

Useful Facts About Sets

1 Preliminaries and Notation

1.1 Standard abbreviations

Remark 1.1. Throughout this chapter we use several common abbreviations:

- “If …, then …” may be written as $P \Rightarrow Q$.
- The converse implication is written $Q \Leftarrow P$.
- “If and only if” is abbreviated as “iff”, or $P \Leftrightarrow Q$.
- “Therefore” may be written as \therefore .

Remark 1.2 (Negation rule). We apply negation directly to symbols:

$$x = y \implies x \neq y, \quad x \in A \implies x \notin A.$$

This convention extends to any symbol introduced later. For example, once $\Sigma \models \tau$ is defined, its negation is written $\Sigma \not\models \tau$.

2 Sets and the Extensionality Principle

2.1 What is a set?

Definition 2.1. A *set* is a collection of objects, called its *elements*. We use:

$t \in A$ to mean that t is an element of A , $t \notin A$ to mean that t is not an element of A ,

and

$x = y$ to mean that x and y name the same object.

Remark 2.2. If $A = B$, then for every object t ,

$$t \in A \iff t \in B.$$

2.2 Fundamental characterization

Proposition 2.3 (Extensionality)

If A and B are sets such that

$$\forall t (t \in A \iff t \in B),$$

then $A = B$.

Remark 2.4. This expresses the idea that a set is completely determined by its members.

3 Basic Constructions

3.1 Adjoining an element

Definition 3.1. For any set A and any object t , define

$$A; t := A \cup \{t\}.$$

This is the set whose elements are the elements of A , together with the (possibly new) element t .

Proposition 3.2

For any set A and any object t ,

$$t \in A \iff A; t = A.$$

4 Common Sets and Notation

4.1 Special sets

Definition 4.1. • The *empty set* \emptyset is the set with no elements.

- For any object x , the *singleton* $\{x\}$ is the set whose only element is x .
- For any finite list of objects x_1, \dots, x_n , the set

$$\{x_1, \dots, x_n\}$$

has exactly those objects as elements.

Remark 4.2. For any objects x, y ,

$$\{x, y\} = \{y, x\},$$

since order does not matter for sets.

4.2 Infinite sets

Definition 4.3. We use the standard notations

$$\mathbb{N} = \{0, 1, 2, \dots\}$$

for the set of natural numbers, and

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

for the set of all integers.

5 Sets Specified by Properties

5.1 Set-builder notation

Definition 5.1. Given a property $P(x)$, the set

$$\{x \mid P(x)\}$$

is the set of all objects x such that $P(x)$ holds.

Example 5.2 • $\{x \in A \mid P(x)\}$ is the set of all elements x in A that satisfy $P(x)$.

- $\{(m, n) \mid m < n \text{ in } \mathbb{N}\}$ is the set of all ordered pairs of natural numbers where the first component is smaller than the second.

6 Subsets and Power Sets

6.1 Subsets

Definition 6.1. A set A is a *subset* of a set B , written $A \subseteq B$, if

$$\forall x (x \in A \implies x \in B).$$

Remark 6.2. Every set is a subset of itself, and $\emptyset \subseteq A$ for every set A . The statement $\emptyset \subseteq A$ is often called “vacuously true”.

6.2 Power set

Definition 6.3. For any set A , the *power set* of A is

$$\mathcal{P}(A) = \{x \mid x \subseteq A\},$$

the set of all subsets of A .

Example 6.4

We have

$$\mathcal{P}(\emptyset) = \{\emptyset\}, \quad \mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}.$$

7 Union and Intersection

7.1 Binary operations

Definition 7.1. For sets A and B ,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\},$$

the *union* of A and B , and

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\},$$

the *intersection* of A and B .

Definition 7.2. Sets A and B are *disjoint* if

$$A \cap B = \emptyset.$$

A family of sets is *pairwise disjoint* if any two distinct sets in the family are disjoint.

7.2 Indexed unions and intersections

Definition 7.3. Let \mathcal{A} be a set whose elements are themselves sets. Then

$$\bigcup \mathcal{A} = \{x \mid \exists S \in \mathcal{A}, x \in S\}$$

is the *union* of all members of \mathcal{A} , and

$$\bigcap \mathcal{A} = \{x \mid \forall S \in \mathcal{A}, x \in S\}$$

is the *intersection* of all members of \mathcal{A} (when it makes sense to consider it).

Example 7.4

If

$$\mathcal{A} = \{\{0, 1, 5\}, \{1, 6\}, \{1, 5\}\},$$

then

$$\bigcup \mathcal{A} = \{0, 1, 5, 6\}, \quad \bigcap \mathcal{A} = \{1\}.$$

Remark 7.5. When we have a family $(A_n)_{n \in \mathbb{N}}$ of sets indexed by natural numbers, we often write

$$\bigcup_{n \in \mathbb{N}} A_n$$

instead of $\bigcup\{A_n \mid n \in \mathbb{N}\}$.

8 Ordered Pairs and Sequences

8.1 Ordered pairs

Definition 8.1 (Kuratowski). The *ordered pair* $\langle x, y \rangle$ of objects x and y is defined by

$$\langle x, y \rangle := \{\{x\}, \{x, y\}\}.$$

Proposition 8.2

For any objects x, y, u, v ,

$$\langle x, y \rangle = \langle u, v \rangle \iff (x = u \text{ and } y = v).$$

8.2 n -tuples

Definition 8.3. Define n -tuples recursively by

$$\langle x_1, \dots, x_{n+1} \rangle := \langle \langle x_1, \dots, x_n \rangle, x_{n+1} \rangle$$

for $n \geq 1$. A *finite sequence* of members of a set A is an n -tuple $\langle x_1, \dots, x_n \rangle$ with each $x_i \in A$.

8.3 Segments of a sequence

Definition 8.4. Let $S = \langle x_1, \dots, x_n \rangle$ be a finite sequence. A *segment* of S is a finite sequence

$$\langle x_k, \dots, x_m \rangle$$

with $1 \leq k \leq m \leq n$. A segment is an *initial segment* if $k = 1$. An initial segment is *proper* if it is different from S .

8.4 A technical lemma

Lemma 8.5 (Lemma 0A)

Suppose

$$\langle x_1, \dots, x_m \rangle = \langle y_1, \dots, y_{m+k} \rangle$$

for some $m, k \geq 0$. Then

$$x_1 = \langle y_1, \dots, y_{k+1} \rangle.$$

Remark 8.6. The proof proceeds by induction on m , using the definition of ordered pairs and the basic property of Kuratowski pairs.

Corollary 8.7

Let A be a set such that no element of A is itself a finite sequence of other elements of A . If

$$\langle x_1, \dots, x_m \rangle = \langle y_1, \dots, y_n \rangle$$

and all x_i, y_j belong to A , then $m = n$ and $x_i = y_i$ for all $1 \leq i \leq m$.

9 Cartesian Products and Relations

9.1 Cartesian products

Definition 9.1. For sets A and B , the *Cartesian product* $A \times B$ is the set

$$A \times B = \{\langle x, y \rangle \mid x \in A, y \in B\}.$$

More generally, for a set A and integer $n \geq 1$,

$$A^n = \underbrace{A \times \cdots \times A}_{n \text{ times}}$$

is the set of all n -tuples of elements of A .

9.2 Binary relations

Definition 9.2. A (*binary*) *relation* R is a set of ordered pairs.

Example 9.3

The strict ordering on the set $\{0, 1, 2, 3\}$ is represented by the relation

$$\{\langle 0, 1 \rangle, \langle 0, 2 \rangle, \langle 0, 3 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle\}.$$

9.3 Domain, range, and field

Definition 9.4. For a relation R , the *domain* of R is

$$\text{dom}(R) = \{x \mid \exists y, \langle x, y \rangle \in R\},$$

the *range* of R is

$$\text{ran}(R) = \{y \mid \exists x, \langle x, y \rangle \in R\},$$

and the *field* of R is

$$\text{fld}(R) = \text{dom}(R) \cup \text{ran}(R).$$

9.4 n -ary relations and restriction

Definition 9.5. Let A be a set. An n -ary relation on A is a subset of A^n .

- When $n = 1$, an n -ary relation is just a subset of A (a unary relation).
- When $n \geq 2$, this recovers the usual notion of a relation of arity n .

The equality relation on A is the binary relation

$$\{\langle x, x \rangle \mid x \in A\}.$$

Definition 9.6 (Restriction of a relation). Let $R \subseteq A^n$ be an n -ary relation on A and let $B \subseteq A$. The *restriction* of R to B is

$$R \upharpoonright B := R \cap B^n.$$

Example 9.7

Let R be the usual strict order on \mathbb{N} , and let $B = \{0, 1, 2, 3\}$. Then $R \upharpoonright B$ is exactly the strict order relation on the set $\{0, 1, 2, 3\}$ shown earlier.

10 Functions and Operations

10.1 Functions

Definition 10.1. A relation F is a *function* if for each $x \in \text{dom}(F)$ there is exactly one y such that $\langle x, y \rangle \in F$.

We write

$$F : A \rightarrow B$$

to mean that F is a function with domain $\text{dom}(F) = A$ and range $\text{ran}(F) \subseteq B$.

The function F maps A onto B (or is *surjective*) if $\text{ran}(F) = B$.

The function F is *one-to-one* (or *injective*) if for each $y \in \text{ran}(F)$ there is exactly one x such that $\langle x, y \rangle \in F$.

10.2 n -ary operations and their restriction

Definition 10.2. An n -ary operation on a set A is a function

$$f : A^n \rightarrow A.$$

Example 10.3

Addition is a binary operation on \mathbb{N} , and the successor mapping $S(n) = n + 1$ is a unary operation on \mathbb{N} .

Definition 10.4 (Restriction of an operation). Let $f : A^n \rightarrow A$ be an n -ary operation on A , and let $B \subseteq A$. Define

$$g := f \cap (B^n \times A).$$

Then g has domain B^n , and

$$g(b_1, \dots, b_n) = f(b_1, \dots, b_n)$$

whenever $b_1, \dots, b_n \in B$.

Proposition 10.5

The restriction g is itself an n -ary operation on B if and only if B is *closed under* f ; that is, whenever $b_1, \dots, b_n \in B$, one has $f(b_1, \dots, b_n) \in B$.

Example 10.6

Consider addition on \mathbb{R} . Its restriction to \mathbb{N} is the usual addition on \mathbb{N} . The set \mathbb{N} is closed under this operation, so the restriction is again a binary operation on \mathbb{N} .

11 Standard Classes of Relations

11.1 Basic properties

Definition 11.1. Let R be a relation on a set A .

- R is *reflexive on* A if $\langle x, x \rangle \in R$ for all $x \in A$.
- R is *symmetric* if whenever $\langle x, y \rangle \in R$, then $\langle y, x \rangle \in R$.
- R is *transitive* if whenever $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$, then $\langle x, z \rangle \in R$.
- R satisfies *trichotomy on* A if for every $x, y \in A$, exactly one of the following holds:

$$\langle x, y \rangle \in R, \quad x = y, \quad \langle y, x \rangle \in R.$$

11.2 Equivalence relations and orderings

Definition 11.2. A relation R on A is an *equivalence relation* on A if it is reflexive on A , symmetric, and transitive.

Definition 11.3. A relation R on A is an *ordering relation* (or *strict order*) on A if it is transitive and satisfies trichotomy on A .

12 Equivalence Classes

Definition 12.1. Let R be an equivalence relation on a set A . For each $x \in A$, the *equivalence class* of x is

$$[x] = \{y \in A \mid \langle x, y \rangle \in R\}.$$

Proposition 12.2

Let R be an equivalence relation on A .

1. The equivalence classes $[x]$ (for $x \in A$) form a partition of A ; that is, every element of A lies in at least one equivalence class, and no element lies in two distinct classes.
2. For any $x, y \in A$,

$$[x] = [y] \iff \langle x, y \rangle \in R.$$