## Data Representation

The CalcLib software uses generic data types to represent values manipulated in calculations so that the algorithm can be expressed without being specific to the representation of those values. Algorithms expressed on these generic types use interfaces based on the core Java interface within CalcLib:

```java
package net.myorb.data.abstractions;

public interface SpaceDescription<T>
{

    /**
     * compute internal representation for scalar
     * @param x simple integer value to convert to i
     * @return internal representation for value
     */
    T newScalar (int x);

    /**
     * determine value of opposite sign
     * @param x value to be negated
     * @return negated value
     */
    T negate (T x);

    /**
     * is the value zero
     * @param x value to be checked
     * @return TRUE for zero value
     */
    boolean isZero (T x);

    /**
     * is the value negative
     * @param x value to be checked
     * @return TRUE for negative value
     */
    boolean isNegative (T x);

    /**
     * compute sum of pair of terms
     * @param x the value of the left side of computation
     * @param y the value of the right side of computation
     * @return result of computation
     */
    T add (T x, T y);

    /**
     * compute product x*y
     * @param x the value of the left side of computation
     * @param y the value of the right side of computation
     * @return result of computation
     */
    T multiply (T x, T y);

    /**
     * compute division 1/x
     * @param x the value of the divisor of the computation
     * @return result of computation
     */
    T invert (T x);

    /**
     * get the conjugate of the specified value
     * @param x the value to be used to compute conjugate
     * @return the conjugate of the parameter
     */
    T conjugate (T x);

    /**
     * compute comparison x LT y
     * @param x the value of the left side of computation
     * @param y the value of the right side of computation
     * @return result of computation
     */
    boolean lessThan (T x, T y);

    /**
     * get the value zero in this representation (additive identity)
     * @return the value zero
     */
    T getZero ();

    /**
     * get the value one in this representation (multiplicative identity)
     * @return the value zero
     */
    T getOne ();

}
```

This collection of methods defines a mathematical field operating on the data representation specified by the generic parameter <T>

## Implemented Types

The following representations are implemented in the core of the CalcLib implementation:

### Default

The most primitive form represents values as real numbers stored as Java Double objects supplying a default of 16 digit precision, common 64 bit floating point

### Complex

Java objects that hold real and imaginary parts as separate Java Double values

### Fraction

Java objects that hold numerator and denominator values as separate BigInteger values supplying arbitrary precision

### Factorization

Java objects that hold prime numbers described as BigInteger each with an integer exponent which together represent an integer ratio with numerator and denominator held as prime factors

## Generic Algorithms

Algorithms implemented using the generic <T> type and using the `SpaceDescription` interface have the advantage of fully working with any data type provided in the CalcLib build. The only requirement is that all arithmetic done in the implementation is provided by reference to the interface methods.

For Example:

```
/**
 * compute x^n for larger exponents
 * @param x base of the exponentiation
 * @param n exponent for the computation
 * @return computed result
 */
public T pow (T x, int n)
{
    if (n < 0) return inverted (pow (x, -n));
    else if (n == 0) return discrete (1);
    else if (n == 1) return x;

    Value<T> v = forValue (x);
    Value<T> square = v.squared ();
    Value<T> result = square;

    while ((n -= 2) >= 2)
    {
        result = result.times (square);
    }

    if (n == 1)
        return result.times (v).getUnderlying ();
    else return result.getUnderlying ();
}
public Value<T> pow (Value<T> x, int exponent)
{
    return forValue (pow (x.getUnderlying (), exponent));
}
```

Algorithm implementations can extend the class net.myorb.math.Arithmetic which constructs with an implementer of `SpaceDescription` providing simple and more readable coding of arithmetic operations in the implementation.

The obvious great advantage of this implementation style is that polynomials can be expressed with only the most simple arithmetic operations. With Taylor expansions most trig and power functions can be made available to any field described with a SpaceDescription implementation.

Consider Newton's root method, using fractions or factors one can get a SQRT approximation expressed as an integer ratio and can go to arbitrary precision.

## FloatDemoAiry.txt

READ JavaMath.txt

INFINITY = 100

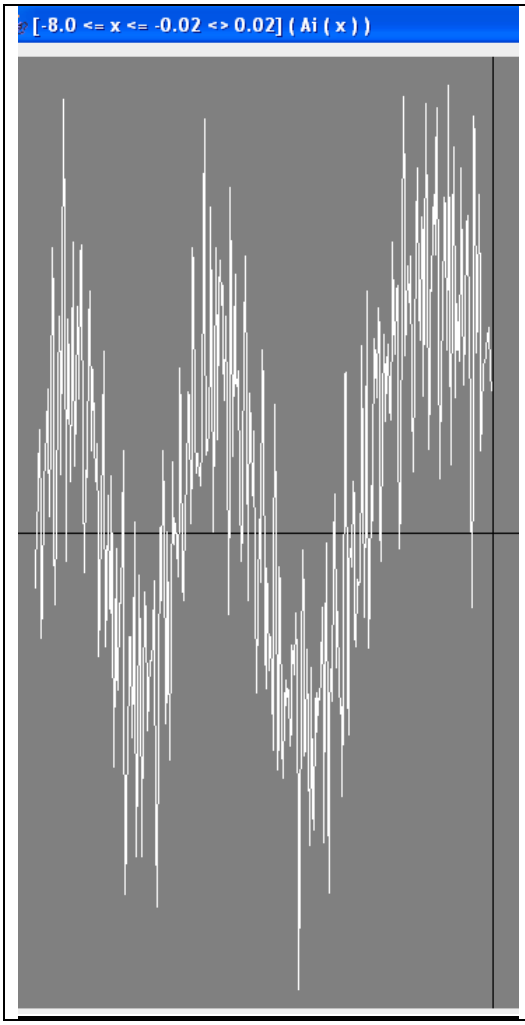
dt = 1 / INFINITY

// airy

!! Ai(x) = 1/pi * INTEGRAL  [0 <= t <= INFINITY <> dt] ( COS (POW (t,3) /3 + x*t) * dt )

!! Bi(x) = 1/pi * INTEGRAL  [0<=t<=INFINITY <> dt] (( EXP (-POW (t,3) /3 + x*t) + SIN (POW (t,3) /3 + x*t)) * dt)

graph [-8 <= x < 0 <> 0.02] (Ai(x))




This is a good demonstration of the difficulty of finding means of computing functions that have extremely long convergence requirements.  The combination of the infinite integral and 'dt' having the implication of 'as small as possible' along with terms that have no diminishing factor all contrive to disallow any 'brute force' attempt to achieve a useful approximation of the value of the series.

A later test shows the attempt to evaluate Airy in the complex space using Bessel functions. While that result is more successful at showing a smooth curve, the further the evaluation is attempted from the 0 of the x-axis, the more error is seen to complicate the approximation

**Functions**

Ai (x) = 1 ÷ π * ∫ [ 0 <= t <= ∞ Δ dt ] ( COS ( POW ( t , 3 ) ÷ 3 + x * t ) * dt )
Bi (x) = 1 ÷ π * ∫ [ 0 <= t <= ∞ Δ dt ] ( ( EXP ( - POW ( t , 3 ) ÷ 3 + x * t ) + SIN ( POW ( t , 3 ) ÷ 3 + x * t ) ) * dt )

## Airy.txt

```
LIBRARY AiFunc
net.myorb.math.specialfunctions.AiryFunctions

!+ AI_IMPORT(x) = AiFunc.Ai
!! Ai(x) = AI_IMPORT x

!+ AIJ_IMPORT(x) = AiFunc.AiJ
!! AiJ(x) = AIJ_IMPORT x

!+ BI_IMPORT(x) = AiFunc.Bi
!! Bi(x) = BI_IMPORT x

!! Ai2(x) = Ai''(x <> 0.001)

!! err(x) = Ai2(x) - x*Ai(x)
```

A spline created from Stokes equation and the Bessel identities is shown here producing a smooth plot

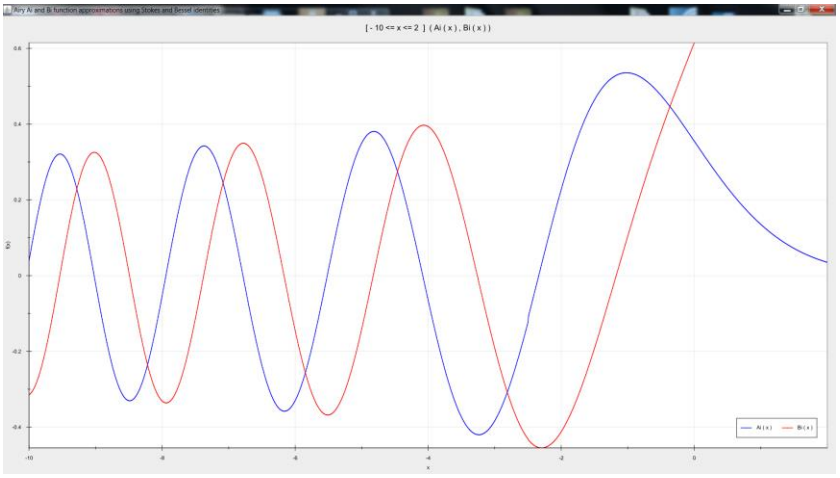
The knot is at x = -2.5 where a small jump in the plot can be seen

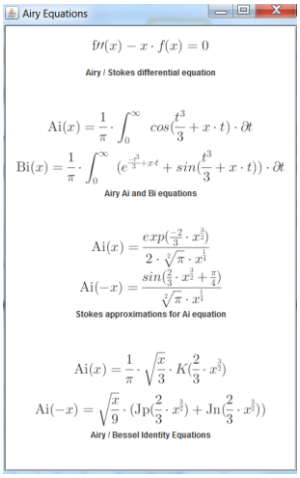
The error plots are done separately on either side of the knot since x < -2.5 is an approximation and for x > -2.5 identities with Bessel J and K functions are used

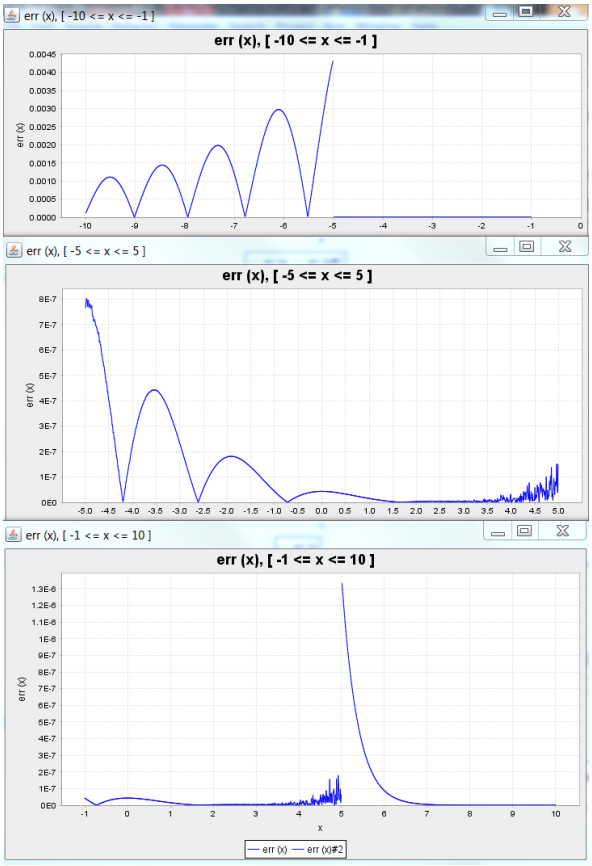
note that the error is so small for x > -2.5 that the graphing software could not even generate the error tick values

max_err = 1.8135048351908267E-7
min_err = 8.545179841501493E-11

### Equations and Plots for Ai(x) and Bi(x)





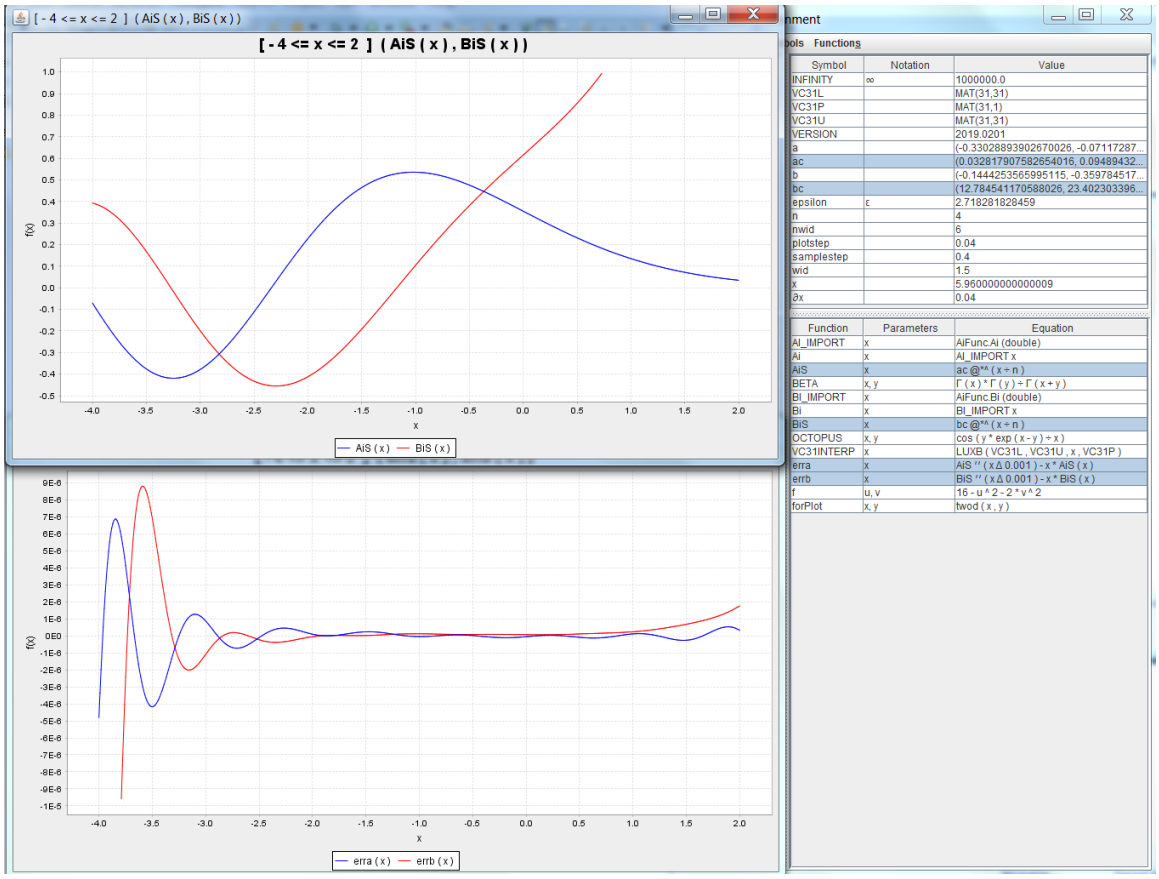

The domain below the low knot at -5 shows the expected errors coming from the Stokes approximation function

Between the low knot at -5 and the hi knot at +5 the error stays very small at < 1E-6 where the computation is done using the Bessel identities based on Jp and Ka. A knot exists at x=0 but shows no jump at that location in the plot

The hi knot at +5 separates the use of Bessel functions [ -5 .. +5] and the Stokes decay approximation used for x > +5. Being an exponential decay the jump at +5 is very small and the error drops very rapidly

# AirySpline.txt

```
READ VC31Prep.txt

LIBRARY AiFunc net.myorb.math.specialfunctions.AiryFunctions

n = 4
wid = 1.5
nwid = n * wid
samplestep = n * 0.1
plotstep = n * 0.01


// Ai(x)

!+ AI_IMPORT(x) = AiFunc.Ai
!! Ai(x) = AI_IMPORT x

a = [-nwid <= x <= nwid+0.05 <> samplestep] ( Ai(x) )
calc LENGTH a ; graph a ; ac = VC31INTERP (a)

!! AiS(x) = ac @*^ (x/n)
!! erra(x) = AiS''(x<>0.001) - x * AiS(x)

PLOTF AiS  [  -nwid <=   x    <= nwid  <>  plotstep  ]
PLOTF erra  [ -nwid <    x    <  nwid  <>  plotstep  ]


// Bi(x)

!+ BI_IMPORT(x) = AiFunc.Bi
!! Bi(x) = BI_IMPORT x

b = [-nwid <= x <= nwid+0.05 <> samplestep] ( Bi(x) )
calc LENGTH b ; graph b ; bc = VC31INTERP (b)

!! BiS(x) = bc @*^ (x/n)
!! errb(x) = BiS''(x<>0.001) - x * BiS(x)

PLOTF BiS   [  -nwid <=   x    <= nwid  <>  plotstep  ]
PLOTF errb  [  -nwid <    x    <  nwid  <>  plotstep  ]
```



## FloatDemoBeta.txt

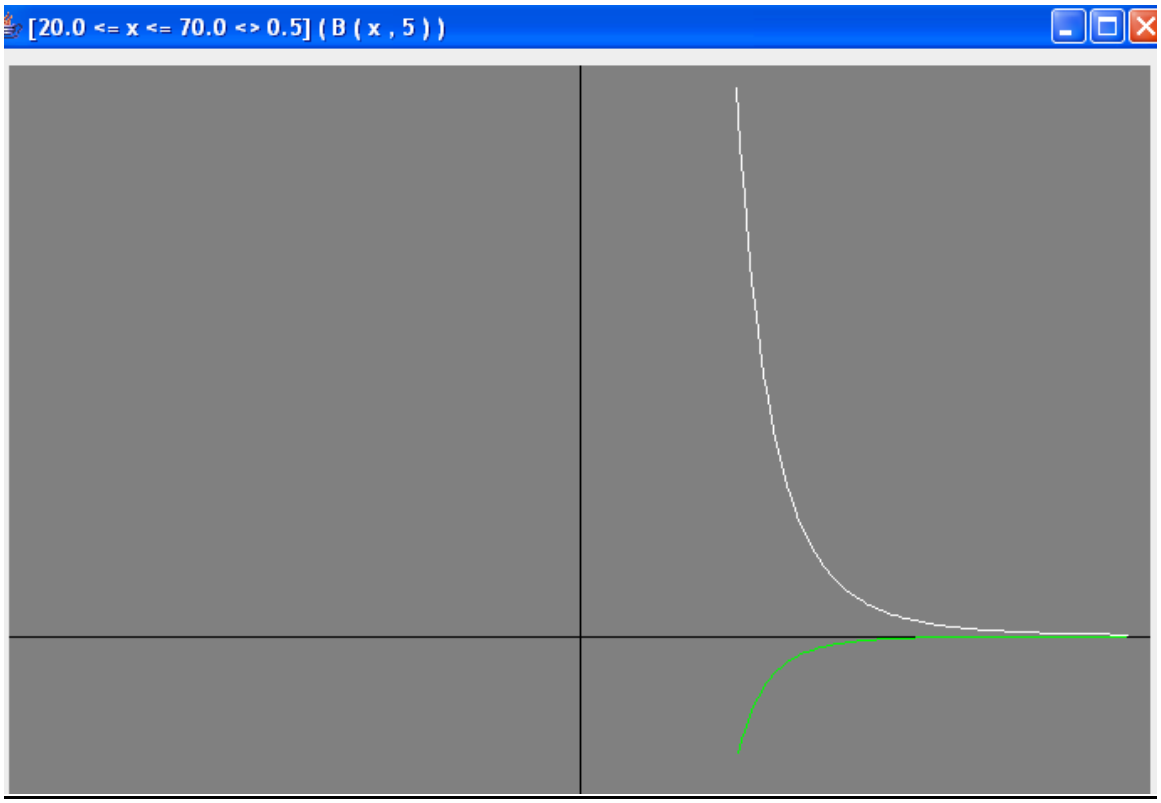
INFINITY = 100

dt = 1 / INFINITY

// beta / beta incomplete / inv / ln

!! B(x,y) = INTEGRAL  [0 < t < 1 <> dt] ( t^(x-1) * (1-t)^(y-1) * dt )

graph [20 <= x <= 70 <> 0.5] (B(x,5))




The integration over the 0 to 1 range provides an easier approximation task.  The computational value of ‘dt’ can easily be seen as a trade-off for the user with the understanding that the reduction of the value increases the precision of the result and requires extra time for the extra calculations.

### Functions

B (x, y) = ∫ [ 0 < t < 1 Δ dt ] ( t ^ ( x - 1 ) * ( 1 - t ) ^ ( y - 1 ) * dt )

**FloatDemoCalculus.txt**

READ FloatDemoCalculus.txt
Reading... C:\workspace\MathFields\scripts\FloatDemoCalculus.txt

SCRIPTPRINT FloatDemoCalculus.txt

```
// computations of polynomial derivative (POLYDER) and integral (POLYINT) are not
// approximations but are actual term-by-term integrals and derivative applied to the
// coefficients of the polynomial

p = (2, -3, 4, -5, 6)

pder = POLYDER p
pint = POLYINT p

POLYPRINT p
POLYPRINT pder
POLYPRINT pint

// by convention the value of 'delta' is assigned as a constant.
// this is a literal selection of precision made by the user with an
// understanding that the smaller delta results in higher calculation time
dx = 0.005

// starting the integral at 0 reduces the result by the -INFINITY to 0 INTEGRAL
e_minus1_approximation = INTEGRAL [0 <= x <= 1 <> dx](exp(x) * dx)
PRETTYPRINT e_minus1_approximation

!! f(x) = 2\(1 - x^2)
// this is the approximation of the area of the unit circle
pi_approximation = 4 * INTEGRAL [0 <= x <= 1 <> dx] ( f(x) * dx )
PRETTYPRINT pi_approximation
```

```
// this is an approximation of the
// Euler-Masheroni gamma value using the INTEGRAL of the H(x) series

gamma_approximation = INTEGRAL [0 <= x <= 1 <> dx] (HARMONIC(x) * dx)
PRETTYPRINT gamma_approximation


// infinite series approximation.
// expanding the Taylor series for EXP(1) give a fast convergence to the value of e

e_approximation = SIGMA [0 <= i <= 10] (1/i!)
PRETTYPRINT e_approximation


// another infinite series approximation.
// slower convergence so number of terms significantly higher

!! zeta(x) = SIGMA [1 <= n <= 1000] (n^(-x))

Apery = zeta(3)
// the Apery constant is equivalent to zeta(3)
PRETTYPRINT Apery


// CALCLIB accepts the F'(x) notation for requesting derivative approximation.
// the addition of the specification of the delta value allows the user to choose precision.
// again the trade-off is increased precision for increased calculation time
// as smaller delta is chosen

fprime = f'( 0.5 <> dx )
PRETTYPRINT fprime
```

**OUTPUT:**

```
// the polynomial

p =

        2 - 3 * x + 4 * x^2 - 5 * x^3 + 6 * x^4


// actual term-by-term derivative

pder =

         - 3 + 8 * x - 15 * x^2 + 24 * x^3


// and actual term-by-term integral

pint =

        2 * x - 1.5 * x^2 + 1.3333333333333333 * x^3
                - 1.25 * x^4 + 1.2000000000000002 * x^5
```

// results of integral approximations by brute force area computation

e_minus1_approximation =

1.7139897036402167

pi_approximation =

3.1511769448395244

gamma_approximation =

0.5746638357147642

// approximations made from infinite series

e_approximation =

2.7182818011463845

Apery =

1.2020564036593433

// output of the approximated derivative requested as F’(x)

fprime =

-0.5773534767267385

**FloatDemoCombo.TXT**

READ FloatDemoCombo.txt
Reading... C:\workspace\MathFields\scripts\FloatDemoCombo.txt
Reading... C:\workspace\MathFields\scripts\FloatDemoCombo.TXT

// Bernoulli numbers are notoriously difficult to calculate.
// brute force calculation is beyond basic capabilities of most computers

SCRIPTPRINT FloatDemoCombo.TXT

n=1
SCRIPTPRINT ComputeNthBernoulli.txt
ITERATE 10 ComputeNthBernoulli.txt

// note the odd numbered values which should be 0 as evidence that even
// the low numbered results introduce error quickly, higher ones become impossible.

// in a later demo script the Bernoulli sequence is computed using prime fractions.
// the first 30 values are computed precisely as can be verified with online sites.

// the GAMMA function evaluated at positive integers GAMMA(n) = (n-1)!

x = 6
SCRIPTPRINT GammaCompute.txt
SCRIPTPRINT GammaIteration.txt
READ GammaCompute.txt

// the correct value of GAMMA(6) = 5! = 120, so this approximation is off by 2%

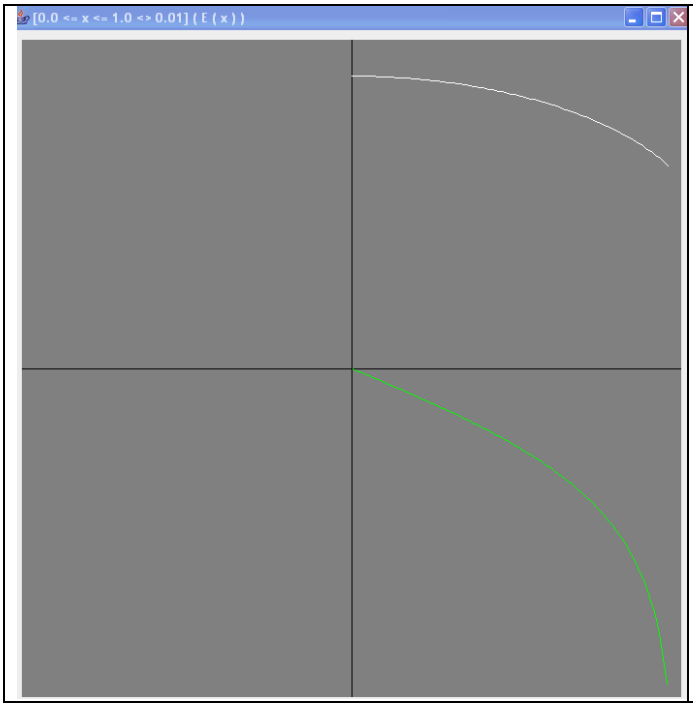
**OUTPUT:**

```
Reading... C:\workspace\MathFields\scripts\ComputeNthBernoulli.txt
bn = BERNOULLI(n)
PRETTYPRINT bn
n = n + 1

bn =  0.5
bn =  0.16666666666666663
bn =  -2.220446049250313E-16
bn =  -0.03333333333333399
bn =  1.7763568394002505E-15
bn =  0.02380952380956758
bn =  3.794742298168785E-13
bn = -0.03333333333154209
bn = -1.0449863196981823E-11
bn =  0.07575757526731397

Iteration 10 has completed
*** Maximum iteration count exceeded
```

```
Reading... C:\workspace\MathFields\scripts\GammaCompute.txt

!! GammaFactor (t,n) = n / (t+n) * (n / (n-1))^t

g = x^2 + x
g = 1 / g
n = 2

ITERATE 1000 GammaIteration.txt
PRETTYPRINT g

// the correct value of GAMMA(6) = 5! = 120, so this approximation is off by 2%
// and this is at 1000 iterations, at 5000 it is not much closer
// this is very slow convergence

ITERATE 4000 GammaIteration.txt
PRETTYPRINT g

ITERATE 5000 GammaIteration.txt
PRETTYPRINT g

ITERATE 3000 GammaIteration.txt
PRETTYPRINT g

// IMPORTANT NOTE:  look at the formula for the Nth GAMMA factor.
// note that for integer t the computation can be done as integer exponentiation.
// if t is real, computation becomes exp(ln(n)*t), much more expensive
Reading... C:\workspace\MathFields\scripts\GammaIteration.txt

g = g * GammaFactor(x,n); n=n+1

Iteration 1000 has completed
g =  117.51405987584201

Iteration 5000 has completed
g =  119.49737453500546

Iteration 10000 has completed
g =  119.74834401649464

Iteration 13000 has completed
g =  119.80635745853313
```

**FloatDemoEllip.txt**

Again the integration range being 0 – 1 allows a reasonable approximation using INFINITY = 100 and dx = 0.01 resulting in a smooth curve in the plot, but how accurate?




INFINITY = 100

dt = 1 / INFINITY

// ellipj ellipke

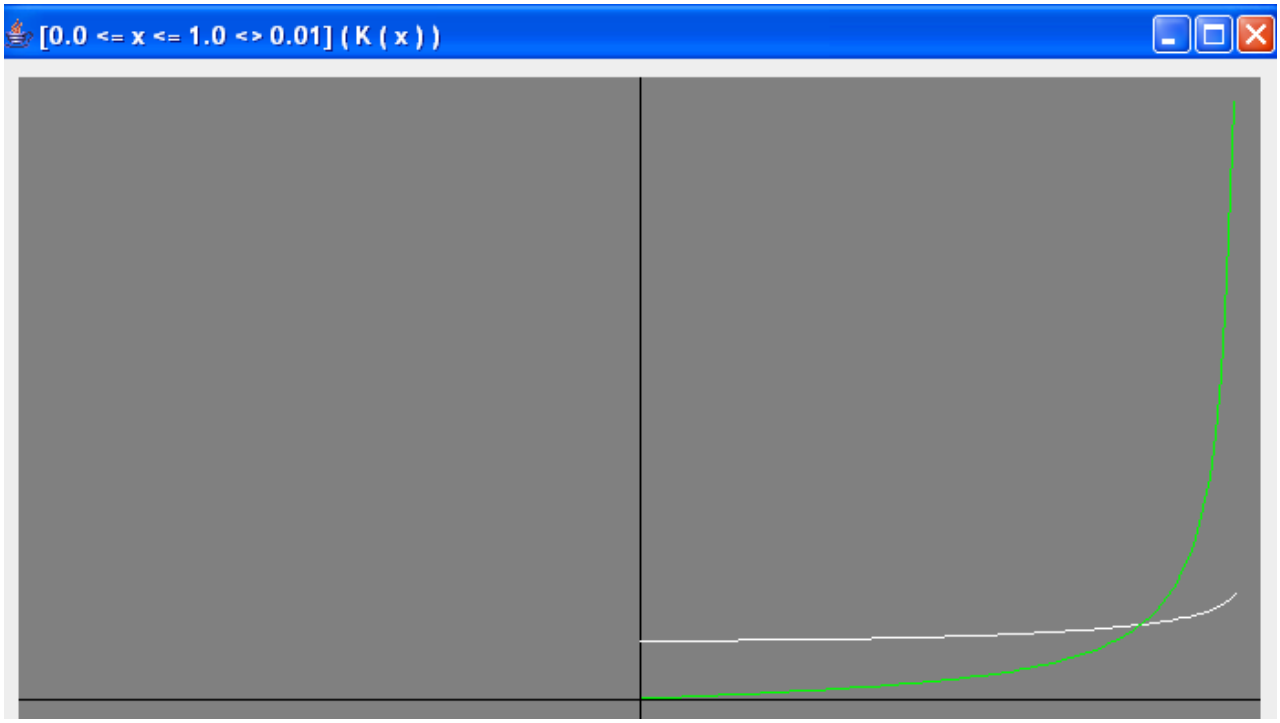
!! K(k) = INTEGRAL  [0<=t<=1<>dt] ( 1 / sqrt ((1-t^2) * (1-k^2*t^2)) * dt )

!! E(k) = INTEGRAL  [0 <= t <= 1 <> dt] ( sqrt ((1-k^2*t^2) / (1-t^2)) * dt )

graph [0 <= x <= 1 <> dt] (E(x))
graph [0 <= x <= 1 <> dt] (K(x))

## Functions

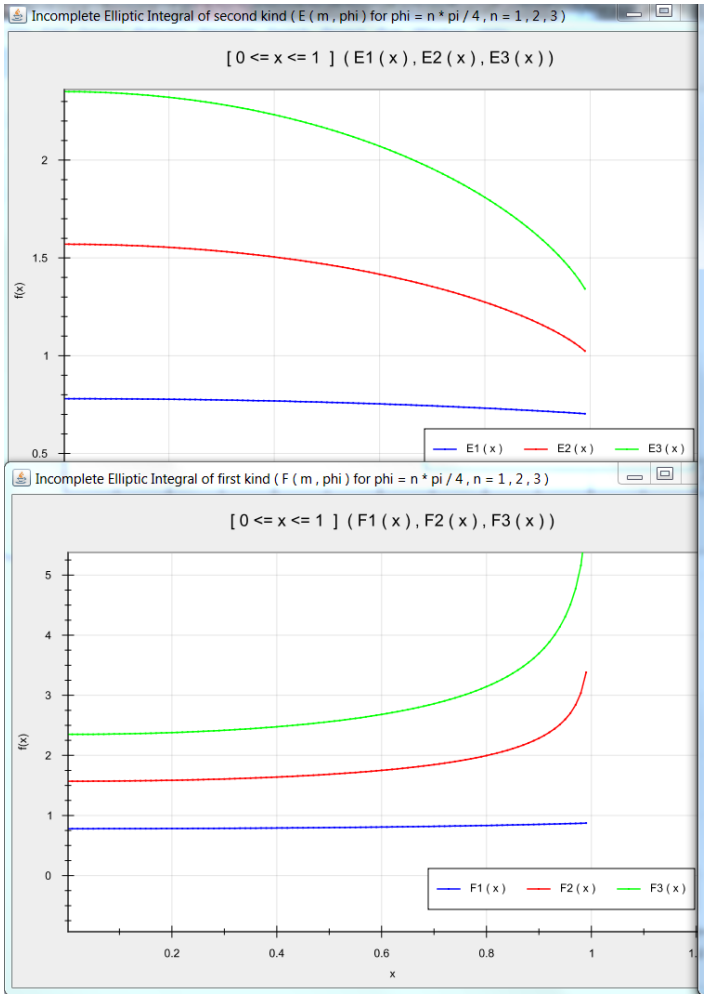
E (k) = ∫ [ 0 <= t <= 1 Δ dt ] ( √ ( ( 1 - k ^ 2 * t ^ 2 ) ÷ ( 1 - t ^ 2 ) ) * dt )
K (k) = ∫ [ 0<=t<=1 Δ dt ] ( 1 ÷ √ ( ( 1 - t ^ 2 ) * ( 1 - k ^ 2 * t ^ 2 ) ) * dt )

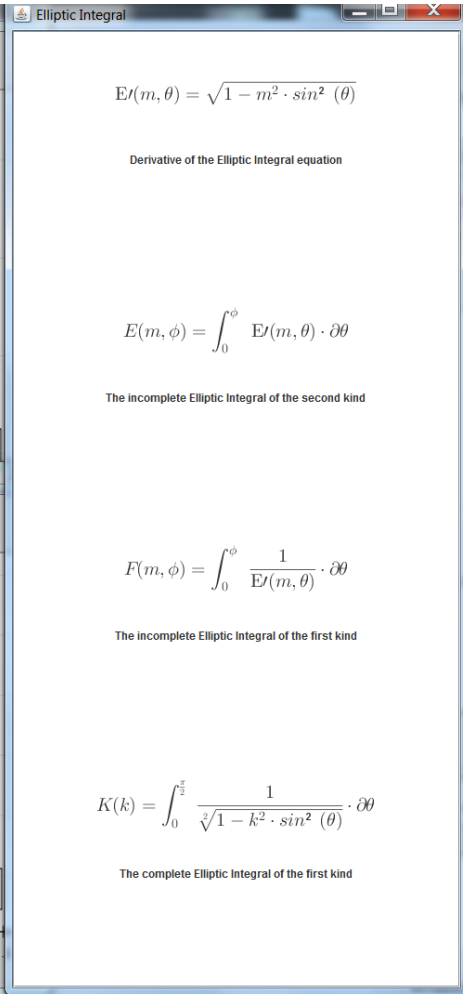


# EllipticInt.txt

```
INFINITY = 100

dt = 1 / INFINITY

qPi = pi #/# 4


// complete first kind

!! K(k) = INTEGRAL [0 < theta <= pi/2 <> dt]
     ( 1 / 2\(1 - k^2 * sinsq theta) * <*> theta )

graph [0 <= x <= 1 <> dt] ( K(x) )
ENTITLED Complete Elliptic Integral of first kind ( K(k) )


// complete second kind

!! E'(m, theta) = sqrt (1 - m^2 * sinsq theta)
!! E(m, phi) = INTEGRAL [0 < theta <= phi <> dt] ( E'(m,theta) * <*> theta )
!! F(m, phi) = INTEGRAL [0 < theta <= phi <> dt]
     ( 1 / E'(m,theta) * <*> theta )

SELECT "Elliptic Integral"

RENDERF E'
RENDER " Derivative of the Elliptic Integral equation " TOP

RENDERF E
RENDER " The incomplete Elliptic Integral of the second kind " TOP

RENDERF F
RENDER " The incomplete Elliptic Integral of the first kind " TOP

RENDERF K
RENDER " The complete Elliptic Integral of the first kind " TOP

// degenerate case is simple circle
// graph [0 < x < 1 <> dt] ( E'(x, 2 * qPi) )

!! E1(x) = E(x, qPi)
!! E2(x) = E(x, 2 * qPi)
!! E3(x) = E(x, 3 * qPi)

!! F'(m, theta) = 1 / E'(m, theta)
!! F(m, phi) = INTEGRAL [0 < theta <= phi <> dt]
     ( F'(m, theta) * <*> theta )

!! F1(x) = F(x, qPi)
!! F2(x) = F(x, 2 * qPi)
!! F3(x) = F(x, 3 * qPi)

graph [0 <= x <= 1 <> dt] ( F1(x),F2(x), F3(x) )
ENTITLED Incomplete Elliptic Integral of first kind
      ( F(m,phi) for phi = n*pi/4, n=1,2,3 )

graph [0 <= x <= 1 <> dt] ( E1(x), E2(x), E3(x) )
ENTITLED Incomplete Elliptic Integral of second kind
     ( E(m,phi) for phi = n*pi/4, n=1,2,3 )

// !! Er(k) = INTEGRAL [0 < t < 1 <> dt]
     ( sqrt ((1-k^2*t^2) / (1-t^2)) * dt )

// !! dif2(x) = E2(x) - Er(x)
// !! dif3(x) = E3(x) - Er(x)

// graph [0 < x < 1 <> dt] ( dif2(x), dif3(x) )
```

```
Description of integrals of first and second kind

Equation renders

Prepare plots for phi = n * pi/4


Incomplete first kind


N = 1
N = 2
N = 3


Incomplete second kind
```


Incomplete Elliptic Integral of second kind ( E ( m , phi ) for phi = n * pi / 4 , n = 1 , 2 , 3 )
[ 0 <= x <= 1 ] ( E1 ( x ) , E2 ( x ) , E3 ( x ) )
f(x)
2
1.5
1
0.5
E1 ( x )
E2 ( x )
E3 ( x )
Incomplete Elliptic Integral of first kind ( F ( m , phi ) for phi = n * pi / 4 , n = 1 , 2 , 3 )
[ 0 <= x <= 1 ] ( F1 ( x ) , F2 ( x ) , F3 ( x ) )
f(x)
5
4
3
2
1
0
0.2
0.4
0.6
0.8
1
x
F1 ( x )
F2 ( x )
F3 ( x )


Elliptic Integral
$$\mathrm{E}\prime(m,\theta) = \sqrt{1 - m^2 \cdot sin^2\ (\theta)}$$
Derivative of the Elliptic Integral equation
$$E(m,\phi) = \int_0^{\phi} \mathrm{E}\prime(m,\theta) \cdot \partial\theta$$
The incomplete Elliptic Integral of the second kind
$$F(m,\phi) = \int_0^{\phi} \frac{1}{\mathrm{E}\prime(m,\theta)} \cdot \partial\theta$$
The incomplete Elliptic Integral of the first kind
$$K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt[2]{1 - k^2 \cdot sin^2\ (\theta)}} \cdot \partial\theta$$
The complete Elliptic Integral of the first kind

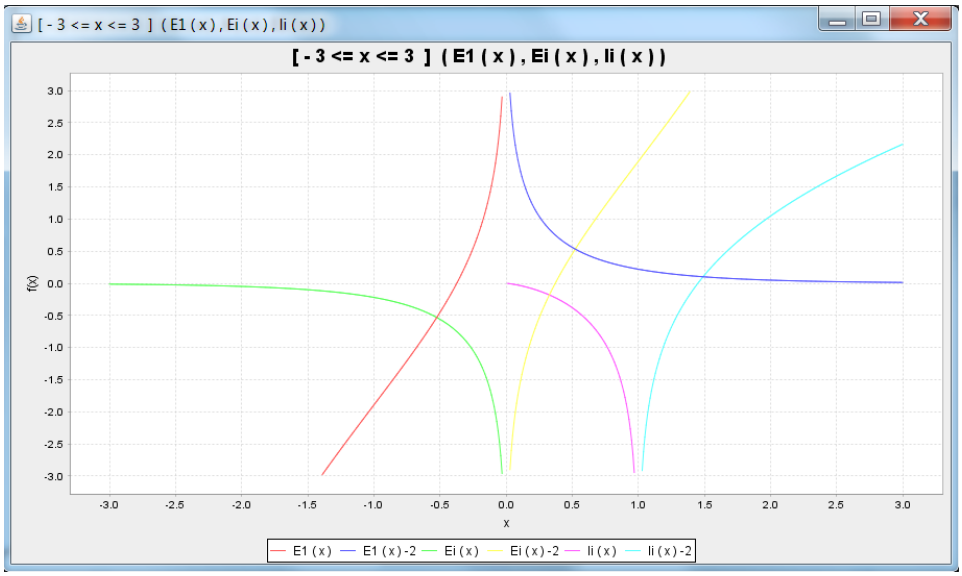
## FloatDemoExpIntegral.txt

READ JavaMath.txt

INFINITY = 70
dt = 1 / INFINITY

// exp integral
!! Ei(x) = - INTEGRAL [-x <= t <= INFINITY <> dt] ( EXP(-t) / t * dt )
!! E1(x) = INTEGRAL [x <= t <= INFINITY <> dt] ( EXP(-t) / t * dt )

graph [0.2 <= x <= 1.3 <> 0.01] (E1(x))




Exponential decay term-by-term, so we would expect fast convergence

## Functions

$$E1\ (x) = \int [\ x <= t <= \infty\ \Delta\ dt\ ]\ (\ EXP\ (\ -\ t\ ) \div t\ *\ dt\ )$$

$$Ei\ (x) = -\int [\ -\ x <= t <= \infty\ \Delta\ dt\ ]\ (\ EXP\ (\ -\ t\ ) \div t\ *\ dt\ )$$

## FloatDemoRegression.txt

```
READ FloatDemoRegression.txt
Reading... C:\workspace\MathFields\scripts\FloatDemoRegression.txt


SCRIPTPRINT FloatDemoRegression.txt
READ regress_data.txt


PRETTYPRINT y_line
line_coefficients = FITLINE (x,y_line)
!! line(x) = line_coefficients +*^ x
PRETTYPRINT line_coefficients


PRETTYPRINT y_exp
exp_coefficients = FITEXP (x,y_exp)
!! aebx(x) = exp_coefficients *^# x
PRETTYPRINT exp_coefficients


PRETTYPRINT y_poly
poly_coefficients = FITPOLY (x,y_poly)
!! poly(x) = poly_coefficients +*^ x
PRETTYPRINT poly_coefficients


PRETTYPRINT y_har
har_coefficients = FITHARMONIC (x_har,y_har,0.1)
!! har(x) = har_coefficients +#* x
PRETTYPRINT har_coefficients

SHOW Functions
```

FloatDemoRegression.txt

x = (-3, -2, 1, 4, 5)

y_poly = (4, -7.1, 9.5, -13.5, 16.2)

y_line = (4, 7.1, 9.5, 13.5, 16.2)

y_exp = (0.1, 0.3, 6, 23.5, 66.2)

y_har = (4, -7.1, 9.5, -13.5, 16.2)

x_har = (1.1, 3.3, 3.9, 5.1, 6.3)

// linear regression using least squares methodology

Reading... C:\workspace\MathFields\scripts\regress_data.txt

// prepare a sample of data

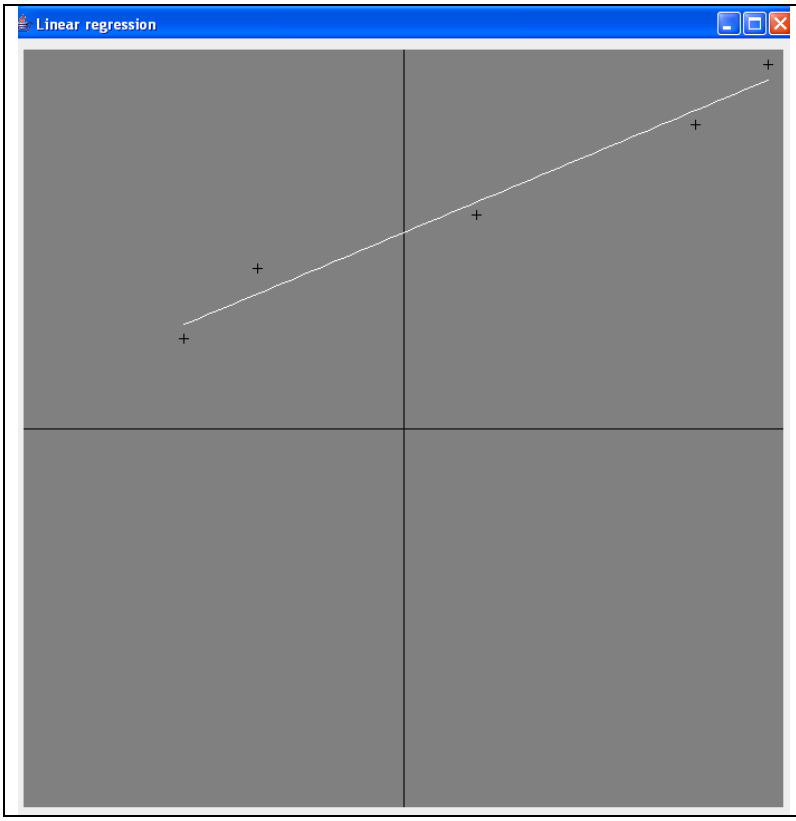
y_line =

[
4
7.1
9.5
13.5
16.2
]

// the result is a set of coefficients describing a line, specifically a slope and a y-intercept

Linear regression

Y = 8.700000000000001 + 1.36*x




// also provided is a set of metrics
// describing the qualities of the
// regression

Linear regression correlation
r^2 = 0.9700834976712961

standard deviation
STD = 0.9750213672872332

covariance
COV = 0.0969206130504208

Mean square error
MSE = 0.9506666666666657

Sum squared residuals
SSR = 92.48

Sum squared error
SSE = 2.851999999999997

Sum square total (for series)
SST = 95.332

Pearson product moment
PC = 0.9849281687876015

// Non-linear regression uses least squares between Ln Y as a function of X

y_exp =

[
0.1
0.3
6
23.5
66.2
]

Non-Linear regression

Ln Y = 0.3456968531967063 + 0.7812796380035235 * X

// The linear solution taking the form

// Ln Y = a + b * X

// Solving for Y gives:

// Y = EXP ( a + b * X ) = EXP(a) * EXP ( b * X )

// A solution of the form Y = C * EXP ( b * X ) { where C = EXP(a) }


r^2 = 0.9795614356940878
STD = 0.46072336115906243
COV = 0.40881363964246686
MSE = 0.21226601551770388
SSR = 30.519893637945838
SSE = 0.6367980465531117
SST = 31.15669168449896
PC = 0.9897279600344046

// Polynomial interpolation uses linear algebra to solve for polynomial coefficients.
// Vandermonde matrix is built from sample data with coefficients as the variables.
// solution for coefficients realized using matrix inversion for small sample sets,
// or by application of Gaussian elimination for larger matrices

READ regress_data.txt
Reading... C:\workspace\MathFields\bin\scripts\regress_data.txt

y_poly =

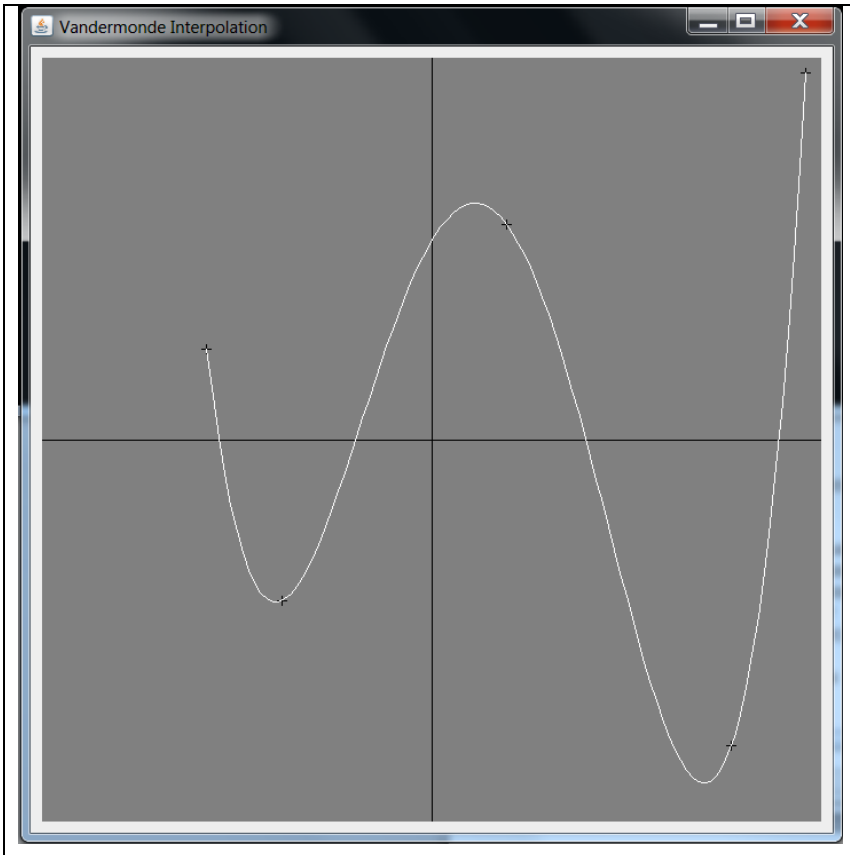
[
4
-7.1
9.5
-13.5
16.2
]

x_FITPOLY_y_poly = FITPOLY (x, y_poly)

// the solution is a set of coefficients making up a polynomial

Vandermonde Interpolation

Y = 8.771428571428572 + 5.586904761904762 * x - 4.269642857142857 * x^2
- 0.9083333333333333 * x^3 + 0.3196428571428571 * x^4




// the computation of the analysis
// of the curve fit still applies

r^2 = 1.0
STD = 1.1659632469897484E-14
COV = 6.4063914669766386E-15
MSE = 1.3594702933308769E-28
SSR = 584.7879999999997
SSE = 4.078410879992631E-28
SST = 584.788

// harmonic regression using linear algebra to solve for Fourier series coefficients

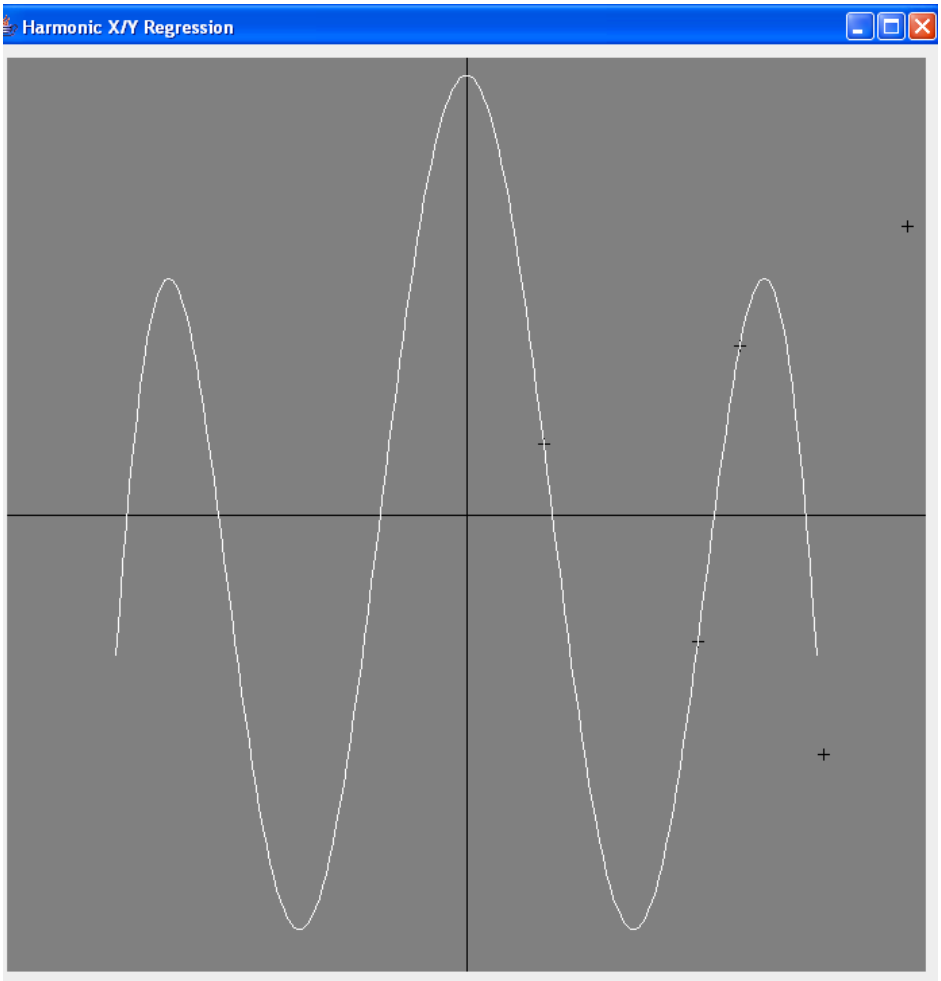
y_har =

[
	4
	-7.1
	9.5
	-13.5
	16.2
]

har_coefficients =

[
	510461.2340641936
	-1215404.4917130603
	1098328.6886200758
	-478250.9996191938
	84890.2593673543
]

// fitting the series Y = c0 + c1 cos wX + c2 cos 2wX + c3 cos 3wx + + +
// where w (actually omega is common) is the base period, each a different solution



// Lagrange Interpolation derives the polynomial best fitting the data
// using computation made directly on the data sample with no matrix intermediates.
// this is an alternative to Vandermonde matrix construction

x_LAGRANGE_y_poly = LAGRANGE  (x,  y_poly)

Lagrange Interpolation

Y = 8.771428571428572 + 5.586904761904761 * x - 4.269642857142857 * x^2
- 0.9083333333333332 * x^3 + 0.3196428571428571 * x^4


r^2 = 1.0
STD = 5.02429586778808E-15
COV = 2.7606021251582853E-15
MSE = 2.5243548967072378E-29
SSR = 584.7879999999999
SSE = 7.573064690121713E-29
SST = 584.788

Note that Vandermonde and Lagrange produced nearly identical polynomials
(in this case it is seen, perhaps others will show differences?)
Also, SSE comes up different, interesting

// The functions created in the regression script:

**Functions**

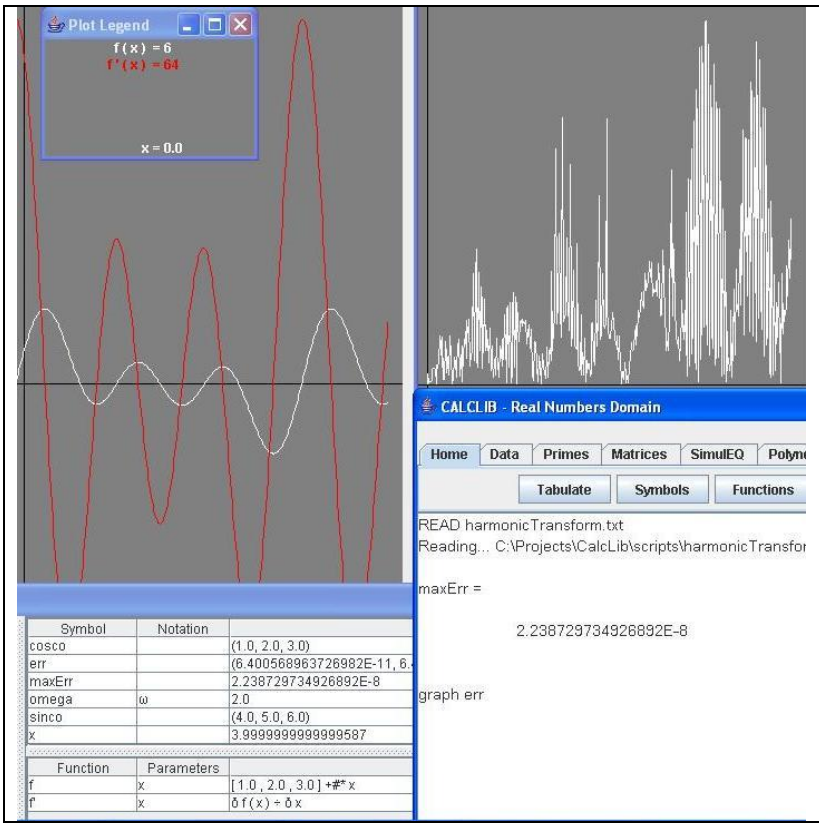
| | |
|---|---|
| aebx (x) | exp_coefficients *^# x |
| har (x) | har_coefficients +#* x |
| line (x) | line_coefficients +*^ x |
| poly (x) | poly_coefficients +*^ x |

4 symbols found and displayed

**HarmonicTransform.txt**

Harmonic Transform describes the Fourier series form

F(x) = c1 cos wx + c2 cos 2wx + c3 cos 3wx + … + s1 sin wx + s2 sin 2wx + …




// coefficients of cos terms
cosco = (1,2,3)

// coefficients of sin terms
sinco = (4,5,6)

// period multiplier
omega = 2

// transform declared
!% f(x) +#*= (cosco, sinco, omega)

graph [-4<=x<=4<>0.01] (f(x), f'(x))

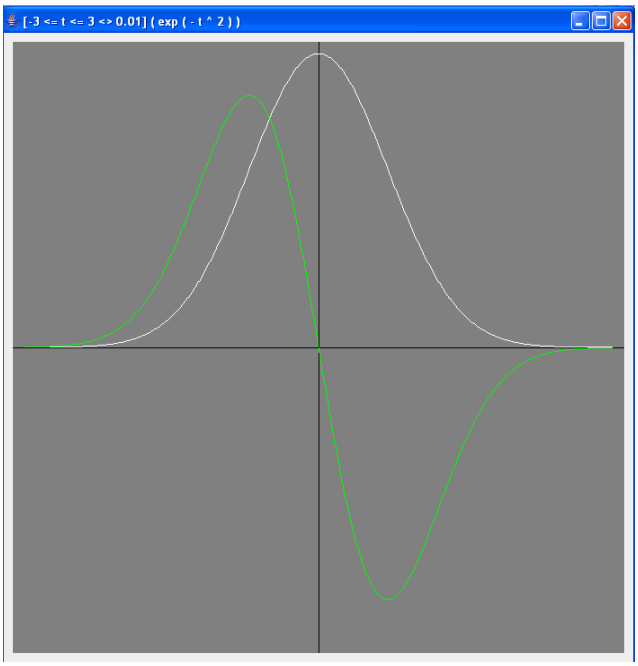
// approximation error analysis
err = [0<=x<=4<>0.01] (abs (f'(x) - f'(x <> 0.000001)))
maxErr = MAX err; PRETTYPRINT maxErr
// 2.2E-08 is highest spike

// display of error
graph err

the error analysis shows the difference between the computations of the function derivative from the function transform as opposed to the rise/run approximation calculation at each point. Using a delta of 1E-06 the maximum error peak found in the difference is 2.2E-08

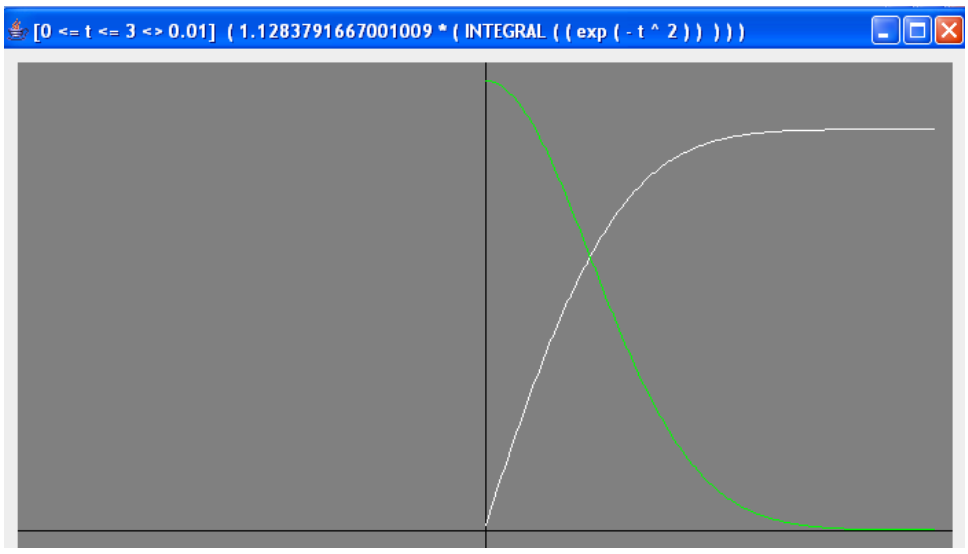
## FloatDemoErf.txt

Again, short range on the integral, but now in addition we have exp (-x) as a factor giving extremely fast convergence since term values now diminish exponentially




```
SCRIPTPRINT FloatDemoErf.txt

// constants for approximation purposes

INFINITY = 100
dt = 1 / INFINITY

// create an array function describing the curve
curve = [-3 <= t <= 3 <> dt] ( exp ( - t ^ 2 ) )

// show a plot of the curve
graph curve

// show a plot of the integral of the curve
graph  2 / sqrt pi * ARRAYINT  (curve, 0, 3)

// capture integral as array function
erf = 2 / sqrt pi * ARRAYINT (INTERVAL (curve, 0, 3))

// define erfc function
!! erfc(x) = 1 - ( erf @# x )
graph [0 <= t <= 3 <> dt] ( erfc (t) )
```

(continued next page)




## Functions

$$2 \div \sqrt{}( \pi ) * \int [\ 0 <= t <= 3\ \Delta\ 0.01\ ] ( \exp ( - t \wedge 2 ) )$$

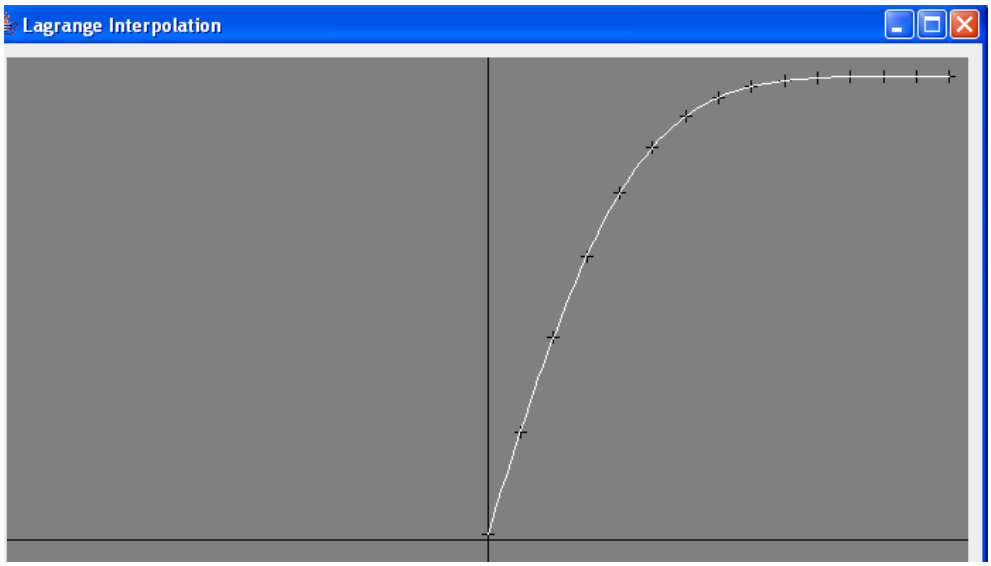
## Polynomial Interpolation

```
// produce a polynomial interpolation of erfx (t)

// reduce the domain to provide proper size sample set
samples = [0 <= t <= 3 <> 0.3] ( erf @# t )

// perform polynomial interpolation
erfpoly = INTERPOLATE samples
!! erfx (x) = erfpoly +*^ x

GRAPH [0 <= t <= 3 <> dt] ( erfx (t) )
```

Lagrange Interpolation

Y = 0.001128 + 1.131666 * x - 0.031665 * x^2 –
0.257873 * x^3 - 0.239321 * x^4 + 0.396119 * x^5 –
0.195003 * x^6 + 0.039859 * x^7 - 5.018E-4 * x^8 –
0.001058 * x^9 + 1.1998855568731703E-4 * x^10

r^2 = 0.9999999999999938
STD = 2.61643420047861E-8
COV = 3.3337018563070156E-8
MSE = 6.845727925434144E-16
SSR = 1.4281441558144792
SSE = 8.899446303064386E-15
SST = 1.4281441240103792




// the full set of generated polynomial coefficients

PRETTYPRINT p

p =
[

| | | | |
|---|---|---|---|
| 0 | 0.0011283791667001009 | 6 | -0.19500366487283216 |
| 1 | 1.1316668575887787 | 7 | 0.0398595045915364 2 |
| 2 | -0.03166565082125494 | 8 | -5.018255826056972E-4 |
| 3 | -0.2578731233342175 | 9 | -0.0010581619105540785 |
| 4 | -0.23932128353545323 | 10 | 1.1998855568731703E-4 |
| 5 | 0.3961193902536131 | | |

]

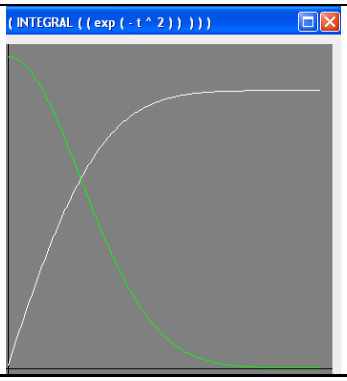
## Applied Approximation Theory

Full expression of a function can be used to calculate points along a curve. With complicated expression using other complex function evaluations the computation time can become unacceptable where many evaluations are necessary. Polynomial interpolation used within approximation theory seeks the optimal polynomial (or polynomials of Spline describing the function in segments) that will minimize the maximum value of the function

e = | P(x) – f(x) | where P(x) is the approximation function and f(x) is the correct value of the function

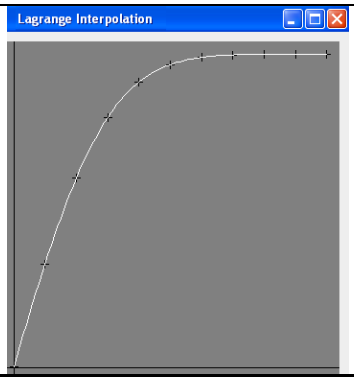
### Function Analysis

The full expression of the function is used to create an array function. The long expression tends to be expensive to compute and the array function introduces errors where extrapolation is required



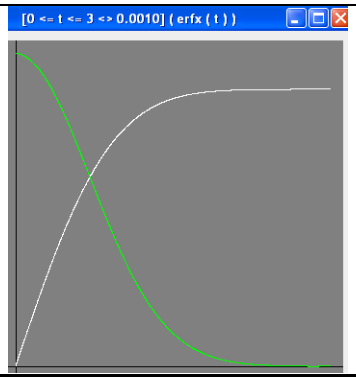

### Interpolation

The interpolation step produces a set of polynomial coefficients that will approximate the function over the domain equivalent to the domain specified in the array function definition

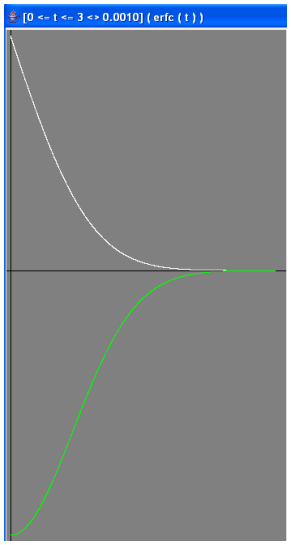



### Publishing

The interpolated polynomial coefficients can be used to define a function based on a polynomial. This function will be necessarily continuous and the accuracy needs to be evaluated to determine usefulness




The definition of erfc(t) was done in the script based on an array function. This would be a relatively fast function but extrapolation is used to compute values between elements and this may introduce unwanted errors




Interpolation could also be done on erfc(x) and a polynomial approximation published for this function

## Functions

erfc (x) = 1 - ( erf @# x ) // array function evaluation based on the 'erf' array
erfx (x) = erfpoly +*^ x // polynomial based on 'erfpoly' coefficients

2 symbols found and displayed

**FloatDemoSpline.txt**

```
// Divide the domain of a function into segments.
// Polynomial interpolation is done on each segment separately.
// A “spline” is defined based on the set of polynomials. All function evaluation
// is then done based on the segment referenced.


READ FloatDemoSpline.txt
Reading... C:\workspace\MathFields\scripts\FloatDemoSpline.txt

SCRIPTPRINT FloatDemoSpline.txt

// constants for approximation purposes
INFINITY = 1000; dt = 1 / INFINITY

// create an array function describing the curve
curve = [-3 <= t <= 3 <> dt] ( exp ( - t ^ 2 ) )

// select segment domains and collect samples

seg1 = [0 <= t <= 1 <> 0.1] ( curve @# t )
seg2 = [1 <= t <= 2 <> 0.1] ( curve @# t )
seg3 = [2 <= t <= 3 <> 0.1] ( curve @# t )

s1poly = INTERPOLATE seg1
s2poly = INTERPOLATE seg2
s3poly = INTERPOLATE seg3

// define a spline of 3 segments
!$ bell(x) = (s1poly, s2poly, s3poly)

GRAPH  [0 <= x <= 3 <> 0.001] (bell (x))

// evaluation of the error curve

!! P(x) = bell(x)
!! f(x) = curve @# x

err = [0 <= x <= 3 <> 0.001] (abs (P(x) - f(x)))

maxError = MAX err
PRETTYPRINT maxError
GRAPH err
```

**OUTPUT:**

// interpolation of seg1

Lagrange Interpolation

Y = 0.9999999999999999 - 8.463247910261984E-7 * x - 0.9999750038276538 * x^2 - 3.0337950116177126E-4 * x^3 + 0.5020299854145378 * x^4 - 0.008398561960802908 * x^5 - 0.14401729277506092 * x^6 - 0.04054632573752315 * x^7 + 0.08925391123739246 * x^8 - 0.03450164736568695 * x^9 + 0.004338601056701918 * x^10

// (captured as s1poly)

r^2 = 1.0
STD = 3.459645151802116E-10
COV = 4.671725253261122E-10
MSE = 1.1969144576387888E-19
SSR = 0.5120456723098887
SSE = 1.0772230118749099E-18
SST = 0.5120456713082211
PC = -0.9858678255264112

// interpolation of seg2

Lagrange Interpolation

Y = 0.8286200879697234 + 0.9736284837995299 * x - 3.200753017808287 * x^2 + 2.470769299660251 * x^3 - 0.8012268409784729 * x^4 + 0.09684142852894684 * x^5

// (captured as s2poly)

r^2 = 1.0
STD = 2.2102786549547502E-13
COV = 1.5062381178003443E-12
MSE = 4.88533173254858E-26
SSR = 0.08995776735634785
SSE = 1.954132693019432E-25
SST = 0.08995776735615145
PC = -0.9582660424707687

// interpolation of seg3

Lagrange Interpolation

Y = - 2.389979813072614 + 13.52847449275123 * x - 24.492748482913157 * x^2 + 22.94192603632837 * x^3 - 13.011997020365925 * x^4 + 4.7541003534429365 * x^5 - 1.133624246020645 * x^6 + 0.17138242630778677 * x^7 - 0.014972771910288785 * x^8 + 5.7754667090959E-4 * x^9

// (captured as s3poly)

r^2 = 0.9999999996805409
STD = 1.1429547976886718E-7
COV = 2.242888597175639E-5
MSE = 1.3063456695595527E-14
SSR = 3.2714460471502753E-4
SSE = 1.0450765356476422E-13
SST = 3.2713940689165356E-4
PC = -0.8952813088488132

// the request to define a spline is accepted,
// verification on continuity is done generating appropriate errors;
// the segment table is then displayed in domain order

bell is found to be continuous on the interval 0.0 to 3.0

| | | |
|---|---|---|
| s1poly | 0.0 | 1.0 |
| s2poly | 1.0 | 2.0 |
| s3poly | 2.0 | 3.0 |

maxError =

8.746921299832011E-7

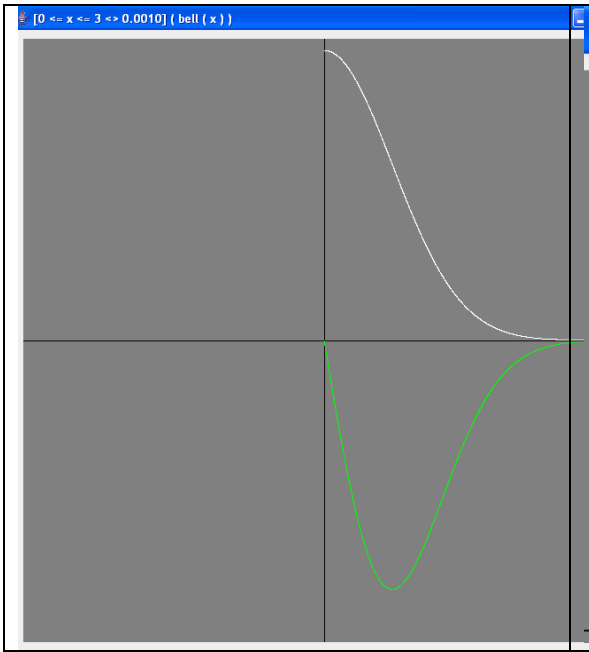
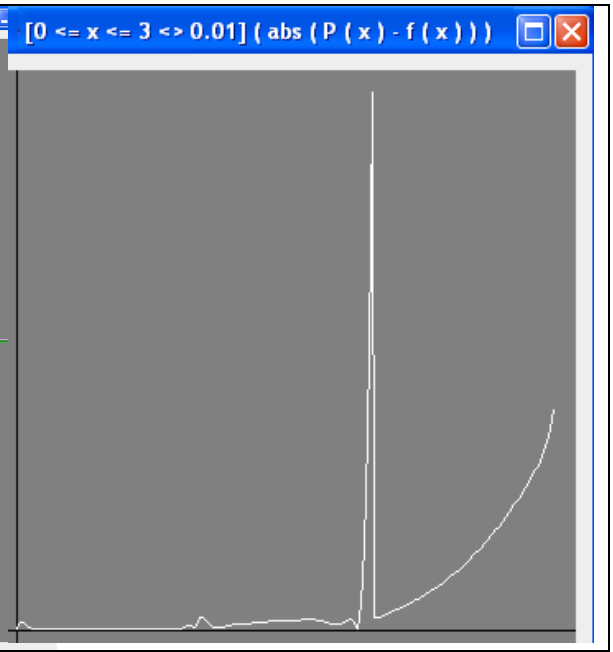
// see the chart, the largest error occurs at the “knot” point
// where segments change (x = 2) as might be expected

// the ENCODE command exports the function to a Java class

ENCODE bell

## Spline Plot

The error plot does show that there are spikes at the knots (segment borders).
The peak at x=2 seems large until one notes that the MAX is 6.5E-07,
So the scale makes the image misleading

| | |
|---|---|
|  |  |
| At a distance the curve looks exactly like the original, it seems impressive that there are no readily noticeable differences | The MAX error value at 6.5E-07 for a function that goes from 0 – 1 makes it reasonable that the error is not obvious |

The increased error on the interval 2 < x < 3 indicates some difficulty for the interpolation maintaining accuracy as the function value becomes very small relative to the full range of function values

## Spline Export

Splines can be exported producing a Java class. This class can be compiled and added to the CLASSPATH allowing the function to be imported in the same way as external library functions. JAR files can be built making transport of function libraries as simple as any Java product. Used in conjunction with the CALCLIB JAR the function library JAR can make the functions available in any Java program.

```
public class Bell extends net.myorb.math.expressions.symbols.SplineInRealDomain
{
        public void initialize ()
        {
                setName ("Bell");
                setSegmentLoConstraints
                (
                        0, 1, 2
                );
                setHiConstraint (3);
                addSegmentPolynomial
                (
                                0.9999999999999999,
                                -8.463247910261984E-7,
                                -0.9999750038276538,
                                -3.0337950116177126E-4,
                                0.5020299854145378,
                                -0.008398561960802908,
                                -0.14401729277506092,
                                -0.04054632573752315,
                                0.08925391123739246,
                                -0.03450164736568695,
                                0.004338601056701918
                );
                addSegmentPolynomial
                (
                                1.0891112255399094,
                                -0.6151861959406233,
                                0.8686902279332571,
                                -3.2547740892405272,
                                4.029346242787142,
                                -2.3866178912721807,
                                0.7430333815646009,
                                -0.1088799341887352,
                                0.002376469596924835,
                                7.800020653121464E-4
                );
                addSegmentPolynomial
                (
                                -2.389979813072614,
                                13.52847449275123,
                                -24.492748482913157,
                                22.94192603632837,
                                -13.011997020365925,
                                4.7541003534429365,
                                -1.133624246020645,
                                0.17138242630778677,
                                -0.014972771910288785,
                                5.7754667090959E-4
                );
        }
}
```

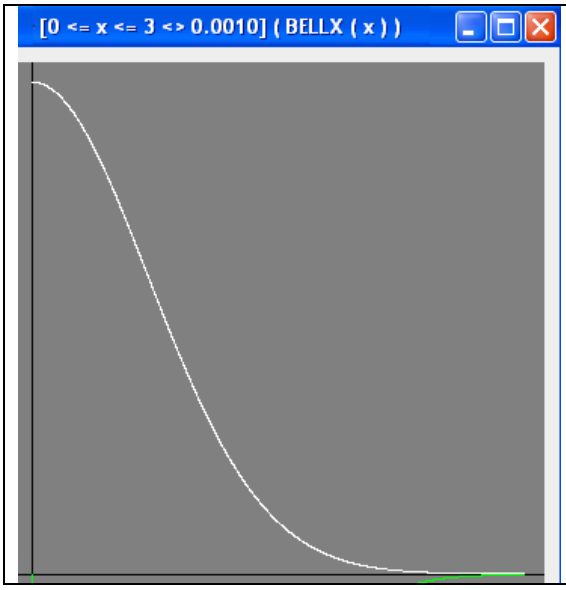
## Access To Exported Functions

// with the exported “Bell” class, an interface can be coded:

```
package a.sample.library;
import net.myorb.math.MultiDimensional;

/**
 * provide static interface to Bell function
 */
public class ExportedFunctions
{
	public static double bell (double x)
	{
		if (bell == null) bell = new Bell ().getFunction ();
		return bell.f (x);
	}
	static MultiDimensional.Function<Double> bell = null;
}
```

// this can be referenced in any Java program:

```
package a.sample.library;

/**
 * unit test for Bell class using static interface
 */
public class ExpFunTest
{
	public static void main(String[] args)
	{
		System.out.println (ExportedFunctions.bell(2));
	}
}
```




// library import can be used allowing
// CALCLIB access to the compiled version

LIBRARY XPF a.sample.library.ExportedFunctions
!+ BELLX(x) = XPF.bell

// now BELLX is available as any other function in the system

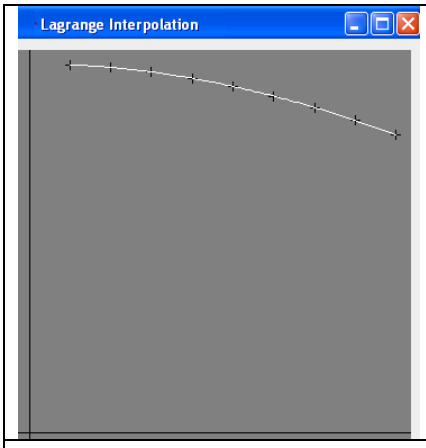
GRAPH [0 <= x <= 3 <> 0.001] ( BELLX(x) )

// the result should be faster execution,
// improved organization and portability

Library
XPF a.sample.library.ExportedFunctions
Functions
**BELLX (x)** **XPF.bell (double)**

**FloatDemoVanChe.txt**

Comparison of LaGrange polynomial interpolation against Chebyshev T function interpolation




```
// Bessel J0(x) over interval [0 < x < 1]
x = (0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9)
jx = (0.997501, 0.990024, 0.9776262,
0.960398, 0.9384, 0.912, 0.8812008, 0.84628,
0.8075237)

// LaGrange interpolation produces
x_LAGRANGE_jx = LAGRANGE(x,jx)
!! lf(x) = x_LAGRANGE_jx +*^ x
```

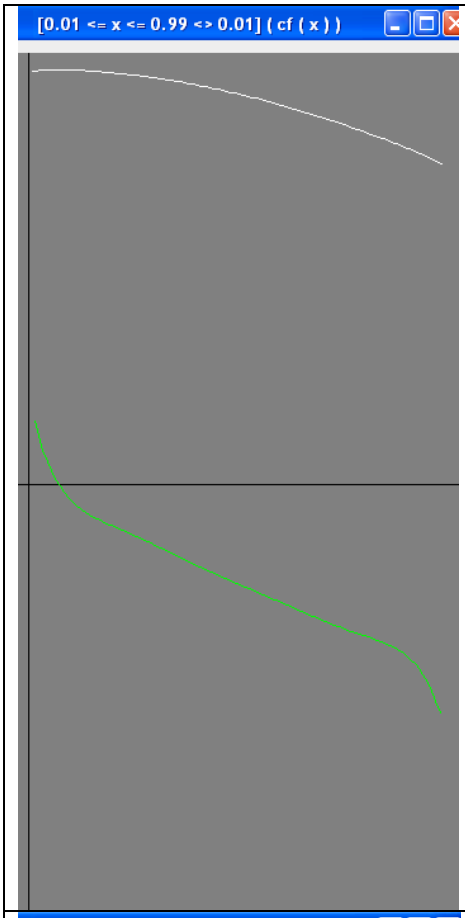
```
// Vandermonde matrix of Chebyshev polynomials
vc = VANCHE x ;  vcInv = INV vc;  jm = MATRIX(jx,9,1)
vcInv_MATMUL_jm = MATMUL(vcInv,jm)
jxc = vcInv_MATMUL_jm|#1

// producing the Chebyshev polynomial interpolation of jx
// using the Clenshaw operator allows an
// efficient function to be defined
!! cf(x) = jxc @*^ x

// look at the error function between LaGrange and Chebyshev

!! P(x) = cf(x);  !! f(x) = lf(x)
!! err(x) = abs ( P(x) - f(x) )

// evaluate the error function across the interval
erf = [0 < x < 1 <> 0.01] (err(x))

// query maximum value of error
maxErf = MAX erf;  PRETTYPRINT maxErf

// show the plot
GRAPH erf
```

**OUTPUT:**

```
Reading...
C:\Projects\CalcLib\scripts\FloatDemoVanChe.txt

Lagrange Interpolation

            Y = 0.9917303000000143 +
0.21729748928555637 * x - 2.4849073442426004 * x^2 +
11.988874097151438 * x^3 - 37.1869706594598 * x^4 +
69.12134722175688 * x^5 - 75.7669097214548 * x^6 +
45.13501984081631 * x^7 - 11.258184523720729 * x^8

r^2 = 1.0
STD = 3.4329155185658895E-11
COV = 3.717532401497196E-11
MSE = 1.178490895761051E-21
SSR = 0.03574957820309609
SSE = 8.249436270327357E-21
SST = 0.03574957822936
PC  = -0.9780255354339753

maxErf =

            1.9089778279246516E-6
```

```
jxc =

[
            -40.95140998426814
            77.09301545187009
            -60.27713885060075
            39.40757141249924
            -21.317397422270005
            9.256728076843729
            -3.071351473209347
            0.7052343269186707
            -0.08795456729584572
]
```

```
x_LAGRANGE_jx =

[
            0.9917303000000143
            0.21729748928555637
            -2.4849073442426004
            11.988874097151438
            -37.1869706594598
            69.12134722175688
            -75.7669097214548
            45.13501984081631
            -11.258184523720729
]
```

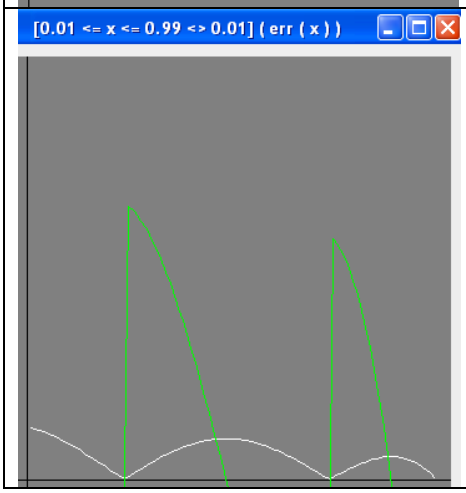



```
// Vandermonde matrix of Chebyshev polynomials
vc = VANCHE x ;  vcInv = INV vc;  jm = MATRIX(jx,9,1)
vcInv_MATMUL_jm = MATMUL(vcInv,jm)
jxc = vcInv_MATMUL_jm|#1

// producing the Chebyshev polynomial interpolation of jx
// using the Clenshaw operator allows an efficient function to be defined
!! cf(x) = jxc @*^ x
```

Functions

| | |
|---|---|
| P(x) | cf ( x ) |
| cf(x) | jxc @*^ x |
| err(x) | abs ( P ( x ) - f ( x ) ) |
| f(x) | lf ( x ) |
| lf(x) | x_LAGRANGE_jx +*^ x |

5 symbols found and displayed

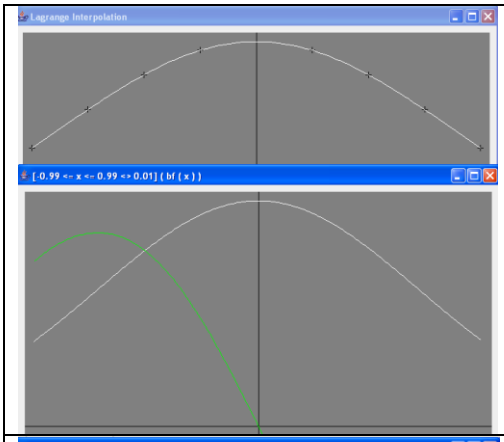



```
// look at the error function between LaGrange and Chebyshev

!! P(x) = cf(x);  !! f(x) = lf(x)
!! err(x) = abs ( P(x) - f(x) )

// evaluate the error function across the interval
erf = [0 < x < 1 <> 0.01] (err(x))

// query maximum value of error
maxErf = MAX erf;  PRETTYPRINT maxErf

// show the plot
GRAPH erf
```

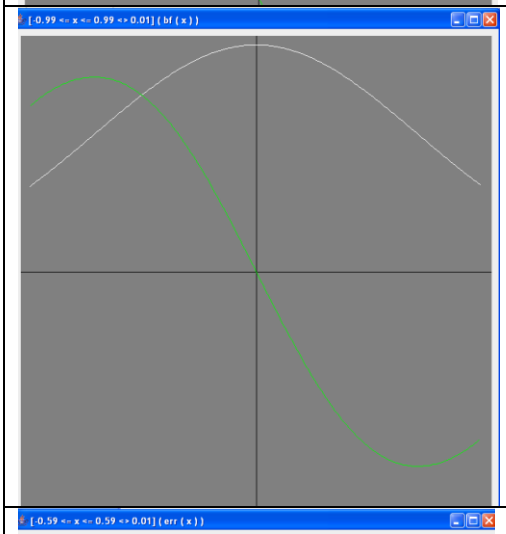
---

```
maxErf =

            1.9089778279246516E-6
```

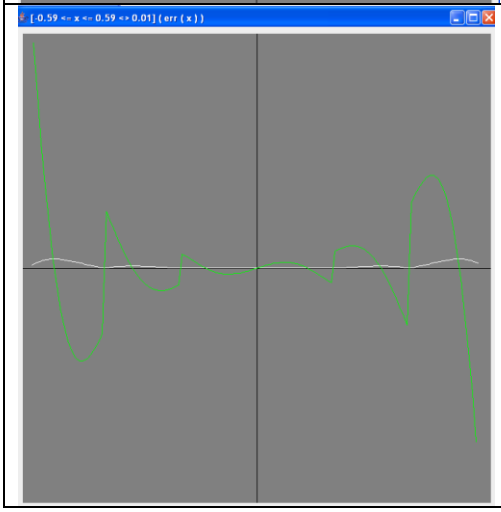
## Chebyshev interpolation applied to Bell curve




```
xaxis = [-1 < x < 1 <> 0.2] (x)
bell = [-1 < x < 1 <> 0.2] ( exp ( - x ^ 2 ) )
graph bell

xaxis_LAGRANGE_bell = LAGRANGE (xaxis, bell)


vc = VANCHE (xaxis)
vcInv = INV vc
```




```
bm = MATRIX (bell, 9, 1)
mm = MATMUL (vcInv, bm)
bcpc = mm|#1

!! bf(x) = bcpc @*^ x

graph [-1 < x < 1 <> 0.01] ( bf (x) )
```




```
!! P(x) = bcpc @*^ x   ;  !! f(x) = exp (-x^2)
!! err(x) = abs ( P(x) - f(x) )

erf = [-1 < x < 1 <> 0.01] ( err (x) )

maxErf = MAX erf

PRETTYPRINT maxErf

maxErf =

                1.5179754979666171E-6


GRAPH erf
```

## The VANCHE Concept Extended

The Chebyshev polynomial are constrained to the interval [-1, 1] in both domain and range. Interpolations for functions that are difficult (expensive) to compute are done making function evaluation and calculus on these functions easier and less (time) expensive. For these purposes a quick tool for the interpolation process is useful. Given the Chebyshev polynomial constraints and the need to calculate points to build the interpolation, the process can be made a simple set of constants. Taking the constraint of -1<=x<=1 the function can be evaluated at equidistant points of 0.1 making a constant Vandermonde matrix of Chebyshev T[n] polynomials calculated at values of [-1 <= x <= 1 <> 0.1]:

```
X = [-1 <= xi <= 1 <> 0.1] (xi)

VC21 = VANCHE X
```

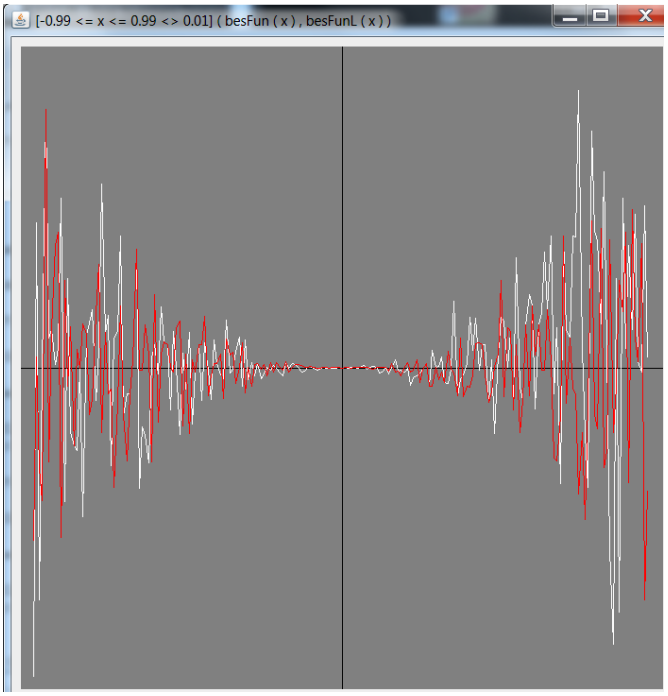
This gives the constant Vandermonde Chebyshev matrix for T[n] over -1 <= x <= 1 with delta 0.1 Using the Colt library to perform decomposition takes it to the next step:

```
READ   ColtLib.txt
CALC   MatRpt   VC21
VC21L  =  GetLudL()
VC21U  =  GetLudU()
```

VC21L and VC21U are constants that can be used with any sequence of 21 equally spaced function evaluation points to produce a Chebyshev polynomial interpolation; Coordinate translation for the [-1,1] interval must be done and the coefficient solution becomes a simple application of LUx = b but the heavy lifting is embedded in the constants.

As a simple test of this concept, the Colt library J0 function was used to capture data on the interval [-1, 1] with 0.1 as increments. The VC21 constant matrix was used to interpolate the function as a Chebyshev polynomial. This function was introduced to the system as “j0lcl”. The Bessel function was used to evaluate J0 and j0lcl since a correct J0 function would be constant 0 return from the Bessel function if J0 provides proper function values. The Bessel function was plotted over the [-1, 1] interval and the error values were collected. The MAX value of the collected errors was calculated. The script and the plotted and printed results are on the following page.

## Extended Concept Test




Calculations of Bessel equation for J0 imported from Colt library versus local interpolation of J0

WHITE
J0 as calculated
Using Colt library

RED
Local Interpolation
Chebyshev Polynomial
Using Vandermonde matrix
And LU decomposition

(outer-most red peaks should be considered likely Runge phenomenon)
(White is plotted first)

```
READ BesselFunctions.txt
jVals = [-1<=x<=1<>0.1](J0(x))
mj = MATRIX(jVals,21,1)

j0coef = SolveWithLUD(VC21,mj)
j0c = j0coef|#1

!! j0lcl(x) = j0c @*^ x

// // // //

alpha=0
dx = 0.00001

!! besFun (x) = x^2 * J0''(x <> dx) + x * J0'(x <> dx) + (x^2 - alpha^2)* J0(x)
GRAPH [-1<x<1<>0.01](besFun(x)) ;  err = [-1 < x < 1 <> 0.01] (abs(besFun(x)))
CALC MAX err ;    // 2.455309003113193E-6

!! besFunL (x) = x^2 * j0lcl''(x <> dx) + x * j0lcl'(x <> dx) + (x^2 - alpha^2)* j0lcl(x)
GRAPH [-1<x<1<>0.01](besFunL(x)) ;   errL = [-1 < x < 1 <> 0.01] (abs(besFunL(x)))
CALC MAX errL  ;   // 2.0533536184697E-6

GRAPH [-1<x<1<>0.01](besFun(x), besFunL(x));  // ( actual graph displayed above )
```

Max error of the interpolation is smaller than that of the original function, interesting little mystery

# VC21 decomposed as L and U matrices

Upper Triangular Matrix

PRETTYPRINT VC21L 5

VC21L =

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.5 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.8 | 0.64 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.15 | 0.51 | -0.92969 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.3 | 0.84 | -0.875 | 0.72398 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.95 | 0.19 | 0.44531 | 0.11054 | 0.81429 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.05 | 0.19 | -0.44531 | 0.55268 | -0.50893 | 0.24038 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.65 | 0.91 | 0.71094 | -0.17647 | -0.8125 | -0.53728 | -0.74503 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.9 | 0.36 | 0.75 | 0.12411 | 0.85714 | 0.97166 | 0.22456 | -0.427 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.4 | 0.96 | -0.5 | 0.33096 | 0.7619 | 0.14395 | 0.36595 | -0.28653 | -0.67102 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.1 | 0.36 | -0.75 | 0.86878 | -0.4 | 0.15115 | 0.59384 | -0.06642 | -0.34222 | 0.816 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.75 | 0.75 | 0.97656 | -0.0808 | -0.44643 | -0.37955 | -0.35088 | 0.54945 | -0.51471 | 0.23011 | -0.329 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.2 | 0.64 | -1 | 0.99289 | 0.45714 | -0.08637 | -0.29942 | 0.10047 | 0.42353 | -0.88364 | -0.72193 | 0.33758 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.85 | 0.51 | 0.92969 | 0.07692 | 0.49583 | 0.51524 | 0.23815 | -0.42621 | 0.79852 | -0.119 | 0.21875 | -0.76718 | 0.41319 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.55 | 0.99 | 0.25781 | -0.10666 | -0.39286 | -0.18556 | -0.34308 | 0.38374 | 0.35948 | -0.375 | 0.22978 | -0.48352 | -0.52083 | -0.67873 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.25 | 0.75 | -0.97656 | 0.88882 | 0.81845 | -0.07732 | -0.25016 | 0.11193 | 0.41939 | -0.8125 | -0.49786 | 0.34921 | 0.94039 | 0.17507 | -0.51587 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.7 | 0.84 | 0.875 | -0.1448 | -0.73333 | -0.55421 | -0.64042 | 0.93122 | -0.43617 | 0.26 | -0.31863 | 0.89397 | 0.24074 | 0.44818 | -0.33016 | -0.32 | 1 | 0 | 0 | 0 | 0 |
| 1 | 0.35 | 0.91 | -0.71094 | 0.52941 | 0.975 | 0.09211 | 0.25544 | -0.17143 | -0.48176 | 0.78975 | 0.16131 | -0.18857 | -0.40625 | -0.22689 | 0.55714 | -0.72 | 0.5 | 1 | 0 | 0 | 0 |
| 1 | 0.6 | 0.96 | 0.5 | -0.16548 | -0.68571 | -0.38866 | -0.62877 | 0.77363 | 0.36235 | -0.324 | 0.26471 | -0.6189 | -0.5 | -0.74467 | 0.91429 | 0.29538 | -0.71795 | -0.41026 | 1 | 0 | 0 |
| 1 | 0.45 | 0.99 | -0.25781 | 0.14932 | 0.4125 | 0.1169 | 0.27018 | -0.24176 | -0.45294 | 0.6075 | -0.12408 | 0.20308 | 0.32812 | 0.30543 | -0.6 | 0.38769 | -0.53846 | -0.76923 | 0.75 | 1 | 0 |

Pivot Table

VC31P =

```
[
    0,  30,  15,  24,   5,  10,  28,   2,  20,  29,   1,  12,  26,   7,  18,   3,  22,
   27,   4,  14,  25,   8,  17,   6,  23,  11,  21,   9,  19,  13,  16
]
```

Lower Triangular Matrix

PRETTYPRINT VC21U 5

VC21U =

```
    1    -1     1    -1     1    -1     1    -1     1    -1     1    -1     1    -1     1    -1     1    -1     1    -1     1
    0     2    -0     2    -0     2    -0     2    -0     2     0     2     0     2    -0     2    -0     2     0     2    -0
    0     0    -2     0     0     0    -2     0     0     0    -2    -0    -0     0    -2     0     0    -0    -2    -0     0
    0     0     0    -1.536  -1.8432  -0.67584  1.03219  0.37847  -0.57803  -1.0721  -0.7085  -1.31409  -0.86841   0.272   1.19481  -0.37422  -1.64388  -1.59843  -0.27424  -0.26665  -0.04575
    0     0     0     0    -3.7128   0.74256  1.03958  0.29702  -0.54058  -0.93741  -0.73852   0.25877  -2.80016   1.56234   1.27035  -0.45275  -1.54107  -1.36331   -0.414   1.28003   -2.0226
    0     0     0     0     0    -1.6128   1.6128   0.77414  -2.10309   0.67351   1.15709  -1.91998   0.4913   0.31058  -0.0611   0.29013  -0.36767  -0.66541   0.3078   0.23934   0.05676
    0     0     0     0     0     0    -3.41453  -2.73162   0.13658  -1.47508  -1.36581   1.46633  -0.05966  -0.53356  -0.24321  -0.03359   0.79701   0.68978  -0.96418  -1.77365  -1.72168
    0     0     0     0     0     0     0     2.95488  -2.95488   2.24571   0.18911  -1.40416   1.06943  -0.46919   0.23684   0.25334  -0.66379   0.25177  -0.38632   1.27799  -0.37917
    0     0     0     0     0     0     0     0    -5.28353   2.11341   2.5361   -0.9299   -0.5833   -1.3323   0.62287   1.28252  -0.30635  -0.27942  -0.31149   0.78329  -1.50367
    0     0     0     0     0     0     0     0     0     2.25608   2.70729   0.99267  -0.72194  -1.89871  -1.74711  -1.16262  -0.66072   0.23627   1.15807   1.74965   1.3069
    0     0     0     0     0     0     0     0     0     0     3.02776   2.42221  -2.66443  -2.32532   1.04639  -0.43794  -2.43151   0.52382   2.53907   0.01474  -0.19686
    0     0     0     0     0     0     0     0     0     0     0    -2.96479   2.37183   1.77887  -2.94107   1.90221   0.32636  -0.04345  -0.76976  -1.21119   2.69323
    0     0     0     0     0     0     0     0     0     0     0     0     2.5361   0.50722  -3.55053  -1.54195   0.38143   1.6442   1.29994   0.3142   0.05032
    0     0     0     0     0     0     0     0     0     0     0     0    -1.88351  1.88351   2.26021  -3.45059   1.59722   0.31281  -2.93804   2.97948
    0     0     0     0     0     0     0     0     0     0     0     0     0    -2.02347  -0.80939   3.31848   1.74827   0.845   -0.22922  -4.32225
    0     0     0     0     0     0     0     0     0     0     0     0     0     0    -1.64807  -0.98884   4.15314   3.24341  -2.81623  -3.99334
    0     0     0     0     0     0     0     0     0     0     0     0     0     0     0    -1.02024   0.40809   3.18313  -1.5834   -3.34311
    0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0    0.58766   0.23506  -2.23309  -1.10949
    0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0    -0.41136   0.08227   1.92516
    0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0    0.16876   0.03375
    0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0    -0.07594
```

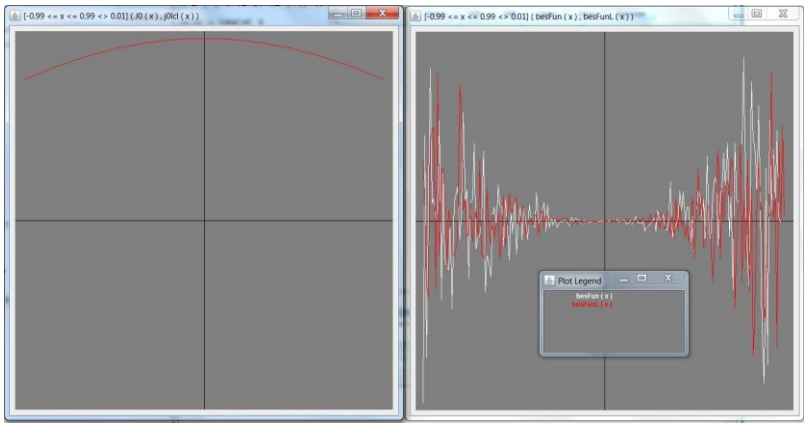
## VC31 Spline Organization

```
                        start                                                 low                 knot                  high
  0   1   2   3   4     5   6   7   8   9   10  11  12  13  14  15   16  17  18  19  20  21  22  23  24   25    26  27  28  29  30
  .   .   .   .   .     .   .   .   .   .   .   .   .   .   .   .   .   .   .   .   .   .   .   .   .    .    .   .   .   .   .
-1.5 .4  .3  .2  .1   -1   .9  .8  .7  .6  .5  .4  .3  .2  .1   0   .1  .2  .3  .4  .5  .6  .7  .8  .9    1    .1  .2  .3  .4  1.5
 [  low end knot  ]       [ standard Chebyshev constraints specify interval of [-1, 1] for best behavior ]
 [    not used    ]
 [Runge phenomenon]

30   31   32  33   34   35   36   37   38   39   40  41  42  43  44  45   46  47  48  49  50
 .   .    .    .    .    .    .    .    .    .    .   .   .   .   .   .    .   .   .   .   .
1.5  .6  .7   .8   .9   2    .1   .2   .3   .4   .5  .6  .7  .8  .9   3   .1  .2  .3  .4  .5
                                                 low              knot              high
```

Note positions of knots and overlap points of segments around knots

## In Search Of: The Perfect Spline

Restricted to the [-1, 1] interval the knots of a spline made of these polynomials would force the useful portion of the interval to perhaps ½ leaving the other ½ for overlay in the knots. This opens the question of "what happens if we drift a bit past the [-1, 1] interval to perhaps [-1.5, 1.5]?




These are plots of the same J0 experiment as before on the interval [-1.5, 1.5] using 31 points of 0.1 increments building VC31, VC31U, and VC31L as was done with VC21. The left side plot is J0 in white plotted first and then j0lcl in red plotted after; note on left that red completely over writes the white plot indicating that the difference is smaller than the plot precision.

The error analysis documents:

errL = [-1.2 < x < 1.2 <> 0.01] (abs(besFunL(x))); CALC MAX errL
3.3008504037868036E-6

err = [-1.2 < x < 1.2 <> 0.01] (abs(besFun(x))); CALC MAX err
4.9065843416018495E-6

Showing that the error evaluation as we step beyond [-1, 1], while larger, is less than the increase in the original function. On this point, interesting to note:

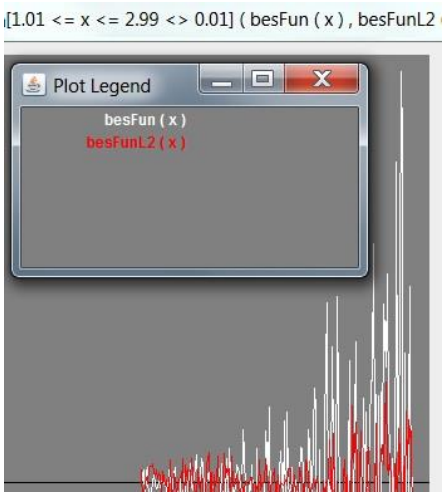
errL = [-1 < x < 1 <> 0.01] (abs(besFunL(x))); CALC MAX errL
1.994155862661273E-6

err = [-1 < x < 1 <> 0.01] (abs(besFun(x))); CALC MAX err
2.455309003113193E-6

The gap between errors on interval [-1, 1] widened using VC31, while the original J0 plot retains a fairly constant error, the interpolation error drops…

Conclusions:
This method may produce useful spline segments in 2 unit increments. Three units of equally spaced points should be sampled into 31 points. The VC31 triangular matrices can be used to solve the LUx = b equation for the X coefficients of the Chebyshev polynomial. This polynomial should perform very well on the [-1, 1] interval with the real X coordinates of the segment translated on to that interval. The lower 5 and upper 5 points of the interpolation are being discarded as being intended overlays of the knot between the segments

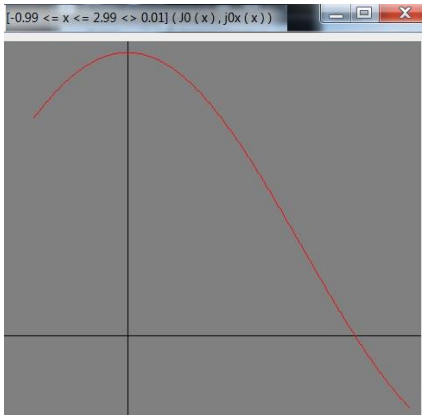
## Building A Spline



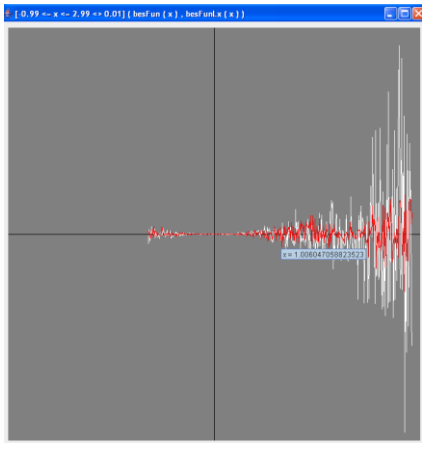

The script for the second segment:

```
j0xx = [0.5<=xx<=3.6<>0.1](J0(xx))
mjxx = MATRIX(j0xx,31,1)

j0xxcoef = SolveWithLUD(VC31,mjxx)
j0xxc = j0xxcoef|#1

!! j0xxlcl(x) = j0xxc @*^ x
!! j0lcl2(x) = j0xxlcl(x-2)

calc j0xxlcl(0) ;    //  0.22389078190857192
calc j0lcl2(2);      //  0.22389078190857192
calc J0(2);          //  0.2238907819085722

!! besFunL2 (x) = x^2 * j0lcl2"(x <> dx) + x * j0lcl2'(x <> dx) + (x^2 - alpha^2)* j0lcl2(x)
graph [1<x<3<>0.01] (besFun (x), besFunL2 (x))
```

Looks like RED wins again




```
err =  [1<x<3<>0.01](besFun(x))
errL =  [1<x<3<>0.01](besFunL2(x))

calc MAX err;     // 6.013422098871679E-5
calc MAX errL;  //  1.464845459953068E-5  //  so far, so good

// the function implementing the knot
!! j0x(x) = (x>1)? j0lcl2(x) : j0lcl(x)

// a graph of the overlay
graph [-1 < x < 3 <> 0.01] (J0(x),  j0x(x))
```

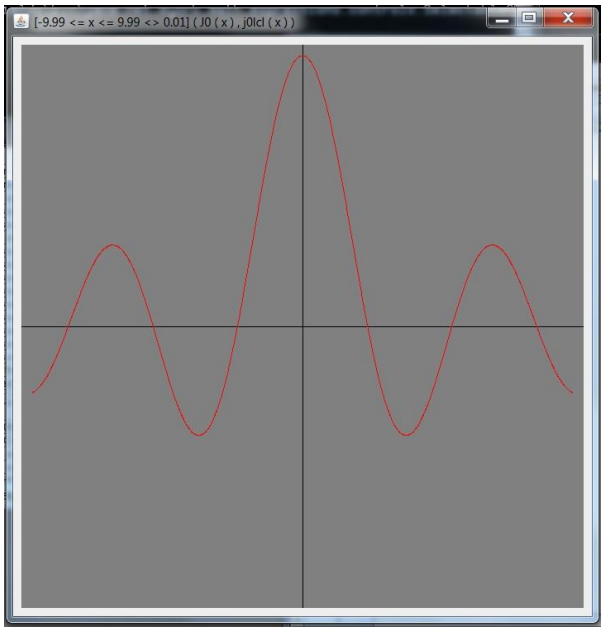


```
// the Bessel equation for testing the spline
!! besFunLx (x) = x^2 * j0x"(x <> dx) + x * j0x'(x <> dx) + (x^2 - alpha^2)* j0x(x)

// a graph to compare the error as before
graph [-1 < x < 3 <> 0.01] ( besFun (x), besFunLx (x) )

// RED Chebyshev polynomial interpolation RULES !!!
// note the lack of error spike at the knot (x = 1)

err  =  [-1 < x < 3 <> 0.01](abs (besFun (x)))
errL =  [-1 < x < 3 <> 0.01](abs (besFunLx (x)))

calc MAX err;   //  5.182550320093071E-5
calc MAX errL;  //  1.4990911813006491E-5
```

## One Additional Feature (an obvious notion)

The CalcLib tools offer function definition commands. The ones that export VC31 spline functions publish function definitions from matrix objects requesting a name for the function and a low knot value.

Optionally a type of function can be selected (Odd, Even) which forces the assumption that the low knot value is 0 and the input parameter is absolute function (abs(x))




```
READ VC31Prep.txt

READ BesselFunctions.txt

// the knot is assumed to be 0
// the -0.5 knot overlap must be present
jVals  = [-0.5<=x<=15<>0.1](J0(x))

// the interpolation remains the same
j0co =   VC31   jVals

// function exported as EVEN type
// the definition shows the abs(x) & 0 knot
!! j0lcl(x) = EVALSPLINE (j0co, 0, abs(x))

// the graph shows the y-axis mirror
graph [-10 < x < 10 <> 0.01] (J0(x),  j0lcl(x))
```

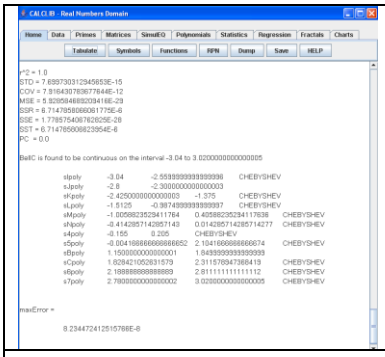
Alternatively:

Odd and Even functions can use the GENKNOT function to produce the knot value to be pre-pended to a data sequence.
The VC31 interpolation is based on data sequences having a 0.1 increment between x-axis points, but scaling can be used.

```
// function evaluation that starts at 0
// the GENKNOT function reflects low 5 values
jVals  = [0.0 <= x <= 12 <> 0.1] (J0(x))
knot  =  GENKNOT  jVals

// append the knot values to the front of the sequence
// the interpolation now works just as it did in the previous example
jv  =  APPEND (knot,  jVals) ;    j0co  =  VC31   jv

!! j0lcl (x)  =  EVALSPLINE (j0co, 0, abs(x))
graph   [-10 < x < 10 <> 0.01]  ( J0(x),   j0lcl(x) )
```

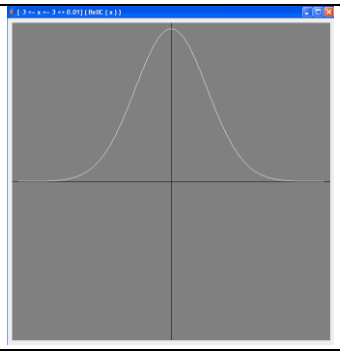
```
// x-axis scaling allows more of the curve per segment
// increment of 0.2 requires ½ as many samples
jVals2  = [0.0 <= x <= 12 <> 0.2] ( J0(x) )

// combined sub-expressions reduce the spline generation to:
j0co2 = VC31 ( APPEND (GENKNOT  jVals2,  jVals2)  )

// evaluation done on x/2 scales the function back
!! j0x(x) = EVALSPLINE (j0co2, 0, abs(x/2))

graph [-10 < x < 10 <> 0.1] (J0(x),  j0x(x))
```

## Mix and Match using High-Order Polynomials



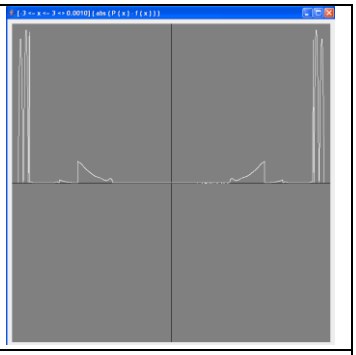
### Interpolation

Segments are chosen and the interpolation produces a polynomial for the segment. Refinements are made reducing the maximum error evaluation. The final max-error on this attempt was 8.2E-08 with the error spikes remaining at the extreme ends of the x-axis. The middle of the range show errors in the 1E-11 up to 1E-09



### The Bell Curve

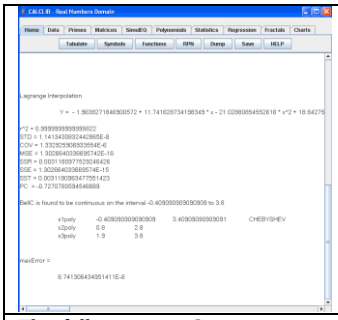
The plot of the produced function is very smooth and appears the well known shape of the Bell curve. No obvious errors are seen and even the ends appear to approach zero but remain just far enough above that the x-axis can still be seen under the plot of the curve



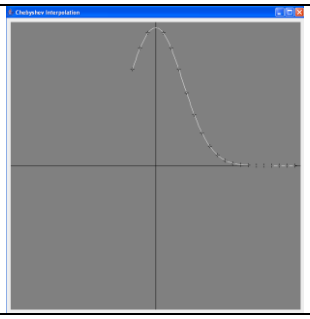
### The Error Plot

In the range -1.5 to 1.5 on the x-axis no error registers at all. The peak at 1.81 show an error of 1E-08 which is 1/8 the size of the highest peak. The end of curve error peaks start at 2.7 and reach values of 8.2E-08. the value of the function is 6.8E-04, so even at worst the error is 1 part in 8000 while at best we get 1 in 100,000,000

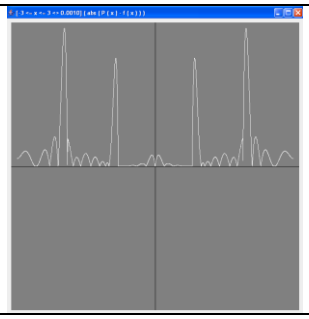
The building blocks for this test can be seen in scripts/FloatDemoRungeSegmented.txt where a test is built using Chebyshev polynomial interpolation. This produced results with errors in the 1E-05 range. The errors anticipated by Carl Runge were being seen as well as problems for |x| > 1. An additional series of tests was built found in scripts/FloatDemoGRungeSegmented.txt where other types of interpolation where used to generate functions with smaller errors in the areas found to be problematic. This result documented here shows errors in the 1E-11 to 8E-08 range. Another set of tests is attempting to get the error down to the 1E-15 optimal range.



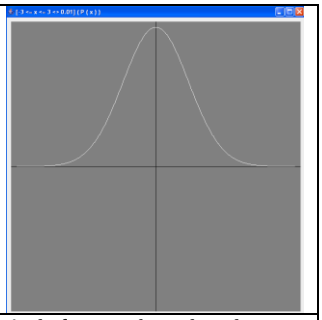
This follow-up uses 3 segments, first a high order Chebyshev Interpolation followed by smaller Lagrange polynomials



The Chebyshev Interpolation tracks very well in the 0-1 domain section, but changing the interpolation for 1-2 and 2-3 segments produces far better results on the error plot



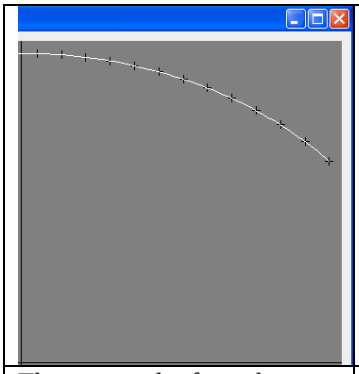
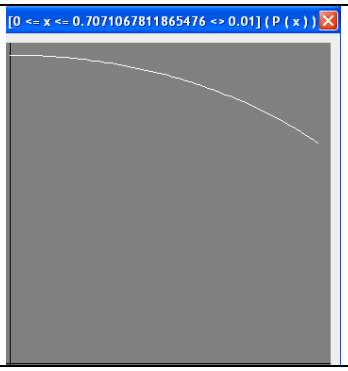
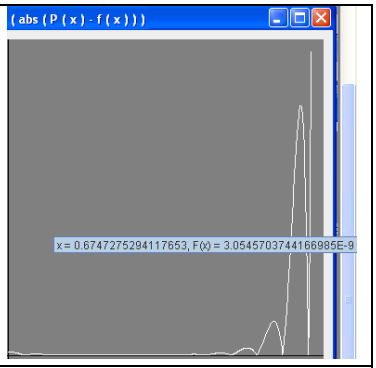
The error curve shows peaks at the knots (stitch points), but outside these the error is far smaller. The highest peak is 6.7E-08, other than the stitch points the peaks are in 1E-10 to 1E-09, at 0 the error is 1E-16



And of course the end product curve appears to have exceptionally smooth continuous shape

## ChebCircle8th.txt

Clenshaw Quadrature is based on Chebyshev Polynomial Interpolation, so with a successful interpolation in place a numerical integration can easily be done using the generated polynomial. The function used here is sqrt(1-x^2) plotted from 0 to sqrt(2)/2 which is 1/8 of the circle. To make it a slice there is a shared piece of the area that calculates as 0.25, this is called excess in the variables in the script

| | | |
|---|---|---|
|  |   |   |
| The scatter plot from the Chebyshev interpolation of the data | A plot of the Chebyshev polynomial generated | The error plot showing 4E-09 as the maximum value of the error over the interval |

```
// equation for a unit circle
!! f(x) = 2\ (1 - x^2)

// 45DEG is 1/8 of circle, sin=cos=sqrt(2)/2
r2o2 = 2\2 / 2

// area of the shared portion of the slice 0 - pi/4
excessArea = r2o2 ^ 2 / 2

// map out the slice to be interpolated
c00  = [-0.20 <= x <= 0.8 <> 0.06] ( f (x) )
c00poly = CHEBINTERP c00


// the interpolated Chebyshev polynomial
!! P(x) = c00poly @*^ x

// the error calculated at 1000 points
err = [0 <= x <= r2o2 <> 0.001] (abs (P(x) - f(x)))
maxError = MAX err

// maximum error found
PRETTYPRINT maxError
```

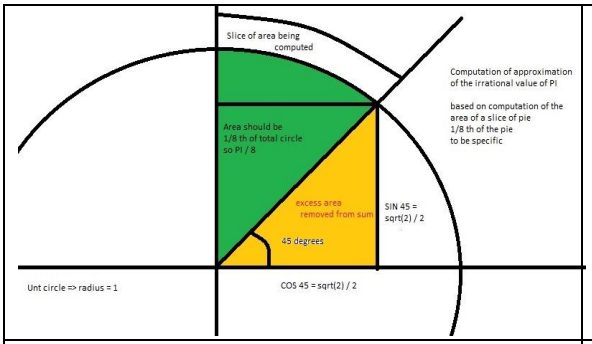
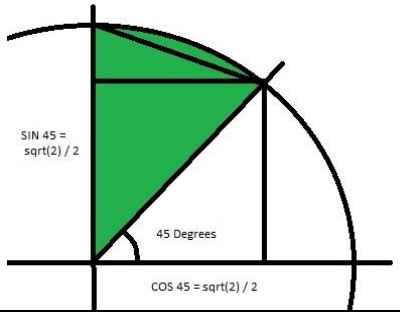
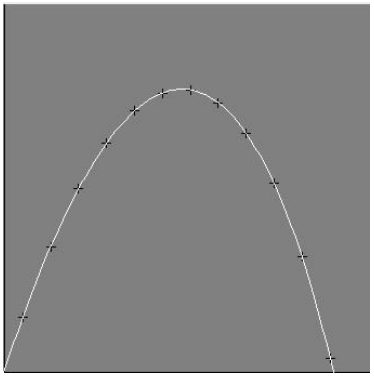
```
// use Clenshaw quadrature to formulate
// the anti-derivative and calculate the area

area = CLENQUAD (c00poly, r2o2) ;
area0 = CLENQUAD (c00poly, 0)

calculatedArea = area - (area0 + excessArea)

PRETTYPRINT calculatedArea

// this should be pi / 8 since area of full unit circle = pi r^2 = pi
approximationOfPi = 8 * calculatedArea
PRETTYPRINT approximationOfPi

// plots of interpolation and error curve
GRAPH [0 <= x <= r2o2 <> 0.01] (P(x))
GRAPH err
```

| Symbol | Notation | |
|---|---|---|
| approximationOfPi | | 3.1415926542866544 |
| area | | 4.453178685699542 |
| area0 | | 3.81047960391371 |
| c00 | | (0.9797958971132712, 0.990 |
| c00poly | | (-8.512669289364831, 17.69 |
| calculatedArea | | 0.3926990817858318 |
| err | | (4.5718095975644246E-11, 4 |
| excessArea | | 0.25000000000000006 |
| maxError | | 4.073859471986907E-9 |
| r2o2 | | 0.7071067811865476 |

Note the approximation of PI is correct to 9 decimal places with maxError printing at 4E-09

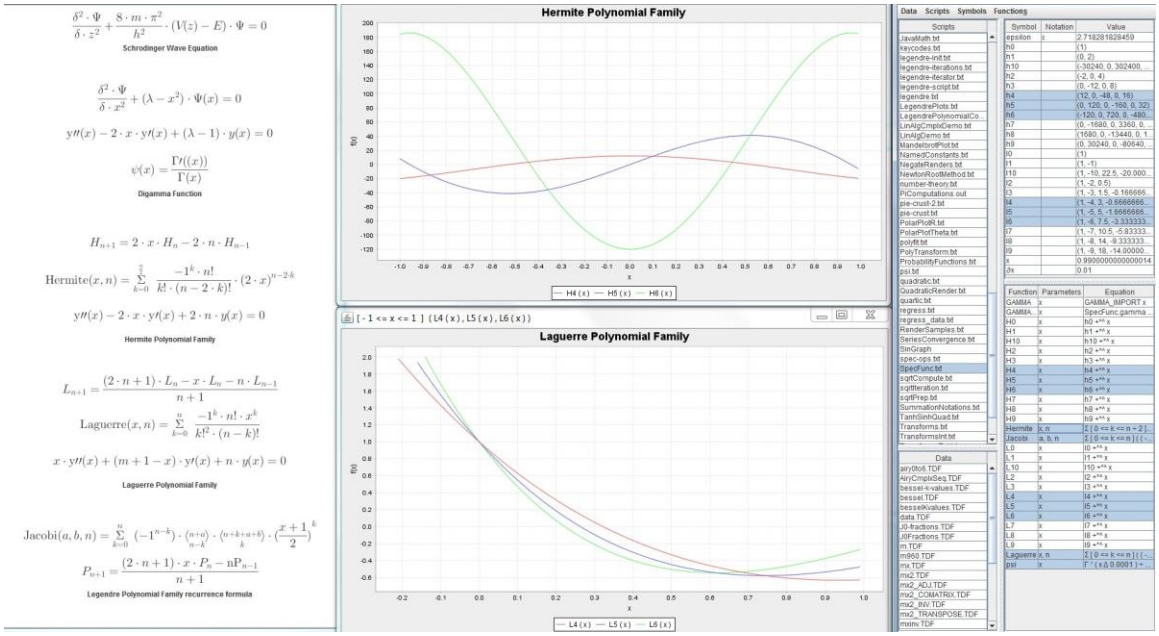
The integral was actually computed using the anti-derivative of the Chebyshev polynomial, but of course the polynomial was an interpolation of data (with a nice small max error)

# Visualization of the realized solution

<table>
<tr>
<td> 
</td>
<td> 
</td>
</tr>
<tr>
<td>The integral 0 – cos (pi/4) of the full circle equation gives the area of the slice but includes the full area of the square. Subtracting half the square area leaves just the slice</td>
<td>An alternative is to break the area down and compute areas of sections using simple geometry and this leaves a smaller integral<br><br>CrustLineSlope = - (1 – 2\2 / 2) / ( 2\2 / 2) = 1 – sqrt(2)</td>
</tr>
<tr>
<td>  </td>
<td>// equation for a unit circle<br>!! f(x) = sqrt (1 - x^2)<br><br>// 45DEG is 1/8 of circle, sin=cos=sqrt(2)/2<br>r2 = sqrt(2) ; r2o2 = r2 / 2<br><br>crustLineSlope = 1 – r2<br>!! crustLine(x) = 1 + crustLineSlope * x<br>!! crust (x) = f(x) - crustLine(x)<br><br>c00 = [-0.20 <= x <= 0.8 <> 0.06] ( crust (x) )<br>c00poly = CHEBINTERP c00<br>!! P(x) = c00poly @*^ x<br><br>areaCrust= CLENQUAD (c00poly, 0, r2o2)</td>
</tr>
<tr>
<td>PRETTYPRINT areaCrust<br>areaCrust = 0.03914569119254674<br>CombinedAreaOfTriangles = r2 / 4<br>sliceArea = areaCrust + CombinedAreaOfTriangles</td>
<td>// LowTriArea = (r2 /2) ^ 2 / 2<br>// HiTriArea = [ (1 – r2) * r2 / 2 ] / 2<br>// CombinedAreaOfTriangles = (r2 – 1) / 4 + 1 / 4 = r2 / 4</td>
</tr>
<tr>
<td>PRETTYPRINT sliceArea<br>sliceArea = 0.3926990817858205</td>
<td>piApprox = sliceArea*8<br>piApprox = 3.14159265<br>// So the approximations are nearly identical</td>
</tr>
<tr>
<td>Compare the interpolated area approximation</td>
<td>To use of Tanh-Sinh quadrature built-in as $| function suffix</td>
</tr>
<tr>
<td>// try built-in TanhSinh quadrature<br>tsq = f $| (0, r2o2, 1E-8) // lo, hi, dx<br><br>// this produces 5 additional digits of precision<br>calc 8 * (tsq - excessArea)<br>3.1415926535897</td>
<td>// using TanhSinh on the crust equation<br>crustTsq = crust $| (0,r2o2,1E-8) // lo, hi, dx<br><br>// this produces 2 more digits than on the left<br>calc 8 * (crustTsq + 2\2 /4)<br>3.141592653589793</td>
</tr>
</table>

# Special Functions for Engineering Applications

Polynomial Families of functions can be introduced in configuration. The FAMILY command can then be used to post the polynomial coefficient arrays and the functions into the symbol table




The SpecFunc script demonstrates this functionality

```
// Schrodinger

render delta^2 * PSI / (delta*z^2) + 8*m*pi^2/h^2 * (V(z) - E) * PSI = 0

RENDER "Schrodinger Wave Equation" TOP

render delta^2 * PSI / (delta*x^2) + (lambda - x^2) * PSI(x) = 0

render y''(x) - 2*x*y'(x) + (lambda-1)*y(x) = 0

LIBRARY SpecFunc net.myorb.math.specialfunctions.Library

// GAMMA spline

!+ GAMMA_IMPORT(x) = SpecFunc.gamma

!! GAMMA(x) = GAMMA_IMPORT x

!! psi(x) = GAMMA'(x <> 0.0001)/GAMMA(x)

RENDERF psi

RENDER "Digamma Function" TOP

// Hermite Polynomials

!! Hermite(x,n) = SIGMA [ 0 <= k <= n/2 ] ( (-1)^k * n! /
                    ( k! * (n - 2*k)! ) * (2*x)^(n - 2*k) )

RENDER H#(n+1) = 2*x*H#n - 2*n*H#(n-1)

RENDERF Hermite

render y''(x) - 2*x*y'(x) + 2*n*y(x) = 0

FAMILY  Hermite 10

GRAPH  [  -1 <=    x    <= 1  <>  0.01  ]   ( H4(x), H5(x), H6(x) )

RENDER "Hermite Polynomial Family" TOP
```

```
// Laguerre Polynomials

!! Laguerre(x,n) = SIGMA [ 0 <= k <= n ] ( (-1)^k * n! * x^k / ( (k!)^2 * (n-k)! ) )

RENDER L#(n+1) = ( (2*n+1)*L#n - x*L#n - n*L#(n-1) ) / (n+1)

RENDERF Laguerre

render x*y''(x) + (m + 1 - x)*y'(x) + n*y(x) = 0

FAMILY  Laguerre 10

GRAPH  [  -1 <=    x    <= 1  <>  0.01  ]   ( L4(x), L5(x), L6(x) )

RENDER "Laguerre Polynomial Family" TOP

// Jacobi Polynomials

!! Jacobi(a,b,n) = SIGMA [ 0 <= k <= n ] ( (-1)^(n-k) *
                    ((n+a)##(n-k)) * ((n+k+a+b)##k) * ((x+1)/2)^k )

RENDERF Jacobi

// Legendre Polynomials

render P#(n+1) = ( (2*n+1)*x*P#n - nP#(n-1) ) / (n+1)

RENDER "Legendre Polynomial Family recurrence formula" TOP
```

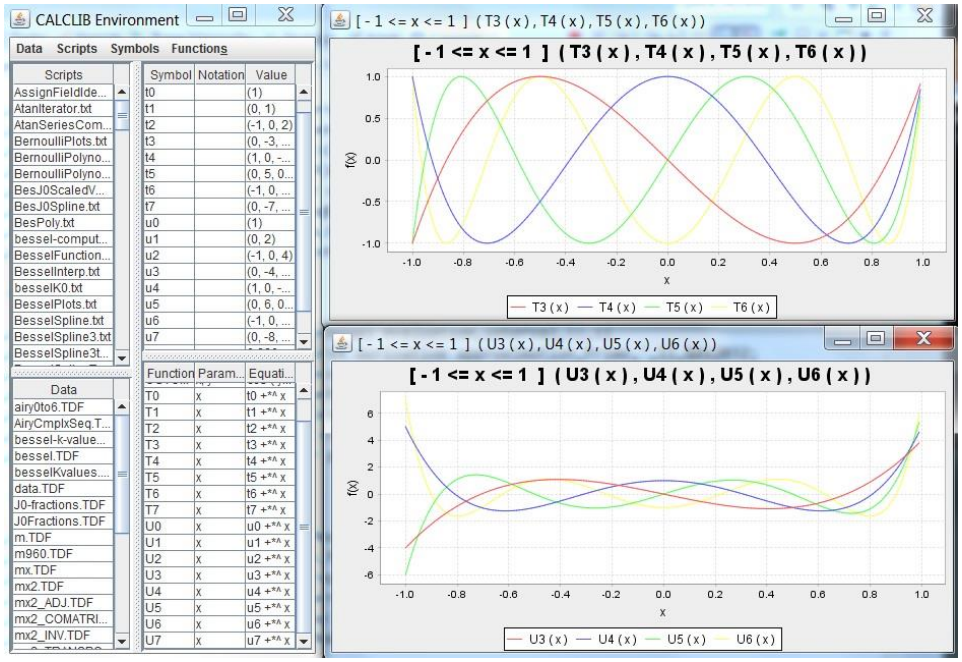
The interface for a class that introduces a polynomial family is net.myorb.math.PolynomialFamily

```java

package net.myorb.math;

import java.util.List;

/**
 * properties of a family of polynomials
 * @param <T> the data type to operate on
 * @author Michael Druckman
 */
public interface PolynomialFamily<T>
{

    /**
     * @return the name of the family
     */
    String getName ();

    /**
     * @param identifier the identifier for the kind of functions
     * @param upTo the highest order requested
     * @return the list of power functions
     */
    List<Polynomial.PowerFunction<T>> getFunctions (String identifier, int upTo);

    /**
     * @param kind a name for the kind (typically first &amp;amp; second, null if only one)
     * @return the letter which typically identifies the functions
     */
    String getIdentifier (String kind);

    /**
     * @param manager data type manager
     */
    void init (SpaceManager<T> manager);

}

```

Typically such a class would contain a recurrence formula coded to produce the function of order 0..n which would be called by the getFunctions method passing parameter upTo as the value for n.

The Chebyshev Polynomial class implements the PolynomialFamily interface. This allows the recurrence formula to be invoked from the FAMILY command and a number of order of the polynomials can be imported into the execution environment. Note that First kind ("T") and Second kind ("U") polynomials can be individually or both imported by use of the KIND parameter on the FAMILY command. Note also that the polynomial coefficient arrays (t0..t7, u0..u7) and the polynomial functions (T0..T7, U0..U7) appear in the Environment table

```
// ChebyshevFamily.txt


FAMILY Chebyshev 7 First

GRAPH [-1 <= x <= 1 <> 0.01] ( T3(x), T4(x), T5(x), T6(x) )


FAMILY Chebyshev 7 Second

GRAPH [-1 <= x <= 1 <> 0.01] ( U3(x), U4(x), U5(x), U6(x) )
```



# Spline for Gamma Function

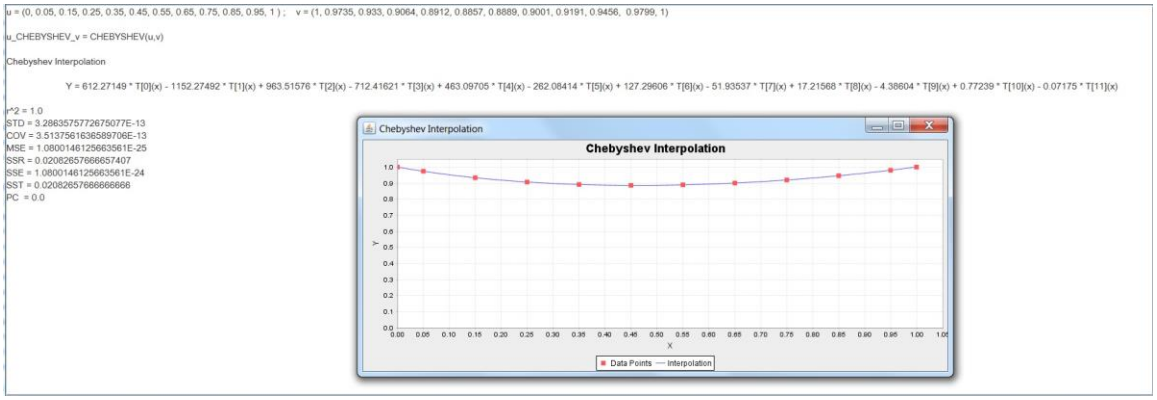
CalcLib was used to build a spline for the Gamma Function. The core interval is the domain 1 to 2. Special treatment is given to x=1, x=1.5, and x=2. Other parts of the domain are calculated by the generating functions

GAMMA (x + 1) = x * GAMMA (x)

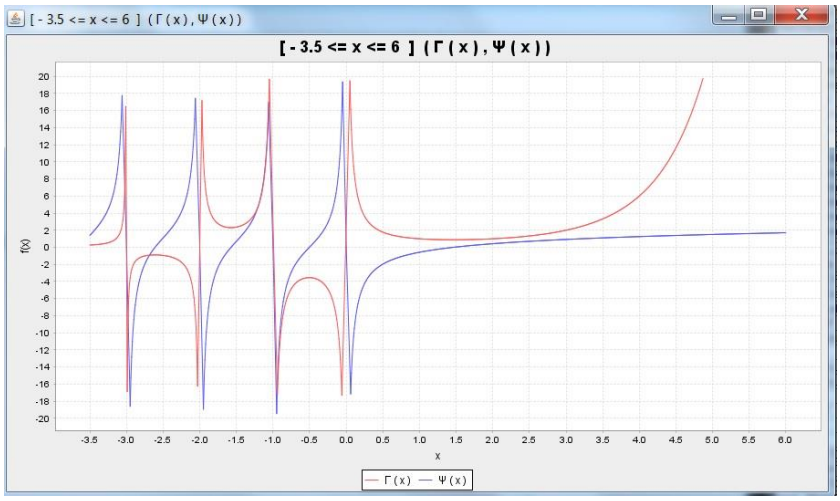
Or

GAMMA (x) = GAMMA (x + 1) / x

**Chebyshev interpolation provided the coefficients for the polynomial**

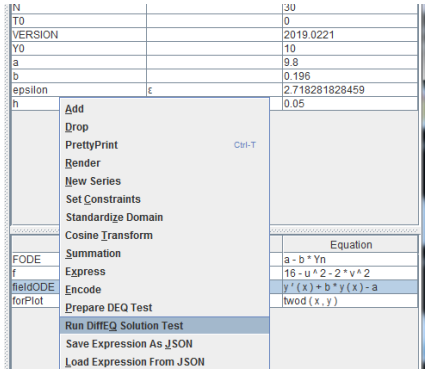
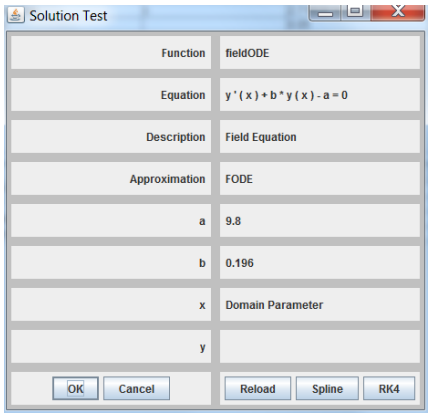
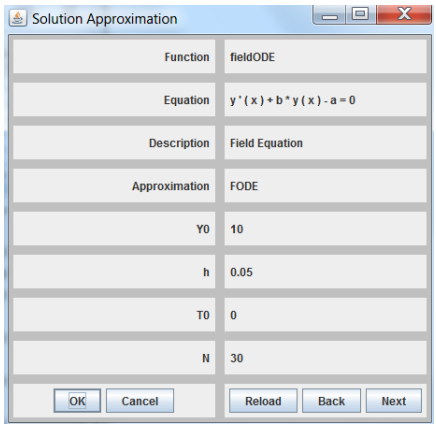


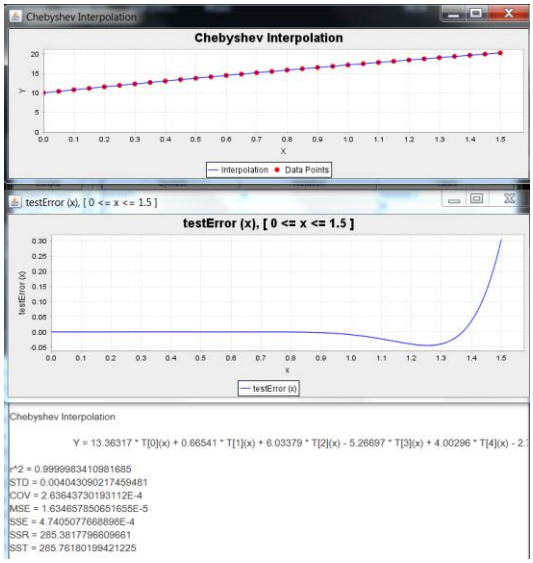
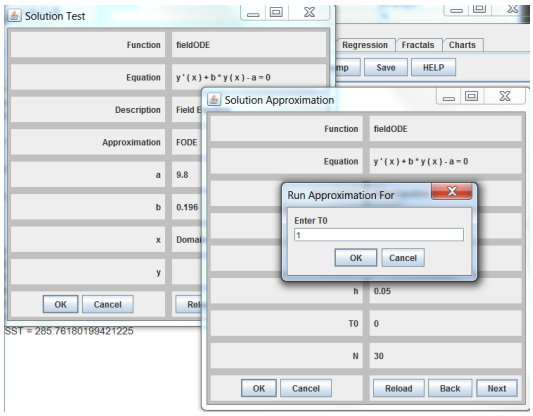
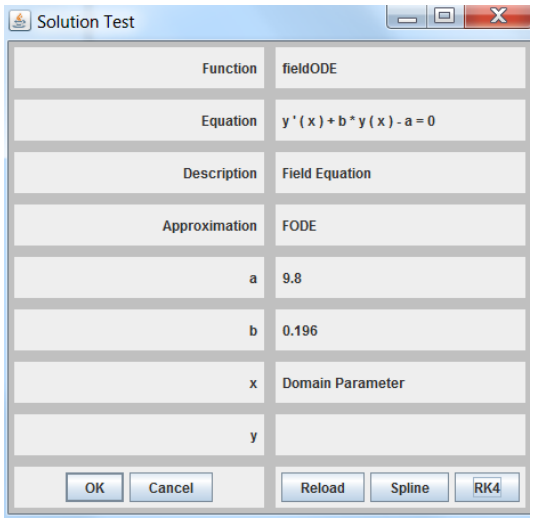

```
LIBRARY SpecFunc net.myorb.math.specialfunctions.Library

!+ GAMMA(x) = SpecFunc.gamma

!! PSI(x) = GAMMA'(x <> 0.0001)/GAMMA(x)

GRAPH  [  -3.5 <=    x    <= 6  <>  0.01  ]   ( GAMMA(x), PSI(x) )
```

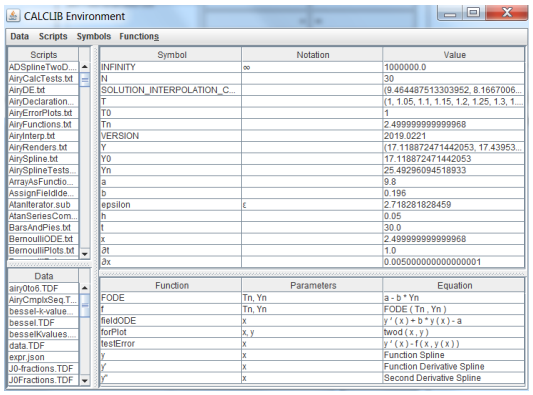


```java
    /**
     * establish Chebyshev polynomial coefficients
     *  collected by regression over domain 1:2
     * spline will use Clenshaw to compute results
     * requested in this domain
     */
    public Gamma ()
    {
        super (new DoubleFloatingFieldManager ());
        coefficients = (new GeneratingFunctions<Double>(manager)).newCoefficients
        (
                  612.2714891668269,
                -1152.274915868479,
                  963.5157625310783,
                 -712.4162057423769,
                  463.09704516096866,
                 -262.0841412593909,
                  127.2960555329889,
                  -51.93536578366297,
                   17.215678004111524,
                   -4.386044338750759,
                    0.7723942678398508,
                   -0.07175167115412054
        );
    }
    GeneratingFunctions.Coefficients<Double> coefficients;


    /**
     * use Clenshaw generating function
     *  to evaluate Chebyshev spline for domain [1-2]
     * @param x value of parameter within domain of [1-2]
     * @return GAMMA(x) as calculated by spline
     */
    public Double eval12 (Double x)
    {
        if (x == 1.5)
            return GAMMA_HALF / 2.0;
        Value<Double> v = forValue (x - 1);                          // spline has shifted x-axis
        Value<Double> result = evaluatePolynomialV (coefficients, v);   // Chebyshev requires [0-1] domain
        return result.getUnderlying ();
    }


    /**
     * recursive evaluation for x &gt; 1.5
     * @param x value of function parameter
     * @return value of function at parameter
     */
    public Double gammaGt (Double x)
    {
        if (x == 2.0) return 1.0;
        else if (x > 2.0) return (x - 1) * gammaGt (x - 1);
        else return eval12 (x);
    }


    /**
     * recursive evaluation for x &lt; 1.5
     * @param x value of function parameter
     * @return value of function at parameter
     */
    public Double gammaLt (Double x)
    {
        if (x == 0) asymptotic ();
        else if (x == 0.5) return GAMMA_HALF;
        else if (x < 1) return gammaLt (x + 1) / x;
        else if (x == 1) return 1.0;
        return eval12 (x);
    }
    void asymptotic () { throw new RuntimeException ("Gamma asymptotic for Integer <= 0"); }


    /* (non-Javadoc)
     * @see net.myorb.math.Function#eval(java.lang.Object)
     */
    public Double eval (Double x)
    {
        if (x == 1.5) return GAMMA_HALF / 2.0;
        else if (x > 1.5) return gammaGt (x);
        else return gammaLt (x);
    }

```

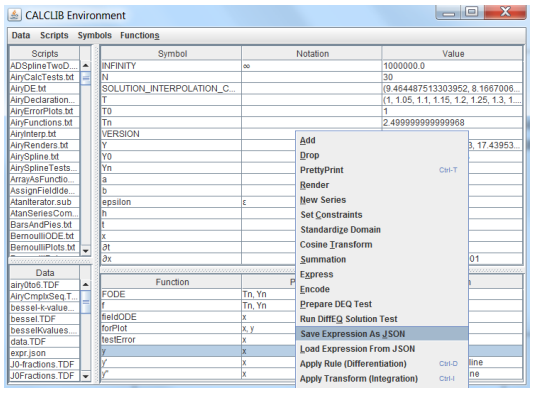
# Differential Equations

| | |
|---|---|
|  | Select an equation<br><br>"Run DiffEQ Solution Test"<br>From the Function drop-down menu |
|  | The "Solution Test" tool is shown<br><br>The information about the equation is shown<br><br>The required parameters are shown with their values<br><br>The RK4 button starts the Runge-Kutta tool |
|  | The "Solution Approximation" screen shows the set values of the RK4 formula parameters<br><br>Press the OK button to run the T0 approximation |

<table>
<tr>
<td> 
</td>
<td>The Interpolation displays show the quality of the best fit attempt made for the approximation<br><br>In addition to the interpolation displays a “testError” plot is shown<br><br>From the error plot the knot point should be chosen</td>
</tr>
<tr>
<td> 
</td>
<td>Use the Next button on the RK4 Solution Approximation screen which will present a dialog requesting the T0 for the knot<br><br>This will present the next Interpolation<br><br>This repeats for as many knots as are required</td>
</tr>
<tr>
<td> 
</td>
<td>Now the Solution Test “Spline” button will generate the spline for the solution approximation</td>
</tr>
</table>



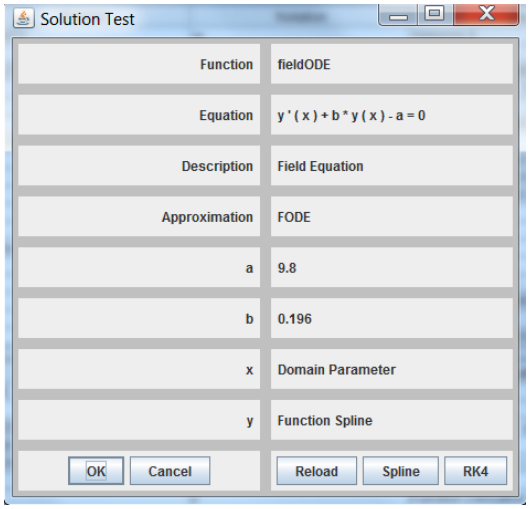
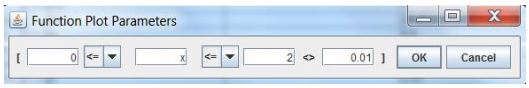
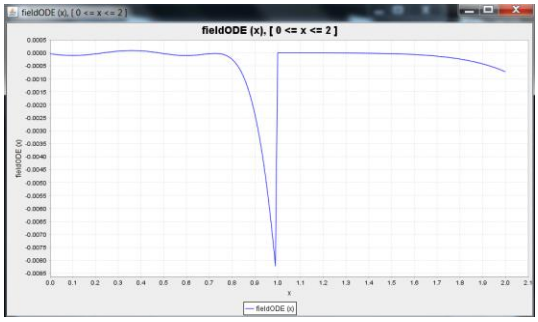

Now the solution function “y” can be found in the functions list along with the derivatives y’ and y’’



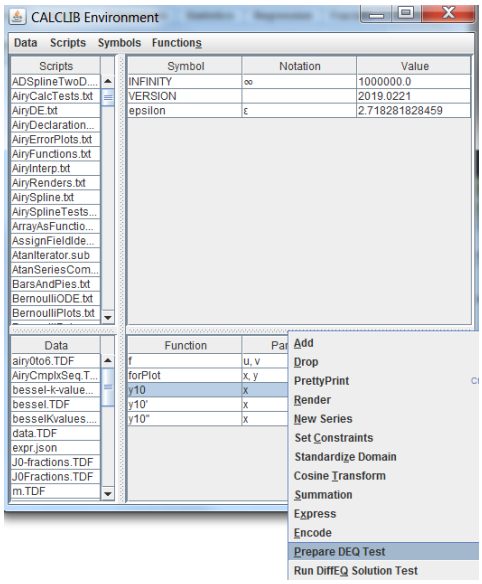
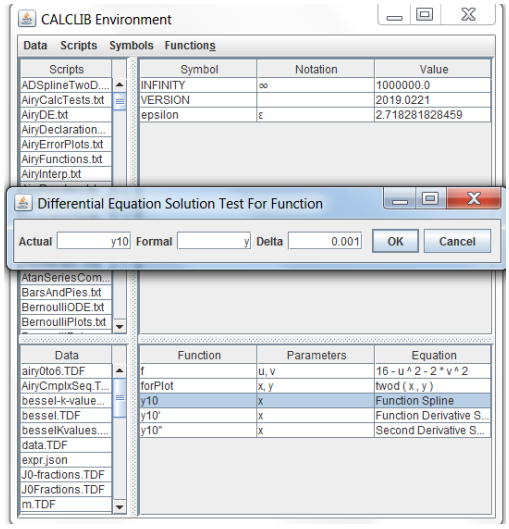
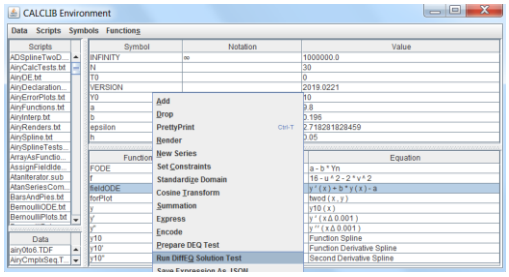

The function can be saved as a JSON expression for later import as a common singleton function

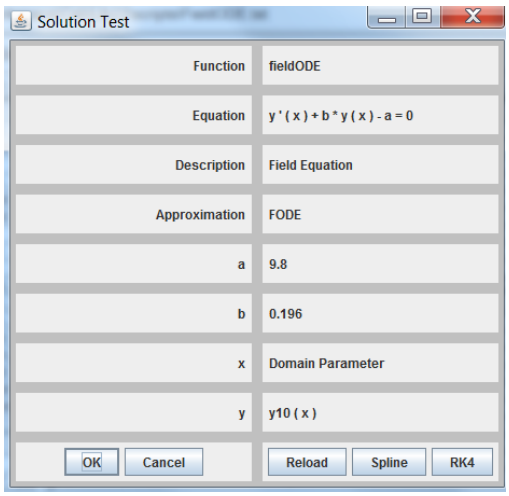
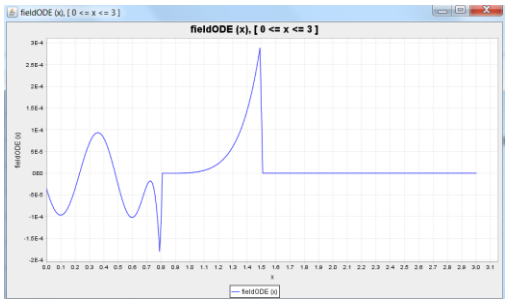
Hence the solution is available as a function and the accuracy across the domain is know from the error plot

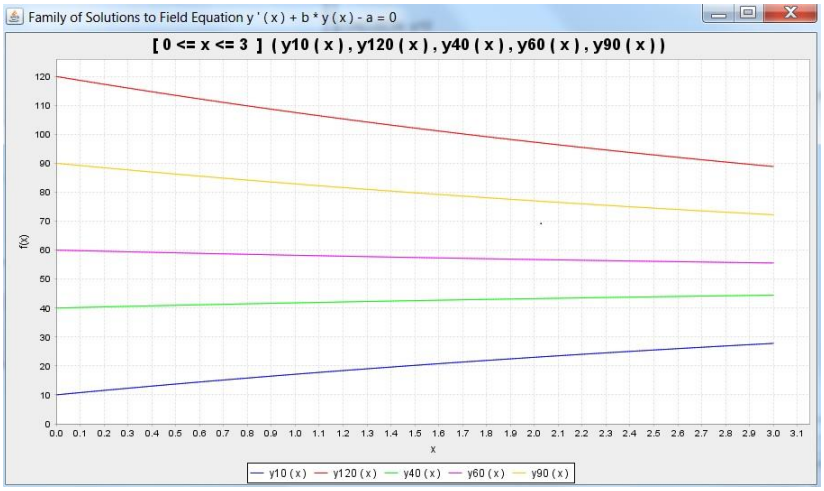
# Test the Solution Quality

| | |
|---|---|
|  | The Solution Test screen now shows the solution “y” has a “Function Spline” available for test |
|  | The OK button show a form for entering domain parameters |
|  | A plot of the solution error for the domain is shown<br><br>An error spike is typically seen at each knot |

# Test function as Solution

| | |
|---|---|
|  | Select function<br><br>“Prepare DEQ Test” from function menu |
|  | A dialog will request the formal parameter name for the function within the differential equation |
|  | Select the differential equation<br><br>“Run DiffEQ Solution Test”<br>from the Function drop-down menu |

| | |
|---|---|
|  | The solution test form comes up<br><br>Now the formal parameter is shown to reference the selected actual parameter<br><br>Function y = y10 (x)<br><br>Press OK button |
|  | Enter domain description |
|  | The error plot is displayed |



**RungeKutta.txt**

Use Runge-Kutta approximation and check the error computed against solution vector

```
// Runge-Kutta error test

// generate interpolation
T  =  [ 0 <= t <= N ]  ( T0 + t * h )
solution = CHEBYSHEV (T, Y)

// generate alias for solution and for derivative
!! y  (x) = solution  @*^   x
!! y' (x) = solution  @*^'  x

// improve polynomial use efficiency
OPTIMIZE  y ; OPTIMIZE  y'

// post error test
!! testError (x) = y' (x) - f ( x, y(x) )

// plot error against domain described by test parameters
PLOTF testError  [  T0 <=  x  <= T0 + h * N  <>  h/10  ]

SIDEBYSIDE "Regression Versus Error Plot"
```

**RungeKuttaRiccati.txt**

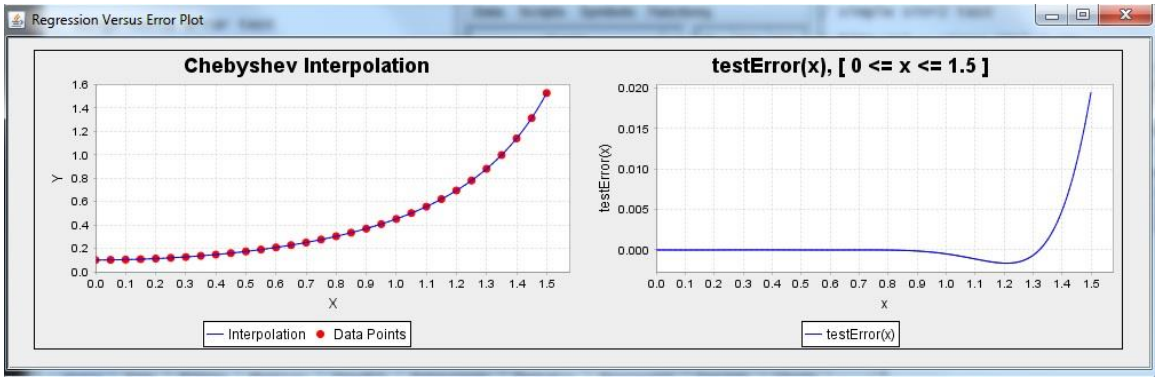
A simple example of use of **RungeKutta.txt**

// Riccati

!! u(x) = x / 2
Y0 = 0.1 ; T0 = 0 ; h = 0.05 ; N = 30
!! f(Tn,Yn) = Yn^2 + Tn * u(Tn) * Yn + u(Tn)

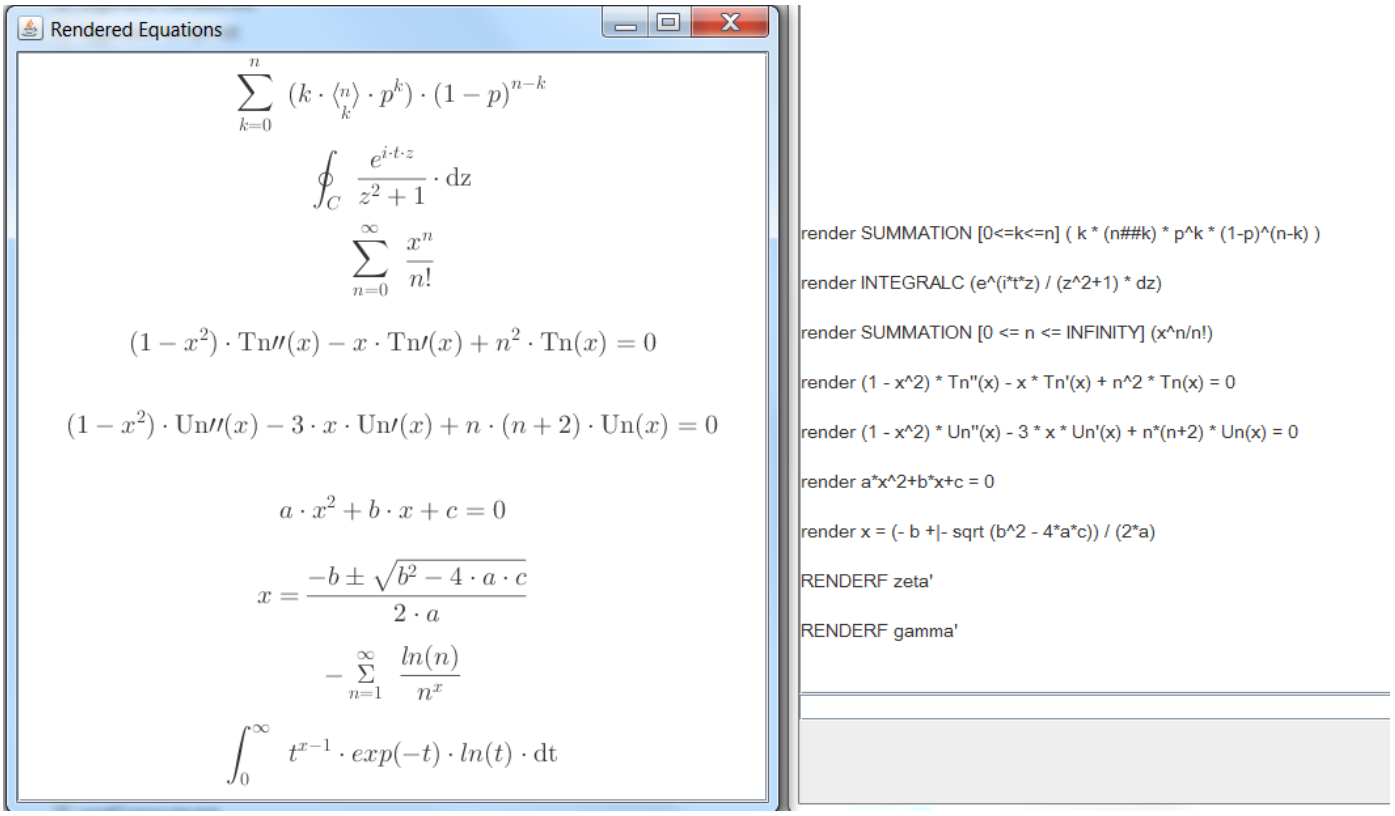
Y = (0.1, 0.10113171857626335, 0.10354424073431215, 0.10726542307837783,
0.11232678123700574, 0.11876514779425473, 0.12662446126207821,
0.1359577239358084, 0.14682917975976403, 0.15931677951018194,
0.1735150208399999, 0.18953827665792933, 0.20752475923532693,
0.22764131262151488, 0.250089287107206, 0.2751118334633071,
0.303003071635968, 0.3341197537209428, 0.36889627761184174,
0.40786425183380254, 0.45167832094096894, 0.5011507269701952,
0.5572982587771222, 0.6214070869571133, 0.6951239492603989,
0.7805870513163335, 0.8806183876003942, 0.999013878270224,
1.1409946276569765, 1.3139341820639356, 1.5285808552014561)

READ RungeKutta.txt



# Equation Renders

RENDER commands are available to supply MathML / LaTeX type renders of sophisticated equations.




Functions found in the Functions list can be rendered using:

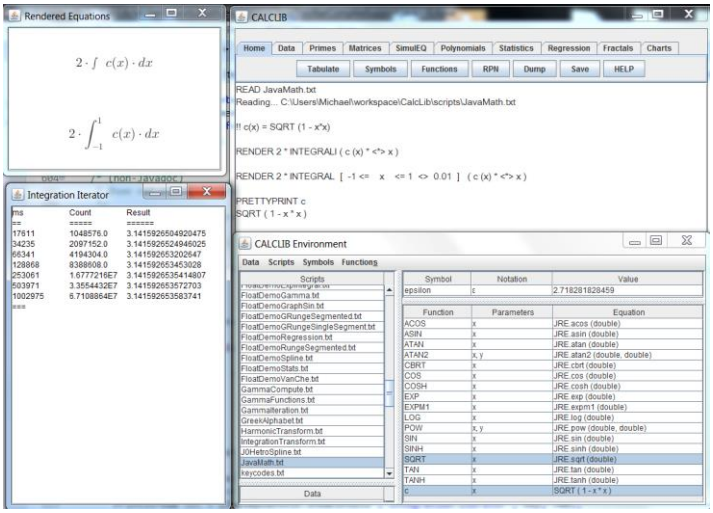
RENDERF function-name

As seen above equations can be rendered directly from the command line e.g.

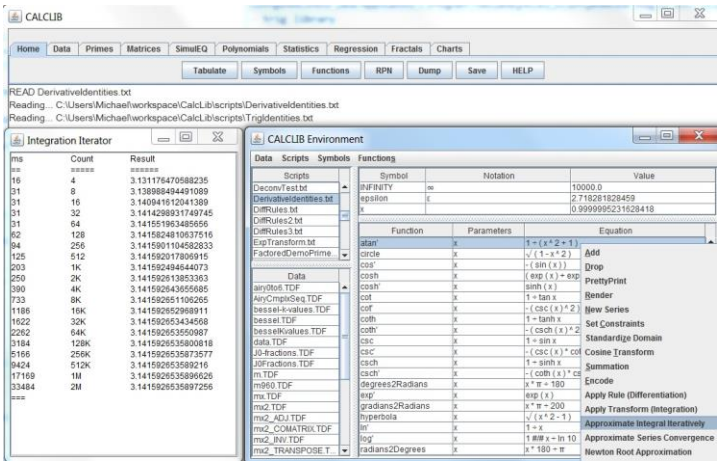
RENDER `(1 - x^2) * Tn''(x) - x * Tn'(x) + n^2 * Tn(x) = 0`

# Numeric Integration

Brute force calculation of the sum of function values over a specified range can result in an approximate value for a definite integral. Doing the process iteratively allows the pattern of convergence to be identified and the efficiency of the method to be evaluated in a time VS accuracy assessment.

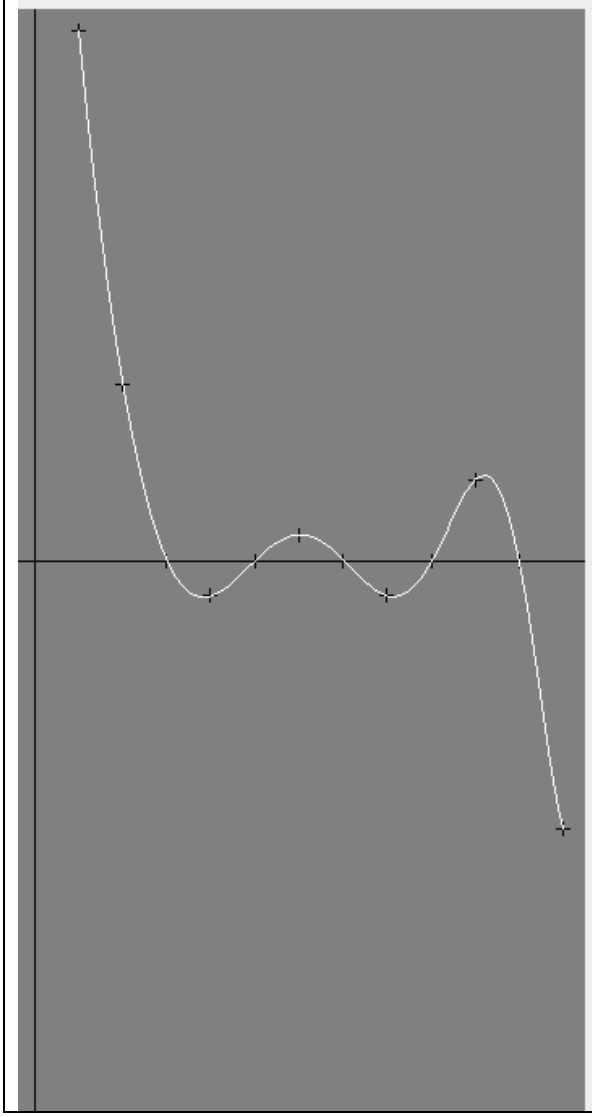



The Integration Approximation of the function selection menu produces the above output. A multiplier can be selected to allow the output to be more intelligible (e.g. pi VS pi/2 in the illustration) and the low to hi bounds of the integration interval are entered into the GUI. The indefinite and then the definite form of the integral are rendered. The iteration count is requested at which point the request can be canceled. The iteration count is relative to the log of the count of calculations. 2^iterations evaluations of the function will be computed. The display runs in the background and will show time, function evaluation count, and the approximation at that point of the integral.



**BernoulliPolynomialRegression.txt**

// this is just a fun exercise to see how well regression can fit the Bernoulli sequence



READ BernoulliPolynomialRegression.txt
Reading... BernoulliPolynomialRegression.txt

a_FITPOLY_b = FITPOLY(a,b)

Polynomial regression

Y = 1.2531135531134323 - 1.3467492296060017*x + 1.0447195450047913*x^2 - 0.6801772589547119*x^3 + 0.30259420381926283*x^4 - 0.0890846611133917*x^5 + 0.01781262969283056*x^6 - 0.00244056651571436*x^7 + 2.2449744919974645E-4*x^8 - 1.3145933284817513E-5*x^9 + 4.393846159717669E-7*x^10 - 6.341028166422841E-9*x^11

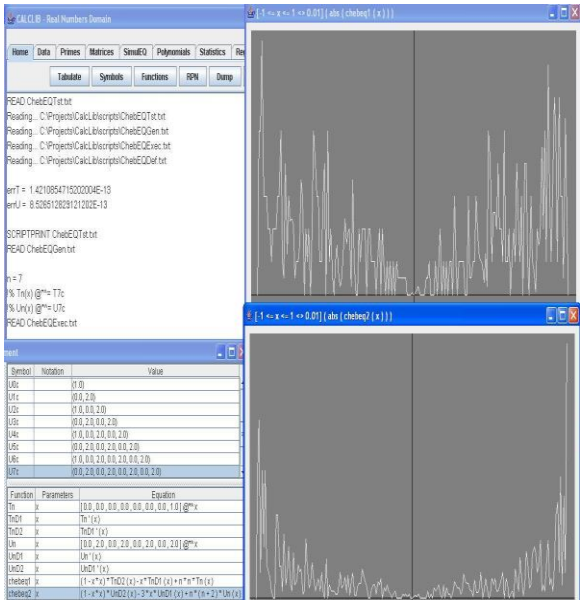
// note the coefficients decay fairly rapidly

r^2 = 1.0 // looks pretty good
STD = 1.840308006797505E-12
COV = 4.946471196610345E-11
MSE = 3.386733559883005E-24
SSR = 0.3337625103863371
SSE = 3.386733559883005E-23
SST = 0.33376251038338967

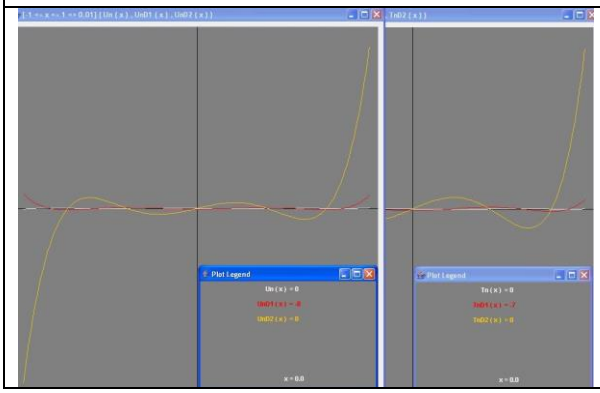
// solve for the roots of this polynomial, it shows:
//  (3, 5, 7, 9, 11) which is pretty impressive

## ChebEQTst.txt

Chebyshev differential equations being used to demonstrate the implemented concept of transform functions. One of the transform types is 'Calculus' providing for integration (anti-derivative) and derivation (derivative function determination).



```
// ChebEQExec.txt:

!% TnD1(x) <>= Tn
!% TnD2(x) <>= TnD1

// display of plots of Tn, Tn', & Tn"
GRAPH [-1 <= x <= 1 <> 0.01] (Tn(x), TnD1(x), TnD2(x))

!% UnD1(x) <>= Un
!% UnD2(x) <>= UnD1

// display of plots of Un, Un', & Un"
GRAPH [-1 <= x <= 1 <> 0.01] (Un(x), UnD1(x), UnD2(x))

READ ChebEQDef.txt

// display MAX error and graph of error of first kind
GRAPH [-1 <= x <= 1 <> 0.01] (abs(chebeq1(x)))
err = [-1 <= x <= 1 <> 0.001] (abs(chebeq1(x)))
errT = MAX err; PRETTYPRINT errT

// display MAX error and graph of error of second kind
GRAPH [-1 <= x <= 1 <> 0.01] (abs(chebeq2(x)))
err = [-1 <= x <= 1 <> 0.001] (abs(chebeq2(x)))
errU = MAX err; PRETTYPRINT errU
```

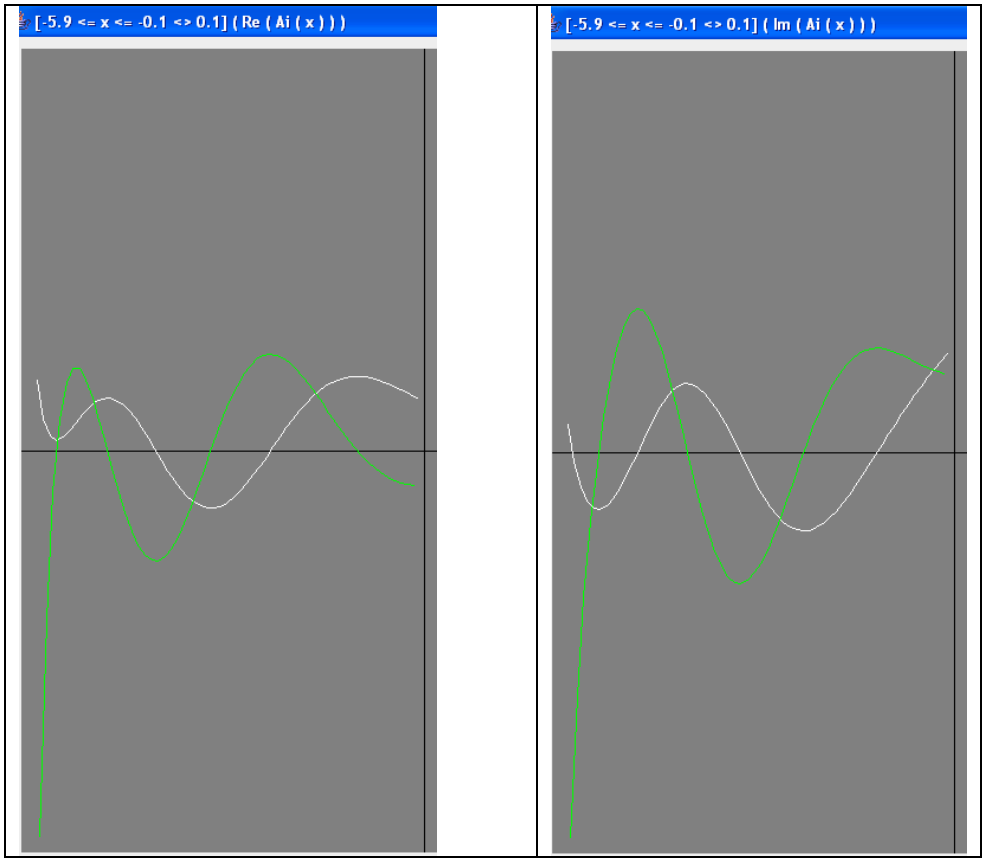


Graphs of Un and Tn overlaid with their first and second derivative functions

## CmplxDemoBesselAiry.txt

```
INFINITY = 10

LIBRARY JRE java.lang.Math;   !+ POW(x,y)   = JRE.pow;  !+ SIN(x)     = JRE.sin
LIBRARY Gamma cern.jet.stat.Gamma;  !+ gamma(x)   = Gamma.gamma

// modified Bessel Ialpha and Kalpha
!! Ialpha(alpha,x) = SIGMA [0<=m<=INFINITY] ( (x/2)^(2*m+alpha) / m! / gamma (m+alpha+1) ) )
!! Kalpha(alpha,x) = pi/2 * ( Ialpha(-alpha,x) - Ialpha(alpha,x) ) / SIN (pi*alpha)

// Airy
!! Ai(x) = 1/pi * sqrt(x/3) * Kalpha( 1/3, 2/3 * sqrt(POW(x,3)) )
!! Bi(x) = sqrt(x/3) * ( Ialpha( 1/3, 2/3 * sqrt(POW(x,3)) ) + Ialpha( -1/3, 2/3 * sqrt(POW(x,3)) ) )

aivals = [-10<x<0<>0.1](Ai(x))
graph [-6<x<0<>0.1](Re (Ai(x)) )
graph [-6<x<0<>0.1](Im (Ai(x)) )
```

// as mentioned before this is the complex version using Bessel functions



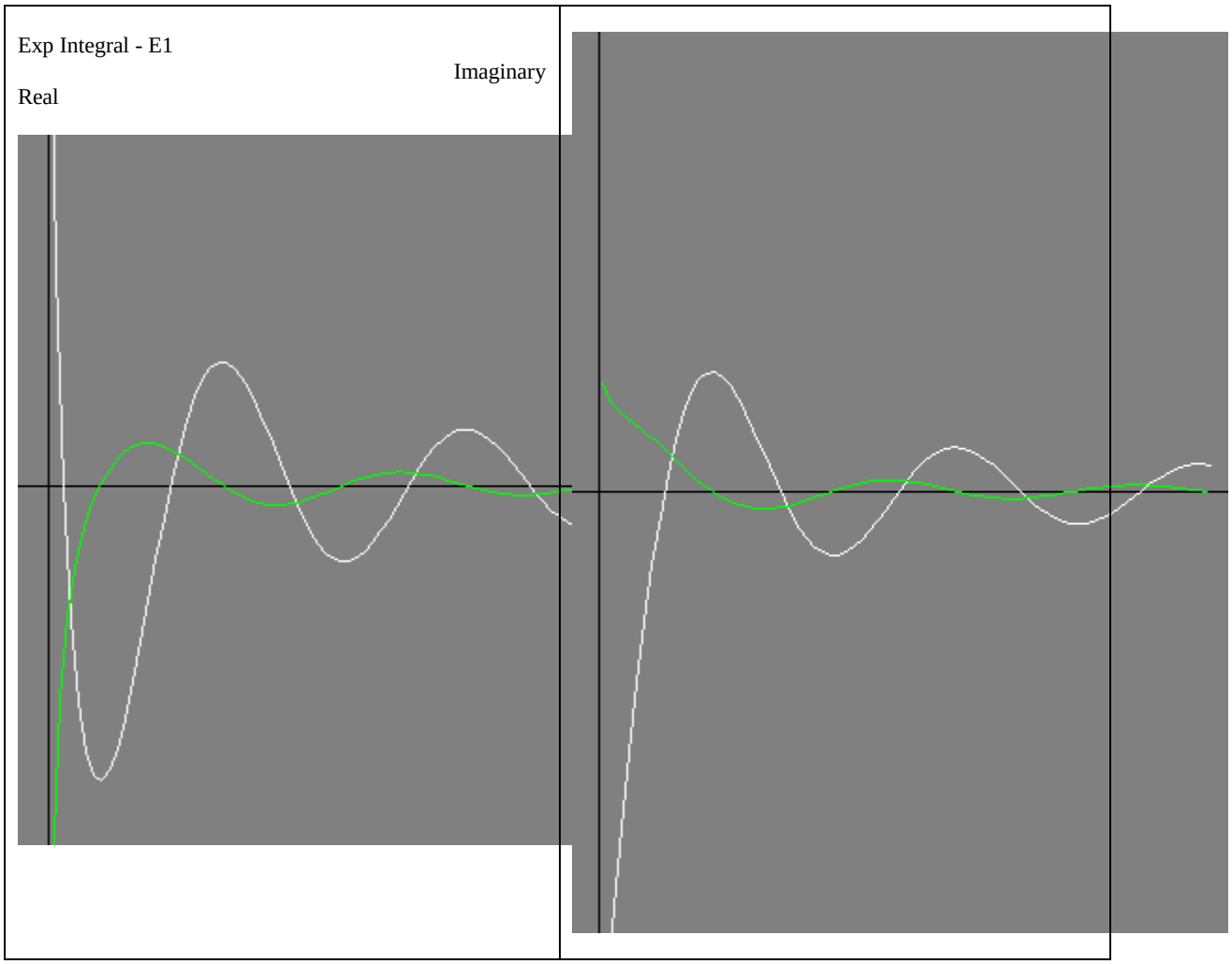

## Functions

Ai (x) = 1 ÷ π * √ ( x ÷ 3 ) * Kalpha ( 1 ÷ 3 , 2 ÷ 3 * √ ( POW ( x , 3 ) ) )
Bi (x) = √ ( x ÷ 3 ) * ( Ialpha ( 1 ÷ 3 , 2 ÷ 3 * √ ( POW ( x , 3 ) ) ) + Ialpha ( - 1 ÷ 3 , 2 ÷ 3 * √ ( POW ( x , 3 ) ) ) )
Ialpha (α, x) = Σ [ 0 <= m <= ∞ ] ( ( x ÷ 2 ) ^ ( 2 * m + α ) ÷ m ! ÷ γ ( m + α + 1 ) ) )
Kalpha (α, x) = π ÷ 2 * ( Ialpha ( - α , x ) - Ialpha ( α , x ) ) ÷ SIN ( π * α )

## CmplxDemoExpInt.txt

```
INFINITY = 20

dt = 1 / INFINITY

// exp integral

!! Ei(x) = - INTEGRAL [-x <= t <= INFINITY <> dt] ( exp(-t) / t * dt )
!! E1(x) = INTEGRAL [x <= t <= INFINITY <> dt] ( exp(-t) / t * dt )

E1vals = [0.2 <= x <= 16 <> 0.2] (E1(i*x))

graph [0<=ii<LENGTH(E1vals)]( Re(E1vals#ii) )
graph [0<=ii<LENGTH(E1vals)]( Im(E1vals#ii) )
```

again with the infinite integrals, but being exponential and complex they become diminishing cyclic functions




## Functions

E1 (x) = ∫ [ x <= t <= ∞ Δ dt ] ( exp ( - t ) ÷ t * dt )

Ei (x) = - ∫ [ - x <= t <= ∞ Δ dt ] ( exp ( - t ) ÷ t * dt )

## CmplxDemoMatrix.txt

```
// simple examples of the matrix primitives


READ CmplxDemoMatrix.txt
Reading... C:\workspace\MathFields\scripts\CmplxDemoMatrix.txt


SCRIPTPRINT CmplxDemoMatrix.txt


IMPORT mx2 MX2.TDF
PRETTYPRINT mx2


mx2_DET = DET(mx2)
PRETTYPRINT mx2_DET

mx2_TRANSPOSE = TRANSPOSE(mx2)
PRETTYPRINT mx2_TRANSPOSE

mx2_COMATRIX = COMATRIX(mx2)
PRETTYPRINT mx2_COMATRIX

mx2_ADJ = ADJ(mx2)
PRETTYPRINT mx2_ADJ

mx2_INV = INV(mx2)
PRETTYPRINT mx2_INV

mx2_TR = TR(mx2)
PRETTYPRINT mx2_TR


mx2_CHARACTERISTIC = CHARACTERISTIC(mx2)

mx2_EIGENVALUE = EIG(mx2,mx2_EIGENVECTOR)
```

**OUTPUT:**

```
Import from file:  C:\workspace\MathFields\data\MX2.TDF
Import to matrix:  mx2
 Columns per row:  3
      Rows read:  3


data read:
     (-7 - 11*i)     (-11 + 10*i)      (14 - 5*i)
    (-11 - 16*i)     (-13 + 11*i)      (19 - 4*i)
     (18 + 12*i)      (12 - 18*i)     (-15 + 12*i)

=EOD=



mx2 =

     (-7 - 11*i)     (-11 + 10*i)      (14 - 5*i)
    (-11 - 16*i)     (-13 + 11*i)      (19 - 4*i)
     (18 + 12*i)      (12 - 18*i)     (-15 + 12*i)



mx2_DET =

              (-369 + 108*i)


mx2_TRANSPOSE =

     (-7 - 11*i)     (-11 - 16*i)      (18 + 12*i)
    (-11 + 10*i)     (-13 + 11*i)      (12 - 18*i)
     (14 - 5*i)      (19 - 4*i)      (-15 + 12*i)



mx2_COMATRIX =

    (-93 + 69*i)      (33 + 48*i)     (-54 - 36*i)
     (33 - 30*i)      (-75 + 3*i)     (-36 + 54*i)
    (-42 + 15*i)     (-57 + 12*i)             -69
```

mx2_ADJ =

| | | |
|---|---|---|
| (-93 + 69*i) | (33 - 30*i) | (-42 + 15*i) |
| (33 + 48*i) | (-75 + 3*i) | (-57 + 12*i) |
| (-54 - 36*i) | (-36 + 54*i) | -69 |

mx2_INV =

| | | |
|---|---|---|
| (0.28256 - 0.10429*i) | (-0.10429 + 0.05078*i) | (0.1158 - 0.00676*i) |
| (-0.04731 - 0.14393*i) | (0.18941 + 0.04731*i) | (0.15105 + 0.01169*i) |
| (0.10849 + 0.12932*i) | (0.12932 - 0.10849*i) | (0.17224 + 0.05041*i) |

mx2_TR =

(-35 + 12*i)

Characteristic polynomial : - (369 - 108*i) + (237 - 72*i) * x - (35 - 12*i) * x^2 - x^3

Von Mises dominant Eigen-Pair
Dominant Eigenvalue is: 42.720018726587654
Dominant Eigenvector is:
[
(0.45054741724475555 + 0.06575787393774088*i),
(0.5946656941881084 + 0.005590057346825491*i),
(-0.6128582806028051 - 0.25185138105731264*i)
]

## CmplxDemoRoots.txt

```
READ CmplxDemoRoots.txt
Reading... C:\workspace\MathFields\scripts\CmplxDemoRoots.txt
Reading... C:\workspace\MathFields\scripts\CmplxDemoRoots.TXT

SCRIPTPRINT CmplxDemoRoots.TXT


x = 5
SCRIPTPRINT sqrtCompute.txt
SCRIPTPRINT sqrtIteration.txt
READ sqrtCompute.txt



x=(1, 1, -1)
SCRIPTPRINT QUADRATIC.txt
READ QUADRATIC.txt

calc ROOTS x


x=(2,-3,4,-5)
SCRIPTPRINT CUBIC.txt
READ CUBIC.txt

calc ROOTS x


x=(2,-3,4,-5,1)
SCRIPTPRINT QUARTIC.txt
READ QUARTIC.txt

calc ROOTS x

Reading... C:\workspace\MathFields\scripts\sqrtCompute.txt

// looking for 10 decimal places of precision

TOLERANCE = 10^(-10)

// polynomial solutions "c - x^2 = 0" gives sqrt(c) as a solution

sqrt_polynomial = (x,0,-1)

polyprint sqrt_polynomial

// Newton's methods uses the first derivative for the solution

sqrt_poly_derivative = POLYDER sqrt_polynomial

polyprint sqrt_poly_derivative

// initial approximation

sqrtx = 1

ITERATE 10 SqrtIteration.txt
```

## OUTPUT:

```
Reading... C:\workspace\MathFields\scripts\sqrtIteration.txt

// compute polynomial value

f = sqrt_polynomial +*^ sqrtx

// compute derivative value

fprime = sqrt_poly_derivative +*^ sqrtx

// iteration contribution computed

sqrt_iteration = f/fprime

// new value for this iteration

sqrtx = sqrtx - sqrt_iteration

// check status of result

sqrtx_squared = sqrtx ^ 2

// display intermediate results

PRETTYPRINT sqrt_iteration
PRETTYPRINT sqrtx_squared
PRETTYPRINT sqrtx

// look for convergence

ASSERT "Convergence Complete" TOLERANCE >|| sqrt_iteration

sqrt_polynomial =

            5 - x^2


sqrt_poly_derivative =

             - 2*x
```

Reading... C:\workspace\MathFields\scripts\SqrtIteration.txt

sqrt_iteration =

-2

sqrtx_squared =

9

sqrtx =

3

sqrt_iteration =

0.6666666666666666

sqrtx_squared =

5.4444444444444455

sqrtx =

2.3333333333333335

sqrt_iteration =

0.09523809523809545

sqrtx_squared =

5.009070294784581

sqrtx =

2.238095238095238

sqrt_iteration =

0.002026342451874501

sqrtx_squared =

5.000004106063731

sqrtx =

2.2360688956433634

sqrt_iteration =

9.181433852065489E-7

sqrtx_squared =

5.000000000000843

sqrtx =

2.236067977499978

sqrt_iteration =

1.8847399291451707E-13

sqrtx_squared =

5.000000000000001

sqrtx =

2.23606797749979

Script interrupted in iteration 6
Assertion "Convergence Complete" has been validated, TOLERANCE >|| sqrt_iteration
*** Script has terminated

```
Reading... C:\workspace\MathFields\scripts\QUADRATIC.txt

a = x#2; b = x#1; c = x#0

quad_common = -b / (2 * a)

quad_d = b^2 - 4*a*c; quad_sqrt = 2\quad_d

quad_roots = quad_common +|- quad_sqrt  / (2 * a)

PRETTYPRINT quad_roots


quad_roots =

[
        -0.6180339887498949
        1.618033988749895
]


// computed real roots
(-0.6180339887498949, 1.618033988749895)



Reading... C:\workspace\MathFields\scripts\CUBIC.txt

delta#3 = 0

a = x#3; b = x#2; c = x#1; d = x#0

delta#0 = b^2 - 3*a*c; delta#1 = 2*b^3 - 9*a*b*c + 27*a^2*d

delta#3 = delta#1 ^ 2  - 4 * delta#0 ^ 3

C = 3\( ( delta#1 + 2\delta#3 ) / 2 )

u = APPEND (-1/2 +|- i*2\3 / 2, 1)

cubic_roots = [0 <= k <= 2] (- (b + u#k * C + delta#0 / (u#k * C)) / (3*a) )

PRETTYPRINT cubic_roots


cubic_roots =

[
        (0.03533842849427097 + 0.7397330112915744*i)
        (0.03533842849427097 - 0.7397330112915744*i)
        0.7293231430114581
]


// computed real roots
(0.7293231430114581)
```

```
Reading... C:\workspace\MathFields\scripts\QUARTIC.txt

delta#3 = 0

a = x#4; b = x#3; c = x#2; d = x#1; e = x#0

delta#0 = c^2 - 3*b*d + 12*a*e

delta#1 = 2*c^3 - 9*b*c*d + 27*b^2*e + 27*a*d^2 - 72*a*c*e

delta#3 = delta#1 ^ 2  - 4 * delta#0 ^ 3

p = (8*a*c - 3*b^2) / (8*a^2)

q = (b^3 - 4*a*b*c + 8*a^2*d) / (8*a^3)

Q = 3\( (delta#1 + 2\(delta#3)) / 2)

S = 2\( (Q + delta#0 / Q) / (3*a) - 2*p/3 ) / 2

common = -b / (4*a)

// quartic_roots_12 = common - S +|- 2\(q/S - 4*S^2 - 2*p) /2
// quartic_roots_34 = common + S +|- 2\(-q/S - 4*S^2 - 2*p) /2
// quartic_roots = APPEND (quartic_roots_12, quartic_roots_34)

CMPS = common -|+ S
quartic_roots = APPEND ( CMPS#0 +|- 2\(q/S - 4*S^2 - 2*p) /2 , CMPS#1 +|- 2\(-q/S - 4*S^2 - 2*p) /2 )
PRETTYPRINT quartic_roots


quartic_roots =

[
        (0.004423084318876391 + 0.7714190717315003*i)
        (0.004423084318876391 - 0.7714190717315003*i)
        4.188847029536467
        0.8023068018257802
]


// computed real roots
(0.8023068018257804, 4.188847029536467)
```

# Iterations Improving Accuracy

Using FACTORIZATION mode, values are maintained as ratios of prime factors. By selecting algorithms that are based on series of integer ratios it is possible to compute irrational values out to arbitrary numbers of significant digits providing that the numerator and denominator are allowed to grow to arbitrary numbers of digits. Having implemented this methodology the remaining issue is how fast the series converges since this impacts how many terms are required to give an accurate number of how many digits. The sample below uses the Ramanujan series to calculate 1/pi. This series converges very rapidly. Computation of 10 terms of the series gives a calculation of 85 digits of pi. The representation of the ratio has numerator and denominator values that are significantly longer than 85 digits

## Script Source

```
SCRIPTPRINT ComputePiRamanujan.txt
Reading... C:\Users\Michael\workspace\CalcLib\scripts\ComputePiRamanujan.txt


x = 2

READ sqrtCompute.txt

// requires 7 iterations of Newton's method
// sqrt(2) = 1.4142_13562_37309_50488_01688_72420_96980_78569_67187_53769_48073_17667_97379_90732_47846_21070_38850_38753_43276 // 95 digits
//         = 1.4142_13562_37309_50488_01688_72420_96980_78569_67187_53769_48073_17667_97379_90732_47846_21070_38850_38753_43276 // per OEIS A002193
//         = 4946041176255201878775086487573351061418968498177 / 34973792557579411720208518520705629194379642126 08
//         = ( 7681 * 1492993 * 431302713980890947612633357964569696769 ) / ( 2^7 * 3 * 17 * 257 * 577 * 1409 * 11777 * 665857 * 2393857 * 2448769 * 55780318173953 )

radical2   = sqrtx


// 1 / pi = ( 2 * sqrt(2) / 9801 ) * SIGMA [0 <= k <= INFINITY] ( (4*k)! * (1103 + 26390*k) / ((k!)^4 * 396 ^ (4*k)) )


!! series (n) = SIGMA [0 <= k <= n] ( (4*k)! * (1103 + 26390*k) / ((k!)^4 * 396 ^ (4*k)) )


s = series(10)

// s = 1103.0000268319745734638134088821330563639360546388967
//   = 3872430076450268064456794609054361181459412148133735536353909359385271078098493094151622107164797146827637310820416697 7
                  2060777126802719277591020187485184860708456341122994882635648594364069184119994702 75
//     / 3510815940388163209292975628161374616891448276678347860097892396840833464993082052835202165786327128761649621423675210573 98
                  5612671493685721632947900318202137813691668941941229874985670718736078940230272
//   = ( 5^2 * 7^3 * 13 * 23 * 29 * 31 * 37 * 41 * 43 * 47 * 53 * 59 * 61 * 67 * 71 * 73 * 79 * 83 * 191 * 1451 *
                  4552468928477187574050463918838755530501788164295356574114208913461935568005590671309567043
                        4403864087806034470850422555296422129968289083320181488913830968277491841459 )
//     / ( 2^154 * 3^153 * 11^77 )
PRETTYPRINT s 50


c = 2 * radical2 / 9801

// c = 0.000288585565222547709172878017387960020114207096 30
//   = 4946041176255201878775086487573351061418968498177 / 171389070428417907134881845010717935867057436238855 04
//   = ( 7681 * 1492993 * 431302713980890947612633357964569696769 ) / ( 2^6 * 3^5 * 11^2 * 17 * 257 * 577 * 1409 * 11777 * 665857 * 2393857 * 2448769 *
55780318173953 )
PRETTYPRINT c 50


piApproximation = 1 / (c*s)

PRETTYPRINT piApproximation  86
// piApproximation = 3.14159_26535_89793_23846_26433_83279_50288_41971_69399_37510_58209_74944_59230_78164_06286_20899_86280_3
//                 = 1063508829885643955702523868829133008241125216586197637972442088222758794531096590230104814300659761592544166161243665275176690233286697474699317739 52
//                / 3385253745963557226072872916040436591184224783071313742055861437609455035446281079201455765711868066989195674473287243928724460161273140237616646768 5
//                 = ( 2^80 * 3^83 * 11^39 * 257 * 577 * 1409 * 11777 * 665857 * 2393857 * 2448769 * 55780318173953 )
//                  / ( 5 * 7^2 * 13 * 19 * 23 * 29 * 31 * 37 * 41 * 43 * 7681 * 1492993 * 431302713980890947612633357964569696769 *
                    838549873976734283142691189373245057507371516097566767496640304132808663454176616756 21 )
//              pi =
                  3.14159_26535_89793_23846_26433_83279_50288_41971_69399_37510_58209_74944_59230_78164_06286_20899_86280_34825_
                        34211_70679_82148_08651_32823_06647_09384_46095_50582_23172_53594_08128
//                  5    10   15   20   25   30   35   40   45   50   55   60   65   70   75   80   85   90   95   100  105  110  115  120  125  130  135  140  145  150
```

# Script Execution

```
READ ComputePiRamanujan.txt
Reading... C:\Users\Michael\workspace\CalcLib\scripts\sqrtCompute.txt

sqrt_polynomial =

                2 - x^2


sqrt_poly_derivative =

                 - 2 * x

Reading... C:\Users\Michael\workspace\CalcLib\scripts\SqrtIteration.txt

sqrt_iteration =

                -0.5000000000000000 = -1 / 2 = (  - 1 ) / ( 2 )


sqrtx_squared =

                2.2500000000000000 = 9 / 4 = ( 3^2 ) / ( 2^2 )


sqrtx =

                1.5000000000000000 = 3 / 2 = ( 3 ) / ( 2 )


sqrt_iteration =

                0.0833333333333334 = 1 / 12 = ( 1 ) / ( 2^2 * 3 )


sqrtx_squared =

                2.0069444444444445 = 289 / 144 = ( 17^2 ) / ( 2^4 * 3^2 )


sqrtx =

                1.4166666666666667 = 17 / 12 = ( 17 ) / ( 2^2 * 3 )


sqrt_iteration =

                0.0024509803921569 = 1 / 408 = ( 1 ) / ( 2^3 * 3 * 17 )


sqrtx_squared =

                2.0000060073048828 = 332929 / 166464 = ( 577^2 ) / ( 2^6 * 3^2 * 17^2 )


sqrtx =

                1.4142156862745099 = 577 / 408 = ( 577 ) / ( 2^3 * 3 * 17 )


sqrt_iteration =

                0.0000021238998199 = 1 / 470832 = ( 1 ) / ( 2^4 * 3 * 17 * 577 )


sqrtx_squared =

                2.0000000000045110 = 443365544449 / 221682772224 = ( 665857^2 ) / ( 2^8 * 3^2 * 17^2 * 577^2 )


sqrtx =

                1.4142135623746900 = 665857 / 470832 = ( 665857 ) / ( 2^4 * 3 * 17 * 577 )


sqrt_iteration =

                1.5948618246059560E-12 = 1 / 627013566048 = ( 1 ) / ( 2^5 * 3 * 17 * 577 * 665857 )


sqrtx_squared =

                2.0000000000000001 = 786292024016459316676609 / 393146012008229658338304 = ( 257^2 * 1409^2 * 2448769^2 ) / ( 2^10 * 3^2 * 17^2 * 577^2 * 665857^2 )


sqrtx =

                1.4142135623730951 = 886731088897 / 627013566048 = ( 257 * 1409 * 2448769 ) / ( 2^5 * 3 * 17 * 577 * 665857 )


sqrt_iteration =

                8.9929283216504540E-25 = 1 / 1111984844349868137938112 = ( 1 ) / ( 2^6 * 3 * 17 * 257 * 577 * 1409 * 665857 * 2448769 )


sqrtx_squared =

                2.0000000000000001 = 247302058812760093938754324378667553070948424908 9 / 123651029406380046969377162189333776535474212454 4 =
                                ( 11777^2 * 2393857^2 * 557803181739 53^2 ) / ( 2^12 * 3^2 * 17^2 * 257^2 * 577^2 * 1409^2 * 665857^2 * 2448769^2 )


sqrtx =

                1.4142135623730951 = 1572584048032918633353217 / 1111984844349868137938112 =
                                ( 11777 * 2393857 * 55780318173953 ) / ( 2^6 * 3 * 17 * 257 * 577 * 1409 * 665857 * 2448769 )
```

sqrt_iteration =

2.8592838433339520E-49 = 1 / 3497379255757941172020851852070562919437964212608 =
( 1 ) / ( 2^7 * 3 * 17 * 257 * 577 * 1409 * 11777 * 665857 * 2393857 * 2448769 * 55780318173953 )

sqrtx_squared =

2.0000000000000001 =
24463323317211940977293404419928347279246091129334268572773850449273611455484253634058690852323329 /
12231661658605970488646702209964173639623045564667134286386925224636805727742126817029345426161664 =
( 7681^2 * 1492993^2 * 43130271398089094761263335796456969676 9^2 ) /
( 2^14 * 3^2 * 17^2 * 257^2 * 577^2 * 1409^2 * 11777^2 * 665857^2 * 2393857^2 * 2448769^2 * 55780318173953^2 )

sqrtx =

1.4142135623730951 = 4946041176255201878775086487573351061418968498177 / 3497379255757941172020851852070562919437964212608 =
( 7681 * 1492993 * 43130271398089094761263335796456969676 9 ) / ( 2^7 * 3 * 17 * 257 * 577 * 1409 * 11777 * 665857 * 2393857 * 2448769 * 55780318173953 )

Script interrupted in iteration 7
Assertion "Convergence Complete" has been validated, TOLERANCE >|| sqrt_iteration
*** Script has terminated

s =

1103.000026831974573463813408882133056336936405463889 67 =
116354295547844200479625540962705305445031010498388307062857519290687784871920308555177681218916232885 /
10548893265400529621533893458912595114524795065503030854065384561385667359563565492440811343288729 6 =
( 5 * 7^2 * 13 * 17 * 19 * 23 * 29 * 31 * 37 * 41 * 43 * 838549873976734283142691189373245057507371516097566767496640304132808663454176616756 21 ) / ( 2^74 * 3^78 * 11^37 )

c =

0.000288585565222547709172878017387960020114207096 30 = 4946041176255201878775086487573351061418968498177 /
171389070428417907134881845010717935867057436238855 04 =
( 7681 * 1492993 * 43130271398089094761263335796456969676 9 ) / ( 2^6 * 3^5 * 11^2 * 17 * 257 * 577 * 1409 * 11777 * 665857 * 2393857 * 2448769 * 55780318173953 )

piApproximation =

3.1415926535897932384626433832795028841971693993751058209749445923078164062862089986280 4 =
1063508829885643955702523868829133008241252165861976379724420882227587945310965902301048143006597615925441661612436652751766902332866974746993177395 2 /
3385253745963557226072872916040436591184224783071313742055861437609455035446281079201455765711868066989195674473287243928724460161273140237616646768 5 =
( 2^80 * 3^83 * 11^39 * 257 * 577 * 1409 * 11777 * 665857 * 2393857 * 2448769 * 55780318173953 ) /
( 5 * 7^2 * 13 * 19 * 23 * 29 * 31 * 37 * 41 * 43 * 7681 * 1492993 * 43130271398089094761263335796456969676 9 *
838549873976734283142691189373245057507371516097566767496640304132808663454176616756 21 )

# Represent Irrational Numbers with Primes

Compute irrational values out to arbitrary numbers of significant digits. This can be seen as an alternate way to describe an irrational number that carries additional attributes such as the specification of the equation and the number of terms that have been used to compute it. The sample below uses the Ramanujan series to calculate 1/pi. This series converges very rapidly. Using just 2 iterations of the summation results in an accurate double float equivalent calculation.

## Script Source

```
1/pi = 2*\2 / 9801 * ( SUMMATION [0 <= k <= INFINITY] ( (4*k)! * (1103 + 26390*k) / ((k!)^4 * 396 ^ (4*k)) ) )


!! series (n) = SUMMATION [0 <= k <= n] ( (4*k)! * (1103 + 26390*k) / ((k!)^4 * 396 ^ (4*k)) )

calc series 1
// 1103.0000268319743490 = 1130173253125 / 1024635744 = ( 5^5 * 7 * 4423 * 11681 ) / ( 2^5 * 3^7 * 11^4 )


calc sqrt 2
// 1.4142135623730951 = 886731088897 / 627013566048 = ( 257 * 1409 * 2448769 ) / ( 2^5 * 3 * 17 * 577 * 665857 )


c = 2 * sqrt 2 / 9801


calc 1 / (c * series 1)
```

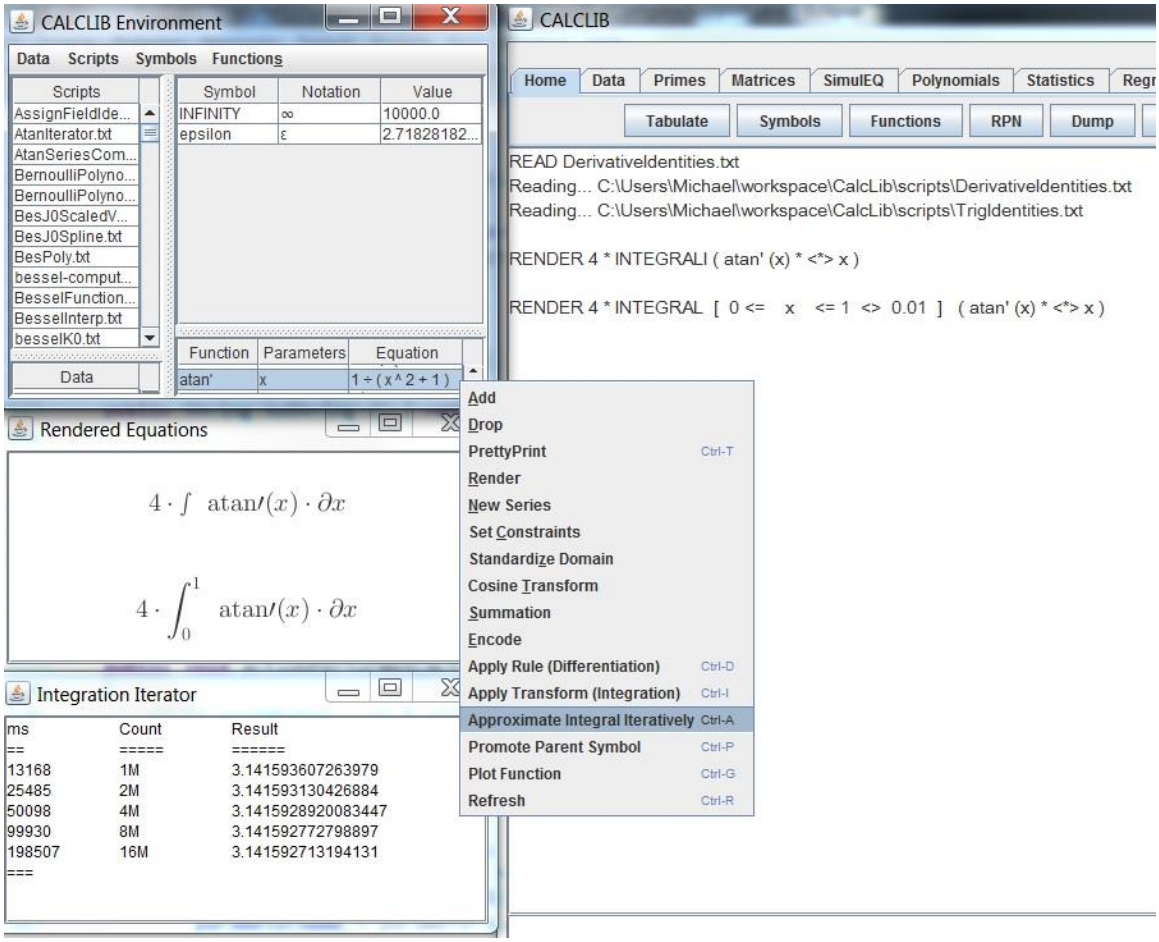
## Script Execution (ComputePrimePI.txt)

```
3.141592653589793 =

        3148377773780973237939865 6 / 1002159759385796058053125 =

        (
                2^9 * 3^12 * 11^6 * 17 * 577 * 665857
        )
                /
        (
                5^5 * 7 * 257 * 1409 * 4423 * 11681 * 2448769
        )
```

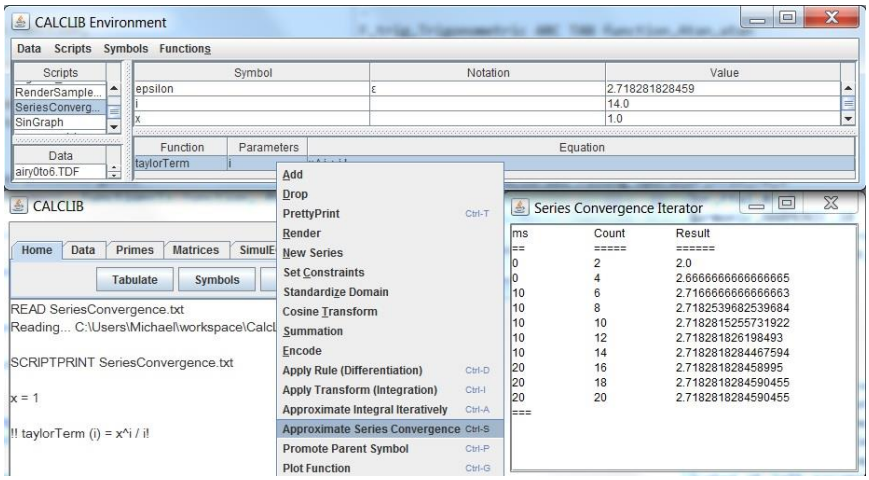
As mentioned above we can attribute this result as a 2 term approximation of the calculation using the Ramanujan series. This can be taken to imply that additional precision can be accomplished by using this as an interim result to which additional terms can be added

# Approximate Integral Iteratively

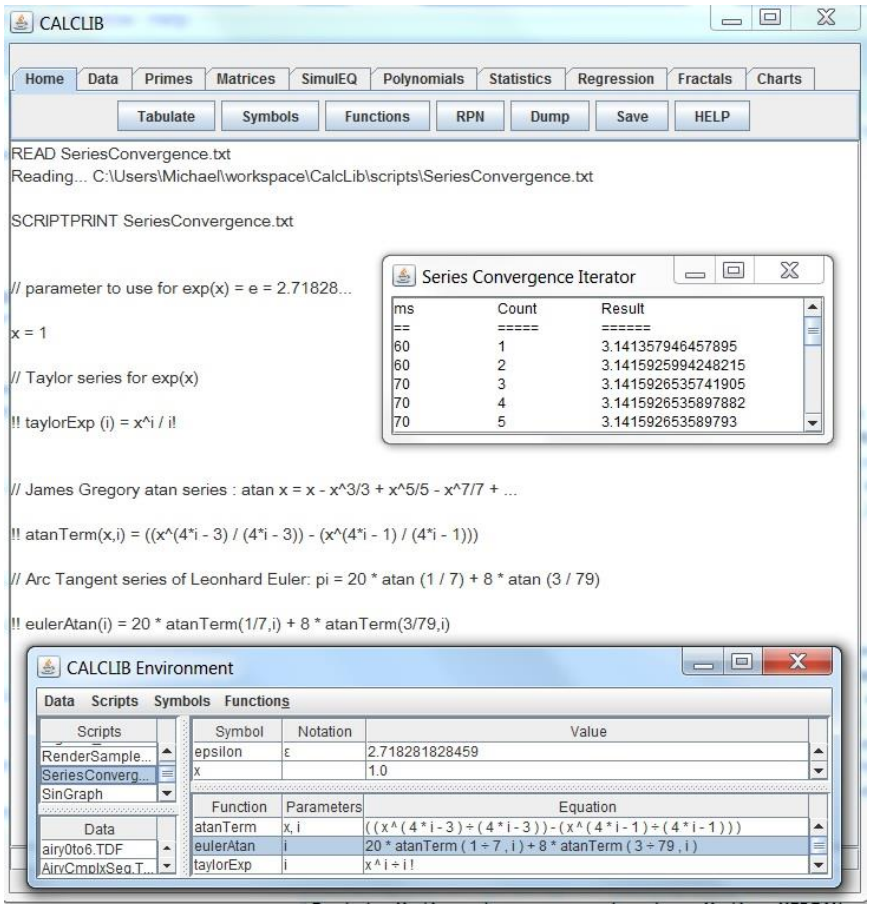
Integral approximation using the trapezoidal rule is the implemented mechanism of the "Approximate Integral" functionality found in the Function List pop-up menu. Each iteration of the approximation doubles the count of function evaluations being used to compute the integral.



# Approximate Series Convergence




A tool for monitoring series convergence is also found in the function menu. A function that computes a series term based on the iteration number must be defined in the Function table. The iterator will accept parameters for the range and increment of the iteration number. The table built will then show the iteration count and the resulting sum generated from that count of iterations.



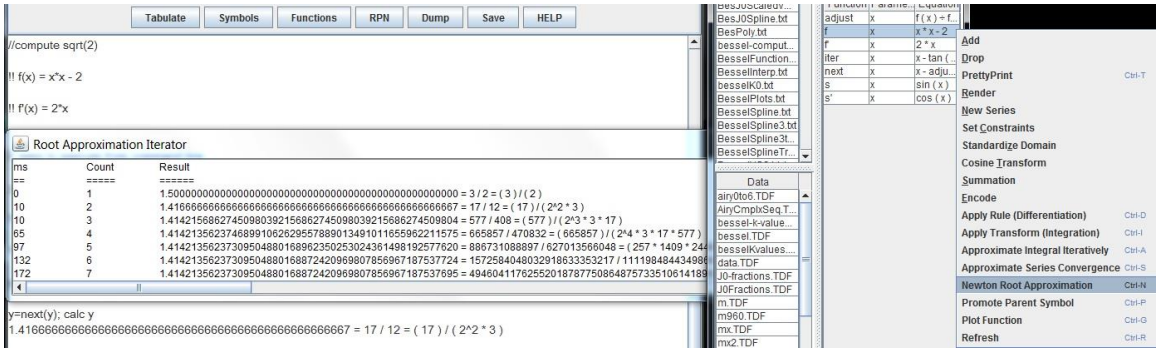

Another Example

This one shows the James Gregory ATAN series being used with the Leonhard Euler PI computation from integer ratios

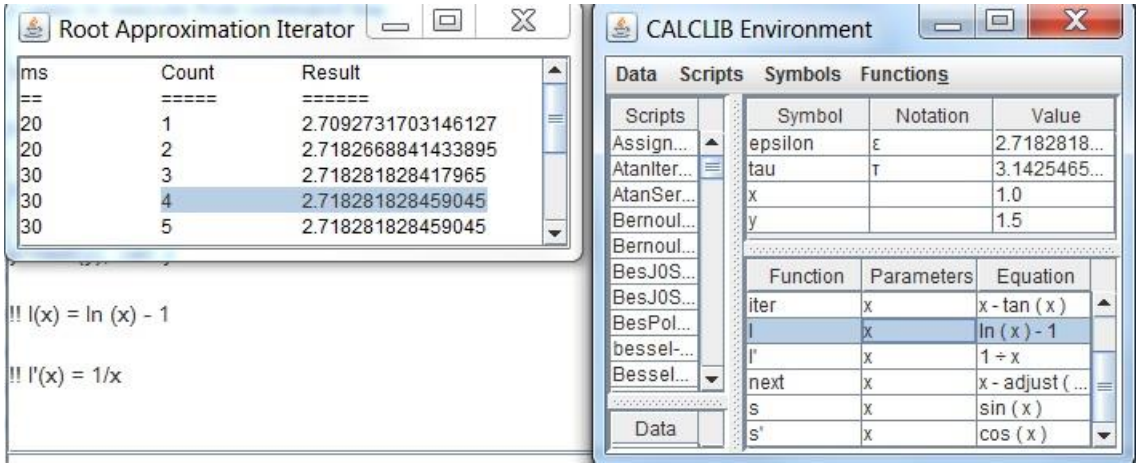
This shows that the convergence of 5 terms of the series (10 alternating sign terms) is adequate to generate 16 digits of PI correctly

# Root Approximation

Root approximation using the Newton's rule is the implemented mechanism of the "Newton Root Approximation" functionality found in the Function List pop-up menu. The selected function must have a corresponding derivative function declaration (F'(x) for a selected function declared as f(x)), the iterative adjustments are then f(x)/f'(x). The user is prompted for the initial value of the approximation. This selection will drive the root that is found when multiple roots exist for the equation.




The example below shows the computation of "e" from solution of the root of the equation "ln(x) - 1 = 0". Note that the solution is found in only four terms providing 16 correct digits of the value of the constant.



## LinAlgDemo.txt

```
// this is an example of functional import from an external library.
// Colt is a library of advanced mathematical functionality published as open source by CERN.
// one simple class is added to each package to make desired functions available for import.
// the LIBRARY command is used to recognize the items as available for import.
// the functions can then be used exactly as if they were internally realized.


IMPORT simple simple.TDF

// this script imports the Colt library functions
READ ColtLib.txt

// MatRpt is the interface entry to the Colt Algebra class
calc MatRpt simple

// having used the Colt library to execute SV decomposition,
// the by-products of the decomposition can be used as internal objects
// Verification of SVD matrices generated by ColtLib

S = GetSvdS()
V = GetSvdV()
U = GetSvdU()


PRETTYPRINT S 5
PRETTYPRINT V 5
PRETTYPRINT U 5


// to reconstitute the matrix the transpose of V is needed
Vt= TRANSPOSE(V)
prd = U * S * Vt

PRETTYPRINT prd

// the product should match the original
PRETTYPRINT simple
```

## OUTPUT:

```
READ ColtLib.txt
Reading... C:\workspace\MathFields\scripts\ColtLib.txt

READ LinAlgDemo.txt
Reading... C:\workspace\MathFields\scripts\LinAlgDemo.txt
Import from file:  C:\workspace\MathFields\data\simple.TDF
Import to matrix:  simple
 Columns per row:  3
      Rows read:  3

data read:
     1    -3     2
     8     6    -3
    -2     1     7

=EOD=
```

## OUTPUT FROM COLT LIBRARY:

A = 3 x 3 matrix
 1 -3  2
 8  6 -3
-2  1  7

cond : 3.395431804039953
det : 235.0
norm1 : 12.0
norm2 : 11.221729908134112
normF : 13.30413469565007
normInfinity : 17.0
rank : 3
trace : 14.0

density : 1.0
isDiagonal : false
isDiagonallyDominantByColumn : false
isDiagonallyDominantByRow : false
isIdentity : false
isLowerBidiagonal : false
isLowerTriangular : false
isNonNegative : false
isOrthogonal : false
isPositive : false
isSingular : false
isSkewSymmetric : false
isSquare : true
isStrictlyLowerTriangular : false
isStrictlyTriangular : false
isStrictlyUpperTriangular : false
isSymmetric : false
isTriangular : false
isTridiagonal : false
isUnitTriangular : false
isUpperBidiagonal : false
isUpperTriangular : false
isZero : false
lowerBandwidth : 2
semiBandwidth : 3
upperBandwidth : 2

```
--------------------------------------------------------------------------
LUDecompositionQuick(A) --> isNonSingular(A), det(A), pivot, L, U, inverse(A)
--------------------------------------------------------------------------
isNonSingular = true
det = 235.0
pivot = [1, 0, 2]

L = 3 x 3 matrix
 1     0        0
 0.125  1        0
-0.25 -0.666667 1

U = 3 x 3 matrix
8  6    -3
0 -3.75  2.375
0  0     7.833333

inverse(A) = 3 x 3 matrix
 0.191489 0.097872 -0.012766
-0.212766 0.046809  0.080851
 0.085106 0.021277  0.12766

---------------------------------------------------------------------
QRDecomposition(A) --> hasFullRank(A), H, Q, R, pseudo inverse(A)
---------------------------------------------------------------------
hasFullRank = true

H = 3 x 3 matrix
 1.120386 0        0
 0.963087 1.942239 0
-0.240772 0.334942 2

Q = 3 x 3 matrix
-0.120386  0.826813 -0.549442
-0.963087 -0.231508 -0.137361
 0.240772 -0.512624 -0.824163

R = 3 x 3 matrix
-8.306624 -5.176592  4.333891
 0        -4.382111 -1.24022
 0         0        -6.455947

pseudo inverse(A) = 3 x 3 matrix
 0.191489 0.097872 -0.012766
-0.212766 0.046809  0.080851
 0.085106 0.021277  0.12766

--------------------------------------------------------------------------
CholeskyDecomposition(A) --> isSymmetricPositiveDefinite(A), L, inverse(A)
--------------------------------------------------------------------------
isSymmetricPositiveDefinite = false

L = 3 x 3 matrix
 1        0 0
 8        0 0
-2 Infinity 0

inverse(A) = 3 x 3 matrix
 Infinity -Infinity NaN
-Infinity  Infinity NaN
 Infinity -Infinity NaN
```

```
--------------------------------------------------------------------
EigenvalueDecomposition(A) --> D, V, realEigenvalues, imagEigenvalues
--------------------------------------------------------------------
realEigenvalues = 1 x 3 matrix
3.870848 3.870848 6.258303
imagEigenvalues = 1 x 3 matrix
4.750436 -4.750436 0

D = 3 x 3 matrix
 3.870848 4.750436 0
-4.750436 3.870848 0
 0        0        6.258303

V = 3 x 3 matrix
 0.106036 -1.021459 0.381526
-1.342451  0.779749 0.009024
 0.564717 -0.044746 1.016627

--------------------------------------------------------------------
SingularValueDecomposition(A) --> cond(A), rank(A), norm2(A), U, S, V
--------------------------------------------------------------------
cond = 3.395431804039953
rank = 3
norm2 = 11.221729908134112

U = 3 x 3 matrix
-0.161259 -0.054457  0.985409
 0.898826 -0.420443  0.123855
-0.407564 -0.905683 -0.116748

S = 3 x 3 matrix
11.22173 0       0
 0       6.33641 0
 0       0       3.304949

V = 3 x 3 matrix
 0.699043 -0.253556  0.668616
 0.487373 -0.515272 -0.704956
-0.523265 -0.81866   0.236621
```

CALCLIB OUTPUT:

// Verification of SVD matrices generated by ColtLib

S =

```
   11.22173          0          0
         0    6.33641          0
         0          0    3.30495
```

V =

```
    0.69904   -0.25356    0.66862
    0.48737   -0.51527   -0.70496
   -0.52327   -0.81866    0.23662
```

U =

```
   -0.16126   -0.05446    0.98541
    0.89883   -0.42044    0.12385
   -0.40756   -0.90568   -0.11675
```

prd =

```
    0.9999999999999998   -3.000000000000001                    2
    7.9999999999999964    5.999999999999999   -2.9999999999999973
   -1.999999999999998     1.0000000000000042   7.0000000000000036
```

simple =

```
    1    -3     2
    8     6    -3
   -2     1     7
```

## LinAlgCmplxDemo.txt

```
// example of exchange of data with external libraries understanding representation within each


IMPORT simple simple.TDF

READ ColtLib.txt

calc MatRpt simple


D = GetEvdD()
V = GetEvdV()
Dr = GetEvdDreal()
Di = GetEvdDimag()


PRETTYPRINT D 5
PRETTYPRINT V 5

// real and imaginary parts are separated in the Colt library
// this simple expression allows further expression to use complex values
eigenvalues = Dr + i * Di

PRETTYPRINT eigenvalues  5
```

**OUTPUT:**

```
D =

    3.87085    4.75044         0
   -4.75044    3.87085         0
          0          0    6.2583



V =

    0.10604   -1.02146    0.38153
   -1.34245    0.77975    0.00902
    0.56472   -0.04475    1.01663



eigenvalues =

[
        (3.87085 + 4.75044*i)
        (3.87085 - 4.75044*i)
        6.2583
]
```

## legendre-iterator.txt

(run in domain of real numbers)

// using real numbers to build equations has the disadvantage of lost decimal places

READ legendre-iterator.txt
Reading... C:\workspace\MathFields\scripts\legendre-iterator.txt
Reading... C:\workspace\MathFields\scripts\legendre-script.txt

pn =

- 0.5 + 1.5 * x^2

pn =

- 1.5 * x + 2.5 * x^3

pn =

0.375 - 3.75 * x^2 + 4.375 * x^4

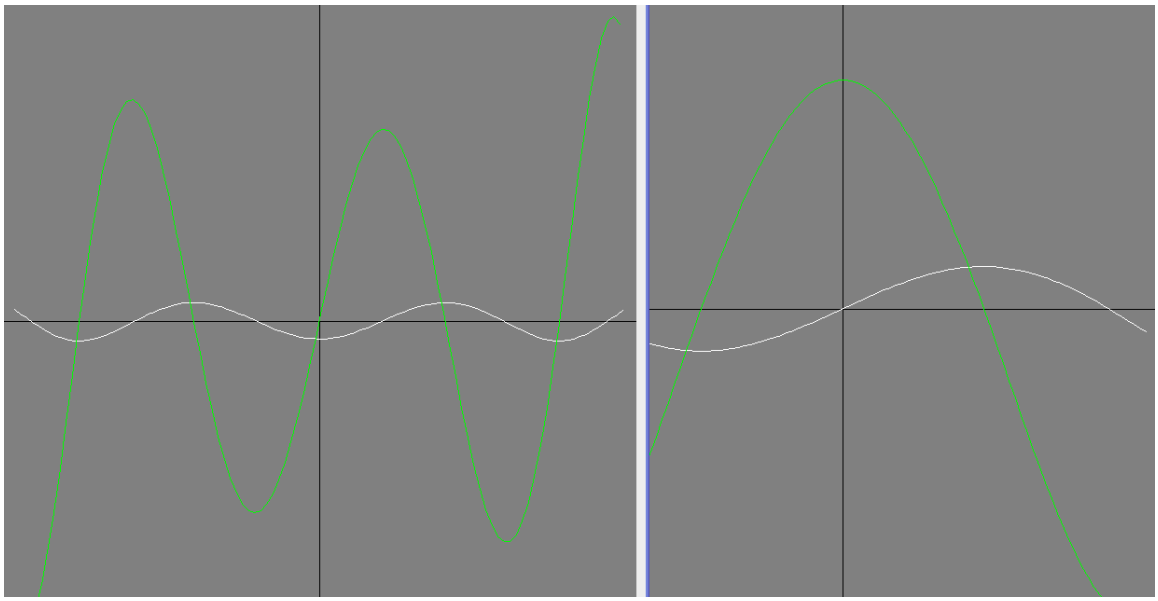
pn =

1.875 * x - 8.75 * x^3 + 7.875 * x^5

pn =

- 0.3125 + 6.5625 * x^2 - 19.6875 * x^4 + 14.4375 * x^6

Iteration 5 has completed
*** Maximum iteration count exceeded



## legendre-iterator.txt

(run in domain of prime factored number fractions)

// this example shows the advantage of lossless computation using prime factored integer ratios

READ legendre-iterator.txt
Reading... C:\workspace\MathFields\scripts\legendre-iterator.txt
Reading... C:\workspace\MathFields\scripts\legendre-script.txt

pn =

- 2^(-1) + 2^(-1) * 3 * x^2

pn =

- 2^(-1) * 3 * x + 2^(-1) * 5 * x^3

pn =

2^(-3) * 3 - 2^(-2) * 3 * 5 * x^2 + 2^(-3) * 5 * 7 * x^4

pn =

2^(-3) * 3 * 5 * x - 2^(-2) * 5 * 7 * x^3 + 2^(-3) * 3^2 * 7 * x^5

pn =

- 2^(-4) * 5 + 2^(-4) * 3 * 5 * 7 * x^2 - 2^(-4) * 3^2 * 5 * 7 * x^4 + 2^(-4) * 3 * 7 * 11 * x^6

Iteration 5 has completed
*** Maximum iteration count exceeded

// the loss of decimal places in this small example may seem unimportant,
// but the point is to consider extrapolation to larger systems of equations

## FactoredDemoPrimes.txt

// this demonstration shows the value of evaluating complicated expressions using ratios of integers

READ FactoredDemoPrimes.txt
Reading... C:\workspace\MathFields\scripts\FactoredDemoPrimes.txt
Reading... C:\workspace\MathFields\scripts\FactoredDemoPrimes.TXT

SCRIPTPRINT FactoredDemoPrimes.TXT

// the first example will compute SQRT(7) as a ratio of integer prime factors

x = 7
SCRIPTPRINT sqrtCompute.txt
SCRIPTPRINT sqrtIteration.txt
READ sqrtCompute.txt

n=1
SCRIPTPRINT ComputeNthBernoulli.txt
ITERATE 30 ComputeNthBernoulli.txt

Reading... C:\workspace\MathFields\scripts\sqrtCompute.txt

// looking for 10 decimal places of precision

TOLERANCE = 10^(-10)

// polynomial solutions "c - x^2 = 0" gives sqrt(c) as a solution

sqrt_polynomial = (x,0,-1)

polyprint sqrt_polynomial

// Newton's methods uses the first derivative for the solution

sqrt_poly_derivative = POLYDER sqrt_polynomial

polyprint sqrt_poly_derivative

// initial approximation

sqrtx = 1

ITERATE 10 SqrtIteration.txt

Reading... C:\workspace\MathFields\scripts\sqrtIteration.txt

// compute polynomial value

f = sqrt_polynomial +*^ sqrtx

// compute derivative value

fprime = sqrt_poly_derivative +*^ sqrtx

// iteration contribution computed

sqrt_iteration = f/fprime

```
// new value for this iteration

sqrtx = sqrtx - sqrt_iteration

// check status of result

sqrtx_squared = sqrtx ^ 2

// display intermediate results

PRETTYPRINT sqrt_iteration
PRETTYPRINT sqrtx_squared
PRETTYPRINT sqrtx

// look for convergence

ASSERT "Convergence Complete" TOLERANCE >|| sqrt_iteration

sqrt_polynomial =

            7 - x^2


sqrt_poly_derivative =

             - 2*x
```

Reading... C:\workspace\MathFields\scripts\SqrtIteration.txt

sqrt_iteration =

-3 = -3 / 1 = (  - 3 ) / ( 1 )

sqrtx_squared =

16 = 16 / 1 = ( 2^4 ) / ( 1 )

sqrtx =

4 = 4 / 1 = ( 2^2 ) / ( 1 )

sqrt_iteration =

1.125 = 9 / 8 = ( 3^2 ) / ( 2^3 )

sqrtx_squared =

8.265625 = 529 / 64 = ( 23^2 ) / ( 2^6 )

sqrtx =

2.875 = 23 / 8 = ( 23 ) / ( 2^3 )

sqrt_iteration =

0.220108695652174 = 81 / 368 = ( 3^4 ) / ( 2^4 * 23 )

sqrtx_squared =

7.048447837901701 = 954529 / 135424 = ( 977^2 ) / ( 2^8 * 23^2 )

sqrtx =

2.654891304347826 = 977 / 368 = ( 977 ) / ( 2^4 * 23 )

sqrt_iteration =

0.0091242601575364 = 6561 / 719072 = ( 3^8 ) / ( 2^5 * 23 * 977 )

sqrtx_squared =

7.000083252123423 = 3619494835009 / 517064541184 = ( 1902497^2 ) / ( 2^10 * 23^2 * 977^2 )

sqrtx =

2.64576704419029 = 1902497 / 719072 = ( 1902497 ) / ( 2^5 * 23 * 977 )

sqrt_iteration =

1.57330789204E-5 = 43046721 / 2736064645568 = ( 3^16 ) / ( 2^6 * 23 * 977 * 1902497 )

sqrtx_squared =

7.00000000024753 = 524023482149430384231502O9 / 74860497447271454620426 24
= ( 127^2 * 449^2 * 126947839^2 ) / ( 2^12 * 23^2 * 977^2 * 1902497^2 )

sqrtx =

2.6457513111113693 = 7238946623297 / 2736064645568
= ( 127 * 449 * 126947839 ) / ( 2^6 * 23 * 977 * 1902497 )

sqrt_iteration =

4.67788E-11 = 1853020188851841 / 396124518543135534331953 92
= ( 3^32 ) / ( 2^7 * 23 * 127 * 449 * 977 * 1902497 * 126947839 )

sqrtx_squared =

7 = 109840243933721649450816460877537675845612553993249 29 /
1569146341910309277868316057704782153153799650033664
= ( 31231^2 * 313727^2 * 10696531080798721^2 ) /
( 2^14 * 23^2 * 127^2 * 449^2 * 977^2 * 1902497^2 * 126947839^2 )

sqrtx =

2.6457513110645907
= 104804696428033056657448577 /
396124518543135534331953 92
= ( 31231 * 313727 * 10696531080798721 ) /
( 2^7 * 23 * 127 * 449 * 977 * 1902497 * 126947839 )

Script interrupted in iteration 6
Assertion "Convergence Complete" has been validated, TOLERANCE >|| sqrt_iteration
*** Script has terminated

```
// this example will compute the first 30 Bernoulli sequence numbers.
// using prime factored fractions complicated values can be computed without loss.


Reading... C:\workspace\MathFields\scripts\ComputeNthBernoulli.txt
bn = BERNOULLI(n)
PRETTYPRINT bn
n = n + 1


bn =

        0.5 = 1 / 2 = ( 1 ) / ( 2 )


bn =

        0.1666666666666667 = 1 / 6 = ( 1 ) / ( 2 * 3 )


bn =

        0


bn =

        -0.0333333333333334 = -1 / 30 = (  - 1 ) / ( 2 * 3 * 5 )


bn =

        0


bn =

        0.0238095238095239 = 1 / 42 = ( 1 ) / ( 2 * 3 * 7 )


bn =

        0


bn =

        -0.0333333333333334 = -1 / 30 = (  - 1 ) / ( 2 * 3 * 5 )


bn =

        0


bn =

        0.0757575757575758 = 5 / 66 = ( 5 ) / ( 2 * 3 * 11 )
```

bn =

0

bn =

-0.2531135531135532 = -691 / 2730 = (  - 691 ) / ( 2 * 3 * 5 * 7 * 13 )

bn =

0

bn =

1.1666666666666667 = 7 / 6 = ( 7 ) / ( 2 * 3 )

bn =

0

bn =

-7.092156862745098 = -3617 / 510 = (  - 3617 ) / ( 2 * 3 * 5 * 17 )

bn =

0

bn =

54.971177944862156 = 43867 / 798 = ( 43867 ) / ( 2 * 3 * 7 * 19 )

bn =

0

bn =

-529.1242424242424 = -174611 / 330 = (  - 283 * 617 ) / ( 2 * 3 * 5 * 11 )

bn =

0

bn =

6192.123188405797 = 854513 / 138 = ( 11 * 131 * 593 ) / ( 2 * 3 * 23 )

bn =

0

bn =

-86580.25311355312 = -236364091 / 2730 = (  - 103 * 2294797 ) / ( 2 * 3 * 5 * 7 * 13 )

bn =

0

bn =

1425517.1666666667 = 8553103 / 6 = ( 13 * 657931 ) / ( 2 * 3 )

bn =

0

bn =

-2.7298231067816094E7 = -23749461029 / 870 = (  - 7 * 9349 * 362903 ) / ( 2 * 3 * 5 * 29 )

bn =

0

bn =

6.015808739006424E8 = 8615841276005 / 14322 = ( 5 * 1721 * 1001259881 ) / ( 2 * 3 * 7 * 11 * 31 )

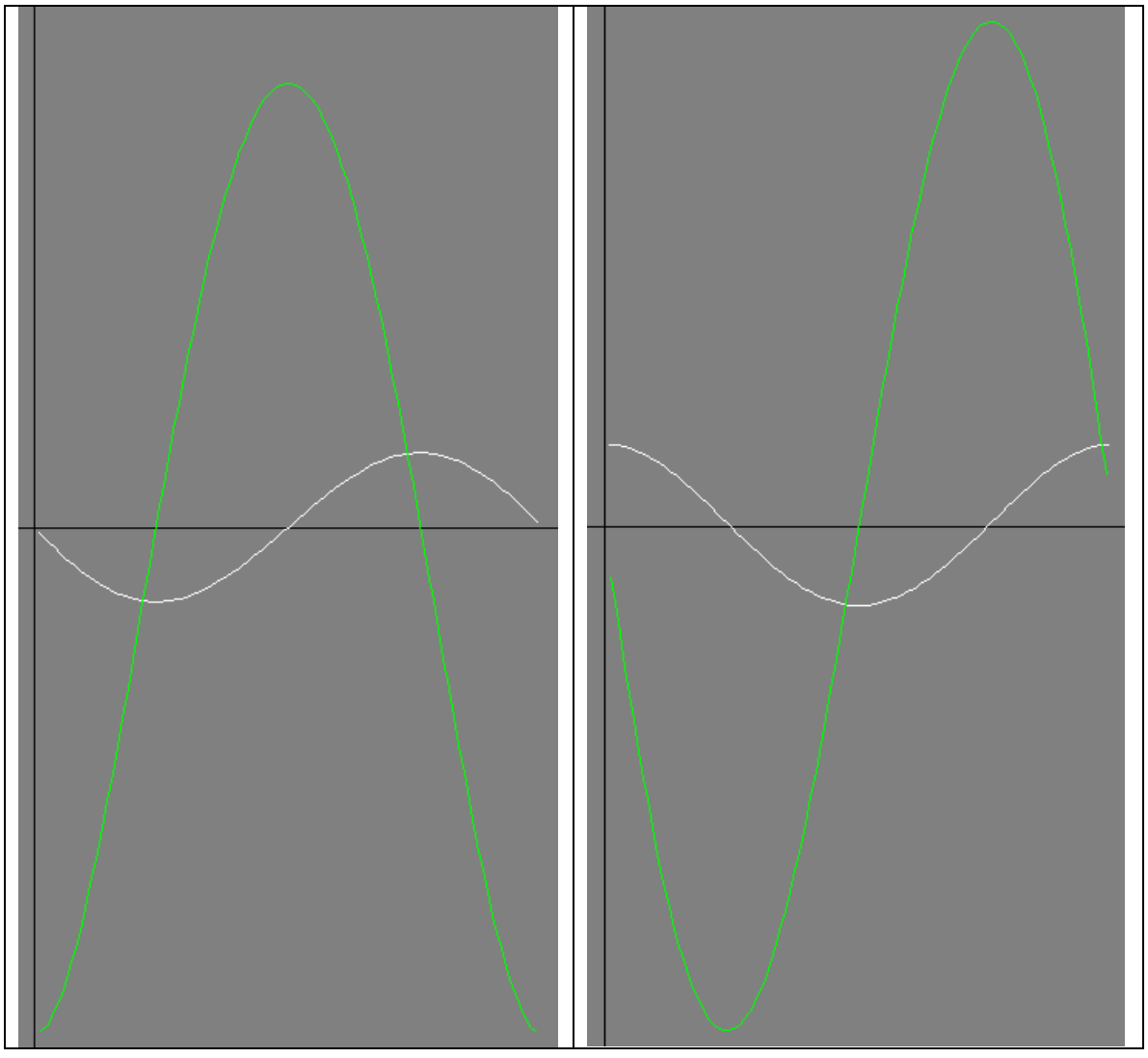
Iteration 30 has completed
*** Maximum iteration count exceeded

// the correct values of the numerators and denominators of these ratios are published online:
// https://oeis.org/ (A027641 / A027642 in OEIS);

## BernoulliPolynomialDerivation.txt

```
// compute a coefficient of the Nth Bernoulli polynomial
!! BPolyCoef  ( n,  k )  =  ( n ## k ) * BERNOULLI  (0,  n - k)

// compute complete set of Nth Bernoulli polynomial coefficients
!! BernPoly  ( n )  =  [0 <= k <= n]  ( BPolyCoef  (n,  k)  )
```

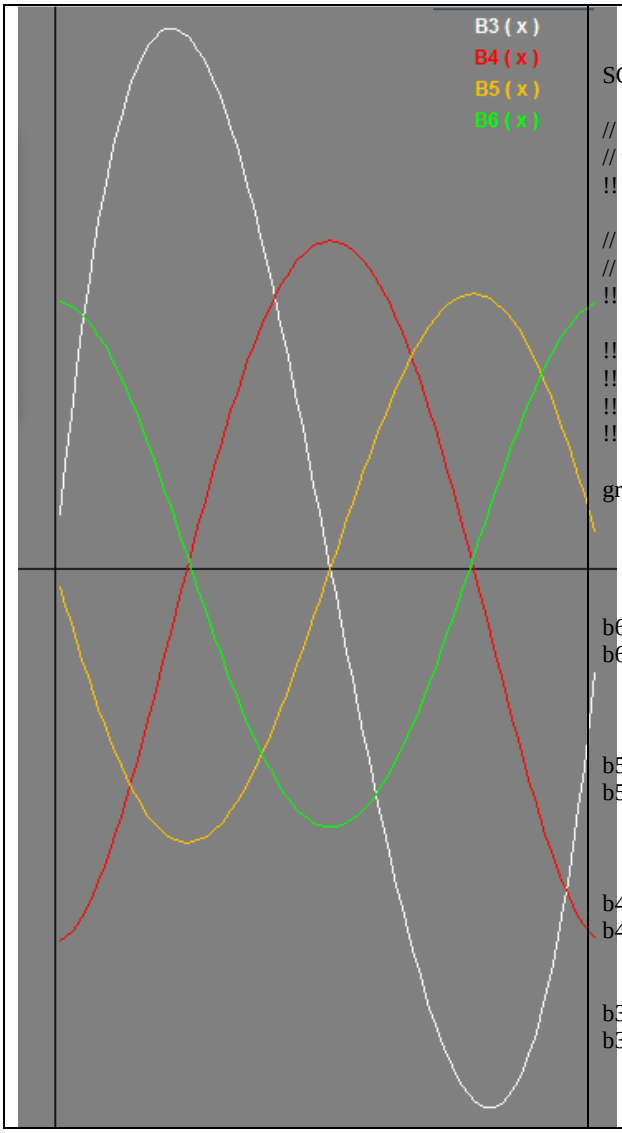
| graph [ 0 <  x <  1 <> 1/100 ]  ( BernPoly(5)  +*^ x  ) | graph [ 0 <  x <  1 <> 1/100 ]  (  BernPoly(6)  +*^ x  ) |
|---|---|



## Functions

B6 (x)  = 2^(-1) * 3^(-1) * 7^(-1) - 2^(-1) * x^2 + 2^(-1) * 5 * x^4 - 3 * x^5 + x^6
B5 (x)  = - 2^(-1) * 3^(-1) * x + 3^(-1) * 5 * x^3 - 2^(-1) * 5 * x^4 + x^5

## Bernoulli Polynomials 3, 4, 5, and 6 overlaid




```
SCRIPTPRINT BernoulliPolynomialDerivation.txt

// compute a coefficient of
// the Nth Bernoulli polynomial
!! BPolyCoef (n, k) = (n##k) * BERNOULLI (0, n-k)

// compute complete set of
// Nth Bernoulli polynomial coefficients
!! BernPoly (n) = [0 <= k <= n] (BPolyCoef (n, k))

!! B3(x) = BernPoly(3) +*^ x
!! B4(x) = BernPoly(4) +*^ x
!! B5(x) = BernPoly(5) +*^ x
!! B6(x) = BernPoly(6) +*^ x

graph [ 0 < x < 1<>1/100] (B3(x), B4(x), B5(x), B6(x))
```

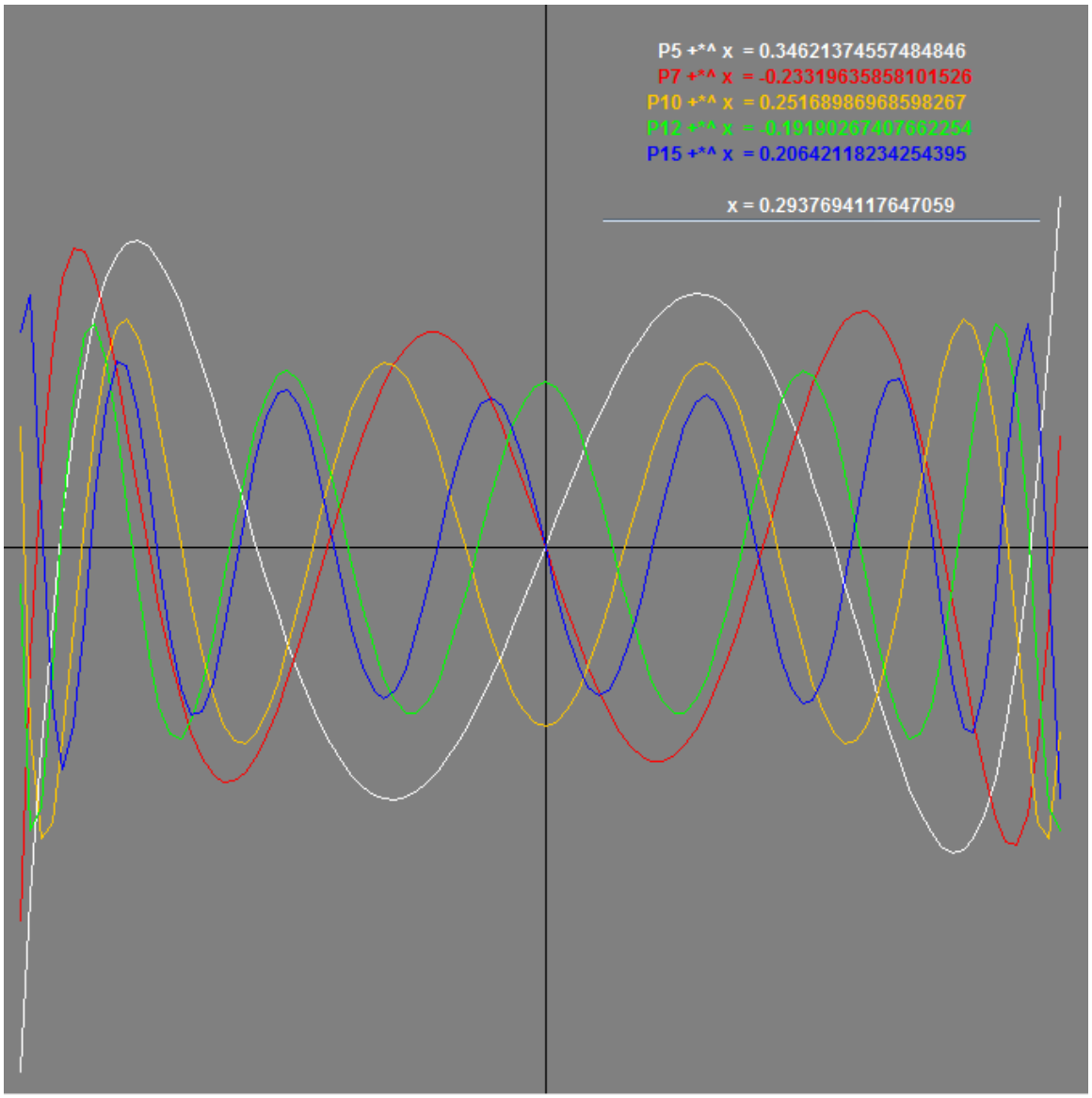
```
b6 = BernPoly(6) ;  POLYPRINT b6
b6 =  2^(-1) * 3^(-1) * 7^(-1) - 2^(-1) * x^2 +
          2^(-1) * 5 * x^4 - 3 * x^5 + x^6
```

```
b5 = BernPoly(5) ;  POLYPRINT b5
b5 = - 2^(-1) * 3^(-1) * x + 3^(-1) * 5 * x^3
          - 2^(-1) * 5 * x^4 + x^5
```

```
b4 = BernPoly(4) ;  POLYPRINT b4
b4 =   - 2^(-1) * 3^(-1) * 5^(-1) + x^2 - 2 * x^3 + x^4
```

```
b3 = BernPoly(3) ; POLYPRINT b3
b3 =  2^(-1) * x - 2^(-1) * 3 * x^2 + x^3
```

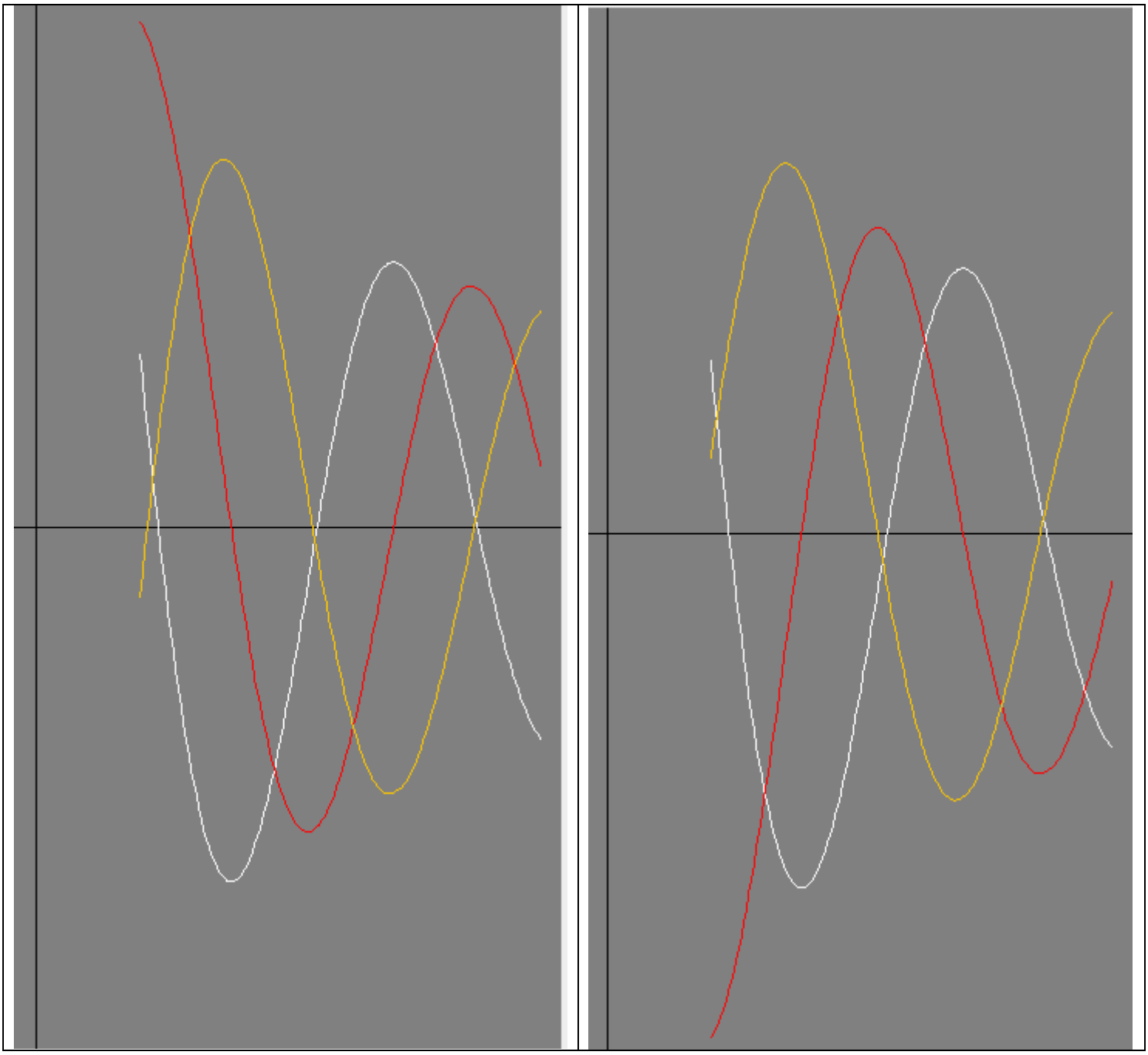
Other Overlaid Series Plots



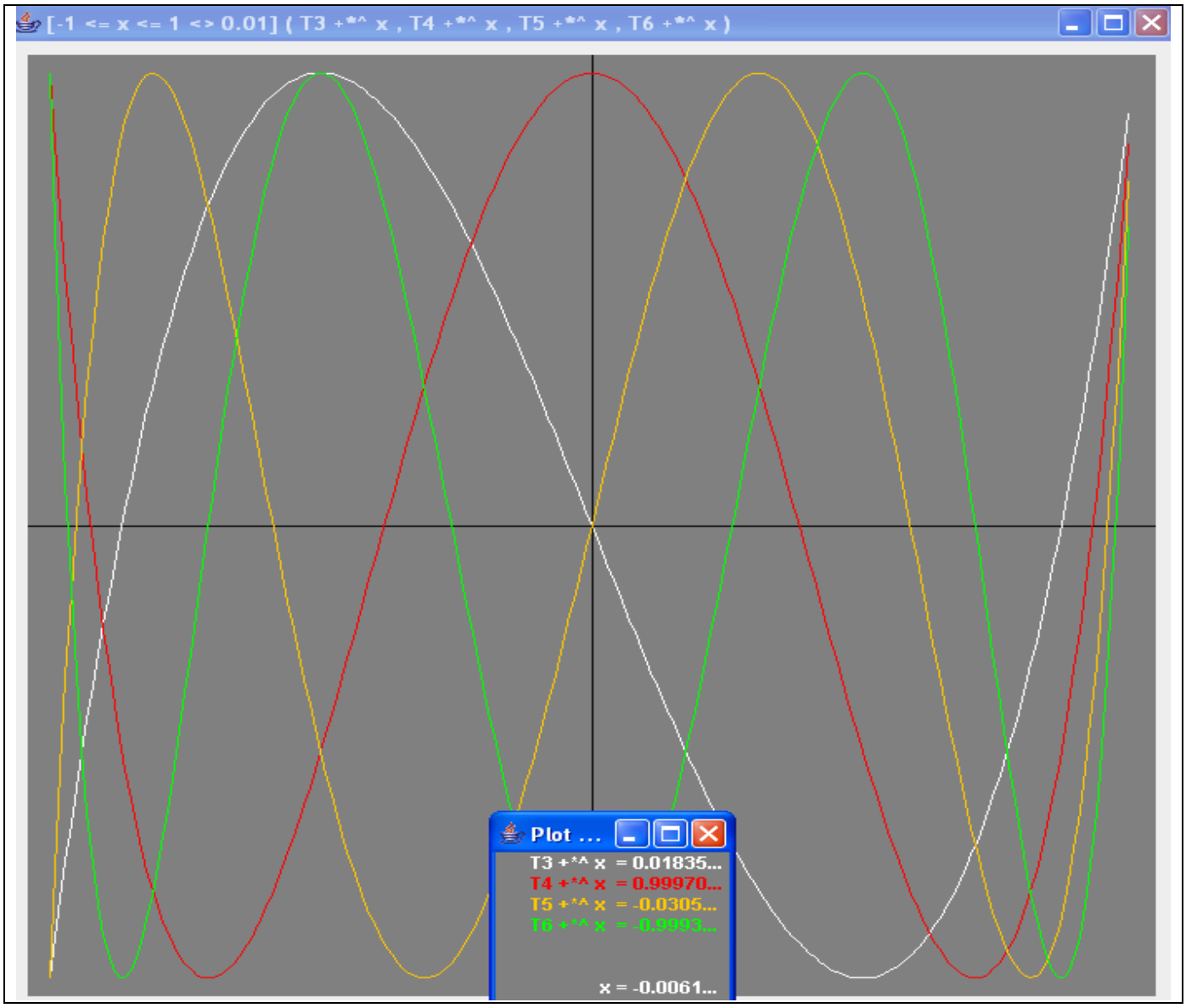

**Legendre Polynomials 5, 7, 10, 12, and 15**

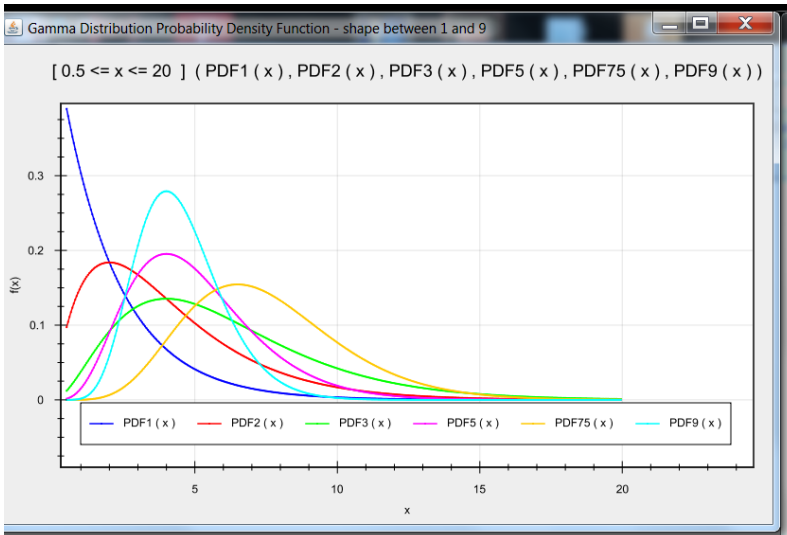
| Legendre-Iterations.txt | Legendre-Init.txt |
|---|---|
| READ legendre-init.txt<br>READ legendre-script.txt; P2 = pn<br>READ legendre-script.txt; P3 = pn<br>READ legendre-script.txt; P4 = pn<br>READ legendre-script.txt; P5 = pn<br>READ legendre-script.txt; P6 = pn<br>READ legendre-script.txt; P7 = pn<br>READ legendre-script.txt; P8 = pn | x = (0,1); p0 = (1); p1 = x; n = 1<br>pn = p1; pnm1 = p0<br><br>Legendre-Script.txt<br>m = 2*n + 1; invNp1 = 1 / (n + 1); pnp1 = m*CONV(x,pn)<br>pnm1 = APPEND (pnm1, 0, 0); pnm1 = n * pnm1<br>pnp1 = pnp1 - pnm1; pnp1 = pnp1 * invNp1<br>pnm1 = pn; pn = pnp1; n = n + 1<br>POLYPRINT pn |

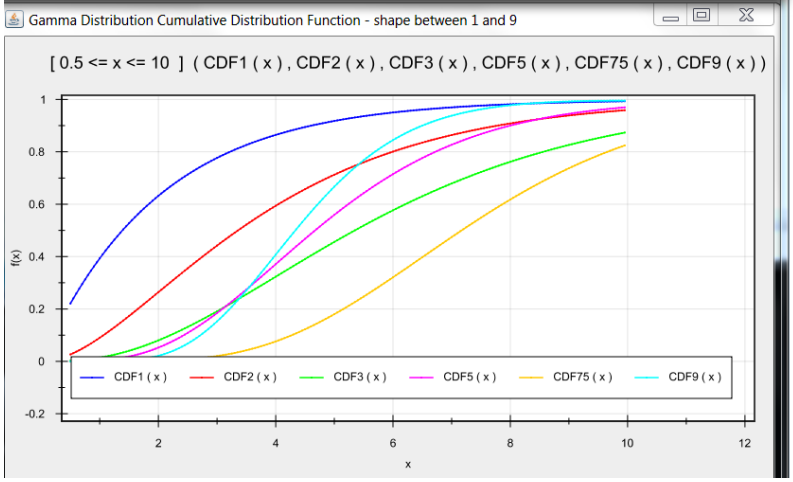
| Bessel J0, J1, and Y1<br>(using imported Colt library functions) | graph [2<x<10 <> 0.05]<br>(J0(x), J0'(x <> 0.01), J0"(x <> 0.01)) |
| --- | --- |



## Chebyshev Polynomials






Gamma Equations

$$\Gamma(s) = \int_0^{\infty} t^{s-1} \cdot e^{-t} \cdot \partial t$$

GAMMA as an integral

$$\gamma(s,x) = \int_0^{x} t^{s-1} \cdot e^{-t} \cdot \partial t$$

Lower incomplete gamma as an integral

$$\Gamma(s+1,x) = s \cdot \Gamma(s,x) + x^s \cdot e^{-x}$$

$$\gamma(s+1,x) = s \cdot \gamma(s,x) - x^s \cdot e^{-x}$$

$$\gamma(s,x) = \Gamma(s) - \Gamma(s,x)$$

GAMMA identities

$$\mathrm{Gstar}(s,z) = e^{-z} \cdot \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(s+k+1)}$$

gamma* limiting function

$$\gamma(s,z) = z^s \cdot \Gamma(s) \cdot \mathrm{Gstar}(s,z)$$

Lower incomplete gamma as an infinite series

$$E(X) = k \cdot \theta$$

$$E(ln(X)) = \psi(k) + ln(\theta)$$

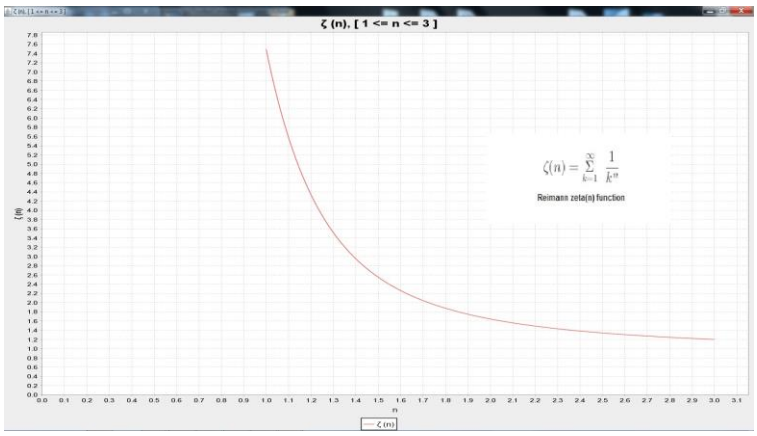
Gamma Distribution of random variable X, with shape k and scale theta

$$\mathrm{PDF}(x,k,\theta) = \frac{x^{k-1} \cdot exp(\frac{-x}{\theta})}{\Gamma(k) \cdot \theta^k}$$

$$\mathrm{CDF}(x,k,\theta) = \frac{\gamma(k,\frac{x}{\theta})}{\Gamma(k)}$$

Probability Density and Cumulative Distribution Functions for Gamma Distribution

# Zeta Function Study




Simple plot of Zeta on Real numbers domain

**// ZetaGeneric.txt**

```
N = 1000

// log of first N integers
// saves repeated calculations

logN = [0 < n <= N] ( ln n )
```

**// implementation of exp comes from configuration file**

**!! GenericZeta (z) = SIGMA [0 <= n < N] ( exp (- z * logN#n) )**

**// ZetaGenericPlot.txt**

```
READ ZetaGeneric.txt

//
// simple plot of specified domain
//
graph [1.1 < t < 5 <> 0.1 ] ( GenericZeta (t) )
```

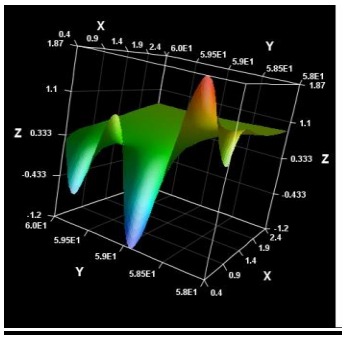
**// ZetaComplex.txt**

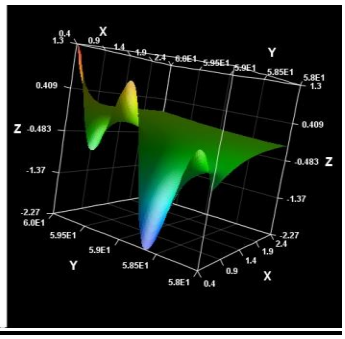
```
READ ZetaGeneric.txt

//
// domain of Reimann hypothesis
//
!! dom (t) = 1/2 + t*i

//
// use zeta implemented in generic script
//
rng = [0 <= x <= 30 <> 0.1] ( GenericZeta (dom (x)) )

//
// use code from CalcLib build
//
//rng = [-10 <= x <= 10 <> 0.1] ( zeta (dom (x)) )
```

```
//
// real and imaginary parts
//   separated into arrays
//
RE = [0 <= n < LENGTH rng] ( Re (rng#n) )
IM = [0 <= n < LENGTH rng] ( Im (rng#n) )

//
// dump vectors of real and imag parts
//
calc RE
calc IM

//
// plot both parts
//
graph [0 <= n < LENGTH rng] (RE#n,IM#n)
```

Plots of Zeta on Complex Plane


RI plot over domain
(1/2 + x*i)  [13<x<26]

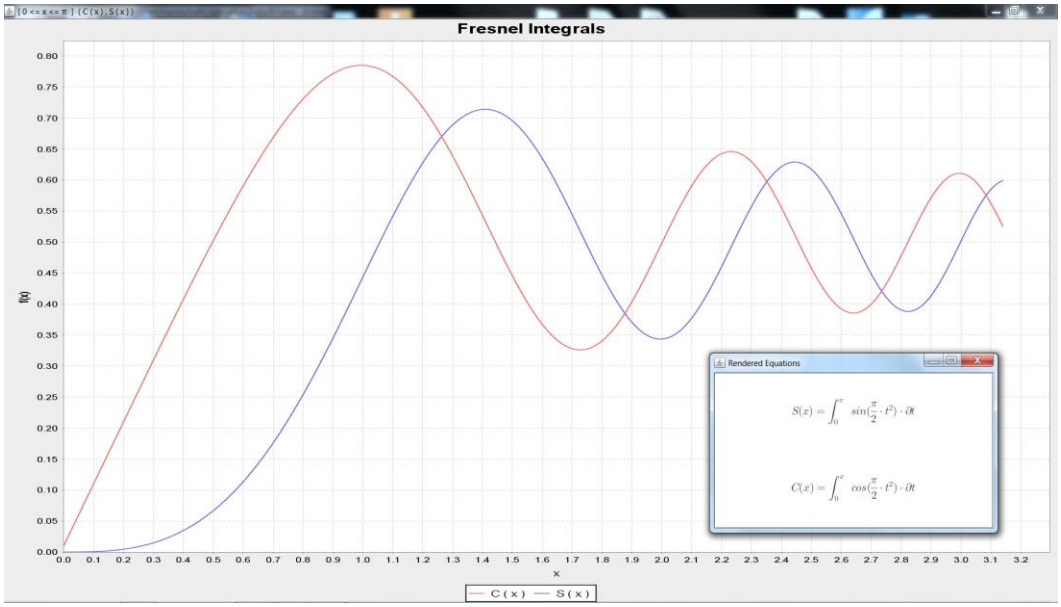


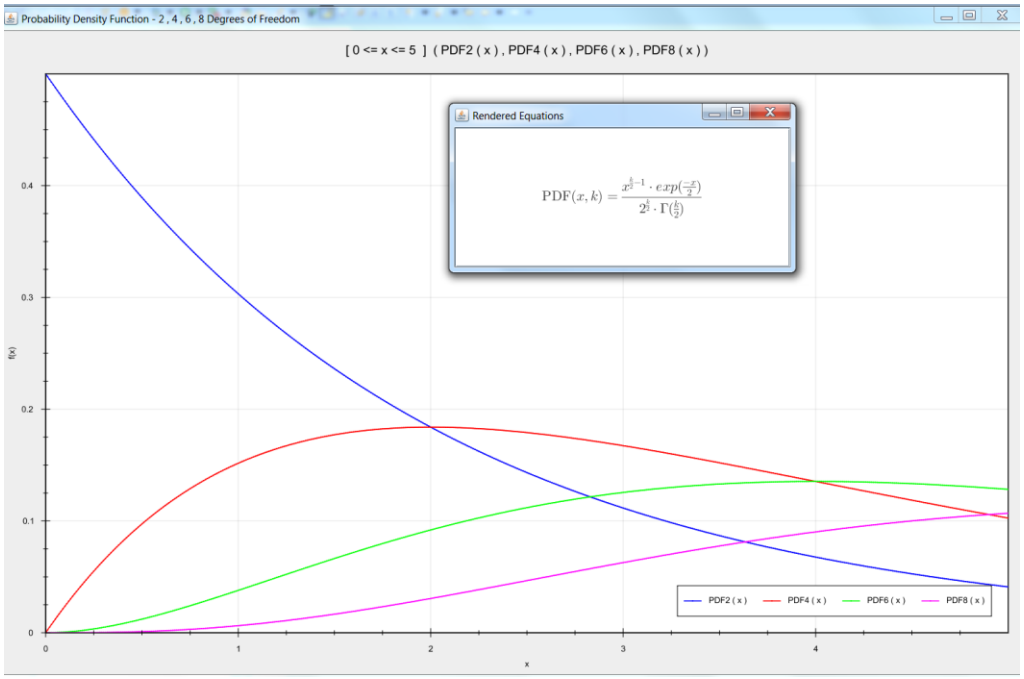




3D contour plots of

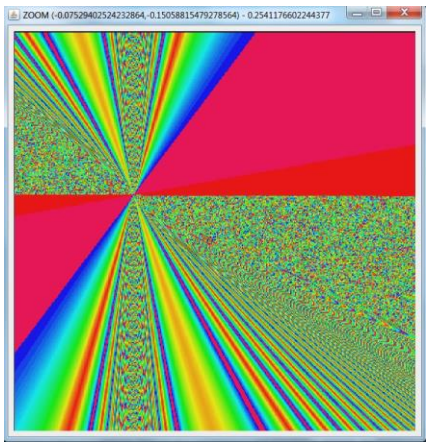
Re (zeta (x + i*y))

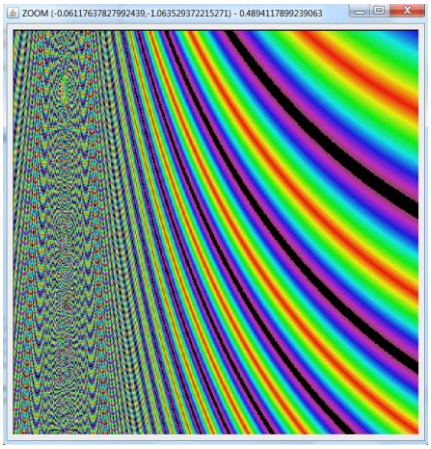
Im (zeta (x + i*y))

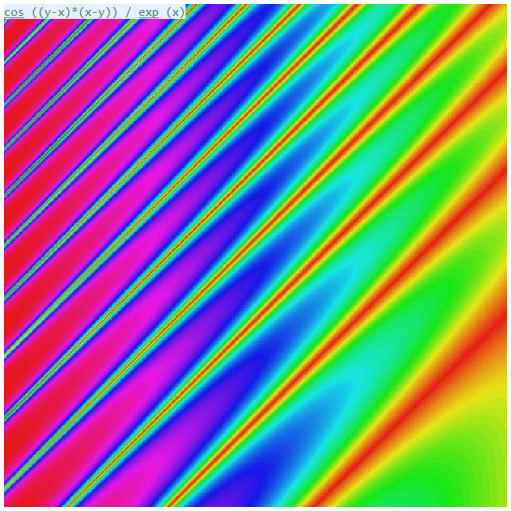

[ 0 <= x <= π ] ( C ( x ) , S ( x ) )
Fresnel Integrals
f(x)
x
C ( x )
S ( x )
Rendered Equations
$S(x) = \int_0^x sin(\frac{\pi}{2} \cdot t^2) \cdot \partial t$
$C(x) = \int_0^x cos(\frac{\pi}{2} \cdot t^2) \cdot \partial t$

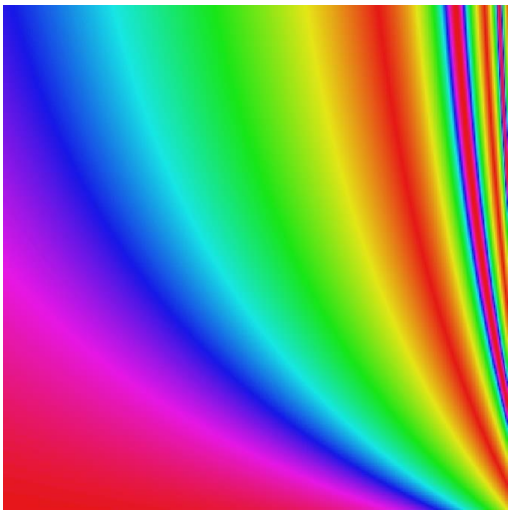

Probability Density Function - 2 , 4 , 6 , 8 Degrees of Freedom
[ 0 <= x <= 5 ] ( PDF2 ( x ) , PDF4 ( x ) , PDF6 ( x ) , PDF8 ( x ) )
f(x)
x
PDF2 ( x )
PDF4 ( x )
PDF6 ( x )
PDF8 ( x )
Rendered Equations
$\mathrm{PDF}(x,k) = \frac{x^{\frac{k}{2}-1} \cdot exp(\frac{-x}{2})}{2^{\frac{k}{2}} \cdot \Gamma(\frac{k}{2})}$

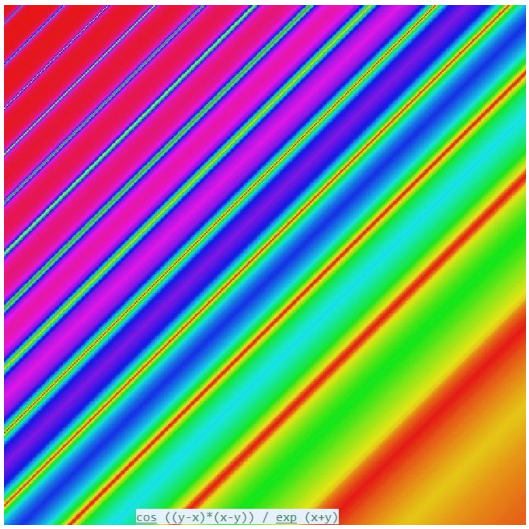
# Contour Plots



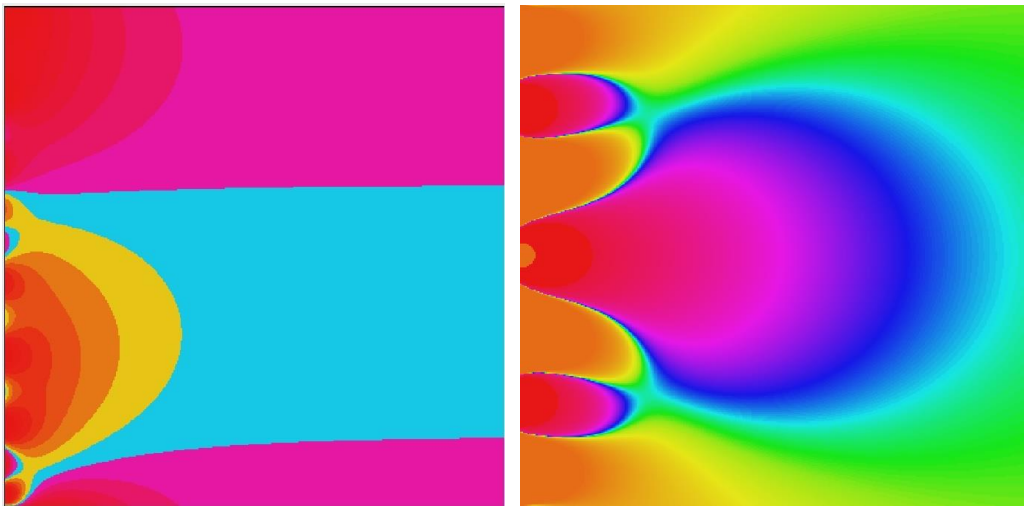



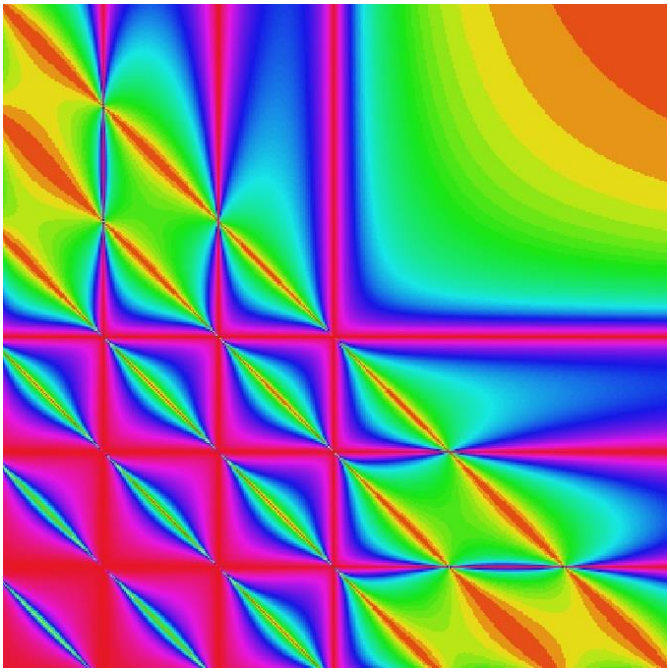





## Zeta Function Translation of Complex plane

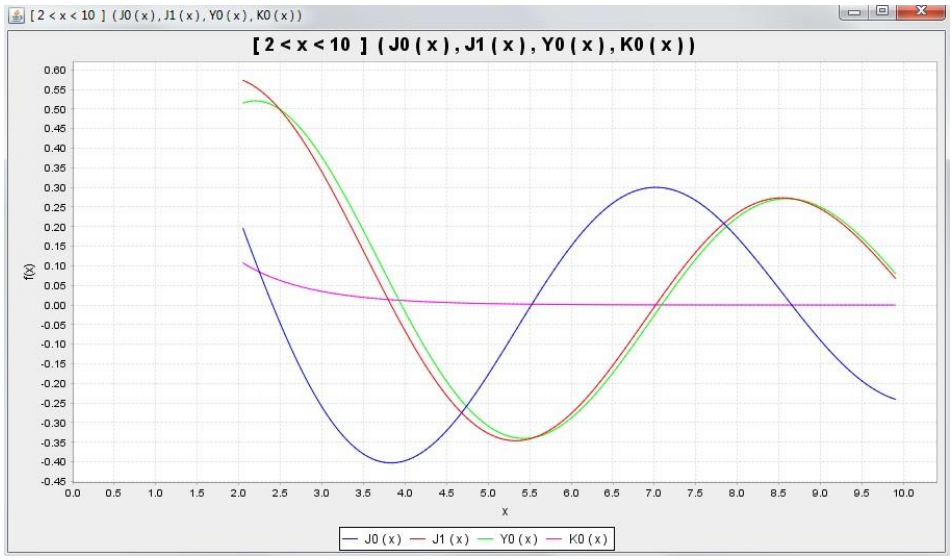


## Beta Function Contour Plot

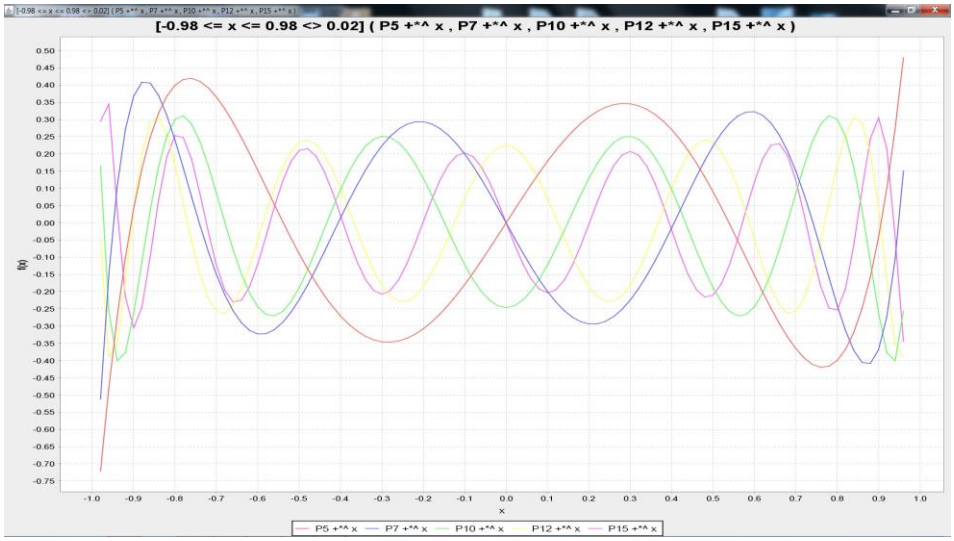


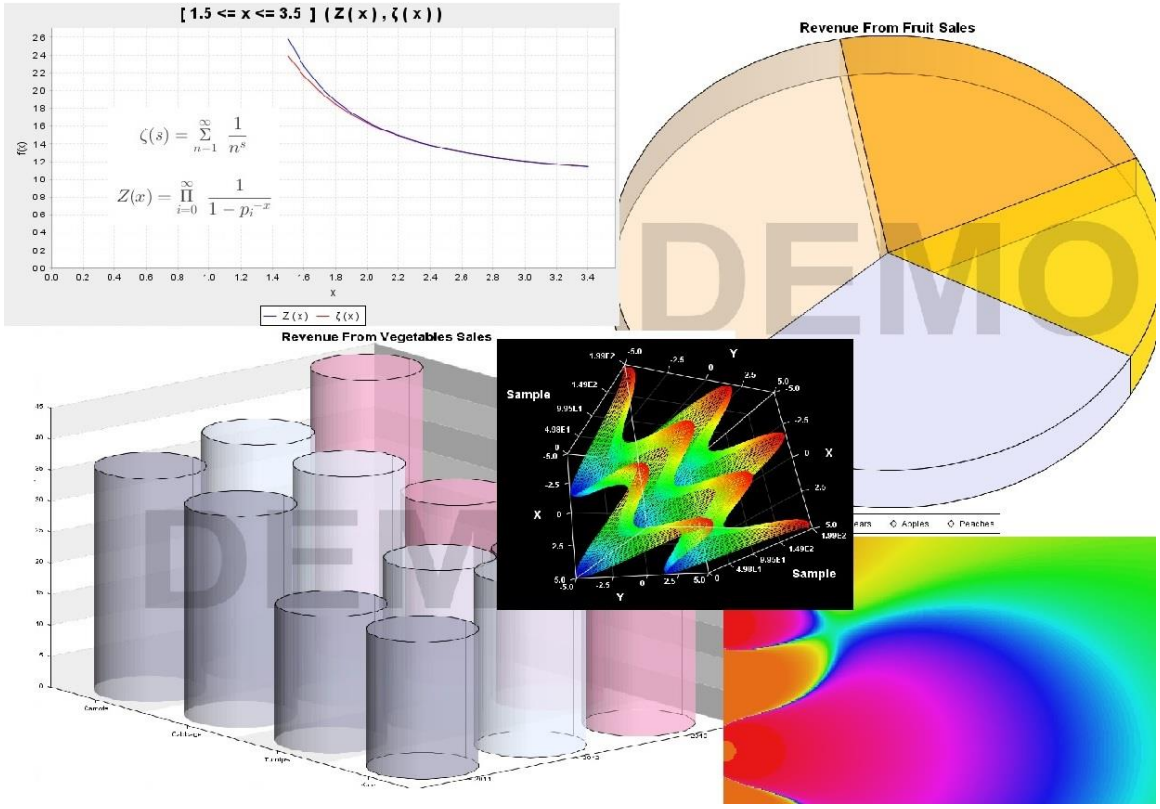
# Alternative Graphics Engines

The CalcLib engine has public interfaces for external calculation libraries, MathML rendering engines, and for graphics libraries. A library that can generate plots can be referenced via the application configuration and the interface to provide for substitution of the plot engine. The plot of the Bessel functions below was generated by JFreeChart using the CalcLib graphics library interface.



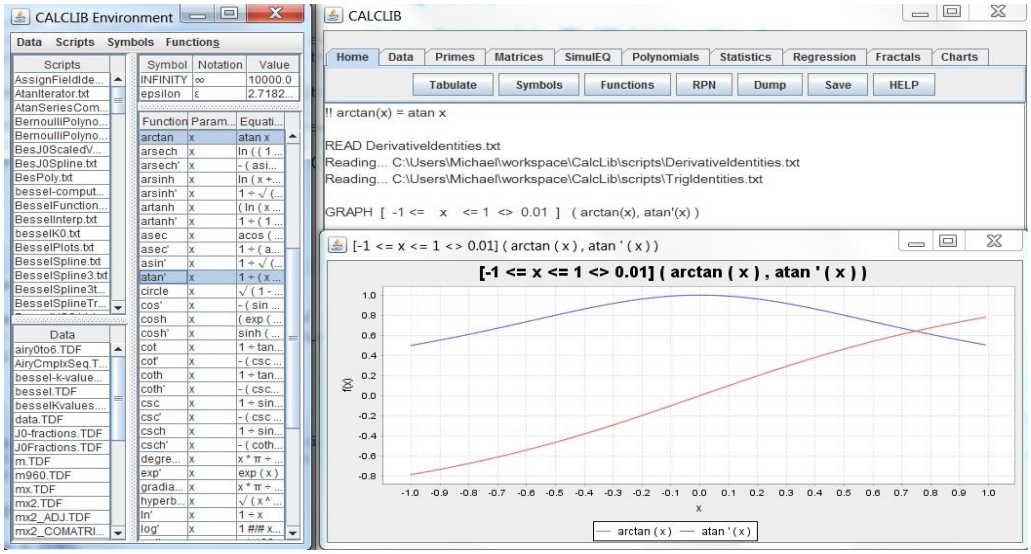

This is the same plot request as shown above **(Legendre Polynomials 5, 7, 10, 12, and 15)** with the CalcLib configuration set to use the JFreeChart implementation. The override for each type of chart is available in the charting interface, implementations can choose to override some and optionally not others.



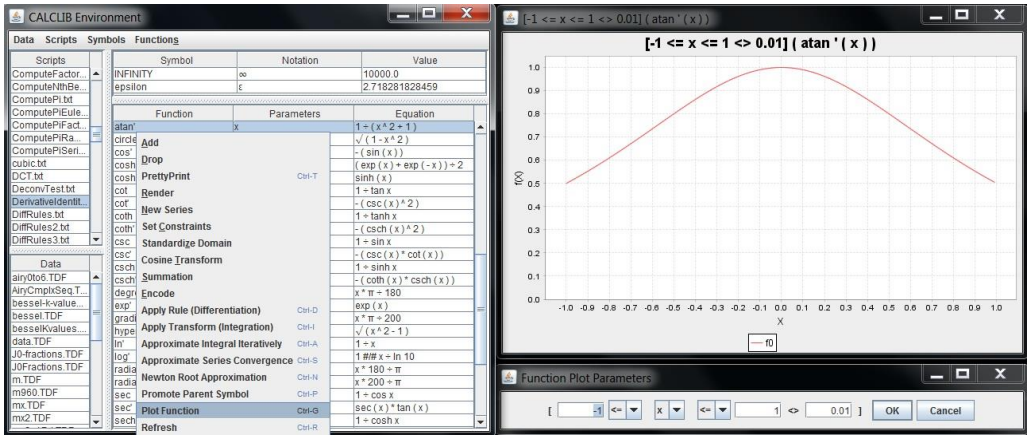



Diverse forms of sophisticated graphics engines

The plotting engine can be invoked from the Function menu or directly from the command line

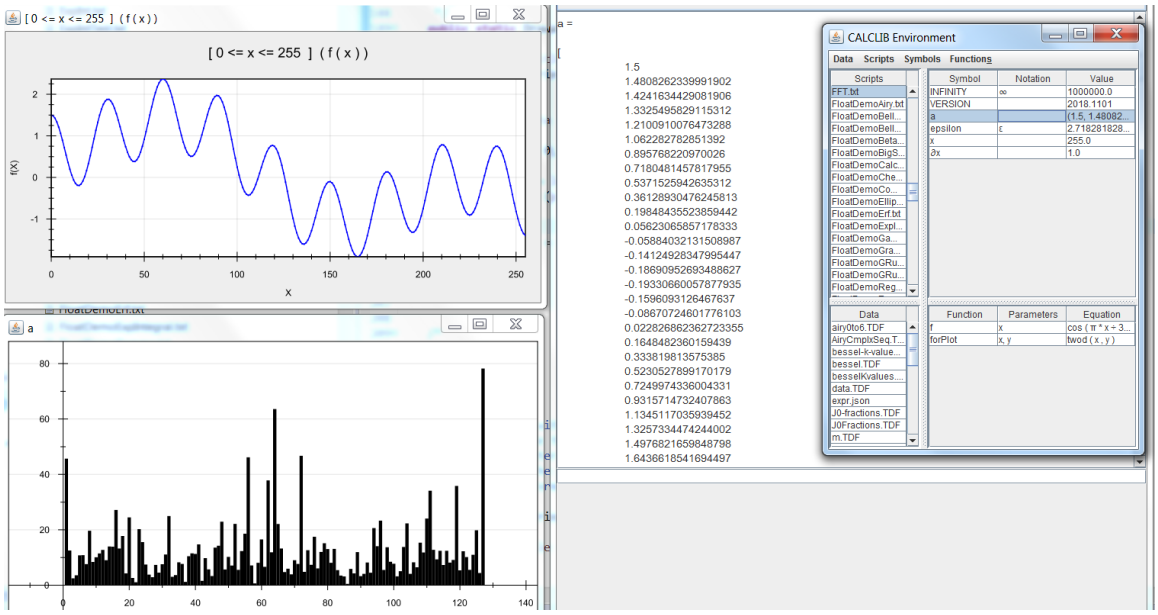



Using Plot Function menu item causes the Function Plot Parameters form to be displayed.
The Low bound, Hi bound, and Increment are taken from the form and used to parameterize the function plot.
The Increment value will determine the count of points calculated for the display, and hence the smoothness.
Typical trade-offs are seen between plot smoothness versus memory usage and calculation time.



FFT – Fast Fourier Transform

$$f(x) = cos(\frac{\pi \cdot x}{300}) + 0.3 \cdot cos(\frac{\pi \cdot x}{150}) - 0.7 \cdot cos(\frac{\pi \cdot x}{75}) + 0.9 \cdot cos(\frac{\pi \cdot x}{15})$$

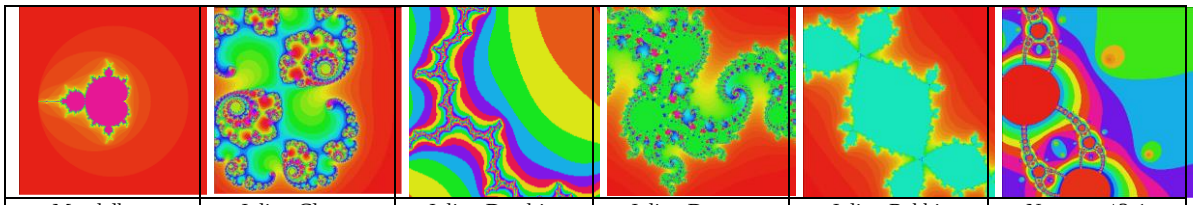



FFT.txt

```
!!f(x) = cos (pi*x/300) + 0.3 * cos(pi*x/150)
             - 0.7 * cos (pi*x/75) + 0.9 * cos (pi*x/15)

RENDERF f

calc f 1

a = [0 <= x <= 255] (f(x))

PRETTYPRINT a

GRAPH a

FFT a
```

## Other Plot Types

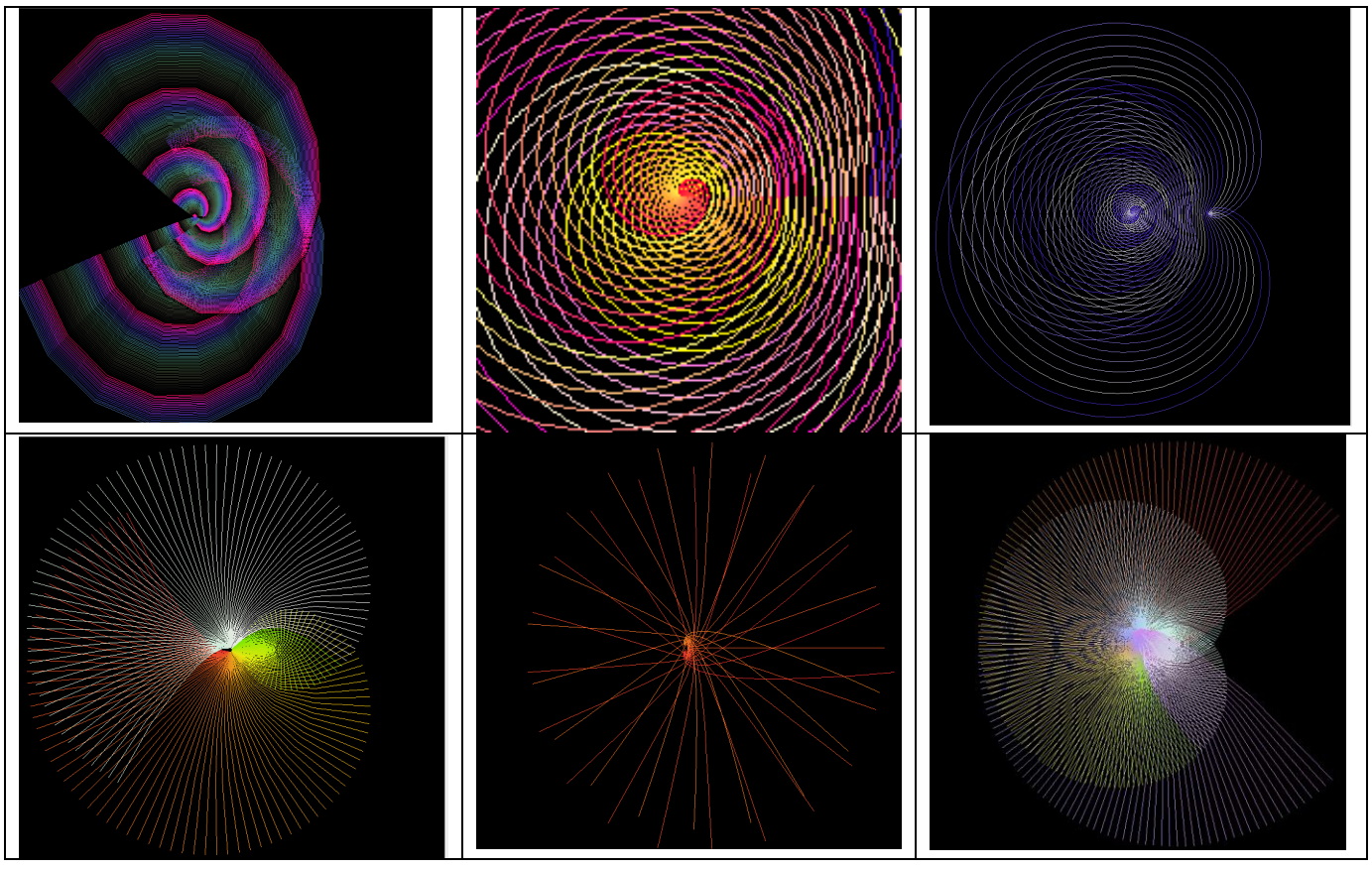
### Fractal Plots



| Mandelbrot | Julia - Clown | Julia - Dendrite | Julia - Dragon | Julia - Rabbit | Newton z^3-1 |
|---|---|---|---|---|---|

### Polar Plots

Angular – with radial distance held constant the angle moves from –pi to pi



Radial – with the angle held constant the distance extends from -2 to 2

# Rendering Expression Notations

The CalcLib engine provides for generation of MathML from command line expression notations. A public interface is provided which enables the graphic rendering of the MathML to common display formats. The examples below were generated with the JMTeX library running from an implementation of the public interface.

Samples of CalcLib RENDER displays

$$a \cdot x^2 + b \cdot x + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$$

Order 2 polynomial and the Quadratic equation

render a*x^2+b*x+c = 0
render x = (- b +|- sqrt (b^2 - 4*a*c)) / (2*a)

(Source: QuadraticRender.txt)

RENDER "Order 2 polynomial and the Quadratic equation" TOP

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

The SIGMA notation for series of terms (Taylor series for e^x)

render SIGMA [0 <= n <= INFINITY] (x^n/n!)

RENDER "The SIGMA notation for series of terms (Taylor series for e^x)" TOP

$$\sum_{x=0}^{1\Delta 0.1} \sqrt{1 - x^2}$$

An added convention for non-standard increment (delta other than standard 1 default)

render SIGMA [0 <= x <= 1 <> 0.1] ( sqrt (1-x^2) )

RENDER "An added convention for non-standard increment (delta other than standard 1 default)" TOP

$$\prod_{i=j+1}^{n+1} x_i$$

The PI notation for series of factors

render PI [j+1 <= i <= n+1] (x#i)

RENDER "The PI notation for series of factors" TOP

$$\left\langle {n \atop k} \right\rangle = \frac{n!}{k! \cdot (n-k)!}$$

$$\sum_{k=0}^{n} \left(k \cdot \left\langle {n \atop k} \right\rangle \cdot p^k\right) \cdot (1-p)^{n-k}$$

Use of Binomial Coefficient expressed as N choose K

render n ## k = n! / ( k! * (n-k) ! )

render SIGMA [0<=k<=n] ( k * (n##k) * p^k * (1-p)^(n-k) )

RENDER "Use of Binomial Coefficient expressed as N choose K" TOP

$$\int asin(\theta) \cdot \partial\theta$$

$$\int\int y \cdot asin(x) \cdot \partial x \cdot \partial y$$

Simple indefinite integrals

render INTEGRALI (asin(theta) * <*> theta)

render INTEGRALD (y * asin(x) * <*> x * <*> y)

RENDER "Simple indefinite integrals" TOP

$$\int_{-1}^{1} asin(x) \cdot \partial x$$

Simple definite integral

render INTEGRAL [-1 <= x <= 1 <> dx] (asin(x) * <*> x)

RENDER "Simple definite integral" TOP

$$\oint_C \frac{e^{i \cdot t \cdot z}}{z^2 + 1} \cdot \partial z$$

Cauchy's Formula

render INTEGRALC (e^(i*t*z) / (z^2+1) * <*> z)

RENDER "Cauchy's Formula" TOP

$$(1 - x^2) \cdot \mathrm{Tn}''(x) - x \cdot \mathrm{Tn}'(x) + n^2 \cdot \mathrm{Tn}(x) = 0$$

$$(1 - x^2) \cdot \mathrm{Un}''(x) - 3 \cdot x \cdot \mathrm{Un}'(x) + n \cdot (n + 2) \cdot \mathrm{Un}(x) = 0$$

Differential Equations (in this case Chebyshev first and second order)

render (1 - x^2) * Tn''(x) - x * Tn'(x) + n^2 * Tn(x) = 0

render (1 - x^2) * Un''(x) - 3 * x * Un'(x) + n*(n+2) * Un(x) = 0

RENDER "Differential Equations (in this case Chebyshev first and second order)" TOP

$$\Gamma'(x) = \int_0^{\infty} t^{x-1} \cdot exp(-t) \cdot ln(t) \cdot \partial t$$

Derivative of the Gamma function

!! GAMMA'(x) = INTEGRAL [0 <= t <= INFINITY <> dt] ( t^(x-1) * exp(- t) * ln(t) * <*> t )

RENDERF GAMMA'

RENDER "Derivative of the Gamma function" TOP

$$\zeta'(x) = -\sum_{n=1}^{\infty} \frac{ln(n)}{n^x}$$

Derivative of the Zeta function

!! zeta'(x) = - ( SIGMA [1 <= n <= INFINITY] ( ln (n) / n^x ) )

RENDERF zeta'

RENDER "Derivative of the Zeta function" TOP

## Historical methods for computation of the value of pi

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$$

Reimann zeta(n) function

$$\pi = \sqrt{6 \cdot \zeta(2)}$$

Basel problem computing the convergence of zeta(2)

$$\text{asin}\prime(x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\pi = 6 \cdot \int_0^{0.5} \text{asin}\prime(x) \cdot \partial x$$

Isaac Newton asin series

$$\text{SegmentArea} = \int_0^{0.25} \sqrt{x - x^2} \cdot \partial x$$

$$\pi = 3/4 \cdot \sqrt{3} + 24 \cdot \text{SegmentArea}$$

Off center circle segment area used to compute pi

$$\text{atan}\prime(x) = \frac{1}{1 + x^2}$$

$$\pi = 4 \cdot \int_0^{1} \text{atan}\prime(x) \cdot \partial x$$

integration of arc tangent derivative to compute pi

```
!!zeta(n) = SIGMA [1 <= k <= INFINITY] ( 1 / k^n )
RENDERF zeta; RENDER "Reimann zeta(n) function" TOP

// zeta uses parameter for exponent which is 2
//  instead using a function which parameterizes the term count
// sum of inverted squares gets small fast but all positive terms
//   pi^2/6 = 1.6449340668482262, sumInvSq(100000) = 1.6449240668982423
//   pi^2/6 - sumInvSq(100000) = 9.999949983852119E-6, large error for 100,000 terms
!!sumInvSq(n) = SIGMA [1 <= k <= n] ( 1/(k^2) )
invSqRez = SQRT ( 6 * sumInvSq(INFINITY) )
//   10000 terms = 3.141497

// very slow convergence
RENDER pi = sqrt ( 6 * zeta(2) )
RENDER " Basel problem computing the convergence of zeta(2) " TOP



// declaration of symbols that can be used with integral approximation

RENDER asin'(x) = 1 / sqrt (1 - x^2)
//  6 * INTEGRAL [0 <= x <= 0.5 <> dx] (asn'(x)) * dx
RENDER pi = 6 * INTEGRAL [0 <= x <= 0.5 <> dx] ( asin'(x) )  * <*>x
RENDER "Isaac Newton asin series" TOP
!!asn'(x) = 1 / SQRT (1 - x*x)



// or the circular segment equation
//  circle'(x) = SQRT (x - x*x) is shown to have circular segment
//  pi = 3*SQRT(3)/4 + 24 * area => area = 0.07677310616304730284654410946459


RENDER SegmentArea = INTEGRAL [0 <= x <= 0.25 <> dx] ( sqrt (x - x^2) ) * <*>x
areaOfCircularSegment = 0.07677310616304730284654410946459
//   = INTEGRAL [0 <= x <= 0.25 <> dx] (circle'(x)) * dx

RENDER pi = 3 #/# 4 * sqrt(3) + 24 * SegmentArea
RENDER "Off center circle segment area used to compute pi" TOP
!!circle'(x) = SQRT (x - x*x)


// James Gregory atan series
// atan x = x - x^3/3 + x^5/5 - x^7/7 + ...
!!atn(x,n) = SIGMA [1 <= i <= n <> 4] (x^i/i - x^(i+2)/(i+2))



RENDER atan'(x) = 1 / (1 + x^2)
RENDER pi = 4 * INTEGRAL [0 <= x <= 1 <> LIM] ( atan'(x) ) * <*> x
RENDER "integration of arc tangent derivative to compute pi" TOP
```

$$\text{atan}(x) = \sum_{i=1}^{\infty} \frac{x^{4 \cdot i - 3}}{4 \cdot i - 3} - \frac{x^{4 \cdot i - 1}}{4 \cdot i - 1}$$

James Gregory atan series : atan x = x - x^3/3 + x^5/5 - x^7/7 + ...

$$\pi = 4 \cdot (4 \cdot \text{atan}(\frac{1}{5}) - \text{atan}(\frac{1}{239}))$$

Arc Tangent series of John Machin

$$\pi = 20 \cdot \text{atan}(\frac{1}{7}) + 8 \cdot \text{atan}(\frac{3}{79})$$

Arc Tangent series of Leonhard Euler

$$\frac{1}{\pi} = \frac{2 \cdot \sqrt{2}}{9801} \cdot \sum_{k=0}^{\infty} \frac{(4 \cdot k)! \cdot (1103 + 26390 \cdot k)}{k!^4 \cdot 396^{4 \cdot k}}$$

Series of Srinivasa Ramanujan

```
LIM = 1/INFINITY
!!atn'(x) = 1 / (1 + x*x)

//  4 * INTEGRAL [0 <= x <= 1 <> dx] (atn'(x)) * dx
atnIntRez = INTEGRAL [0 <= x <= 1 <> LIM] (4 * atn'(x) * <*> x)


PRETTYPRINT LIM
PRETTYPRINT invSqRez
PRETTYPRINT atnIntRez


// arc tangent function used to compute pi
polyTerms = 8; polyDegree = polyTerms * 2 - 1; countMax = polyDegree - 2
RENDER atan(x) = SUMMATION [1 <= i <= INFINITY] ( ( ((x^(4*i - 3) / (4*i - 3)) MINUS (x^(4*i - 1) / (4*i - 1))) ) )
RENDER "James Gregory atan series : atan x = x - x^3/3 + x^5/5 - x^7/7 + ..." TOP


RENDER pi = 4 * ( 4 * atan (1 / 5) - atan (1 / 239) )
RENDER "Arc Tangent series of John Machin" TOP


RENDER pi = 20 * atan (1 / 7) + 8 * atan (3 / 79)
RENDER "Arc Tangent series of Leonhard Euler" TOP


RENDER 1/pi = 2 * sqrt(2) / 9801 * ( SUMMATION [0 <= k <= INFINITY] ( (4*k)! * (1103 + 26390*k) / ((k!)^4 * 396 ^ (4*k)) ) )
RENDER "Series of Srinivasa Ramanujan" TOP
```

# CalcLib Notation, MathML, and rendered expression

## MathML

Generated by CalcLib

```
<?xml version="1.0"?>
<math>
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    <mi>x</mi>
    <mo>=</mo>
    <mfrac>
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          <mrow>
            <mo>-</mo>
            <mrow>
              <mrow>
                <mi>b</mi>
              </mrow>
            </mrow>
            <mo>±</mo>
            <msqrt>
              <mrow>
                <mrow>
                  <mrow>
                    <msup>
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                        <mi>b</mi>
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                    <mo>-</mo>
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                    <mo>·</mo>
                    <mi>c</mi>
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      <mrow>
        <mrow>
          <mn>2</mn>
          <mo>·</mo>
          <mi>a</mi>
        </mrow>
      </mrow>
    </mfrac>
  </mrow>
</math>
```

## CalcLib Notation

**render x = (- b +|- sqrt (b^2 - 4*a*c)) / (2*a)**

## Renders As

$$x = \frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$$

Using the public interface implementation connected to the JMTeX library