1.1 Elementary Set Theory

What is a set? (Question)

- In everyday language we often talk about *collections*:
  - the students in this class,
  - the books on a shelf,
  - the prime numbers less than 20.
- Mathematicians call such a collection a set.

Think:

1. Can we treat $\{3, 7, 11, \pi\}$ as a single mathematical object?
2. Is 7 "in" this set? Is 2?
3. When we write a set, does the *order* of the elements matter?

What is a set? (Answers)

- A **set** is a collection of distinct objects, called its **elements**.
- We write $a \in A$ if a is an element of a set A, and $a \notin A$ if it is not.
- For $A = \{3, 7, 11, \pi\}$:

$$7 \in A, \quad \pi \in A, \quad 2 \notin A.$$

- The order and repetition do not matter:

$$\{3, 7, 11, \pi\} = \{11, 3, \pi, 7\}.$$

Next question: What sets of numbers show up so often that we give them special symbols?

Familiar number systems (Question)

We often see symbols like $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

Guess from examples:

- Where should each of the following live?

$$0, \quad 5, \quad -3, \quad \frac{3}{2}, \quad \pi$$

Questions

1. Which of these sets should contain $0, 1, 2, 3, \ldots$?
2. Which should also include negative whole numbers?
3. Which should contain fractions like $\frac{3}{2}$?
4. Which should contain "all" points on the number line?

Familiar number systems (Answers)

- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ is the set of **natural numbers**.
- $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is the set of **integers**.
- $\mathbb{Q}$ is the set of **rational numbers** (numbers that can be written as p/q with $p, q \in \mathbb{Z}$, $q \neq 0$).
- $\mathbb{R}$ is the set of **real numbers**.

Classification examples:

$$0 \in \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}, \quad -3 \in \mathbb{Z}, \quad \frac{3}{2} \in \mathbb{Q}, \quad \pi \in \mathbb{R} \setminus \mathbb{Q}.$$

Next question: How do we compare two sets? When do we say they are "the same" or that one is "contained in" the other?

Comparing sets (Question)

Consider

$$A = \{1, 2, 3\}, \quad B = \{3, 1, 2\}, \quad C = \{1, 2, 3, 4\}, \quad D = \varnothing.$$

Think:

1. Are A and B the same set?
2. Is A "contained in" C? Is C contained in A?
3. Can we say that D is contained in A? in C?
4. What should it mean for a set to be a *proper* subset of another?

Comparing sets (Answers)

- Two sets are **equal**, $A = B$, if they have exactly the same elements. Thus $A = B$ above.
- We say A is a **subset** of B, written $A \subseteq B$, if every element of A is also an element of B.
- A **proper subset** is a subset that is not equal:

$$A \subset B \quad \text{means} \quad A \subseteq B \text{ and } A \neq B.$$

  In our example, $A \subset C$ and $A \neq C$, so $A \subset C$.
- The **empty set** $\varnothing$ is the set with no elements.

New puzzle: Why should $\varnothing \subseteq A$ hold for *every* set A?

Why is $\varnothing \subseteq A$? (Question)

Suppose someone doubts that $\varnothing \subseteq A$ for a given set A.

Try a proof-by-contradiction:

1. What would it mean to say $\varnothing \not\subseteq A$?
2. That would mean there exists some $x \in \varnothing$ with $x \notin A$.
3. But what elements does $\varnothing$ have?

Also:

- When should we call two sets *disjoint*?
- Are $E = \{1, 2, 3\}$ and $F = \{4, 5\}$ disjoint?
  What about E and $\{3, 4\}$?

Empty subset and disjoint sets (Answers)

- If we assume $\varnothing \not\subseteq A$, then there must be some $x \in \varnothing$ with $x \notin A$.
- But $\varnothing$ has *no* elements, so such an x cannot exist.
- Therefore the assumption is impossible, and

$$\varnothing \subseteq A \quad \text{for every set } A.$$

- Two sets A and B are **disjoint** if they have no elements in common. Equivalently, $A \cap B = \varnothing$.
- Example: $\{1, 2, 3\}$ and $\{4, 5\}$ are disjoint, but $\{1, 2, 3\}$ and $\{3, 4\}$ are not.

Next question: How can we use a property $P(x)$ to carve out a subset of a given set?

Building subsets from properties (Question)

Let A be a set and $P(x)$ a property about elements x of A.

We can form the set

$$\{x \in A : P(x)\},$$

called the **truth set** of P in A.

Try these:

1. $A_1 = \{x \in \mathbb{N} : 3 < x < 11\}$.
2. $B_1 = \{y \in \mathbb{Z} : y^2 = 4\}$.
3. $C_1 = \{z \in \mathbb{N} : z \text{ is a multiple of } 3\}$.

Write out the elements of each set.

Building subsets from properties (Answers)

- $A_1 = \{4, 5, 6, 7, 8, 9, 10\}$.
- $B_1 = \{2, -2\}$ (both square to 4).
- $C_1 = \{0, 3, 6, 9, 12, \dots\}$, the nonnegative multiples of 3.

Next question: How can we describe "all real numbers between a and b" using set notation?

Intervals on the real line (Question)

Fix real numbers $a < b$.

Think about these descriptions:

1. All real numbers strictly between a and b.
2. All real numbers between a and b, including the endpoints.
3. All real numbers between a and b that include a but not b.
4. All real numbers between a and b that include b but not a.

How might we abbreviate each of these as a set? What inequalities should each one correspond to?

Intervals on the real line (Answers)

For real numbers $a < b$:

- **Open interval**:

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

- **Closed interval**:

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

- **Left-closed, right-open**:

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}.$$

- **Left-open, right-closed**:

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}.$$

Next question: How do we write sets that extend infinitely in one direction, like "all real numbers bigger than a"?

Rays and half-lines (Question)

For a real number a, consider these verbal descriptions:

1. All real numbers greater than a.
2. All real numbers greater than or equal to a.
3. All real numbers less than a.
4. All real numbers less than or equal to a.

Questions

- What notation should we use for each of these sets?
- What role should the symbol ∞ play? Is ∞ itself a real number?

Rays and half-lines (Answers)

For $a \in \mathbb{R}$:

- $(a, \infty) = \{x \in \mathbb{R} : a < x\}$.
- $[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$.
- $(-\infty, a) = \{x \in \mathbb{R} : x < a\}$.
- $(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$.

- The symbol ∞ means "without a finite bound on the right."
- It is *not* a real number; it marks that the interval has no endpoint in that direction.

Next question: Given a set A, what is the set of all its subsets?

Power set (Question)

Let $A = \{1, 2, 3\}$.

Think:

1. List as many subsets of A as you can.
2. Try to organize them by size: subsets with 0 elements, with 1 element, with 2 elements, etc.
3. How many subsets does A have in total?
4. If a set has n elements, what pattern do you suspect for the number of its subsets?

Power set (Answers)

- The **power set** of A is the set of all subsets of A:

$$\mathcal{P}(A) = \{X : X \subseteq A\}.$$

- For $A = \{1, 2, 3\}$ we get

$$\mathcal{P}(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

- There are 8 subsets in total.
- In general, if A has n elements, then $\mathcal{P}(A)$ has 2^n elements (each element can be either "in" or "out" of a subset).

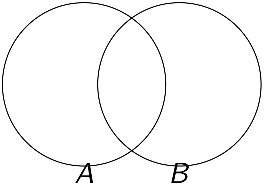
Next question: How do we combine two sets into new ones using operations such as "or", "and", and "difference"?

Union, intersection, difference (Question)

Let A be the set of students who play a school sport, and B the set of students who are in the school band.

Questions

1. What students should belong to $A \cup B$?
2. What students should belong to $A \cap B$?
3. What students should belong to $A \setminus B$?



Use the picture to help you reason about which region(s) each operation selects.

Union, intersection, difference (Definitions)

Given sets A and B:

- **Union**:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

- **Intersection**:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

- **Set difference**:

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

Next question: Let's practice these operations with a concrete numerical example.

Example: computing set operations (Question)

Let

$$A = \{1, 2, 3, 4, 5, 6\}, \qquad B = \{2, 4, 6, 8, 10, 12\}.$$

Compute:

1. $A \cup B$
2. $A \cap B$
3. $A \setminus B$ and $B \setminus A$
4. $(A \setminus B) \cup (B \setminus A)$
5. $(A \setminus B) \cap (B \setminus A)$

What pattern do you notice in the last two answers?

Example: computing set operations (Answers)

With $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8, 10, 12\}$:

$$
\begin{aligned}
A \cup B &= \{1, 2, 3, 4, 5, 6, 8, 10, 12\}, \\
A \cap B &= \{2, 4, 6\}, \\
A \setminus B &= \{1, 3, 5\}, \\
B \setminus A &= \{8, 10, 12\}, \\
(A \setminus B) \cup (B \setminus A) &= \{1, 3, 5, 8, 10, 12\}, \\
(A \setminus B) \cap (B \setminus A) &= \varnothing.
\end{aligned}
$$

Here

- $(A \setminus B) \cup (B \setminus A)$ consists of elements that belong to exactly one of A or B (the *symmetric difference*).
- $(A \setminus B) \cap (B \setminus A)$ is empty because no element can be in A but not B and also in B but not A.

Summary

We have introduced:

- Sets and membership ($\in, \notin$).
- Standard number sets: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.
- Subsets, proper subsets, the empty set, disjoint sets.
- Truth sets built from properties.
- Intervals and rays on the real line.
- Power sets and the idea of 2^n subsets.
- Set operations: union, intersection, difference, and a taste of symmetric difference.

These ideas form the basic language that the rest of mathematics will speak.