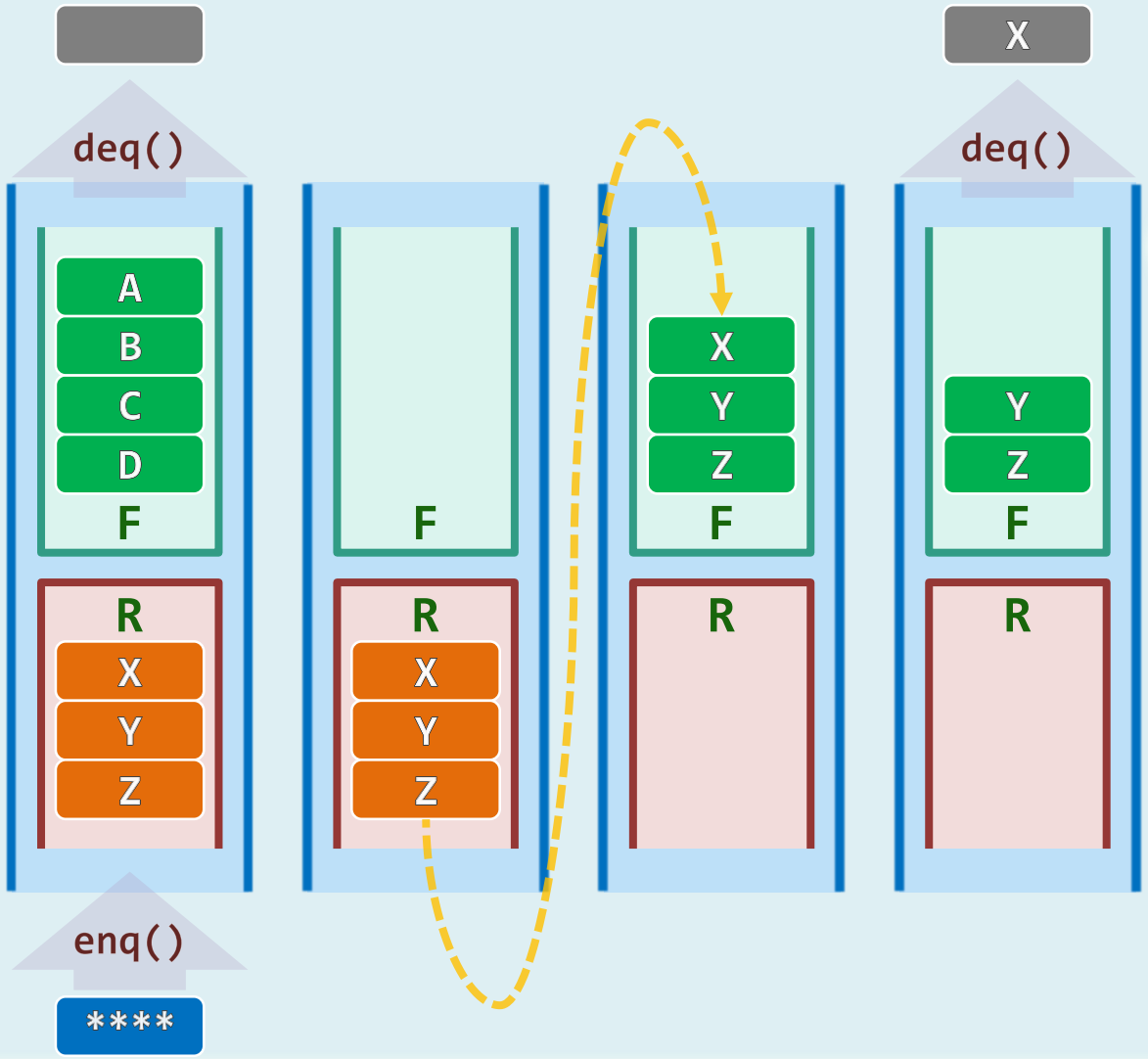
  $$\leq 4 \cdot d + 3 \cdot (e - d) = 3e + d$$

- The amortized cost

  for each OPERATION is

  $$\frac{3e + d}{e + d} < 3$$

Amortization By Potential



- Consider the k^{th} operation
- Define $\Phi_k = |R_k| - |F_k|$
- Then $A_k = T_k + \Phi_k - \Phi_{k-1} \equiv 2$
- Hence

$$2n \equiv \sum_{k=1}^{n} A_k = \sum_{k=1}^{n} T_k + \Phi_n - \Phi_0$$

$$2n = T(n) + \Phi_n - \Phi_0 > T(n) - n$$

$$T(n) < 3n = \mathcal{O}(n)$$