04-E

栈与队列

栈混洗

邓俊辉
deng@tsinghua.edu.cn

Stack Permutation

- 考查栈 $\mathcal{A} = \langle\, a_1,\ a_2,\ a_3,\ \ldots,\ a_n\,]$

  $\mathcal{B} = \mathcal{S} = \varnothing$

- 只允许
  - 将$\boxed{\text{A}}$的顶元素弹出并压入$\boxed{\text{S}}$，或
  - 将$\boxed{\text{S}}$的顶元素弹出并压入$\boxed{\text{B}}$

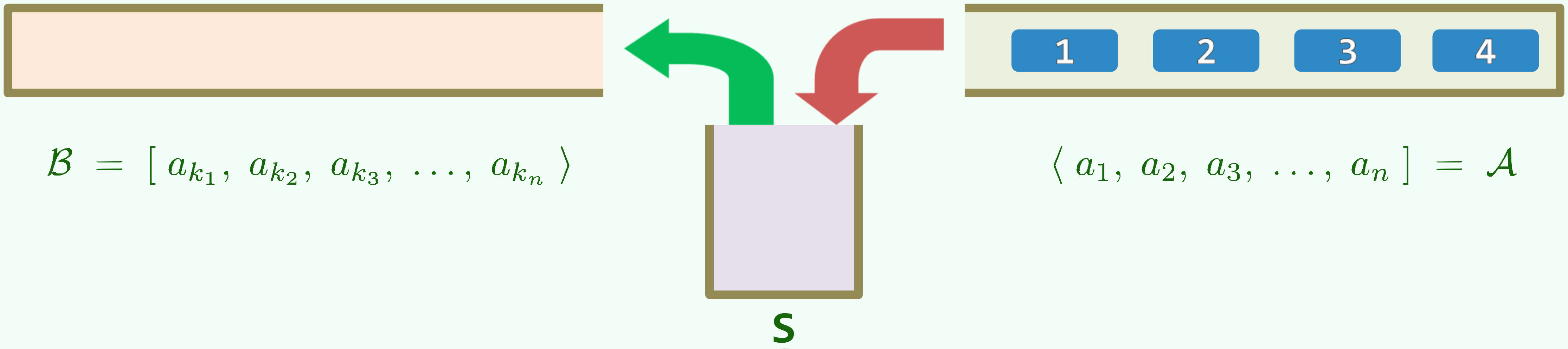
- 亦即 $\mathcal{S}.\text{push}(\mathcal{A}.\text{pop}())$

  $\mathcal{B}.\text{push}(\mathcal{S}.\text{pop}())$

- 若经一系列以上操作后，A中元素全部转入B中

  $\mathcal{B} = [\, a_{k_1},\ a_{k_2},\ a_{k_3},\ \ldots,\ a_{k_n}\, \rangle$

  则称为A的一个**栈混洗**



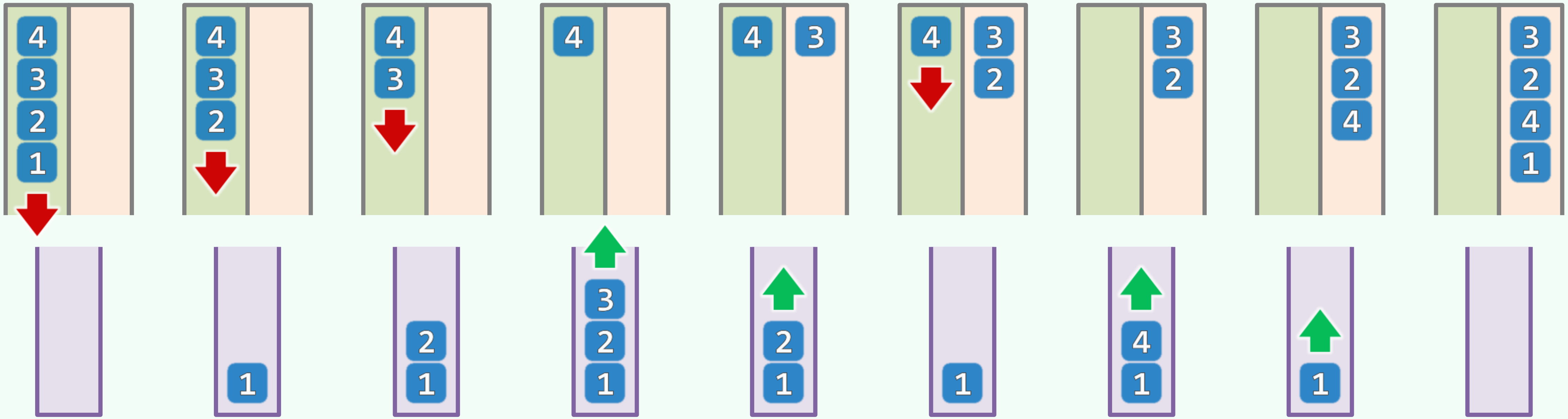
$\mathcal{B} = [\, a_{k_1},\ a_{k_2},\ a_{k_3},\ \ldots,\ a_{k_n}\, \rangle$

$\langle\, a_1,\ a_2,\ a_3,\ \ldots,\ a_n\,] = \mathcal{A}$

计数：SP(n)

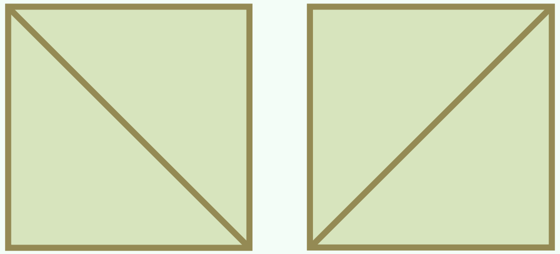
❖ 同一输入序列，可有多种栈混洗：[1, 2, 3, 4 >, [4, 3, 2, 1 >, [3, 2, 4, 1 > ...

❖ 一般地，对于长度为n的序列，混洗总数SP(n) = ?



❖ 显然，SP(n) <= n!；更准确地呢？

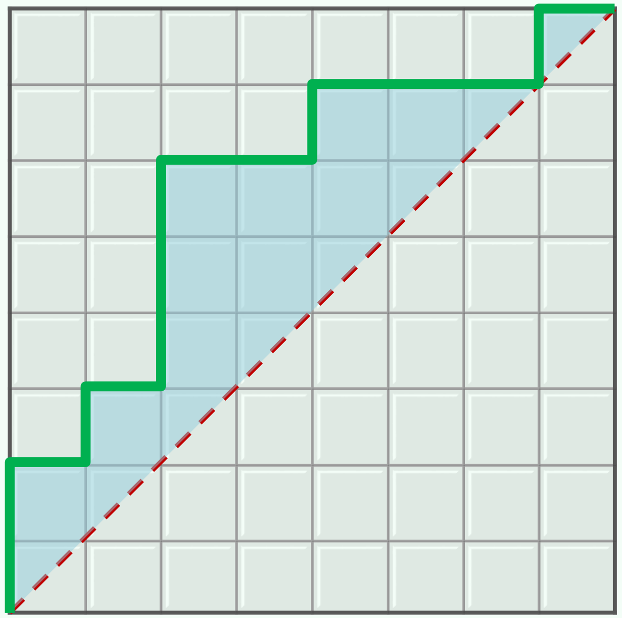
计数：catalan(n)

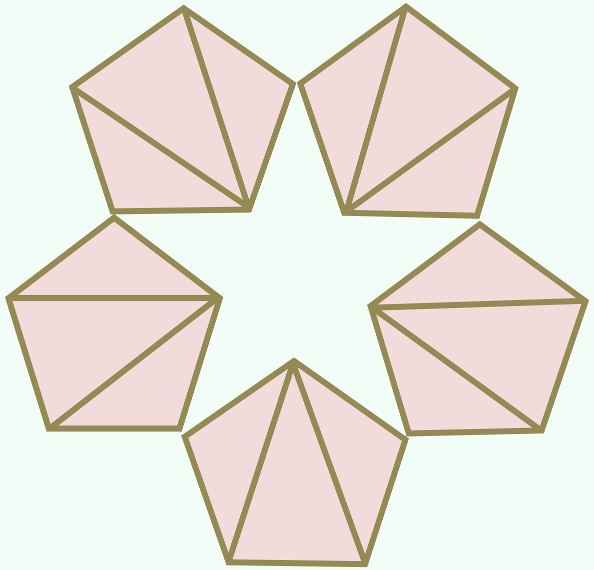


- $SP(1) = 1$
- **考查S再度变空（A首元素从S中弹出）的时刻，无非n种情况：**

$$SP(n) = \sum_{k=1}^{n} SP(k-1) \cdot SP(n-k)$$

$$= catalan(n) = \frac{(2n)!}{(n+1)! \cdot n!}$$





$$SP(2) = 4!/3!/2! = 2$$

$$SP(3) = 6!/4!/3! = 5$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$SP(6) = 12!/7!/6! = 132$$

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$$SP(n) = \sum_{k=1}^{n} SP(k-1) \cdot SP(n-k) = catalan(n) \doteq \frac{(2n)!}{n!\,(n+1)!}$$

k-1

B

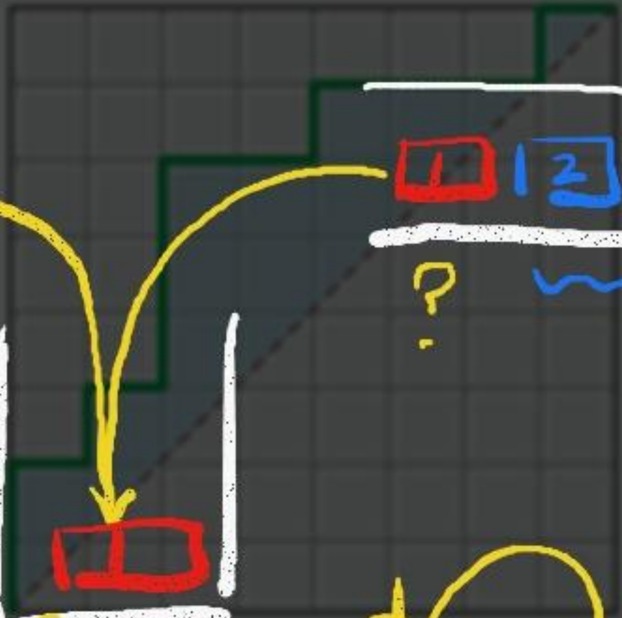
A

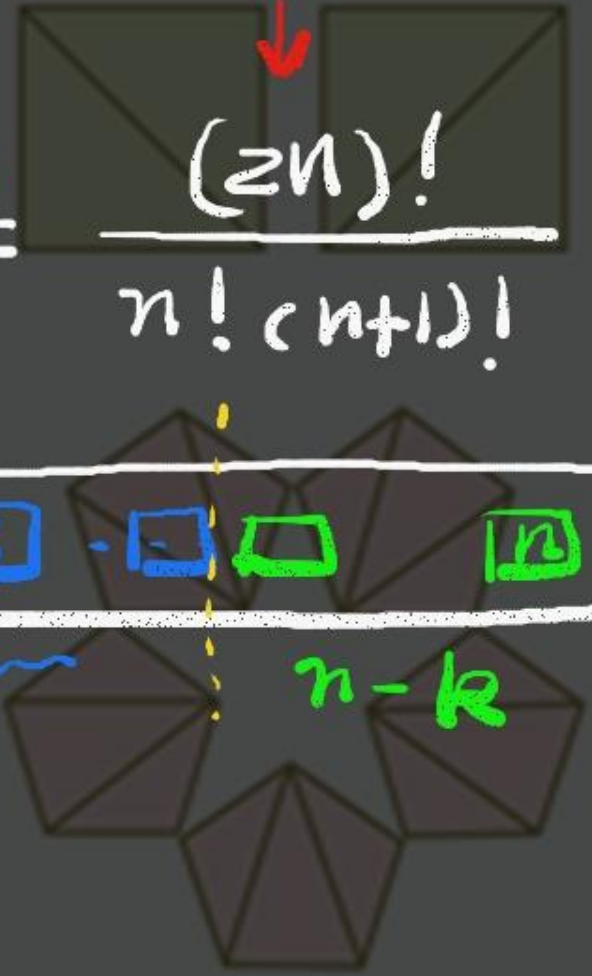
n-k

B.size() → k

?

S → ∅





甄别：检测禁形

- 输入序列< 1, 2, 3, ..., n]的任一排列[p_1, p_2, p_3, ..., p_n >是否为栈混洗?

- 先考查简单情况：n = 3, A = < 1, 2, 3]

  - 栈混洗共 6! / 4! / 3! = 5 种；全排列共 3! = 6 种 //少了一种...

- $\boxed{[\ 3,\ 1,\ 2\ >}$ //为什么是它?

- 观察：任意三个元素能否按某相对次序出现于混洗中，与其它元素无关 //故可推而广之...

- 禁形：对任何1 ≤ i < j < k ≤ n, [..., $\boxed{k}$, ..., $\boxed{i}$, ..., $\boxed{j}$, ... > **必非**栈混洗

- 反过来，不存在“$\boxed{312}$”模式的序列，一定是栈混洗吗?

甄别：直接模拟

- **充要性:** A permutation is a stack permutation **iff**

  (Knuth, 1968) it does NOT involve the permutation **312** //习题[4-3]

- **如此，可得一个$\mathcal{O}(n^3)$的甄别算法** //进一步地...

- $[\ p_1,\ p_2,\ p_3,\ \ldots,\ p_n\ \rangle$是$\langle\ 1,\ 2,\ 3,\ \ldots,\ n\ ]$的栈混洗，**当且仅当**

  对于任意$i < j$，不含模式$[\ \ldots,\ j+1,\ \ldots,\ i,\ \ldots,\ j,\ \ldots\ \rangle$

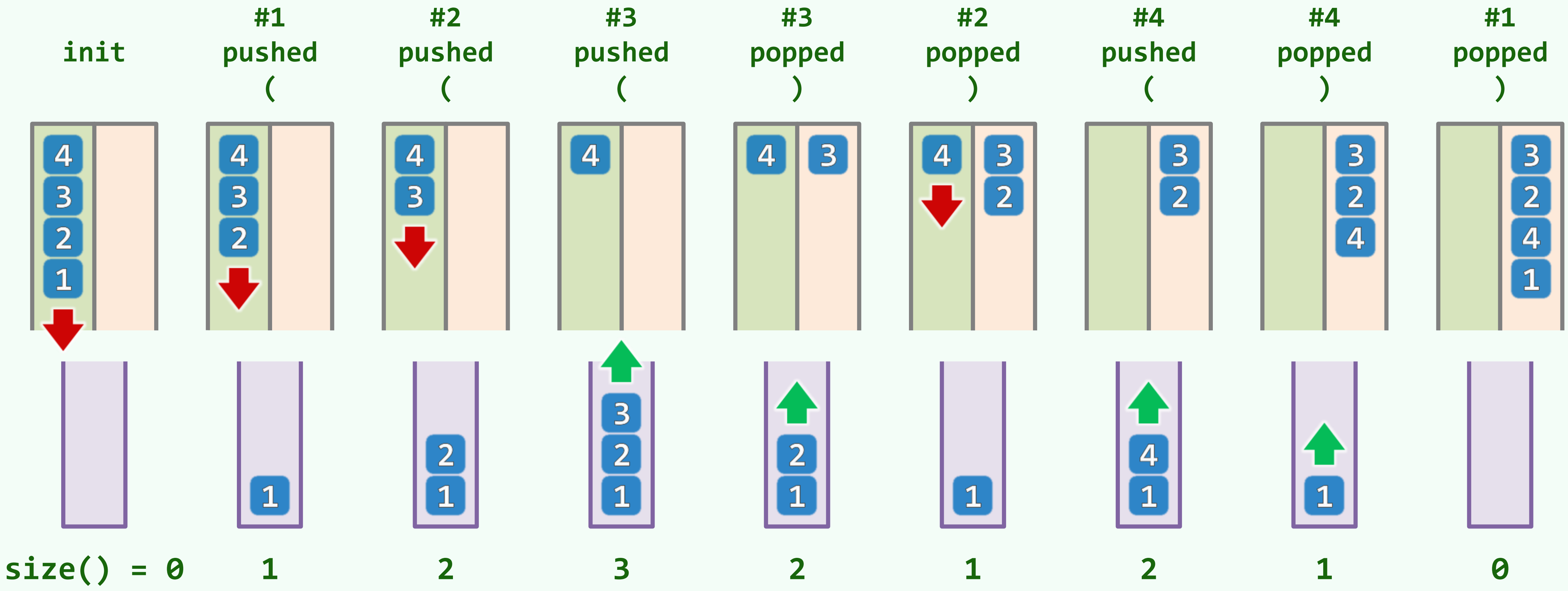
- **如此，可得一个$\mathcal{O}(n^2)$的甄别算法** //再进一步地...

- **$\mathcal{O}(n)$算法：** 直接借助栈A、B和S，模拟混洗过程 //为何可行?

  每次S.pop()之前，检测S是否已空；或需弹出的元素在S中，却非顶元素

括号匹配

❖ 观察：每一栈混洗，都对应于栈S的n次push与n次pop操作构成的某一序列；反之亦然



❖ n个元素的栈混洗，等价于n对括号的匹配；二者的组合数，也自然相等