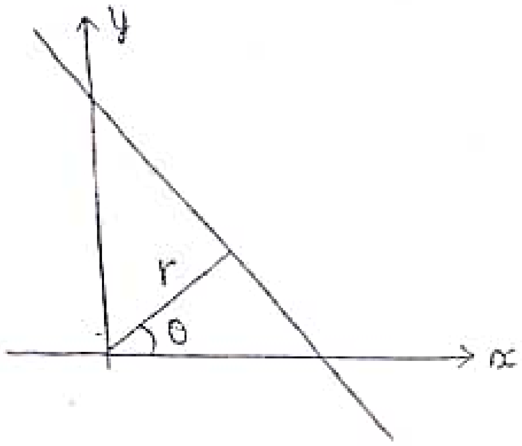
Digital Image Processing: Image Reconstruction from Projections

October 26 and 30, 2023

Naren Akash. R.J.

Parameterization of a line



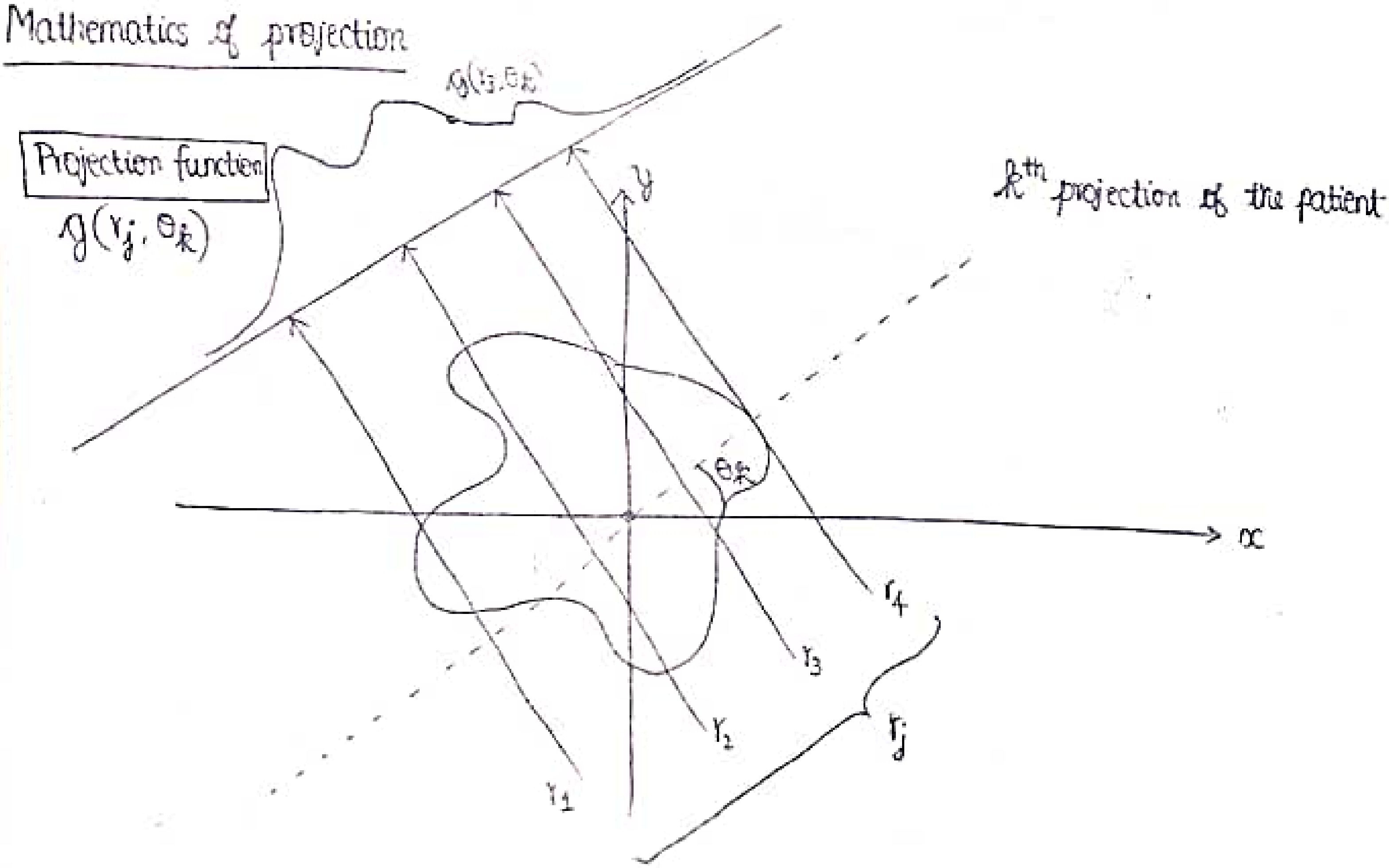
* Slope-intercept form

$$y = ax + b$$

* Normal representation

$$x\cos\theta + y\sin\theta = r$$

Mathematics of projection



How do we get the projection? — **The Radon Transform**

It's an integral along a line going through the patient $\longrightarrow$ "Line integral".

$$g(r,\theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underbrace{f(x,y)}_{\text{Image fn.}} \; \underbrace{\delta(x\cos\theta + y\sin\theta - r)}_{\text{delta fn. (If inner condition true, it fires up, otherwise not)}} \; \overbrace{dx\,dy}^{\text{integral across x-y plane}}$$

taking an image and only integrating it along the line.

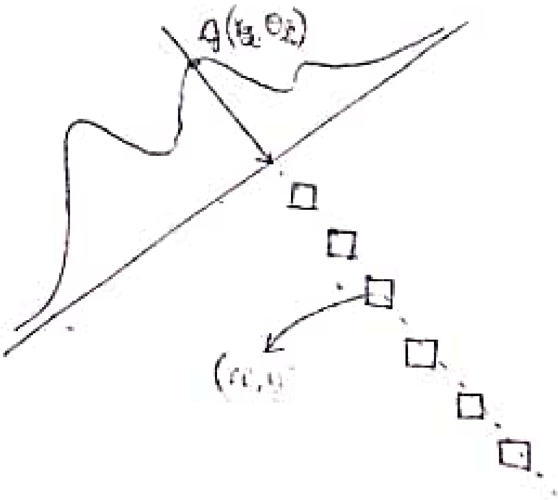
Sinogram $\longrightarrow$ visualize $g(r,\theta)$ as an image.

$\boxed{\text{METHOD \#01 · BACK PROJECTION}}$

What would happen if we take these projections from diff. angles, smear them (backproject) and add them up?

We have $g(r,\theta_k)$ for a set of angles $\{\theta_k\}$ between $\overbrace{\theta \text{ and } \pi}^{\text{not } 2\pi \text{ (same projection 180° opp.)}}$.

(Ex.)



For a fixed value $g(r_j,\theta_k)$, just copy this value into the image along the line $x\cos\theta_k + y\sin\theta_k = r_j$

ckprojected image at θ

$$b_\theta(x,y) = g(x\cos\theta + y\sin\theta, \theta) = g(r,\theta)$$

Sum of backprojections $\longrightarrow$ $\int_0^\pi b_\theta(x,y)\, d\theta$ Laminogram

Problem: blur

METHOD #02 DIRECT FOURIER RECONSTRUCTION

Fourier-Slice-Theorem (aka. Projection-Slice Theorem)

$\left.\begin{array}{l}\text{Image has a 2D FT.} \\ \text{Each projection has a 1D FT.}\end{array}\right\}$ how are they related?

Consider a fixed angle θ's projection and take its Fourier transform w.r.t. r.

$$\underset{\text{"time domain"}}{g(r,\theta)} \longrightarrow \underset{\text{"frequency domain"}}{G(k,\theta)}$$

Fourier transform

$$G(k,\theta) = \int_{-\infty}^{+\infty} g(r,\theta)\, e^{-2\pi j k r}\, dr$$

From Radon transform,

$$= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\, \delta(x\cos\theta + y\sin\theta - r)\, dx\, dy \right] e^{-2\pi j k r}\, dr$$

Taking x and y out,

$$G(k,\theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y) \left[\int_{-\infty}^{+\infty} \delta(x\cos\theta + y\sin\theta - r)\, e^{-2\pi j k r}\, dr \right] dx\,dy$$

Delta fn. fires up only when inner condition is true. $(r = x\cos\theta + y\sin\theta)$

$$G(k,\theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\, e^{-2\pi j k (x\cos\theta + y\sin\theta)}\, dx\, dy$$

Consider $k_x = k\cos\theta$ and $k_y = k\sin\theta$.

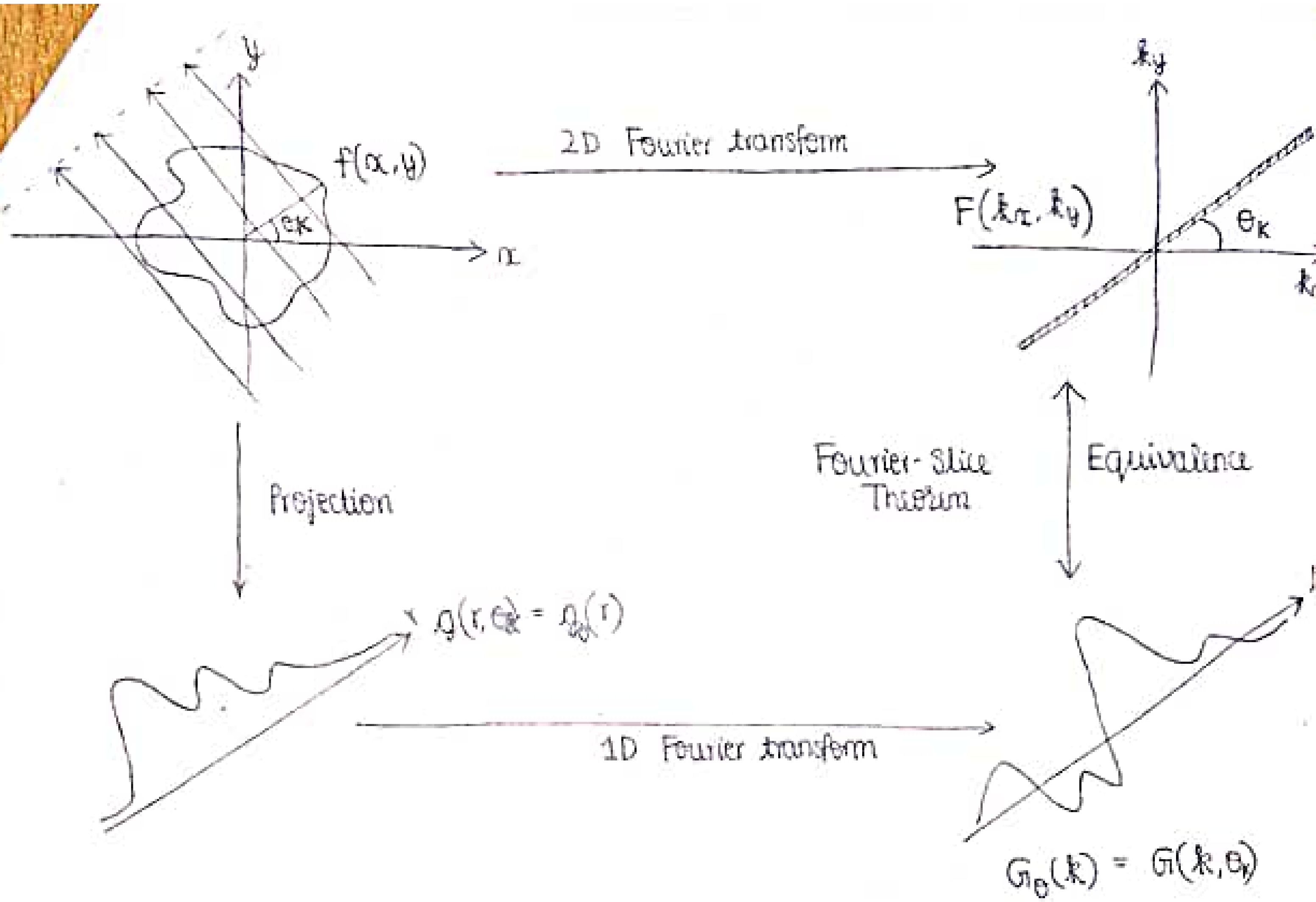
$$G(k,\theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\, e^{-2\pi j (k_x x + k_y y)}\, dx\, dy$$

This looks like a 2D Fourier transform! FT of original image along the line $k_x = k\cos\theta$ and $k_y = k\sin\theta$.

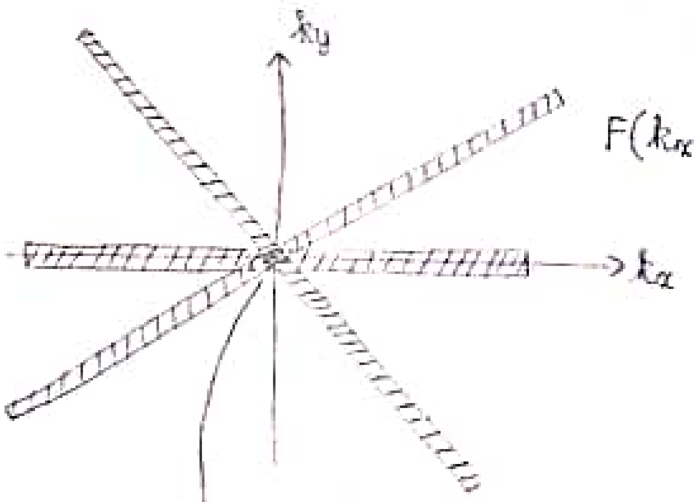
1D FT of projection	=	slice through the 2D FT of image at angle θ

Every time we are getting a projection, we are getting another slice through the image's Fourier transform.

→ gives intuition why we should be able to reconstruct from projections and why we need a lot of θ's to make it work.



Why do we have the bluriness problem?



Adding low frequency stuff to the image that shouldn't be there causes blurring.

Middle of 2D FT is <u>oversampled</u>.

Summing up backprojections directly is inaccurate.

→ downweight the middle s.t. when we add them up, we get exactly the contribution we need in the middle.

Algorithm - Direct Fourier Reconstruction

1. Calculate 1D FT of all projections $g_\theta(r) \equiv G_\theta(r)$.

   put all values of 1D fn. $G_\theta(r)$ on a polar grid to obtain 2D fn. $G(r,\theta)$

2. Interpolate polar grid data samples to obtain $F(k_x, k_y)$ data on Cartesian grid.

3. Calculate 2D IFT of $F(k_x, k_y) = f(x,y)$.

   Problem: Interpolation.

METHOD #03 - FILTERED BACK PROJECTION

Is it possible to avoid interpolation in reconstructing $f(x,y)$? Yes.

Fourier transform - recap

Let k and r be conjugate variables in the Fourier domain and the original domain, respectively. 'r' represents spatial position, 'k' represents spectral frequency.

Forward Fourier transform
$$F(\vec{k}) = \int_{-\infty}^{+\infty} f(\vec{r})\, e^{-2\pi j \vec{k}\cdot\vec{r}}\, d\vec{r}$$

Inverse Fourier transform
$$f(\vec{r}) = \int_{-\infty}^{+\infty} F(\vec{k})\, e^{+2\pi j \vec{k}\cdot\vec{r}}\, d\vec{k}$$

form of Fourier transform – recap

Given a fn. in polar form, obtain its FT in Cartesian form.

Using polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$.

$$F(k_x, k_y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\, e^{-2\pi j(k_x x + k_y y)}\, dx\, dy$$

$$= \int_{0}^{2\pi}\int_{0}^{+\infty} f(r,\theta)\, e^{-2\pi j(k_x r\cos\theta + k_y r\sin\theta)}\, r\, dr\, d\theta$$

since $f(k, \theta+\pi) = f(-k, \theta)$,

$$= \int_{0}^{\pi}\int_{-\infty}^{+\infty} f(r,\theta)\, e^{-2\pi j(k_x r\cos\theta + k_y r\sin\theta)}\, |r|\, dr\, d\theta$$

Analogously, <u>polar form of inverse Fourier transform.</u>

Let $k_x = k\cos\theta$ and $k_y = k\sin\theta$.

$$f(x,y) = \int_{0}^{\pi}\int_{-\infty}^{+\infty} \underbrace{F(k,\theta)}\, e^{+2\pi j(xk\cos\theta + yk\sin\theta)}\, |k|\, dk\, d\theta.$$

from Fourier-Slice theorem.

<u>We want original image $f(x,y)$</u>

$$f(x,y) = \int_{0}^{\pi} \underbrace{\left[\int_{-\infty}^{+\infty} |k|\, \overbrace{G(k,\theta)}\, e^{+2\pi j k r}\, dk \Bigg|_{r = x\cos\theta + y\sin\theta} \right]}\, d\theta$$

Looks like inverse FT of each of the projections – but with $|k|$.

It tells us that it <u>modifies</u> each of the projections before adding them up.

$$\int_{-\infty}^{+\infty} |k|\, G(k,\theta)\, e^{+2\pi j k r}\, dk$$ is the IFT of $G(k,\theta)$, but ...

multiplied by a filter function $|k|$.

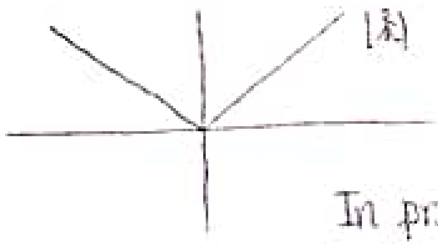
→ These are, thus, filtered back projections.

$\boxed{\text{Algorithm - Filtered Back Projection}}$

1. Compute 1D FT of each projection $g_\theta(r) \equiv G_\theta(r)$.
2. Multiply each FT $G(k,\theta_x)$ by $|k|$.
3. Take 1D IFT.
4. Integrate over all angles to get the original image $f(x,y)$.

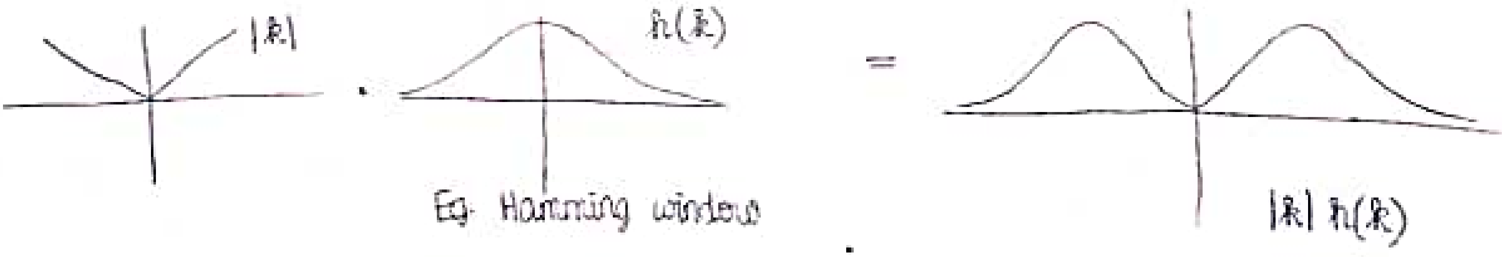
Assumes we have infinite number of projections.
In practice, we need to have enough projections to get close to ideal result.

What does the fn. $|k|$ look like?



Can't take FT of a fn. like this
→ it integrates to ∞.

In practice, we multiply it w/ some sort of fn. like window.



- to prevent possible FT numerical issues.