Exercise class: propositional logic (solutions)

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Week 5

1. **Logical equivalence**

   We consider the following truth table:

a	b	$a \wedge \neg b$	$a \vee (\neg a \wedge b)$	$a \wedge (\neg a \vee \neg b)$	$\neg(a \rightarrow b)$	$a \vee b \rightarrow \neg a$
true	*true*	*false*	*true*	*false*	*false*	*false*
true	*false*	*true*	*true*	*true*	*true*	*false*
false	*true*	*false*	*true*	*false*	*false*	*true*
false	*false*	*false*	*false*	*false*	*false*	*true*

   It follows that $a \wedge \neg b$ is equivalent to $a \wedge (\neg a \vee \neg b)$ and $\neg(a \rightarrow b)$ but not to $a \vee (\neg a \wedge b)$ or $a \vee b \rightarrow \neg a$.

2. **Conjunctive normal form**

   We find

$$
\begin{aligned}
&\neg(a \vee b) \vee (c \rightarrow \neg d) \\
&\qquad \equiv \neg(a \vee b) \vee (\neg c \vee \neg d) \\
&\qquad \equiv \quad (\neg a \wedge \neg b) \vee (\neg c \vee \neg d) \\
&\qquad \equiv (\neg a \vee \neg c \vee \neg d) \wedge (\neg b \vee \neg c \vee \neg d)
\end{aligned}
$$

$$
\begin{aligned}
&(a \rightarrow (b \rightarrow c)) \rightarrow d \\
&\qquad \equiv \neg(\neg a \vee (\neg b \vee c)) \vee d \\
&\qquad \equiv (\neg\neg a \wedge \neg\neg b \wedge \neg c) \vee d \\
&\qquad \equiv (a \wedge b \wedge \neg c) \vee d \\
&\qquad \equiv (a \vee d) \wedge (b \vee d) \wedge (\neg c \vee d)
\end{aligned}
$$

$$
\begin{aligned}
&a \wedge \neg(b \wedge (c \vee \neg(d \vee e))) \\
&\qquad \equiv a \wedge (\neg b \vee (\neg c \wedge \neg\neg(d \vee e))) \\
&\qquad \equiv a \wedge (\neg b \vee (\neg c \wedge (d \vee e))) \\
&\qquad \equiv a \wedge (\neg b \vee \neg c) \wedge (\neg b \vee d \vee e)
\end{aligned}
$$

3. **Refutation**
Let us first convert $\neg(\neg(a \to b \lor c) \to \neg(d \land e))$ into CNF:

$$\begin{aligned}
&\neg(\neg(a \to b \lor c) \to \neg(d \land e)) \\
&\qquad \equiv \neg(\neg\neg(\neg a \lor b \lor c) \lor \neg(d \land e)) \\
&\qquad \equiv \neg(\neg a \lor b \lor c \lor \neg(d \land e)) \\
&\qquad \equiv (\neg\neg a \land \neg b \land \neg c \land \neg\neg(d \land e)) \\
&\qquad \equiv a \land \neg b \land \neg c \land d \land e
\end{aligned}$$

- We need to show that $\{a, \neg b, \neg c, d, e, \neg a\}$ is unsatisfiable, by showing that the empty clause can be derived using resolution. From a and $\neg a$ we immediately obtain the empty clause using resolution.
- We need to show that $\{a, \neg b, \neg c, d, e, \neg a \lor b \lor \neg e\}$ is unsatisfiable. From $\neg a \lor b \lor \neg e$ and a we derive
$$b \lor \neg e$$
which combined with e gives
$$b$$
which together with $\neg b$ allows us to derive the empty clause.

4. **Refutation II**
$K \cup \{\neg \alpha\}$ contains the following clauses:

$$\begin{array}{ll}
(1) & a \lor e \\
(2) & a \lor \neg f \\
(3) & b \lor d \\
(4) & c \\
(5) & \neg d \\
(6) & \neg e \lor f \\
(7) & \neg a \lor \neg b \\
(8) & \neg a \lor \neg c
\end{array}$$

Using resolution we find:

$$\begin{array}{rll}
(9) & \neg b \lor e & (1+7) \\
(10) & d \lor e & (3+9) \\
(11) & e & (5+10) \\
(12) & \neg c \lor \neg f & (2+8) \\
(13) & \neg f & (4+12) \\
(14) & \neg e & (6+13)
\end{array}$$

Combining (11) and (14) we derive the empty clause.