MATH 155 HWK 4

Shiqi Liang 305117507

February 2021

Question 1

To show the transformation is linear, we only need to show

$$\nabla^2(cf+g) = c\nabla^2 f + \nabla^2 g \tag{1}$$

We can show this by

$$\nabla^2(cf+g) = \frac{\partial}{\partial x^2}\left[cf+g\right] + \frac{\partial}{\partial y^2}\left[cf+g\right] \tag{2}$$

$$= c\frac{\partial}{\partial x^2}f + \frac{\partial}{\partial x^2}g + c\frac{\partial}{\partial y^2}f + \frac{\partial}{\partial y^2}g \tag{3}$$

$$= c\left[\frac{\partial}{\partial x^2}f + \frac{\partial}{\partial y^2}f\right] + \left[\frac{\partial}{\partial x^2}g + \frac{\partial}{\partial y^2}g\right] \tag{4}$$

$$= c\nabla^2 f + \nabla^2 g \tag{5}$$

As we can see, the transformation is linear as it is closed under addition and scalar multiplication.

Question 2

It is known that for $u(x,y) = u(p(x,y), q(x,y))$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} \tag{6}$$

First we find $\frac{\partial f}{\partial x'}$:

$$\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'} \tag{7}$$

$$= \frac{\partial f}{\partial x}\frac{\partial}{\partial x'}(x'\cos\theta - y'\sin\theta) + \frac{\partial f}{\partial y}\frac{\partial}{\partial x'}(x'\sin\theta + y'\cos\theta) \tag{8}$$

$$= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \tag{9}$$

Now, we need to find $\frac{\partial^2 f}{\partial x'^2}$:

$$\frac{\partial^2 f}{\partial x'^2} = \frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial x'}\right) \tag{10}$$

$$= \frac{\partial}{\partial x'}\left[\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right] \tag{11}$$

$$= \frac{\partial^2 f}{\partial x^2}\left[\frac{\partial x}{\partial x'}\cos\theta\right] + \frac{\partial}{\partial x}\frac{\partial f}{\partial y}\left[\frac{\partial x}{\partial x'}\sin\theta\right] + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\left[\frac{\partial y}{\partial x'}\cos\theta\right] + \frac{\partial^2 f}{\partial y^2}\left[\frac{\partial y}{\partial x'}\sin\theta\right] \tag{12}$$

$$= \frac{\partial^2 f}{\partial x'^2} = \frac{\partial^2 f}{\partial x^2}\cos^2\theta + \frac{\partial}{\partial x}\frac{\partial f}{\partial y}\sin\theta\cos\theta + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\sin\theta\cos\theta + \frac{\partial^2 f}{\partial y^2}\sin^2\theta \tag{13}$$

For $\frac{\partial f}{\partial y'}$

$$\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y'} \tag{14}$$

$$= -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta \tag{15}$$

For $\frac{\partial^2 f}{\partial y'^2}$

$$\frac{\partial^2 f}{\partial y'^2} = \frac{\partial^2 f}{\partial x^2}\sin^2\theta - \frac{\partial}{\partial x}\frac{\partial f}{\partial y}\sin\theta\cos\theta - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\sin\theta\cos\theta + \frac{\partial^2 f}{\partial y^2}\cos^2\theta \tag{16}$$

Adding them together

$$\frac{\partial^2 f}{\partial x^2}\cos^2\theta + \frac{\partial}{\partial x}\frac{\partial f}{\partial y}\sin\theta\cos\theta + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\sin\theta\cos\theta + \frac{\partial^2 f}{\partial y^2}\sin^2\theta \tag{17}$$

$$+\frac{\partial^2 f}{\partial x^2}\sin^2\theta - \frac{\partial}{\partial x}\frac{\partial f}{\partial y}\sin\theta\cos\theta - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\sin\theta\cos\theta + \frac{\partial^2 f}{\partial y^2}\cos^2\theta \tag{18}$$

$$= \frac{\partial^2 f}{\partial x^2}\left(\sin^2\theta + \cos^2\theta\right) + \frac{\partial^2 f}{\partial y^2}\left(\sin^2\theta + \cos^2\theta\right) \tag{19}$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \tag{20}$$

so:

$$\frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Question 3

Question 3a

From the previous questions, we can see that

$$\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

and that

$$\frac{\partial f}{\partial y'} = -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta$$

From this we can see that

$$\sqrt{\left[\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right]^2 + \left[-\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta\right]^2} \tag{21}$$

$$= \sqrt{\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right](\sin^2\theta + \cos^2\theta) + 2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)\sin\theta\cos\theta - 2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)\sin\theta\cos\theta} \tag{22}$$

$$= \sqrt{\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]} \tag{23}$$

So it is isotropic.

Question 3b

Using this approximation we have

$$\left|\frac{\partial f}{\partial x'}\right| + \left|\frac{\partial f}{\partial y'}\right| \tag{24}$$

$$= \left|\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right| + \left|-\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta\right| \tag{25}$$

which is obviously not the same as the $|\frac{\partial f}{\partial x}| + |\frac{\partial f}{\partial y}|$ With this approximation, we loses the isotropic property.

Question 4

From class,we know that the composite sharpening mask is in the form of

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

This is because we want to put more "weight" on the original point to strengthen the distinctions. We would apply this filter by multiplying it with the area around a point and taking the sum of all the resulting matrix.

From the results we can see that now the image is a lot clearer and more distinct.

```
%% Question 4

%% Question 4

image=im2double(imread('Fig3.40(a).jpg'));

lap_filter=[1,1,1;
            1,-8,1;
            1,1,1];

output=zeros(size(image));

for x = 2:size(image,1)-1
    for y=2:size(image,2)-1
        temp=image(x-1:x+1,y-1:y+1).*lap_filter;
        output(x,y)=image(x,y)-sum(temp(:));
    end
end

figure();
imagesc(output);colormap(gray(256));
```


before filter

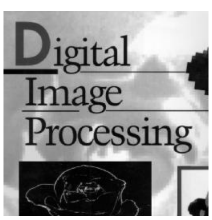

after filter

Question 5

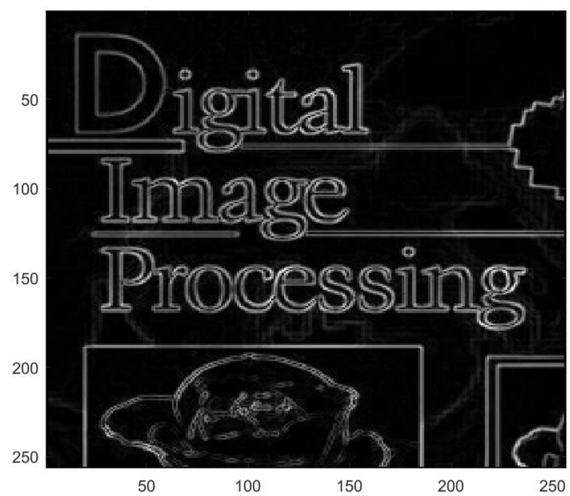
For the equation provided in the homework, we can approximate $|\frac{\partial f(x,y)}{\partial x}|$ with $|f(x+1,y)-f(x-1,y|+|f(x,y+1)-f(x,y-1)|$

We would then have the matlab code with

```
image=im2double(imread('Fig5.26a.jpg'));

output=zeros(size(image));

for x = 2:size(image,1)-1
    for y=2:size(image,2)-1
       Gx=abs(image(x+1,y)-image(x-1,y));
       Gy=abs(image(x,y+1)-image(x,y-1));
       output(x,y)=Gx+Gy;
    end
end

imagesc(output);colormap(gray(256))
```



before



after We first have the sobel operator which has

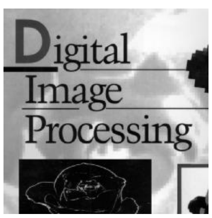
$$fx = \begin{bmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

and

$$fy = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

with $|\nabla f| = |fx| + |fy|$. So we have the program

```
%% Question 5a

image=im2double(imread('Fig5.26a.jpg'));

MX=[-1,0,1;-2,0,2;-1,0,1];
MY=[-1,-2,-1;0,0,0;1,2,1];
output=zeros(size(image));

for x = 2:size(image,1)-1
    for y=2:size(image,2)-1
       Gx=sum(sum(MX.*image(x-1:x+1,y-1:y+1)));
       Gy=sum(sum(MY.*image(x-1:x+1,y-1:y+1)));
       output(x,y)=sqrt(Gx.^2+Gy.^2);
    end
end

imshow(output)
```



before sobel

after sobel

We can see that both methods can detect edges, while the sobel one is a lot more "stark". Personally I think the sobel method detects more edges and is more clear.