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Quantum Theory of Nuclear Scattering

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Contents

1 Introduction

When we want to approach a phenomenon, we usually start from its simplest version in order to understand the basic principles that govern it, as well as to gain intuition about it. In that spirit, we'll do the same for nuclear scattering, which is an extremely complex phenomenon since it is Quantum Mechanical in principle and it contains a large amount of degrees of freedom due to the many-body nature of the nucleus itself. In many cases the Rutherford scattering picture is enough for describing the scattering phenomenon but this is just a lucky coincidence since it is the only scattering case whose cross section has the same functional form both in classical and quantum mechanical description.

We will start from the simplest possible case which is the elastic scattering, where both the total energy and momentum as well as the number of scattered particles are conserved. Apart from the simplicity, we shall analyze the elastic case because it is present in all nuclear scattering processes and also highlights its quantum mechanical aspect. We know that when two nuclei collide there are a lot of different phenomena that may happen, like inelastic scattering, knock-out reaction, transfer reaction, compound nucleus reaction etc. The only of these processes that is always present is the elastic one. Thus, it is the first that we have to study both theoretically and experimentally

2 Partial Wave Analysis

A first and simple theoretical approach to the quantum scattering problem in general and therefore to the nuclear scattering problem as well, is called *Partial Wave Analysis*. The choice of this precise name will become clear very shortly. Since we want to describe scattering in quantum mechanical terms and in the non-relativistic regime, we will use the Time Independent Schrodinger Equation.[1] Also, in experiments we place detectors around the sample that we are studying so it is convenient to work in terms of angles, thus we will work with angles and spherical coordinates.

It is important to emphasize the assumptions under which this method is valid. First and foremost, we will not take into consideration the effects of spin but we do not imply that it is unimportant to scattering problems in general. We will ignore the internal structure of the nuclei and instead we will consider a radial interaction potential between them. A consequence of ignoring the internal structure is that we ignore all the possible reactions of the two colliding nuclei and as a result we can only treat elastic collisions where both the energy and the number of particles are conserved. We shall also assume that, in an experimental setup, the target is so thin that any secondary scattering phenomena can be neglected.

Assumptions
- Low energy
- Elastic
- Well-Defined in-out energy
- "Spinless"
- Radial Interaction Potential
- Ignore Internal Structure
- V(r) faster than $1/r$

! **CAUTION**
- Not for Coulomb Scattering

[1] Time dependence vanishes if we consider solutions with well-defined energy, $\Psi(r,t) = \psi(r)e^{-iEt}/\hbar$. If we substitute this expression in the Time Dependent Schrodinger Equation we obtain the time independent one.

2.1 *Building the Solution*

We have assumed that the interaction potential is radial, so it depends only on the relative distance between the colliding particles and not their orientation in space, so we can factor out the radial part of the wave function

$$\Psi = R(r)Y_{lm}(\theta, \phi) \tag{1}$$

There is only the radial part of the Schrodinger equation left

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right] u = Eu \tag{2}$$

where $u(r) = rR(r)$, V(r) is the interaction potential and the second term in the parenthesis is a called *centrifugal term* and tends to throw the particle away from the origin of the potential, similarly to the centrifugal inertial force in classical mechanics.

Now, we have to solve this second order differential equation for u(r) and as a result for R(r). Many times when we try to approach a physical problem, prior to solving it, we want to highlight some properties of the solution that contain physical meaning and provide information about some particular regions of the problem. An obvious one is its asymptotic behavior, or in other words its behavior for $r \to \infty$. It is natural to consider a localized potential which means that in the region of infinite distance it will be $V(r \to \infty) \to 0$ and there the certifugal term will be 0 as well. That is, the scattered particles will be free at $r \to \infty$ and consequently their dispersion relation is $E \simeq p^2/2m = \hbar^2 k^2/2m$. From (2) we have

$$\begin{aligned} &\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} \simeq -k^2 u \Rightarrow \\ &u(r) = \underbrace{ce^{ikr}}_{\text{scattered particle}} + \underbrace{de^{-ikr}}_{\text{incoming particle}} \end{aligned} \tag{3}$$

So for the incoming particle we have $d = 0$ and the asymptotic solution is

$$R(r) \simeq c\frac{e^{ikr}}{r} \tag{4}$$

Which resembles to a monochromatic spherical EM wave. Now we will discuss the origin of the term *"partial"* in the title of the method.

If it is not yet clear, this method best describes a nuclear scattering in which the incoming particle or nucleus has much smaller mass than the other nucleus which we consider to be stationary (target) and the "source" of the interacting potential.[2] So, now we will approach an other region, the Intermediate region, in order to obtain other properties of the full solution. In Figure (1) we see the different regions that we divide space in order to solve the problem. In the intermediate region we assume that $V(r) \to 0$. For the validity of this, the interacting potential between he two nuclei must

[2] Of course that is not exactly accurate but an intuitive way to think about this method as seen by the lab frame of reference. If we wanted to be more precise, the problem we are trying to solve is the problem of two colliding particles transformed to the center of mass frame. This reduced problem can be interpreted as the scattering of a single particle with the reduced mass from an external potential V(r).

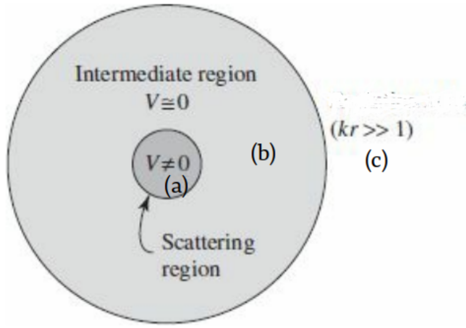

Figure 1: Roughly the regions that we separate space in order to approach the scattering problem. (a.) Asymptotic Region, $r \to \infty$ (b.) Intermediate region V=0 and (c.) Scattering region

go to zero quicker than the centrifugal which goes as $1/r^2$. Now that the Potential term is much smaller than the centrifugal, we have

The radial equation reads

$$\frac{d^2u}{dr^2} - \frac{l(l+1)}{r^2}u = -k^2u \Rightarrow$$

The solutions of this equation are well-known in bibliography[3] and they are associated with the *Spherical Bessel*, $j_l(r)$ and the *Spherical Neuman*, $n_l(r)$ functions

[3] For example *Gasiorovich S., Quantum Physics, 3rd Edition*

$$r \cdot j_l(kr) = r(-kr)^l \left(\frac{1}{kr}\frac{d}{dkr}\right)^l \frac{sinkr}{kr} \tag{5}$$

$$r \cdot n_l(kr) = -r(-kr)^l \left(\frac{1}{kr}\frac{d}{dkr}\right)^l \frac{coskr}{kr} \tag{6}$$

But it is obvious that the solutions in the above form, cannot reproduce spherical waves as obtained in the $r \to \infty$ region. Therefore, we can manipulate them however it suits us in order to reproduce our spherical waves. We observe that a linear combination would be sufficient for that cause [4]

[4] From the linearity of the ODE we have that a linear combination of two solutions is also a solution.

$$h_l^{(1)}(kr) := j_l(kr) + in_l(kr) \tag{7}$$

$$h_2^{(2)}(kr) := j_l(kr) - in_l(kr) \tag{8}$$

These are called *spherical Hankel* functions. But from a physical point of view, if we make a comparison with the $r \to \infty$ region, one of them will represent the incoming and the other the scattered particle. Thus, we must reject one of them. If we do our algebra, from (7) & (8) we have

$$h_l^{(1)}(kr) = -i(-1)^l \left(\frac{1}{kr}\frac{d}{dkr}\right)^l \frac{e^{ikr}}{kr} \tag{9}$$

$$h_l^{(2)}(kr) = -i(-1)^l \left(\frac{1}{kr}\frac{d}{dkr}\right)^l \frac{e^{-ikr}}{kr} \tag{10}$$

So we immediately reject the 2nd Spherical Hankel function as the one which corresponds to the incoming particle whose wave function is known.

If we suppose that in incident waves are plane and travel along z axis then $\psi_{in} = e^{ikz}$. The full solution, including the angular part, is the wave function of the incident plus the wave function of the scattered. The latter, being a general solution of the radial Schrodinger equation is the sum of the partial solutions (9)

$$\psi(r,\theta,\phi) = A\left(e^{ikz} + \sum_{l,m} C_{l,m} h_l^{(1)}(kr) Y_{l,m}(\theta,\phi)\right) \tag{11}$$

Solution in the Intermediate region where V(r)$\simeq$ 0

If we kill the ϕ-dependence which means that we assume a spherically symmetric scattered wave, then we have the following identity for the spherical harmonics

$$Y_{l,0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(cos\theta) \tag{12}$$

So the solution reads

$$\psi(r,\theta) = A\left(e^{ikz} + \sum_{l=0} C_{l,0} h_l^{(1)}(kr) \sqrt{\frac{2l+1}{4\pi}} P_l(cos\theta)\right) \tag{13}$$

Now, we can define the following parameter α_l

$$\alpha_l = \frac{C_{l,0}}{i^{l+1} k \sqrt{4\pi(2l+1)}} \tag{14}$$

Why we do we choose this as α_l? ...

The solution becomes

$$\psi(r,\theta) = A\left(e^{ikz} + k \sum_{l=0} i^{l+1}(2l+1)\alpha_l h_l^{(1)}(kr) P_l(cos\theta)\right) \tag{15}$$

So the solution in the intermediate region is a sum of *partial waves* with different angular momentum l. Each wave has its own amplitude or in other words its own probability of rising, α_l

2.2 The Scattering Amplitude

We have solved the Schrodinger equation in regions of $V(r) = 0$. Now we wanna examine how the solution (15) behaves in larger r. In nuclear physics experiments this means a few mm or cm away from the target. If we use the property of 1st Hankel function for large r which is shown in the blue box, we have

$h_l^{(1)}(kr) \xrightarrow{kr>>1} \frac{1}{kr}(-i)^{l+1} e^{ikr}$

$$\begin{aligned}\psi(r,\theta) \simeq & A\left(e^{ikz} + \cancel{k} \sum_{l=0} \cancel{i^{l+1}}(2l+1)\alpha_l \frac{1}{\cancel{k}r}(\cancel{-i})^{l+1} e^{ikr} P_l(cos\theta)\right) \Rightarrow \\ \simeq & A\left(\underbrace{e^{ikz}}_{in} + \underbrace{f(\theta)\frac{e^{ikr}}{r}}_{scattered}\right)\end{aligned} \tag{16}$$

Where

$$f(\theta) = \sum_{l=0} \underbrace{(2l+1)\alpha_l}_{f_l} P_l(\cos\theta) \tag{17}$$

is called *scattering amplitude*.

2.3 *And why do we care about the Scattering Amplitude?*

To quote a professor of ours, clearly experimentalist :), "*Experiment is the God of physics*". In that spirit, every theory that we build should be stimulated by experiments and must have some channels that can be experimentally confirmed. In a higher level, this also applies to the solutions that we give inside our theories. So, randomly defining and labeling parameters and variables is of no use, unless they are related to a measurable quantity.

Which gives rise to the question: Why do we care about the Scattering Amplitude?

The answer can be given in three ways. A mathematically formal one, starting from the Lippmann-Schwinger equation and using integral equations and Green's functions. Due to lack of space and time and clearly... not due to lack of understanding of Green's functions, we will start from the second one which is more intuitive and stems from a more classical point of view. However, it works, and since we know the correct answer, we will act as if it were flawless.

Say that we have a target and an incident particle travelling with velocity u which passes through the infinitesimal area $d\sigma$ in time dt. The probability of the particle passing through an infinitesimal volume dV which has $d\sigma$ as the normal surface and udt as the longitudinal length is

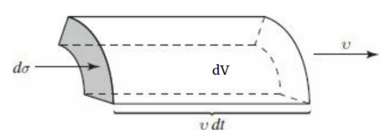

Figure 2: The infinitesimal volume dV through which the nucleus passes

$$\begin{aligned} dP_{in} =& |\psi_{in}|^2 dV = |Ae^{-ikz}|^2 (udt) d\sigma \Rightarrow \\ =& |A|^2 (udt) d\sigma \end{aligned} \tag{18}$$

The infinitesimal volume can also be written in terms of solid angle $dV = r^2 dr d\Omega = r^2(udt)d\Omega$. So, the probability of being scattered into the volume dV is

$$\begin{aligned} dP_{out} =& |\psi_{scatt}|^2 dV \xrightarrow[r>>1]{(16)} \\ =& \left| \frac{Af(\theta)e^{ikr}}{\cancel{r^2}} \right|^2 ud\cancel{t} d\Omega \Rightarrow \\ =& |A|^2 |f(\theta)|^2 udtd\Omega \end{aligned} \tag{19}$$

Naturally a particle that enters a volume dV will certainly be scattered into the same volume dV. So the two probabilites are equal and from (18). (19) we have

$$\begin{aligned} dP_{in} =& dP_{out} \Rightarrow \\ \cancel{|A|^2}\cancel{(udt)}d\sigma =& \cancel{|A|^2}|f(\theta)|^2 \cancel{udt} d\Omega \Rightarrow \\ d\sigma =& |f(\theta)|^2 d\Omega \Rightarrow \end{aligned}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta)|^2 \tag{20}$$

The significance of the Scattering Amplitude is now obvious. It is directly related to the differential cross section, which expresses the probability of being scattered in the θ direction and it can be experimentally measured. This probability is theoretically expressed via the scattering amplitude which connects it to each partial waves' amplitude. The importance of this method can also be found in the fact that usually just a few of the partial waves are enough to calculate the differential cross section.[5]

All we have to do is "just"... to calculate the partial wave amplitudes.

Now, that we validated the importance of the scattering amplitude via the intuitive way, we can follow the third route, which is not formally heavier, although it's more precise.

In 1-d Quantum Mechanics, for example in the tunneling problem through a simple square barrier potential, we express the transmission coefficient (percentage of the flux of the initial wave function transmitted past the potential) as the ratio of the in- going probability current over the transmitted. Accordingly, we can express the differential cross section as the ratio of the probability current that flows through surface dA over the probability current of the incoming particles in solid angle $d\Omega$ which corresponds to dA

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} =& \frac{|\boldsymbol{J}_{scattered} \cdot \boldsymbol{dA}|}{|\boldsymbol{J}_{in}|d\Omega} \xrightarrow[Appendix A]{\boldsymbol{J}=-\hbar/m \cdot Im[\psi\nabla\psi^*]} \\ =& \frac{|Im\left[\psi_{scattered}\nabla\psi^*_{scattered}\right] \cdot \boldsymbol{dA}|}{|Im\left[\psi_{in}\nabla\psi^*_{in}\right]|d\Omega} \end{aligned} \tag{21}$$

From (16) we have

$$\begin{aligned} \psi_{in} =& e^{ikz} \Rightarrow \\ \psi_{in}\nabla\psi^*_{in} =& e^{ikz} \cdot ike^{-ikz}\hat{z} \Rightarrow \\ =& ik\hat{\boldsymbol{z}} \end{aligned}$$

[5] Krane K. *Introductory Nuclear Physics*. Wiley, 3rd edition edition, 1987

and

$$\begin{aligned}
\psi_{scattered} =& f(\theta)\frac{e^{ikr}}{r} \xRightarrow{\nabla=\partial_r\hat{r}+1/r\partial_\theta\hat{\theta}} \\
\psi_{scattered}\nabla\psi^*_{scattered} =& f(\theta)\frac{e^{ikr}}{r}\left(\frac{-ike^{-ikr}}{r}f^*(\theta)\hat{\boldsymbol{r}} - \frac{e^{-ikr}}{r^2}f^*(\theta)\hat{\boldsymbol{r}} + \frac{e^{-ikr}}{r^2}f'(\theta)\hat{\boldsymbol{\theta}}\right) \Rightarrow \\
=& f(\theta)\frac{1}{r^2}\left(-ikf^*(\theta)\hat{\boldsymbol{r}} - \frac{1}{r}f^*(\theta)\hat{\boldsymbol{r}} + \frac{f'(\theta)}{r}\hat{\boldsymbol{\theta}}\right) \xRightarrow[1/r^2 \text{ dominates}]{r\to\infty} \\
=& -|f(\theta)|^2\frac{ik}{r^2}\hat{\boldsymbol{r}}
\end{aligned}$$

So for r which are a lot larger than the radius of the target nucleus, we have that the probability current is radial, thus the surface through which it flows has surface element $\boldsymbol{dA} = \hat{\boldsymbol{r}}dA = \hat{\boldsymbol{r}}r^2d\Omega$. From (20)

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} =& \frac{|f(\theta)|^2k/r^2 \cdot r^2d\Omega}{kd\Omega} \Rightarrow \\
=& |f(\theta)|^2
\end{aligned} \tag{22}$$

We derived the same equation to connect the scattering amplitude with an easy to measure quantity. However, using these arguments we it became clear that is holds only for larger distances r from the target nucleus.

2.4 *Remaining part of the solution*

The furthest that we have gone into solving the problem is equation (15). But it's a little irrational to express some terms in Cartesian and some others in Spherical coordinates. In order to express e^{ikz} in spherical we can use the Rayleigh formula which expands a plane wave in spherical coordinates

$$e^{ikz} = \sum_{l=0} i^l(2l+1)j_l(kz)P_l(cos\theta) \tag{23}$$

Where $P_l(cos\theta)$ are the Legendre polynomials. Now, if we substitute (23) in (15) we obtain

$$\begin{aligned}
\psi(r,\theta) =& A\left(\sum_{l=0} i^l(2l+1)j_l(kr)P_l(cos\theta) + k\sum_{l=0} i^{l+1}(2l+1)\alpha_l h_l^{(1)}(kr)P_l(\theta)\right) \Rightarrow \\
=& A\sum_{l=0} i^l(2l+1)\left(j_l(kr) + ik\alpha_l h_l^{(1)}(kr)\right)P_l(\cos\theta)
\end{aligned} \tag{24}$$

Here we see that we chose these α_l in (2.1) in order to factor out some terms.

Again, this is the solution of Schrodinger equation in the intermediate regime, where $V(r) = 0$ and the centrifugal term is nonzero. This is where the solution of the equation ends. From now on, in order to approach the scattering region (V$\neq$ 0), we will use (24) as an Ansatz that has as unknown parameters the partial wave amplitudes α_l. By substituting (24) into the Schrodinger equation we will, in general, obtain these amplitudes and as a result an approximate solution.

Giving some physical intuition about (24), we see that the out-coming *partial waves* are uncoupled. This means that they scatter independently and add up in different areas of space to create some peculiar phenomena of diffraction as we will see later. Moreover, each one of them carries a certain amount of angular momentum l, which stays constant during the process of the scattering.

2.5 *What about the phase?*

As we know from Classical and Quantum mechanics, when a wave hits an object its phase changes. For example, during the reflection of a wave from a wall, the wave undergoes a phase change of π. In Electromagnetism, we use the change of phase of a scattered laser beam in order to measure the distance and even the shape of the scattering particles (LIDAR) and in Quantum Mechanics the change in phase shows up even in elementary problems such as the scattering of a particle on a simple square barrier potential. Naturally, we wait that the phase of the scattered particle will change during the collision and all we aim to do is to describe this phenomenon with mathematical arguments.

We will follow the conventional path to give rise to the phase difference which is through the Schrodinger equation. One thing that is important to mention is the fact that in this text, we do not care to answer the questions of how and why the phase changes in fundamental terms. We just care that it changes. This leads us to ignore the evolution of phase in the scattering regime and discuss only about the initial and the final phase of the particle or nucleus. After all, that's what we are measuring in experiments.

As mentioned we will work in the region where the scattering potential V is practically zero relatively to the centrifugal. Then as we have seen, the solutions of the radial Schrodinger equation are the Spherical Bessel $j_l(kr)$ and Spherical Neumann $n_l(kr)$ functions which are given by equations (5) and (6). So, the solution which describes a partial wave with angular momentum l is a linear combination of these functions

$$\begin{aligned} R_l =& A_l j_l(kr) + B_l n_l(kr) \xRightarrow{kr>>1} \\ =& \frac{1}{kr}\left(A_l sin\left(kr - l\frac{\pi}{2}\right) + B_l cos\left(kr - l\frac{\pi}{2}\right)\right) \Rightarrow \\ =& \frac{1}{kr} C_l sin\left(kr - l\frac{\pi}{2} + \delta_l\right) \end{aligned} \tag{25}$$

For $x >> 1$

- $j_l(x) \to \frac{sin(x - l\pi/2)}{x}$
- $n_l(x) \to \frac{cos(x - l\pi/2)}{x}$

A phase has appeared but so far it doesn't help us or give any physical insight. In the intermediate region we have shown that the total solution is given by (16). So, if we massage it a little bit and compare it with (25) maybe we will end up with something useful.

$$\begin{aligned}\psi(r,\theta) =& e^{ikr} + f(\theta)\frac{e^{ikr}}{r} \xRightarrow{(17)} \\ &\simeq \sum_l (2l+1)i^l \frac{sin(kr - l\pi/2)}{kr} P_l(cos\theta) + \sum_l f_l P_l(cos\theta)\frac{e^{ikr}}{r} \Rightarrow \\ &= \sum_l \underbrace{\left\{ (2l+1)i^l \frac{sin(kr - l\pi/2)}{kr} + f_l \frac{e^{ikr}}{r} \right\}}_{R_l} P_l(cos\theta) \Rightarrow \\ R_l =& (2l+1)i^l \frac{sin(kr - l\pi/2)}{kr} + f_l \frac{e^{ikr}}{r} \end{aligned} \tag{26}$$

Rayleigh formula for $kr >> 1$

$$\begin{aligned} e^{ikr} &= \sum_l (2l+1)i^l j_l(kr) P_l \Rightarrow \\ &\simeq \sum_l (2l+1)i^l \frac{sin(kr - l\pi/2)}{kr} P_l \end{aligned}$$

Now we have two different expressions of R_l, so we can compare them (25), (26). After some unpleasing calculations[6]we derive the following

$$\begin{aligned} C_l =& (2l+1)i^l e^{i\delta_l} \\ f_l =& \frac{2l+1}{2ik}\left(e^{2i\delta_l} - 1\right) \end{aligned} \tag{27}$$

[6] The unpleasing calculations in B

At this point we can claim that we have achieved something important: we expressed the scattering amplitude in terms of a single phase δ_l.

$$\begin{aligned} f(\theta) =& \sum_l f_l P_l(cos\theta) \Rightarrow \\ =& \sum_l \frac{2l+1}{2ik}\left(e^{2i\delta_l} - 1\right) P_l(cos\theta) \Rightarrow \\ =& \sum_l \frac{2l+1}{2ik} e^{i\delta_l}(e^{i\delta_l} - e^{-i\delta_l}) P_l(cos\theta) \Rightarrow \\ =& \sum_l \frac{2l+1}{2ik} e^{i\delta_l} 2isin\delta_l P_l(cos\theta) \Rightarrow \\ =& \sum_l \underbrace{\frac{2l+1}{k} e^{i\delta_l} sin\delta_l}_{f_l} P_l(cos\theta) \end{aligned} \tag{28}$$

So we have seen that the phase difference of the incoming and scattered wave is the only parameter that we have to compute in order to find the scattering amplitude and as a result the differential cross section as a function of θ. We have to emphasize again that δ_l refers only to the phase difference between the final and the initial waves, which means that (28) is valid only for a region where the phase of the out-going wave is constant. As a result, it is valid when the whole scattering process is over which is the region that the interaction potential V is zero.

But up to this point, the physical meaning of δ_l which is the phase difference before and after the scattering of the partial wave, with angular momentum l, is not absolutely clear. With the Second Path, we will clarify this physical meaning.

2.6 Optical Theorem

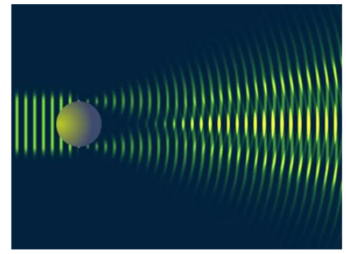

Figure 3: Optical Perspective of Scattering

In scattering phenomena, the central physical quantity is the differential cross section. Thus far we have seen that it is directly connected with the scattering amplitude and as a result it depends only on the phase difference of each partial wave before and after the scattering. So, a reasonable question one might have is what happens with the total cross section.

Luckily, in the elastic scattering case, the answer is rather simple. We just have to substitute (28) into the integral of (20), we obtain

$$\int_{-1}^{+1} P_l(cos\theta)P_{l'}(cos\theta)d(cos\theta) = \\ = \frac{2}{2l+1}\delta_{l,l'}$$

$$\begin{aligned}
\sigma_{tot} &= \int \frac{d\sigma}{d\Omega}d\Omega \xRightarrow{(20)} \\
&= \int |f(\theta)|^2 d\Omega \Rightarrow \\
&= 2\pi \int_{-1}^{+1} f(\theta)f^*(\theta)d(cos\theta) \xRightarrow{(28)} \\
&= 2\pi \int_{-1}^{+1} \sum_{l,l'} f_l f_{l'}^* P_l(cos\theta)P_{l'}(cos\theta)d(cos\theta) \Rightarrow \\
&= 2\pi \sum_{l,l'} f_l f_{l'}^* \frac{2}{2l+1}\delta_{l,l'} \Rightarrow \\
&= 2\pi \sum_{l} f_l f_l^* \frac{2}{2l+1} \Rightarrow \\
&= 2\pi \sum_{l} \frac{(2l+1)^2}{k^2} sin^2\delta_l \frac{2}{2l+1} \Rightarrow \\
&= \frac{4\pi}{k^2} \sum_{l} (2l+1) sin^2\delta_l
\end{aligned} \tag{29}$$

We see, as we may have expected that the total cross section is dependent only on the phase difference δ_l of each partial wave.

An interesting fact about the total cross section is that it can be related to the scattering amplitude for $\theta = 0$. From (28) we have

$$\begin{aligned}
f(\theta = 0) &= \frac{1}{k}\sum_l (2l+1)e^{i\delta_l} sin\delta_l \Rightarrow \\
&= \frac{1}{k}\sum_l (2l+1)(cos\delta_l sin\delta_l + i sin^2\delta_l) \Rightarrow \\
Im[f(\theta = 0)] &= \frac{1}{k}\sum_l (2l+1) sin^2\delta_l \Rightarrow \\
\frac{4\pi}{k} Im[f(\theta = 0)] &= \frac{4\pi}{k^2}\sum_l (2l+1) sin^2\delta_l \xRightarrow{(29)} \Rightarrow \\
\sigma_{tot} &= \frac{4\pi}{k} Im[f(\theta = 0)]
\end{aligned} \tag{30}$$

This relation is known as the *Optical Theorem* and it connects the total cross section with the scattering amplitude at $\theta = 0$. In a Classical point of view, $f(\theta = 0)$ would have been 0 since there is no chance of finding the scattered particle just behind the target nucleus. Thus, there is only Quantum Mechanical explanation of this fact which is experimentally confirmed. The connection with optics is the optical diffraction due to which there light is detected just behind am opaque object (e.g. Fresnel spot).

A Appendix A: Probability Density Current

In a rough approach, it is natural to demand from Quantum Theory to conserve the number of particles in the physical phenomena that it describes, as well as in scattering. So, the following will not be valid for collisions that lead to nuclear reactions where the number, or the type of particles change.

In other words, we want a law similar to the continuity equation in fluid mechanics which states that matter does not disappear or appear out of nowhere, but instead, if it flows out of a region of space then it's imperative that it will appear in an other region following the Navier-Stokes equation for its transfer. The Quantum Mechanical analogue is that instead of mass density we have probability density $|P(\boldsymbol{r},t)|^2 = \psi^*(\boldsymbol{r},t)\psi(\boldsymbol{r},t)$ which is conserved. In order to ensure its transfer around the space we need a current $\boldsymbol{J}$, which is called *probability current*. In order to calculate $\boldsymbol{J}$ we shall start from the continuity equation for probability (conservation of probability)

$$\begin{aligned}
\frac{\partial P}{\partial t} + \nabla \boldsymbol{J} &= 0 \Rightarrow \\
\frac{\partial \psi^*}{\partial t}\psi + \psi^*\frac{\partial \psi}{\partial t} + \nabla \boldsymbol{J} &= 0 \xrightarrow[\partial_t\psi = 1/i\hbar \cdot H\psi]{Schrodinger} \\
-\frac{1}{i\hbar}(H\psi)^*\psi + \frac{1}{i\hbar}\psi^*(H\psi) + \nabla \boldsymbol{J} &= 0 \xrightarrow{H=-\hbar^2/2m\nabla^2 + V(\boldsymbol{r})} \\
\frac{1}{i\hbar}\left[-\left(-\frac{\hbar^2}{2m}\nabla^2\psi^* + \cancel{V(\boldsymbol{r})\psi^*}\right)\psi + \psi^*\left(-\frac{\hbar^2}{2m}\nabla^2\psi + \cancel{V(\boldsymbol{r})\psi}\right)\right] + \nabla \boldsymbol{J} &= 0 \Rightarrow \\
-\frac{\hbar}{2mi}\left(\psi\nabla^2\psi^* - \psi^*\nabla^2\psi\right) + \nabla \boldsymbol{J} &= 0 \Rightarrow
\end{aligned}$$

If we use the analogus of $f \cdot g' = (f \cdot g)' - f' \cdot g$, we have

$$\begin{aligned}
\nabla \boldsymbol{J} =& \frac{i\hbar}{2m}\left(\nabla(\psi\nabla\psi^*) - \cancel{\nabla\psi\nabla\psi^*} - \nabla(\psi^*\nabla\psi) + \cancel{\nabla\psi^*\nabla\psi}\right) \Rightarrow \\
=& \frac{i\hbar}{2m}\nabla\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right) \Rightarrow \\
=& \frac{i\hbar}{2m}\nabla\left(\psi\nabla\psi^* - (\psi\nabla\psi^*)^*\right) \Rightarrow \\
=& \frac{i\hbar}{2m}2i\nabla\left(Im\left[\psi\nabla\psi^*\right]\right) \Rightarrow \\
=& -\frac{\hbar}{m}\nabla\left(Im\left[\psi\nabla\psi^*\right]\right) \Rightarrow \\
\boldsymbol{J} =& -\frac{\hbar}{m}Im\left[\psi\nabla\psi^*\right]
\end{aligned}$$

B Appendix B: The unpleasing calculations

We have two different expressions for R_l and we want to obtain C_l and f_l

$$\begin{cases} R_l = & \frac{1}{kr} C_l sin\left(kr - l\frac{\pi}{2} + \delta_l\right) \\ R_l = & (2l+1)i^l \frac{sin(kr - l\pi/2)}{kr} + f_l \frac{e^{ikr}}{r} \end{cases}$$

Now we use $sin\alpha = (e^{i\alpha} - e^{-i\alpha})/2i$ and we set $\alpha = kr - l\pi/2$

$$\frac{C_l}{2ikr}\left(e^{ia}e^{i\delta_l} - e^{-ia}e^{-i\delta_l}\right) = \frac{(2l+1)i^l}{kr} sin\alpha + \frac{f_l e^{il\pi/2}}{r} cos\alpha + \frac{i f_l e^{il\pi/2}}{r} sin\alpha \Rightarrow$$

$$\frac{C_l}{2ikr}\left((cos\alpha + isin\alpha)e^{i\delta_i} - (cos\alpha - isin\alpha)e^{-i\delta_l}\right) = \left(\frac{(2l+1)i^l}{kr} + \frac{i f_l e^{il\pi/2}}{r}\right) sin\alpha + \frac{f_l e^{il\pi/2}}{r} cos\alpha$$

Now if we compare the cos and the sin terms, we have respectively

$$\begin{cases} \frac{C_l}{kr} isin\delta_l = i\frac{f_l}{r} e^{il\pi/2} \\ \\ \frac{C_l}{kr} cos\delta_l = \frac{2l+1}{kr} i^l + i\frac{f_l}{r} e^{il\pi/2} \end{cases}$$

If we subtract we obtain

$$C_l = (2l+1)i^l e^{i\delta_l} \tag{31}$$

And if add them

$$\begin{aligned} \frac{C_l}{kr} e^{i\delta_l} =& \frac{2l+1}{kr} i^l + 2i\frac{f_l}{r} e^{il\pi/2} \overset{(31)}{\Longrightarrow} \\ (2l+1)i^l e^{i\delta_l} e^{i\delta_l} =& (2l+1)i^l + 2i f_l k e^{il\pi/2} \Rightarrow \\ 2i f_l k e^{il\pi/2} =& (2l+1)i^l \left(e^{2i\delta_l} - 1\right) \Rightarrow \\ f_l =& \frac{2l+1}{2ik} i^l \left(e^{2i\delta_l} - 1\right) e^{-il\pi/2} \Rightarrow \\ =& \frac{2l+1}{2ik} i^l \left(e^{2i\delta_l} - 1\right)(-i)^l \Rightarrow \\ =& \frac{2l+1}{2ik}\left(e^{2i\delta_l} - 1\right) \end{aligned}$$

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