图

概述

10-A

黄药师道：“你拿了这图，到临安府找一家客店或是寺观住下，三月之后，我派人前来取回。图中一切，只许心记，不得另行抄录印摹。”

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基本术语

❖ G = (V; E)

vertex: n = |V|

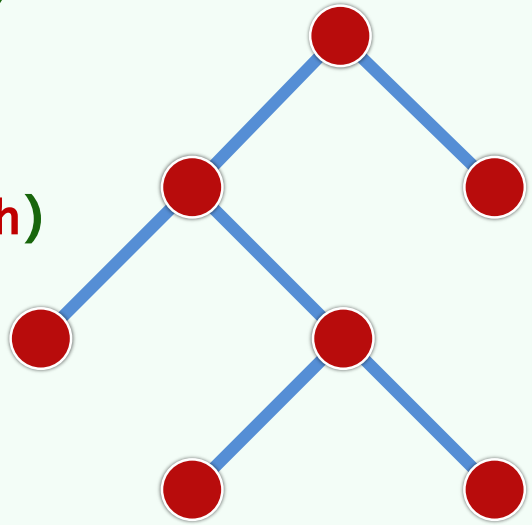
edge|arc: e = |E|

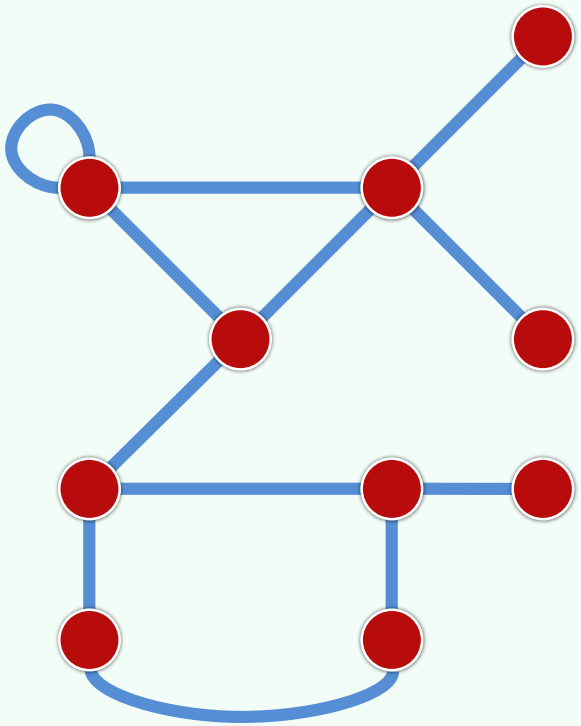
❖ 同一条边的两个顶点，彼此邻接（adjacency）

同一顶点自我邻接，构成自环（self-loop）

不含自环及重边，即为简单图（simple graph）

非简单（non-simple）图，暂不讨论





❖ 顶点与其所属的边，彼此关联（incidence）

度（degree/valency）：与同一顶点关联的边数

无向图 + 有向图

- 若邻接顶点u和v的次序无所谓

  则(u，v)为无向边（undirected edge）

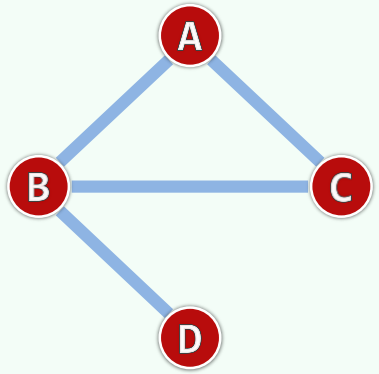
- 所有边均无方向的图，即无向图（undigraph）

- 反之，有向图（digraph）中均为有向边（directed edge）

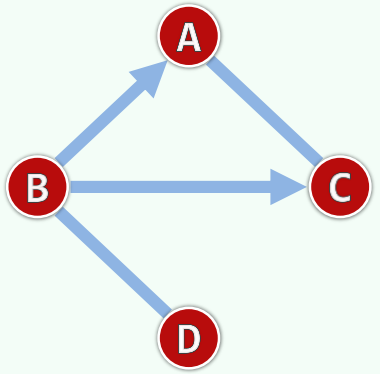
  u、v分别称作边(u，v)的尾（tail）、头（head）

- 无向边、有向边并存的图，称作混合图（mixed graph）

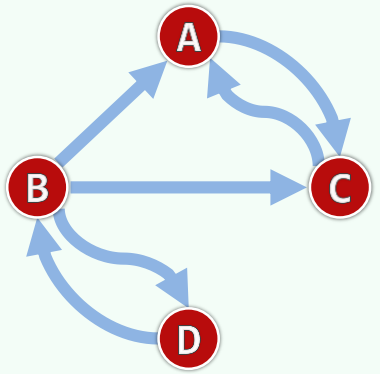
- 有向图通用性更强，故本章主要针对有向图介绍相关结构及算法



(a) undigraph



(b) mixed graph



(c) digraph

路径 + 环路

- 路径$\pi = \langle v_0, v_1, \ldots, v_k \rangle$

  长度$|\pi| = k$

- 简单路径：$v_i \neq v_j$ 除非 $i = j$

- 环/环路：$v_0 = v_k$

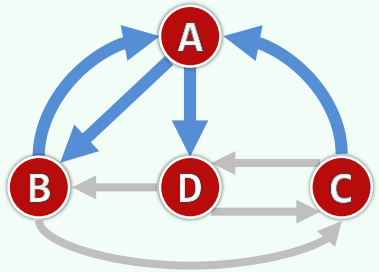
- 有向无环图（DAG）

- 欧拉环路：$|\pi| = |E|$

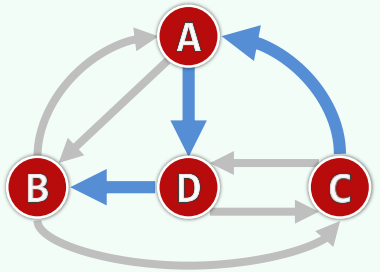
  各边恰好出现一次

- 哈密尔顿环路：$|\pi| = |V|$

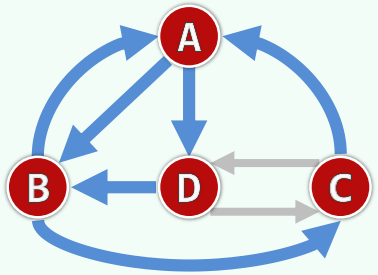
  各顶点恰好出现一次



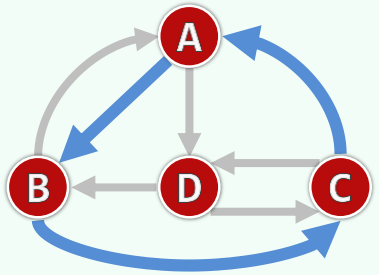
(i) path



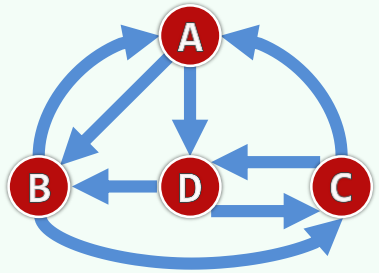
(ii) simple path



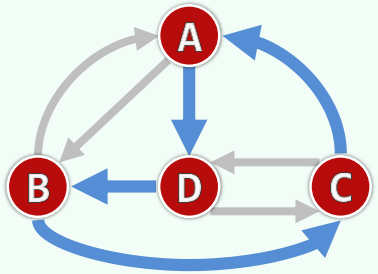
(i) cycle



(ii) simple cycle



(i) Eulerian tour

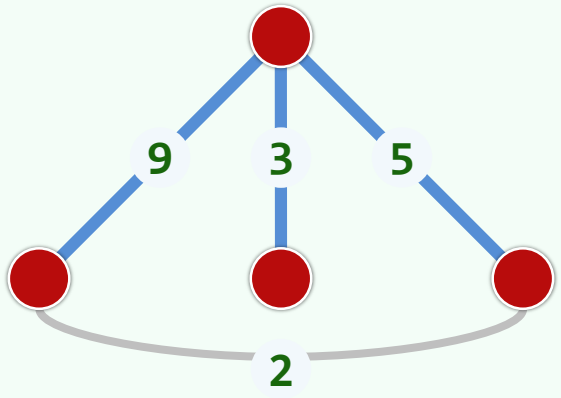


(ii) Hamiltonian tour

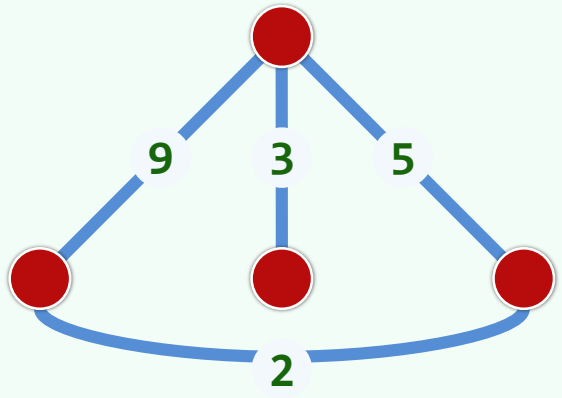
支撑树 + 带权网络 + 最小支撑树

❖ 图G = (V; E)的子图T = (V; F)若是树，即为其支撑树（spanning tree）

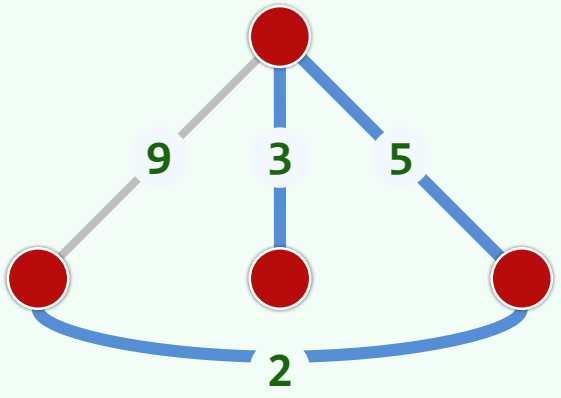
同一图的支撑树，通常并不唯一



spanning tree



weighted network
(triangle inequality?)



minimum spanning tree

❖ 各边e均有对应的权值wt(e)，则为带权网络（weighted network）

❖ 同一网络的支撑树中，总权重最小者为最小支撑树（MST）

10-B1

图

邻接矩阵：构思

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Graph模板类

```
template <typename Tv, typename Te> class Graph {
private: void reset() { //所有顶点、边的辅助信息复位
            for ( Rank v = 0; v < n; v++ ) { //顶点
               status(v) = UNDISCOVERED; dTime(v) = fTime(v) = -1;
               parent(v) = -1; priority(v) = INT_MAX;
               for ( Rank u = 0; u < n; u++ ) //边
                  if ( exists(v, u) ) type(v, u) = UNDETERMINED;
            } //for
         } //reset
public:  Rank n, e; //顶点、边数目
         /* ... 顶点操作、边操作、图算法：无论如何实现，接口必须统一 ... */
} //Graph
```

邻接矩阵 + 关联矩阵

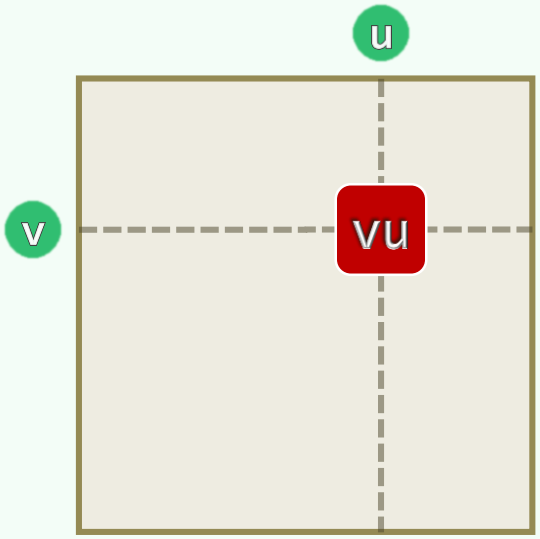
❖ adjacency matrix：记录顶点之间的邻接关系

——一对应：矩阵元素 ⇔ 图中可能存在的边

- A(v, u) = 1　（若顶点v与u之间存在一条边）
- = 0　（否则）

既然只考察简单图，对角线统一设置为0

空间复杂度为Θ(n²)，与图中实际的边数无关

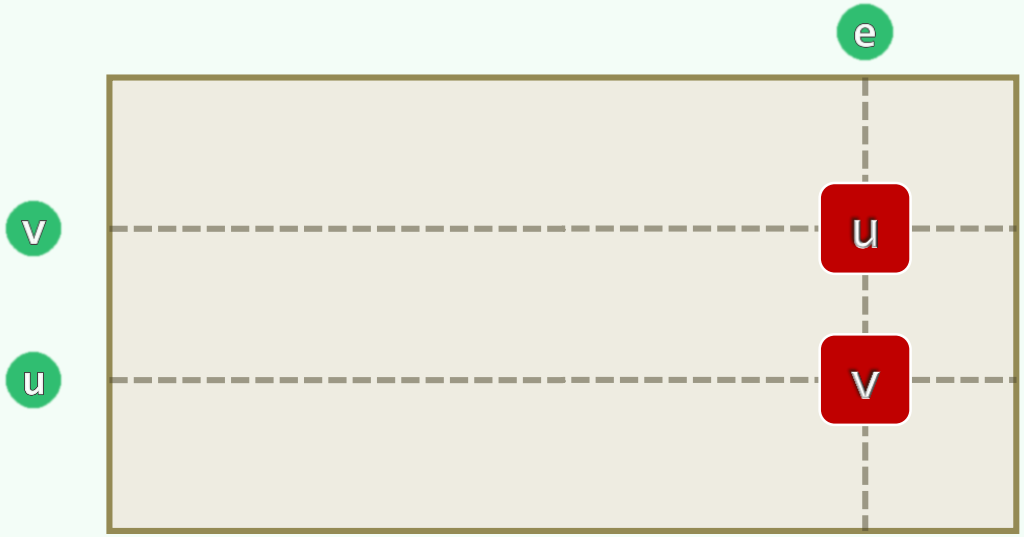


❖ incidence matrix：记录顶点与边之间的关联关系

空间复杂度为Θ(n*e) = 𝒪(n³)

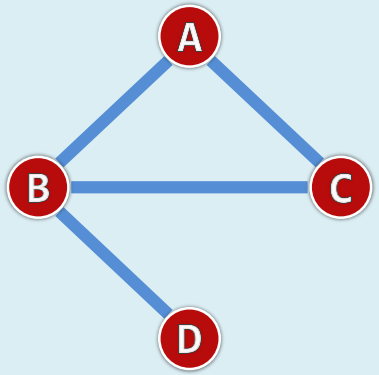
空间利用率 = 2e/ne = 2/n

解决某些问题时十分有效



实例

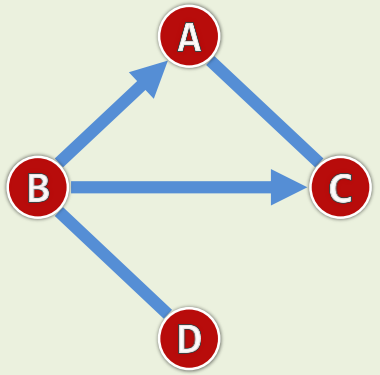
(a) undigraph



redundancy

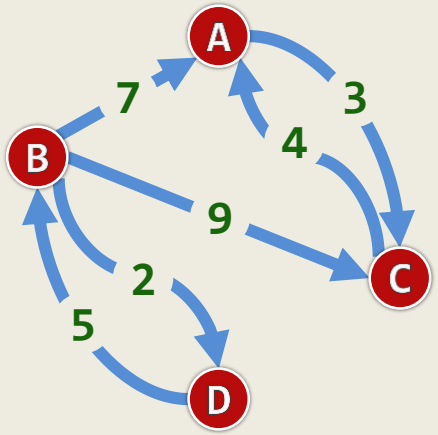
0	A	B	C	D
A		1	1	
B	1		1	1
C	1	1		
D		1		

(b) digraph



0	A	B	C	D
A			1	
B	1		1	1
C	1			
D		1		

(c) network



∞	A	B	C	D
A			3	
B	7		9	2
C	4			
D		5		

10-B2

图

邻接矩阵：模板实现

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Vertex

```
using VStatus = enum { UNDISCOVERED, DISCOVERED, VISITED };
template <typename Tv> struct Vertex { //不再严格封装
   Tv data; int inDegree, outDegree;
   VStatus status; //（如上三种）状态
   int dTime, fTime; //时间标签
   Rank parent; //在遍历树中的父节点
   int priority; //在遍历树中的优先级（最短通路、极短跨边等）
   Vertex( Tv const & d ) : //构造新顶点
      data( d ),  inDegree( 0 ),  outDegree( 0 ),  status( UNDISCOVERED ),
      dTime( -1 ),  fTime( -1 ),  parent( -1 ),    priority( INT_MAX ) {}
};
```

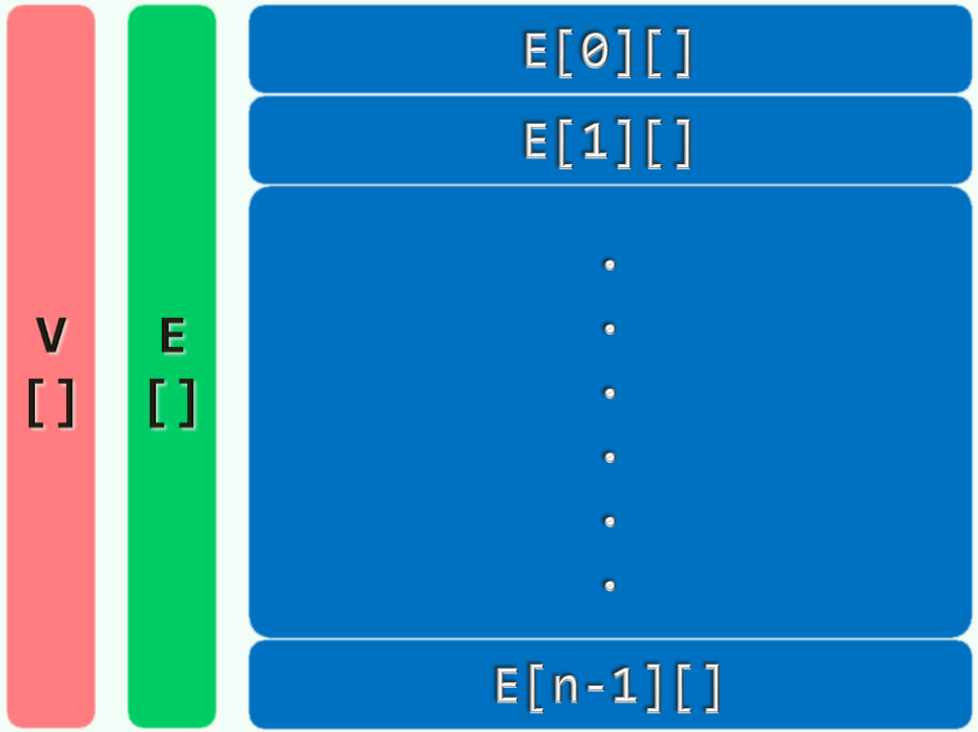
Edge

```
using EType = enum { UNDETERMINED, TREE, CROSS, FORWARD, BACKWARD };

template <typename Te> struct Edge { //不再严格封装
   Te data; //数据
   int weight; //权重
   EType type; //在遍历树中所属的类型
   Edge( Te const & d, int w ) : //构造新边
      data(d), weight(w), type(UNDETERMINED) {}
};
```

GraphMatrix

```
template <typename Tv, typename Te> class GraphMatrix : public Graph<Tv, Te> {
private:
   Vector< Vertex<Tv> > V; //顶点集
   Vector< Vector< Edge<Te>* > > E; //边集
public: // 操作接口：顶点相关、边相关、...
   GraphMatrix() { n = e = 0; } //构造
   ~GraphMatrix() { //析构
      for ( Rank v = 0; v < n; v++ )
         for ( Rank u = 0; u < n; u++ )
            delete E[v][u]; //清除所有边记录
   }
};
```



10-B3

图

邻接矩阵：静态操作

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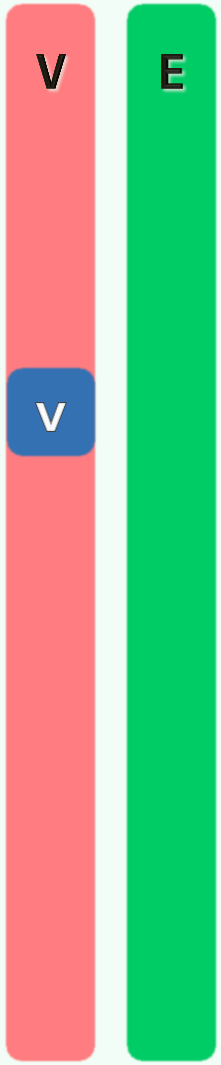
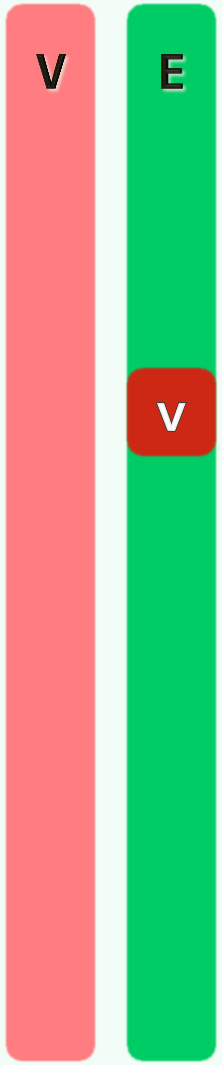
顶点的读写




```
Tv & vertex(Rank v) { return V[v].data; } //数据

int inDegree(Rank v) { return V[v].inDegree; } //入度

int outDegree(Rank v) { return V[v].outDegree; } //出度

VStatus & status(Rank v) { return V[v].status; } //状态

int & dTime(Rank v) { return V[v].dTime; } //时间标签dTime

int & fTime(Rank v) { return V[v].fTime; } //时间标签fTime

Rank & parent(Rank v) { return V[v].parent; } //在遍历树中的父亲

int & priority(Rank v) { return V[v].priority; } //优先级数
```

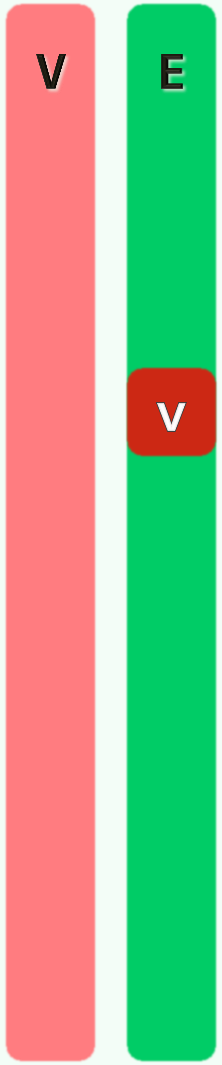
边的读写




```
bool exists( Rank v, Rank u ) { //判断边(v, u)是否存在（短路求值）
   return (v < n) && (u < n) && (E[v][u] != NULL);
} //以下假定exists(v, u) = true
```

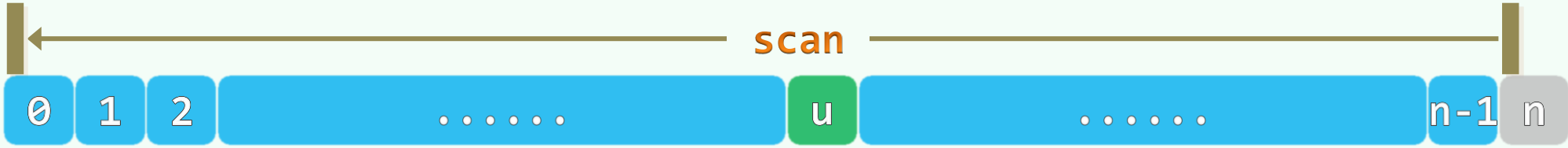

```
Te & edge( Rank v, Rank u ) { return E[v][u]->data; } //数值
EType & type( Rank v, Rank u ) { return E[v][u]->type; } //类型
int & weight( Rank v, Rank u ) { return E[v][u]->weight; } //权重
```

邻点的枚举



对于任意顶点v，如何枚举其所有的邻接顶点（neighbor）？

```
Rank firstNbr( Rank v ) { return nextNbr( v, n ); } //假想哨兵
```




```
Rank nextNbr( Rank v, Rank u ) { //若已枚举至邻居u，则转向下一邻居
   while ( ( -1 != --u ) && ! exists( v, u ) ); //逆向顺序查找
   return u;
} //O(n)——改用邻接表，可提高至O(1 + outDegree(v))
```

10-B4

图

邻接矩阵：动态操作

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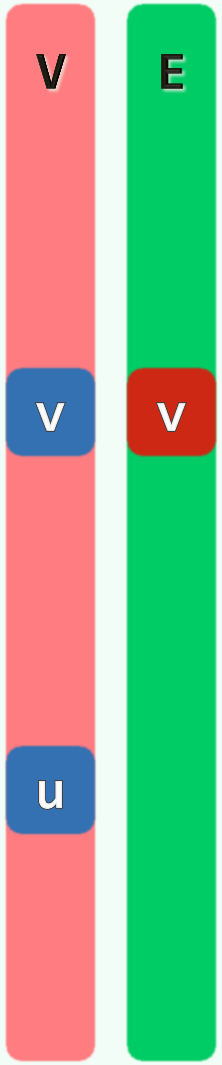
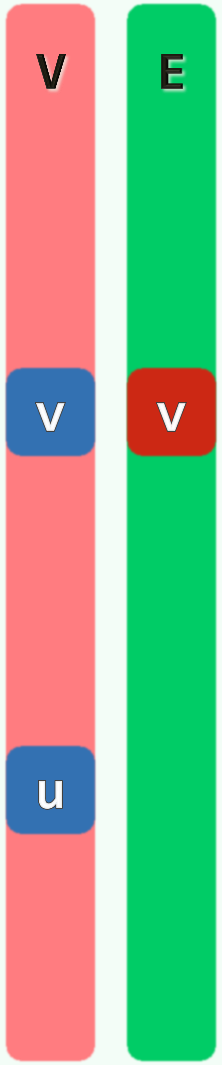
边的插入




```
void insert( Te const & edge, int w, Rank v, Rank u ) {
   if ( exists(v, u) ) return; //忽略已有的边
   E[v][u] = new Edge<Te>( edge, w ); //创建新边（权重为w）
   e++; //更新边计数
   V[v].outDegree++; //更新顶点v的出度
   V[u].inDegree++; //更新顶点u的入度
}
```

边的删除

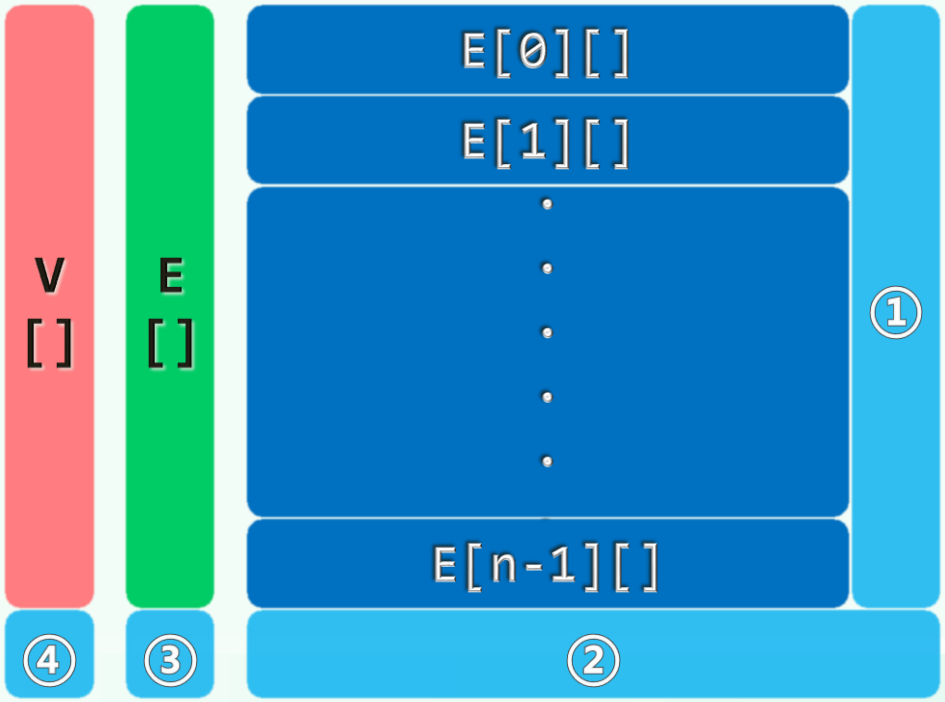


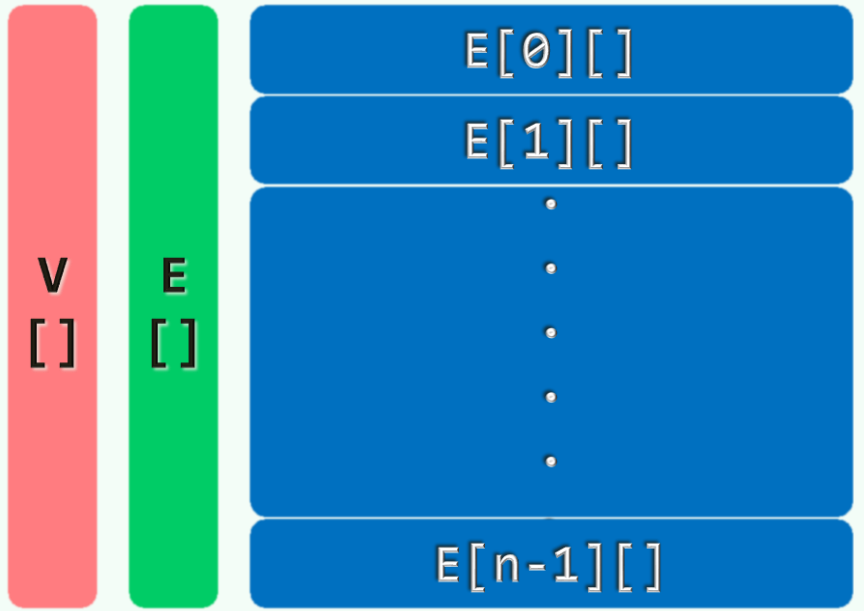

```
Te remove( Rank v, Rank u ) { //删除（已确认存在的）边(v, u)
   Te eBak = edge(v, u); //备份边(v, u)的信息
   delete E[v][u]; E[v][u] = NULL; //删除边(v, u)
```


```
   e--; //更新边计数
   V[v].outDegree--; //更新顶点v的出度
   V[u].inDegree--; //更新顶点u的入度
   return eBak; //返回被删除边的信息
}
```

顶点插入

```
Rank insert( Tv const & vertex ) { //插入顶点，返回编号
  for ( Rank u = 0; u < n; u++ ) E[u].insert( NULL ); n++; //①
  E.insert( Vector< Edge<Te>* >( n, n, NULL ) ); //②③
  return V.insert( Vertex<Tv>( vertex ) ); //④
}
```





顶点删除

```
Tv remove( Rank v ) { //删除顶点及其关联边，返回该顶点信息
   for ( Rank u = 0; u < n; u++ ) //删除所有出边
      if ( exists( v, u ) ) { delete E[v][u]; V[u].inDegree--; e-- }
   E.remove(v); n--; //删除第v行
   Tv vBak = vertex( v ); V.remove( v ); //备份之后，删除顶点v
   for ( Rank u = 0; u < n; u++ ) //删除所有入边及第v列
      if ( Edge<Te> * x = E[u].remove( v ) )
         { delete x; V[u].outDegree--; e--; }
   return vBak; //返回被删除顶点的信息
}
```

10-B5

图

邻接矩阵：性能分析

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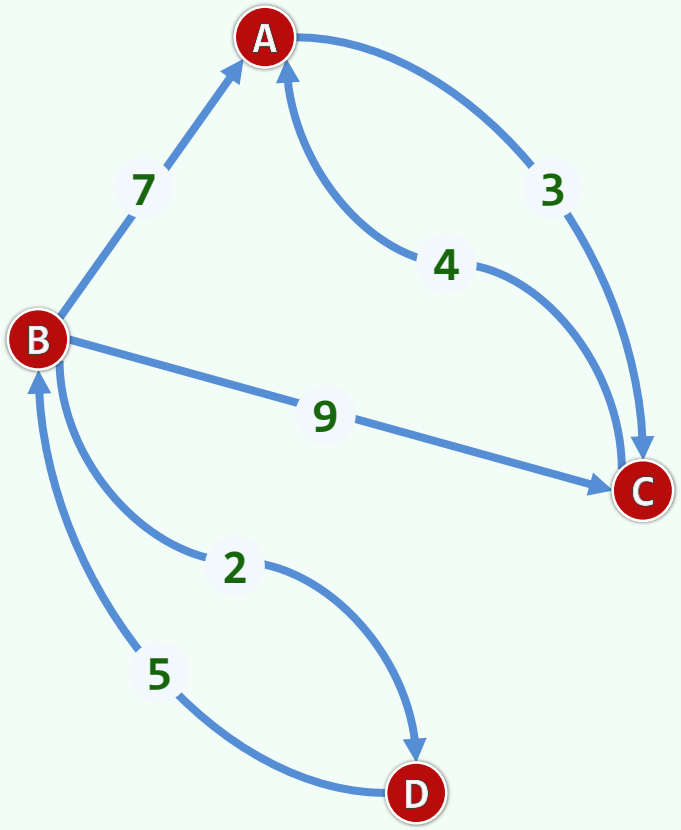
优点

- 直观，易于理解和实现
- 适用范围广泛

  尤其适用于稠密图（dense graph）
- 判断两点之间是否存在联边：$O(1)$
- 获取顶点的（出/入）度数：$O(1)$

  添加、删除边后更新度数：$O(1)$

∞	A	B	C	D
A			3	
B	7		9	2
C	4			
D		5		



- 扩展性（scalability）：得益于Vector良好的控制策略，空间溢出等情况可被“透明地”处理

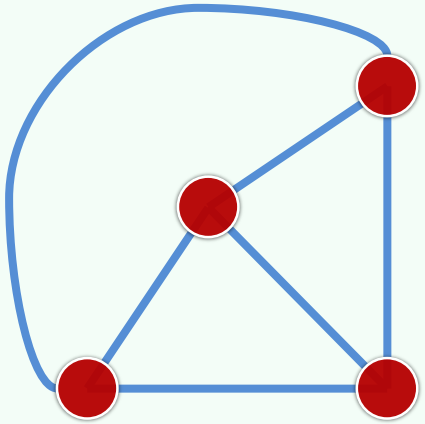
缺点

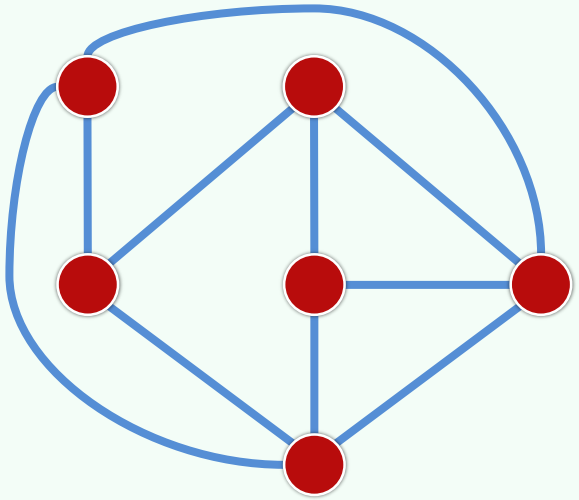
- $\Theta(n^2)$空间，与边数无关！
- 真会有这么多条边吗？不妨考察一类特定的图...
- 平面图（planar graph）：可嵌入于平面的图
- Euler's formula (1750):

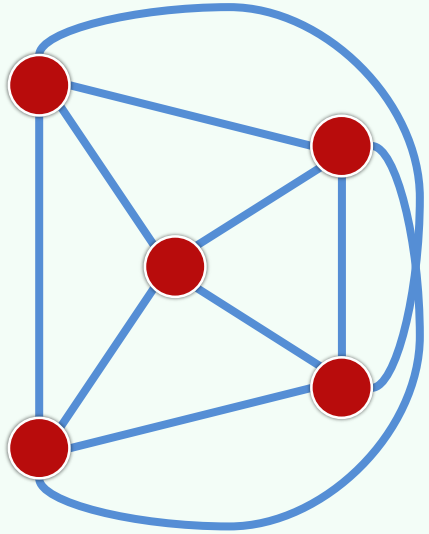
  v - e + f - c = 1, for any PG
- 平面图：e ≤ 3×n - 6 = $\mathcal{O}$(n) << n^2

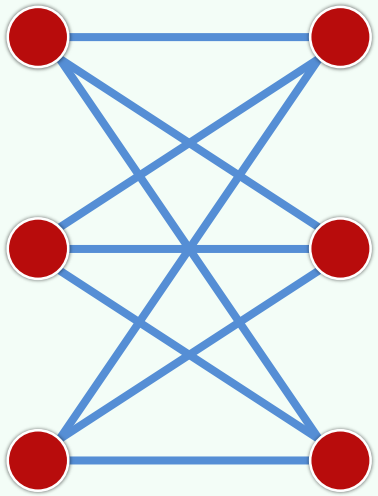
  此时，空间利用率 ≈ 1/n
- 稀疏图（sparse graph）

  空间利用率同样很低，可采用压缩存储技术









10-C

图

邻接表

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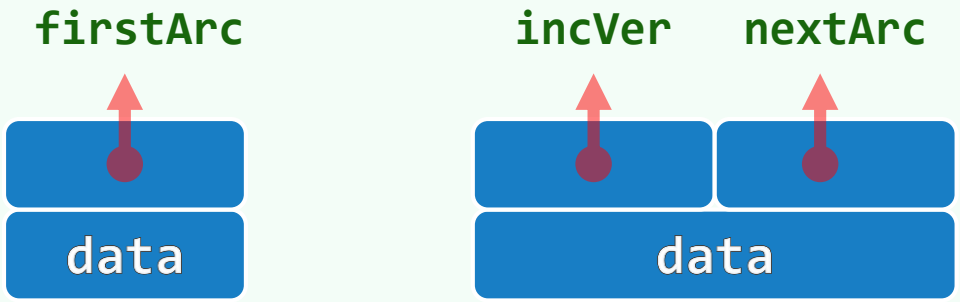
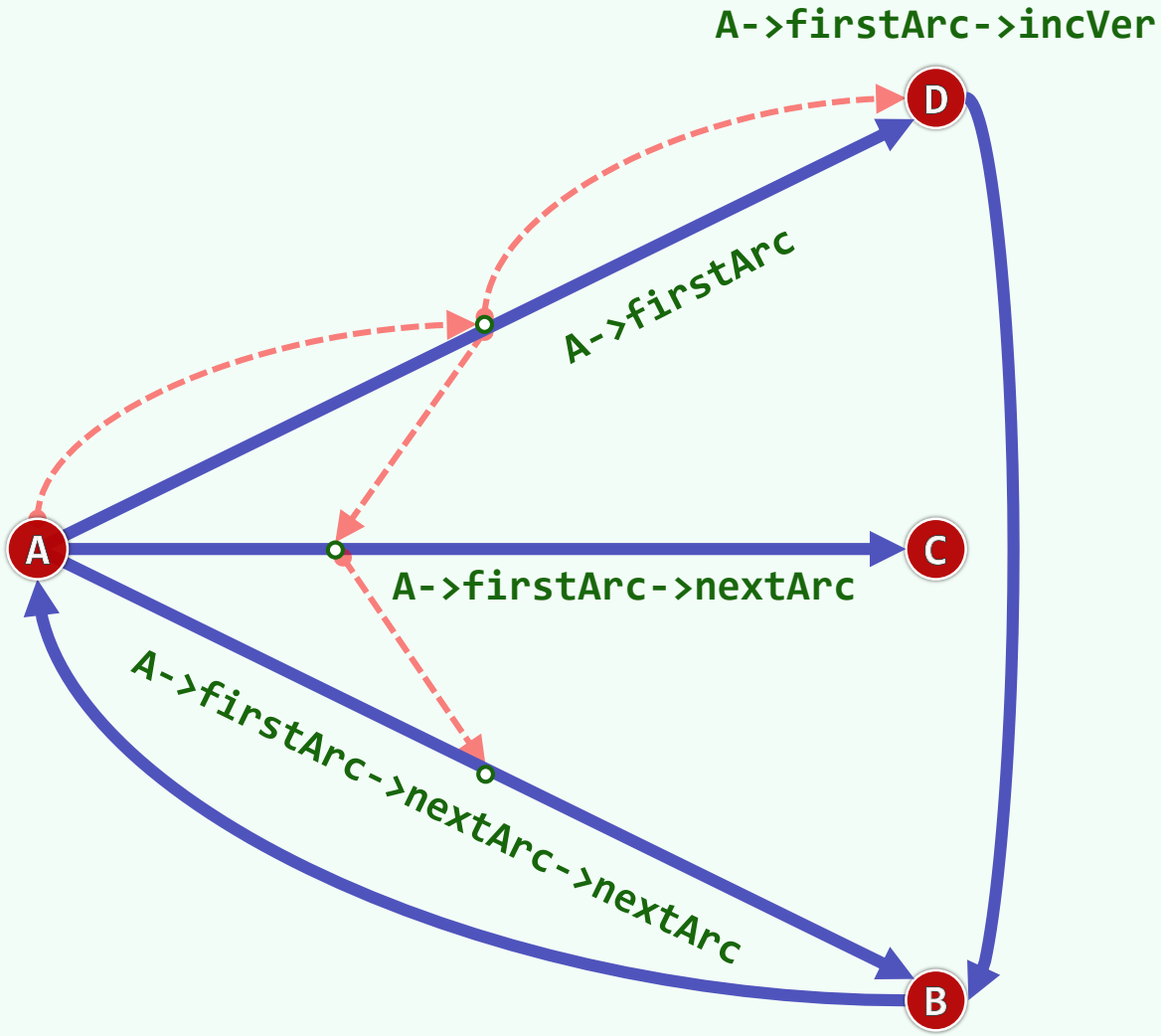
邻接表

- 如何避免邻接矩阵的空间浪费?
- 将邻接矩阵的各行组织为列表，只记录存在的边
- 等效于，每一顶点v对应于列表：

$$L_v = \{ u \mid \langle v, u \rangle \in E \}$$

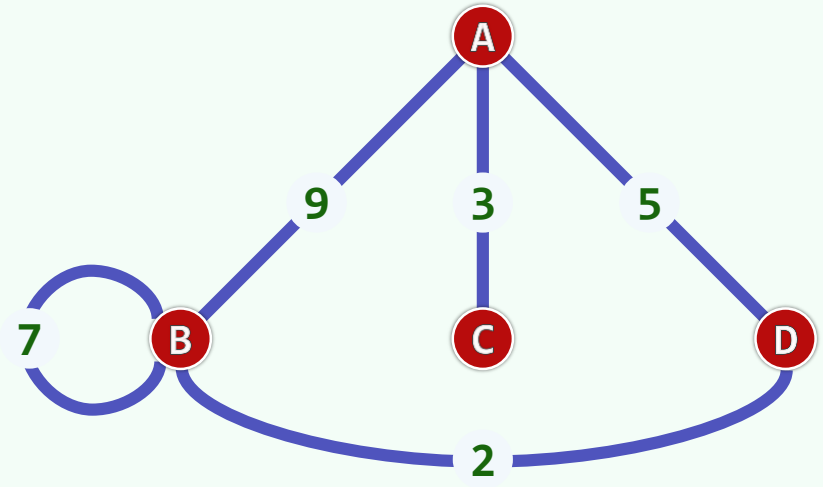




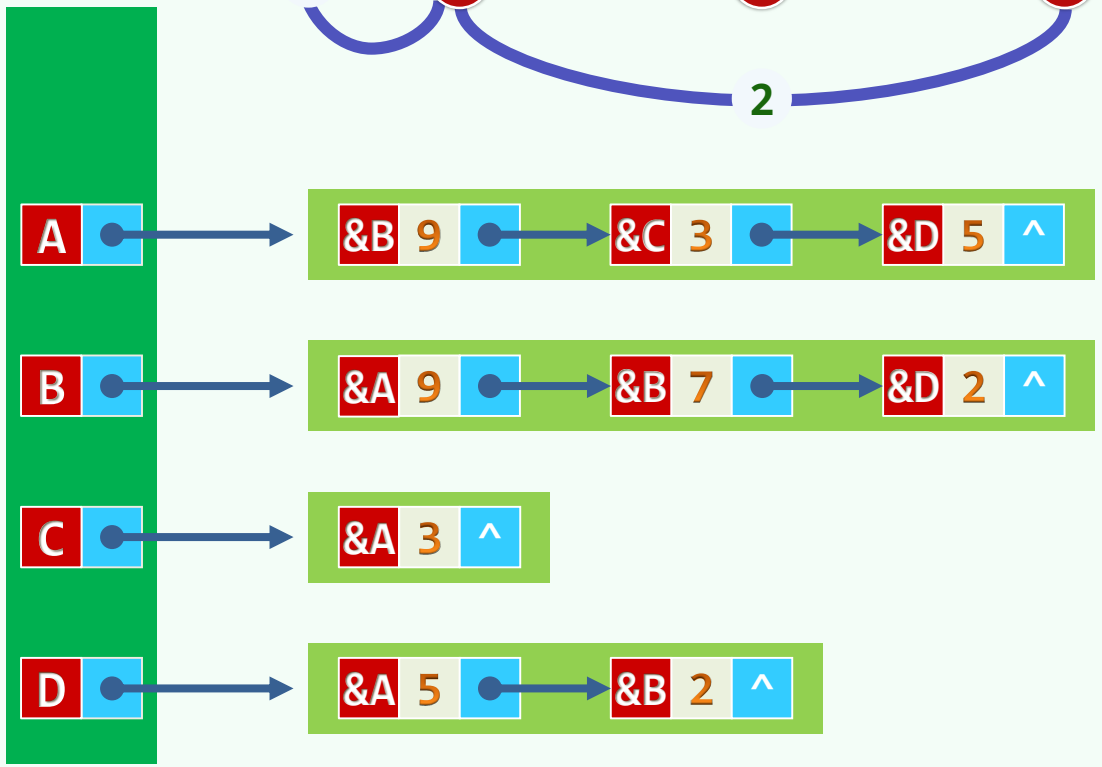
实例

❖ 4个顶点，5条弧：不必占用4 × 4 = 16个单元，但

还是占用了9个单元，另加4个表头



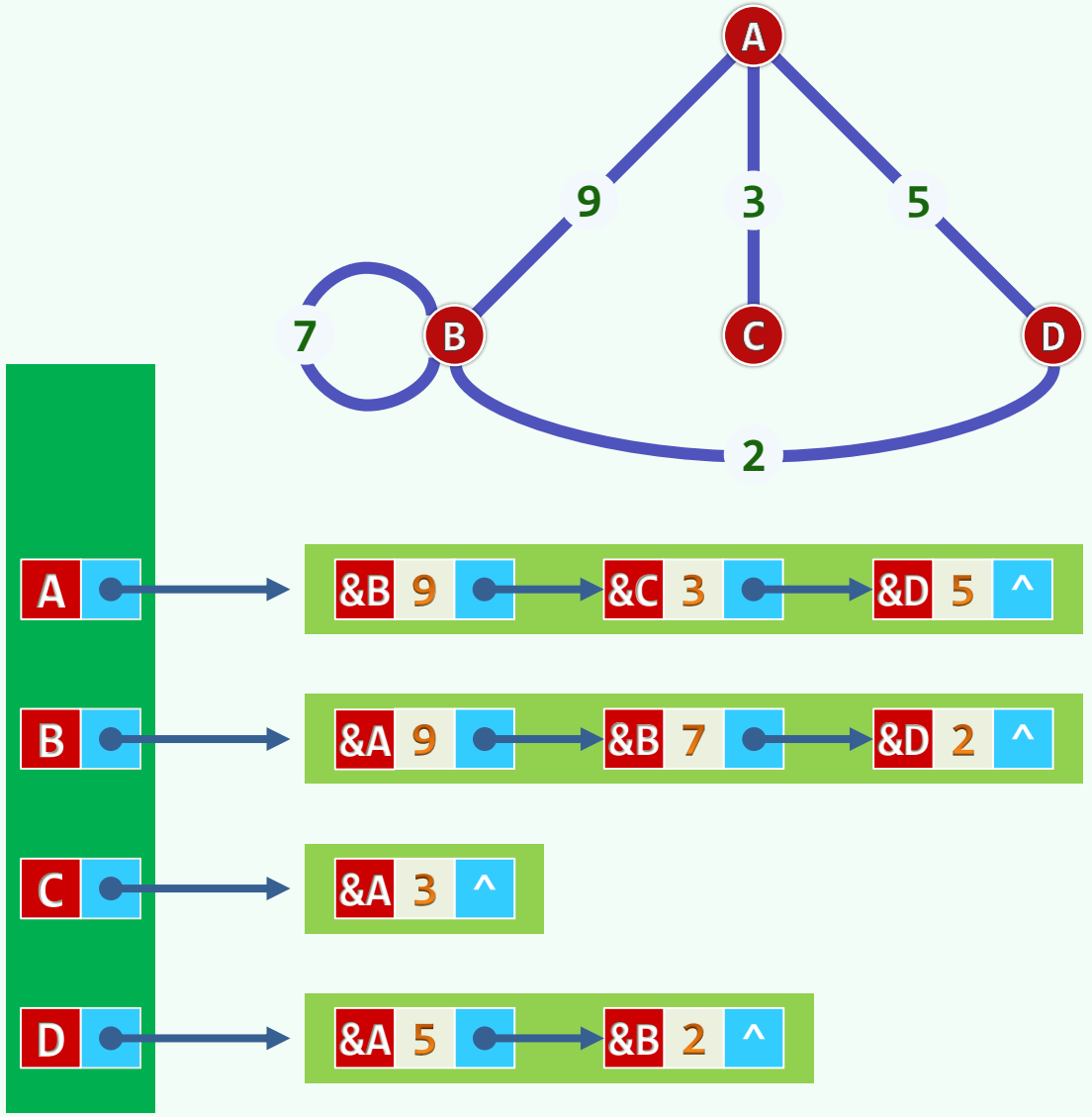
∞	A	B	C	D
A		9	3	5
B	9	7		2
C	3			
D	5	2		



空间复杂度

- ❖ 有向图 = $O(n + e)$
- ❖ 无向图 = $O(n + 2\times e) = O(n + e)$
  - 注意：无向弧被重复存储
  - 问题：如何改进?
- ❖ 适用于稀疏图
- ❖ 平面图 = $O(n + 3\times n) = O(n)$

  较之邻接矩阵，有极大改进



时间复杂度（1/2）

- 建立邻接表（递增式构造）：$O(n+e)$ //如何实现
- 枚举所有以顶点v为尾的弧：$O(1+\deg(v))$ //遍历v的邻接表
- 枚举（无向图中）顶点v的邻居：$O(1+\deg(v))$ //遍历v的邻接表
- 枚举所有以顶点v为头的弧：$O(n+e)$ //遍历所有邻接表

  可改进至$O(1+\deg(v))$ //建立逆邻接表——为此，空间需增加多少？
- 计算顶点v的出度/入度：
  - 增加度数记录域：$O(n)$附加空间
  - 增加/删除弧时更新度数：$O(1)$时间 //总体$O(e)$时间
  - 每次查询：$O(1)$时间！

时间复杂度（2/2）

- 给定顶点u和v，判断是否<u，v> ∈ E
  - 有向图：搜索u的邻接表，$O(\deg(u)) = O(e)$
  - 无向图：搜索u或v的邻接表，$O(\max(\deg(u), \deg(v))) = O(e)$
  - “并行”搜索：$O(2 \times \min(\deg(u), \deg(v))) = O(e)$

  能够达到邻接矩阵的$O(1)$吗？
- 散列！如果装填因子选取得当 //保持兴趣
  - 弧的判定：expected-$O(1)$，与邻接矩阵“相同”
  - 空间：$O(n + e)$，与邻接表相同
- 为何有时仍使用邻接矩阵？仅仅因为实现简单？不，有更多用处！比如，可处理

  Euclidean graph和intersection graph之类的隐式图（implicitly-represented graphs）

取舍原则

- 空间/速度
- 顶点类型
  - bit
  - int
  - float
  - struct
  - class
  - ...
- 弧类型（方向 / 权值）
- 图类型（稀疏 / 稠密）

	邻接矩阵	邻接表
适用场合	经常检测边的存在 经常做边的插入/删除 图的规模固定 稠密图	经常计算顶点的度数 顶点数目不确定 经常做遍历 稀疏图

10-D1

图

广度优先搜索：算法

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Breadth-First Search

❖ 始自顶点s的广度优先搜索

访问顶点s

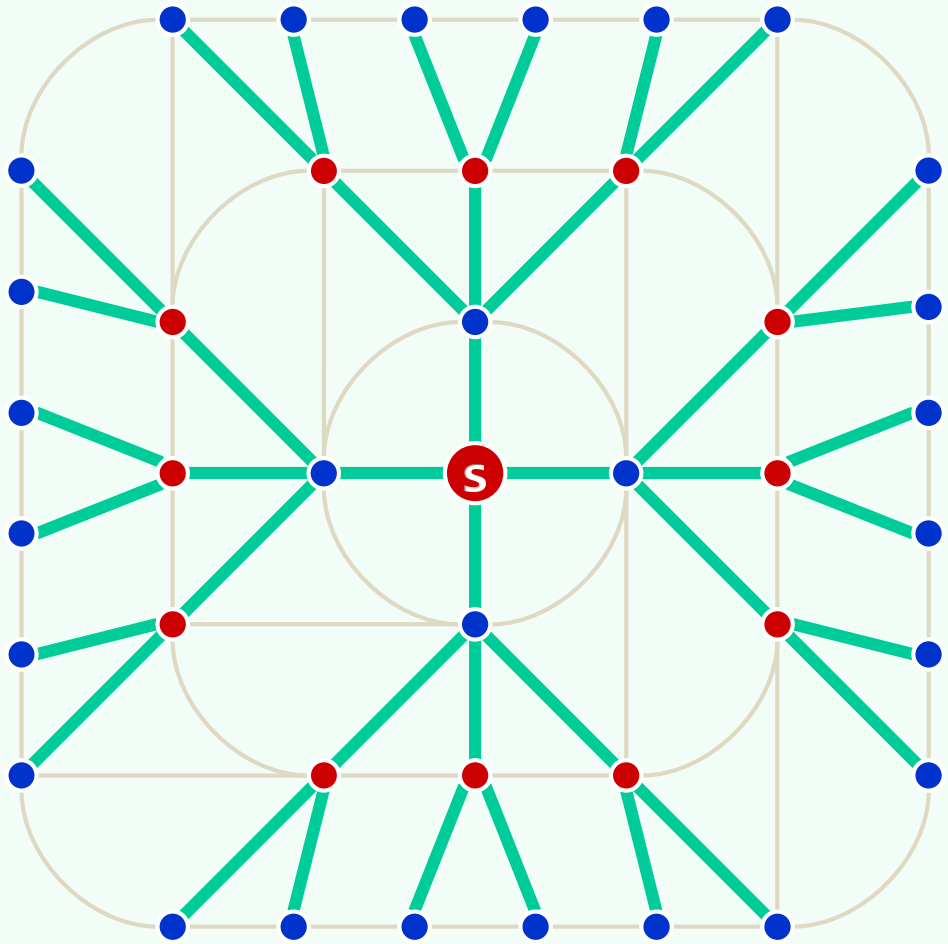
依次访问s所有尚未访问的邻接顶点

依次访问它们尚未访问的邻接顶点

...

如此反复

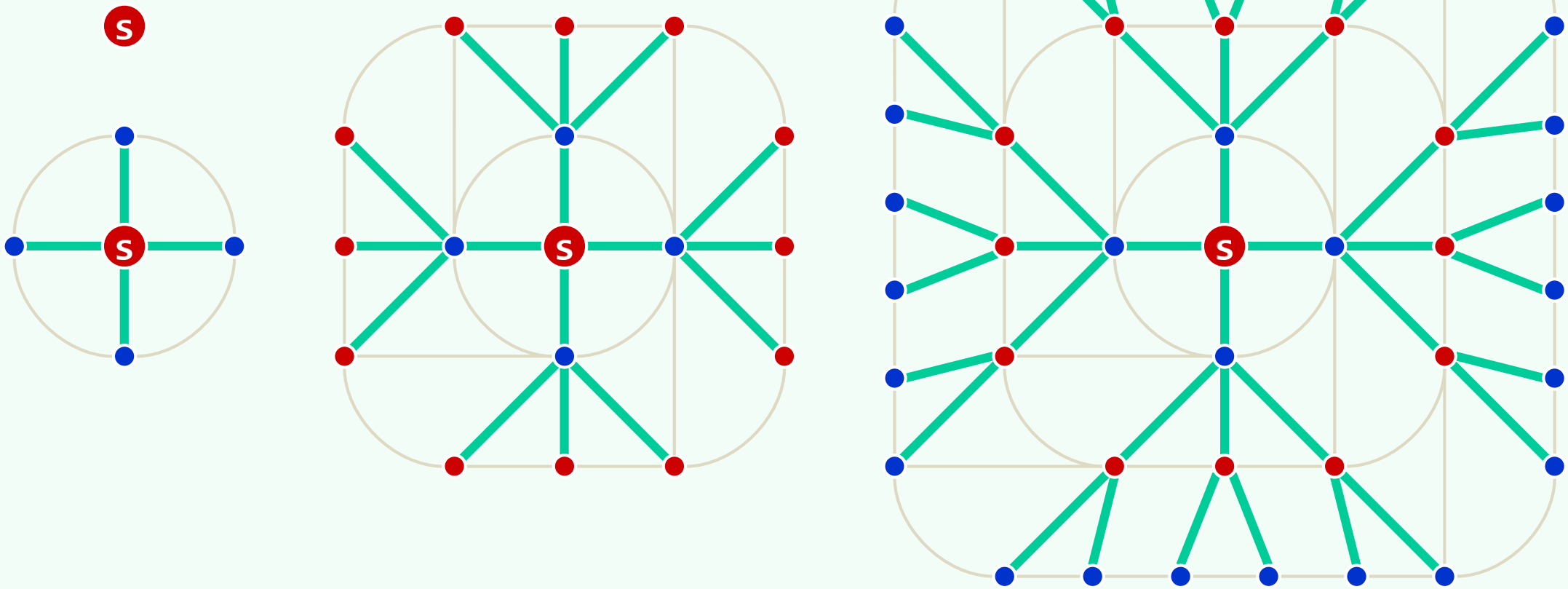
直至没有尚未访问的邻接顶点



❖ 以上策略及过程完全等同于树的层次遍历

❖ 事实上，BFS也的确会构造出原图的一棵支撑树（BFS tree）

譬喻：油气管线 + 绳网 + 涟漪



Graph::BFS() [1/2]

```
template<typename Tv, typename Te> void Graph<Tv, Te>::BFS( Rank v, Rank& dClock ) {
   Queue<Rank> Q; status(v) = DISCOVERED; Q.enqueue(v); dTime(v) = dClock++; //起点
   for ( Rank fClock = 0; !Q.empty(); ) { //在Q变空之前，反复地
      if ( dTime(v) < dTime( Q.front() ) ) //dTime的增加，意味着开启新的一代，因此
         dClock++, fClock = 0; //dTime递增，fTime复位
      v = Q.dequeue(); //取出队首顶点v，并
      for ( Rank u = firstNbr(v); -1 != u; u = nextNbr(v, u) ) //考查v的每一个邻居u
         /* ... 视u的状态分别处理：最终，所有顶点按[dTime,fTime]字典序被遍历 ... */
      status(v) = VISITED; fTime(v) = fClock++; //至此，v访问完毕
   } //for
} //BFS
```

Graph::BFS() [2/2]

```
/* ....... * /
v v = Q.dequeue(); //取出队首顶点v，并
  for ( Rank u = firstNbr(v); -1 != u; u = nextNbr(v, u) ) //考查v的每一个邻居u
u  if ( UNDISCOVERED == status(u) ) { //若u尚未被发现，则发现之
u    status(u) = DISCOVERED; Q.enqueue(u); dTime(u) = dClock; //发现该顶点
       type(v, u) = TREE; parent(u) = v; //引入树边，拓展BFS树
u  } else //若u已被发现（正在队列中），或者甚至已访问完毕（已出队列），则
u    type(v, u) = CROSS; //将(v, u)归类于跨边
v status(v) = VISITED; fTime(v) = fClock++; //至此，v访问完毕
/* ....... * /
```

图

广度优先搜索：实例

10-D2

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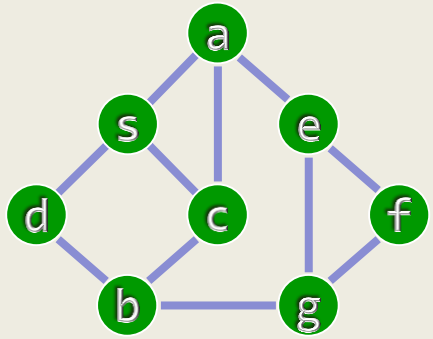
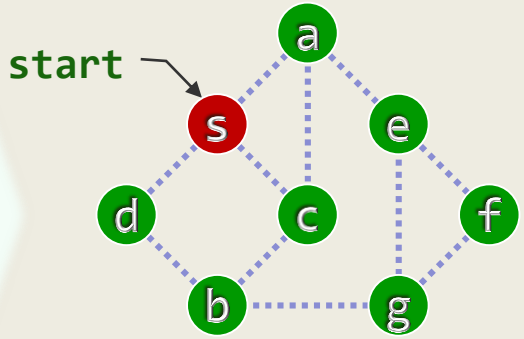
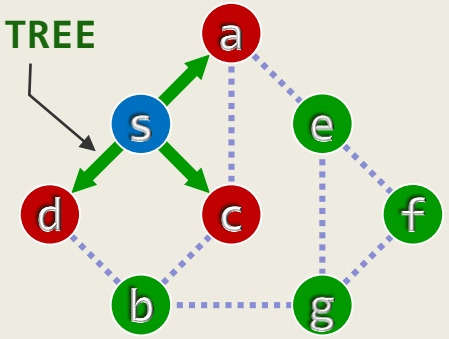
无向图（1/2）

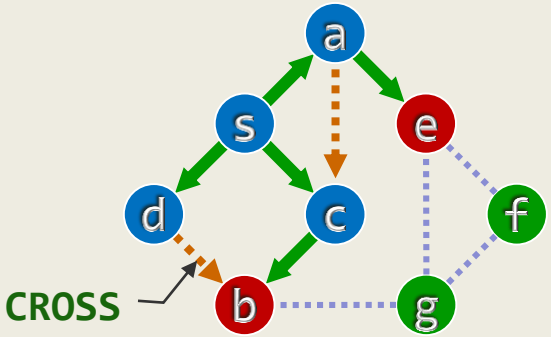




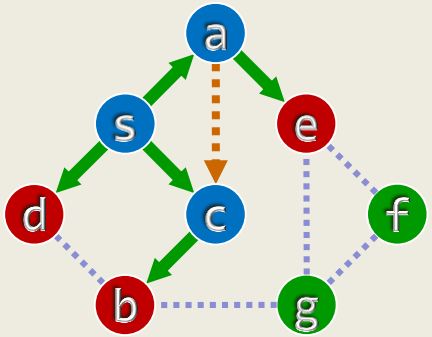
s



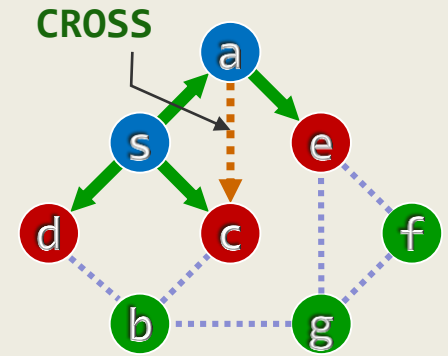
d c a s



b e d c a s

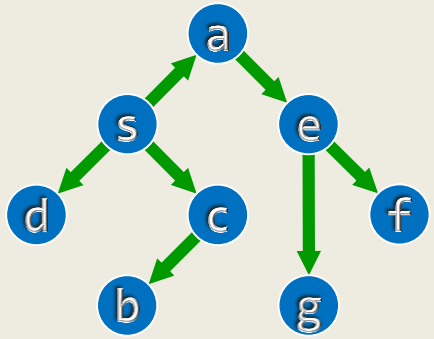


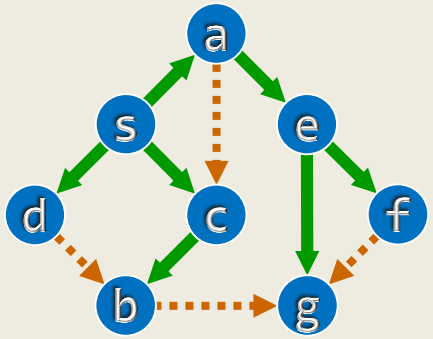
b e d c a s



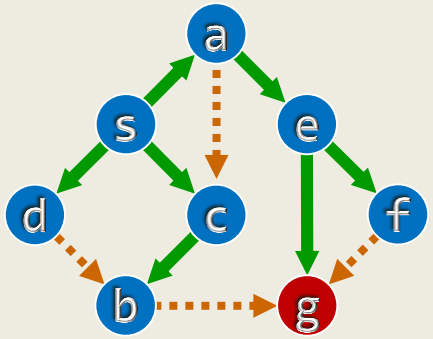
e d c a s

无向图（2/2）

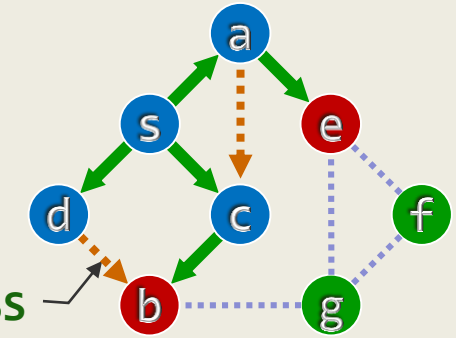




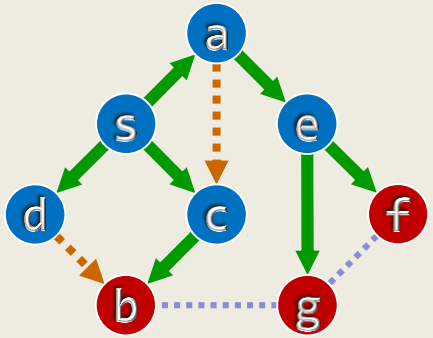
g f b e d c a s



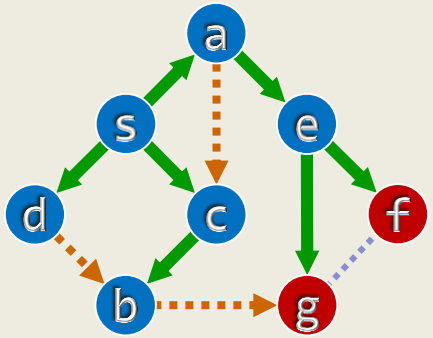
g f b e d c a s



b e d c a s



g f b e d c a s



g f b e d c a s

10-D3

图

广度优先搜索：推广

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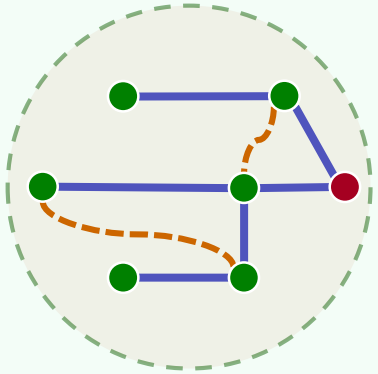
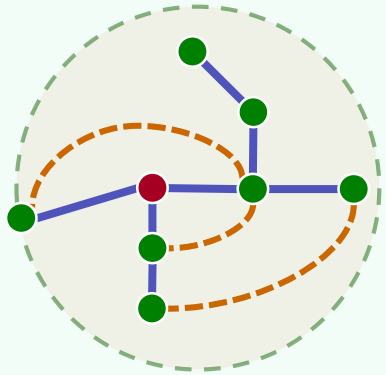
一个人做一件好事并不难，难的是一辈子做好事，不做坏事。

连通分量 + 可达分量

❖ 问题

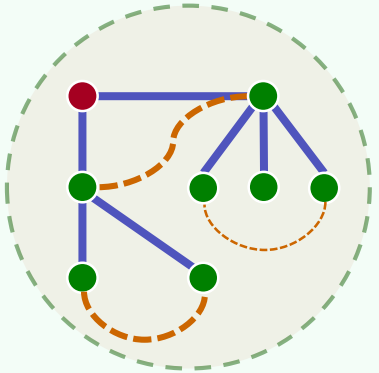
- 给定无向图，找出其中任一顶点s所在的连通图
- 给定有向图，找出源自其中任一顶点s的可达分量





❖ 算法

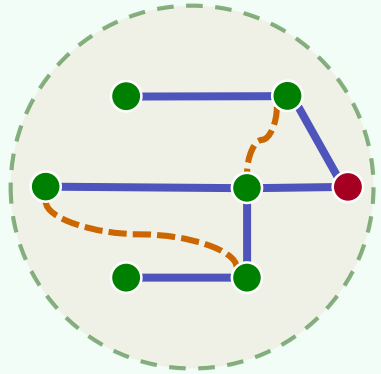
- 从s出发做BFS
- 输出所有被发现的顶点
- 队列为空后立即终止，无需考虑其它顶点

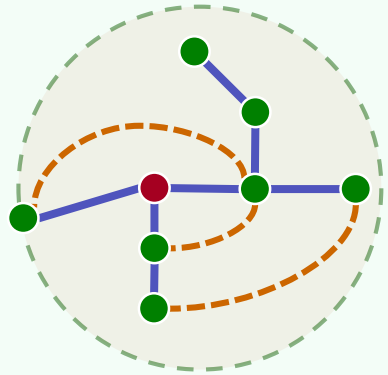


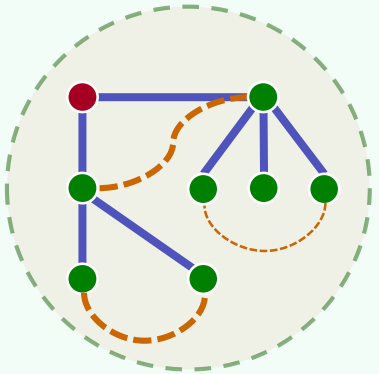
❖ 若图中包含多个连通/可达分量，又该如何保证对全图的遍历呢?

Graph::bfs()

```
template <typename Tv, typename Te>

void Graph<Tv, Te>::bfs( Rank s ) { //s < n

   reset(); Rank dClock = 0; //全图复位

   for ( Rank v = s; v < s + n; v++ ) //从s起顺次检查所有顶点

      if ( UNDISCOVERED == status(v % n) ) //一旦遇到尚未发现者

         BFS( v % n, dClock ); //即从它出发启动一次BFS

} //如此可完整覆盖全图，且总体复杂度依然保持为O(n+e)
```



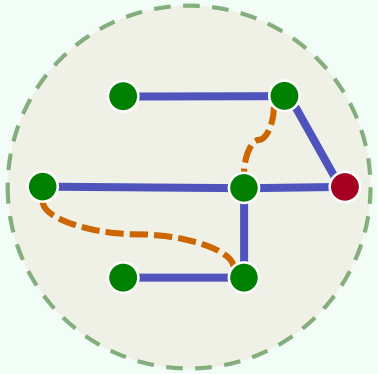


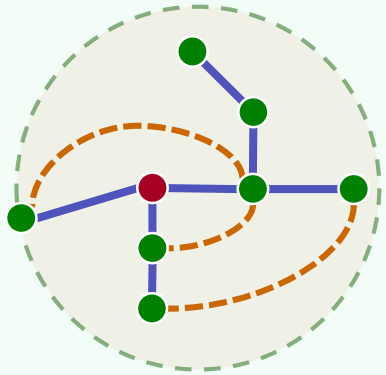


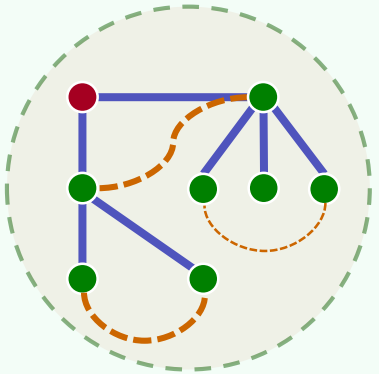
复杂度

- 考查无向图...
- bfs()的初始化（reset()）：$\mathcal{O}(n+e)$
- BFS()的迭代
  - 外循环（while (!Q.empty())）

    每个顶点各进入1次
  - 内循环（枚举v的每一邻居）：$\mathcal{O}(1+deg(v))$ （改用邻接表）
  - 总共：$\mathcal{O}(\sum_{v \in V}(1+deg(v))) = \mathcal{O}(n+2e)$
- 整个算法：$\mathcal{O}(n+e)+\mathcal{O}(n+2e) = \mathcal{O}(n+e)$
- 有向图呢？亦是如此！







图

广度优先搜索：性质及规律

10-D4

邓俊辉

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啊，五环，你比四环多一环；啊，五环，你比六环少一环

边分类

❖ 经BFS后，所有边将确定方向，且被分为两类

❖ (v,u)被标记为TREE时，v为DISCOVERED且u为UNDISCOVERED

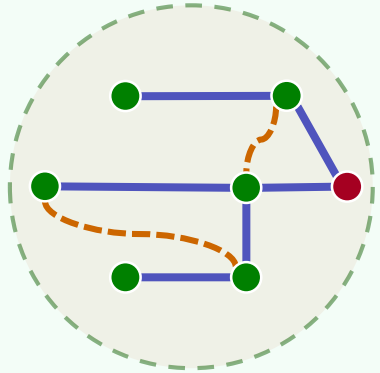
❖ (v,u)被标记为CROSS时，v和u均为DISCOVERED 或者 v为DISCOVERED而u为VISITED

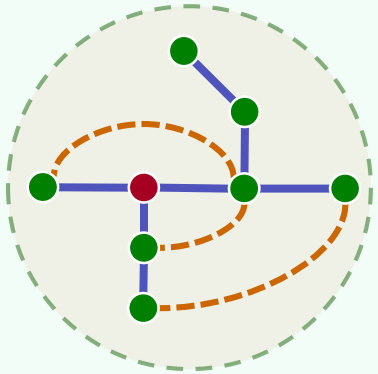
不论(v,u)是有向边或无向边，两种情况均可能出现

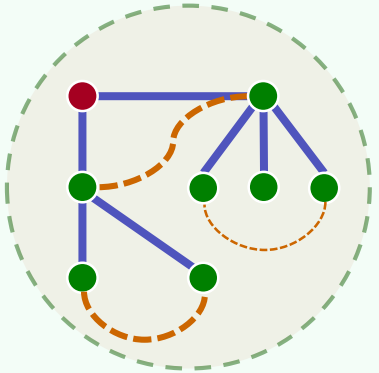
BFS树/森林

- 对于（起始于v的）每一连通/可达分量，bfs()进入BFS(v)恰好1次
- 进入BFS(v)时，队列为空；v所属分量内的每个顶点
  - 迟早会以UNDISCOVERED状态进队1次
  - 进队后随即转为DISCOVERED状态，并生成一条树边
  - 迟早会出队并转为VISITED状态

  退出BFS(v)时，队列为空
- BFS(v)以v为根，生成一棵BFS树
- bfs()生成一个BFS森林包含 c 棵树、n-c 条树边和 e-n+c 条跨边

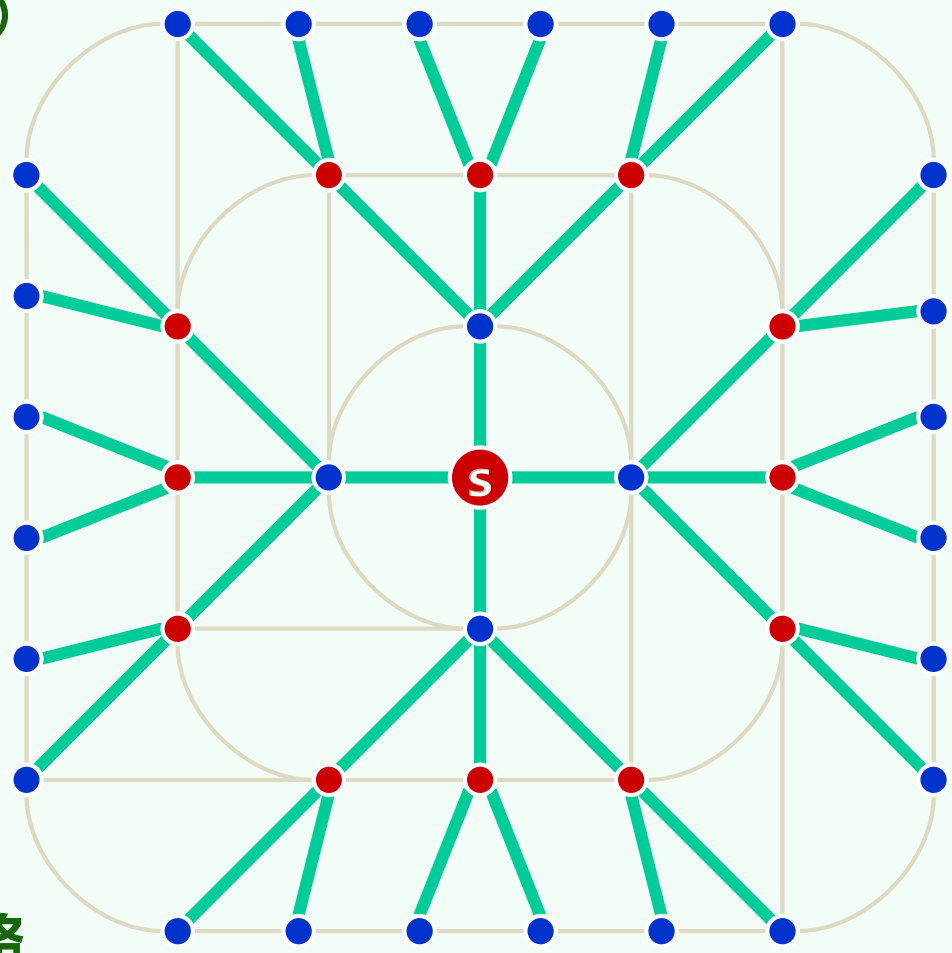






最短路径

- 无向图中，顶点v到u的（最近）距离记作dist(v, u)
- BFS过程中，队列Q犹如一条贪吃蛇
  - 其中的顶点按dist(s)单调排列
  - 相邻顶点的dist(s)相差不超过1
  - 首、末顶点的dist(s)相差不超过1
  - 由树边联接的顶点，dist(s)恰好相差1
  - 由跨边联接的顶点，dist(s)至多相差1
- BFS树中从s到v的路径，即是二者在原图中的最短通路



10-D5

图

广度优先搜索：应用

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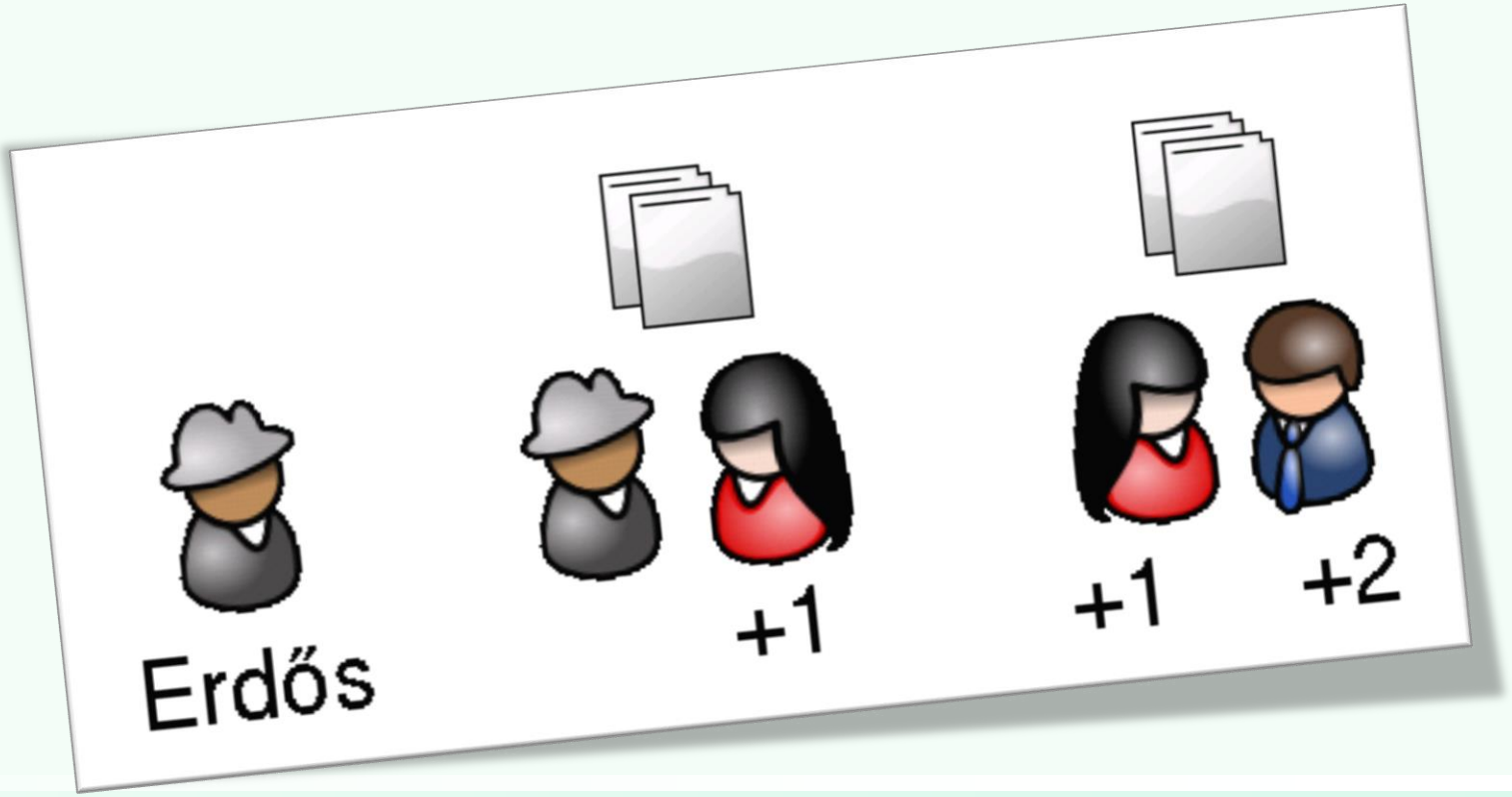
啊，五环，你比四环多一环；啊，五环，你比六环少一环

Erdös Number

- Describes the "collaborative distance"

  between mathematician Paul Erdős and another person,

  as measured by authorship of mathematical papers



Chow Number

- chow（吴孟达）= 1

  [整蛊专家]（1）　*周星驰 +吴孟达　成奎安　刘德华　关之琳　邱淑贞

- chow（葛优）= 2

  [没完没了]（2）　+葛优　*吴倩莲　傅彪

  [97家有喜事]（1）　*周星驰 +吴倩莲　吴镇宇　钟丽缇　伍咏薇　黄百鸣

- chow（姜昆）= 3

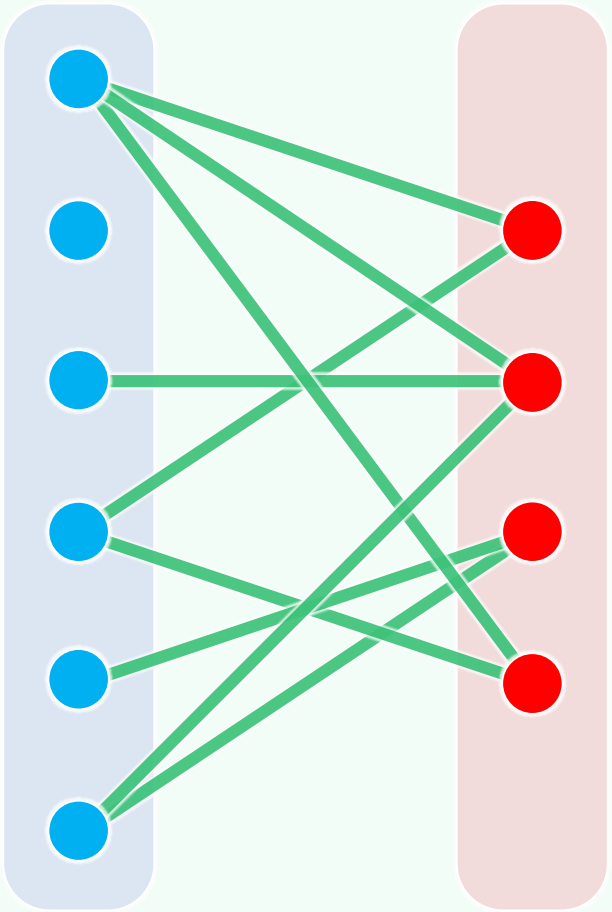
  [京都球侠]（3）　*张丰毅　孙敏　+姜昆　陈佩斯　于绍康　唐杰忠

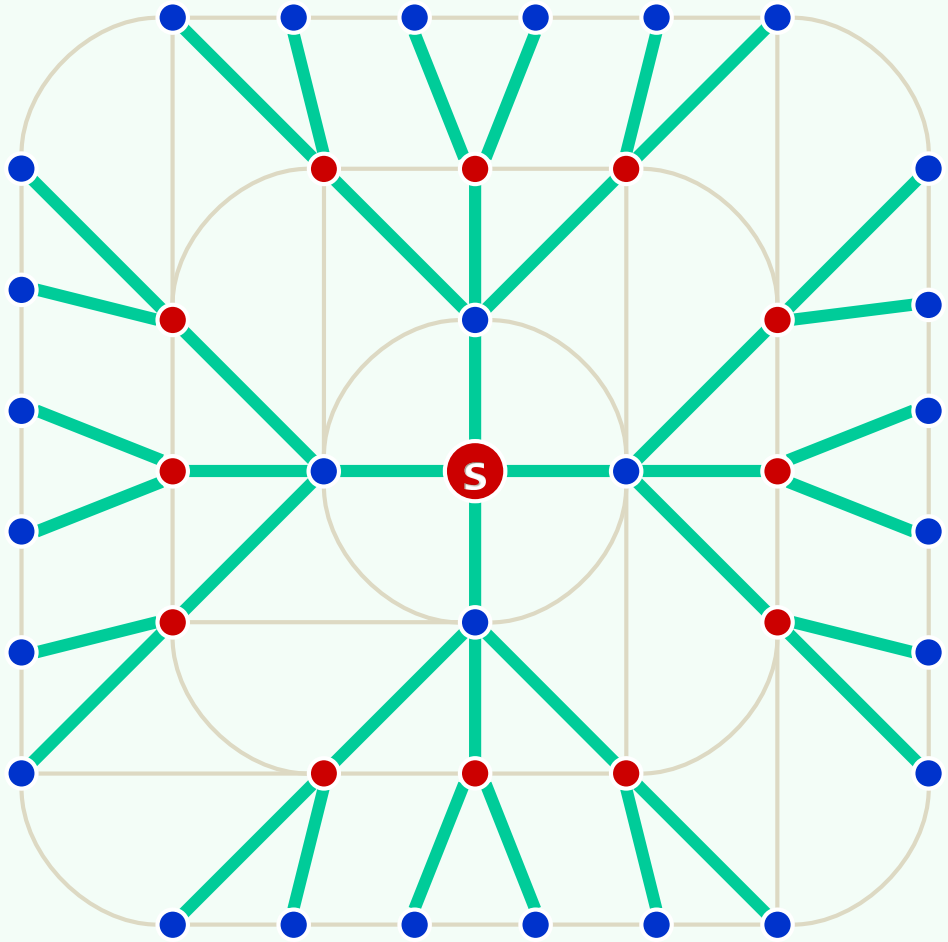
  [热线追击]（2）　+张丰毅　任达华　王馨平 *吴家丽

  [审死官]（1）　*周星驰　吴孟达 +吴家丽　秦沛　朱咪咪　梅艳芳

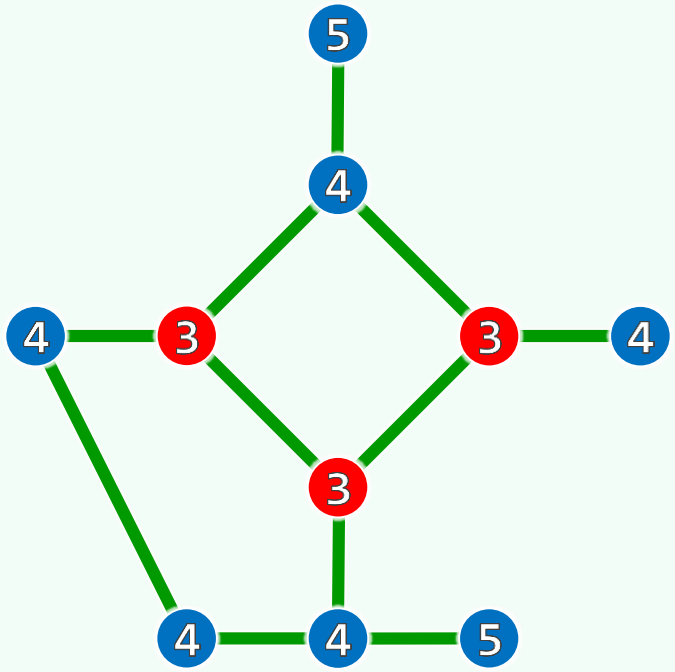
- chow ("Julia Roberts") = Infinity

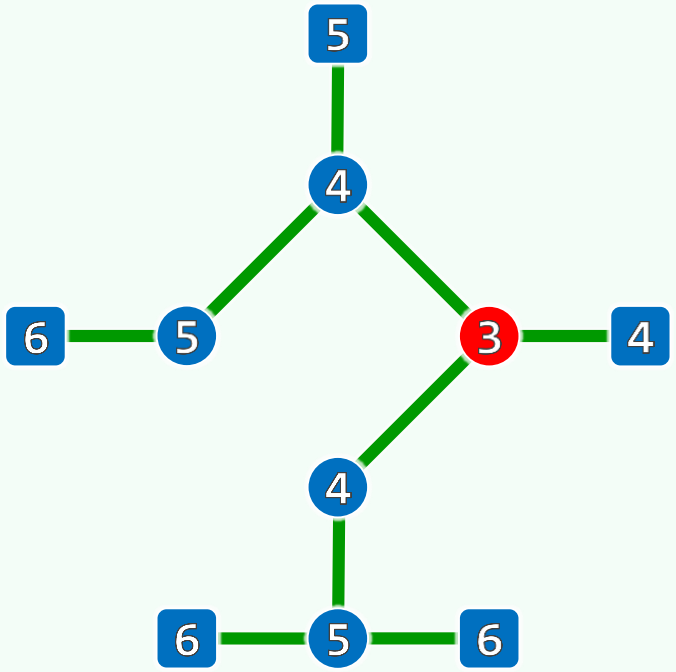
Bipartite Graph (Bigraph)

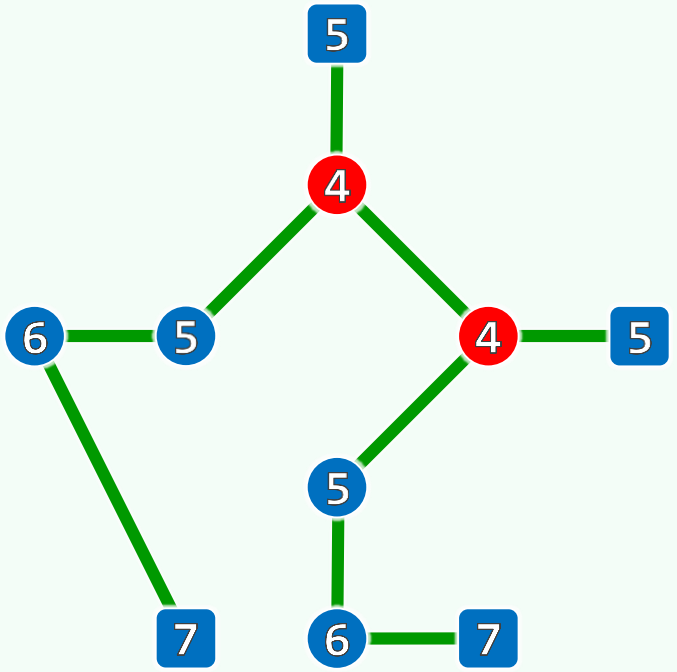




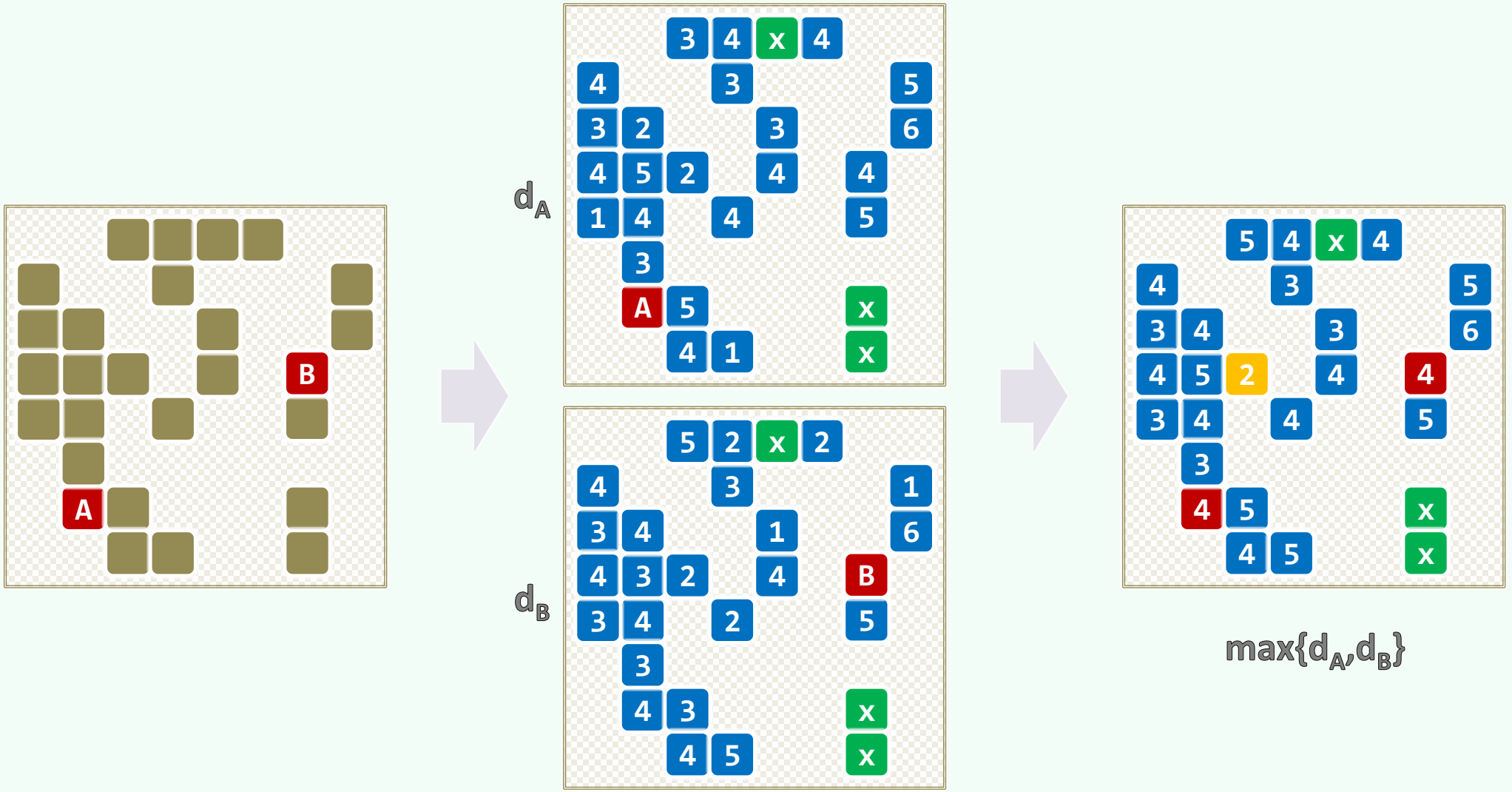
Eccentricity/Radius/Diameter/Center







Knights of the Round Table



10-E1

图

深度优先搜索：算法

Keep it simple, stupid.

- K. Johnson

如右边天气阴沉多雨，我便往左边走；如遇上不宜骑马的地方，我就停下。如此这般行路……我身后是否还有什么东西可看？有，我便返回去，因为那也是我要走的路。

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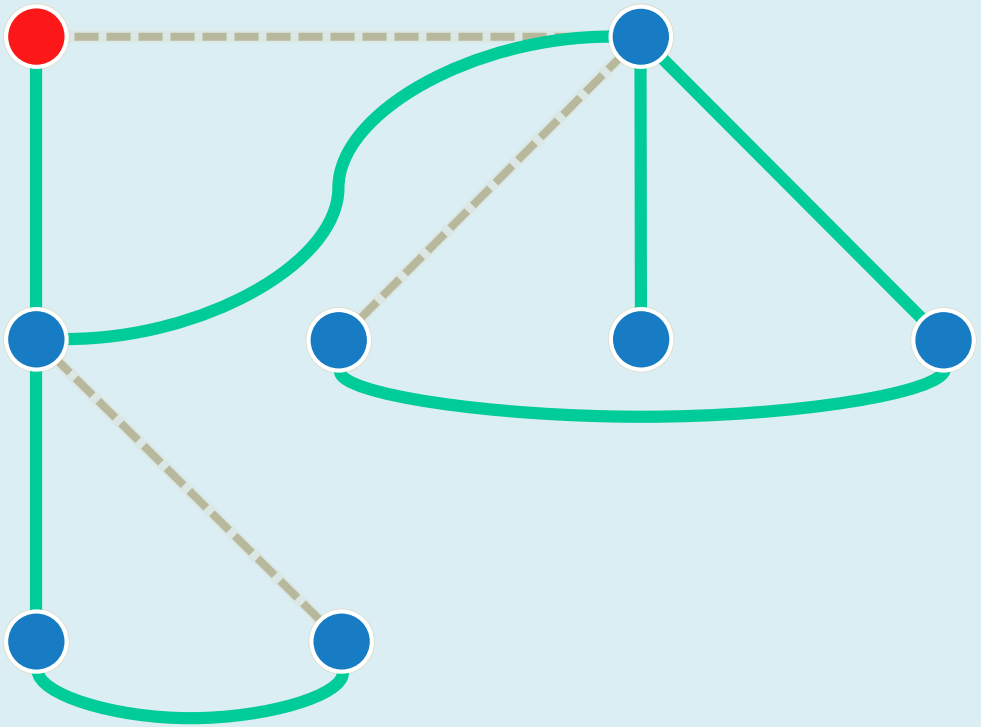
Depth-First Search

- DFS(s) //始自顶点s的深度优先搜索
  - 访问顶点s
  - 若s尚有未被访问的邻居，则

    任取其一u，递归执行DFS(u)
  - 否则，返回
- 若此时尚有顶点未被访问

  任取这样的一个顶点作起始点
- 重复上述过程，直至所有顶点都被访问到
- 对树而言，等效于先序遍历：DFS也的确会构造出原图的一棵支撑树（DFS tree）



Graph::DFS()

```
template<typename Tv, typename Te> void Graph<Tv, Te>::DFS( Rank v, Rank& clock ) {
   dTime(v) = ++clock; status(v) = DISCOVERED; //发现当前顶点v
v  for ( Rank u = firstNbr(v); -1 != u; u = nextNbr(v, u) ) //考察v的每一邻居u
      switch ( status(u) ) { //并视其状态分别处理
u        case UNDISCOVERED: //u尚未发现，意味着支撑树可在此拓展
            type(v, u) = TREE; parent(u) = v; DFS( u, clock ); break; //递归
u        case DISCOVERED: //u已被发现但尚未访问完毕，应属被后代指向的祖先
            type(v, u) = BACKWARD; break;
u        default: //u已访问完毕（VISITED，有向图），则视承袭关系分为前向边或跨边
            type(v, u) = dTime(v) < dTime(u) ? FORWARD : CROSS; break;
      } //switch
v  status(v) = VISITED; fTime(v) = ++clock; //至此，当前顶点v方告访问完毕
}
```

图

深度优先搜索：实例（无向图）

10-E2

悔相道之不察兮，延伫乎吾将反

回朕车以复路兮，及行迷之未远

邓俊辉

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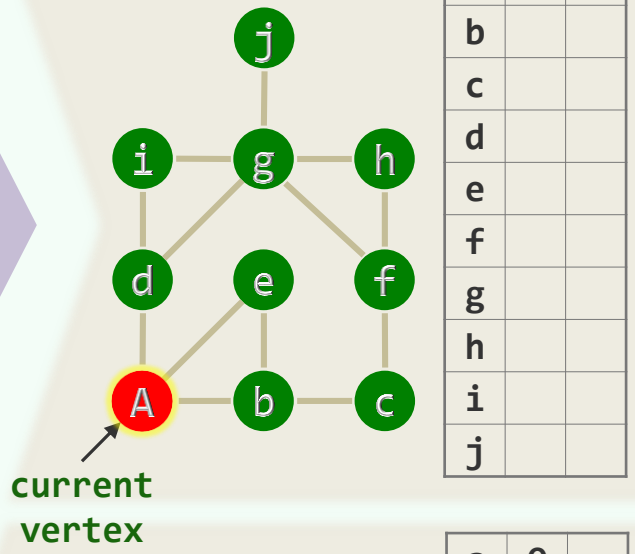

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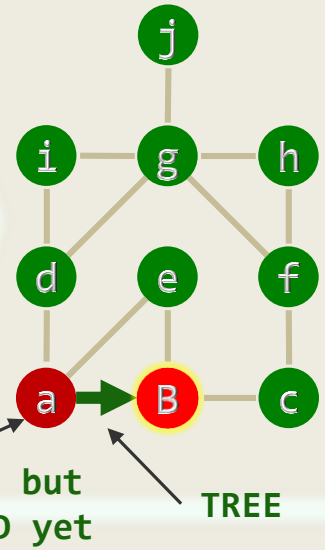

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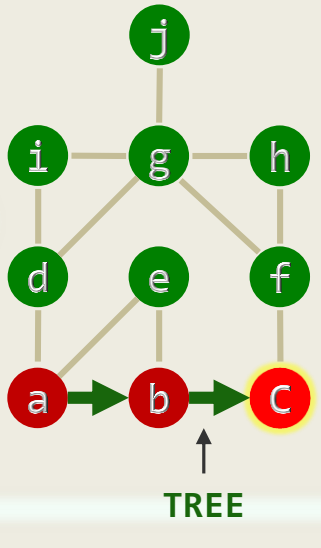

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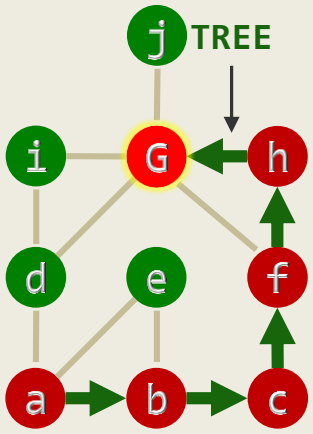

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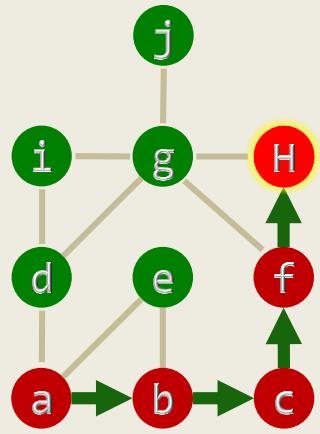

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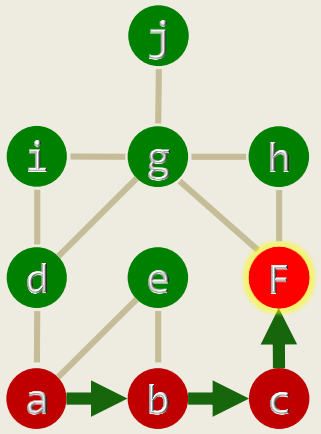

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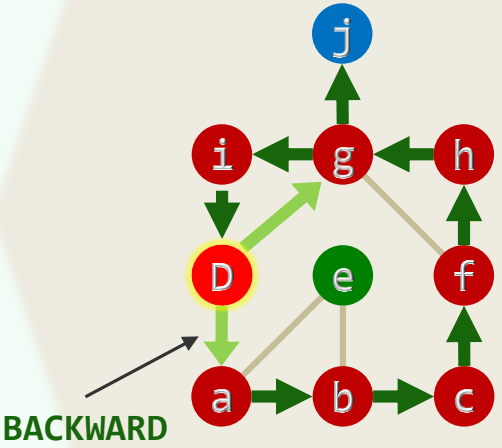

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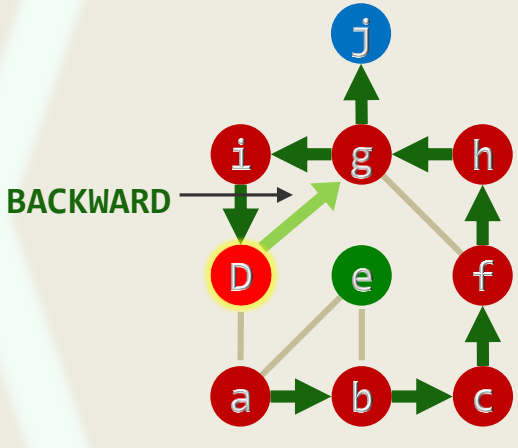

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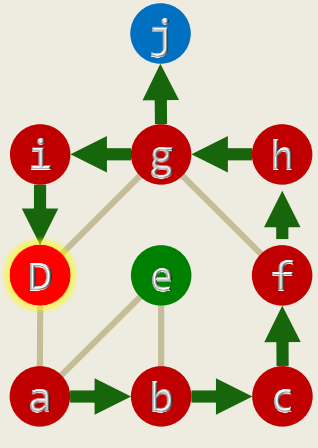

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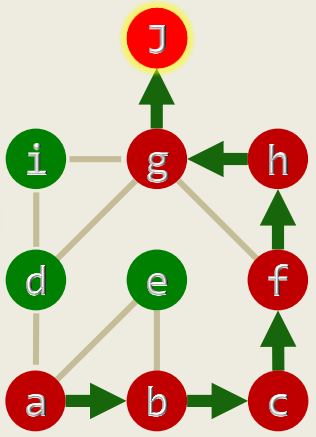

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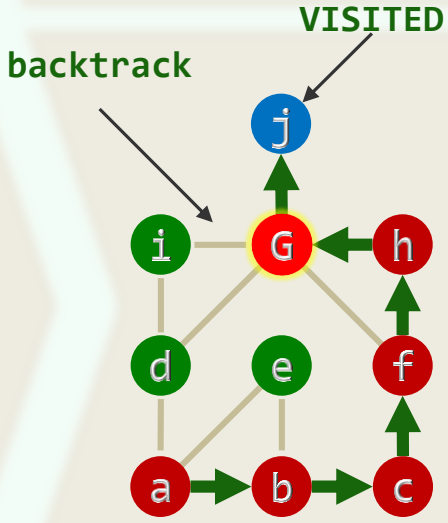

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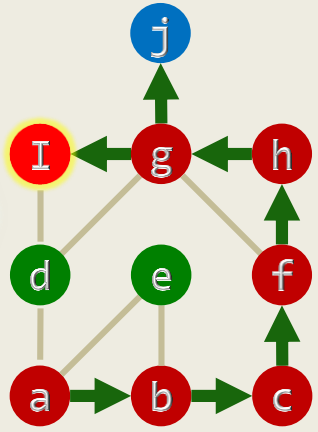

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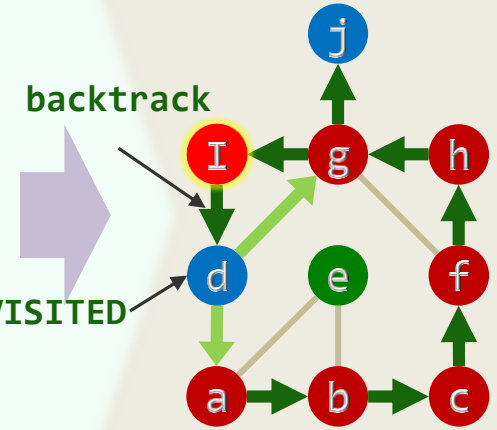
3/4



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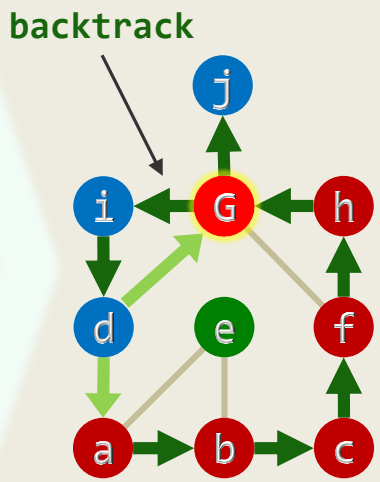

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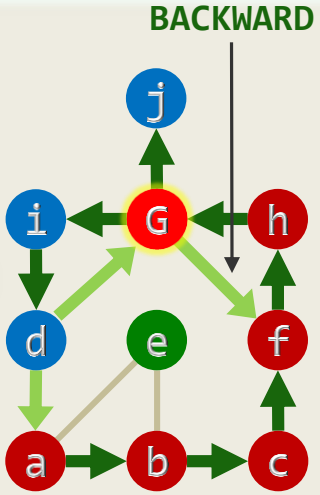

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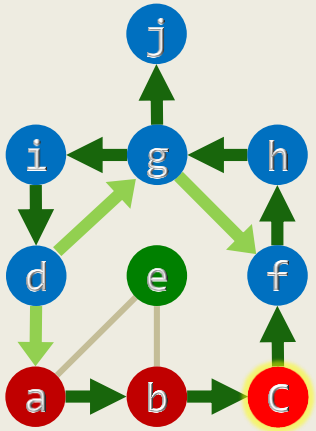

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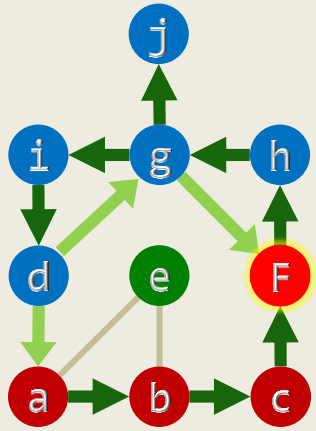

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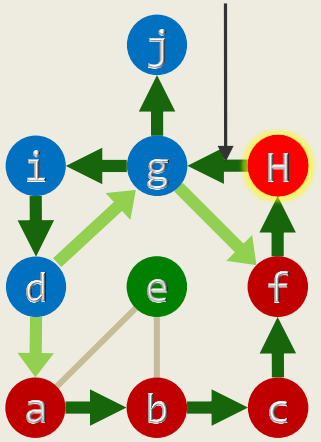

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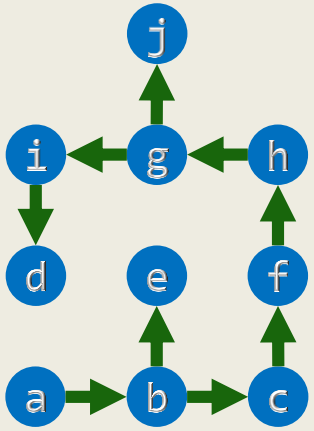

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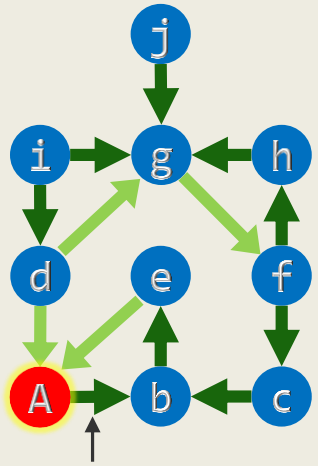

backtrack

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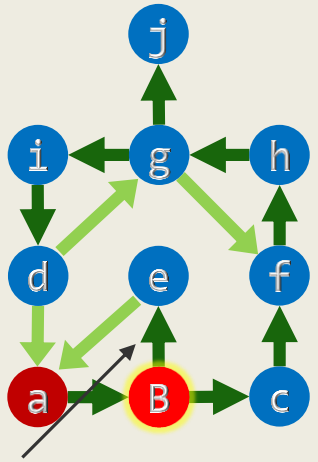
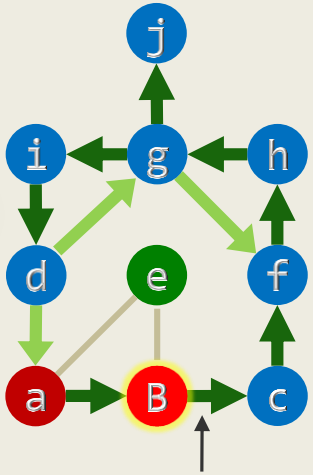
f	3	14
g	5	12
h	4	13
i	8	11
j	6	7



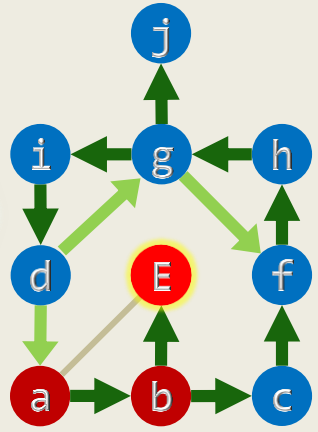
backtrack

a	0	
B	1	
c	2	15
d	9	10
e	16	17
f	3	14
g	5	12
h	4	13
i	8	11
j	6	7

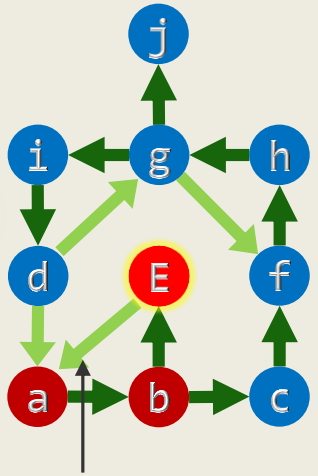


backtrack

a	0	
B	1	
c	2	15
d	9	10
e		
f	3	14
g	5	12
h	4	13
i	8	11
j	6	7



a	0	
b	1	
c	2	15
d	9	10
E	16	
f	3	14
g	5	12
h	4	13
i	8	11
j	6	7



BACKWARD

a	0	
b	1	
c	2	15
d	9	10
E	16	
f	3	14
g	5	12
h	4	13
i	8	11
j	6	7

图

深度优先搜索：推广

10-E3

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我要到N进K学堂去了，仿佛是想走异路，逃异地，去寻求别样的人们。

非连通

❖ 与BFS(v)类似，DFS(v)也可遍历v所属分量

——若含多个分量呢?

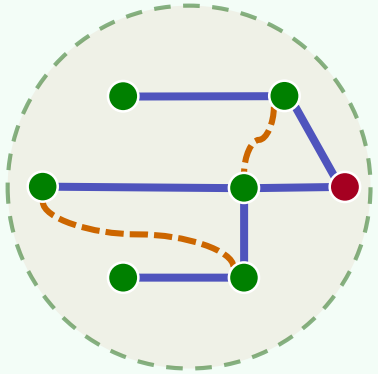
❖ 与bfs(s)类似（采用邻接表）

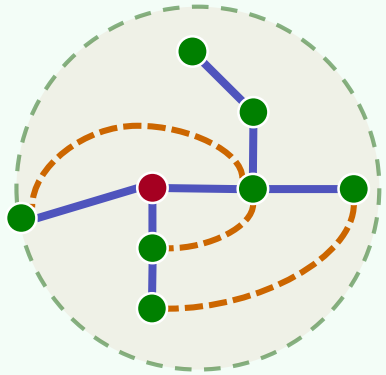
dfs(s)也可在累计$O(n+e)$时间内

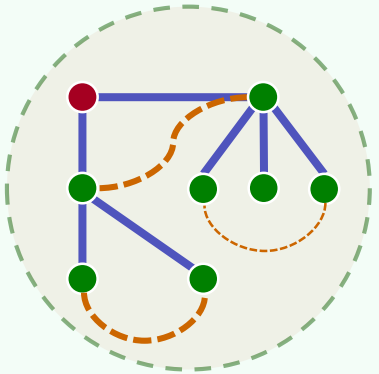
- 对于每一连通/可达分量

  从其起始顶点v进入DFS(v)恰好1次，并

- 最终生成一个DFS森林（包含 c 棵树、n-c 条树边）

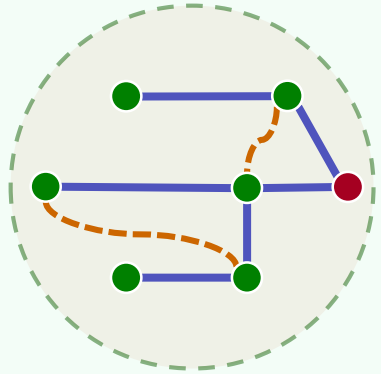


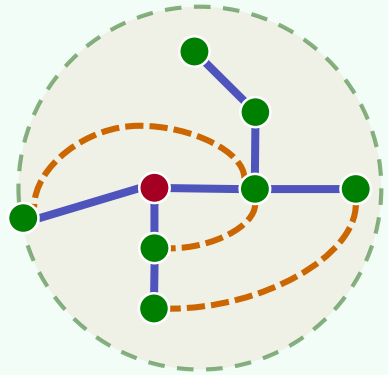


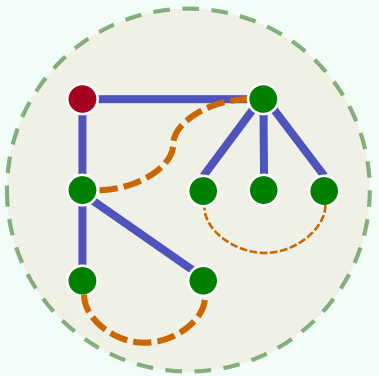


Graph::dfs()

```
template <typename Tv, typename Te>

void Graph<Tv, Te>::dfs( Rank s ) { //s < n

   reset(); Rank clock = 0; //全图复位

   for ( Rank v = s; v < s + n; v++ ) //从s起顺次检查所有顶点

      if ( UNDISCOVERED == status(v % n) ) //一旦遇到尚未发现者

         DFS( v % n, clock ); //即从它出发启动一次DFS

} //如此可完整覆盖全图，且总体复杂度依然保持为O(n+e)
```







图

深度优先搜索：实例（有向图）

10-E4

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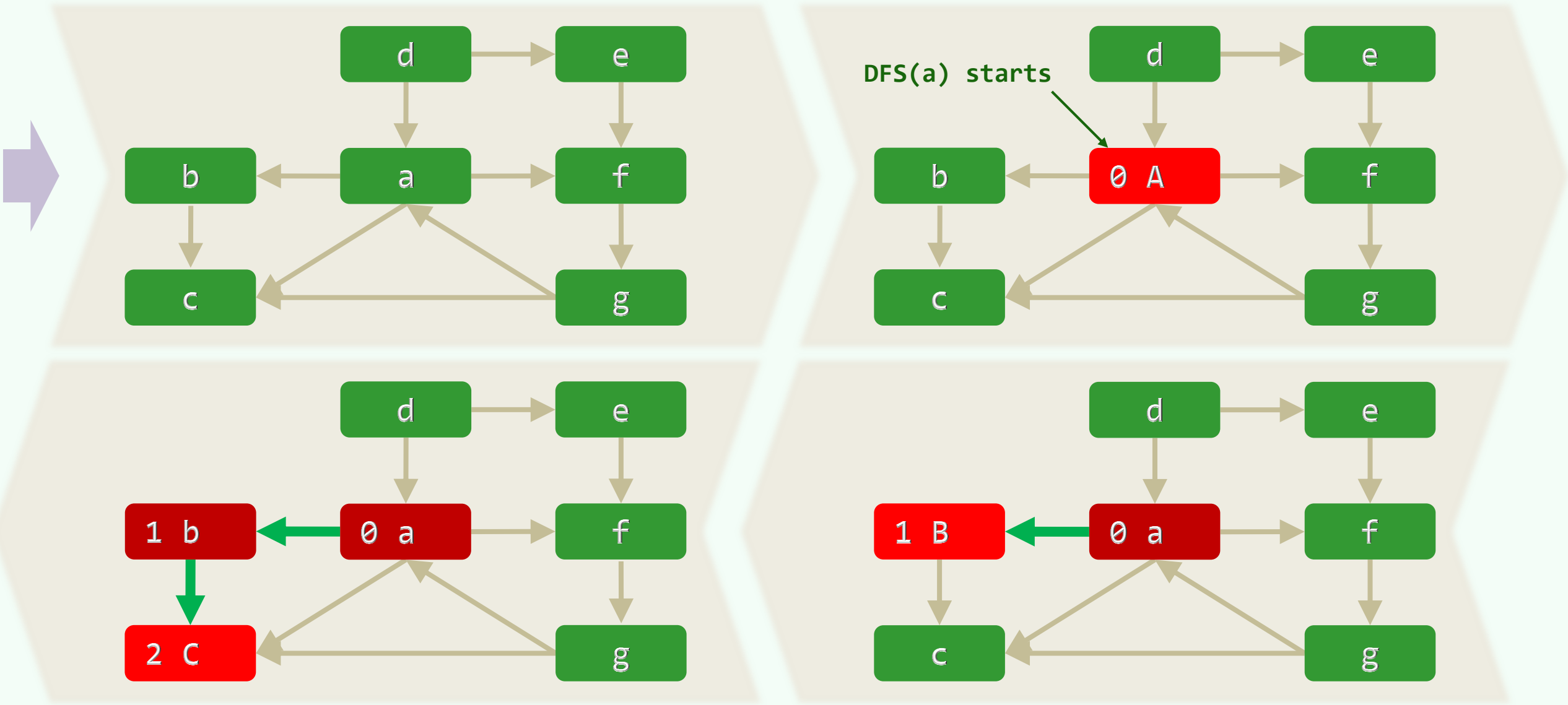
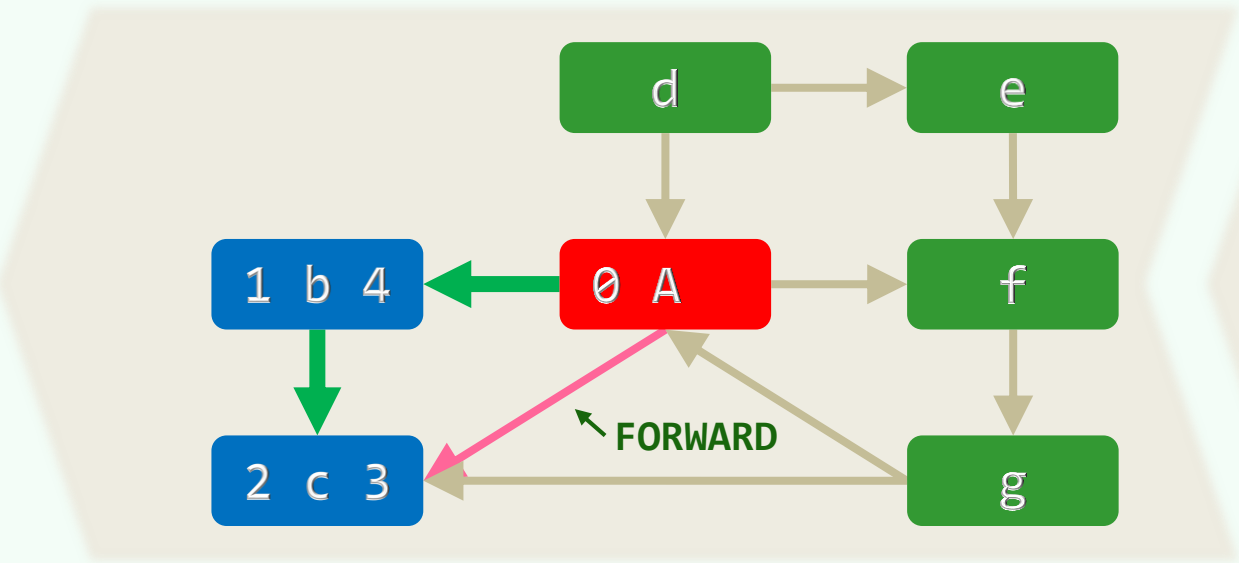
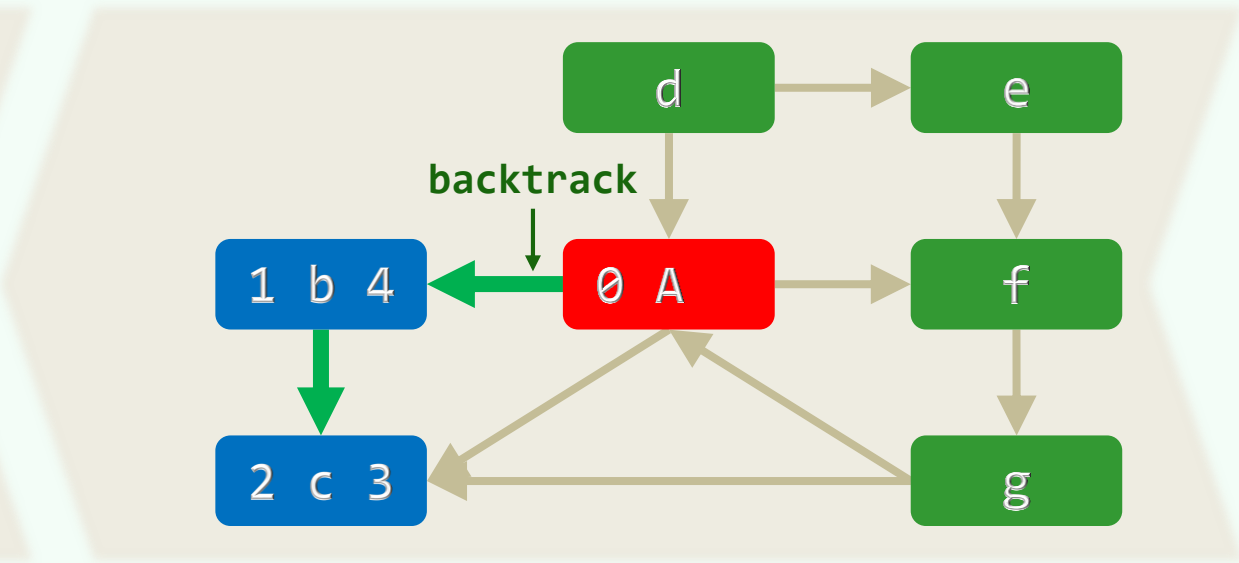
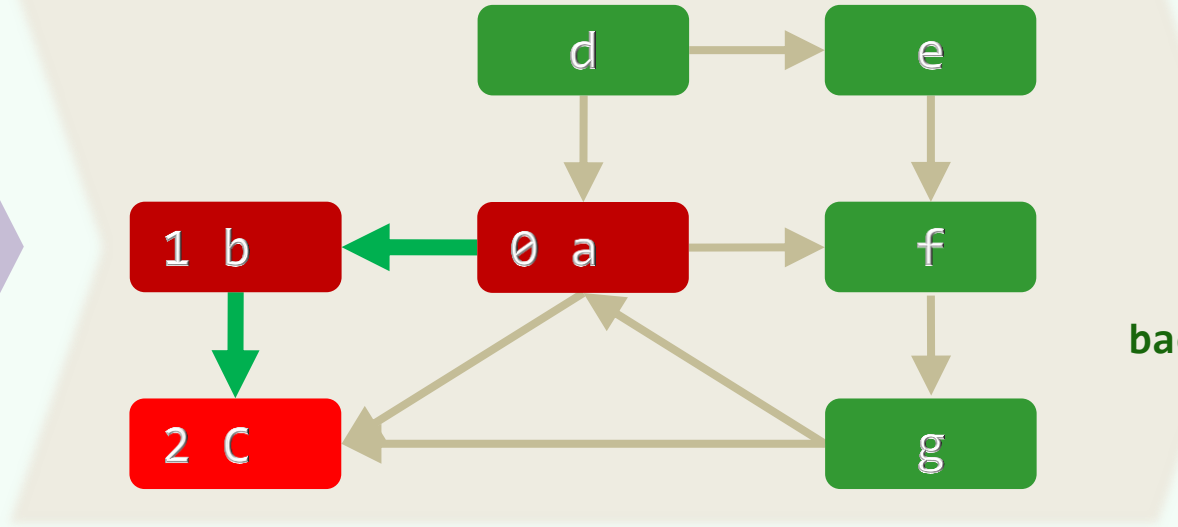
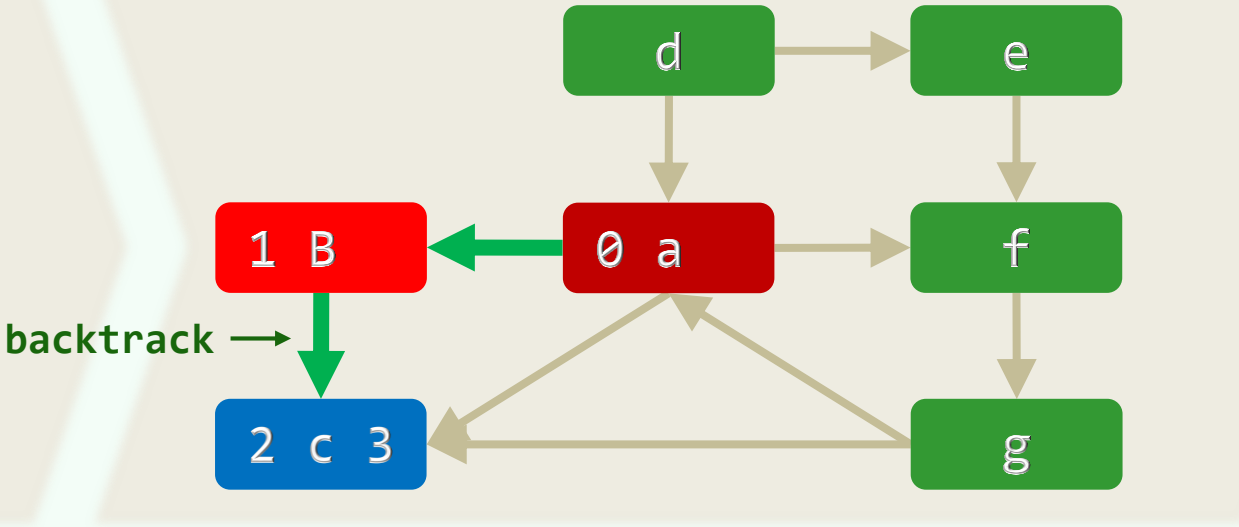
時率意獨駕，不由徑路；車跡所窮，輒慟哭而反

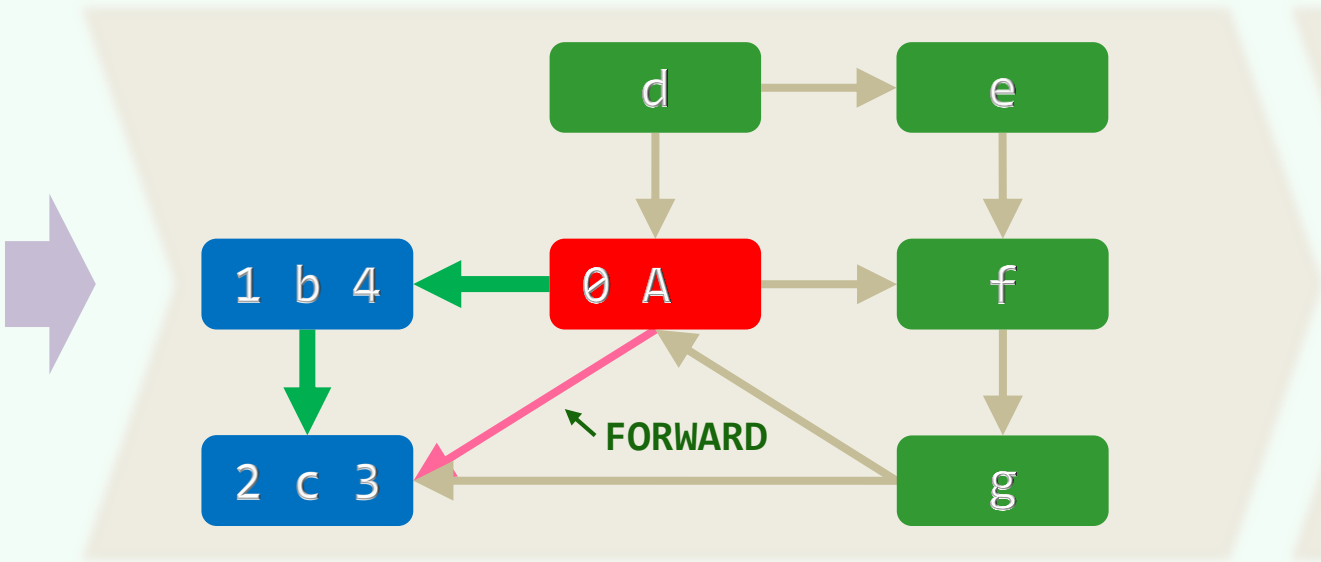

d
e
b
a
f
c
g
DFS(a) starts
0 A
1 b
2 C
1 B
0 a

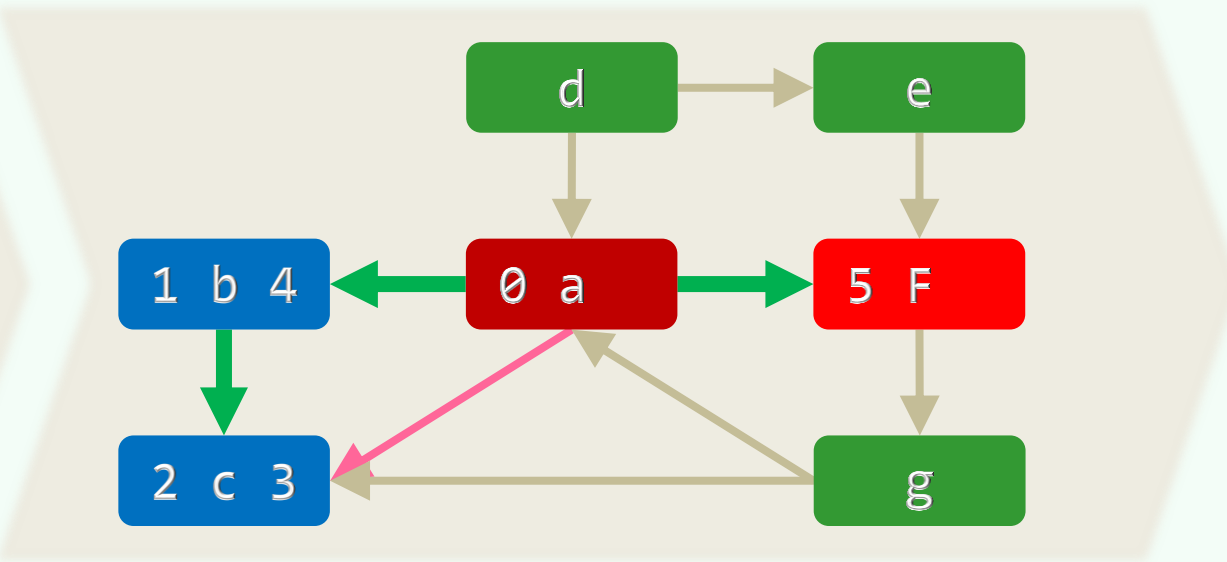

d
e
1 b 4
0 A
f
FORWARD
2 c 3
g

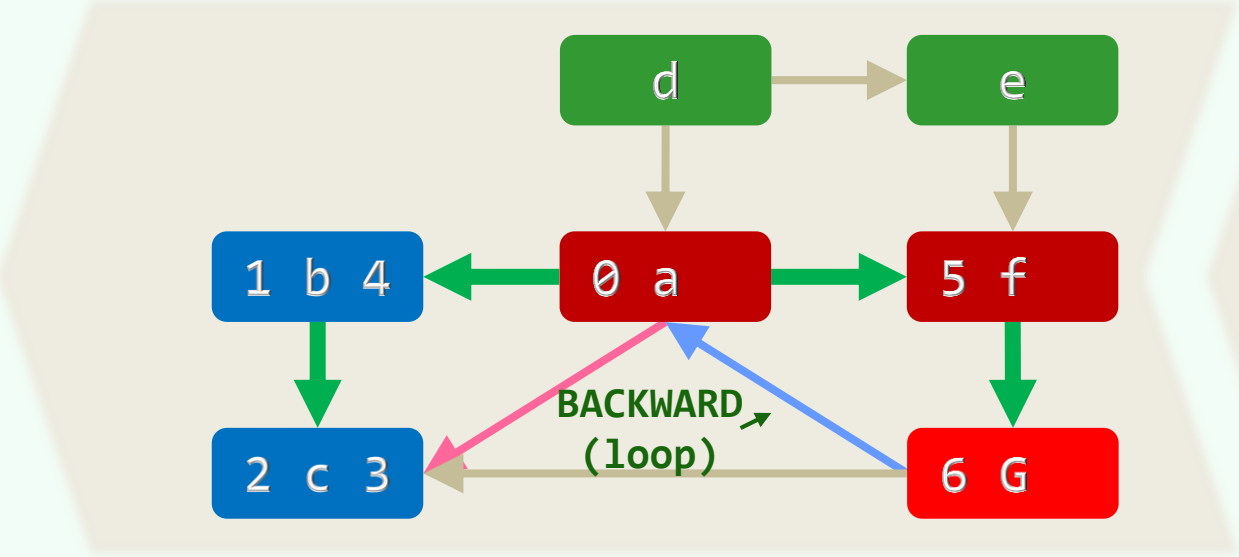

d
e
backtrack
1 b 4
0 A
f
2 c 3
g

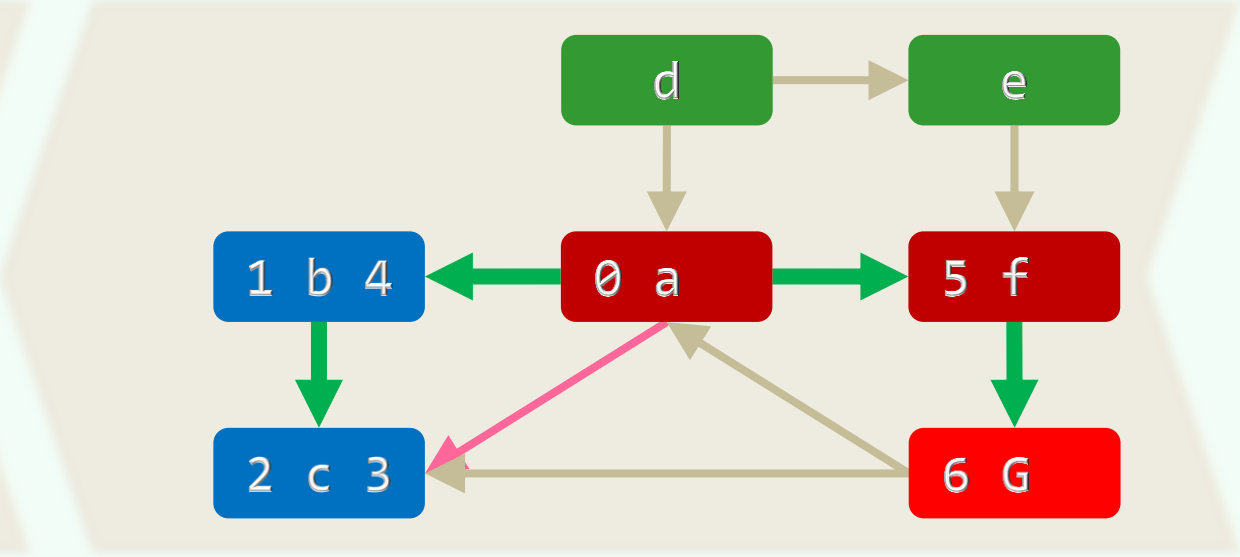

d
e
1 b
0 a
f
2 C
g

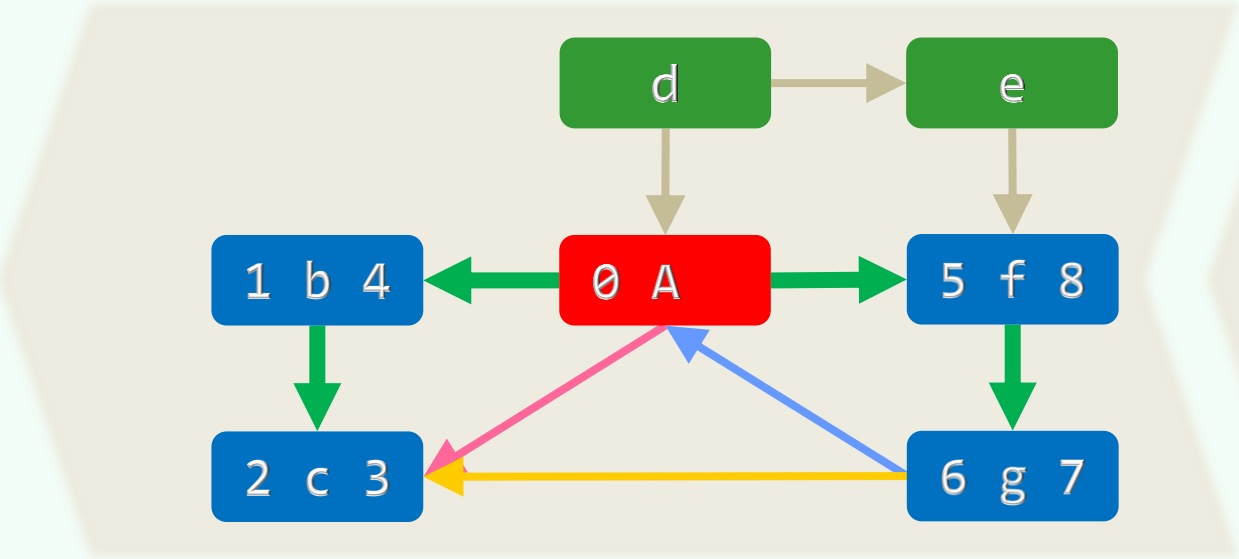

d
e
1 B
0 a
f
backtrack
2 c 3
g

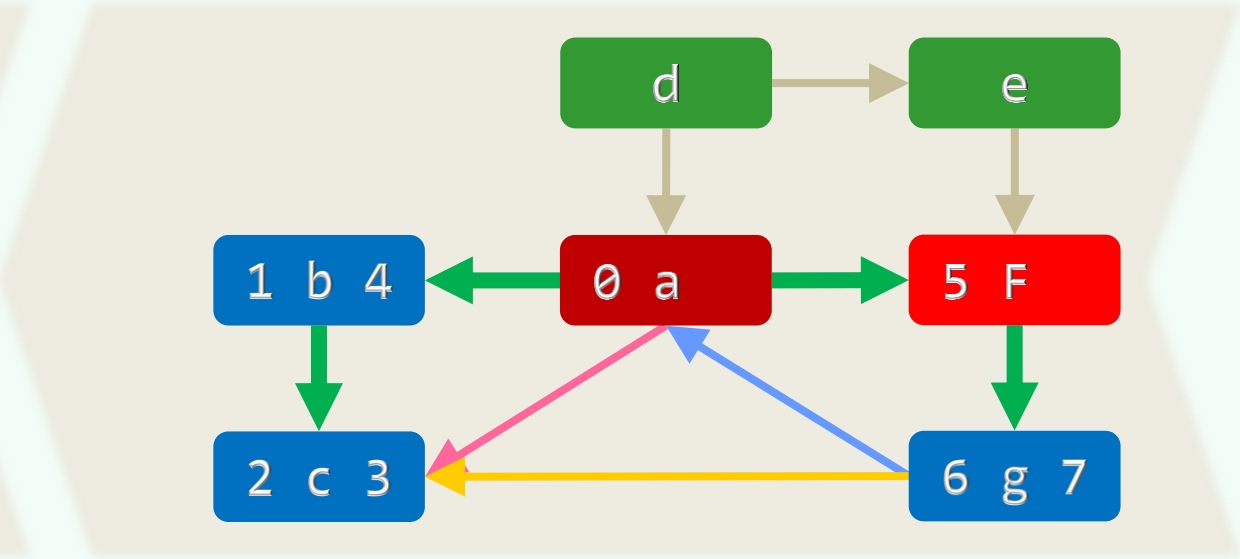

d
e
1 b 4
0 A
f
FORWARD
2 c 3
g

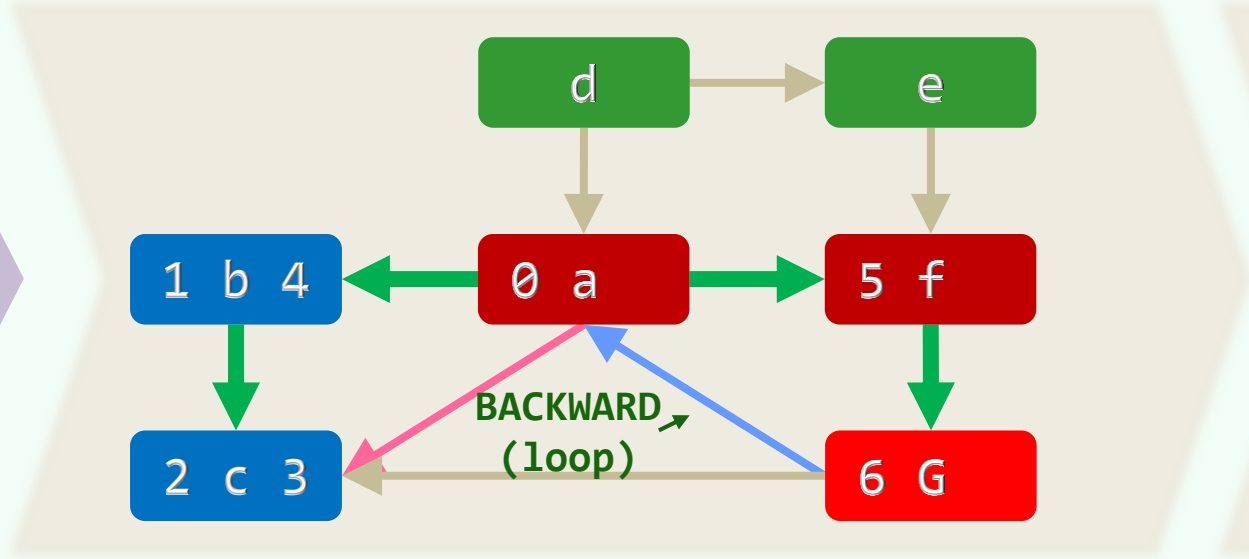

d
e
1 b 4
0 a
5 F
2 c 3
g

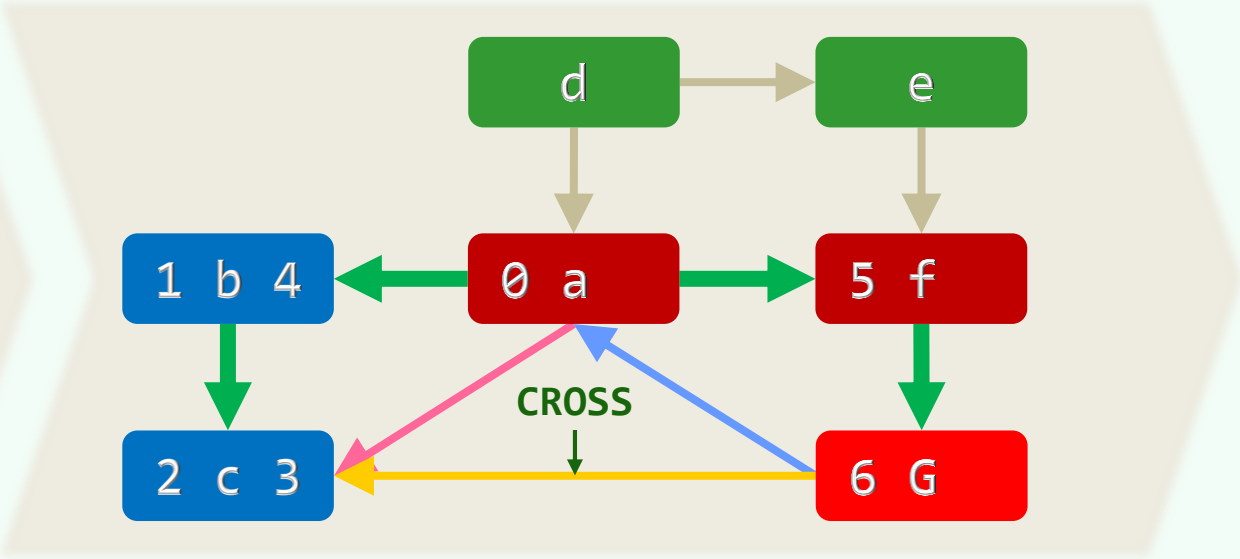

d
e
1 b 4
0 a
5 f
BACKWARD
(loop)
2 c 3
6 G

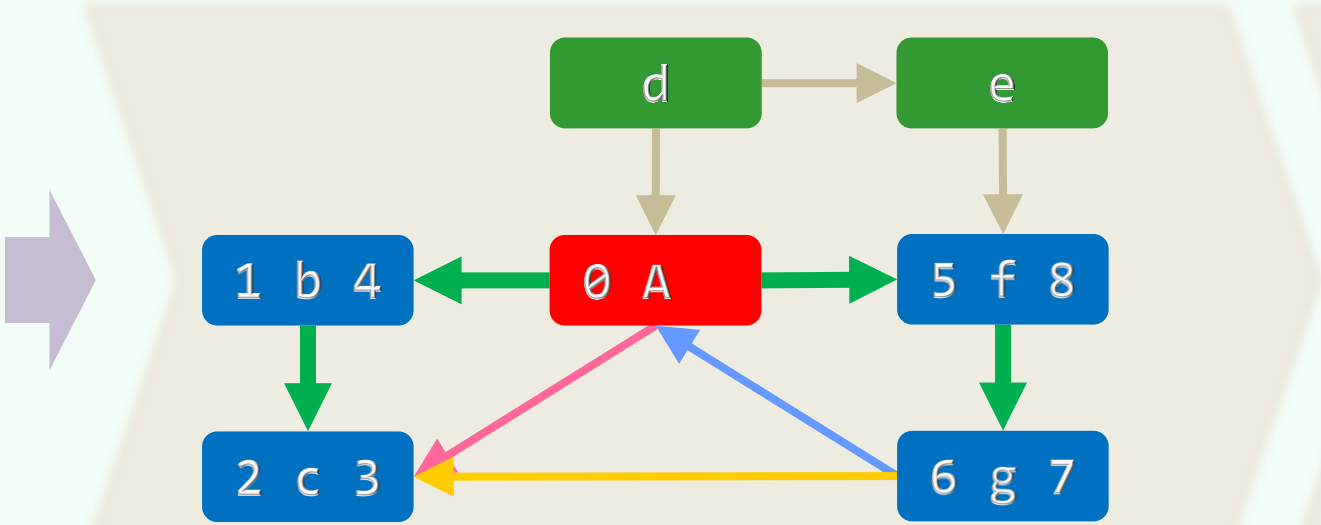

d
e
1 b 4
0 a
5 f
2 c 3
6 G

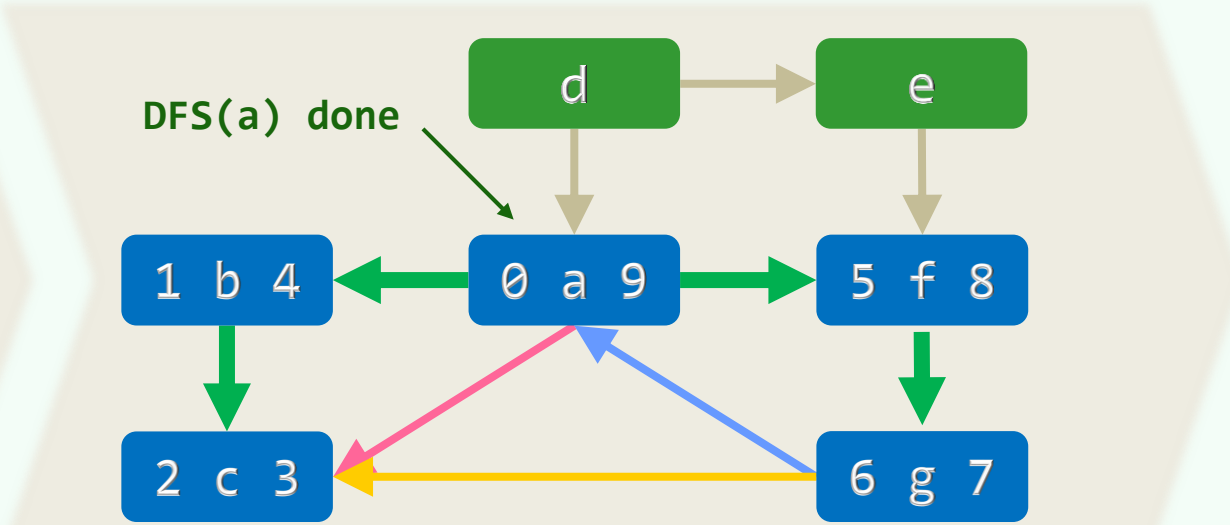

d
e
1 b 4
0 A
5 f 8
2 c 3
6 g 7

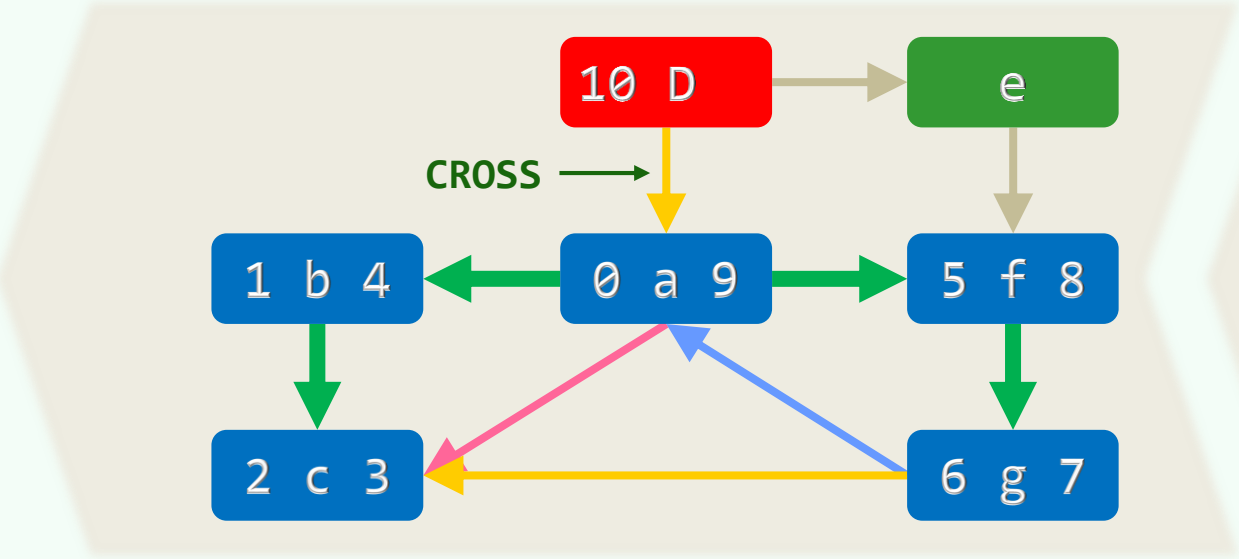

d
e
1 b 4
0 a
5 F
2 c 3
6 g 7

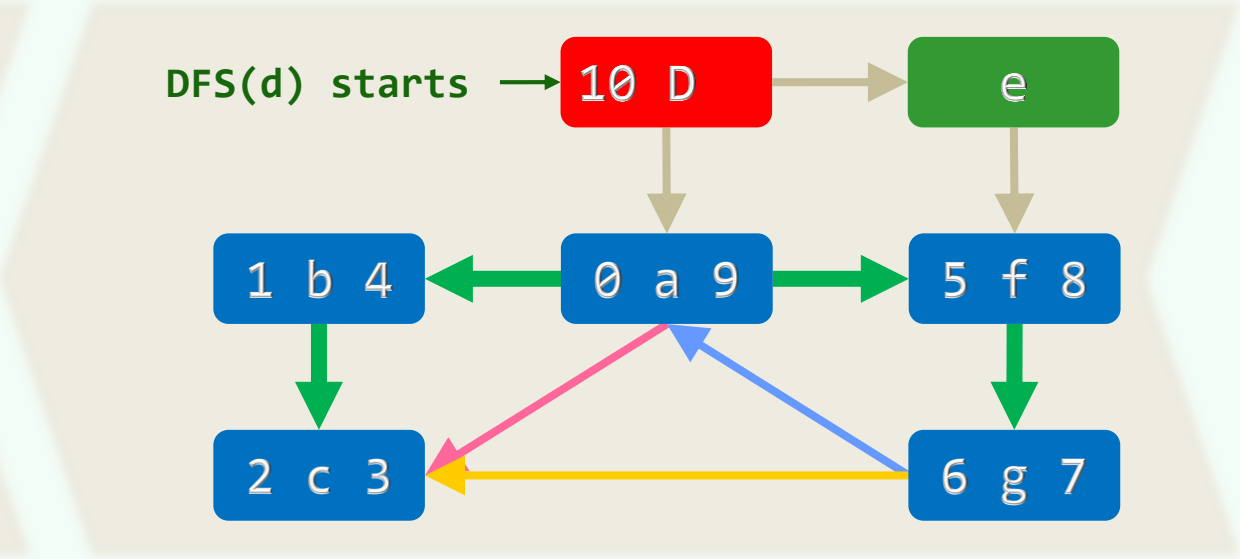

d
e
1 b 4
0 a
5 f
BACKWARD
(loop)
2 c 3
6 G

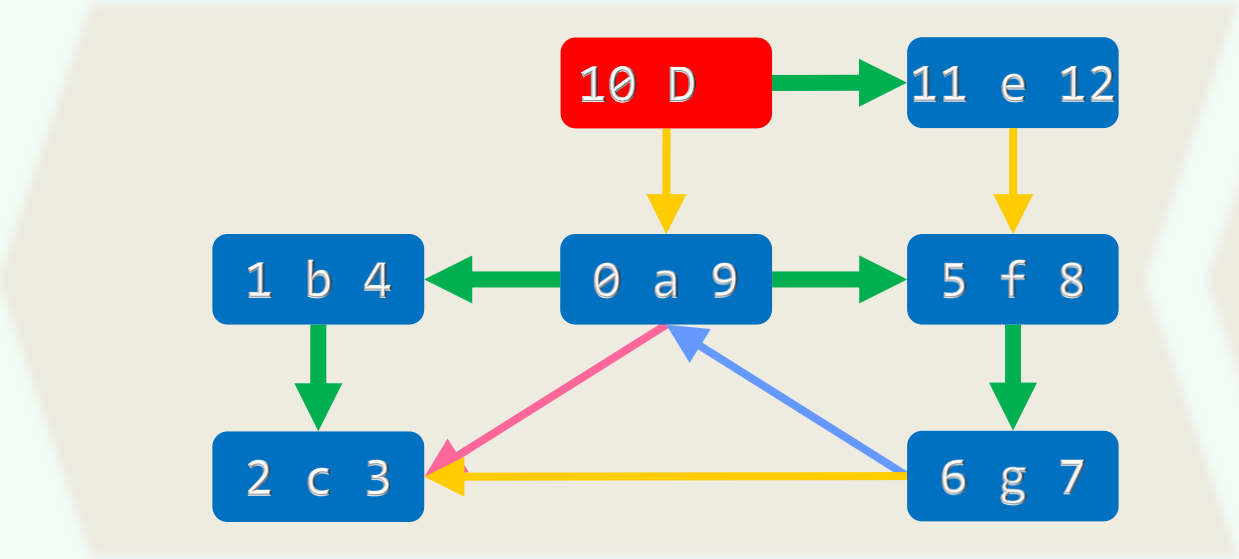

d
e
1 b 4
0 a
5 f
CROSS
2 c 3
6 G

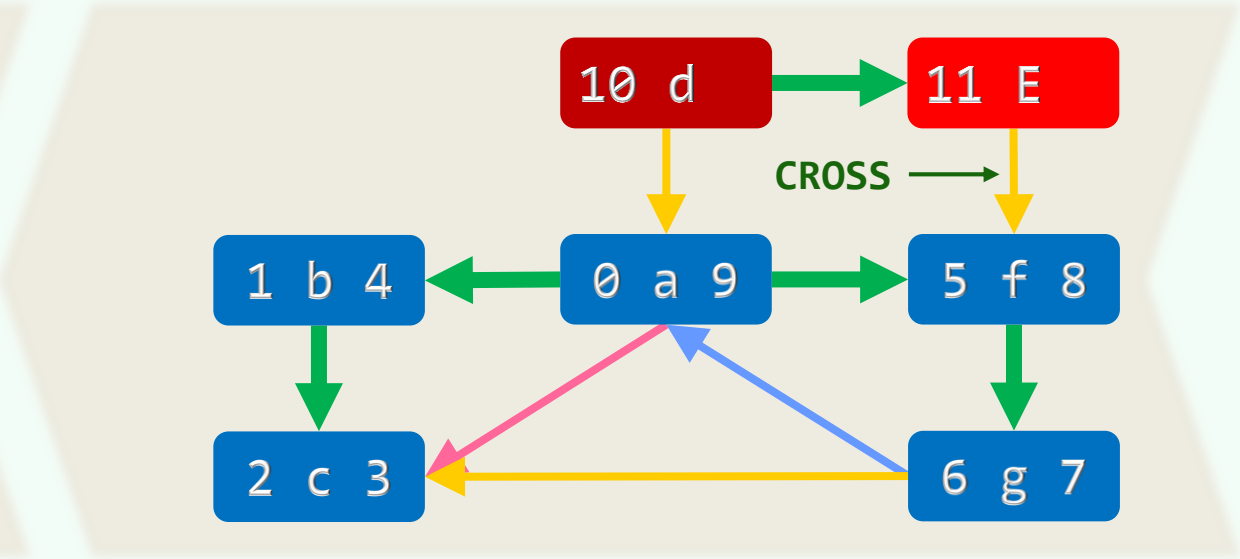

d
e
1 b 4
0 A
5 f 8
2 c 3
6 g 7

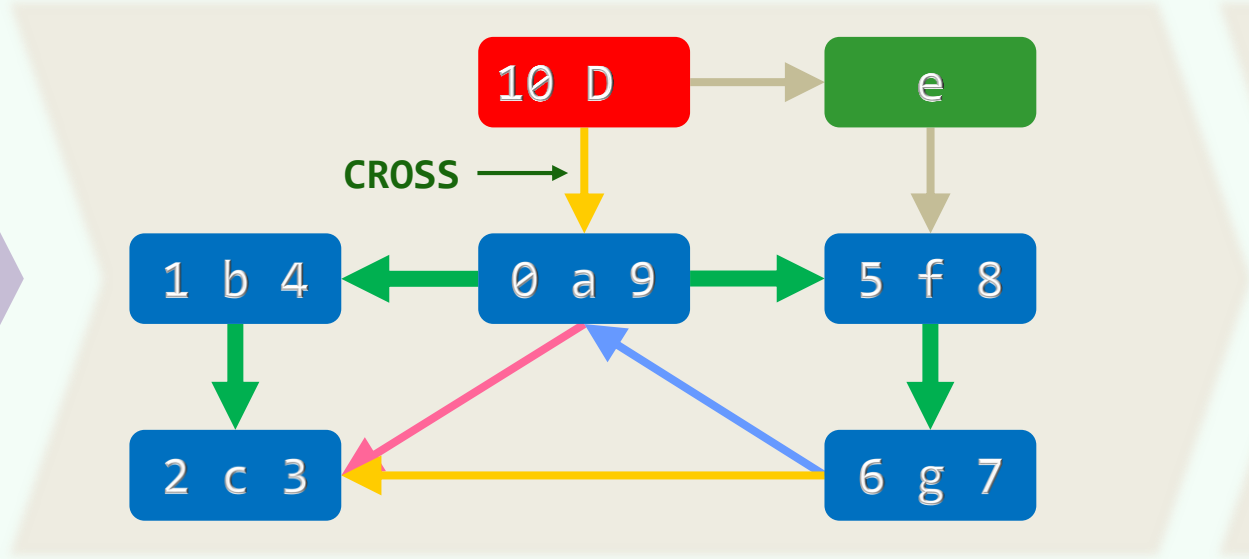

DFS(a) done
d
e
1 b 4
0 a 9
5 f 8
2 c 3
6 g 7

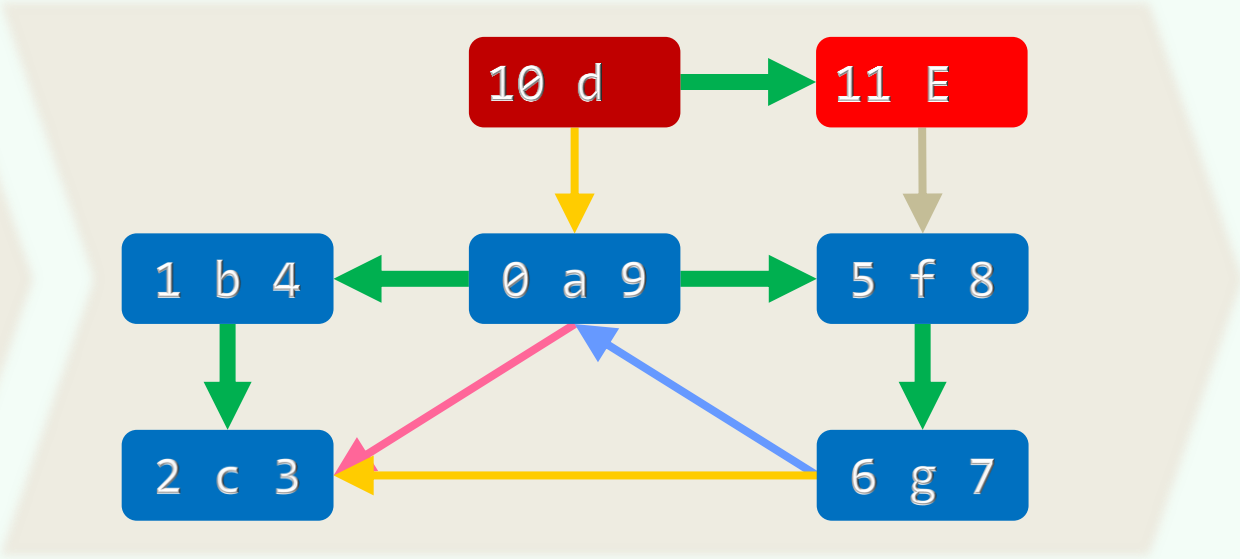

10 D
e
CROSS
1 b 4
0 a 9
5 f 8
2 c 3
6 g 7

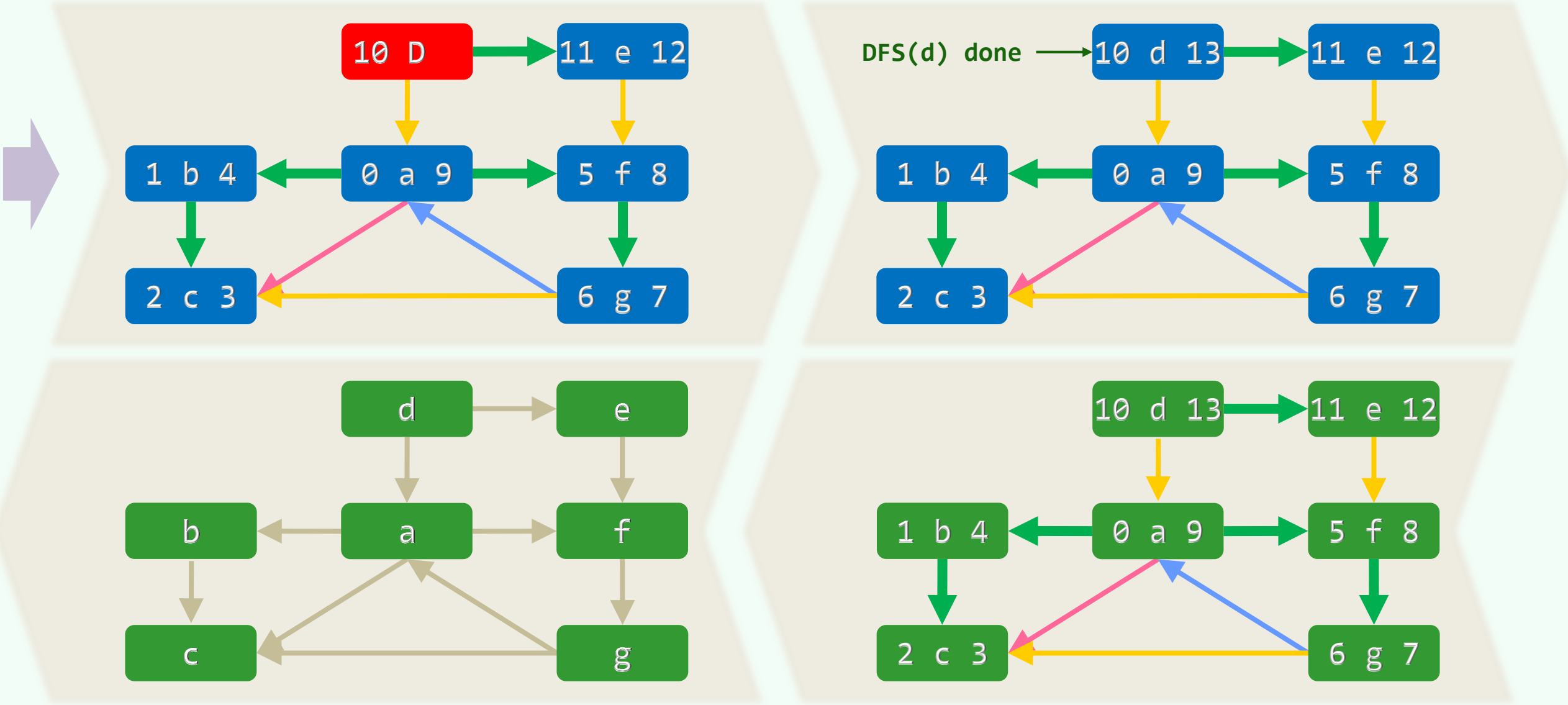

DFS(d) starts
10 D
e
1 b 4
0 a 9
5 f 8
2 c 3
6 g 7


10 D
11 e 12
1 b 4
0 a 9
5 f 8
2 c 3
6 g 7


10 d
11 E
CROSS
1 b 4
0 a 9
5 f 8
2 c 3
6 g 7


10 D
e
CROSS
1 b 4
0 a 9
5 f 8
2 c 3
6 g 7


10 d
11 E
1 b 4
0 a 9
5 f 8
2 c 3
6 g 7


10 D
11 e 12
1 b 4
0 a 9
5 f 8
2 c 3
6 g 7
DFS(d) done
10 d 13
11 e 12
1 b 4
0 a 9
5 f 8
2 c 3
6 g 7
d
e
b
a
f
c
g
10 d 13
11 e 12
1 b 4
0 a 9
5 f 8
2 c 3
6 g 7

图

深度优先搜索：性质

10-E5

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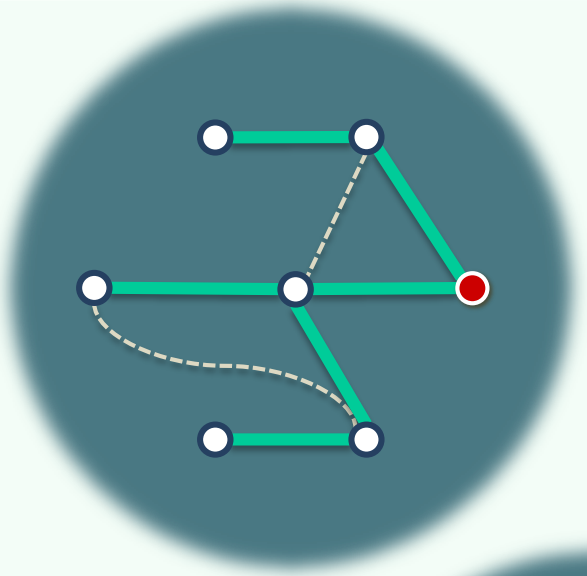
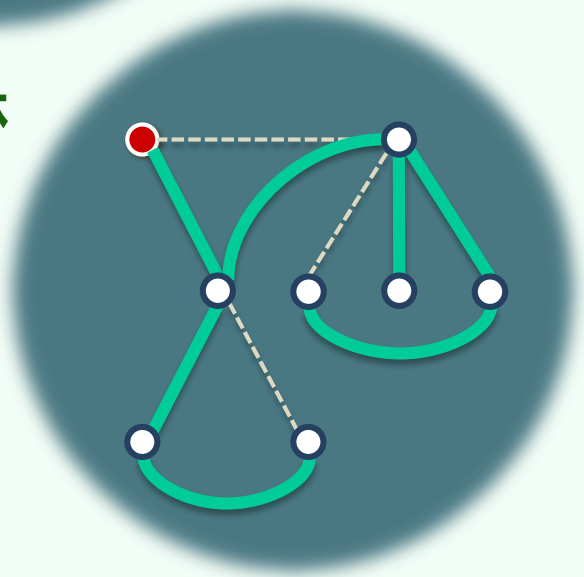
身后有余忘缩手，眼前无路想回头

DFS树/森林

- 从顶点s出发的DFS
  - 在无向图中将访问与s连通的所有顶点（connectivity）
  - 在有向图中将访问由s可达的所有顶点（reachability）



- 经DFS确定的树边，不会构成回路
- 从s出发的DFS，将以s为根生成一棵DFS树；所有DFS树，进而构成DFS森林



- DFS树及森林由parent指针描述（只不过所有边取反向）
- DFS之后，我们已经知道森林乃至原图的全部信息了吗？

  就某种意义而言，是的...

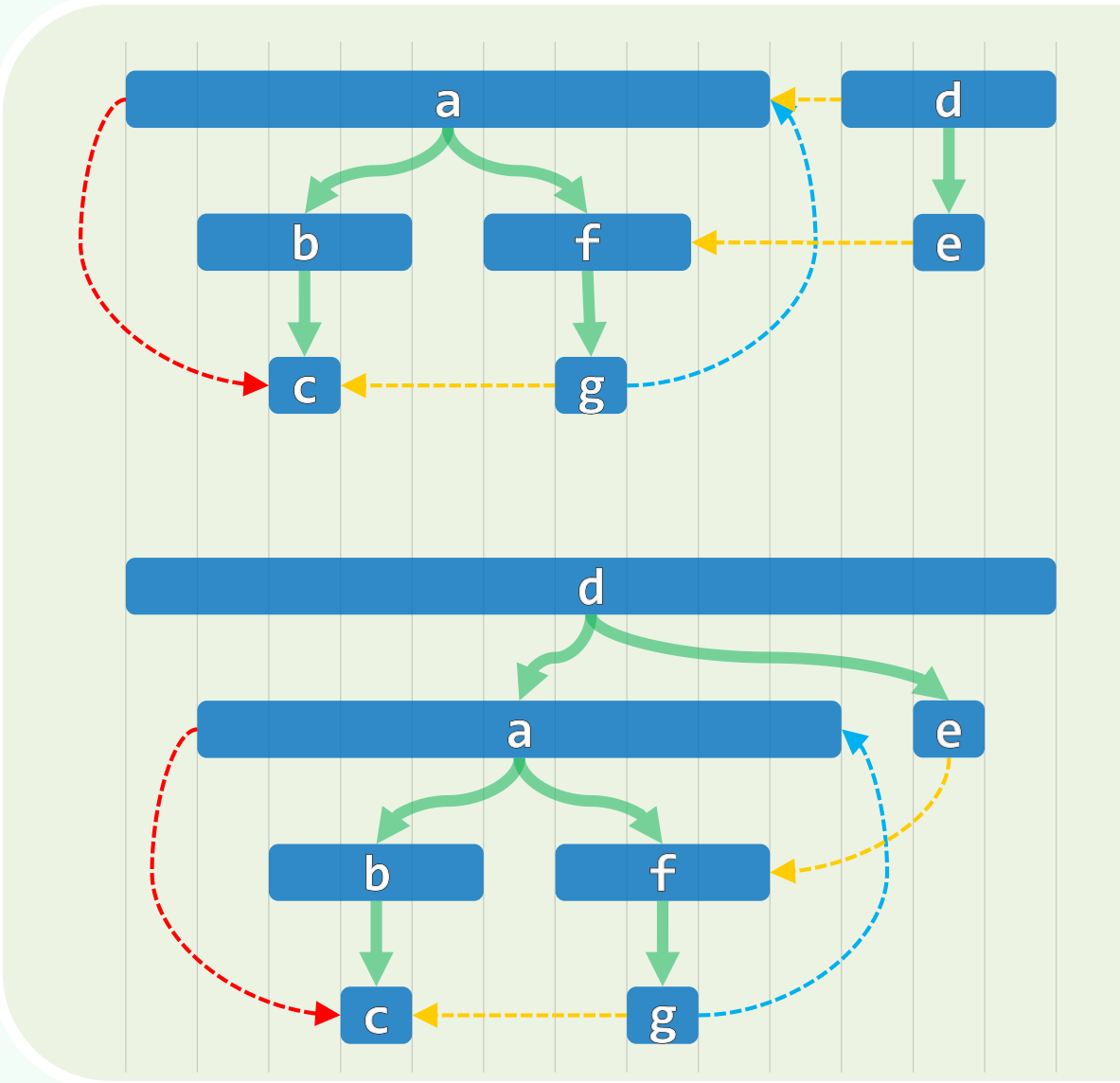
活跃期 & 括号引理

- $active[u] = (dTime[u], fTime[u])$
- 【Parenthesis Lemma】

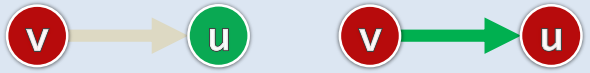
  给定有向图G = (V, E)及其任一DFS森林，则

  - u是v的后代 iff $active[u] \subseteq active[v]$
  - u是v的祖先 iff $active[u] \supseteq active[v]$
  - u与v“无关” iff $active[u] \cap active[v] = \varnothing$
- 仅凭status[]、dTime[]和fTime[]

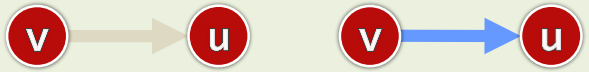
  即可对各边分类...



边分类

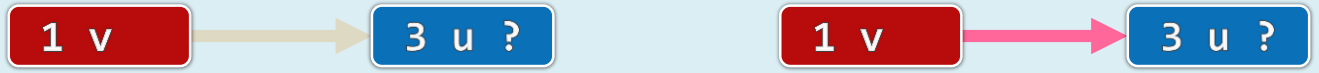
❖ TREE(v, u): 

可从当前v进入处于UNDISCOVERED状态的u

❖ BACKWARD(v, u): 

试图从当前v进入处于DISCOVERED状态的u

DFS发现后向边 iff 存在回路 //后向边数 == 回路数?

❖ FORWARD(v, u): 

试图从当前顶点v进入处于VISITED状态的u，且v更早被发现

❖ CROSS(v, u): 

试图从当前顶点v进入处于VISITED状态的u，且u更早被发现

遍历算法应用举例

连通图的支撑树（DFS/BFS Tree）	DFS/BFS
非连通图的支撑森林	DFS/BFS
连通性检测	DFS/BFS
无向图环路检测/二部图判定	DFS/BFS
有向图环路检测	DFS
顶点之间可达性检测/路径求解	DFS/BFS
顶点之间的最短距离	BFS
直径/半径/围长/中心	BFS
欧拉回路	DFS
拓扑排序	DFS
双连通分量、强连通分量分解	DFS
...	...