TP PCFOL (Propositional Calculus and First Order Logic)
Techniques of AI [INFO-H-410]
Correction

v1.0.0

Source files, code templates and corrections related to practical sessions can be found on the UV or on github (`https://github.com/iridia-ulb/INFOH410`).

Representation and Interpretation of Boolean Functions

Symbol	**Name**
0	FALSE
1	TRUE
$!A$ / $\neg A$	NOT A
$A \wedge B$	A AND B
$A \vee B$	A OR B
$A \oplus B$	A XOR B

Propositional Calculus

Question 1. Use truth tables to prove the following equivalences:

a) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

b) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

c) $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

d) $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

e) $P \Rightarrow Q \equiv \neg P \vee Q$

Answer:

a) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

P	Q	R	$(Q \vee R)$	$(P \wedge Q)$	$(P \wedge R)$	$P \wedge (Q \vee R)$	$(P \wedge Q) \vee (P \wedge R)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	0	1	1	1
1	1	0	1	1	0	1	1
1	1	1	1	1	1	1	1

b) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

P	Q	R	$(Q \wedge R)$	$(P \vee Q)$	$(P \vee R)$	$P \vee (Q \wedge R)$	$(P \vee Q) \wedge (P \vee R)$
0	0	0	0	0	0	0	0
0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

c) $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

P	Q	$(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
0	0	0	1	1	1	1
0	1	0	1	0	1	1
1	0	0	0	1	1	1
1	1	1	0	0	0	0

d) $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

P	Q	$(P \vee Q)$	$\neg P$	$\neg Q$	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
0	0	0	1	1	1	1
0	1	1	1	0	0	0
1	0	1	0	1	0	0
1	1	1	0	0	0	0

e) $P \Rightarrow Q \equiv \neg P \vee Q$

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg P \vee Q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	1	0	1

Question 2. Formulate the following expressions as propositional sentences:

a) If the unicorn is magical, then it is immortal.

b) If the unicorn is not magical, then it is a mortal mammal.

c) If the unicorn is either immortal or a mammal, then it is horned.

Using truth tables, can you prove whether the unicorn is magical? Immortal? Horned?

Answer: We use G for magical, O for mortal, M for mammal and H for horned.

a) $G \Rightarrow \neg O$

b) $\neg G \Rightarrow (O \wedge M)$

c) $(\neg O \lor M) \Rightarrow H$

We can prove that the Unicorn is horned, but not whether it is magical, mortal nor mammal.

First Order propositions

Question 3. Convert those expressions to first order logic expressions.

a) All roads lead to Rome.

b) All that glitters is not gold.

c) The enemy of my enemy is my friend.

d) A dog is a man's best friend.

Answer:

a) $\forall x, Road(x) \Rightarrow GoToRome(x)$

b) $\neg(\forall x, Glitters(x) \Rightarrow Gold(x))$

c) $\forall x, y, Enemy(Me, x) \land Enemy(x, y) \Rightarrow Friend(Me, y)$

d) $\forall x, y, Man(x) \land BestFriend(x, y) \Rightarrow Dog(y)$

Resolution

Question 4. Prove the following using resolution (negate conclusion, convert to CNF, prove contradiction)

a) Given $KB = \{P \land Q\}$, prove that $KB \models P \lor Q$.

b) Given $KB = \{P \lor Q, Q \Rightarrow (R \land S), (P \lor R) \Rightarrow U\}$, prove that $KB \models U$.

Answer:

a) Given $KB = \{P \land Q\}$, prove that $KB \models P \lor Q$.

- Negate conclusion: $\neg P \land \neg Q$
- Four sentences: $P, Q, \neg P, \neg Q$
- Resolve P with $\neg P$, and Q with $\neg Q$ gives $\{\}$. This means we have a contradiction, or $KB \models P \lor Q$ is true.

b) Given $KB = \{P \lor Q, Q \Rightarrow (R \land S), (P \lor R) \Rightarrow U\}$, prove that $KB \models U$.

- Negate conclusion: $\neg U$.

- Convert to CNF: $KB = \{P \vee Q, (\neg Q \vee R) \wedge (\neg Q \vee S), (\neg P \vee U) \wedge (\neg R \vee U)\}$.
- 6 sentences.
- Resolve $\neg P \vee U$ with $\neg U$ which gives $\neg P$. $\neg R \vee U$ with $\neg U$ gives $\neg R$.
- $\neg P \vee Q$ and $\neg P$ resolves in Q.
- Q and $\neg Q \vee R$ resolves in R which contradicts with $\neg R$ from the first resolution. Thus, $KB \models U$ is true.

Question 5. On the island of Knights and Knaves, everything a Knight says is true and everything a Knave says is false. You meet two people, Alice and Bob:

- Alice says "Neither Bob nor I are Knaves"
- Bob says "Alice is a Knave"

Using the proposition A to represent "Alice is a Knight" ($\neg A$ means "Alice is a Knave") and B to represent "Bob is a Knight".

a) Formulate what Alice and Bob said.

b) Formulate that what they said is true if and only if they are knights.

c) Put those into CNF form.

d) Use resolution to prove who is what.

Answer:

a) Formulate what Alice and Bob said.

   Alice: $A \wedge B$, Bob: $\neg A$.

b) Formulate that what they said is true if and only if they are knights.

   Alice: $A \Leftrightarrow A \wedge B$, Bob: $B \Leftrightarrow \neg A$

c) Put those into CNF form.

   - $A \Leftrightarrow A \wedge B$
   - $A \Rightarrow A \wedge B$, $A \wedge B \Rightarrow A$
   - $\neg A \vee (A \wedge B)$, $\neg(A \wedge B) \vee A$
   - $\neg A \vee A$ (tautology), $\neg A \wedge B$, $\neg A \vee A \vee \neg B$(tautology)
   - $B \Leftrightarrow \neg A$
   - $B \Rightarrow \neg A$, $\neg A \Rightarrow B$
   - $\neg B \vee \neg A$, $A \vee B$

d) Use resolution to prove who is what.

   - $\neg A \vee B$, $A \vee B$: B (proof that Bob is a knight)

- Now we know that B is true, so since $B \Rightarrow \neg A$: $\neg A$. (Alice is a Knave)

Question 6. From "Sheep are animals", it follows that "The head of a sheep is the head of an animal." Demonstrate that this inference is valid by carrying out the following steps:

a) Translate the premise and the conclusion into the language of first-order logic. Use three predicates: $H(h, x)$ (meaning "h is the head of x"), $S(x)$ ($Sheep(x)$), and $A(x)$ ($Animal(x)$).

b) Negate the conclusion, and convert the premise and the negated conclusion into conjunctive normal form.

c) Conclude.

Answer:

a) $\forall x, S(x) \Rightarrow A(x)$, and $\forall x, y, H(y, z) \wedge S(z) \Rightarrow H(y, z) \wedge A(z)$

b) Which translate to, once the arrows are transformed: $\forall x, \neg S(x) \vee A(x)$, and $\forall y, z, \neg(H(y, z) \wedge S(z) \vee (H(y, z) \wedge A(z))$

then we negate the conclusion: $\neg(\forall y, z, \neg(H(y, z) \wedge S(z) \vee (H(y, z) \wedge A(z)))$

switch to existential to move the negation: $\exists y, z, \neg(\neg(H(y, z) \wedge S(z) \vee (H(y, z) \wedge A(z)))$

Use De Morgan laws twice: $\exists y, z, H(y, z) \wedge S(z) \wedge (\neg H(y, z) \vee \neg A(z))$

Skolemize: $H(Y0, Z0) \wedge S(Z0) \wedge (\neg H(Y0, Z0) \vee \neg A(Z0))$

Knowledge base of the problem:

- $H(Y0, Z0)$
- $S(Z0)$
- $\neg H(Y0, Z0) \vee \neg A(Z0)$
- $\neg S(x) \vee A(x)$

c) Resolve $H(Y0, Z0)$ with $\neg H(Y0, Z0) \vee \neg A(Z0)$ gives $\neg A(Z0)$

Then resolve $\neg S(x) \vee A(x)$ with $\neg A(Z0)$ with unifier $\{Z0/x\}$

Resolve $\neg S(Z0)$ with $S(Z0)$ gives contradiction.