Useful Facts About Sets

1 I. Preliminaries and Notation

1.1 Standard abbreviations

Remark 1.1. Throughout this chapter we use several common abbreviations:

- End of proof is indicated by a symbol such as $\square$.
- "If ..., then ..." may be written as $P \Rightarrow Q$.
- The converse implication is written $Q \Leftarrow P$.
- "If and only if" is abbreviated as "iff", or $P \Leftrightarrow Q$.
- "Therefore" may be written as $\therefore$.

Remark 1.2 (Negation rule)**.** We apply negation directly to symbols:

$$x = y \quad\Longrightarrow\quad x \neq y, \qquad x \in A \quad\Longrightarrow\quad x \notin A.$$

This convention extends to any symbol introduced later. For example, once $\Sigma \models \tau$ is defined, its negation is written $\Sigma \not\models \tau$.

2 II. Sets and the Extensionality Principle

2.1 What is a set?

Definition 2.1. A *set* is a collection of objects, called its *elements*. We use:

$$t \in A \text{ to mean that } t \text{ is an element of } A, \qquad t \notin A \text{ to mean that } t \text{ is not an element of } A,$$

and

$$x = y \text{ to mean that } x \text{ and } y \text{ name the same object.}$$

Remark 2.2. If $A = B$, then for every object t,

$$t \in A \iff t \in B.$$

2.2 Fundamental characterization

Proposition 2.3 (Extensionality)
If A and B are sets such that

$$\forall t\,(t \in A \iff t \in B),$$

then $A = B$.

Remark 2.4. This expresses the idea that a set is completely determined by its members.

3 III. Basic Constructions

3.1 Adjoining an element

Definition 3.1. For any set A and any object t, define

$$A; t \coloneqq A \cup \{t\}.$$

This is the set whose elements are the elements of A, together with the (possibly new) element t.

Proposition 3.2
For any set A and any object t,

$$t \in A \iff A; t = A.$$

4 IV. Common Sets and Notation

4.1 Special sets

Definition 4.1.

- The *empty set* $\varnothing$ is the set with no elements.
- For any object x, the *singleton* $\{x\}$ is the set whose only element is x.
- For any finite list of objects $x_1, \ldots, x_n$, the set

$$\{x_1, \ldots, x_n\}$$

has exactly those objects as elements.

Remark 4.2. For any objects x, y,

$$\{x, y\} = \{y, x\},$$

since order does not matter for sets.

4.2 Infinite sets

Definition 4.3. We use the standard notations

$$\mathbb{N} = \{0, 1, 2, \dots\}$$

for the set of natural numbers, and

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

for the set of all integers.

5 V. Sets Specified by Properties

5.1 Set-builder notation

Definition 5.1. Given a property $P(x)$, the set

$$\{x \mid P(x)\}$$

is the set of all objects x such that $P(x)$ holds.

Example 5.2

- $\{x \in A \mid P(x)\}$ is the set of all elements x in A that satisfy $P(x)$.
- $\{\langle m, n\rangle \mid m < n \text{ in } \mathbb{N}\}$ is the set of all ordered pairs of natural numbers where the first component is smaller than the second.

6 VI. Subsets and Power Sets

6.1 Subsets

Definition 6.1. A set A is a *subset* of a set B, written $A \subseteq B$, if

$$\forall x\, (x \in A \implies x \in B).$$

Remark 6.2. Every set is a subset of itself, and $\varnothing \subseteq A$ for every set A. The statement $\varnothing \subseteq A$ is often called "vacuously true".

6.2 Power set

Definition 6.3. For any set A, the *power set* of A is

$$\mathcal{P}(A) = \{x \mid x \subseteq A\},$$

the set of all subsets of A.

Example 6.4

We have
$$\mathcal{P}(\varnothing) = \{\varnothing\}, \qquad \mathcal{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}.$$

7 VII. Union and Intersection

7.1 Binary operations

Definition 7.1. For sets A and B,
$$A \cup B = \{x \mid x \in A \text{ or } x \in B\},$$
the *union* of A and B, and
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\},$$
the *intersection* of A and B.

Definition 7.2. Sets A and B are *disjoint* if
$$A \cap B = \varnothing.$$
A family of sets is *pairwise disjoint* if any two distinct sets in the family are disjoint.

7.2 Indexed unions and intersections

Definition 7.3. Let $\mathcal{A}$ be a set whose elements are themselves sets. Then
$$\bigcup \mathcal{A} = \{x \mid \exists S \in \mathcal{A},\ x \in S\}$$
is the *union* of all members of $\mathcal{A}$, and
$$\bigcap \mathcal{A} = \{x \mid \forall S \in \mathcal{A},\ x \in S\}$$
is the *intersection* of all members of $\mathcal{A}$ (when it makes sense to consider it).

Example 7.4

If
$$\mathcal{A} = \{\{0, 1, 5\}, \{1, 6\}, \{1, 5\}\},$$
then
$$\bigcup \mathcal{A} = \{0, 1, 5, 6\}, \qquad \bigcap \mathcal{A} = \{1\}.$$

Remark 7.5. When we have a family $(A_n)_{n\in\mathbb{N}}$ of sets indexed by natural numbers, we often write
$$\bigcup_{n\in\mathbb{N}} A_n$$

instead of $\bigcup\{A_n \mid n \in \mathbb{N}\}$.

8 VIII. Ordered Pairs and Sequences

8.1 Ordered pairs

Definition 8.1 (Kuratowski)**.** The *ordered pair* $\langle x, y\rangle$ of objects x and y is defined by

$$\langle x, y\rangle \coloneqq \{\{x\}, \{x, y\}\}.$$

Proposition 8.2

For any objects x, y, u, v,

$$\langle x, y\rangle = \langle u, v\rangle \quad \Longleftrightarrow \quad (x = u \text{ and } y = v).$$

8.2 n-tuples

Definition 8.3. Define n-tuples recursively by

$$\langle x_1, \ldots, x_{n+1}\rangle \coloneqq \big\langle \langle x_1, \ldots, x_n\rangle, x_{n+1}\big\rangle$$

for $n \geq 1$. A *finite sequence* of members of a set A is an n-tuple $\langle x_1, \ldots, x_n\rangle$ with each $x_i \in A$.

8.3 Segments of a sequence

Definition 8.4. Let $S = \langle x_1, \ldots, x_n\rangle$ be a finite sequence. A *segment* of S is a finite sequence

$$\langle x_k, \ldots, x_m\rangle$$

with $1 \leq k \leq m \leq n$. A segment is an *initial segment* if $k = 1$. An initial segment is *proper* if it is different from S.

8.4 A technical lemma

Lemma 8.5 (Lemma 0A)

Suppose

$$\langle x_1, \ldots, x_m\rangle = \langle y_1, \ldots, y_{m+k}\rangle$$

for some $m, k \geq 0$. Then

$$x_1 = \langle y_1, \ldots, y_{k+1}\rangle.$$

Remark 8.6. The proof proceeds by induction on m, using the definition of ordered pairs and the basic property of Kuratowski pairs.

Corollary 8.7

Let A be a set such that no element of A is itself a finite sequence of other elements of A. If

$$\langle x_1, \ldots, x_m \rangle = \langle y_1, \ldots, y_n \rangle$$

and all x_i, y_j belong to A, then $m = n$ and $x_i = y_i$ for all $1 \leq i \leq m$.

9 IX. Cartesian Products and Relations

9.1 Cartesian products

Definition 9.1. For sets A and B, the *Cartesian product* $A \times B$ is the set

$$A \times B = \{\langle x, y \rangle \mid x \in A,\ y \in B\}.$$

More generally, for a set A and integer $n \geq 1$,

$$A^n = \underbrace{A \times \cdots \times A}_{n \text{ times}}$$

is the set of all n-tuples of elements of A.

9.2 Binary relations

Definition 9.2. A *(binary) relation* R is a set of ordered pairs.

Example 9.3

The strict ordering on the set $\{0, 1, 2, 3\}$ is represented by the relation

$$\{\langle 0, 1\rangle, \langle 0, 2\rangle, \langle 0, 3\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}.$$

9.3 Domain, range, and field

Definition 9.4. For a relation R, the *domain* of R is

$$\mathrm{dom}(R) = \{x \mid \exists y,\ \langle x, y \rangle \in R\},$$

the *range* of R is

$$\mathrm{ran}(R) = \{y \mid \exists x,\ \langle x, y \rangle \in R\},$$

and the *field* of R is

$$\mathrm{fld}(R) = \mathrm{dom}(R) \cup \mathrm{ran}(R).$$

9.4 n-ary relations and restriction

Definition 9.5. Let A be a set. An *n-ary relation* on A is a subset of A^n.

- When $n = 1$, an n-ary relation is just a subset of A (a unary relation).
- When $n \geq 2$, this recovers the usual notion of a relation of arity n.

The equality relation on A is the binary relation

$$\{\langle x, x \rangle \mid x \in A\}.$$

Definition 9.6 (Restriction of a relation)**.** Let $R \subseteq A^n$ be an n-ary relation on A and let $B \subseteq A$. The *restriction* of R to B is

$$R \restriction B := R \cap B^n.$$

Example 9.7

Let R be the usual strict order on $\mathbb{N}$, and let $B = \{0, 1, 2, 3\}$. Then $R \restriction B$ is exactly the strict order relation on the set $\{0, 1, 2, 3\}$ shown earlier.

10 X. Functions and Operations

10.1 Functions

Definition 10.1. A relation F is a *function* if for each $x \in \mathrm{dom}(F)$ there is exactly one y such that $\langle x, y \rangle \in F$.

We write

$$F : A \to B$$

to mean that F is a function with domain $\mathrm{dom}(F) = A$ and range $\mathrm{ran}(F) \subseteq B$.

The function F *maps A onto B* (or is *surjective*) if $\mathrm{ran}(F) = B$.

The function F is *one-to-one* (or *injective*) if for each $y \in \mathrm{ran}(F)$ there is exactly one x such that $\langle x, y \rangle \in F$.

10.2 n-ary operations and their restriction

Definition 10.2. An *n-ary operation* on a set A is a function

$$f : A^n \to A.$$

Example 10.3

Addition is a binary operation on $\mathbb{N}$, and the successor mapping $S(n) = n + 1$ is a unary operation on $\mathbb{N}$.

Definition 10.4 (Restriction of an operation)**.** Let $f : A^n \to A$ be an n-ary operation on A, and let $B \subseteq A$. Define

$$g \coloneqq f \cap (B^n \times A).$$

Then g has domain B^n, and

$$g(b_1, \ldots, b_n) = f(b_1, \ldots, b_n)$$

whenever $b_1, \ldots, b_n \in B$.

Proposition 10.5

The restriction g is itself an n-ary operation on B if and only if B is *closed under* f; that is, whenever $b_1, \ldots, b_n \in B$, one has $f(b_1, \ldots, b_n) \in B$.

Example 10.6

Consider addition on $\mathbb{R}$. Its restriction to $\mathbb{N}$ is the usual addition on $\mathbb{N}$. The set $\mathbb{N}$ is closed under this operation, so the restriction is again a binary operation on $\mathbb{N}$.

11 XI. Standard Classes of Relations

11.1 Basic properties

Definition 11.1. Let R be a relation on a set A.

- R is *reflexive on* A if $\langle x, x\rangle \in R$ for all $x \in A$.
- R is *symmetric* if whenever $\langle x, y\rangle \in R$, then $\langle y, x\rangle \in R$.
- R is *transitive* if whenever $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then $\langle x, z\rangle \in R$.
- R satisfies *trichotomy on* A if for every $x, y \in A$, exactly one of the following holds:

$$\langle x, y\rangle \in R, \qquad x = y, \qquad \langle y, x\rangle \in R.$$

11.2 Equivalence relations and orderings

Definition 11.2. A relation R on A is an *equivalence relation* on A if it is reflexive on A, symmetric, and transitive.

Definition 11.3. A relation R on A is an *ordering relation* (or *strict order*) on A if it is transitive and satisfies trichotomy on A.

12 XII. Equivalence Classes

Definition 12.1. Let R be an equivalence relation on a set A. For each $x \in A$, the *equivalence class* of x is

$$[x] = \{y \in A \mid \langle x, y \rangle \in R\}.$$

Proposition 12.2

Let R be an equivalence relation on A.

1. The equivalence classes $[x]$ (for $x \in A$) form a partition of A; that is, every element of A lies in at least one equivalence class, and no element lies in two distinct classes.

2. For any $x, y \in A$,
$$[x] = [y] \iff \langle x, y \rangle \in R.$$