1 Solving the KKT system

1.1 KKT system without inequality constraints

If there are no inequality constraints, the KKT system has the form

$$Hx + A'\nu = -q \tag{1}$$
$$Ax = b \tag{2}$$

Here H is symmetric and positive semidefinite (convexity!) and A satisfies $ncol(A) = ncol(H)$. First we assume that H is balanced and positive definite. Then we proceed as follows: do a Cholesky factorization $H = LL'$. Solve

$$HX = LL'X = [A', q] \quad \text{via} \quad LY = [A', q] \text{ then } L'X = Y.$$

Obtain $H^{-1}A'$ and $H^{-1}q$. From the first equation get $x = -H^{-1}(q + A'\nu)$. Stick this into the second equation to obtain

$$AH^{-1}A'\nu = -(b + AH^{-1}q).$$

Here the matrix $S = AH^{-1}A'$ is symmetric and positive semdefinite. It is also invertible since A has full rank. Thus it is positive definite and can be solved with a Cholesky factorization

$$S = R'R.$$

Moreover this matrix is small ($p \times p$) and so this effort is negligible. We set $\mu = R\nu$ and solve for ν in 2 steps via

$$R'\mu = -(b + AH^{-1}q) \text{ then } R\nu = \mu.$$

Next we treat the general case. First we equilibrate the matrix H by passing to $H \to DHD$, for a suitable diagonal matrix $D = diag(d)$ with $d_i \neq 0$, for all i. Setting $x = Dy$ we can rewrite (1) as

$$HDy + A'\nu = -q$$
$$ADy = b$$

Multiply the first equation with D on the left to get

$$My + B'\nu = -Dq \tag{3}$$
$$By = b \tag{4}$$

with $M = DHD$ and $B = AD$. This system is of the same form but with a balanced matrix M instead of H. I.e. we apply the transformation

$$x \to y = D^{-1}x, \quad H \to DHD, \quad A \to AD, \quad q \to Dq. \tag{5}$$

After solving this new system as outlined above we have to undo the transformation $x \to D^{-1}x$ by replacing $x \to Dx$.

If now DHD is not positive definite (this happens exactly if H is not positive definite since the d_i are all nonzero), we pass to the equivalent system

$$DHD \to DHD + rA'A, \quad -q \to -q + rA'b. \tag{6}$$

Here $DHD + rA'A$ is positive definite if and only if the KKT system (3) itself is nonsingular. If it is singular, we stop. Assume now it is nonsingular. Then the new matrix H is positive definite.