Initial report of 1976 U.S. Atmospheric Model methodology

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1 Background

"The USSA mathematical model divides the atmosphere into layers with an assumed linear distribution of absolute temperature T against geopotential altitude h. The other two values (pressure P and density ρ) are computed by simultaneously solving the equations resulting from:

$$\frac{dP}{dh} = -\rho g \tag{1}$$

$$P = \rho R_{\text{specific}} T \tag{2}$$

where R_{specific} is the specific gas constant for dry air and g is the standard gravitational acceleration." Equation (1) evaluates the vertical pressure variation with the mean sea level as a boundary condition ($P_0 = 101{,}325$ Pa) and Equation (2) is the ideal gas law in molar form [1].

Combining equations (1) and (2) to find an expression for the pressure:

$$\ln P \Big|_{P_0}^{P_1} = -\frac{g}{R_{\text{specific}}} \int_{h_0}^{h_1} \frac{dh}{T(h)} \tag{3}$$

where the minimum geopotential altitude above sea level is set to $h_0 = -1{,}524$ m. From The Engineer ToolBox, various temperatures have been given for their corresponding geopotential altitudes in Table 1 [2].

Table 1: USSA1976 Temperature for Geopotential Altitude - Imperial (BG) Units

h [ft]	T [°F]
−5000	76.84
0	59
5000	41.17
10000	23.36
15000	5.55
20000	−12.26
25000	−30.05
30000	−47.83
35000	−65.61
40000	−69.70
45000	−69.70
50000	−69.70
60000	−69.70
70000	−67.42
80000	−61.98
90000	−56.54
100000	−51.10
150000	19.40
200000	−19.78
250000	−88.77

2 The Implemented Model

2.1 Calculating Temperature, Pressure, and Density

To find an expression for temperature as a function of geopotential altitude, use the data from Table 1. Start by converting the data into SI units, plot T against h, and then fit a curve to the data as in Figure 1. And so, $T(h) =$

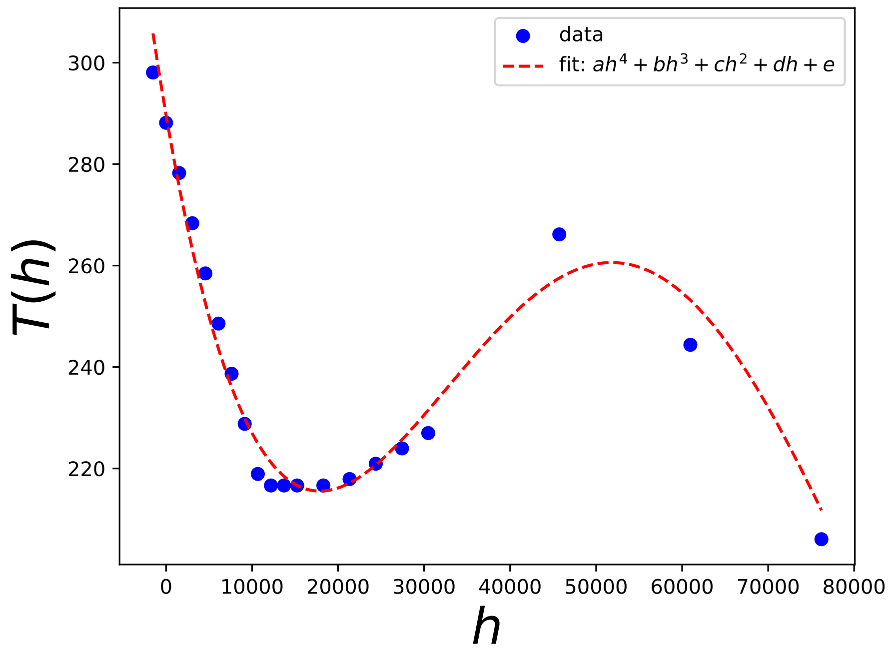

Figure 1: $T(h)$ vs. h; A scatter plot of temperature as a function of geopotential altitude given from the data in Table 1. A fourth degree polynomial has been fitted to the data in order to find a mathematical expression for $T(h)$. Please note that the fit would have more (and actual) physical meaning if it was not a generic polynomial since a polynomial of a large enough degree will virtually always fit to any function. The parameters which are found from the fit are: $a = 0.000000000000000027$, $b = -0.000000000006074933$, $c = 0.000000422889669793$, $d = -0.00991893662487261$, and $e = 289.661267963331567898$.[1]

$ah^4 + bh^3 + ch^2 + dh + e$ with the coefficients given in the caption of Figure 1. With the data given in Table 1, this expression is assumed to only work if $-1{,}524\ \mathrm{m} \leq h \leq 76{,}200\ \mathrm{m}$. Equation (3) can now be written as:

$$P(h) = P_0 \exp\left[-\frac{g}{R_{\text{specific}}} \int_{h_0}^{h \leq 76200} \frac{dh}{T(h)}\right] \tag{4}$$

where $R_{\text{specific}} = 287.052874\ \mathrm{J \cdot kg^{-1} \cdot K^{-1}}$ and g is negative [3,4]; with the currently given expression for the integrand this equation must be solved numerically.[2] Once the pressure is found then the density can be obtained with Equation (2) for each geopotential altitude given into the fourth degree polynomial for $T(h)$.

2.2 The Instantaneous Drag Force

The drag force formula is given by,

$$\vec{F}_{\mathrm{d}} = \frac{1}{2} \rho \vec{v}^2 C_{\mathrm{d}} A \tag{5}$$

[1]Here, please do not mix up e with Euler's constant.

[2]It is interesting to note that if the maximum value of h is fed into (4) then the numerical value for the integral in the RHS becomes $323.7972921077598 \pm 1.173024627261422 \times 10^{-10}$.

where C_d is the drag coefficient and A is the cross-sectional area. The method which computes drag takes in ECEF position,[3] velocity, and ballistic parameters (C_d, A, m) to output an acceleration due to force vector. The acceleration due to drag can be found using Newton's second law; $\vec{a}_\mathrm{d} = \vec{F}_\mathrm{d}/m$.

3 An Example

An object of mass $m = 100\,\mathrm{kg}$ is traveling with a velocity of $\vec{v} = (v_x, v_y, v_h) = (20, 100, 3000)\,\mathrm{m/s}$ at an ECEF position of $(0, 0, 10000)$ m where the latter component is the altitude. The drag coefficient is 1 and the cross-sectional area of the object is $100\,\mathrm{m}^2$. From the applied model for atmosphere here and the fourth degree polynomial for $T(h)$ its temperature is 226.956 K and it follows that its pressure is 21846.097 Pa and the density of air at that altitude is $0.335\,\mathrm{kg/m}^3$. The applied drag force at that altitude would be $\vec{F}_\mathrm{d} = (1.677 \times 10^5, 1.509 \times 10^8, 6.707 \times 10^3)$ N; ergo, the length of the acceleration due to drag would be $|\vec{a}_\mathrm{d}| = 1508978.973\,\mathrm{m/s}^2$.[4]

Bibliography

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[3] Wikipedia, *Gas constant*, retrieved on 01/18/2024 from `https://en.wikipedia.org/wiki/Gas_constant`.

[4] Wikipedia, *Vertical pressure variation*, retrieved on 01/18/2024 from `https://en.wikipedia.org/wiki/Vertical_pressure_variation`.

[5] National Oceanic and Atmospheric Administration, National Aeronautics and Space Administration, United States Air Force, *U.S Standard Atmosphere, 1976*, US Department of Commerce/NOAA/NESDIS, October 1976.

[6] Ralph L. Carmichael, Public Domain Aeronautical Software, *Geopotential & Geometric Altitude*, retrieved on 01/22/2024 from `https://www.pdas.com/hydro.pdf`.

[7] Wikipedia, *Earth-centered, Earth-fixed coordinate system*, retrieved on 01/23/2024 from `https://en.wikipedia.org/wiki/Earth-centered,_Earth-fixed_coordinate_system`.

[3] The altitude is the only relevant coordinate when it comes to this specific calculation as Equation (5)'s only parameter dependent on position is ρ.
[4] Although it does change with altitude, $g = -9.81\,\mathrm{m/s}^2$ is used for all altitudes in this implementation.