Some Topics in S-matrix Theory

A project report

submitted in partial fulfillment for the award of the degree of

Master of Science

in

Theoretical Physics

by

Saroj Prasad Chhatoi

under the guidance of

Dr. V. S. Nemani

Department of Theoretical Physics

Institute Of Mathematical Science

Chennai 600 113, India

CERTIFICATE

This is to certify that the project titled **Some Topics in S-matrix Theory** is a bona fide record of work done by **Saroj Prasad Chhatoi** towards the partial fulfillment of the requirements of the Master of Science degree in Theoretical Physics at the Institute Of Mathematical Sciences, Madras, Chennai 600 113, India.

(V. S. Nemani, Project supervisor)

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ABSTRACT

S-matrix theory was an ambitious program to constrain the possible scattering amplitudes of Poincare invariant QFTs based on a list of basic tenets expected on physical grounds. The techniques developed in this program have become relevant again because of their adaptation to correlators of CFTs. In this project we have studied some aspects of S-matrix theory that have found recent applications. In this report we review some of technical details of the basic assumptions, singularity structures, high energy behaviour, analyticity in angular momentum etc of scattering amplitudes.

Contents

1 Introduction

Quantum field theory provides the basic framework for studying high energy physics. In a given QFT one computes scattering amplitudes which can be directly related to observables quantities in particle colliders like scattering amplitudes (cross-section). In classroom setup we learn QFTs through feynman diagrams which requires a lagrangian/action to describe the theory.

In the 1960s, there had been extensive work in the field of 'Analytic S-matrix' which was based on the idea to calculate the S-matrix elements directly from a few basic properties of the S-matrix rather than demanding for the existence of some underlying Lagrangian theory. This approach which concerned with direct study of S-matrix, without having to introduce fields was first suggested by Heisenberg in 1943 in an attempt to avoid the divergence difficulties of the field theory, which later was tackled using renormalisation techniques. By the late 1970s the community moved away from this approach once it was recognised that quantum chromodynamics could solve the problems of strong interactions within the framework of field theory.

Recently these techniques have found their importance in conformal bootstrap program. Especially a recent work 'Analyticity in Spin in Conformal Theories' by Simon Caron-Huot uses the analogy from Froissart Gribov formula to analytically continue the conformal data in spin.

The final destination of this project report will be to understand the Froissart-Gribov projection formula

$$f_n = \frac{1}{\pi}\int_{z_1}^{\infty} Q_n(z)A_1(z,t)dz + \frac{(-1)^n}{\pi}\int_{-z_2}^{-\infty} Q_n(z_u)A_2(-z_u,t)dz_u{}^*$$

We start by first reviewing some basic concepts of non-relativistic scattering and arrive at the cross-section for hard sphere scattering or more commonly called the black disc model. In the subsequent analysis we see the black disc model is incompatible with the double discontinuity formula as it was famously being proved by Gribov. To arrive at this we will review many important results, one of which is the famous Froissart bound, which are all derived using some basic axioms of scattering matrix. The property of unitarity $SS^\dagger = \mathbb{1}$

*Here I refrain from elaborating on the details of the equation as it will be discussed later.

states the probability is conserved which is a crucial requirement for any physical system. The other important property we use is the analyticity of S-matrix.

This report is arranged as follows. In section 1 we review concepts in non relativistic scattering and we briefly sketch the derivation of cross section of black disc model. In section 2 we start with stating the principles of scattering matrix which forms the cornerstone based on which we review the derivation of a few important results. Then we use the unitarity condition to arrive at the optical theorem. The analyticity imposes further singularities in the partial waves in s-channel which are called the Karplus curves. We use these ideas to work through Froissart theorem.

Thereafter, in Section 3 we move on to discuss the unitarity condition in t-channel then analytically continue it to large s region † to obtain the double discontinuity relation. And finally we knit all the ideas to arrive at the Froissart-Gribov projection formula.

In the last section we briefly mention the recent applications of these techniques to conformal field theory data.

† s and t here refer to mandelstam variables.

2 Lessons from Non-relativistic Scattering theory

In this section we review some of the general aspects of scattering problems in non-relativistic quantum mechanics which will be be useful for later sections.

2.1 Free Solution to Radial equation

Schrodinger equation in radial coordinates is:

$$\Big[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+p^2-2V-\frac{l(l+1)}{r^2}\Big]R(r)=0 \tag{1}$$

assuming the wavefunction could be written as $\Psi(r,\theta,\phi) = R(r)Y_{lm}(\theta,\phi)$ as there is spherical symmetry.

The formalism of scattering theory is based on the existence of asymptotic states, which are the states at $r\to\infty$ and $V(r)\to 0$. We look first for the $l=0$ case and find the asymptotic solution

$$\begin{aligned}\Big[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+p^2\Big]R(r)&=0\\ \implies \frac{d^2}{dr^2}(rR(r))+p^2&=0\\ \implies rR(r)&=\exp(\pm ipr)\end{aligned}$$

The solution which is regular at $r=0$ is $rR(r)=\sin(pr)=j_0(pr)$. Now we consider the $l\neq 0$ case

$$\Big[\frac{d^2}{dr^2}+2\frac{d}{rdr}+p^2-\frac{l(l+1)}{r^2}\Big]R_l(r)=0$$

Consider $X_l=r^{-l}R_l(r)$, then in terms of X_l the above differential equation takes the form,

$$\frac{d^2}{dr^2}(r^{-l}R_l)+\frac{2(l+1)}{r}\frac{d}{dr}(r^{-l}R_l)+p^2(r^{-l}R_l)=0$$

If we differentiate X_l and substitute $\frac{1}{r}\frac{d}{dr}X_l$ in the schrodinger equation, we find

$$\frac{d^2}{dr^2}(\frac{1}{r}\frac{d}{dr}X_l)+\frac{2(l+2)}{r}\frac{d}{dr}(\frac{1}{r}\frac{d}{dr}X_l)+p^2(\frac{1}{r}\frac{d}{dr}X_l)=0$$

which is the equation of the same form as the differential equation for X_l with $l \to l+1$. So X_l satisfies a recurrence relation

$$X_{l+1} = \frac{1}{r}\frac{d}{dr}X_l \implies X_l = \Big(\frac{1}{r}\frac{d}{dr}\Big)^l X_0$$

or substituting the form of X_l

$$rR(r) = (-1)^l \frac{r^{l+1}}{p^l}\Big(\frac{1}{r}\frac{d}{dr}\Big)^l\Big[\frac{1}{r}exp(\pm ipr)\Big] \tag{2}$$

The regular solution at r$\to 0$ is

$$rR(r) = (-1)^l \frac{r^{l+1}}{p^l}\Big(\frac{1}{r}\frac{d}{dr}\Big)^l\Big[\frac{\sin(pr)}{r}\Big] = \hat{j}_l(pr) \tag{3}$$

The singular solution is called the riccati-neumann function.

$$\begin{aligned} n(z) &= \frac{1}{2}(h_l^{(+)}(z) + h_l^{(-)}(z)) \\ j(z) &= \frac{1}{2i}(h_l^{(+)}(z) - h_l^{(-)}(z)) \end{aligned}$$

where, $h^{(\pm)}$ are the ricaati-hankel functions.

For large r (r$\to \infty$) dominant term is that in which each derivative hits exponential,

$$\begin{aligned} &h_l^{(\pm)}(pr) \to \Big(\frac{-d}{dr}\Big)^l \exp(\pm ipr) = \exp(\pm i(pr - \frac{1}{2}l\pi)) \\ &j_l(pr) \to \sin(pr - \frac{1}{2}l\pi) \end{aligned}$$

The asymptotic r$\to 0$ behaviour of regular solution

$$\begin{aligned} j_l(pr) &= \frac{r^{l+1}}{p^l}\Big(\frac{-1}{r}\frac{d}{dr}\Big)^l \frac{\sin(pr)}{r} \\ &= \frac{r^{l+1}}{p^l}\Big(-\frac{1}{r}\frac{d}{dr}\Big)^l \sum_{n=0}^{\infty}\frac{p^{2n+1}r^{2n}}{(2n+1)!}(-1)^n \\ &\approx (pr)^{l+1}\frac{2^n n!}{(2n+1)!} \end{aligned}$$

2.2 Partial Wave expansion

For spherically symmetric potential V $\neq 0$ the transition operator, T, commutes with L^2 and $\vec{L}$ so T is a scalar operator. Using the Wigner Eckart Theorem in the spherical wave basis($|E,l,m\rangle$)

$$\langle E',l',m'|T|E,l,m\rangle = T_l(E)\delta_{ll'}\delta_{mm'} \tag{4}$$

We know, scattering amplitude has the form

$$f(\vec{k'},\vec{k}) = \frac{1}{4\pi}\frac{2m}{\hbar^2}L^3\langle k'|T|k\rangle^{\ddagger} \tag{5}$$

If we choose $\hat{k}$ as the z-axis then we can easily express $f(\vec{k'},\vec{k})$ using the spherical wave basis and the T operator as,

$$f(\vec{k'},\vec{k}) = -\frac{4\pi^2}{k}\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}P_l(\cos\theta)T_l(E)\Big|_{E=\frac{\hbar^2k^2}{2m}} \tag{6}$$

The outgoing wavefunction in (spherical) position basis at large r is

$$\langle x|\psi^+\rangle \to \frac{1}{(2\pi)^{3/2}}\Big(e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\Big) \tag{7}$$

Using the spherical wave expansion for e^{ikz} § and $f(\theta)$, we obtain

$$\langle x|\psi^+\rangle \to \frac{1}{2\pi^{3/2}}\Big(\sum_l (2l+1)P_l(\cos\theta)(\frac{e^{ikr}-e^{-i(kr-l\pi)}}{2ikr}) \tag{8}$$

$$+\sum_l (2l+1)f_l(k)P_l(\cos\theta)\frac{e^{ikr}}{r}\Big) \tag{9}$$

$$= \frac{1}{(2\pi)^{3/2}}\sum_l (2l+1)\frac{P_l}{2ik}\Big((1+2ikf_l(k))\frac{e^{ikr}}{r} - \frac{e^{-i(kr-l\pi)}}{r}\Big) \tag{10}$$

As we know the current conservation in time independent formalism,

$$\nabla.\vec{j} = -\frac{\partial|\psi|^2}{\partial t} = 0 \tag{11}$$

‡the differential cross section $\frac{d\sigma}{d\Omega} = \left(\frac{mL^3}{2\pi\hbar^2}|T_{ni}^2|\right)$ where L is the length of the 'big box' and T_{ni} are the transition matrix elements.

§the partial wave expansion of plane waves can be found in any standard text on quantum mechanics and here I have used the results.

So, the outgoing flux must be equal to the incoming flux and because of angular momentum conservation (manifestation of spherical symmetry) this must hold for each partial wave separately. Lets denote

$$S_l(k) = 1 + 2ikf_l(k) \tag{12}$$

$\implies |S_l(k)| = 1$ (by equating the incoming and outgoing flux) $\implies S_l = \exp(2i\delta_l(k))$

Now $f_l = \frac{S_l - 1}{2ik}$

$$\begin{aligned} f_l = \frac{e^{2i\delta_l} - 1}{2ik} &= \frac{e^{i\delta_l}\sin(\delta_l)}{k} \\ &= \frac{1}{k\cot\delta_l - ik} \end{aligned}$$

Substituting this into the full scattering amplitude

$$f(\theta) = \sum_{l=0}(2l+1)\big(\frac{e^{2i\delta_l} - 1}{2ik}\big)P_l(\cos\theta) \tag{13}$$

$$= \frac{1}{k}\sum_{l=0}(2l+1)e^{i\delta_l}\sin(\delta_l)P_l(\cos\theta) \tag{14}$$

Substituting the $f(\theta)$ into differential cross section $\sigma_{tot} = \int |f(\theta)|^2 d\Omega$ and using the orthonormality condition of the Legendre polynomials we obtain the form of cross section as

$$\sigma_{tot} = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l \tag{15}$$

Consider now the case of forward scattering *i.e.* $\theta = 0$. The imaginary part of $f(\theta)$ then takes the form

$$Im(f(\theta=0)) = \sum_l \frac{(2l+1)Im(e^{i\delta_l}\sin\delta_l)P_l(cos\theta)}{k}\Big|_{\theta=0} = \sum_l \frac{(2l+1)}{k}\sin^2\delta_l \tag{16}$$

From which we can easily see that

$$\sigma_{tot} = Im(f(\theta=0)) \tag{17}$$

This is the famous Optical theorem!!

2.3 Hard Sphere Scattering

The potential in this case is of the form

$$V = \begin{cases} 0 & r > R \\ \infty & r < R \end{cases}$$

The full wavefunction at any r can be written as

$$\langle x|\psi^{(+)}\rangle = \frac{1}{(2\pi)^{3/2}} \sum i^l (2l+1) A_l(r) P_l(cos\theta)$$

where,

$$A_l(r) = c_l^{(1)} h_l^{(+)}(kr) + c_l^{(2)} h_l^{(-)}(kr)$$

At large r this can be compared to (14) *i.e.*

$$\langle x|\psi^{(+)}\rangle = \frac{1}{(2\pi)^{3/2}} \sum_l (2l+1) P_l \cos(\theta) \Big(\frac{e^{(i2\delta_l)} \exp(ikr)}{2ikr} - \frac{e^{(-i(kr-l\pi))}}{2ikr} \Big) \tag{18}$$

So we obtain the coefficients as,

$$c_l^{(1)} = \frac{1}{2} e^{2i\delta_l} \quad c_l^{(2)} = \frac{1}{2}$$

$$\implies A_l(r) = e^{i\delta_l} (\cos\delta_l j_l(kr) - \sin\delta_l n_l(kr))^{¶} \tag{19}$$

Wavefunction vanishes at $r = R \implies A_l(r)|_{r=R} = 0$

$$\implies j_l(kR)\cos(\delta_l) - n_l(kR)\sin(\delta_l) = 0$$

$$or \quad \tan\delta_l = \frac{j_l(kR)}{n_l(kR)}$$

For instance for $l = 0$ $\tan\delta_0 = -\tan kR \implies \delta_0 = -kR$

Eikonal approximation : At high energy limit many partial waves contribute to the scattering cross section. Using semiclassical arguements the angular momentum $'l'$ can be written as $l\hbar = bp$, where b is the impact parameter and $p = \hbar k \implies l = bk$ and thus $l_{max} = kR$.

Noting that

$$\sin^2\delta_l = \frac{\tan^2\delta_l}{1+\tan^2\delta_l} = \frac{(j_l(kR))^2}{(j_l(kR))^2 + (n_l(kR))^2} \simeq \sin^2(kR - \frac{\pi l}{2}) \tag{20}$$

¶here the asymptotic properties of the radial wavefunction is used.

From (14) if we split $f(\theta)$ into two terms f_{refl} and f_{shadow} as [‖]

$$f_{refl} = \frac{1}{2ik}\sum_{l=0}^{kR}(2l+1)e^{2i\delta_l}P_l(cos\theta)$$

$$f_{shadow} = \frac{i}{2k}\sum_{l=0}^{kR}(2l+1)P_l(\cos\theta)$$

Then the total cross-section is as follows

$$\sigma_{tot} = \int |f_{refl}|^2 d\Omega + \int |f_{shadow}|^2 d\Omega + \int f^*_{shadow} f_{refl}\, d\Omega \tag{21}$$

The first term in the R.H.S. of the above expression can be calculated as

$$\int |f_{refl}|^2 d\Omega = \frac{2\pi}{4k^2}\sum_{l=0}^{l_{max}}\int_{-1}^{1}(2l+1)^2\Big[P_l(\cos\theta)\Big]^2 d(\cos\theta) \tag{22}$$

$$= \frac{\pi}{l_{max}^2} = \pi R^2$$

The second term f_{shadow} is purely imaginary and we get a maximum contribution from this term in the forward direction *i.e.* $\theta = 0$ as in this limit all the terms in the sum adds up coherently. We can approximate $P_l(\cos\theta) \simeq J_0(l\theta)$ at large l and small $\theta \implies f_{shadow} \simeq \frac{i}{2k}\sum(2l+1)J_0(l\theta)$. Considering many partial waves contribute within the range $l_{max} = kR$ we can substitute the sum with integral over l and substitute $l = kb$, $i.e.$

$$\sum_{l}^{l_{max}=kR} \to k\int db$$

$$f_{shadow} \simeq \frac{i}{2k}\int_0^R 2kb\; kdbJ_0(kb\theta)$$
$$= \frac{iRJ_1(kR\theta)}{\theta}$$

$$\implies \int |f_{shadow}|^2 d\Omega = \int_{-1}^{+1}\int_0^{2\pi} R^2\frac{(J_1(kR\theta))^2}{\theta^2} d(\cos\theta)d(\phi)$$

$$\simeq 2\pi R^2\int_0^{\infty}\frac{J_1(\zeta)}{\zeta} = \pi R^2 \tag{23}$$

[‖] the reason of calling the terms with these specific names will become clearly soon

The last term in (21) vanishes because the of the oscillating phase of f_{refl}, which averages to zero (and f_{shad} being purely imaginary) .

$$Re(f^*_{shad} f_{refl}) \simeq 0$$

Thus,

$$\sigma_{tot} = \pi R^2 + \pi R^2 \tag{24}$$

The second term is called the shadow contribution as intuitively the waves with impact parameter less than R, gets deflected and the probability for finding the particle just behind the scatter would be zero, hence forming a shadow.

3 Relativistic Scattering Theory

In this section we spell out the basic assumptions of S-matrix program. Taking $2 \to 2$ scattering of particles in generic relativistic QFTs as the main illustrative scattering process. We will also find generalisation of non relativistic aspects of previous section to relativistic scattering and study in more detail the aspects of analytic properties and high energy behaviour of such S-matrix theory.

3.1 Basic principles of Scattering Matrix

Rather than working on an explicit lagrangian we would rather try to extract as much information as possible about the S-matrix from the consequences of some postulates.

1. **S-matrix is lorentz invariant**, so it is made up of product of four momentas of incoming and outgoing particles. In the case of two to two particle scattering $1 + 2 \to 3 + 4$ there are two independent invariants considering the momentum conservation and the on shell condition for the particles. These are described easily using the mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$. These three variables refer to three different processes can be understood as different channels which lead to the $1 + 2 \to 3 + 4$ process involving a single intermediate particle.

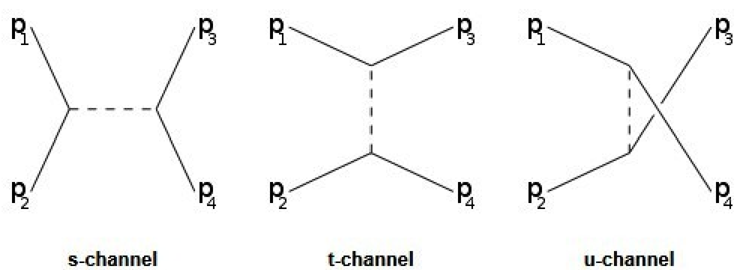

Figure 1: The three channels for 2-2 particle scattering

But these three variables are not independent given the conservation of momentum, which leads to $s + t + u = (p_a + p_b)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2 = m_1^2 + m_2^2 + m_3^2 + m_4^2$. So we can write the amplitude for 2 particle scattering as $A(s, t)$. In the general case of 2 particle scattering into n particles there are $3n - 4$ independent invariants.

2. **Unitarity**: In relativistic mechanics this is just the statement that the sum of the probabilities of incoming particles to be found in any possible outgoing state is unity $i.e.$ $SS^\dagger = S^\dagger S = \mathbb{1}$. Similarly we get the second equality from the fact that the probability for the final state to arise from some initial state gives unity.

3. **Analyticity**: The S-matrix should be analytic function of the invariant quantities and only the unitarity condition imposes the possible singularities. Analyticity is a consequence of causality. As we will see ahead that the analyticity will help us define the imaginary part of the amplitude and then the dispersion relations help us construct the real part of the amplitude.

3.2 Optical theorem

In the unitarity relation $SS^\dagger = 1$ we can substitute S=1+T, to obtain

$$2\,\mathrm{Im}T = T^\dagger T \tag{25}$$

which is the operator form of the unitarity condition.

For the case of $2 \rightarrow\ 2$ particle scattering lorentz invariance leads to the symmetry of the matrix element

$$\langle m|T|n\rangle = \langle n|T|m\rangle$$
$$\implies \langle p_3, p_4|T|p_1, p_2\rangle = \langle p_1, p_2|T|p_3, p_4\rangle$$

which using (25) leads to

$$2Im\langle p_4, p_3|T|p_2, p_1\rangle = \langle p_4, p_3|T^\dagger T|p_2, p_1\rangle \tag{26}$$

Inserting a complete set of intermediate states in the R.H.S. of the above equation we get

$$\langle p_3, p_4|T^\dagger T|p_1, p_2\rangle = \prod_i \int \frac{d^3 q_i}{(2\pi)^3 2E_i} \langle p_3, p_4|T^\dagger|q_i\rangle\langle q_i|T|p_1, p_2\rangle \tag{27}$$

Now using a form of T operator in which the momentum conserving term has been extracted

$$\langle p_1, p_2|T|q_i\rangle = (2\pi)^4\delta^4(\sum_{n=3}^{N} P_n - P_1 - P_2)A(1 + 2 \rightarrow q_i)$$

and the unitarity condition (25), we can write (27) as

$$2ImA(1+2\to 3+4) = \prod_i \int \frac{d^3q_i}{(2\pi)^3 2E_i}(2\pi^4)\delta^4\Big(\sum_i Q_i - P_1 - P_2\Big) \tag{28}$$

$$\times A^*(p_3p_4 \to q_i)A(p_1,p_2\to q_i) \tag{29}$$

In the elastic limit the above reduces to

$$\text{Im}A(s,t) = \frac{|k|}{8\pi\sqrt{s}}\int \frac{d\Omega}{4\pi}A(s,\cos\theta_1)A^*(s,\cos\theta_2) \tag{30}$$

note that in going from (28) to (30) we have expressed amplitude as a function mandelstam variables and integrated over the phase space of the third particle. Now we substitute the partial wave expansion of $A(s,t)$

$$A(s,t) = \sum_{l=0}^{\infty} f_l(s)(2l+1)P_l(z), \tag{31}$$

to the R.H.S. of (30), $i.e.$

$$\text{Im}A(s,t) = \frac{|k|}{8\pi\sqrt{s}}\sum_{l_1,l_2} f_{l_1}(s)f^*_{l_2}(s)\ (2l_1+1)(2l_2+1)\int\frac{d\Omega}{4\pi}P_{l_1}(\cos\theta_1)P_{l_2}(\cos\theta_2) \tag{32}$$

Now simplifying the integration part of the above equation,

$$\int\frac{d\Omega}{4\pi}P_{l_1}(\cos\theta_1)P_{l_2}(\cos\theta_2) = \int\frac{d\Omega}{4\pi}P_{l_1}(\cos\theta_1)\frac{4}{2l+1}\sum_{m=-l}^{l}Y_{lm}(\theta,\pi/2)Y_{lm}(\theta_1,\phi)$$

Now, using the orthogonality condition of the Y_{lm}, we can write

$$\int\frac{d\Omega}{4\pi}P_{l_1}(\cos\theta_1)P_{l_2}(\cos\theta_2) = \frac{\delta_{l_1,l_2}}{2l_1+1}P_{l_1}(\cos\theta)$$

substituting this into (30) we obtain,

$$Imf_l(s) = \frac{k_s}{16\pi\omega_s}f_l(s)f_l^*(s) + \Delta, \tag{33}$$

$$where,\ \ k_s = \frac{\sqrt{s-4\mu^2}}{2},\quad \omega_s = \frac{\sqrt{s}}{2}$$

This is the generalised unitarity condition where the R.H.S. contains the contribution , which is represented as Δ, from all possible intermediate state. In the elastic case (33) can be rewritten as

$$\Big|1 + i\frac{k_s}{8\pi\omega_s}f_l\Big|^2 = 1 \implies 1 + \frac{ik_s}{8\pi\omega_s}f_l = e^{2i\delta_l(s)} \implies f_l(s) = \frac{i8\pi\omega_s}{k_s}(1-e^{2i\delta_l(s)})$$

In the general case ($i.e.$ $\Delta \neq 0$),

$$f_l(s) = i\frac{8\pi}{v}[1 - \eta_l e^{2i\delta_l}] \tag{34}$$

where, η_l is the elasticity parameter $\eta_l(s) \leq 1$, $\eta_l^2 = 1 - \frac{v}{4\pi}\Delta$, $v = \frac{k_s}{\omega_s}$. The maximum inelasticity in partial wave corresponds to $\eta_l(s) = 0$. In this case $f_l(s)$ is purely imaginary so the elastic channel is basically the shadow of inelastic channels.

$$f_l = \begin{cases} i\frac{8\pi}{v}, & \eta_l = 0, \text{for } \ l < l_0 = k_s R \\ 0, & \eta_l = 1, \delta_l = 0, \ \text{for } \ l > l_0 \end{cases}$$

At high energies $v = k_s/\omega_s \simeq 1$ and $4k_s^2 \simeq s$

$$A(s,0) = \sum_l (2l+1) f_l \simeq l_0^2 8\pi i \simeq is2\pi R^2 \tag{35}$$

From optical theorem

$$\sigma_{tot} = \frac{\text{Im}A(s,0)}{vs} \simeq 2\pi R^2 \tag{36}$$

This is what we obtain from the eikonal aproximation of hard sphere scattering case.

3.3 Threshold singularities

Above the energy threshold for inelastic scattering , in order to include one extra intermediate state, the generalised unitarity condition gets contribution on the R.H.S. . This implies the scattering matrix element has a singularity at the energy threshold for new allowed physical process. These thresholds, each energy threshold corresponds to a new possible intermediate state, turn out to be branch points of the amplitude $A(s,t)$**. Below the threshold energy imaginary part is zero so there exists a region on the real s-axis where

$$A^*(s,t) = A(s^*,t) \tag{37}$$

As stated above for real s above the threshold we get contribution from higher order intermediate states so we need a cut along the real s axis in order to have a non zero imaginary part. Using the schwartz reflection principle,

$$\text{Im}A(s+i\epsilon,t) = \frac{A(s+i\epsilon,t) - A(s-i\epsilon,t)}{2i} \tag{38}$$

**the proof of the nature of the singularities is mentioned in 'The Analytic S-matrix' , it is cited in the reference section.

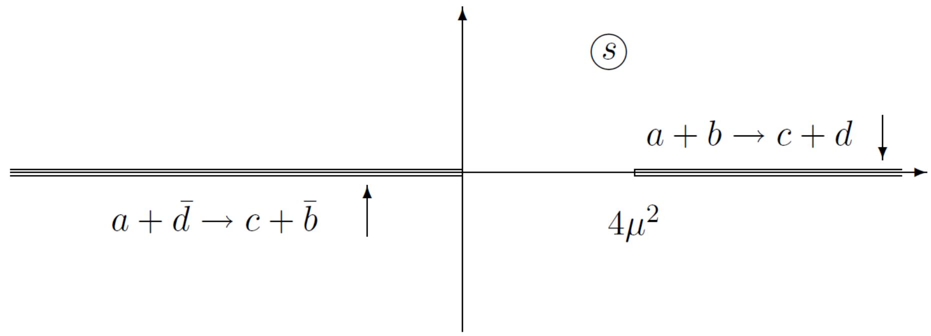

Figure 2: s-plane threshold

in the region where amplitude in analytic. Above the threshold we can define imaginary part of the amplitude as

$$\text{Im}A(s,t) = \lim_{\epsilon\to 0}\frac{A(s+i\epsilon,t)-A(s-i\epsilon,t)}{2i} \tag{39}$$

Now using the analyticity condition on $A(s,t)$ we can analytically continue amplitude from the physical region of s channel $(a+b \to\ c+d)$ to the physical region of u channel $(a+\bar{d} \to\ c+\bar{b})$. Similarly we can continue the same amplitude function to t channel.

$$A_{a+\bar{c}\to\bar{b}+d}(s,t,u) = A_{a+b\to c+d}(t,s,u) \tag{40}$$

Since the amplitude for the u-channel and t-channel have imaginary parts so from the above argument both will have threshold singularities. These singularities manifest itself in the s-plane which is easily seen by using the

$$s+t+u = \sum_i m_i^2 \tag{41}$$

At fixed u, $A(s,t)$ must have a cut along the negative real s-axis with branch point at s= $\sum_i m_i^2 - t_{th} - u$, where t_{th} is the threshold in the t-channel.

3.4 Singularities of partial waves(Karplus Curves)

Now using the unitariry condition we will derive the exact form of singularity of $\text{Im}_s A(s,t)$ for the case of $\Delta = 0$ (elastic case).

We note that for $l \to \infty$

$$P_l(\cosh\alpha) \simeq \frac{e^{(l+\frac{1}{2})\alpha}}{\sqrt{2\pi l \sinh\alpha}}, \quad \cosh\alpha \equiv z = 1+\frac{t}{2k_s^2}$$

In the derivation below, we use the fact the singularity of $\text{Im}_s A(s,t)$ closest to the physical region of s-channel is at t= $4\mu^2$. So, the series should be convergent for $t < 4\mu^2$ and to ensure this

$$f_l \sim \exp(-l\alpha_0), \quad \cosh\alpha_0 = 1 + \frac{4\mu^2}{2k_s^2}$$

Using the unitarity condition we establish that Im $f_l \sim \exp(-2l\alpha_0)$. In the series for $\text{Im}A_s(s,t)$, as t increases the legendre polynomial starts dominating the exponentially falling factor $\exp(-2l\alpha_0)$ and the series eventually becomes divergent. This results to line of singularities which can be evaluated easily as follows

$$\begin{aligned} \alpha = 2\alpha_0 &\implies \cosh\alpha = \cosh 2\alpha_0 \\ &\implies 2\Big(\frac{4\mu^2}{2k_s^2} + 1\Big)^2 - 1 = \frac{t}{2k_s^2} + 1 \end{aligned}$$

Simplifying this we get

$$\frac{t}{16\mu^2} = \frac{s}{s - 4\mu^2}, \quad 4\mu^2 \le s \le 16\mu^2 \tag{42}$$

This is the Karplus curve for the region $4\mu^2 \le s \le 16\mu^2$ and $t \ge 4\mu^2$. Similarly we can obtain the Karplus curve for the region $4\mu^2 \le t \le 16\mu^2, s > 4\mu^2$ as

$$\frac{s}{16\mu^2} = \frac{t}{t - 4\mu^2}\ , \quad 4\mu^2 \le s \le 16\mu^2 \tag{43}$$

3.5 Froissart Theorem

In the last subsection we have shown that partial waves fall exponentially for $k_s^2 \propto s \gg t > 0$.

For α small

$$\cosh\alpha = 1 + \alpha^2/2 + \mathcal{O}(\alpha^4) = 1 + \frac{t^2}{2k_s^2}$$

where the last equality is valid for $t > 0$. $\implies 1 + \frac{\alpha^2}{2} \simeq 1 + \frac{t^2}{2k_s^2} \implies \alpha \simeq \frac{\sqrt{t}}{k_s}$. So, at $t = 4\mu^2$

$$f_l(s) \simeq c(s,l)\exp\Big(\frac{-l}{k_s}\sqrt{4\mu^2}\Big) \tag{44}$$

Here we are trying to estimate f_l at large 's' using the fact that the singularity closest to the physical region of s-channel is at $t = 4\mu^2$, where $c(s,l)$ is slowly (non exponentially) varying with l. Using the unitarity condition derived earlier

$$\begin{aligned} \text{Im } f_l(s) &= \frac{k_s}{16\pi\omega_s} f_l(s) f_l^*(s) + \Delta \\ K_s &= \frac{\sqrt{s - 4\mu^2}}{2} \qquad \omega_s = \frac{\sqrt{s}}{2} \end{aligned}$$

we see that Im$f_l(s) > 0$ and for $t \geq 0,\ P_l(1+\frac{t}{2k_s^2}) > 0$

The asymptotic behaviour of legendre polynomials as $l \to \infty$, is given by

$$P_l(\cosh\alpha) \simeq \frac{\exp((l+1/2)\alpha)}{\sqrt{2\pi l \sinh\alpha}}$$

for small $\alpha, \alpha \approx \frac{\sqrt{4\mu^2}}{k_s}\ at\ t \to 4\mu^2$ and $\sinh\alpha \approx \alpha$

$$\implies P_l(\cosh\alpha) \simeq \exp\Big(l\frac{\sqrt{t}}{k_s}\Big)$$

Now, we assume that as t$\to 4\mu^2$ amplitude grows not faster than $(\frac{s}{s_0})^N$, then the same is valid for Im $c(s,t)$

$$\begin{aligned}\Big(\frac{s}{s_0}\Big)^N &> \text{Im } A(s,t) = \sum_{l=0}^{\infty}\text{Im} f_l(s)(2l+1)P_l\Big(1+\frac{t}{2k_s^2}\Big) \\ &> \text{Im } c(s,t)\Big(2\pi l\frac{\sqrt{t}}{k_s}\Big)^{-\frac{1}{2}}\ \exp\Big(\frac{l}{k_s}(\sqrt{t}-\sqrt{4\mu^2})\Big)\end{aligned}$$

$$\implies \text{Im } f_l(s) \leq \Big(\frac{s}{s_0}\Big)^N \exp\Big(\frac{-2\mu}{k_s}l\Big) \tag{45}$$

The value of partial waves are large till $l = L$ *i.e.* Im$f_l(s) \simeq |f_l| \simeq O(1)$ so we extract the finite sum till $l = L$,

$$\text{Im } A(s,t=0) = \sum_{l=0}^{\infty}\text{Im} f_l(s)(2l+1) \tag{46}$$

$$\leq 8\pi\sum_{l=0}^{L}(2l+1) + \sum_{l=L+1}^{\infty}\text{Im} f_l(s)(2l+1) \tag{47}$$

The value of L at which the above is true is found as follows

$$\Big(\frac{s}{s_0}\Big)^N \exp\Big(-\frac{2\mu}{k_s}L\Big) \simeq 1 \quad \implies L \simeq \frac{k_s}{2\mu}ln\Big(\frac{s}{s_0}\Big)$$

The contribution from the second term in the summation can be estimated by using $f_{L+N} \approx$

$f_L \exp(-2\frac{\mu n}{k_s})$. Now as $f_L \approx 1$ so second term in (47) reduces to

$$\sum_{n=0}^{\infty} \exp\Big(-2\frac{\mu n}{k_s}\Big)\Big(2(L+n)+1\Big) = \sum_{n=0}^{\infty} 2L\exp\Big(\frac{-2\mu n}{k_s}\Big) + 2n\exp\Big(\frac{-2\mu n}{k_s}\Big)$$
$$= 2L\frac{1}{1-\exp(-2\mu/k_s)} + 2\frac{e^{\alpha}}{(1-e^{\alpha})^2}$$
$$= \frac{Lk_s}{\mu} + 2\Big(\frac{k_s}{2\mu}\Big)^2 \ll_{s\to\infty} L^2$$

Thus, Im $A(s,t=0) \propto L^2 \propto sln^2(\frac{s}{s_0})$. But according to optical theorem Im $A(s,t=0) = s\sigma_{tot}(s)$.

The amplitude $A(s,t=0)$ has to grow so that the σ doesn't decrease with increasing energy which leads to many partial waves contributing to the sum. So we can replace the sum with integral over l, this is similar to the eikonal approximation,

$$A(s,t) \simeq \int f_l(s) J_0\Big[(2l+1)\theta/2\Big]$$

Now if we replace l by the impact parameter ρ and $\cos\theta = 1 + \frac{t}{2k_s^2} \implies t \simeq -(k_s\theta)^2$

$$A(s,t) \simeq k_s^2 \int f(\rho,s) J_0(\rho\sqrt{-t}) 2\rho d\rho \tag{48}$$

We see that for finite size potential where ρ doesn't depend on s, $A(s,t)$ can be factorised as $a(s)F(t)$. Further if the dominant $f(\rho,s)$ approach constant value as $s \to \infty$, then $A(s,t) \sim sF(t)$ and thus the total cross section tends to a constant value.

3.6 Pomeranchuk Theorem

When we approach the the right cut in the s-plane the value of analytic function $A(s,t)$ corresponds to the reaction $a+b \to c+d$ *i.e.*

$$A(a+b\to c+d) \to \lim_{\epsilon\to 0} A(s+i\epsilon,t)$$

Similarly if we approach the the right cut in u-plane we obtain the physical amplitude of $a+\bar{d} \to c+\bar{b}$

$$A(a+\bar{d}\to c+\bar{b}) = A(u+i\epsilon,t) = \lim_{\epsilon\to 0} A(-(s-i\epsilon)-t+4\mu^2,t)$$

here we have used the relation $s+t+u=4\mu^2$.

Thus the amplitude of the cross channel reaction is obtained by approaching the left cut from below in s-plane. We have already seen that $A(s - i\epsilon, t < 0) = A^*(s + i\epsilon, t < 0)$ to finally arrive at,

$$A_{a+\bar{d}\to c+\bar{b}} \simeq \left[A_{a+b\to c+d}\right]^* \tag{49}$$

For the finite size potential, in case of elastic reaction, as discussed in last section $A(s,t)$ takes the factorised form $A(s,t) = sF(t)$ so that the total cross section takes on a constant value at $s \to \infty$. So using the relation (49) we obtain $A(a + \bar{b} \to a + \bar{b}) = -sF^*(t)$. Hence we see that the imaginary part of $A(s,t)$ in both processes are the same.

Since the total cross section (σ_{tot}) is calculated using the imaginary part of $A(s,t)$ so we obtain

$$\sigma_{tot}(a + b) = \sigma_{tot}(a + \bar{b}) \tag{50}$$

4 Complex Angular Momentum

In this section we analytically continue the unitarity condition from the physical region in t-channel ($t > 4\mu^2$) to large $s > 0$. One might expect that as the relative distance between $t > 4\mu^2$ and $t < 0$ for large s is small there must be some restriction on the $A(s,t)$ in the physical region of the s-channel($t < 0$) when one analytically continues the unitarity condition from the physical region of t-channel to the region of large positive s. The unitarity condition in the t-channel is

$$\begin{aligned} A_3 &\equiv \text{Im}_t A(s,t) \\ &= \frac{1}{2}\int \frac{d^4k_1}{(2\pi)^3}\frac{d^4k_2}{(2\pi)^3} A(p_2,p_4,k_1,k_2)A^*(p_1,p_3,k_1,k_2) \\ &\times \delta(k_1^2-m^2)\delta(k_2^2-m^2)(2\pi)^4\delta(k_1+k_2-p_2-p_4) \end{aligned} \tag{51}$$

As before in the center of mass frame of the t channel

$$\text{Im}_t A(s,t) = \frac{k_t}{w_t}\int \frac{d\Omega}{64\pi^2} A(z_1,t)A^*(z_2,t) \tag{52}$$

$$where,\ \ k_t = \frac{\sqrt{t-4\mu^2}}{2},\quad \omega_t = \frac{\sqrt{t}}{2}$$

and in (52) we have used

$$s_1 = -2k_t^2(1-z_1)\quad s_2 = -2k_t^2(1-z_2) \tag{53}$$

where z_1 and z_2 are the cosines of the angles between the intermediate particles and the initial and final particles respectively.

Now, we make a change of variables

$$z_1 \to \ z_1 \text{ and } \phi \to \ z_2$$

The jacobian of the transformation can be obtained using the relation

$$z_2 = zz_1 + \sqrt{(1-z_1^2)(1-z^2)}\ \cos\phi$$

$$\binom{z_1}{z_2} = \begin{pmatrix} 1 & 0 \\ z - z_1\frac{\sqrt{1-z^2}\cos\phi}{\sqrt{(1-z_1^2)}} & \sqrt{(1-z_1^2)(1-z^2)}\sin\phi \end{pmatrix}\binom{z_1}{\phi}$$

Thus, the jacobian $|J| = |\sin\phi|\sqrt{(1-z_1^2)(1-z^2)}$. Here the trigonometric relation can be inverted to obtain

$$\sin\phi = \frac{\sqrt{-(z-z_1z_2)^2+(1-z_1^2)(1-z_2^2)}}{\sqrt{(1-z_1^2)(1-z^2)}}$$

$$\implies \int d\Omega = \int \frac{dz_1dz_2}{|J|} = \int \frac{2dz_1dz_2}{\sqrt{-k(z,z_1,z_2)}}$$

where, $K(z,z_1,z_2) = (z-z_1z_2)^2-(1-z_1^2)(1-z_2^2)$. So, we see here that $A_3(z,t)$ is analytic function of z and so can be continued to $s>0(z>1)$ until we hit a singularity $z=z_0(t)$. Such singularities have a branch point nature so one would expect discontinuity across $z>z_0(t)$. This discontinuity can be easily calculated using the form of $K(z,z_1,z_2)$ and one finds,

$$\begin{aligned}\rho(s,t) &\equiv \mathrm{Im}_s A_3(s,t)\\ &= \frac{k_t}{16\pi^2\omega_t}\int\frac{dz_1dz_2}{\sqrt{K(z,z_1,z_2)}}[A_1(z_1)A^*z(_2)+A_2(z_1)A_2^*(z_2)]\end{aligned} \tag{54}$$

where, $A_1(z_1)$ is the imaginary part of the s-channel. As z_1 and z_2 can be negative we have included the u-channel contribution $A_2(z_1)$. The real function $\rho(s,t)$ is double discontinuity of the amplitude which we calculated first across t-channel and then across s-channel. In the mandelstam plane, $\rho(s,t)$ differs from zero only inside the karplus curve $z=z_0(t)$. Inside this region $\rho(s,t)$ gets contribution from each particle threshold.

We have found ρ in two steps, first finding the imaginary part across the branch cut in the t-channel then analytically continued into the $s>4\mu^2$ region and then calculated the imaginary part in s-channel. Similarly one could have done the calculation starting from the s-channel and analytically continuing it to the t-channel and then finding the imaginary part in s-channel. Both the ways lead us to the same results for ρ.

Next using the above double discontinuity relation we show that the black disc model discussed in earlier section is inconsistent with the formula for ρ found here. So A(s,t) cannot have the following form

$$A_1(s,t) = sf(t), \;\; s\to\infty$$

From pomeranchuk theorem we saw that $A_2(s,t)=A_1(s,t)$, then substituting the form of $A_1(s,t)$ in $\rho(s,t)$ we obtain

$$\rho(s,t) = s\,\mathrm{Im}\,f(t)$$

For large z, z $\propto$ s, the dominant contribution to the integral over z_1 and z_2 arise from $z_1 \propto z_2 \propto \sqrt{z} \gg 1$. In this limit we can write K(z,z_1,z_2)

$$\mathrm{K}(z_1,z_2,z) \simeq z(z-2z_1z_2)$$

Substituting the above asymptotic expression for $\mathrm{K}(z, z_1, z_2)$ and using variable $x = 2z_1z_2$ we obtain

$$\rho(s,t) \propto \frac{1}{\sqrt{z}} \int_1^z \frac{dz_1}{z_1} \int_{z_1/2}^z \frac{xdx}{\sqrt{z-x}}$$

Using the lower limit *i.e.* $z_1 \approx 0$ we approximate the above equation. As $z_1 \approx \sqrt{z} \implies z \gg z_1$ for $z_1, z > 1$ we get

$$\simeq \frac{1}{\sqrt{z}} \int_1^z \frac{dz_1}{z_1} \int_0^z \frac{xdx}{\sqrt{z(1-x/z)}}$$

Now taking

$$y = \frac{x}{z}, \quad dy = \frac{dx}{z}, \quad ydy = \frac{xdx}{z^2}$$

the equation for ρ reduces to

$$\begin{aligned} &= \frac{1}{\sqrt{z}} ln(z) \int_0^1 \frac{z^2 ydy}{\sqrt{z}\sqrt{1-y}} \\ &= \frac{z^{3/2}}{\sqrt{z}} ln(z) \int_0^1 \frac{ydy}{\sqrt{1-y}} \end{aligned} \tag{55}$$

Here we observe a contradiction as we started with the assumption that $\rho(s,t) = s\mathrm{Im}f(t)$ (which we obtained by continuing $A_1(s,t)$ into the unphysical region $t > 4\mu^2$) but the R.H.S. in (55) exceeds L.H.S.

4.1 Sommerfeld-Watson representation

Now, suppose one can find an analytic function $f_l(t)$ that does not increase exponentially in any direction in right-half of the complex l-plane and its values coincide with partial wave amplitudes at all integer l *i.e.*

$$f_l|_{l=n} = f_n \tag{56}$$

one can write the expansion of the amplitude in t-channel as

$$A(s,t) = \frac{1}{2i} \int_L \frac{dl}{\sin \pi l} f_l(t) P_l(-z)(2l+1) \tag{57}$$

Here $\sin \pi l$ provides the residue of $(-1)^n$ which is combined with $P_l(z)$ to give $P_n(-z)$ as $P_n(-z) = (-1)^n P_n(z)$. Our goal here will be to analytically continue the expansion to large z.

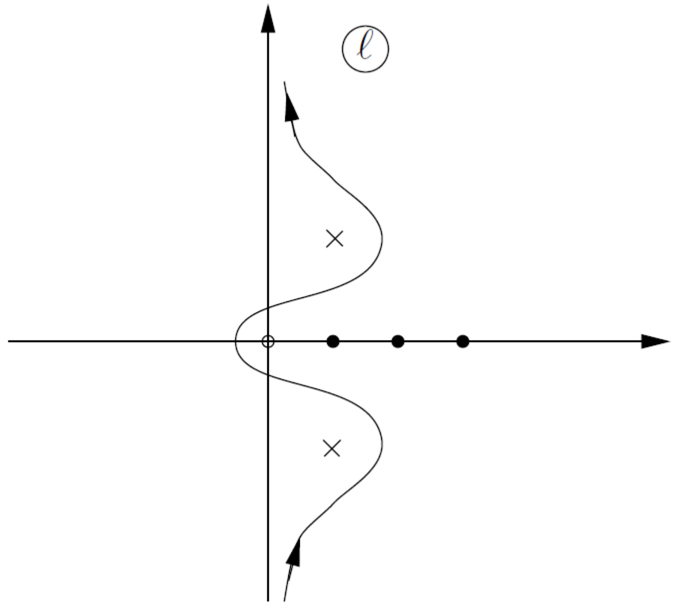

Figure 3: Deformed contour

Now we try to deform the contour of integration to the imaginary l-axis but this will be possible only if the integrand vanishes at $|l| \to \infty$ in the right half plane. Considering the asymptotics of legendre polynomials at large-l ($z > 0$)

$$P_l(z)|_{l\to\infty} \approx e^{il\theta} + e^{-il\theta} \tag{58}$$

$$P_l(-z)|_{l\to\infty} \approx e^{il(\pi-\theta)} + e^{-il(\pi-\theta)} \tag{59}$$

we see that $\frac{P_l(-z)}{\sin \pi l}$ falls exponentially for $0 < \theta < \pi$ and Im$l \to \infty$, so we can deform the contour along the imaginary l-axis and thus the function is defined in the **entire complex z-plane**.

Once we have the amplitude defined for the whole complex z-plane we now consider the two different regions of interest $z > 1$ and $z < -1$. In the region $z < -1$, $P_l(z)$ has the form $\cosh(l\chi)$ where

$$\cosh \chi = z \tag{60}$$

Along the imaginary l-axis $P_l(z)$ oscillates for $0 < \theta < \pi$ so the integrand converges as $\sin \pi l$ grows along imaginary l-axis. Now in the region $z > 1$ we know that $P_l(z) \approx \exp(-il\pi)\exp(l\chi)$. In this case the $\sin \pi l$ in the denominator is cancelled by the $\exp(-il\pi)$

so the convergence here depends on $f_l(z)$ as $l \to \infty$. So, if $\mathrm{f}_l(z)$ has singularities in the right half l-plane it may lead to singularities of $A(z,t)$.

If Im$z \neq 0$, z can be written as $\exp(\text{Re}\theta + i\text{Im}\theta)$. Substituting this into the asymptotic of $P_l(z)$ ($l \to \infty$) we see that $A(z,t)$ always converges as we get a exponentially decaying contribution along imaginary l-axis.

For $z \to \infty$, $\quad P_l(-z) \approx z^l \exp(-il\pi)$ from which we conclude that the asymptotics of $A(z,t)$ is governed by the rightmost singularity in l-plane because of the z^l factor. For say, the rightmost singularity is a pole at $l = \alpha(t)$, then the amplitude in the Sommerfeld-Watson representation takes the form

$$A(s,t) \to \pi \frac{2\alpha+1}{\sin \pi\alpha} P_\alpha(-z) Res f_\alpha(t) + \frac{1}{2i} \int_{Rel<Re\alpha(t)} \frac{dl}{\sin \pi l} f_l(t) P_l(-z)(2l+1) \tag{61}$$

where Res $f_\alpha(t)$ is the residue of $f_l(z)$. Thus we see that the asymptotics of $A(z,t)$ is defined by l plane singularities. This is similar to the results of the last section where the singularities in t-plane determined the asymptotics in $A(s,t)$ for large s. We should note here that θ and l are conjugate variables.

4.2 Gribov-Froissart projection

Inverting the partial wave expansion of $A(z,t)$ we obtain

$$f_n(t) = \frac{1}{2} \int_{-1}^{1} P_n(z) A(t,z) dz \tag{62}$$

We can re-express this in terms of legendre polynomial of second kind for integer n

$$Q_n(z) = \frac{1}{2} \int_{-1}^{1} \frac{P_n(z')dz'}{z-z'}$$

as

$$f_n(t) = \frac{1}{2\pi i} \oint_{(a)} Q_n(z) A(t,z) dz \tag{63}$$

where the contour of integration encloses $[-1,1]$ on the real z-axis in the z-plane. Here we have used the following property of $Q_n(z)$

$$Q_n(z+i\epsilon) - Q_n(z-i\epsilon) = -i\pi P_n(z) \tag{64}$$

which accounts for the discontinuity of the integrand across $-1 \leq z \leq 1$.

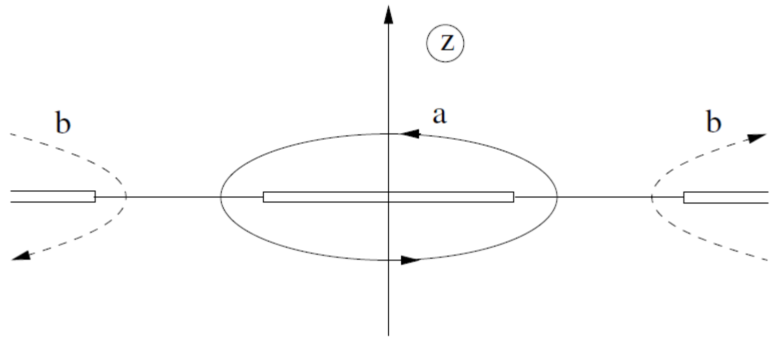

Figure 4: The contour of integration used in the inversion formula above

As $Q_n(z)_{|z|\to\infty} \approx \frac{c}{z^{(n+1)}}$ so the contribution to the integration at $z \to \infty$ can be neglected and thus the integration contour can be deformed into (b) leading to

$$f_n = \frac{1}{\pi}\int_{z_1}^{\infty} Q_n(z)A_1(z,t)dz + \frac{1}{\pi}\int_{-z_2}^{-\infty} Q_n(z)A_2(z,t)dz \tag{65}$$

Now substituting $z_u = -z$ in the second term and using the relation

$$Q_n(-z) = (-1)^{n+1}Q_n(z) \tag{66}$$

We obtain

$$f_n = \frac{1}{\pi}\int_{z_1}^{\infty} Q_n(z)A_1(z,t)dz + \frac{(-1)^n}{\pi}\int_{z_2}^{\infty} Q_n(z_u)A_2(-z_u,t)dz_u \tag{67}$$

The asymptotics of $P_n(z)$ and $Q_n(z)$ at large n are as follows

$$\begin{aligned} P_n(z) \simeq J_0(n\theta) &\sim \cosh(n\chi) \\ Q_n(z) &\sim \exp(-n\chi); \end{aligned} \tag{68}$$

So we see that if we had used $P_n(z)$, the Sommerfeld-Watson representation would have been invalidated as $f_n(z)$ would have grown exponentially.

In (67) the factor $(-1)^n \sim \exp(|\mathrm{Im}n|\pi)$ in the second term gives an oscillating factor which prevents the convergence of the whole term in the right half n plane. Now, we define two analytic functions

$$f_l^{\pm} = \frac{1}{\pi}\int_{z_1}^{\infty} Q_l(z)A_1(z,t)dz \pm \frac{1}{\pi}\int_{z_2}^{\infty} Q_l(z_u)A_2(-z_u,t)dz_u \tag{69}$$

where,

$$f_l^+|_{l=n=2k} = f_n, \quad f_l^-|_{l=n=2k+1} = f_n \tag{70}$$

So we see that in relativistic theory we cannot have a single analytic function which can extrapolate all the partial waves amplitudes, rather we used two functions one for n=even and the other for n=odd.

Statement of Uniqueness (Carlson's Theorem): The theorem states that if one has found a function which is analytic in the region Re $l > l_0$, and its growth is bounded by $\exp(\pm il\pi)$ in the right half plane then the function is unique. And if we have two such functions $f_1$and f_2 then the function $\phi = (f_1(l) - f_2(l))/\sin \pi l$ would be analytic and exponentially decaying in the right half-plane and thus identically zero. This theorem ensures that we have an unique analytic function in (69).

We can represent the scattering amplitude in terms of $f_l^\pm$ as

$$A^+ = \sum_{n=2r} P_n(z)(2n+1)f_n^+, \quad A^- = \sum_{n=2r+1} P_n(z)(2n+1)f_n^- \tag{71}$$

4.3 t-channel partial waves

We rewrite (52) in a different form

$$\frac{1}{2i}[f_n(t+i\epsilon) - f_n(t-i\epsilon)] = \frac{k_t}{16\pi\omega_t} f_n(t+i\epsilon) f_n(t-i\epsilon) \tag{72}$$

Now moving the R.H.S. to the L.H.S. and substituting l in place of n we see that the equation is valid for arbitrary complex l. The equation thus obtained is zero at all even and odd l values and doesn't grow exponentially in the right half $l-$plane, hence it is identically equal to zero:

$$\frac{1}{2i}[f_l^\pm(t+i\epsilon) - f_l^\pm(t-i\epsilon)] = \frac{k_t}{16\pi\omega_t} f_l^\pm(t+i\epsilon) f_l^\pm(t-i\epsilon) \tag{73}$$

Hence the above expression is the continuation of t-channel unitarity condition into the complex $l-$plane.

5 Conclusion

The analytic S-matrix theory is a vast and old subject. We have learnt a few aspects of it which have been reviewed here. Gribov used the projection formula, reviewed in the earlier section, to show that the black disc model is incompatible with the conditions reviewed in earlier sections. This can be seen once we substitute the form of amplitude $sf(t)$ for finite interaction radius which is prescribed by the black disc model. In the limit $|z| \to \infty$ we know

$$Q_n(z) \sim \frac{c}{z^{n+1}}$$

When one substitutes the above asymptotic behaviour of $Q_n(z)$ and the form of $A(s,t)$ into (69) one finds (using the pomeranchuk theorem)

$$f_l^{\pm} \simeq \int_{z_1}^{\infty} \frac{c}{z^{l+1}}\, 4z\, k_t^2 f(t) dz$$

which can be further simplified into

$$f_l^{\pm} \simeq r(t) \int dz \frac{1}{z^l}$$
$$f_l^{\pm} \simeq \frac{r(t)}{l-1}$$

But this contradicts with (73) as the L.H.S has a single pole whereas the R.H.S has double pole at $l \to 1$. Therefore one concludes that hadron cannot be thought as an object with fixed interaction radius.

The aspects covered here have found applications in recent times to observables in conformal field theory(CFT). Our aim was to master the relevant technology in S-matrix theory before turning to its applications to CFTs. We briefly sketch out the CFT applications below which we want to learn in due course.

Conformal field theory is quantum field theory that is invariant under conformal transformations. As in quantum field theory S-matrix elements form the most important data from which one can derive most of the observables,in conformal field theory correlators forms the important data which are characterised by the spectrum of primary operators and three-point function of local operators.

For instance, the 4-point correlator of scalar primary operators is given as

$$\langle O_4(x_4)O_3(x_3)O_2(x_2)O_1(x_1)\rangle = \frac{1}{x_{12}^{2}{}^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}} \times \left(\frac{x_{14}^2}{x_{24}^2}\right)^2\left(\frac{x_{12}^2}{x_{13}^2}\right)\mathcal{G}(z,\bar{z}) \tag{74}$$

where, a= $\frac{1}{2}(\Delta_2-\Delta_1)$, b= $\frac{1}{2}(\Delta_3-\Delta_4)$, and z and $\bar{z}$ are the conformal cross ratios defined as

$$z\bar{z} = \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}, \quad (1-z)(1-\bar{z}) = \frac{x_{23}^2x_{14}^2}{x_{13}^2x_{24}^2} \tag{75}$$

The operator product expansion in the s-channel around the limit where the points(x_1 and x_2) coincide

$$\mathcal{G}(z,\bar{z}) = \sum_{J,\Delta} f_{12\mathcal{O}}\ f_{43\mathcal{O}}\ \ G_{J,\Delta}(z,\bar{z}) \tag{76}$$

where J and Δ are spin and dimension of the exchanged primary operator $\mathcal{O}$ and the conformal blocks G(J,Δ) resum the descendants of $\mathcal{O}$. These blocks are eigenfunctions of the quadratic and quartic Casimir invariants of the conformal group.

$\mathcal{G}(z,\bar{z})$ admits an integral representation over continuous dimensions as is already shown in [9]:

$$\mathcal{G}(z,\bar{z}) = 1_{12}1_{34} + \sum_{J=0}^{\infty}\int_{d/2-i\infty}^{d/2+i\infty}\frac{d\Delta}{2\pi i}c(J,\Delta)F_{J,\Delta}(z,\bar{z}) \tag{77}$$

Here we note that $c(J,\Delta)$ is similar to the partial wave coefficients. The first term in the above expression is identity contribution. In a recent significant work, by Simon Caron-Hout, a lorentzian inverse to this representation has been worked out, which expresses the coefficients $c(J,\Delta)$ analytically in J and in terms of positive lorentzian data.

Quoting the results, the s-channel OPE coefficients

$$c(J,\Delta) = c^t(J,\Delta) + (-1)^J c^u(J,\Delta) \tag{78}$$

where the contribution from each channel is

$$c^t(J,\Delta) = \frac{\kappa_{J+\Delta}}{4}\int_0^1 dzd\bar{z}\mu(z,\bar{z})G_{\Delta+1-d,J+d-1}(z,\bar{z})dDisc[\mathcal{G}(z,\bar{z})] \tag{79}$$

We aim to understand the details of the above stated results. To process the substantial understanding of the background material we have gained so far will be crucial.

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