香港培正中學第二屆數學邀請賽

Pui Ching Middle School 2nd Invitational Mathematics Competition

個人賽（中四組）

Individual Event (Secondary 4)

時限：**1** 小時 **30** 分

Time allowed: 1 hour 30 minutes

參賽者須知：

Instructions to Contestants:

1. 本卷共設 20 題，總分為 100 分。

   There are 20 questions in this paper and the total score is 100.

2. 除特別指明外，本卷內的所有數均為十進制。

   Unless otherwise stated, all numbers in this paper are in decimal system.

3. 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

   Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

4. 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

   Put your answers on the spaces provided on the answer sheet. You are not required to hand in your steps of working.

5. 不得使用計算機。

   The use of calculators is not allowed.

6. 本卷的附圖不一定依比例繪成。

   The diagrams in this paper are not necessarily drawn to scale.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 在一個圓形上畫 2003 條直徑，最多可把圓形分成幾份？

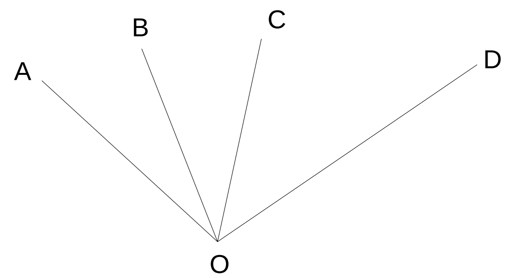
   2003 diameters are drawn on a circle. What is the maximum number of regions formed?

2. 投擲一枚硬幣四次，擲得正面次數為奇數的概率是多少？

   A coin is tossed four times. What is the probability that an odd number of heads is obtained?

3. 圖中，$\angle AOD = 120°$，$3\angle AOB = \angle BOD$，且 $\angle AOC = 2\angle COD$。求 $\angle BOC$。

   In the figure, $\angle AOD = 120°$, $3\angle AOB = \angle BOD$ and $\angle AOC = 2\angle COD$. Find $\angle BOC$.



4. 小嘉的姊妹比兄弟多 10 個，而小嘉每位姊妹的姊妹數目也比兄弟數目多 10 個。小明是小嘉的長兄，他的姊妹比兄弟多 n 個。求 n。

   Chris has 10 more sisters than brothers. Each of Chris's sisters also has 10 more sisters than brothers. Alan, Chris's eldest brother, has n more sisters than brothers. Find n.

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 小強寫下了所有不超過 2003 而至少有一個「2」字的正整數，即 2, 12, 20, 21, …, 2002, 2003。小強共寫下了多少個整數？

   Peter wrote down all positive integers not exceeding 2003 which have at least one '2' in their digits, i.e. 2, 12, 20, 21, …, 2002, 2003. How many integers did Peter write down?

6. 設 x 為順序寫出 1 到 2003 所得的正整數，y 為把 x 的所有數字倒轉所得的正整數，即 $x = 123456789101112\ldots200120022003$ 及 $y = 300220021002\ldots211101987654321$。已知 s 為合成數，且為 $y - x$ 的因數。求 s 的最小可能值。

Let x be the positive integer formed by writing 1 to 2003 in order, and y be the positive integer formed by reversing the digits of x, i.e. $x = 123456789101112\ldots200120022003$ and $y = 300220021002\ldots211101987654321$. Given that s is a composite number and is a factor of $y - x$. Find the smallest possible value of s.

7. 平面上畫有 n 條直線，它們之間剛好有 2003 個交點。求 n 的最小可能值。

On the plane n straight lines are drawn. They produce exactly 2003 points of intersection. Find the smallest possible value of n.

8. 小美參選學生會主席，得票率（準確至小數點後一個位）為 99.3%。問最少有幾人投了票？

Amy joined the presidential election of the Student Union and obtained 99.3% of the votes, correct to 1 decimal place. What is the smallest possible number of voters?

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 求 $1^3 + 6^3 + 11^3 + 16^3 + \ldots + 2001^3$ 除以 2002 時的餘數。

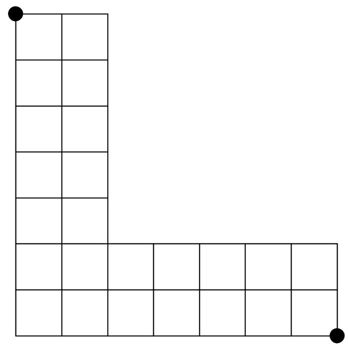
Find the remainder when $1^3 + 6^3 + 11^3 + 16^3 + \ldots + 2001^3$ is divided by 2002.

10. 一張尺寸為 10×10 的氈子被分成 100 個尺寸為 1×1 的方格。氈子左上角和右下角的方格隨後被割去。那麼氈子上有多少個長方形（包括正方形）？

A mat of size 10×10 is divided into 100 squares of size 1×1. The squares at the top left and bottom right corners are then removed. How many rectangles (including squares) can be found on the mat?

11. 如圖所示，我們從一個 7×7 方格陣的右上角移去一個 5×5 的方格陣，從而得到一個 L 形方格陣。若要從這 L 形方格陣的左上角走至右下角，且每步只能沿格線向右或向下走，共有多少種不同的走法？

Given a 7×7 grid, we remove the 5×5 grid at the upper right hand corner to obtain an L-shaped grid as shown. We want to travel from the upper left hand corner to the lower right hand corner of this L-shaped grid, but in each step we can only move rightward or downward along the gridlines. How many different paths are there?



12. 求最大的整數 k，使得 $(2^1)^{2^{2003}}+(2^2)^{2^{2002}}+(2^3)^{2^{2001}}+\cdots+(2^{2001})^{2^3}+(2^{2002})^{2^2}+(2^{2003})^{2^1}$ 可被 2^k 整除。

Find the largest integer k such that

$$(2^1)^{2^{2003}}+(2^2)^{2^{2002}}+(2^3)^{2^{2001}}+\cdots+(2^{2001})^{2^3}+(2^{2002})^{2^2}+(2^{2003})^{2^1}$$

is divisible by 2^k.

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 2003 位小朋友被編成 1 至 2003 號。他們每人最初有 20 顆糖果，然後按以下方法派給或拿走糖果：先派 3 顆糖果給每位編號是 1 的倍數的小朋友，再從每位編號是 2 的倍數的小朋友手上取走 1 顆糖果，然後派 3 顆糖果給每位編號是 3 的倍數的小朋友，再從每位編號是 4 的倍數的小朋友手上取走 1 顆糖果，如此類推，直至最後派 3 顆糖果給每位編號是 2003 的倍數的小朋友。最後有多少個小朋友比最初多了糖果（即有 21 顆糖果或以上）？

2003 children are labelled 1 to 2003. Each of them has 20 candies at the beginning. Candies are then given to or taken away from them as follows. 3 candies are given to each child with a label which is a multiple of 1, then 1 candy is taken away from each child with a label which is a multiple of 2, then 3 candies are given to each child with a label which is a multiple of 3, then 1 candy is taken away from each child with a label which is a multiple of 4, and so on, till finally 3 candies are given to each child with a label which is a multiple of 2003. How many children can get extra candies (i.e. have 21 candies or more) in the end?

14. 若方程 $x^2 - px + p + 2002 = 0$ 的根為非零整數，求 p 的值。

If the roots of the equation $x^2 - px + p + 2002 = 0$ are non-zero integers, find the value of p.

15. 潘先生對小敏和小賢說：「我想了兩個正整數 a 和 b，其中 $a > b$。」然後他秘密地將兩數之差（$a-b$）告訴小敏，及將兩數之積（ab）告訴小賢。以下是他們之後的對話。

潘先生問小敏：「你知道 a 和 b 是甚麼嗎？」

小敏說：「不知道。」

然後潘先生問小賢：「你知道 a 和 b 是甚麼嗎？」

小賢說：「我知道啊。它們是……」

潘先生立即打斷了小賢的說話，並再問小敏：「現在你知道 a 和 b 是甚麼嗎？」

這時小敏回答：「聽過你和小賢剛才的對話後，我知道它們是甚麼了。它們的和 $a+b$ 大於 90，小於 100。潘先生，對嗎？」

潘先生說：「是啊，你們真聰明呢。」

假設小賢和小敏都是誠實和聰明的（即是說只要當答案可以確定時，他們一定知道答案），求 ab。

Mr Poon told Dora and Ken: ‘I thought of two positive integers a and b, where $a > b$.’ He then secretly told Dora the difference of the two numbers (i.e. $a-b$) and Ken the product of the two numbers (i.e. ab). Their subsequent conversation is recorded below.

Mr Poon asked Dora, ‘Do you know what a and b are?’

‘No.’ Dora answered.

Then Mr Poon asked Ken, ‘Do you know what a and b are?’

‘Yes, they are …’ Ken said.

Mr Poon interrupted Ken immediately and asked Dora again, ‘Do you know what a and b are now?’

At that time, Dora said, ‘Now I know what they are after listening to the conversation between Ken and you. Their sum (i.e. $a+b$) is greater than 90 but less than 100. Am I right, Mr Poon?’

‘Yes. You are so clever!’ Mr Poon said.

Assuming that Ken and Dora are honest and intelligent (it means that whenever the answer can be confirmed, they must know the answer), find ab.

16. 設 x、y、z 為滿足以下方程組的實數，且 $z>0$。求 z 的值。

Let x, y, z be real numbers satisfying the following system of equations, and $z>0$. Find the value of z.

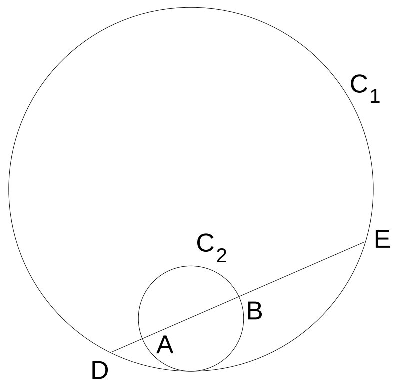
$$\begin{cases} 2x^2+xz+18=0 \\ 2y^2+yz+18=0 \\ \dfrac{1}{x}-\dfrac{1}{y}=\dfrac{8}{9} \end{cases}$$

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 圖中，小圓 C_2 內切於大圓 C_1。C_2 的一條直徑 AB 向兩端延長後交 C_1 於 D 及 E，且 $DA:AB:BE=1:2:4$。若 C_2 的面積為 1，求 C_1 的面積。

In the figure, a smaller circle C_2 is internally tangent to a larger circle C_1. A diameter AB of C_2 is produced on both sides to meet C_1 at D and E, and $DA:AB:BE=1:2:4$. If the area of C_2 is 1, find the area of C_1.

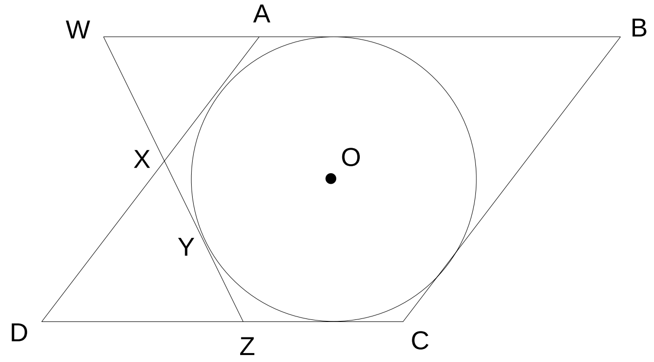


18. 小明和小芳玩一個遊戲。首先小明在一個 13×13 的方格表中放了 5 枚硬幣，使得這 5 枚硬幣放在 5 個位於同一橫行或同一直行的相鄰方格中。然後，小芳開始猜這些硬幣的位置。她每次會選一個方格，如果她選中了一個有硬幣的方格，她便獲勝。小芳最少要猜多少次才可以確保獲勝？

Alan and Betty play a game. First, Alan puts 5 coins on a 13×13 square grid such that the 5 coins are in 5 neighbouring cells lined up in a row or a column. Then Betty starts to guess where the coins are. Every time when she guesses, she chooses a cell. If the cell is occupied by a coin, she wins. What is the minimum number of guesses she has to make to ensure that she wins?

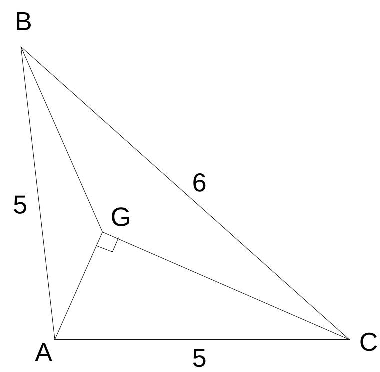
19. 如圖所示，一個圓形內切於菱形 $ABCD$ 內，它與菱形的四條邊皆相切。W、X 及 Z 分別是 BA 延線、AD 及 DC 上的點，使得直線 WXZ 切圓於 Y，且 $WX = 5$、$XY = 3$ 及 $YZ = 2$。求 $ABCD$ 的面積。

In the figure, a circle is inscribed in a rhombus $ABCD$. It touches all four sides of the rhombus. W, X and Z are points on BA produced, AD and DC respectively such that the straight line WXZ is tangent to the circle at Y, $WX = 5$, $XY = 3$ and $YZ = 2$. Find the area of $ABCD$.



20. 圖中，ABC 為等腰三角形，當中 $AB = AC = 5$ 及 $BC = 6$。假設 G 是 ΔABC 內的一點，使得 $\angle AGC = 90°$ 及 $\angle ACG = \angle CBG$，求 $\dfrac{AG}{GC}$。

In the figure, ABC is an isosceles triangle with $AB = AC = 5$ and $BC = 6$. Suppose G is a point in ΔABC such that $\angle AGC = 90°$ and $\angle ACG = \angle CBG$. Find $\dfrac{AG}{GC}$.



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END OF PAPER

個人賽（中四組）答案

Individual Event (Secondary 4) Answers

1. 4006
2. $\frac{1}{2}$
3. 50°
4. 12
5. 546
6. 6
7. 64
8. 134
9. 1001
10. 2826
11. 491
12. 4006
13. 1753
14. 2006
15. 97
16. 20
17. $\frac{121}{4}$
18. 33
19. 80
20. $\frac{2}{3}$