Useful Facts about Sets

Logical Jargon and Symbols – Questions

- When mathematicians write proofs, what kinds of *abbreviations* and symbols do they use instead of full sentences?
- How might we shorten the phrase "If . . . , then . . . "?
- What is the usual shorthand for "if and only if"?
- How do we write "therefore" in symbolic form?
- If $x = y$ is a statement, what should its "denial" look like? What about the denial of $x \in A$?

Logical Jargon and Symbols – Answers

- We use standard mathematical abbreviations to make proofs shorter.
- "If ..., then ..." is often written as

  $$\cdots \Rightarrow \ldots .$$

  The converse implication is written $\Leftarrow$.
- "If and only if" becomes "iff" and is written with

  $$\Leftrightarrow .$$

- "Therefore" is abbreviated by the symbol $\therefore$.
- The *denial* of a statement is written with a slashed symbol:

  $$x \neq y \quad \text{denies } x = y, \qquad x \notin A \quad \text{denies } x \in A.$$

  Similarly, if we write $\Sigma \models \tau$ then its denial is $\Sigma \not\models \tau$.

Next: Once our logical language is settled, what are the basic objects we talk about in set theory?

What is a Set? – Questions

- Informally, what do we mean by a *set*?
- How do we say that an object t is a member of a set A?
- How do we say that t is not a member?
- When are two sets A and B considered to be *the same set*?
- Can you think of real–life examples of sets?

What is a Set? – Answers

- A **set** is a collection of things, called its *members* or *elements*.
- Membership is written

  $$t \in A \quad (\text{“}t \text{ is a member of } A\text{”}).$$

  Non-membership is

  $$t \notin A.$$

- The expression $x = y$ means that x and y are the *same object*.
- Two sets A and B are equal, $A = B$, exactly when they have the same members. This idea is called the **principle of extensionality**.

Question to lead us on: how can we *test* whether two sets are equal just by looking at their members?

Extensionality – Questions

- Suppose $A = B$. What does that tell us about membership of an arbitrary object t?
- Conversely, if every object t satisfies

$$t \in A \quad \text{iff} \quad t \in B,$$

  what can we conclude about A and B?
- Why does this capture the idea that a set is "determined by its members"?

Extensionality – Answers

- If $A = B$, then for every object t we have
$$t \in A \quad \text{iff} \quad t \in B,$$
because A and B share all members.
- Conversely, if for every object t we have
$$t \in A \quad \text{iff} \quad t \in B,$$
then by **extensionality** we conclude that $A = B$.
- Thus a set is completely determined by which objects belong to it.

Next: how can we *adjoin* a new element to an existing set?

Adjoining an Element – Questions

- Given a set A and an object t, how might we describe the set that contains all elements of A together with t?
- What if t is already in A?
- Can we express this "adjoining" operation using the usual set operations?
- What simple test using this operation tells us whether $t \in A$?

Adjoining an Element – Answers

- For a set A and object t, define
  $$A; t$$
  to be the set whose members are:
  1. all members of A, and
  2. possibly the new member t.
- In ordinary notation,
  $$A; t = A \cup \{t\}.$$
- If t already belongs to A, adjoining it does not change the set:
  $$t \in A \quad \text{iff} \quad A; t = A.$$

Next: what special sets play a basic role in set theory?

Special Sets – Questions

- Is there a set with no members at all? What should it be called and how is it written?
- Given an object x, how do we form the set whose only member is x?
- How do we describe a finite set with exactly n distinct members $x_1, \ldots, x_n$?
- Why does $\{x, y\} = \{y, x\}$ hold, even though we wrote the elements in a different order?

Special Sets – Answers

- The **empty set** is the set with no members:

$$\varnothing.$$

  A set that is not empty is called **nonempty**.
- The set containing only x is the **singleton**

$$\{x\}.$$

- For finitely many objects $x_1, \ldots, x_n$ we write

$$\{x_1, \ldots, x_n\}$$

  for the set whose members are exactly those objects.
- Because sets are determined only by which elements they have,

$$\{x, y\} = \{y, x\}.$$

  Order does not matter for sets.

Question: how do we extend this notation to *infinite* sets and to sets defined by conditions?

Infinite Sets and Set-Builder Notation – Questions

- How might we write the set of all natural numbers $0, 1, 2, \ldots$?
- How do we write the set of all integers $\ldots, -2, -1, 0, 1, 2, \ldots$?
- When listing is impossible, how can we describe sets using a *condition* on the elements?
- Can you think of a set of pairs of natural numbers that is conveniently described using such notation?

Infinite Sets and Set-Builder Notation – Answers

- The set of natural numbers is written
$$\{0, 1, 2, \dots\} = \mathbb{N}.$$
- The set of all integers is
$$\{\dots, -2, -1, 0, 1, 2, \dots\} = \mathbb{Z}.$$
- We use **set-builder notation**:
$$\{x \mid \_\_x\_\_\}$$
for the set of all objects x such that the condition between the underscores is true. Restricted form: $\{x \in A \mid \_\_x\_\_\}$.
- Example: the set
$$\{(m, n) \mid m < n \text{ in } \mathbb{N}\}$$
consists of all ordered pairs of natural numbers with first component smaller than the second.

Next: how do we compare sets using the notion of "subset"?

Subsets and Power Sets – Questions

- What does it mean for a set A to be a *subset* of a set B?
- Is every set a subset of itself? Why?
- Is the empty set a subset of every set? Why might this be called "vacuously true"?
- What is the *power set* of a set A?
- Can you compute $\mathcal{P}\varnothing$ and $\mathcal{P}\{\varnothing\}$?

Subsets and Power Sets – Answers

- A is a **subset** of B, written $A \subseteq B$, if every member of A is also a member of B.
- Every set is a subset of itself: $A \subseteq A$.
- The empty set is a subset of every set: checking "for every member of $\varnothing$" is automatically satisfied, since there are no members to violate the condition.
- The **power set** of A is the set of all subsets of A:

$$\mathcal{P}A = \{x \mid x \subseteq A\}.$$

- Examples:

$$\mathcal{P}\varnothing = \{\varnothing\}, \qquad \mathcal{P}\{\varnothing\} = \{\varnothing, \{\varnothing\}\}.$$

New question: how do we combine sets using union and intersection, both for two sets and for whole families of sets?

Union and Intersection – Questions

- Given sets A and B, what should $A \cup B$ represent?
- What about $A \cap B$?
- When are two sets said to be *disjoint* or *pairwise disjoint*?
- If A is a set whose members are themselves sets, how could we define

$$\bigcup A \quad \text{and} \quad \bigcap A?$$

- Try the concrete example

$$A = \{\{0, 1, 5\}, \{1, 6\}, \{1, 5\}\}.$$

  What are $\bigcup A$ and $\bigcap A$?

Union and Intersection – Answers

- The **union** of A and B,
$$A \cup B,$$
is the set of all things that are members of A or B (or both).
- The **intersection** $A \cap B$ is the set of all things that are members of both A and B.
- Sets A and B are **disjoint** if $A \cap B = \varnothing$. A collection is **pairwise disjoint** if any two distinct members are disjoint.
- For a set A of sets,
$$\bigcup A = \{x \mid x \text{ belongs to some member of } A\},$$
and, for nonempty A,
$$\bigcap A = \{x \mid x \text{ belongs to all members of } A\}.$$
- For $A = \{\{0, 1, 5\}, \{1, 6\}, \{1, 5\}\}$ we get
$$\bigcup A = \{0, 1, 5, 6\}, \qquad \bigcap A = \{1\}.$$

Next: how can we talk about *ordered* information, where order matters?

Ordered Pairs and Tuples – Questions

- Sets ignore order, but sometimes order matters. How do we encode the ordered pair $\langle x, y \rangle$?
- What condition must any definition of ordered pair satisfy?
- What standard set-theoretic definition works?
- How can we build ordered triples and n-tuples from ordered pairs?
- What is a finite sequence (or string) of elements of a set A?

Ordered Pairs and Tuples – Answers

- We want
$$\langle x, y\rangle = \langle u, v\rangle \quad \text{iff} \quad x = u \text{ and } y = v.$$
- Any definition with this property is acceptable.
- The standard one is
$$\langle x, y\rangle = \big\{\{x\}, \{x, y\}\big\}.$$
- Ordered triples are defined by
$$\langle x, y, z\rangle = \langle\langle x, y\rangle, z\rangle,$$
and in general
$$\langle x_1, \dots, x_{n+1}\rangle = \langle\langle x_1, \dots, x_n\rangle, x_{n+1}\rangle.$$
- A **finite sequence** of members of A is an n-tuple $\langle x_1, \dots, x_n\rangle$ with each $x_i \in A$.

Question: under what conditions does equality of two finite sequences force them to have the same length?

Sequences and a Lemma – Answers

- If
$$\langle x_1, \ldots, x_n \rangle = \langle y_1, \ldots, y_n \rangle,$$
then clearly $x_i = y_i$ for $1 \leq i \leq n$.
- But if
$$\langle x_1, \ldots, x_m \rangle = \langle y_1, \ldots, y_n \rangle,$$
it need *not* follow in general that $m = n$ (because some x_i could themselves be finite sequences).
- Lemma (0A in the text): if no member of a set A is itself a finite sequence of other members of A, then any equality
$$\langle x_1, \ldots, x_m \rangle = \langle y_1, \ldots, y_n \rangle$$
with each $x_i, y_j \in A$ implies $m = n$ and $x_i = y_i$.
- Proof idea: induction on m, using the defining property of ordered pairs.

Next: how do we combine sets using all possible ordered pairs of their elements, and what structure does that give us?

Cartesian Products and Relations – Questions

- Given sets A and B, how do we form the set of all ordered pairs with first component in A and second in B?
- What is A^n in this context?
- How can we see a relation on a set A as a certain subset of $A \times A$?
- For a relation R, what are the *domain*, *range*, and *field*?
- Can you describe, using ordered pairs, the usual ordering of the numbers 0 to 3?

Cartesian Products and Relations – Answers

- The **Cartesian product**

$$A \times B = \{\langle x, y\rangle \mid x \in A,\ y \in B\}.$$

- A^n is the set of all n-tuples of members of A. For example, $A^3 = (A \times A) \times A$.
- A **relation** R is a set of ordered pairs. A binary relation on A is a subset of $A^2 = A \times A$.
- The **domain** of R:

$$\operatorname{dom} R = \{x \mid \exists y\, \langle x, y\rangle \in R\}.$$

  The **range**:

$$\operatorname{ran} R = \{y \mid \exists x\, \langle x, y\rangle \in R\}.$$

  The **field** is $\operatorname{fld} R = \operatorname{dom} R \cup \operatorname{ran} R$.
- Example: the usual ordering on $\{0, 1, 2, 3\}$ can be represented by

$$\{\langle 0,1\rangle, \langle 0,2\rangle, \langle 0,3\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}.$$

Next: how do we generalize relations and then isolate the special case of *functions*?

n-ary Relations and Restrictions – Questions

- What is an n-ary relation on a set A?
- What is a unary (one-place) relation?
- How is the equality relation on A expressed as a set of ordered pairs?
- Given a relation R on A and a subset $B \subseteq A$, how do we define the *restriction* of R to B?
- How would restricting the ordering relation on $\mathbb{N}$ to the subset $\{0, 1, 2, 3\}$ look?

n-ary Relations and Restrictions – Answers

- An *n-ary* **relation** on A is a subset of A^n.
- A **unary relation** is just a subset of A.
- The equality relation on A is

$$\{\langle x, x\rangle \mid x \in A\}.$$

- If R is an n-ary relation on A and $B \subseteq A$, its **restriction** to B is

$$R \cap B^n.$$

- Restricting the ordering relation on $\mathbb{N}$ to $\{0, 1, 2, 3\}$ gives exactly the set of pairs displayed earlier.

Next: among all relations, which ones qualify as *functions*?

Functions – Questions

- How can a function be viewed as a special kind of relation?
- What does it mean for a relation to be *single-valued*?
- Given a function F, what are its domain and range?
- What does it mean for F to map A into B or *onto* B?
- When is a function *one-to-one*?

Functions – Answers

- A **function** F is a relation with the property of being **single-valued**: for each x in $\operatorname{dom} F$ there is exactly one y with $\langle x, y\rangle \in F$.
- This unique y is denoted $F(x)$.
- We say that F **maps** A **into** B and write

  $$F : A \to B$$

  if $\operatorname{dom} F = A$ and $\operatorname{ran} F \subseteq B$.
- If in addition $\operatorname{ran} F = B$ we say that F maps A **onto** B.
- F is **one-to-one** (injective) iff every y in the range comes from a unique x: if $\langle x, y\rangle \in F$ and $\langle x', y\rangle \in F$ then $x = x'$.
- The notation extends to n-tuples:

  $$F(x_1, \ldots, x_n) = F(\langle x_1, \ldots, x_n\rangle).$$

Question: what is an n-ary *operation* on a set, and how do we restrict such an operation?

Operations and Identity – Questions

- What is an n-ary *operation* on a set A?
- Can you give examples of unary and binary operations on $\mathbb{N}$?
- Given an n-ary operation f on A and a subset $B \subseteq A$, how can we define its restriction to B?
- When do we say that B is *closed under* f?
- What is the identity function on A, and how is it written as a set of ordered pairs?

Operations and Identity – Answers

- An n-**ary operation** on A is a function
$$f : A^n \to A.$$
- Examples on $\mathbb{N}$:
  - Addition is a binary operation.
  - The successor operation $S(n) = n + 1$ is unary.
- If f is an n-ary operation on A and $B \subseteq A$, the **restriction** of f to B is
$$g = f \cap (B^n \times A).$$
- B is **closed under** f if $f(b_1, \ldots, b_n) \in B$ whenever each $b_i \in B$.
- The identity function on A is given by
$$\mathrm{Id}(x) = x \quad \text{for } x \in A,$$
and as a set of pairs
$$\mathrm{Id} = \{\langle x, x\rangle \mid x \in A\}.$$

Next: how do we classify relations according to properties like reflexive, symmetric, and transitive?

Properties of Relations – Questions

- For a relation R on a set A, what does it mean for R to be:
  - reflexive?
  - symmetric?
  - transitive?
- What is the *trichotomy* property on A?
- Combining these properties, how do we recognize:
  - an *equivalence relation*?
  - an *ordering relation*?

Properties of Relations – Answers

- A relation R on A is:
  - **reflexive** on A if $\langle x, x \rangle \in R$ for every $x \in A$.
  - **symmetric** if whenever $\langle x, y \rangle \in R$, then also $\langle y, x \rangle \in R$.
  - **transitive** if whenever $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$, then $\langle x, z \rangle \in R$.
- R satisfies **trichotomy** on A if for every $x, y \in A$ exactly one of the three possibilities holds:
$$\langle x, y \rangle \in R, \quad x = y, \quad \langle y, x \rangle \in R.$$
- R is an **equivalence relation** on A iff it is a binary relation on A that is reflexive, symmetric, and transitive.
- R is an **ordering relation** on A iff it is transitive and satisfies trichotomy on A.

Final topic: how do equivalence relations carve a set into disjoint pieces called *equivalence classes*?

Equivalence Classes – Questions

- Let R be an equivalence relation on a set A. For an element $x \in A$, what set of elements naturally goes with x?
- How do we define the *equivalence class* $[x]$?
- How do the equivalence classes of R sit inside A? Do they overlap? Do they cover A?
- For $x, y \in A$, how can we tell when $[x] = [y]$ purely in terms of the relation R?

Equivalence Classes – Answers

- For an equivalence relation R on A and $x \in A$, the **equivalence class** of x is
$$[x] = \{y \mid \langle x, y \rangle \in R\}.$$
- Each equivalence class is a subset of A. The collection of all equivalence classes **partitions** A:
  - every element of A lies in some class, and
  - no element lies in more than one class.
- For $x, y \in A$, we have
$$[x] = [y] \quad \text{iff} \quad \langle x, y \rangle \in R.$$

These ideas about sets, relations, functions, and equivalence classes form the basic toolkit that will be used throughout the rest of the course.