Initial report of 1976 U.S. Atmospheric Model methodology

Kamyar Modjtahedzadeh

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1 Background

"The USSA mathematical model divides the atmosphere into layers with an assumed linear distribution of absolute temperature T against geopotential altitude h. The other two values (pressure P and density ρ) are computed by simultaneously solving the equations resulting from:

$$\frac{dP}{dh} = -\rho g \tag{1}$$
$$P = \rho R_{\text{specific}} T \tag{2}$$

where R_{specific} is the specific gas constant for dry air and g is the standard gravitational acceleration." Equation (1) evaluates the vertical pressure variation with the mean sea level as a boundary condition ($P_0 = 101{,}325$ Pa) and Equation (2) is the ideal gas law in molar form [1].

Combining equations (1) and (2) to find an expression for the pressure:

$$\ln P \Big|_{P_0}^{P_1} = -\frac{g}{R_{\text{specific}}} \int_{h_0}^{h_1} \frac{dh}{T(h)} \tag{3}$$

where the minimum geopotential altitude above sea level is set to $h_0 = -1{,}524$ m. From The Engineer ToolBox, various temperatures have been given for their corresponding geopotential altitudes in Table 1 [2].

Table 1: USSA1976 Temperature for Geopotential Altitude - Imperial (BG) Units

h [ft]	T [°F]
−5000	76.84
0	59
5000	41.17
10000	23.36
15000	5.55
20000	−12.26
25000	−30.05
30000	−47.83
35000	−65.61
40000	−69.70
45000	−69.70
50000	−69.70
60000	−69.70
70000	−67.42
80000	−61.98
90000	−56.54
100000	−51.10
150000	19.40
200000	−19.78
250000	−88.77

2 The Implemented Model

2.1 Calculating Temperature, Pressure, and Density

To find an expression for temperature as a function of geopotential altitude, use the data from Table 1. Start by converting the data into SI units, plot T against h, and then fit a curve to the data as in Figure 1. And so, $T(h) =$

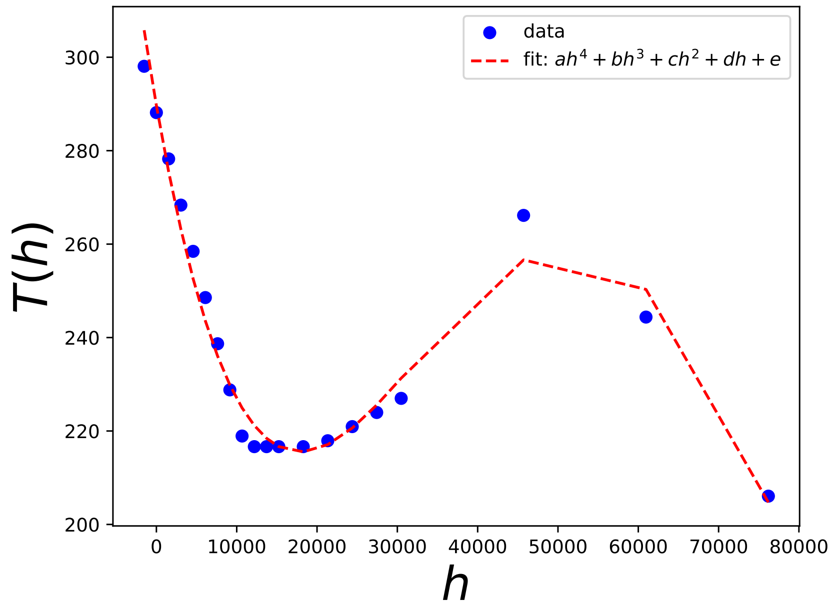

Figure 1: $T(h)$ vs. h; A scatter plot of temperature as a function of geopotential altitude given from the data in Table 1. A fourth degree polynomial has been fitted to the data in order to find a mathematical expression for $T(h)$. Please note that the fit would have more (and actual) physical meaning if it was not a generic polynomial since a polynomial of a large enough degree will virtually always fit to any function. Also note that the fit can be smoothed out if more data points are given to the function once the parameters are known; speaking of which, the parameters found from the fit are: $a = 0.000000000000000027$, $b = -0.000000000006074933$, $c = 0.000000422889669793$, $d = -0.00991893662487261$, and $e = 289.661267963331567898$.[1]

$ah^4 + bh^3 + ch^2 + dh + e$ with the coefficients given in the caption of Figure 1. With the data given in Table 1, this expression is assumed to only work if $-1{,}524$ m $\leq h \leq 76{,}200$ m. Equation (3) can now be written as:

$$P(h) = P_0 \exp\left[-\frac{g}{R_{\text{specific}}} \int_{h_0}^{h \leq 76200} \frac{dh}{T(h)}\right] \tag{4}$$

where $R_{\text{specific}} = 287.052874\ \text{J}\cdot\text{kg}^{-1}\cdot\text{K}^{-1}$ and g is negative [3,4]; with the currently given expression for the integrand this equation must be solved numerically.[2] Once the pressure is found then the density can be obtained with Equation (1) for each geopotential altitude given into the fourth degree polynomial for $T(h)$.

2.2 The Instantaneous Drag Force

The drag force formula is given by,

$$\vec{F}_{\text{d}} = \frac{1}{2}\rho \vec{v}^2 C_{\text{d}} A \tag{5}$$

[1]Here, please do not mix up e with Euler's constant.

[2]It is interesting to note that if the maximum value of h is fed into (4) then the numerical value for the integral in the RHS becomes $323.7972921077598 \pm 1.173024627261422 \times 10^{-10}$.

where C_d is the drag coefficient and A is the cross-sectional area. The method which computes drag takes in ECEF position,[3] velocity, and ballistic parameters (C_d, A, m) to output an acceleration due to force vector. The acceleration due to drag can be found using Newton's second law; $\vec{a}_\mathrm{d} = \vec{F}_\mathrm{d}/m$.

3 An Example

An object of mass $m = 100\,\mathrm{kg}$ is traveling with a velocity of $\vec{v} = (v_x, v_y, v_h) = (20, 100, 3000)\,\mathrm{m/s}$ at an ECEF position of $(0, 0, 10000)$ m where the latter component is the altitude. The drag coefficient is 1 and the cross-sectional area of the object is $100\,\mathrm{m}^2$. From the applied model for atmosphere here and the fourth degree polynomial for $T(h)$ its temperature is 226.956 K and it follows that its pressure is 21846.097 Pa and the density of air at that altitude is $0.335\,\mathrm{kg/m^3}$. The applied drag force at that altitude would be $\vec{F}_\mathrm{d} = (1.677 \times 10^5, 1.509 \times 10^8, 6.707 \times 10^3)$ N; ergo, the length of the acceleration due to drag would be $|\vec{a}_\mathrm{d}| = 1508978.973\,\mathrm{m/s^2}$.[4]

Bibliography

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[2] The Engineering ToolBox, *U.S. Standard Atmosphere vs. Altitude*, retrieved on 01/22/2024 from `https://www.engineeringtoolbox.com/standard-atmosphere-d_604.html`.

[3] Wikipedia, *Gas constant*, retrieved on 01/18/2024 from `https://en.wikipedia.org/wiki/Gas_constant`.

[4] Wikipedia, *Vertical pressure variation*, retrieved on 01/18/2024 from `https://en.wikipedia.org/wiki/Vertical_pressure_variation`.

[5] National Oceanic and Atmospheric Administration, National Aeronautics and Space Administration, United States Air Force, *U.S Standard Atmosphere, 1976*, US Department of Commerce/NOAA/NESDIS, October 1976.

[6] Ralph L. Carmichael, Public Domain Aeronautical Software, *Geopotential & Geometric Altitude*, retrieved on 01/22/2024 from `https://www.pdas.com/hydro.pdf`.

[7] Wikipedia, *Earth-centered, Earth-fixed coordinate system*, retrieved on 01/23/2024 from `https://en.wikipedia.org/wiki/Earth-centered,_Earth-fixed_coordinate_system`.

[3] The altitude is the only relevant coordinate when it comes to this specific calculation as Equation (5)'s only parameter dependent on position is ρ.
[4] Although it does change with altitude, $g = 9.81\,\mathrm{m/s^2}$ is used for all altitudes in this implementation.