

绪论

复杂度分析：级数

$\Theta(1) - D1$

谁校对时间，谁就会突然老去。

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算法分析

- ❖ 两个主要任务 = 正确性 (不变性 x 单调性) + 复杂度
- ❖ 为确定后者, 真地需要将算法描述为RAM的基本指令, 再累计各条代码的执行次数? 不必!
- ❖ C++等高级语言的基本指令, 均等效于常数条RAM的基本指令; 在渐近意义下, 二者大体相当
 - 分支转向: `goto` // 算法的灵魂; 为结构化而被隐藏了而已
 - 迭代循环: `for()`、`while()`、... // 本质上就是 “if + goto”
 - 调用 + 递归 (自我调用) // 本质上也是goto
- ❖ 主要方法: 迭代 (级数求和)、递归 (递归跟踪 + 递推方程)、实用 (猜测 + 验证)

级数

❖ **算术级数：与末项平方同阶** $T(n) = 1 + 2 + \dots + n = \frac{n(n+1)}{2} = \mathcal{O}(n^2)$

❖ **幂方级数：比幂次高出一阶** $\sum_{k=0}^n k^d \approx \int_0^n x^d dx = \frac{x^{d+1}}{d+1} \Big|_0^n = \frac{n^{d+1}}{d+1} = \mathcal{O}(n^{d+1})$

$$T_2(n) = \sum_{k=1}^n k^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} = \mathcal{O}(n^3)$$

$$T_3(n) = \sum_{k=1}^n k^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4} = \mathcal{O}(n^4)$$

$$T_4(n) = \sum_{k=1}^n k^4 = 1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = \mathcal{O}(n^5)$$

❖ **几何级数：与末项同阶**

$$T_a(n) = \sum_{k=0}^n a^k = a^0 + a^1 + a^2 + a^3 + \dots + a^n = \frac{a^{n+1} - 1}{a - 1} = \mathcal{O}(a^n), \quad 1 < a$$

$$T_2(n) = \sum_{k=0}^n 2^k = 1 + 2 + 4 + 8 + \dots + 2^n = 2^{n+1} - 1 = \mathcal{O}(2^{n+1}) = \mathcal{O}(2^n)$$

收敛级数

$$\diamond \quad \sum_{k=2}^n \frac{1}{(k-1) \cdot k} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{(n-1) \cdot n} = 1 - \frac{1}{n} = \mathcal{O}(1)$$

$$\sum_{k=1}^n \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} < 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6} = \mathcal{O}(1)$$

$$\sum_{k \text{ is a perfect power}} \frac{1}{k-1} = \frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \frac{1}{31} + \frac{1}{35} + \dots = 1 = \mathcal{O}(1)$$

$$\diamond \text{ 几何分布: } (1 - \lambda) \cdot [1 + 2\lambda + 3\lambda^2 + 4\lambda^3 + \dots] = 1/(1 - \lambda) = \mathcal{O}(1), \quad 0 < \lambda < 1$$

❖ 有必要讨论这类级数吗？

难道，基本操作次数、存储单元数可能是分数？是的，某种意义上的是！

不收敛，但有限

❖ 调和级数: $h(n) = \sum_{k=1}^n \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} = \ln n + \gamma + \mathcal{O}(\frac{1}{2n}) = \Theta(\log n)$

❖ 对数级数: $\sum_{k=1}^n \ln k = \ln \prod_{k=1}^n k = \ln n! \approx (n + 0.5) \cdot \ln n - n = \Theta(n \cdot \log n)$

❖ 对数 + 线性 + 指数: $\sum_{k=1}^n k \cdot \log k \approx \int_1^n x \ln x dx = \frac{x^2 \cdot (2 \cdot \ln x - 1)}{4} \Big|_1^n = \mathcal{O}(n^2 \log n)$

$$\begin{aligned} \sum_{k=1}^n k \cdot 2^k &= \sum_{k=1}^n k \cdot 2^{k+1} - \sum_{k=1}^n k \cdot 2^k = \sum_{k=1}^{n+1} (k-1) \cdot 2^k - \sum_{k=1}^n k \cdot 2^k \\ &= n \cdot 2^{n+1} - \sum_{k=1}^n 2^k = n \cdot 2^{n+1} - (2^{n+1} - 2) = (n-1) \cdot 2^{n+1} + 2 = \mathcal{O}(n \cdot 2^n) \end{aligned}$$

❖ 如有兴趣，不妨读读: Concrete Mathematics

//ex-2.35, Goldbach Theorem