# Combinatorial Optimization

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## TP 1. Formulations

**Exercise 1.1** Suppose that you are interested in choosing a set of investments $\{1, ..., 7\}$ using boolean variables. Model the following constraints:

1. You cannot invest in all of them.
2. You must choose at least one of them.
3. Investment 1 cannot be chosen if investment 3 is chosen.
4. Investment 4 can be chosen only if investment 2 is also chosen.
5. You must choose either both investments 1 and 5 or neither.
6. You must choose at least one of the investments 1,2,3 or at least two from 4,5,6,7.

### *Correction of Exercise 1.1*

We introduce a binary variable $x_i \in \{0, 1\}$ for each investment $i \in [7]$, such that:

$$\begin{cases} x_i = 1 & \text{if we chose to invest in } i \\ x_i = 0 & \text{otherwise} \end{cases}$$

To model constraint 6, we introduce a binary variable $y$, the objective is:

1. to force $y$ to be equal to 1 if less than two investments have been chosen in $\{4, 5, 6, 7\}$ (I.7),
2. to force $y$ to be equal to 0 if less than one investment has been chose in $\{1, 2, 3\}$ (I.6),
3. to force to chose at least one investment in $\{1, 2, 3\}$ if $y = 1$ (I.6),
4. to force to chose at least two investments in $\{4, 5, 6, 7\}$ if $y = 0$ (I.7).

In other words, $y$ aims at determining which alternative satisfies the constraints.

**Integer Linear Program I.** Exercise 1.1

$$\sum_{i=1}^{7} x_i \leq 6 \tag{I.1}$$

$$\sum_{i=1}^{7} x_i \geq 1 \tag{I.2}$$

$$x_1 + x_3 \leq 1 \tag{I.3}$$

$$x_4 - x_2 \leq 0 \tag{I.4}$$

$$x_5 - x_1 = 0 \tag{I.5}$$

$$x_1 + x_2 + x_3 - y \geq 0 \tag{I.6}$$

$$x_4 + x_5 + x_6 + x_7 - 2(1 - y) \geq 0 \tag{I.7}$$

**Exercise 1.2** Model each of the following constraints:

1. Given a constant $C$, $u = \min\{x_1, x_2\}$, assuming that $0 \leq x_j \leq C$ for $j = 1, 2$
2. $v = |x_1 - x_2|$ with $0 \leq x_j \leq C$ for $j = 1, 2$
3. the set $X \setminus \{x^*\}$ where $X = \{x \in Z^n : Ax \leq b\}$ and $x^* \in X$

*Correction of Exercise 1.2*

1. First, $u \leq \min\{x_1, x_2\}$ is easily given by Equations II.3 and II.4.

   For the $u \geq \min\{x_1, x_2\}$ part, we introduce a variable $y$ such that:

   $$\begin{cases} y = 1 & \text{if } x_1 \geq x_2 \\ y = 0 & \text{otherwise} \end{cases}$$

   Then the idea is to use the upper bound $C$ for each of the variables to "*nullify*" the constraints II.1 (resp. II.2) if $x_1 \geq x_2$ (resp. $x_2 > x_1$).

   Note that, in this case, $y$ is totally bound, *i.e.* $y$ cannot be equal to zero if $x_1 \geq x_2$. Indeed such a scenario would lead to violation of either constraint II.3 or II.2.

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**Integer Linear Program II.** Exercise 1.2

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$$u - x_1 \leq 0 \qquad \text{(II.1)}$$
$$u - x_2 \leq 0 \qquad \text{(II.2)}$$
$$u - x_1 + Cy \geq 0 \qquad \text{(II.3)}$$
$$u - x_2 + C(1 - y) \geq 0 \qquad \text{(II.4)}$$

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2. First $u \geq |x_1 - x_2|$ is easily given by Equations (III.1) and (III.2). To model the $u \leq |x_1 - x_2|$, we use, once again, a binary variable $y \in \{0, 1\}$ such that:

   $$\begin{cases} y = 1 & \text{if } x_1 \geq x_2 \\ y = 0 & \text{otherwise} \end{cases}$$

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**Integer Linear Program III.** Exercise 1.2

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$$v - x_1 + x_2 \geq 0 \qquad \text{(III.1)}$$
$$v - x_2 + x_1 \geq 0 \qquad \text{(III.2)}$$
$$v - x_1 + x_2 + 2C(1 - y) \leq 0 \qquad \text{(III.3)}$$
$$v - x_2 + x_1 + 2Cy \leq 0 \qquad \text{(III.4)}$$

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   In Equations III.3 and III.4, the factor 2 is mandatory since we have to "*nullify*" the action of two variables.

3. As a reminder, given two $n$-dimensional vectors $x, x^* \in \mathbb{Z}^n$, $x \neq x^*$ if and only if there exists a coordinate $i \in [n]$ such that $|x_i - x_i^*| > 0$. In other words there exists a coordinate $i$ such that $x_i > x_i^*$ or $x_i < x_i^*$.

Based on this, we introduce one integer variable $x_i$ (resp. $x_i^*$) for each component of $x$ (resp. $x^*$). We also introduce, for each $i = 1, \dots, n$, the variables $y_i^+$ such that:

$$\begin{cases} y_i^+ = 1 & \text{if } x_i > x_i^* \\ y_i^+ = 0 & \text{otherwise} \end{cases}$$

and the variables $y_i^-$ such that:

$$\begin{cases} y_i^- = 1 & \text{if } x_i < x_i^* \\ y_i^- = 0 & \text{otherwise} \end{cases}$$

These variables will be used to determine whether $x$ differs from $x^*$ since Equation (IV.4) is satisfied if and only if at least a variable $y_i^+$ or $y_i^-$ is not null.

Lastly, we denote $M_i$ as the maximum bound of coordinate $x_i$. Thus Equations (IV.1) ensure that $y_i^- = 1$ if $x_i < x_i^*$. Similarly, Equations (IV.2) ensure that $y_i^+ = 1$ if $x_i > x_i^*$.

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**Integer Linear Program IV.** Exercise 1.2

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$$x_i - x_i^* - y_i^+ + y_i^- M_i \geq 0 \qquad \forall i = 1, \dots, n \qquad \text{(IV.1)}$$

$$x_i^* - x_i - y_i^- + y_i^+ M_i \geq 0 \qquad \forall i = 1, \dots, n \qquad \text{(IV.2)}$$

$$y_i^+ + y_i^- \leq 1 \qquad \forall i = 1, \dots, n \qquad \text{(IV.3)}$$

$$\sum_{i=1}^{n} y_i^+ + y_i^- \geq 1 \qquad \text{(IV.4)}$$

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**Exercise 1.3** Let

$$\begin{aligned} P_1 &= \{ x \in \mathbb{B}^4 : && 97\,x_1 + 32\,x_2 + 25\,x_3 + 20\,x_4 \leq 139 \} \\ P_2 &= \{ x \in \mathbb{B}^4 : && 2\,x_1 + x_2 + x_3 + x_4 \leq 3 \} \\ P_3 &= \{ x \in \mathbb{B}^4 : && x_1 + x_2 + x_3 \leq 2 \\ & && x_1 + x_2 + x_4 \leq 2 \\ & && x_1 + x_3 + x_4 \leq 2 \ \} \end{aligned}$$

Show that $P_1 = P_2 = P_3$.

### *Correction of Exercise 1.3*

By definition, $P_1 = P_2 = P_3$ if they contain the same set of points.

| | | $x_1 = 0$ | | | | $x_1 = 1$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $x_2 = 0$ | | $x_2 = 1$ | | $x_2 = 0$ | | $x_2 = 1$ | | |
| | | $x_3 = 0$ | $x_3 = 1$ | $x_3 = 0$ | $x_3 = 1$ | $x_3 = 0$ | $x_3 = 1$ | $x_3 = 0$ | $x_3 = 1$ | |
| $x_4 = 0$ | $P_1$ | 0 | 25 | 32 | 57 | 97 | 122 | 129 | 154 | $\leq 139$ |
| | $P_2$ | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | $\leq 3$ |
| | $P_3$ | 0 | 1 | 1 | 2 | 1 | 2 | 2 | 3 | $\leq 2$ |
| | | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | $\leq 2$ |
| | | 0 | 1 | 0 | 1 | 1 | 2 | 2 | 2 | $\leq 2$ |
| $x_4 = 1$ | $P_1$ | 20 | 45 | 52 | 77 | 117 | 142 | 149 | 174 | $\leq 139$ |
| | $P_2$ | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 4 | $\leq 3$ |
| | $P_3$ | 0 | 1 | 1 | 2 | 1 | 2 | 2 | 3 | $\leq 2$ |
| | | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | $\leq 2$ |
| | | 1 | 1 | 1 | 1 | 2 | 2 | 3 | 2 | $\leq 2$ |

**Exercise 1.4** John Dupont is attending a summer school where he must take four courses per day. Each course lasts an hour, but because of the large number of students, each course is repeated several times per day by different teachers. Section $i$ of course $k$ denoted $(i, k)$ meets at the hour $t_{ik}$, where courses start on the hour between 10:00 and 19:00. John's preferences for when he takes courses are influenced by the reputation of the teacher, and also the time at which the course begins. Let $p_{ik}$ be his preference for section $(i, k)$. Unfortunately, due to conflicts, John cannot always choose the sections he prefers.

1. Formulate an integer program to choose a feasible course schedule that maximizes the sum of John's preferences.
2. Modify the formulation so that John never has more than two consecutive hours of classes without a break.
3. Modify the formulation so that John chooses a schedule in which he starts his day as late as possible.

*Correction of Exercise 1.4*

We introduce the $x_{i,k}$ binary variables such that:

$$\begin{cases} x_{i,k} = 1 & \text{if the course the } i^{th} \text{ session of course } k \text{ has been chosen} \\ x_{i,k} = 0 & \text{otherwise} \end{cases}$$

The objective function (V.1) maximizes the preferences of John. The Equation (V.2) ensures that four courses are chosen per day. The Equations (V.3) ensure that a course is selected at most once while the Equations (V.4) ensure that two sessions cannot be chosen at the same time.

John can ensure that he does not have more than two consecutive hours of classes without a break with the Equations (V.5).

**Integer Linear Program V.** Exercise 1.4

$$\text{Maximize} \sum_i \sum_k p_{i,k} x_{i,k} \tag{V.1}$$

$$\text{Subject To} \sum_i \sum_k x_{i,k} = 4 \tag{V.2}$$

$$\sum_i x_{i,k} \leq 1 \quad \forall k \tag{V.3}$$

$$\sum_i \sum_{k:t_{i,k}=t} x_{i,k} \leq 1 \quad \forall t \in [10, 19] \tag{V.4}$$

$$\sum_i \sum_{k:t \leq t_{i,k} \leq t+2} x_{i,k} \leq 2 \quad \forall i, \forall t \in [10, 19] \tag{V.5}$$

If John wants a schedule in which he starts as late as possible, he has to introduce a variable $z$ that will be equal to the first hour of course. The objective function becomes max $z$ and he needs to add the following constraints:

$$z \leq x_{i,k} t_{i,k}$$

**Exercise 1.5** The ACME Company must draw up a production program for the next nine weeks. Jobs last several weeks and once started must be carried out without interruption. During each week a certain number of skilled workers are required to work full time on the job. Thus if job $i$ lasts $p_i$ weeks then $l_{i,u}$ workers are required in week $u$ for $u = 1, \ldots, p_i$. The total number of workers available in week $t$ is $L_t$. An example of job data $(i, p_i, l_{i1}, \ldots, l_{ip_i})$ is shown below.

| Job | Length | Week 1 | Week 2 | Week 3 | Week 4 |
|---|---|---|---|---|---|
| 1 | 3 | 2 | 3 | 1 | - |
| 2 | 2 | 4 | 5 | - | - |
| 3 | 4 | 2 | 4 | 1 | 5 |
| 4 | 4 | 3 | 4 | 2 | 2 |
| 5 | 3 | 9 | 2 | 3 | - |

1. Formulate the problem of finding a feasible schedule as an IP.
2. Formulate when the objective is to minimize the maximum number of workers used during any of the nine weeks.
3. Formulate that job 1 must start at least two weeks before job 3.
4. Formulate that job 4 must start not later than one week after job 5.
5. Formulate that job 1 and 2 both need the same machine, and cannot be carried out simultaneously.

*Correction of Exercise 1.5*

**Integer Linear Program VI.** Exercise 1.5

$$
\begin{array}{llrrr}
\text{Minimize} & z & & & \text{(VI.1)} \\
\text{Such that} & \sum_j s_{i,j} & = 1 & \forall i & \text{(VI.2)} \\
& \sum_{j=1}^{9-p_i+1} s_{i,j} & = 1 & \forall i & \text{(VI.3)} \\
& w_{i,j} - \sum_{k=\max(1,j-p_i)}^{j} s_{i,k} l_{i,j-k+1} & = 0 & \forall i,\ \forall j & \text{(VI.4)} \\
& \sum_i w_{i,j} - L_j & \leq 0 & \forall j & \text{(VI.5)} \\
& z - \sum_i w_{i,j} & \geq 0 & \forall j & \text{(VI.6)} \\
& s_{3,j} - \sum_{k=1}^{j-2} s_{1,j} & \leq 0 & \forall j = 3, \ldots, 9 & \text{(VI.7)} \\
& s_{5,j} - \sum_{k=1}^{j+1} s_{4,k} & = 0 & \forall j = 1, \ldots, 9 - \max(p_5, p_4 + 1) & \text{(VI.8)} \\
& s_{1,j} + \sum_{k=\max(1,j-p_2)}^{j+p_1-1} s_{2,k} & \leq 1 & \forall j = 1, \ldots, 10 - p_1 & \text{(VI.9)}
\end{array}
$$

1. To find a feasible solution we introduce two types of variables. We first introduce the binary variables $s_{i,j} \in \{0, 1\}$ such that:

$$
\begin{cases} s_{i,j} = 1 & \text{if job } i \text{ starts at week } j \\ s_{i,j} = 0 & \text{otherwise} \end{cases}
$$

and the integer variables $w_{i,j}$ equal to the number of workers needed by job $i$ at week $j$.

We write then the Equations (VI.2) ensuring that every job is scheduled and Equations (VI.3) ensuring that enough weeks are available before the end of the schedule to complete the job.

It remains to avoid an excess of workers demands on the overall schedule. Equations (VI.4) bind the $s$ and $w$ variables together and Equations (VI.5) prevent from an excess of workforce.

2. To model the objective function, we introduce the $z$ integer variable which will be greater than the maximum demand of workers among the jobs and the weeks thanks to Equations (VI.6).

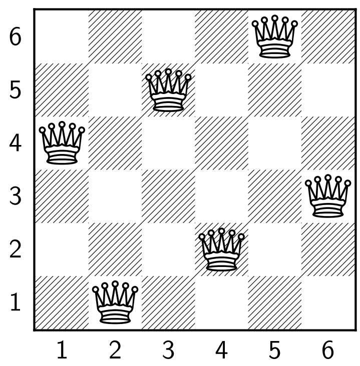
   It remains to minimize $z$.

3. This constraint is modeled by Equation (VI.7).
4. This constraint is modeled by Equation (VI.8).
5. This constraint is modeled by Equation (VI.9).

**Exercise 1.6** $N$ queens have to be placed on a $N \times N$ chessboard in such a way that no two queens share any row, column or diagonal. Formulate an BIP that solve such a problem.

*Correction of Exercise 1.6*

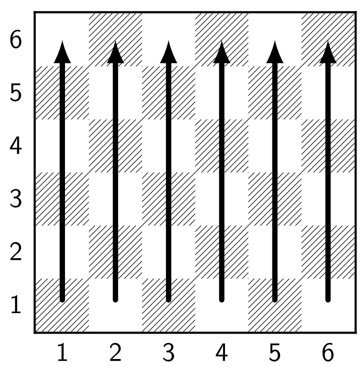
If we consider $N = 6$, we are looking for the following solution.



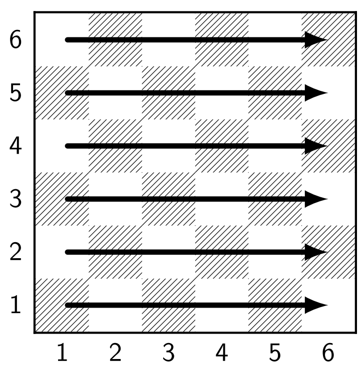

A solution $S = \{x_{14} = 1, x_{21} = 1, x_{35} = 1, x_{42} = 1, x_{56} = 1, x_{63} = 1, x_* = 0\}$ for $N = 6$.

We introduce the binary variables $x_{ij}$ such that:

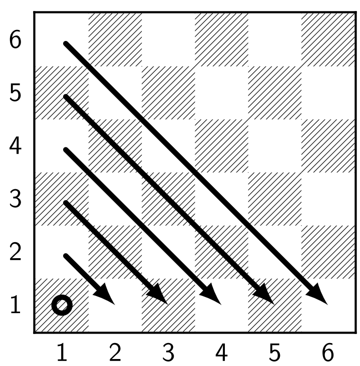
$$\begin{cases} x_{ij} = 1 & \text{if the field } ij \text{ contains a queen} \\ x_{ij} = 0 & \text{otherwise} \end{cases}$$



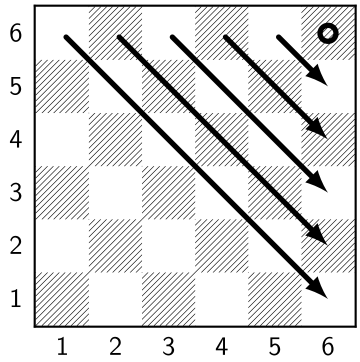

Equations (VII.2)



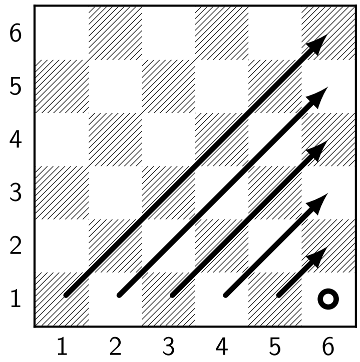

Equations (VII.3)



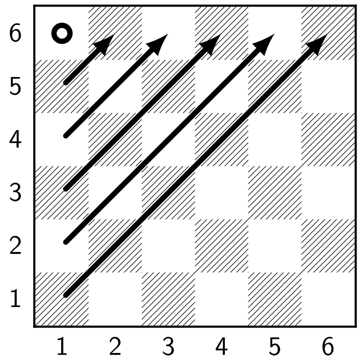

Equations (VII.4)



Equations (VII.5)



Equations (VII.6)



Equations (VII.7)

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**Integer Linear Program VII.** Exercise 1.6

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$$
\begin{array}{llll}
\text{Maximize} & \displaystyle\sum_i \sum_j x_{ij} & & & \text{(VII.1)} \\
\text{Subject To} & \displaystyle\sum_j x_{ij} & \leq 1 & \forall i \in [N] & \text{(VII.2)} \\
& \displaystyle\sum_i x_{ij} & \leq 1 & \forall j \in [N] & \text{(VII.3)} \\
& \displaystyle\sum_{i=1}^{N-j+1} x_{i,j+i-1} & \leq 1 & \forall j \in [N] & \text{(VII.4)} \\
& \displaystyle\sum_{i=1}^{N-j+1} x_{i,n-j-i+2} & \leq 1 & \forall j \in [N] & \text{(VII.5)} \\
& \displaystyle\sum_{j=1}^{N-i+1} x_{i+j-1,j} & \leq 1 & \forall i \in [N] & \text{(VII.6)} \\
& \displaystyle\sum_{j=1}^{N-i+1} x_{i+j-1,n-j+1} & \leq 1 & \forall i \in [N] & \text{(VII.7)}
\end{array}
$$

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**Exercise 1.7** Let $A$ be a set of $n$ real numbers $a_i, i = 1, \ldots, n$. We are looking for a subset $S \subseteq A$ such that $\sum_{a_i \in S} a_i$ is the most fractional, *i.e.* the fractional part of this sum is the closest to $\frac{1}{2}$.

Formulate this problem as a mixed linear program which gives this subset $S$ as the optimal solution.

Example : If $A = \{0,53; 0,43; 0.87; 0,72; 0.63\}$, then $S = \{0,87; 0,63\}$ with $0,87 + 0,63 = 1,5$.

## *Correction of Exercise 1.7*

We first introduce the binary variables $x_i \in \{0, 1\}$ such that:

$$
\begin{cases} x_i = 1 & \text{if the number } a_i \text{ belongs to } S \\ x_i = 0 & \text{otherwise} \end{cases}
$$

We then introduce the variable $y \in \mathbb{N}$ that will be equal to the integer part of the sum over the selected $a_i$. Lastly we introduce the variable $z \in [0, 1/2]$. $z$ will represents the distance between the fractional part of the sum and $1/2$.

The Equations (VIII.2) and (VIII.3) ensure that $y = \lfloor \sum_{s\in S} s \rfloor$. The Equations (VIII.4) and (VIII.5) ensure that $z \geq |{}^1\!/\!{}_2 - \sum_{s\in S} s + y|$, but combined with the objective function, these equations ensure that $z = |{}^1\!/\!{}_2 - \sum_{s\in S} s + y|$.

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**Integer Linear Program VIII.** Exercise 1.7

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$$\begin{array}{lllr} \text{Minimize} & z & & \text{(VIII.1)} \\ \text{Subject to} & y - \sum_{i=1}^{n} x_i a_i & \leq 0 & \text{(VIII.2)} \\ & y - \sum_{i=1}^{n} x_i a_i + 1 & \geq 0 & \text{(VIII.3)} \\ & z - y + \sum_{i=1}^{n} x_i a_i - {}^1\!/\!{}_2 & \geq 0 & \text{(VIII.4)} \\ & z - \sum_{i=1}^{n} + y + {}^1\!/\!{}_2 + y & \geq 0 & \text{(VIII.5)} \end{array}$$

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**Exercise 1.8** A prisoner is standing in front of two doors. Behind one door, there is a beautiful young lady; behind the other, there is a starved tiger. The prisoner must choose a door. If by luck, he picks the doors with the woman behind it, he is set free and may marry her. If he picks the door with the tiger behind it, the hungry beast immediately pounces him and the prisoner will be eaten up.

On each of the doors, there are the following messages that can be true or false :

**Door 1** : *In this room, there is a lady and in the other room, there is a tiger.*
**Door 2** : *In one of the rooms, there is a lady and in the other room, there is a tiger.*

The prisoner is informed that one of the messages is true and the other is false. These informations are sufficient to determine what is hidden behind each door. Model this situation as an IP. Obviously, you must find the lady !

### *Correction of Exercise 1.8*

We introduce the following binary variables:

$$\begin{cases} y_i = 1 & \text{if message on door } i \text{ is true} \\ y_i = 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} x_{ij} = 1 & \text{if reward } j \text{ is behind door } i \\ x_{ij} = 0 & \text{otherwise} \end{cases}$$

We use, as convention, that reward 1 symbolizes the princess while reward 2 stands for the tiger. We try to maximize $y_1$. Note that maximizing $y_2$ is also possible since, as stated in the statement of the exercise, the informations are sufficient to determine which door to chose. This means that the problem is *fully constrained*, in other words, there only exists a unique solution that satisfies all the constraints. Hence, the choice of the objective function does not really matter.

First, Equations (IX.2) models the fact that each reward is affected to a room and that they are affected to distinct rooms. Then, Equation (IX.3) models the assertions "*one of the message is true and the other is false*".

The Equations (IX.4) and (IX.5) encodes the message on Door 1. Indeed, Equation (IX.4) forces the assignment of the rewards (princess to room 1, tiger to room 2) if message 1 is true, while Equation (IX.5) ensures that $y_1$ is set to one, if the previous assignment is respected.

Based on the same scheme, Equations (IX.6) and (IX.7) ensure that if message 2 is true, thus the princess is assigned to a room and the tiger to another.

Combined with the Equations (IX.2), the Equations (IX.8) and (IX.9) ensure that $y_2$ is set to one, if the assertion is verified.

Based on this ILP, we claim that Princess is in Room 2 while tiger is in Room 1.

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**Integer Linear Program IX.** Exercise 1.8

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$$
\begin{array}{llll}
\text{Maximize} & y_1 & & \quad\text{(IX.1)} \\
\text{Subject To} & \displaystyle\sum_{j=1}^{2} x_{ij} = 1 & \forall i = 1,2 & \quad\text{(IX.2)} \\
& y_1 + y_2 = 1 & & \quad\text{(IX.3)} \\
& y_1 - x_{11} - x_{22} + 1 \geq 0 & & \quad\text{(IX.4)} \\
& 2y_1 - x_{11} - x_{22} \leq 0 & & \quad\text{(IX.5)} \\
& y_2 - x_{11} - x_{22} + 1 \geq 0 & & \quad\text{(IX.6)} \\
& y_2 - x_{12} - x_{21} + 1 \geq 0 & & \quad\text{(IX.7)} \\
& y_2 - x_{11} - x_{21} \leq 0 & & \quad\text{(IX.8)} \\
& y_2 - x_{12} - x_{22} \leq 0 & & \quad\text{(IX.9)}
\end{array}
$$

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# TP 2. Optimality, Relaxations and Bounds

**Exercise 2.1** We consider the 0-1 IPs X and XI where $u \in \mathbb{R}^n$.
Show that $P_2$ is a relaxation of $P_1$.

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**Integer Linear Program X.** $P_1$

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$$\begin{aligned}
&\text{Maximize} \quad cx && && \text{(X.1)}\\
&\text{Subject To} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i && \forall i = 1, \dots, m && \text{(X.2)}\\
&x \in \mathbb{B}^n && && \text{(X.3)}
\end{aligned}$$

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**Integer Linear Program XI.** $P_2$

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$$\begin{aligned}
&\text{Maximize} \quad cx && \text{(XI.1)}\\
&\text{Subject To} \quad \sum_{j=1}^{n} \left( \sum_{i=1}^{m} u_i a_{ij} \right) x_j = \sum_{i=1}^{m} u_i b_i && \text{(XI.2)}\\
&x \in \mathbb{B}^n && \text{(XI.3)}
\end{aligned}$$

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*Correction of Exercise 2.1*

Remark that the objective function is the same in both $P_1$ and $P_2$. It follows that, if we denote as $X$ (resp. $T$) the set of admissible solution of $P_1$ (resp. $P_2$), $P_2$ is a relaxation of $P_1$ if and only if $X \subseteq T$.

We consider $x \in X$ a solution of $P_1$, let us now show that $x \in T$. By definition of $P_1$, $x$ satisfies the following constraints:

$$\begin{cases}
\sum_{j=1}^{n} a_{1j} x_j = b_1 & \Leftrightarrow \quad u_1 \sum_{j=1}^{n} a_{1j} x_j = b_1 u_1 \quad \Leftrightarrow \quad \sum_{j=1}^{n} u_1 a_{1j} x_j = b_1 u_1 \\
\sum_{j=1}^{n} a_{2j} x_j = b_2 & \Leftrightarrow \quad u_2 \sum_{j=1}^{n} a_{2j} x_j = b_2 u_2 \quad \Leftrightarrow \quad \sum_{j=1}^{n} u_2 a_{2j} x_j = b_2 u_2 \\
\quad \vdots & \\
\sum_{j=1}^{n} a_{mj} x_j = b_1 & \Leftrightarrow \quad u_m \sum_{j=1}^{n} a_{mj} x_j = b_m u_m \quad \Leftrightarrow \quad \sum_{j=1}^{n} u_m a_{mj} x_j = b_m u_m
\end{cases}$$

If we sum up all the previous constraints, we get that every solution $x \in X$ satisfies:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} u_i a_{ij} x_j = \sum_{i=1}^{m} u_i b_i \tag{1}$$

and thus also satisfies:

$$\sum_{j=1}^{n} \left( \sum_{i=1}^{m} u_i a_{ij} \right) x_j = \sum_{i=1}^{m} u_i b_i \tag{2}$$

Hence every $x \in X$ also satisfies $x \in T$.

**Exercise 2.2** We consider the knapsack problems XII, XIII and XIV.

1. Show that $P_2$ is a relaxation of $P_1$.
2. Show that $P_3$ is a relaxation of $P_2$.

**Integer Linear Program XII.** $P_1$

$$\begin{aligned} \text{Maximize} \quad & \sum_{j=1}^{5} c_j x_j && \text{(XII.1)} \\ \text{Subject To} \quad & \frac{7}{4}x_1 - \frac{2}{3}x_2 + \frac{5}{2}x_3 - \frac{5}{12}x_4 + \frac{19}{6}x_5 = \frac{8}{3} && \text{(XII.2)} \\ & x \in \mathbb{Z}_+^5 && \text{(XII.3)} \end{aligned}$$

**Integer Linear Program XIII.** $P_2$

$$\begin{aligned} \text{Maximize} \quad & \sum_{j=1}^{5} c_j x_j && \text{(XIII.1)} \\ \text{Subject To} \quad & \frac{3}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 + \frac{7}{12}x_4 + \frac{1}{6}x_5 = \frac{2}{3} + w && \text{(XIII.2)} \\ & x \in \mathbb{Z}_+^5 && \text{(XIII.3)} \\ & w \in \mathbb{Z}_+^1 && \text{(XIII.4)} \end{aligned}$$

**Integer Linear Program XIV.** $P_3$

$$\begin{aligned} \text{Maximize} \quad & \sum_{j=1}^{5} c_j x_j && \text{(XIV.1)} \\ \text{Subject To} \quad & \frac{3}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 + \frac{7}{12}x_4 + \frac{1}{6}x_5 \geq \frac{2}{3} && \text{(XIV.2)} \\ & x \in \mathbb{R}_+^5 && \text{(XIV.3)} \end{aligned}$$

## *Correction of Exercise 2.2*

1. Let us remark that $P_1$ and $P_2$ share the same objective function. Thus, if we denote the solution set of $P_1$ (resp. $P_2$) as $X_1$ (resp. $X_2$), $P_2$ is a relaxation of $P_1$ if and only if $X_1 \subseteq X_2$.

We consider a solution of $P_1$ $x = \{x_1, x_2, x_3, x_4, x_5\} \in X_1$. Obviously, $x$ satisfies:

$$\frac{7}{4}x_1 - \frac{2}{3}x_2 + \frac{5}{2}x_3 - \frac{5}{12}x_4 + \frac{19}{6}x_5 = \frac{8}{3} \tag{3}$$

We rewrite the latter as follows:

$$\left(x_1 + \frac{3}{4}x_1\right) + \left(-x_2 + \frac{1}{3}x_2\right) + \left(2x_3 + \frac{1}{2}x_3\right) + \left(-x_4 + \frac{7}{12}x_4\right) + \left(3x_5 + \frac{1}{6}x_5\right) = 2 + \frac{2}{3} \quad (4)$$

The latter is equivalent to:

$$\frac{3}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 + \frac{7}{12}x_4 + \frac{1}{6}x_5 = \frac{2}{3} + (2 - x_1 + x_2 - 2x_3 + x_4 - 3x_5) \quad (5)$$

Let us remark that Equation 5 looks very similar to Equation XIII.2. Indeed, if we can prove that $2 - x_1 + x_2 - 2x_3 + x_4 - 3x_5 \in \mathbb{Z}^1_+$, thus we are able to prove that every solution of $P_1$ is a solution of $P_2$ with $w = 2 - x_1 + x_2 - 2x_3 + x_4 - 3x_5$.

Let us denote as $w' = 2 - x_1 + x_2 - 2x_3 + x_4 - 3x_5$. Since $x \in \mathbb{Z}^5$, $w' \in \mathbb{Z}$. Furthermore, let us remark that:

$$\frac{3}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 + \frac{7}{12}x_4 + \frac{1}{6}x_5 \geq 0 \quad (6)$$

Leading to:

$$\frac{2}{3} + w' \geq 0 \quad (7)$$

And:

$$w' \geq -\frac{2}{3} \quad (8)$$

Since $w' \in \mathbb{Z}$, the previous equation implies:

$$w' \geq 0 \quad (9)$$

Said differently, we can rewrite $P_1$ as follows:

---

**Integer Linear Program XV.** Modified $P_1$

---

$$\begin{array}{lll}
\text{Maximize} & \displaystyle\sum_{j=1}^{5} c_j x_j & \text{(XV.1)} \\
\text{Subject To} & \frac{3}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 + \frac{7}{12}x_4 + \frac{1}{6}x_5 = \frac{2}{3} + w & \text{(XV.2)} \\
& w = 2 - x_1 + x_2 - 2x_3 + x_4 - 3x_5 & \text{(XV.3)} \\
& w \in \mathbb{Z}^1_+ & \text{(XV.4)} \\
& x \in \mathbb{Z}^5_+ & \text{(XV.5)}
\end{array}$$

---

$P_2$ is obviously a relaxation of $P_1$ since $P_2 = P_1$ without constraint XV.3.

2. Once again the two formulations share the same objective function, we thus have to prove that $X_2 \subseteq X_3$. Let us consider a solution $x = \{x_1, x_2, x_3, x_4, x_5, w\} \in \mathbb{Z}_+^6 \in X_2$.

By definition, $x$ verifies:

$$\frac{3}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 + \frac{7}{12}x_4 + \frac{1}{6}x_5 = \frac{2}{3} + w \tag{10}$$

Since $w \geq 0$, we can write:

$$\frac{3}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 + \frac{7}{12}x_4 + \frac{1}{6}x_5 \geq \frac{2}{3} \tag{11}$$

Furthermore $\mathbb{Z}_+^5 \subset \mathbb{R}_+^5$, this prove that $P_3$ is a relaxation of $P_2$.

**Exercise 2.3** Find primal and dual bounds for the integer knapsack problem XVI.

---

**Integer Linear Program XVI.** $P_1$

---

$$\begin{array}{rll} \text{Maximize} & 42x_1 + 26x_2 + 35x_3 + 71x_4 + 53x_5 & \quad \text{(XVI.1)} \\ \text{Subject To} & 14x_1 + 10x_2 + 12x_3 + 25x_4 + 20x_5 \leq 69 & \quad \text{(XVI.2)} \\ & x \in \mathbb{Z}_+^5 & \quad \text{(XVI.3)} \end{array}$$

---

*Correction of Exercise 2.3*

We are dealing with a maximization problem, it follows that:

1. a primal bound is a lower bound for the optimal value,
2. a dual bound is an upper bound for the optimal value.

To get a lower bound, one needs to find a feasible solution for XVI. For instance, the vector $x = (4, 0, 1, 0, 0)$ is an admissible solution of profit 203. Thus $p(x^*)$, the profit of an optimal solution $x^*$ verifies $p(x^*) \geq 203$.

To get an upper bound for $p(x^*)$, several tools can be used:

1. solving optimally a relaxation of the problem,
2. find a feasible solution of the dual problem.

Let us consider the linear relaxation of Problem XVI given by I.

---

**Linear Program I.** Linear relaxation of $P_1$

---

$$\begin{array}{rll} \text{Maximize} & 42x_1 + 26x_2 + 35x_3 + 71x_4 + 53x_5 & \quad \text{(I.1)} \\ \text{Subject To} & 14x_1 + 10x_2 + 12x_3 + 25x_4 + 20x_5 \leq 69 & \quad \text{(I.2)} \\ & x \in \mathbb{R}_+^5 & \quad \text{(I.3)} \end{array}$$

---

Let us remark that $\frac{c_i}{a_i}$ is maximal for $i = 1$. It follows that an optimal solution is given by $x_r^* = \left(\frac{69}{14}, 0, 0, 0, 0\right)$. The derived upper bound is thus given by $p(x_r^*) = 207$.

On the other hand, we can use the duality principle to get an upper bound. Indeed, any feasible solution for the dual problem is an upper bound for the primal problem. Let us consider the dual problem II.

---

**Linear Program II.** $P_1$

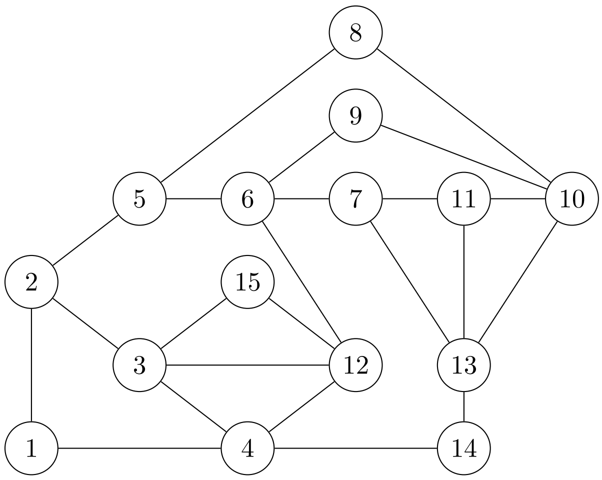
---

$$
\begin{aligned}
\text{Minimize} \quad & 69y & & \qquad \text{(II.1)}\\
\text{Subject To} \quad & 14y & \geq 42 & \qquad \text{(II.2)}\\
& 10y & \geq 26 & \qquad \text{(II.3)}\\
& 12y & \geq 35 & \qquad \text{(II.4)}\\
& 25y & \geq 71 & \qquad \text{(II.5)}\\
& 20y & \geq 53 & \qquad \text{(II.6)}\\
& y \in \mathbb{R}_+ & & \qquad \text{(II.7)}
\end{aligned}
$$

---

The optimal solution is $y^* = 3$, implying an upper bound equal to 207.

**Exercise 2.4** A **stable set** is a set of nodes $U \subseteq V$ such that there are no edges between any two nodes of $U$. A **clique** is a set of nodes $U \subseteq V$ such that there is an edge between every pair of node in $U$. Show that the problem of finding a maximum cardinality stable set is dual to the problem of finding a minimum cover of the nodes by cliques. Use this information to find bounds on the maximum size of a stable set in the graph shown below.



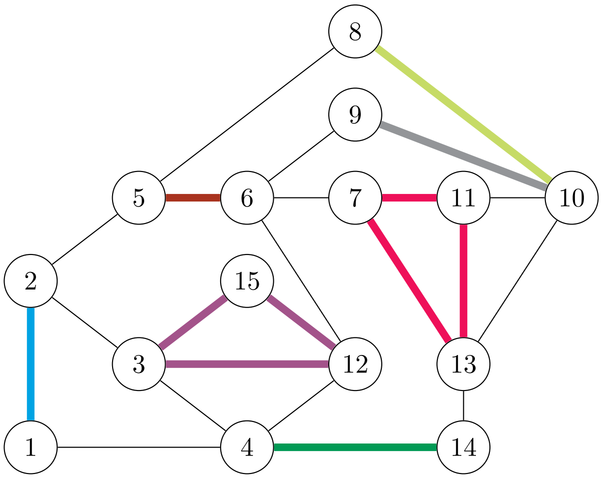

*Correction of Exercise 2.4*

1. To prove that Maximal Stable Set is the dual of Minimum Clique Cover, we have to prove that $|VS|$ is an upper bound of $|cover|$.

Let us consider $C = \{c_1, c_2, \ldots, c_k\}$ a set of $k$ cliques such that every node of $V$ belongs to at least one clique of $C$. By definition of a stable set, each stable set $S$ contains at most one vertex per clique, since two vertices of a same clique define a edge of $E$.

This leads to $|S| \leq k$, thus each Minimum Clique Cover is an upper bound for every stable set $S$.

2. From the following Clique Cover with 7 cliques, we can deduce that every stable set will contain at most 7 vertices.




**Exercise 2.5** Consider a directed graph $D = (V, A)$ with arc lengths $c_e \geq 0$ for $e \in A$. Taking two distinct nodes $s, t \in V$, consider the problem of finding a shortest path from $s$ to $t$. Show that LP III is a strong dual problem.

---

**Linear Program III.** Z

---

$$
\begin{array}{rlcr}
\text{Maximize} & \pi_t & & \text{(III.1)} \\
\text{Subject To} & \pi_j - \pi_i \leq c_{ij} & \forall e = (i, j) \in A & \text{(III.2)} \\
& \pi_s = 0 & & \text{(III.3)} \\
& \pi \in \mathbb{R}_+^{|V|} & & \text{(III.4)}
\end{array}
$$

---

*Correction of Exercise 2.5*

A way to prove the strong duality is to proceed as follows. First we consider the classical ILP formulation for the $s, t-$path problem given by ILP XVII where $x_{ij} = 1$ if and only if arc $(i, j) \in A$ is part of the solution.

---

**Integer Linear Program XVII.** Z

---

$$
\begin{array}{rlcr}
\text{Minimize} & c_{ij}x_{ij} & & \text{(XVII.1)} \\
\text{Subject To} & \sum_{i \in V} x_{ji} - \sum_{i \in V} x_{ij} \leq 0 & \forall j \in V \setminus \{s, t\} & \text{(XVII.2)} \\
& \sum_{i \in V} x_{si} - \sum_{i \in V} x_{is} \leq 1 & & \text{(XVII.3)} \\
& \sum_{i \in V} x_{ti} - \sum_{i \in V} x_{it} \leq -1 & & \text{(XVII.4)} \\
& x \in \mathbb{B}_+^{|V|} & & \text{(XVII.5)}
\end{array}
$$

---

The dual problem is thus given by LP IV, which is a relaxation of LP III. This implies that, given $\pi_1^*$ (resp.

$\pi_2^*$) an optimal solution for LP III (resp. IV), $p(\pi_2^*) \geq p(\pi_1^*)$. Furthermore, since every solution of LP IV can be turned into a solution of LP III of same cost, we have that $p(\pi_1^*) \geq p(\pi_2^*)$, and thus: $p(\pi_1^*) = p(\pi_2^*)$.

---

**Linear Program IV.** Z

---

$$\begin{aligned} \text{Maximize} \quad & \pi_t - \pi_s && && \text{(IV.1)} \\ \text{Subject To} \quad & \pi_j - \pi_i && \leq c_{ij} && \forall e = (i,j) \in A && \text{(IV.2)} \\ & \pi \in \mathbb{R}_+^{|V|} && && && \text{(IV.3)} \end{aligned}$$

---

Thus LP III is a strong dual problem for the shortest $s, t-$path problem.

**Definition 1.** *A matrix A is totally unimodular (denoted TU) if every square submatrix has determinant +1, -1 or 0.*

**Property 1.** *A matrix A that respects the following properties is TU:*

1. $a_{ij} \in \{1, 0, -1\}$ *for all i, j.*
2. *Each column of A contains at most two nonzero coefficients* ($\sum_{i=1}^{m} |a_{ij}| \leq 2$).
3. *There exists a partition* $(M_1, M_2)$ *of the set of rows of matrix A such that each column j containing two nonzero coefficients satisfies:*

$$\sum_{i \in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} = 0.$$

## Exercise 2.6

Determine whether the following matrices are totally unimodular.

$$A_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

### *Correction of Exercise 2.6*

The matrix $A_1$ is not totally unimodular as the submatrix $B$ selected as follows, has determinant equal to 2.

$$\begin{array}{cc|ccc|} & & & B & \\ \cline{3-5} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ \cline{3-5} \multicolumn{1}{c}{1} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{0} & \multicolumn{1}{c}{0} & \multicolumn{1}{c}{0} \end{array}$$

The matrix $A_2$ is totally unimodular, we can for instance highlight the following partition of the rows: $M = (\{1,2\}, \{3,4,5\})$.

**Property 2.** *A matrix that respects the following properties is totally unimodular.*

*1.* $a_{ij} \in \{1, 0, -1\}$ *for all* $i$, $j$

*2. For any subset* $M$ *of the rows, there exists a partition* $(M_1, M_2)$ *of* $M$ *such that each column* $j$ *satisfies:*

$$\mid \sum_{i \in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} \mid \leq 1.$$

**Definition 2.** *A* $0-1$ *matrix* $A$ *has the* consecutive 1's property *if for any column* $j$, $a_{ij} = a_{i'j} = 1$ *with* $i < i'$ *implies* $a_{lj} = 1$ *for* $i < l < i'$.

**Exercise 2.7** Prove that a matrix with the consecutive 1's property is TU.

## *Correction of Exercise 2.7*

Let us consider $\mathcal{M}$ a $0-1$ matrix that has the consecutive ones property. Since $\mathcal{M}$ is a $0-1$ matrix, then $a_{ij} \in \{0, 1\} \subset \{-1, 0, 1\}$ for all $i, j$.

Let us now consider the folowing partitions of the rows:

$$\begin{cases} M_1 & = & \text{odd lines} \\ M_2 & = & \text{even lines} \end{cases}$$

Note that non zero elements in $\mathcal{M}$ are consecutive on each columns, it follows that:

1. if the number of non-zero elements is equal to $2k$ ($k \in \mathbb{N}$), thus there are exactly $k$ elements in $M_1$ and $k$ elements in $M_2$, thus:

$$\sum_{i \in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} = 0 \quad \text{for each column} j$$

2. if the number of non-zero elements is equal to $2k + 1$ ($k \in \mathbb{N}$), thus there are either exactly $k + 1$ elements in $M_1$ and $k$ elements in $M_2$ or $k$ elements in $M_1$ and $k + 1$ elements in $M_2$. Leading to

$$\sum_{i \in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} \in \{-1, 1\} \quad \text{for each column} j$$

Thus $\mathcal{M}$ is totally unimodular.

**Exercise 2.8** Prove that the polyhedron $P = \{(x_1, \ldots, x_m, y) \in \mathbb{R}_+^{m+1} : y \leq 1,\ x_i \leq y \text{ for } i = 1, \ldots, m\}$ has integer vertices.

## *Correction of Exercise 2.8*

To show that a polyhedron is an integral polyhedron, we can simply show that the constraints matrix is totally unimodular.

The constraints matrix is given by:

$$A = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & -1 \\ 0 & 1 & \ldots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -1 \end{pmatrix}$$

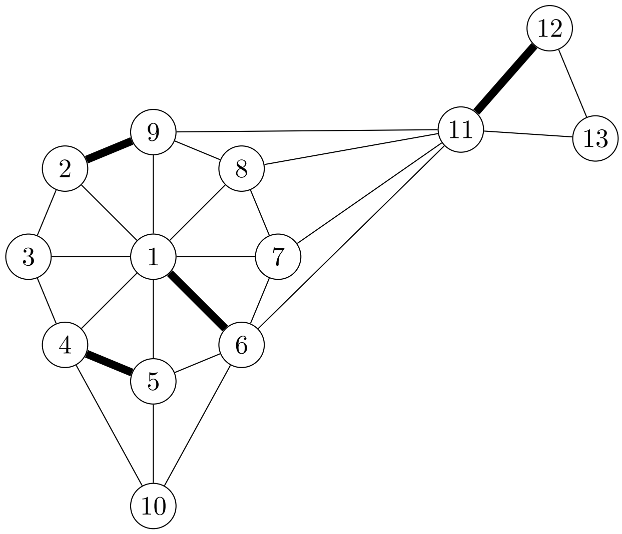
Note that a matrix $A$ is totally unimodular if and only if $A^T$ is totally unimodular.

$$A^T = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & -1 & -1 & \dots & -1 \end{pmatrix}$$

We can prove that $A^T$ is totally unimodular by providing the partition of the rows $M = (\{1, 2, \dots, m+1\}, \emptyset)$.

# TP 3. Matchings and Assignments

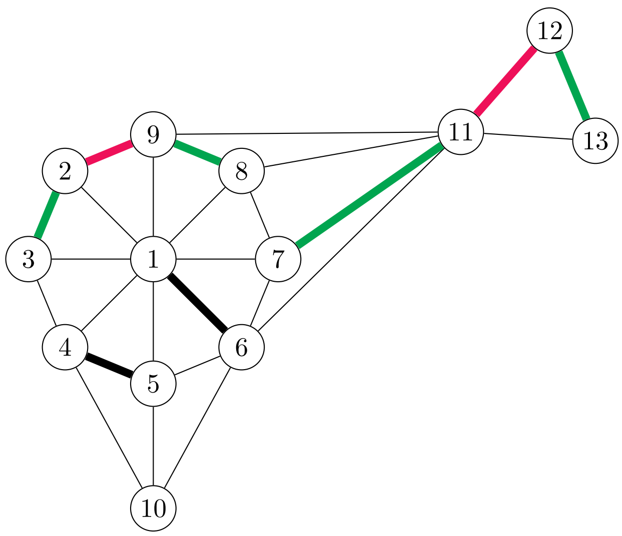
**Exercise 3.1** Consider the following graph $G_1$ and an associate matching $M = \{\{1,6\},\{2,9\},\{4,5\},\{11,12\}\}$.




1. Find two augmenting path for $M$.
2. Is the matching obtained after augmentation the optimal matching?
3. Why?

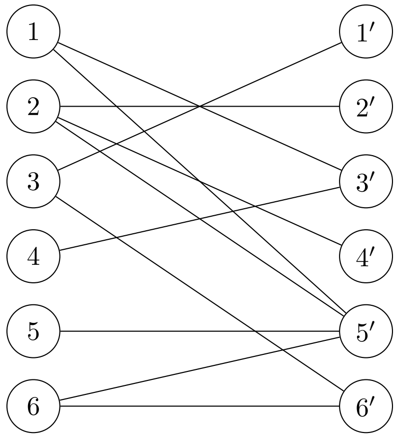
*Correction of Exercise 3.1*

We consider the two following augmenting paths: $3,2,9,8$ and $7,11,12,13$.




The constructed matching is an optimal matching, indeed 12 of the 13 vertices of the graphs are covered by $M$, thus it is impossible to add another edge that does contain only uncovered vertices.

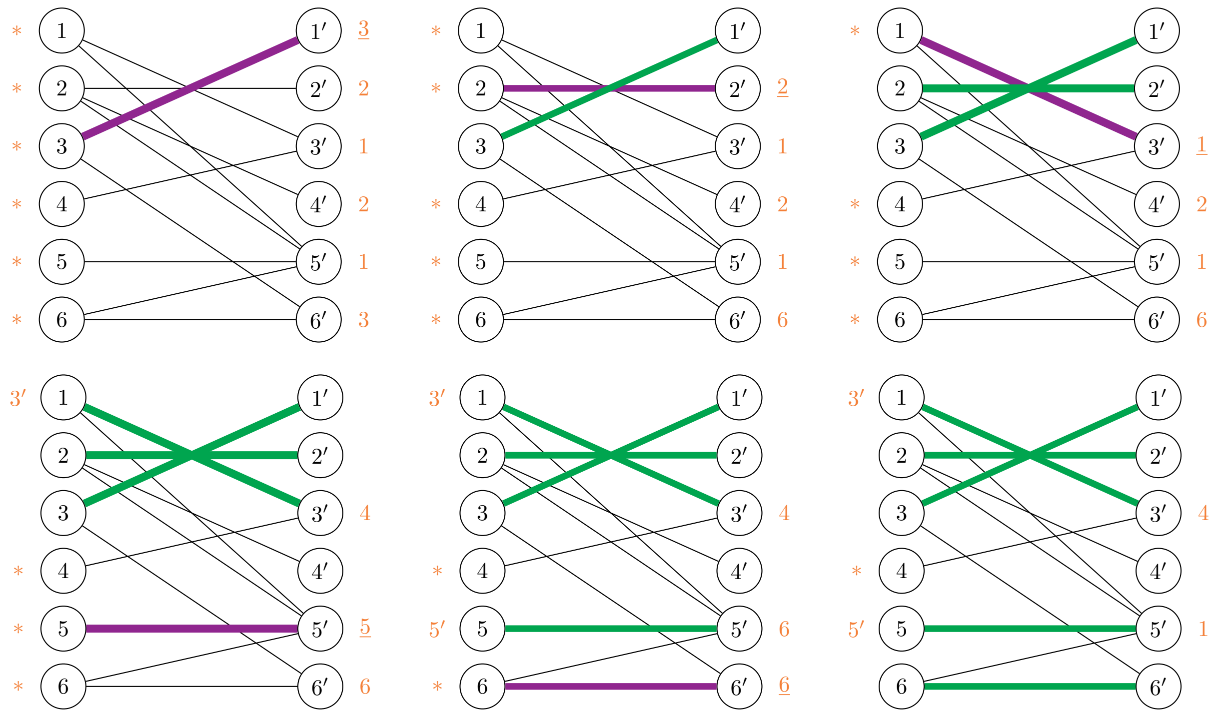
**Exercise 3.2** Consider the following graph $G_2$.



1. Find a maximum cardinality matching.
2. Prove that solution is optimal.

*Correction of Exercise 3.2*

Let us apply the maximum cardinality matching algorithm on $G_2$.



The proof of correctness of the algorithm proves that the returned solution is optimal. However, one can also remark that vertices $2'$ and $4'$ can only be covered with edge have 2 as other terminal node.

It follows that, an upper bound to the cardinality of the matching is given by 5. Since the cardinality of the solution returned by the algorithm is equal to the upper bound, the given matching is an optimal one.

**Exercise 3.3** If a graph has $n = 2k$ nodes, a matching with $k$ edges is called *perfect*. Show that the graph $G_2$ does not contain a perfect matching.

*Correction of Exercise 3.3*

The previous remark concerning nodes $2, 2', 4'$ shows that $G_2$ does not admit a perfect matching.

**Exercise 3.4** Show how a maximum flow algorithm can be used to find a maximal cardinality matching in a bipartite graph.

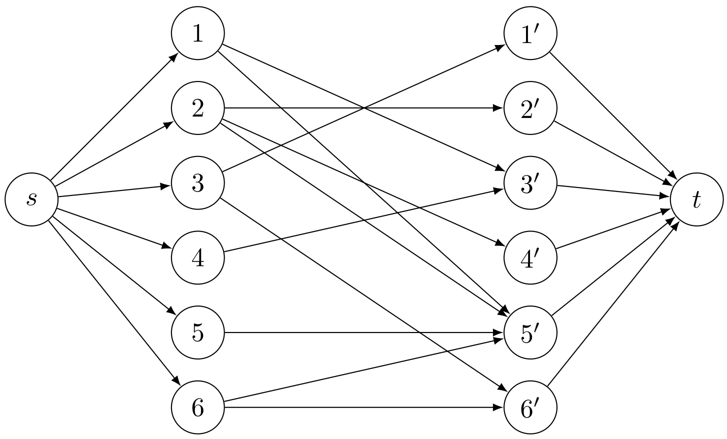
*Correction of Exercise 3.4*

Every Maximum Cardinality Matching instance can be turned into a flow instance by following the following procedure. Given a bipartite graph $G = (A, B, E)$:

1. transform each edge $\{ij\} \in E$, with $i \in A$ and $j \in B$ into an arc $(i, j)$,
2. create a source vertex $s$ as such as an arc $(s, i)$ for every vertex $i \in A$,
3. create a sink vertex $t$ as such as an arc $(j, t)$ for every vertex $j \in B$,
4. assign capacity 1 to every arc of the newly created directed graph.

The edge of the matching corresponds to arcs between $A$ and $B$ carrying a non zero flow.

As an example let us consider $G_2$ of Exercise TP 3.. The constructed flow instance is the following one, in which every arc has capacity equal to one:




**Exercise 3.5** Find a maximum weight assignment with the following weight matrix :

$$(c_{ij}) = \begin{pmatrix} 6 & 2 & 3 & 4 & 1 \\ 9 & 2 & 7 & 6 & 0 \\ 8 & 2 & 1 & 4 & 9 \\ 2 & 1 & 3 & 4 & 4 \\ 1 & 6 & 2 & 9 & 1 \end{pmatrix}$$

*Correction of Exercise 3.5*

Let us use the primal dual algorithm to solve this problem. First let us recall the the LP V [1] that characterizes the assignment problem and its dual LP VI. Let us recall that the principle of a dual-algorithm is to successively compute increasing primal (resp. decreasing dual) solution using previous dual (resp.

[1]Remember that this formulation is totally unimodular, thus the polyhedra associated to the LP relaxation is integral.

primal) one. The idea is to strengthen iteratively lower and upper bound until both are equal (leading then to optimal solutions).

Applied to the assignment problem on a bipartite graph $G = (V_1, V_2, E)$, the intuition behind this algorithm is to keep record of a subgraph $\overline{G}$ of $G$. The feasible dual solution is used to determine the subset of *the most interesting edges* of $E$ that will be contained in $\overline{G}$. The primal solution is computed on $\overline{G}$ and will be mainly of interest to identify the sets of vertices for which alternate matchings exist and thus that could potentially increase the profit of the matching by adding new *interesting edges*.

**Linear Program V.** Assignment Problem

$$\begin{array}{llll}
\text{Maximize} \quad z = & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} & & & \text{(V.1)}\\
\text{Subject To} & \displaystyle\sum_{j=1}^{n} x_{ij} & = 1 & \forall i = 1,\ldots,n & \text{(V.2)}\\
& \displaystyle\sum_{i=1}^{n} x_{ij} & = 1 & \forall j = 1,\ldots,n & \text{(V.3)}\\
& x_{ij} & \geq 0 & \forall i,j = 1,\ldots,n & \text{(V.4)}
\end{array}$$

**Linear Program VI.** Assignment Problem (Dual)

$$\begin{array}{lllll}
\text{Minimize} \quad w = & \displaystyle\sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j & & & \text{(VI.1)}\\
\text{Subject To} & u_i + v_j & \geq c_{ij} & \forall i,j = 1,\ldots,n & \text{(VI.2)}
\end{array}$$

To initiate the algorithm, we need to find a feasible dual solution, id est two $n$-dimensional vectors such that Equations VI.2 are verified. Thus fixing $u_i = 0$ for every $i = 1,\ldots,n$ and $v_j = \max_{i=1,\ldots,n} c_{ij}$ is a feasible dual solution. Indeed:

$$\begin{array}{rcll}
u_i + v_j & = & v_j & \forall i,j = 1,\ldots,n\\
& = & \max_{i=1,\ldots,n} c_{ij} & \forall i,j = 1,\ldots,n\\
& \geq & c_{ij} & \forall i,j = 1,\ldots,n
\end{array}$$

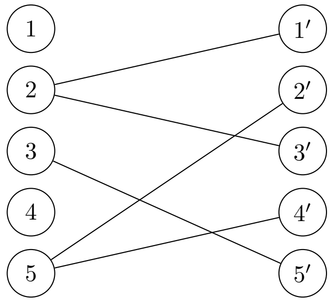
Thus we select, for each column, the maximal coefficient in the cost matrix.
We consider now the dual solution given by $u = (0,0,0,0,0)$ and $v = (9,6,7,9,9)$.

$$c_{ij} = \begin{pmatrix} 6 & 2 & 3 & 4 & 1\\ 9 & 2 & 7 & 6 & 0\\ 8 & 2 & 1 & 4 & 9\\ 2 & 1 & 3 & 4 & 4\\ 1 & 6 & 2 & 9 & 1 \end{pmatrix}$$

$$\overline{c_{ij}}^1 = \begin{pmatrix} -3 & -4 & -4 & -5 & -8\\ 0 & -4 & 0 & -3 & -9\\ -1 & -4 & -6 & -5 & 0\\ -7 & -5 & -4 & -5 & -5\\ -8 & 0 & -5 & 0 & -8 \end{pmatrix}$$

Let us recall that, for all values $\{u_i\}_{i=1}^n$ and $\{v_j\}_{j=1}^n$, the value of any assignment with weights $c_{ij}$ differs by a constant amount from its value with weights $\overline{c_{ij}} = c_{ij} - u_i - v_j$. Thus we may update the coefficient matrix to get $\overline{c_{ij}}^1$.

Each 0 coefficient of the coefficient matrix $\overline{c_{ij}}^1$ corresponds to a an edge of the subgraph $\overline{G}^1$ in which the primal part of the algorithm will be looking for a maximal matching. $\overline{G}^1$ is depicted at left. Selecting the higher coefficient for each column lead to select, for each node of $V_2$, the edge with the better profit. However, we can easily see, that with this selection, no edge has been selected for vertices 1 and 4 (there is no 0 coefficient on first and fourht row).

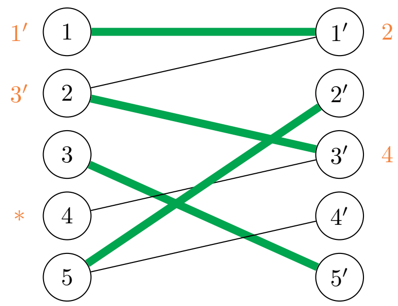



$$\overline{c_{ij}}^1 = \begin{pmatrix} -3 & -4 & -4 & -5 & -8 \\ 0 & -4 & 0 & -3 & -9 \\ -1 & -4 & -6 & -5 & 0 \\ -7 & -5 & -4 & -5 & -5 \\ -8 & 0 & -5 & 0 & -8 \end{pmatrix}$$

Let us remark that, given $i, j \in \{1, \ldots, n\}$, the coefficient $c_{ij} \in \overline{c_{ij}}^1$ represents the value of the slack variable associated to the inequation $u_i + v_j \geq c_{ij}$. As long as every coefficient remain non positive, no constraint is violated by the dual solution.
Thus to select an edge covering vertex 1 (resp. 4) without violating related constraints, it suffices to take the maximum coefficient of the corresponding row. It gives us the dual solution $u = (-3, 0, 0, -4, 0)$, $v = (9, 6, 7, 9, 9)$.
This obviously gives us a dual solution with cost $w = -3 - 4 + 9 + 6 + 7 + 9 + 9 = 33$.

Since we modified the $u_i$, we can update the coefficient matrix to get $\overline{c_{ij}^2}$.

$$\overline{c_{ij}}^2 = \begin{pmatrix} 0 & -1 & -1 & -2 & -5 \\ 0 & -4 & 0 & -3 & -9 \\ -1 & -4 & -6 & -5 & 0 \\ -3 & -1 & 0 & -1 & -1 \\ -8 & 0 & -5 & 0 & -8 \end{pmatrix}$$



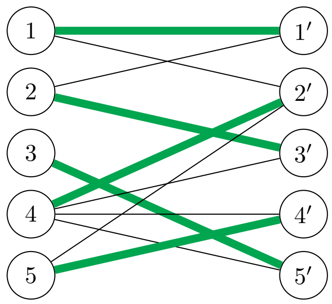

This gives us the subgraph $G^2$ in which a maximum cardinality matching is found with the appropriate algorithm.
We instantly see that this is not an optimal solution for $G$ as the number of edges in the matching is only 4. But we can use the strong duality principle to formally prove. Indeed, recall that the value of the dual solution is $w = 33$, while the profit of this matching $z = 6 + 7 + 9 + 6 = 28 \neq 33$.
Note that the algorithm gives us two interesting sets of vertices: $V_1^+ = \{1, 2, 4\}$ (the labeled node in $V_1$) and $V_2^- = \{2', 4', 5'\}$, the unlabeled node in $V_2$.

Remark that thes two sets define a matrix in which no coefficient is set to zero. These sets highlight the set of nodes for which adding an edge could decrease the value of the dual solution.
We select the maximal coefficient in this submatrix ($\delta = -1$), and we update the $u_i$ and the $v_j$ so that no constraint is violated. In other words for $i \in V_1^+$ we set $u_i = u_i + \delta$, and for all $j \in V_2^+ = V_2 \backslash V_2^-$ we set $v_j = v_j - \delta$.
Done in this way, we ensure that coefficient that are not part of the submatrix are not modified.

$$\begin{pmatrix} -1 & -2 & -5 \\ -4 & -3 & -9 \\ -1 & -1 & -1 \end{pmatrix}$$

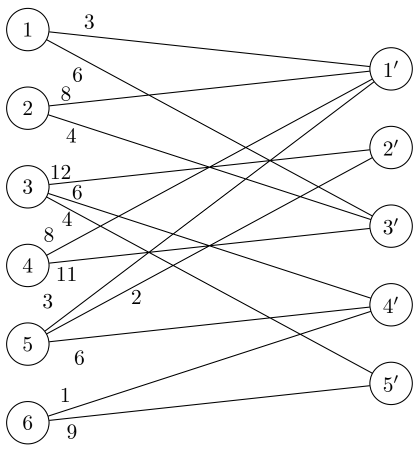
$$\overline{c_{ij}}^3 = \begin{pmatrix} 0 & 0 & -1 & -1 & -4 \\ 0 & -3 & 0 & -2 & -8 \\ -1 & -4 & -6 & -5 & 0 \\ -3 & 0 & 0 & 0 & 0 \\ -8 & 0 & -5 & 0 & -8 \end{pmatrix}$$

This gives us dual solution $u = (-4, -1, 0, -5, 0)$ and $v = (10, 6, 8, 9, 9)$ of cost $w = 32$.
We can then update the coefficient matrix $\overline{c_{ij}}^2$ to get $\overline{c_{ij}}^3$.




By applying the maximal cardinality matching on $\overline{G}^3$ the subgraph of $G$ associated to coeeficient matrix $\overline{c_{ij}}^3$, we get a perfect matching of profit $z = 6 + 7 + 9 + 1 + 9 = 32 = w$. The solution is thus optimal.

**Exercise 3.6** Find a maximum weight matching in the following weighted bipartite graph $G_3$.




*Correction of Exercise 3.6*

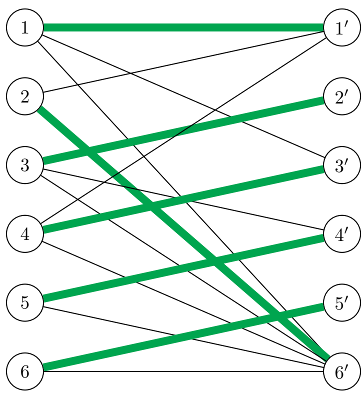
The first is to *transform* this problem into an assignment problem. To do so, we add a dummy vertex in $V_2$ and for each couple of vertices $(i, j) \in V_1 \times V_2$ such that $(ij) \notin E$, we create an edge $(ij)$ of profit 0.

This gives us the following coefficient matrix:

$$c_{ij} = \begin{pmatrix} 3 & 0 & 6 & 0 & 0 & 0 \\ 8 & 0 & 4 & 0 & 0 & 0 \\ 0 & 12 & 0 & 6 & 4 & 0 \\ 8 & 0 & 11 & 0 & 0 & 0 \\ 3 & 2 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 9 & 0 \end{pmatrix}$$

We apply the previous algorithm on this matrix. We first select the maximum value for each column and select an edge for vertex 1. This gives us the following matrix and associated graph:

$$\overline{c_{ij}}^2 = \begin{pmatrix} 0 & -7 & 0 & -1 & -4 & 0 \\ 0 & -12 & -7 & -6 & -9 & 0 \\ -8 & 0 & -11 & 0 & -5 & 0 \\ 0 & -12 & 0 & -6 & -9 & 0 \\ -5 & -10 & -11 & 0 & -9 & 0 \\ -8 & -12 & -11 & -5 & 0 & 0 \end{pmatrix}$$



At the end of the algorithm we get the dual solution $u = (-5, 0, 0, 0, 0, 0)$, $v = (8, 12, 11, 6, 9, 0)$ of cost $w = 41$ and a matching of profit $z = w = 41$.

**Exercise 3.7** Consider the statement of Exercise TP 1.:

> John Dupont is attending a summer school where he must take four courses per day. Each course lasts an hour, but because of the large number of students, each course is repeated several times per day by different teachers. Section $i$ of course $k$ denoted $(i, k)$ meets at the hour $t_{ik}$, where courses start on the hour between 10:00 and 19:00. John's preferences for when he takes courses are influenced by the reputation of the teacher, and also the time at which the course begins. Let $p_{ik}$ be his preference for section $(i, k)$. Unfortunately, due to conflicts, John cannot always choose the sections he prefers.
>
> Formulate an integer program to chose a feasible course schedule that maximizes John's preferences.
>
> 1. Propose an algorithm to solve this problem

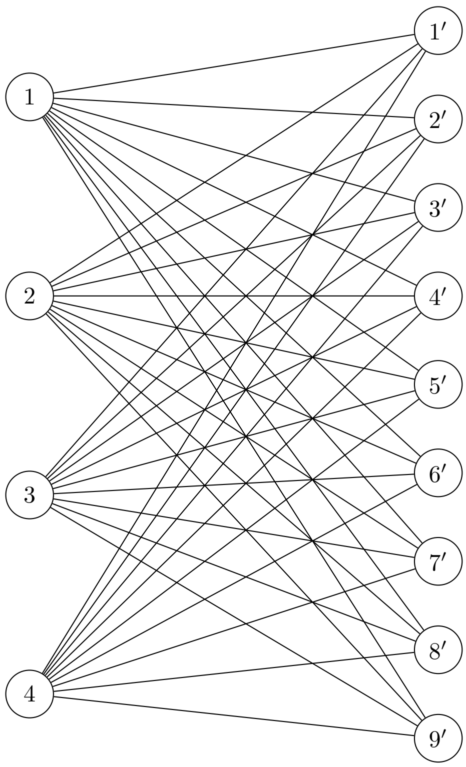
*Correction of Exercise 3.7*

We just have to represent the problem under the form of a bipartite graph in which we are looking for a maximum matching.

We construct the graph as follows:

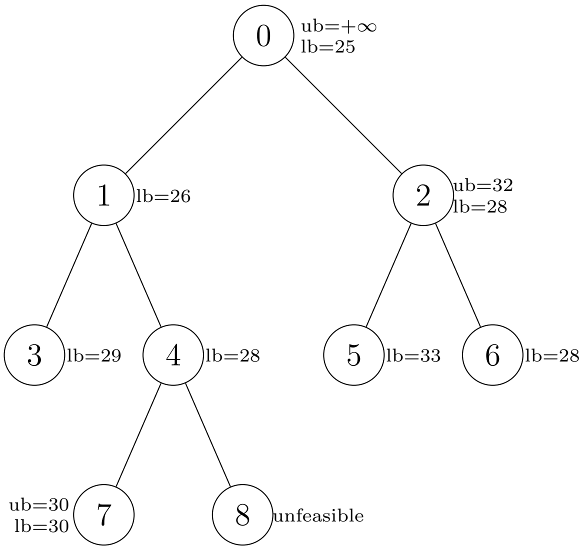
1. we create a vertex in $V_1$ for each of the four courses John has to chose,
2. we create a vertex in $V_2$ for each of the session,
3. we make the graph complete by adding edge with profit equal to the preference of John.

This gives us the following graph:



# TP 4. Branch and Bound

**Exercise 4.1** Consider the following enumeration tree of a minimization problem:




1. Give the tightest possible lower and upper bounds on the optimal value z.
2. Which nodes can be pruned and which must be explored further? Provide a justification for each node.

*Correction of Exercise 4.1*

To get the best possible lower and upper bounds on the optimal value, for a minimization problem, we need to make a *walk* from the leaves to the root node. At each node, we only keep the minimum value for each of the bounds:

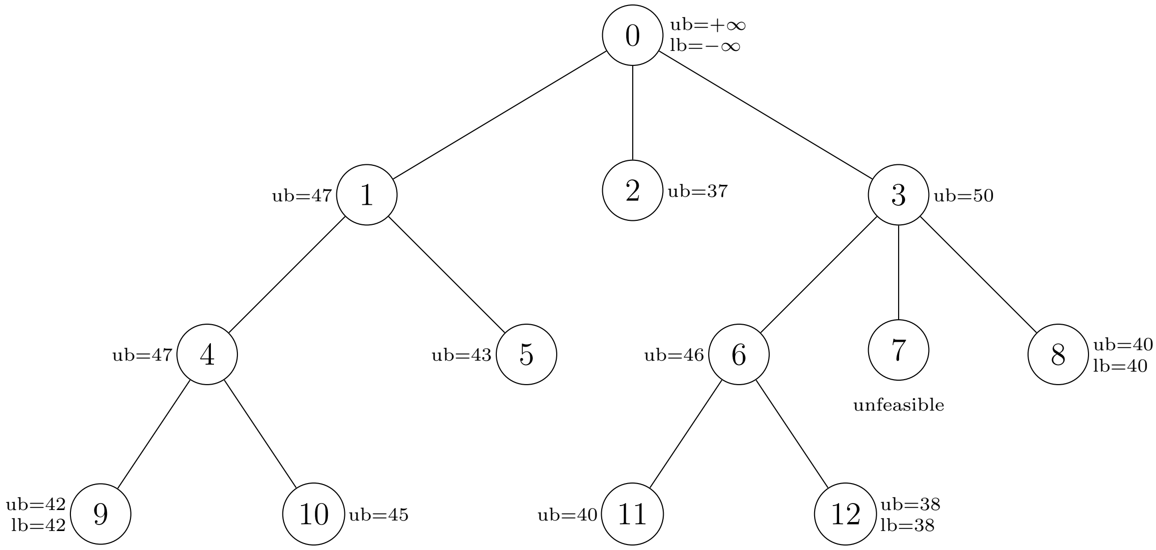
1. Let us start at node 4, the upper bound is given by $\min\{ub_7, ub_8\}$ (where $ub_7$ denotes the upper bound known for node 7). Thus $ub_4$ can be updated to value 30. The same with the lower bound gives us $lb_4 = \min\{lb_7, lb_8\} = 28$.
2. We consider node 1, $lb_1 = \min\{lb_3, lb_4\} = 28$ and $ub_1 = \min\{ub_3, ub_4\} = 30$.
3. For node 2, we have $lb_2 = 28$ and $ub_2 = 32$.
4. To finish at node 0 we have $ub_0 = 30$ and $lb_0 = 28$.

Thus we can deduce that $28 \leq z \leq 30$.

Obviously node 8 can be pruned, since no feasible solution can be found. Node 7 could be pruned, if and only if an integral solution has been found, *id est* every basic variable in the found solution is integral, otherwise, we need to branch on the non integral basic variables. To finish node 5 can also be pruned since $lb_5 \geq ub_0$. This is due to the fact that, for a minimization problem, the more you get low in the enumeration tree, the more the value of the objective function of the constrained problem increases.

**Exercise 4.2** Consider the following enumeration tree of a maximization problem:

1. Give the tightest possible lower and upper bounds on the optimal value z.
2. Which nodes can be pruned and which must be explored further? Provide a justification for each node.




*Correction of Exercise 4.2*

Same problem as before, except that we are considering a maximization problem. Thus the bounds are computed by keeping the maximum of the children at each node.

This gives us $ub_0 = 45$ and $lb_0 = 42$. Based on these bounds, on unfeasibility and on optimality, the nodes $2, 7, 8, 9, 11$ and $12$ can be pruned while nodes 5 and 10 must be explored further.

**Exercise 4.3** Consider the 0-1 knapsack problem XVIII with $a_j, c_j > 0$ for $j = 1, ..., n$.

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**Integer Linear Program XVIII.** 0/1 knapsack formulation

---

$$\text{Maximize} \quad \sum_{j=1}^{n} c_j x_j \tag{XVIII.1}$$

$$\text{Subject To} \quad \sum_{j=1}^{n} a_j x_j \leq b \tag{XVIII.2}$$

$$x \in \mathbb{B}^n \tag{XVIII.3}$$

---

Show that if $\frac{c_1}{a_1} \geq ... \geq \frac{c_n}{a_n} > 0$, $\sum_{j=1}^{r-1} a_j \leq b$ and $\sum_{j=1}^{r} a_j > b$, the solution of the linear relaxation of the problem above is:

$$\begin{cases} x_j^* = 1 & \forall j = 1, ..., r-1 \\ x_r^* = \frac{\left(b - \sum\limits_{j=1}^{r-1} a_j u_j\right)}{a_r} & \\ x_j^* = 0 & \forall j = r+1, ..., n \end{cases}$$

*Correction of Exercise 4.3*

Let us consider a solution $x$ to the linear relaxation of ILP XVIII. Suppose that there exists a couple of indices $i, j$ such that $i < j$ and such that $x_i = 1 - \alpha$, with $0 < \alpha \leq 1$ and $x_j = \beta$ with $0 < \beta \leq 1$.

We now show that increasing $x_i$ while decreasing $x_j$ does not violate the constraint and improves the profit of the solution. Let us consider two cases:

**Case 1:** $a_i\alpha \leq a_j\beta$ Intuitively, it means that the space freed by removing $x_j$ from the sack is enough to entirely pack $x_i$. This can be formally described by defining a new solution $x'$ such that:

$$x'_k = \begin{cases} 1 & \text{if } k = i \\ \beta - \frac{\alpha a_i}{a_j} & \text{if } k = j \\ x_k & \text{otherwise} \end{cases}$$

Let us show that the capacity constraint is not violated by this change:

$$\begin{aligned} a_i x'_i + a_j x'_j &= a_i + a_j\beta - \alpha a_i \\ &= a_i(1-\alpha) + a_j\beta \\ &= a_i x_i + a_j x_j \end{aligned}$$

Since $x$ is a feasible solution, thus $x'$ is also a feasible solution.

Now let us focus on the profit of $x'$:

$$\begin{aligned} c_i x'_i + c_j x'_j &= c_i + c_j + \left(\beta - \frac{\alpha a_i}{a_j}\right) \\ &= c_i + c_j\beta - \frac{c_j}{a_j}\alpha a_i \qquad \text{since } i < j, \text{ then } \frac{c_j}{a_j} < \frac{c_i}{a_i} \\ &> c_i + c_j\beta - \frac{c_i}{a_i}\alpha a_i \\ &= c_i x_i + c_j x_j \end{aligned}$$

**Case 2:** $a_i\alpha > a_j\beta$ In this case, the space freed by removing every part of $x_j$ is not sufficient to pack completely $x_i$. More formally, we define a new solution $x'$ as follows:

$$x'_k = \begin{cases} 1 - \alpha + \beta\frac{a_j}{a_i} & \text{if } k = i \\ 0 & \text{if } k = j \\ x_k & \text{otherwise} \end{cases}$$

Let us show that, once again, the capacity constraint is not violated by this change:

$$\begin{aligned} a_i x'_i &= a_i(1-\alpha) + \beta a_j \\ &= a_i x_i + a_j x_j \end{aligned}$$

Since $x$ is a feasible solution, thus $x'$ is also a feasible solution.

As previously, let us now focus on the profit of $x'$:

$$\begin{aligned} c_i x'_i &= c_i\,(1-\alpha) + \beta\frac{a_j}{a_i} \\ &= c_i(1-\alpha) + \frac{c_i}{a_i}\beta a_j \qquad \text{since } i < j, \text{ then } \frac{c_j}{a_j} < \frac{c_i}{a_i} \\ &> c_i(1-\alpha) + c_j\beta \\ &= c_i x_i + c_j x_j \end{aligned}$$

## Exercise 4.4

Solve the instance XIX by branch-and-bound. Use the linear relaxation to get dual bounds.

**Integer Linear Program XIX.** Instance

$$\begin{aligned}
\text{Maximize} \quad w = \;& 14x_1 + 10x_2 + 10x_3 + 3x_4 && && \text{(XIX.1)}\\
\text{Subject To} \quad & 6x_1 + 4x_2 + 5x_3 + 2x_4 && \leq 10 && \text{(XIX.2)}\\
& x_1, x_2 \in \mathbb{Z}_+ && && \text{(XIX.3)}\\
& x_3, x_4 \in \{0, 1\} && && \text{(XIX.4)}
\end{aligned}$$

*Correction of Exercise 4.4*

Let us first change the indices of variables such that $\frac{c_1}{a_1} > \frac{c_2}{a_2} > \frac{c_3}{a_3} > \frac{c_4}{a_4}$. This gives us the ILP XX.

**Integer Linear Program XX.** Modified Instance

$$\begin{aligned}
\text{Maximize} \quad w = \;& 10x_1 + 14x_2 + 10x_3 + 3x_4 && && \text{(XX.1)}\\
\text{Subject To} \quad & 4x_1 + 6x_2 + 5x_3 + 2x_4 && \leq 10 && \text{(XX.2)}\\
& x_1, x_2 \in \mathbb{Z}_+ && && \text{(XX.3)}\\
& x_3, x_4 \in \{0, 1\} && && \text{(XX.4)}
\end{aligned}$$

As proved in Exercise TP 4., the solution of the linear relaxation is:

$$\begin{cases}
x_j^* = 1 & \forall j = 1, ..., r-1\\
x_r^* = \dfrac{\left(b - \sum\limits_{j=1}^{r-1} a_j u_j\right)}{a_r} & \\
x_j^* = 0 & \forall j = r+1, ..., n
\end{cases}$$

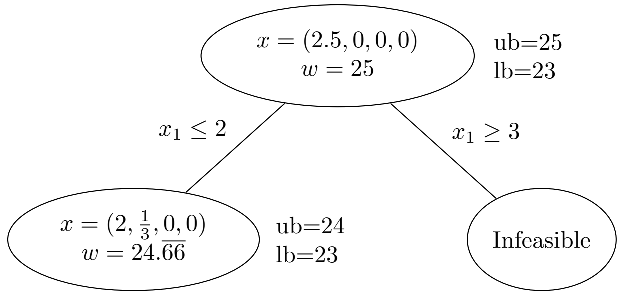
Then we find a feasible solution for this instance with a greedy assignment which first packs the maximum amount of object 1, then the maximum amount of object 2 and so on until object 4. This gives us $x = (2, 0, 0, 1)$ with $w = 23$. This gives us an initial lower bound.

Let us begin the branch-and-bound algorithm. The first step is to solve the linear relaxation of the root node.

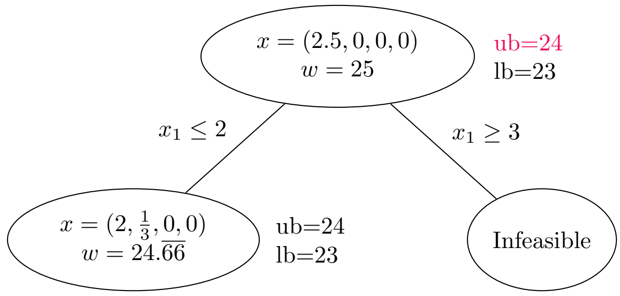
$x = (2.5, 0, 0, 0)$
$w = 25$

ub=25
lb=23

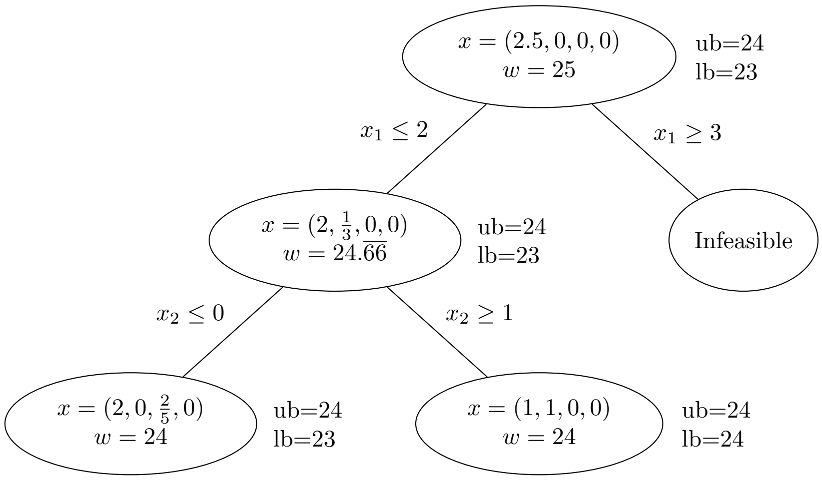
The optimal solution of the linear relaxation is not integral, we have to branch. This is done by introducing constraint $x_1 \leq 2$ on the left child and $x_1 \geq 3$ on the right child. The linear relaxation of children node is then optimally solved.



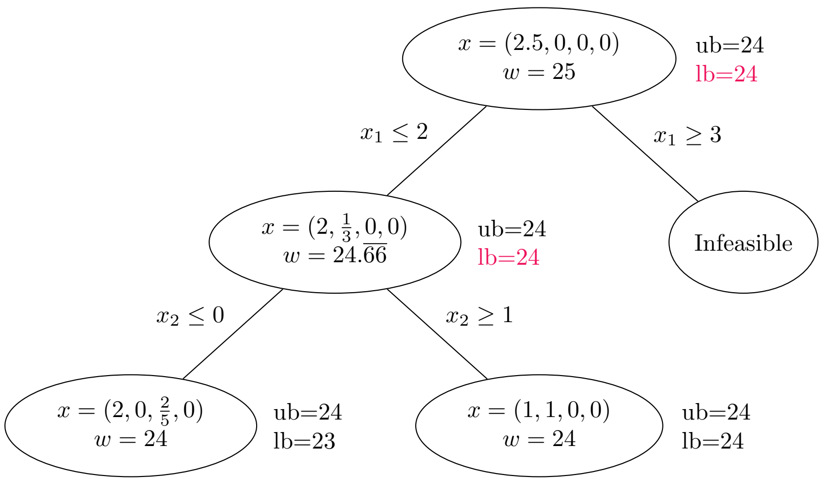

At this point it is possible to update the upper bound for the root node.




We continue by exploring the left child (since the right one is infeasible). We now branch on variable $x_2$ by adding constraint $x_2 \leq 0$ on the left child and $x_2 \geq 1$ on the right one.




The solution returned by the linear relaxation of the right child is integral. Thu, it is a feasible solution of the integer problem, and it gives a new lower bound. We update hence the lower bounds.



We do not need to continue exploring the left child, since every solution found in its children will verify $z \leq ub = 24$, which is the value of the integral solution found.

Thus the optimal integral solution is given by $x^* = (1, 1, 0, 0)$.

**Exercise 4.5** Consider the following STSP instance with $n = 5$ and the following distance matrix:

$$(c_e) = \begin{pmatrix} - & 10 & 2 & 4 & 6 \\ - & - & 9 & 3 & 1 \\ - & - & - & 5 & 6 \\ - & - & - & - & 2 \end{pmatrix}$$

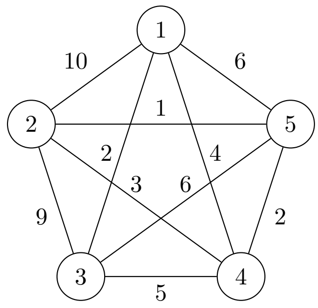
Solve it by branch-and-bound using a 1-tree relaxation to obtain bounds, a 1-tree being a subgraph consisting of two edges adjacent to node 1, plus the edges of a tree on nodes $\{2, ..., n\}$.

### *Correction of Exercise 4.5*

Let us first recall that an optimal 1-tree is computed as follows:

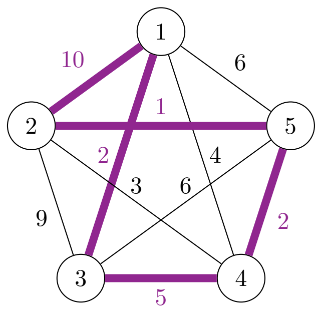
1. compute a minimum spanning on the subgraph induced by set of vertices $\{2, 3, 4, 5\}$,
2. add the two edges adjacent to 1 with the smallest cost.

We consider then the graph described by $(c_e)$:

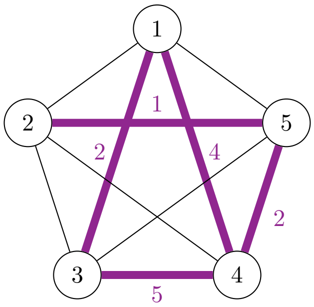


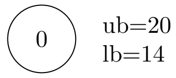

To start the branch-and-bound algorithm, we compute a feasible solution with a greedy algorithm starting from an arbitrary point (let us say 1) and adding to the tour the smallest edge adjacent to 1. It then iterates on the newly reached vertex, add the smallest edge to an unvisited vertex. Once every vertex has been visited, it just closes the turn.

This algorithm then first select edge $(1, 3)$, then $(3, 4)$, $(4, 5)$, $(5, 2)$ and closes the tour with $(2, 1)$. This solution of cost 20 is depicted in the following figure.

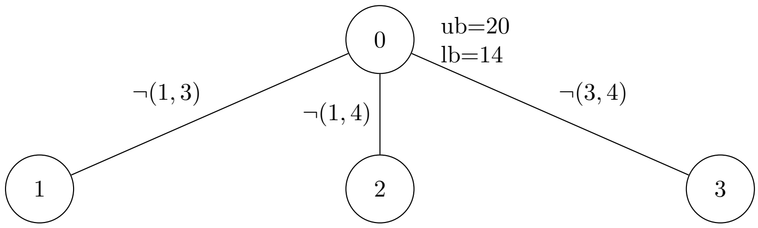



We can now start the branch-and-bound algorithm. At the root node, the optimal 1-tree of cost 14 is the following one, with the associated node in branching tree:

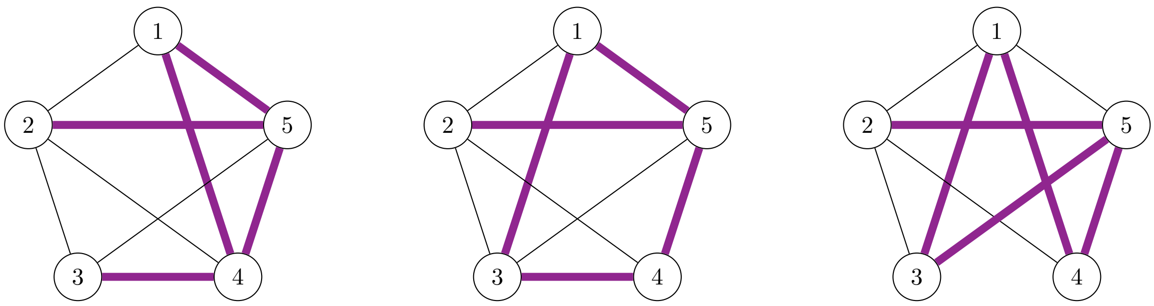



The solution being not a tour, we need to branch. We could branch on each edge by determining whether it is part of the solution or not, but we can also branch on the edges being part of subtours in the optimal one-tree: edges $(1,3)$, $(1,4)$ and $(3,4)$ cannot be chosen together, as they define a tour of size three. This gives us the following branching tree.

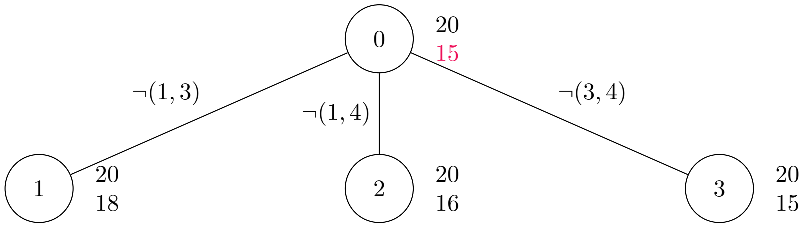



For each of nodes $1, 2$ and $3$, the optimal one-tree is described below:

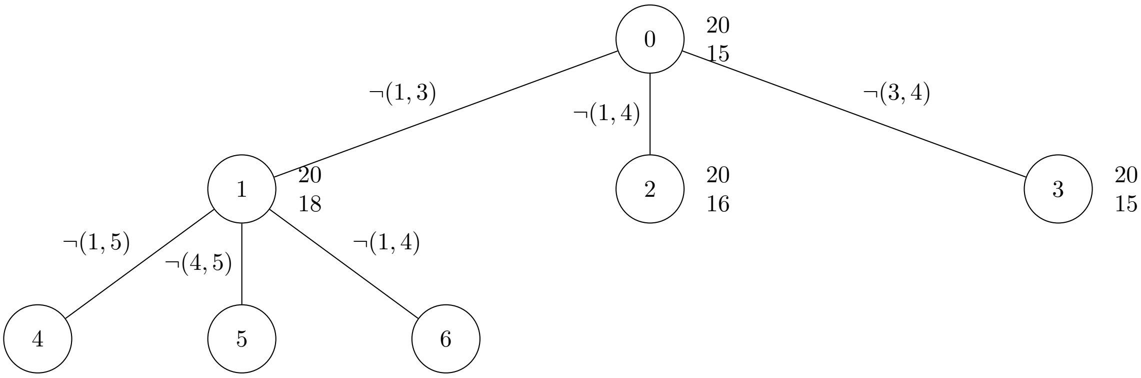



$z = 18$ $z = 16$ $z = 15$

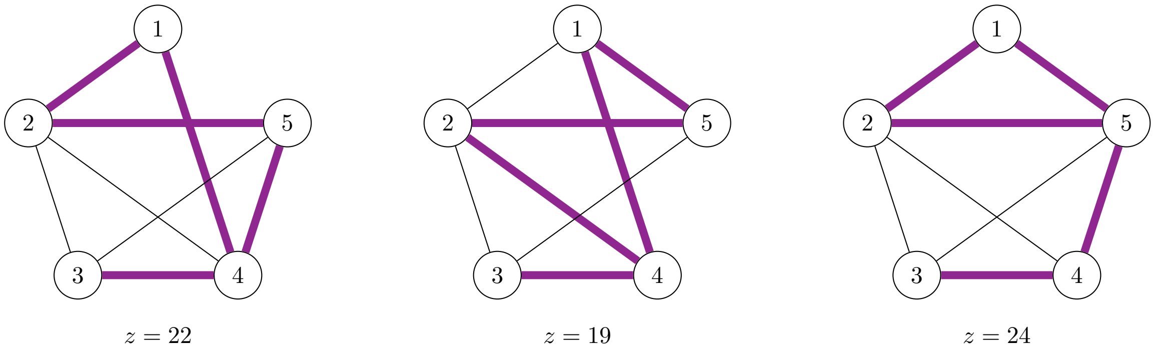
This gives us the following branching tree.



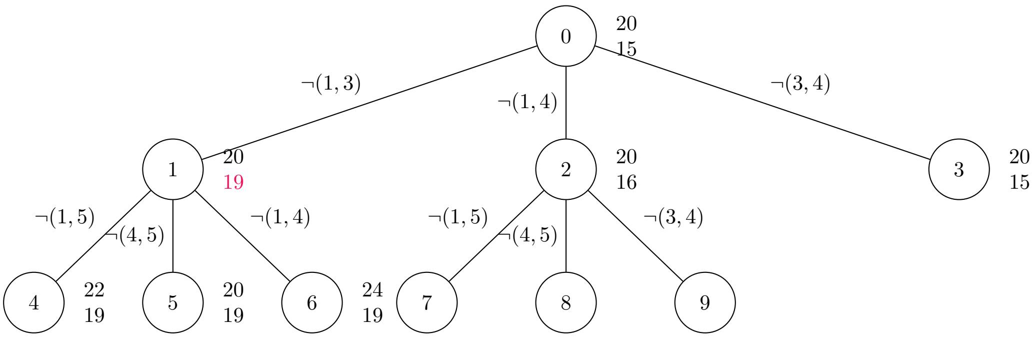

We branch on node 1 on the subtour $(1, 4, 5)$.



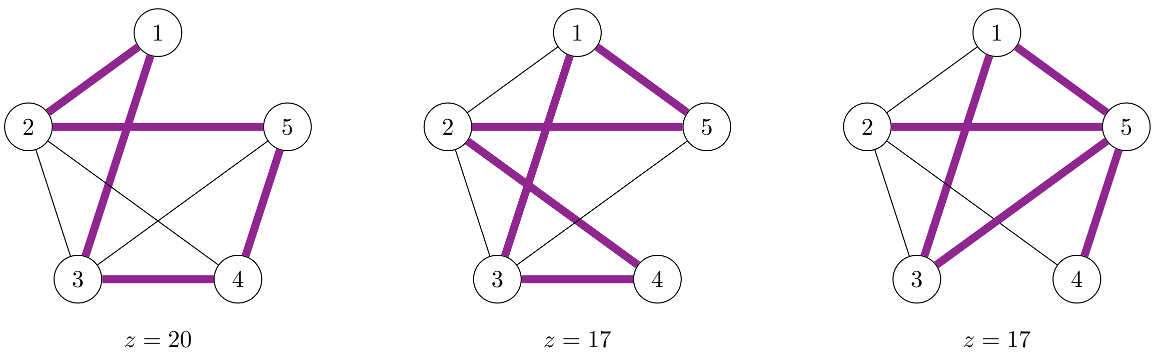

And the one-trees associated to each child.



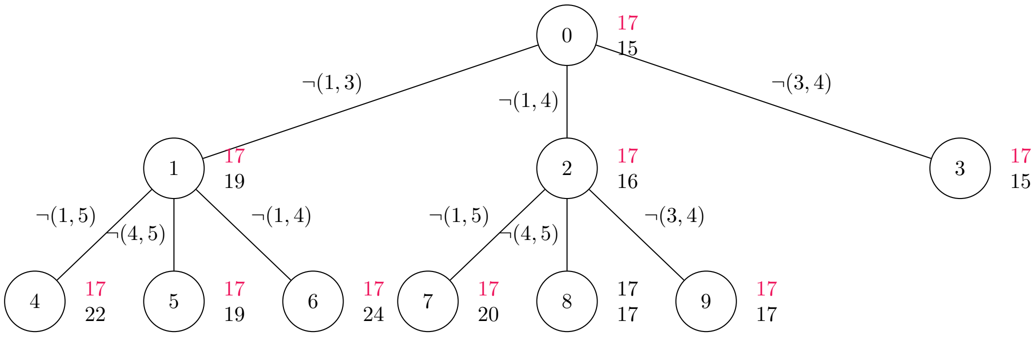

We update the branching tree and branch, for node 2 on the subtour (1543). However we do not need to branch on edge $(1, 3)$ since it is the same as node 6.



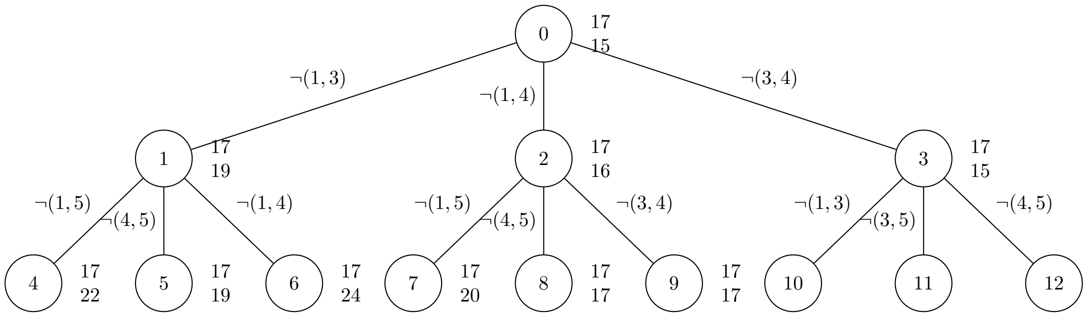

For each child, the lower is computed as follows:



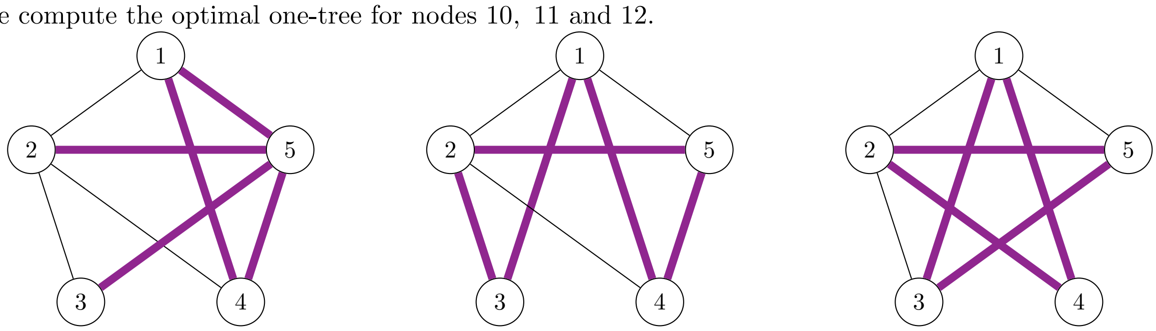

Let us remark that node 8 yields a better tour than the one initially found, we can update the bounds.




We can remark that nodes 7 and 9 will be pruned. We branch for node 3 on subtour (1354). Here again, we do not need to branch on edge $(1, 4)$ since it leads to the exact same node as 9.




We compute the optimal one-tree for nodes 10, 11 and 12.




$z = 19$ $z = 18$ $z = 16$

Note that node 12 yields a better subtour, and after having updated the bounds, no node can be explored. Thus the optimal solution is given by node 12 with a minimum hamiltonian cycle of cost 16.

**Exercise 4.6** Consider the following TSP instance with $n = 4$ and the following distance matrix:

$$(c_e) = \begin{pmatrix} - & 7 & 6 & 3 \\ 3 & - & 6 & 9 \\ 2 & 3 & - & 1 \\ 7 & 9 & 4 & - \end{pmatrix}$$

Solve it by branch-and-bound using an assignment relaxation to obtain bounds.

### *Correction of Exercise 4.6*

This one is left as homework. If you have any questions, please contact me.

**Exercise 4.7** Consider the two-variables integer program:

**Integer Linear Program XXI.** Two-variables integer program

$$
\begin{array}{rll}
\text{Maximize} & 9x_1 + 5x_2 & \\
\text{Subject To} & x_1 & \leq 6 \\
& x_1 - 3x_2 & \geq 1 \\
& 3x_1 + 2x_2 & \leq 19 \\
& x \in Z_+^2. &
\end{array}
\qquad
\begin{array}{r}
\text{(XXI.1)} \\ \text{(XXI.2)} \\ \text{(XXI.3)} \\ \text{(XXI.4)} \\ \text{(XXI.5)}
\end{array}
$$

Solve it by branch-and-bound graphically and algebraically.

## *Correction of Exercise 4.7*

Same as previous exercise. Left as homework.

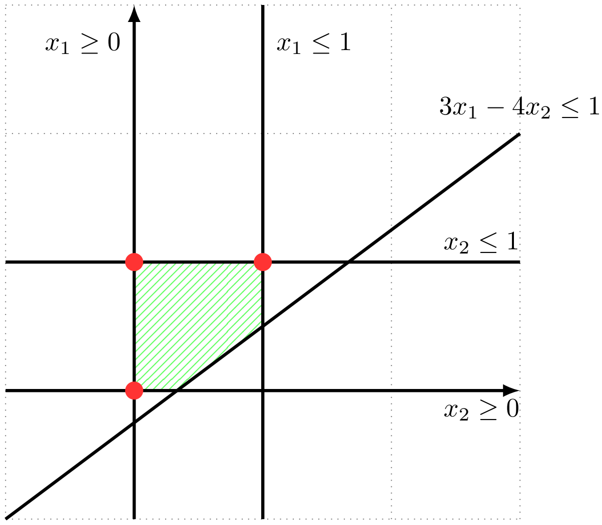
# TP 5. Cutting Plane Algorithms

**Exercise 5.1** For each of the three sets below, find a missing valid inequality and verify graphically that its addition to the formulation gives $conv(X)$.

1. $X_1 = \{x \in \mathbb{B}^2 \,:\, 3\,x_1 - 4\,x_2 \leq 1\}$
2. $X_2 = \{(x, y) \in \mathbb{R}^1_+ \times \mathbb{B}^1 \,:\, x \leq 20\,y,\ x \leq 7\}$
3. $X_3 = \{(x, y) \in \mathbb{R}^1_+ \times \mathbb{Z}^1_+ \,:\, x \leq 6\,y,\ x \leq 16\}$

*Correction of Exercise 5.1*

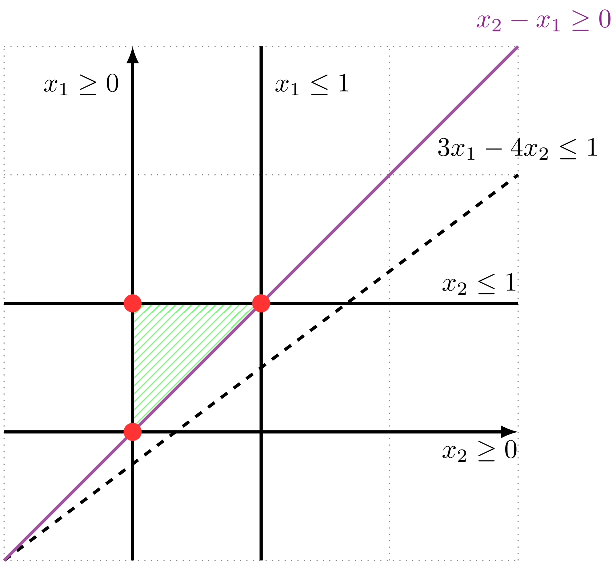
Let us consider the first set $X_1 = \{x \in \mathbb{B}^2 \,:\, 3\,x_1 - 4\,x_2 \leq 1\}$. $X_1$ can be graphically represented as follows:



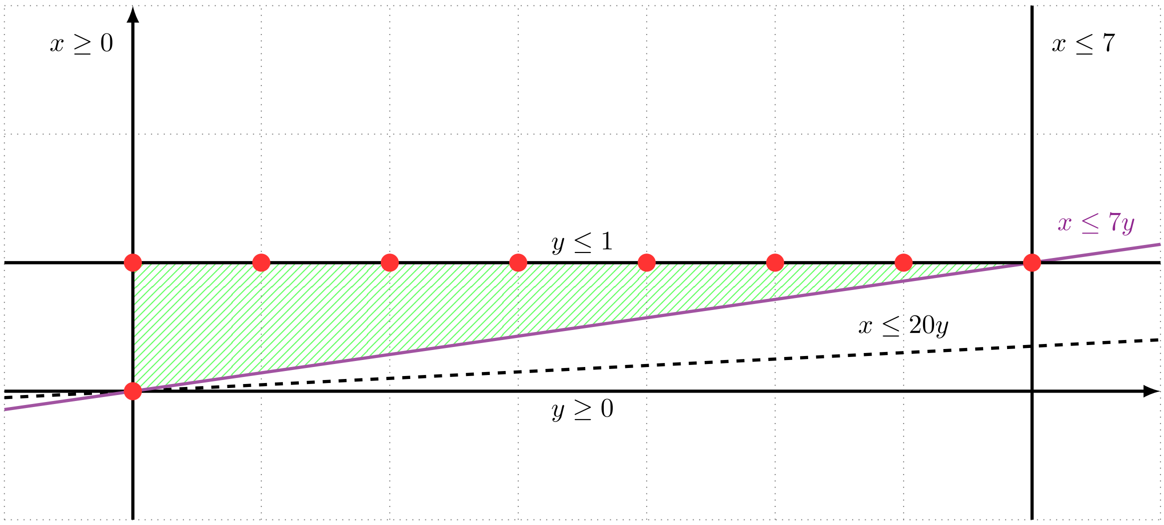

We can remark that the feasible solutions are the following ones: $(0, 0)$, $(0, 1)$ and $(1, 1)$. Informally, $x_1$ can be equal to 1 only if $x_2 = 1$, this can be modeled by the following inequality:

$$x_2 - x_1 \geq 0$$

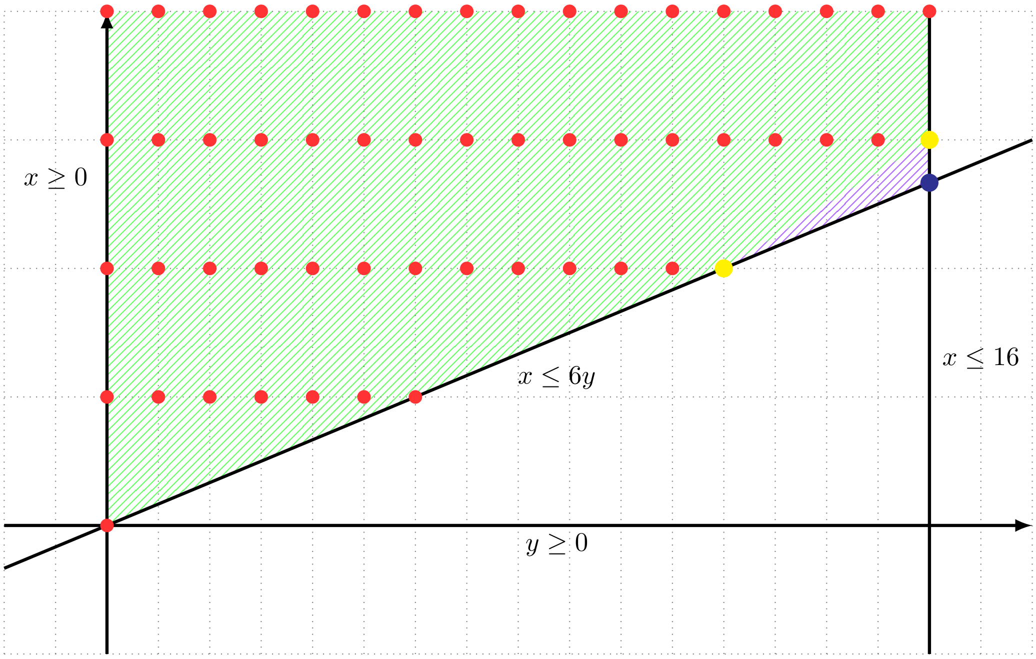
We can graphically verify that adding this inequality to the set of constraints gives $conv(X)$.



Let us now consider the set $X_2 = \{(x, y) \in \mathbb{R}^1_+ \times \mathbb{B}^1 \,:\, x \leq 20\,y,\, x \leq 7\}$. Obviously, $x$ being bounded by 7, the inequality $x \leq 20y$ can be strengthened to the following one: $x \leq 7y$. This can be graphically represented as follows:

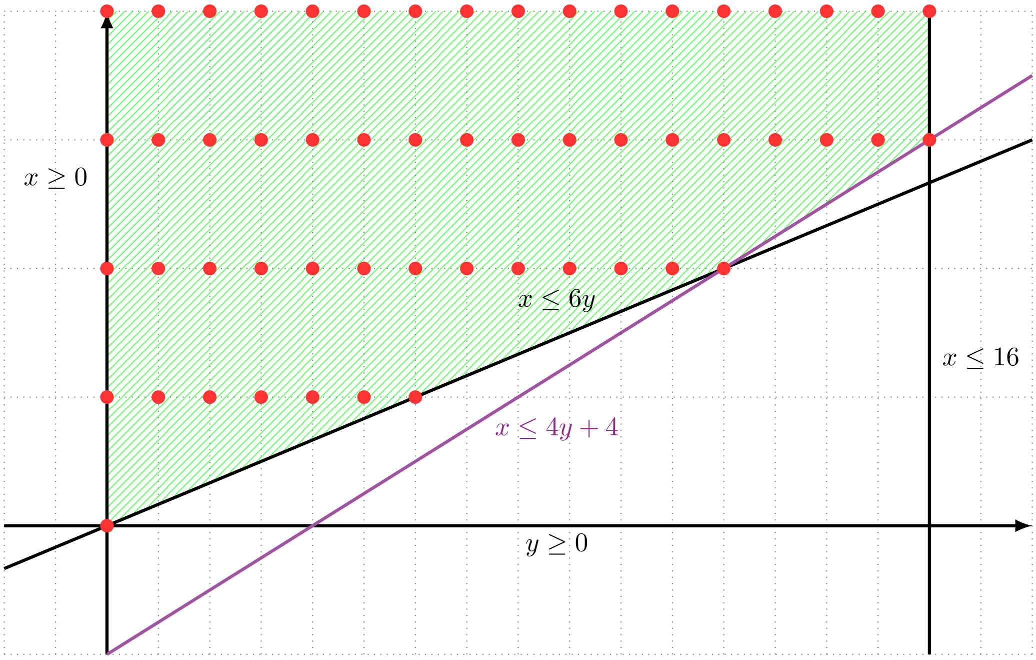



Finally, let us consider the set $X_3 = \{(x, y) \in \mathbb{R}^1_+ \times \mathbb{Z}^1_+ \,:\, x \leq 6\,y,\, x \leq 16\}$. The latter can be graphically represented as follows.




Let us remark that the extremal point of the polyhedra $(16, {}^{16}\!/\!_6)$ (in blue) is not an integral point. Thus we can define a new cut going through both of the integral feasible solutions $(12, 2)$ and $(16, 3)$ (in yellow).

The straight lines joining the yellow point has the following equation $y = {}^{x}\!/\!_4 - 1$. We can then derive the following inequality: $x \leq 4y - 4$. Leading to the following graphical representation:




Another way to find such a valid inequality is to remark that 16/6 is not an integer. Let us recall that if:

$$X \subseteq \left\{(x, y) : x \leq Cy, 0 \leq x \leq b, y \in \mathbb{Z}^+\right\}$$

thus the following inequality is a valid inequality:

$$x \leq b - \left(b - \left(\left\lceil \frac{b}{C} - 1 \right\rceil\right) C\right)\left(\left\lceil \frac{b}{C} \right\rceil - y\right)$$

In other words, we get:

$$x \leq 16 - \left(16 - \left(\left\lceil \frac{16}{6} \right\rceil - 1\right) \times 4\right)\left(\left\lceil \frac{16}{6} \right\rceil - y\right) = 16 - 4(3 - y) = 4y + 4$$

**Exercise 5.2** In each of the examples below a set $X$ and a point $x$ or $(x, y)$ are given. Find a valid inequality for $X$ cutting off the point.

1. $X = \{(x, y) \in \mathbb{R}^2_+ \times \mathbb{B}^1 \ : \ x_1 + x_2 \leq 2\,y,\ x_j \leq 1 \ \ for \ \ j = 1, 2\}$

$(x_1,\ x_2,\ y) = (1,\ 0,\ \frac{1}{2})$

2. $X = \{(x, y) \in \mathbb{R}^1_+ \times \mathbb{Z}^1_+ \ : \ x \leq 9,\ x \leq 4\,y\}$

$(x,\ y) = (9,\ \frac{9}{4})$

3. $X = \{(x, y) \in \mathbb{R}^2_+ \times \mathbb{Z}^1 \ : \ x_1 + x_2 \leq 25,\ x_1 + x_2 \leq 8\,y\}$

$(x_1,\ x_2,\ y) = (20,\ 5,\ \frac{25}{8})$

4. $X = \{x \in \mathbb{Z}^5_+ \ : \ 9\,x_1 + 12\,x_2 + 8\,x_3 + 17\,x_4 + 13\,x_5 \geq 50\}$

$(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (0,\ \frac{25}{6},\ 0,\ 0,\ 0)$

5. $X = \{x \in \mathbb{Z}^4_+ \ : \ 4\,x_1 + 8\,x_2 + 7\,x_3 + 5\,x_4 \leq 33\}$

$(x_1,\, x_2,\, x_3,\, x_4) = (0,\ 0,\ \frac{33}{7},\, 0)$

## *Correction of Exercise 5.2*

1. Let us remark that, if $y = 1$ then $x_1 \leq 1$ and $x_2 \leq 1$ and if if $y = 0$ then $x_1 = x_2 = 0$. Thus the inequality $x_1 \leq y$ is valid. Furthermore, the latter cuts the point.
2. We can remark that 4 does not divide 9, thus we can use the formula:

$$x \leq b - \left(b - \left(\left\lceil \frac{b}{C} - 1 \right\rceil\right) C\right) \left(\left\lceil \frac{b}{C} \right\rceil - y\right)$$

This gives us:

$$\begin{array}{rcl} x & \leq & 9 - \left(9 - 4\left(\lceil \frac{9}{4} - 1 \rceil\right)\right) \left(\lceil \frac{9}{4} \rceil - y\right) \\ & = & 9 - 3 + y \\ & = & 6 + y \end{array}$$

Or we can also make the following remarks:

**if** $y \geq 3$**:** the constraint $x \leq 4y$ is useless,
**if** $y = 0$**:** $x = 0$
**if** $y = 1$**:** $x \leq 4 < 7$
**if** $y = 2$**:** $x \leq 8$

Thus $x \leq 6 + y$ is a valid constraint.

3. Once again we can use the formula since constraints depends both on $x_1 + x_2$. This gives us:

$$\begin{array}{rcl} x_1 + x_2 & \leq & 25 - \left(25 - 8\left(\lceil \frac{25}{8} - 1 \rceil\right)\right) \left(\lceil \frac{25}{8} \rceil - y\right) \\ & = & 25 - 4 + y \\ & = & 21 + y \end{array}$$

We can also proceed as previously:

**if** $y \geq 4$**:** the constraint $x_1 + x_2 \leq 8y$ is useless,
**if** $y = 0$**:** $x_1 = x_2 = 0$
**if** $y = 1$**:** $x_1 + x_2 \leq 8$
**if** $y = 2$**:** $x_1 + x_2 \leq 16$
**if** $y = 3$**:** $x_1 + x_2 \leq 24$

Thus $x_1 + x_2 \leq 21 + y$ is a valid constraint.

4. We divide both sides of the inequality by 12. This leads to the following inequality:

$$\frac{3}{4}x_1 + x_2 + \frac{2}{3}x_3 + \frac{17}{12}x_4 + \frac{13}{12}x_5 \geq \frac{26}{6}$$

Since $x \in \mathbb{Z}^+_5$, the following inequality is valid:

$$\left\lceil \frac{3}{4} \right\rceil x_1 + x_2 + \left\lceil \frac{2}{3} \right\rceil x_3 + \left\lceil \frac{17}{12} \right\rceil x_4 + \left\lceil \frac{13}{12} \right\rceil x_5 \geq \left\lceil \frac{26}{6} \right\rceil$$

Giving then:

$$x_1 + x_2 + x_3 + 2x_4 + 2x_5 \geq 5$$

5. As previously, we divide both sides of the inequality by 7:

$$\frac{4}{7}x_1 + \frac{8}{7}x_2 + x_3 + \frac{5}{7}x_4 \leq \frac{33}{7}$$

However, we take the floor instead of the ceil because of the sign of the inequality.
This gives us:

$$x_2 + x_3 \leq 4$$

Note that the straightforward inequality $x_3 \leq 4$ also cuts the point.

**Exercise 5.3** Prove that $y_2+y_3+2\,y_4 \leq 6$ is valid for the set $X = \{y \in \mathbb{Z}^4_+ : 4y_1+5y_2+9y_3+12y_4 \leq 34\}$.

*Correction of Exercise 5.3*

Let us first remark that $y \in \mathbb{Z}^4_+$, hence each point that satisfies $4y_1 + 5y_2 + 9y_3 + 12y_4 \leq 34$ also satisfies $5y_2 + 9y_3 + 12y_4 \leq 34$.
We divide both sides by 5, and take the floor to get the following valid inequality:

$$y_2 + \left\lfloor \frac{9}{5} \right\rfloor y_3 + \left\lfloor \frac{12}{5} \right\rfloor y_4 \leq \left\lfloor \frac{34}{5} \right\rfloor$$

This gives us the following inequality:

$$y_2 + y_3 + 2y_4 \leq 6$$

**Exercise 5.4** Consider the Problem XXII

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**Integer Linear Program XXII.** Exercise 5.4

---

$$\begin{aligned}
\text{Minimize} \quad & x_1 + 2x_2 & & \text{(XXII.1)}\\
\text{Subject To} \quad & x_1 + x_2 & \geq 4 & \quad \text{(XXII.2)}\\
& \frac{1}{2}x_1 + \frac{5}{2}x_2 & \geq \frac{5}{2} & \quad \text{(XXII.3)}\\
& x \in Z^2_+. & & \text{(XXII.4)}
\end{aligned}$$

---

Show that $x^* = \left(\frac{15}{4}, \frac{1}{4}\right)$ is the optimal linear programming solution and find an inequality cutting off $x^*$.

*Correction of Exercise 5.4*

We first prove that $\left(\frac{15}{4}, \frac{1}{4}\right)$ is an optimal solution. We will us the primal-dual complementarity rules to proceed. Let us recall that given $x^*$ an optimal solution of the primal problem and $y^*$ an optimal solution of the dual, then $x_i\left(a_i^T y - c_i\right) = 0$.

The dual problem is given by VII.

---

**Linear Program VII.** Dual Problem

---

$$
\begin{aligned}
\text{Maximize} \quad & 4y_1 + \frac{5}{2}y_2 & & \text{(VII.1)}\\
\text{Subject To} \quad & y_1 + \frac{1}{2}y_2 & \leq 1 \quad & \text{(VII.2)}\\
& y_1 + \frac{5}{2}y_2 & \leq 2 \quad & \text{(VII.3)}\\
& y_1, y_2 & \geq 0 \quad & \text{(VII.4)}
\end{aligned}
$$

---

From this we can write the following complementarity rules:

$$x_1^*(y_1^* + \frac{1}{2}y_2^* - 1) = 0 \tag{12}$$

$$x_2^*(y_1^* + \frac{5}{2}y_2^* - 2) = 0 \tag{13}$$

$$y_1^*(x_1^* + x_2^* - 4) = 0 \tag{14}$$

$$y_2^*(\frac{1}{2}x_1^* + \frac{5}{2}x_2^* - 4) = 0 \tag{15}$$

We use Equations 12 and 13 to find $(y_1^*, y_2^*)$. Since $x_1^* \neq 0$ and $x_2^* \neq 0$, we can write the following system:

$$\begin{cases} y_1^* + \frac{1}{2}y_2^* - 1 & = & 0 \\ y_1^* + \frac{5}{2}y_2^* - 2 & = & 0 \end{cases}$$

By solving the system, we get $y_1^* = 3/4$ and $y_2^* = 1/2$. The objective primal value of the given solution ($17/4$) being equal to the objective dual value of the computed dual solution, the given solution is an optimal one.

Let us now show how to find an inequality that will cut off the point $\left(\frac{15}{4}, \frac{1}{4}\right)$. To do so we will use a Gomory cut. Let us detail how to find such a cut.

First we introduce slack variables:

$$\begin{cases} x_1 + x_2 - x_3 = 4 & (16)\\ \frac{1}{2}x_1 + \frac{5}{2}x_2 - x_4 = \frac{5}{2} & (17) \end{cases}$$

Let us remark that $x_1^* = 15/4$ and $x_2^* = 1/4$ are in base variables (as they are non null) while variables $x_3^* = 0$ and $x_4^* = 0$ are out base variables.

Then we isolate $x_1$ and $x_2$:

$$\begin{cases} x_1 = 4 - x_2 + x_3 & (18)\\ x_2 = 1 - \frac{1}{5}x_1 + \frac{2}{5}x_4 & (19) \end{cases}$$

We replace $x_2$ in Equation (18) by the value given by Equation (19).

$$
\begin{aligned}
&& x_1 &= 4 - \left(1 - \frac{1}{5}x_1 + \frac{2}{5}x_4\right) + x_3 \\
\Leftrightarrow && \frac{4}{5}x_1 &= 3 - \frac{2}{5}x_4 + x_3 \\
\Leftrightarrow && x_1 &= \frac{15}{4} - \frac{1}{2}x_4 + \frac{5}{4}x_3 \qquad (20)
\end{aligned}
$$

We replace $x_1$ in Equation (18):

$$
\begin{aligned}
&& x_2 &= 1 - \frac{1}{5}\left(\frac{15}{4} - \frac{1}{2}x_4 + \frac{5}{4}x_3\right) + \frac{2}{5}x_4 \\
\Leftrightarrow && x_2 &= \frac{1}{4} - \frac{1}{4}x_3 + \frac{1}{2}x_4 \qquad (21)
\end{aligned}
$$

We derive from Equations (20) and (21) the following equations:

$$
\begin{cases}
x_1 \quad - \frac{5}{4}x_3 + \frac{1}{2}x_4 = \frac{15}{4} & (22) \\
\quad x_2 + \frac{1}{4}x_3 - \frac{1}{2}x_4 = \frac{1}{4} & (23)
\end{cases}
$$

Since $x_1^*$ is not an integer, we can use Equation (22) to generate a cut as follows:

$$
\sum_{j=3,4} (\overline{a}_{1j} - \lfloor \overline{a}_{1j} \rfloor) \geq \overline{a}_{10} - \lfloor \overline{a}_{10} \rfloor \qquad (24)
$$

This gives us:

$$
\left(-\frac{5}{4} - \left\lfloor -\frac{5}{4} \right\rfloor\right) x_3 + \left(\frac{1}{2} - \left\lfloor \frac{1}{2} \right\rfloor\right) x_4 \geq \frac{15}{4} - \left\lfloor \frac{15}{4} \right\rfloor
$$

Giving us the following cut:

$$
\frac{3}{4}x_3 + \frac{1}{2}x_4 \geq \frac{3}{4} \qquad (25)
$$

Similarly we can generate a cut using Equation (23) since $x_2^*$ is not an integer.

$$
\left(\frac{1}{4} - \left\lfloor \frac{1}{4} \right\rfloor\right) x_3 + \left(-\frac{1}{2} - \left\lfloor -\frac{1}{2} \right\rfloor\right) x_4 \geq \frac{1}{4} - \left\lfloor \frac{1}{4} \right\rfloor
$$

Leading to the following cut:

$$
\frac{1}{4}x_3 + \frac{1}{2}x_4 \geq \frac{1}{2} \qquad (26)
$$

We still need to get back to variables $x_1$ and $x_2$. We use Equations (18) and (19) to isolate $x_3$ and $x_4$:

$$
\begin{cases}
x_3 = \quad x_1 + \quad x_2 - 4 & (27) \\
x_4 = \frac{1}{2}x_1 + \frac{5}{2}x_2 - \frac{5}{2} & (28)
\end{cases}
$$

We replace in the cuts (25) and (26) we previously found:

$$\begin{cases} x_1 + 2x_2 \geq 5 & (29) \\ \frac{1}{2}x_1 + \frac{3}{2}x_2 \geq \frac{5}{2} & (30) \end{cases}$$

On can easily verify that these inequalities cut off the point $x^* = ({}^{15}/_4, {}^{1}/_4)$.

**Exercise 5.5** Solve the Integer Linear Program XXIII

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**Integer Linear Program XXIII.** Exercise 5.5

---

$$\begin{aligned} &\text{Minimize} && 5x_1 + 9x_2 + 23x_3 && && \text{(XXIII.1)} \\ &\text{Subject To} && 20x_1 + 35x_2 + 95x_3 && \geq 319 && \text{(XXIII.2)} \\ & && x \in Z_+^3. && && \text{(XXIII.3)} \end{aligned}$$

---

### *Correction of Exercise 5.5*

Let us first apply some preprocessing steps to simplify the problem. We remark that the coefficient of the left side of Constraint (XXIII.2) can be divided by 5:

$$\begin{aligned} & 4x_1 + 7x_2 + 19x_3 \geq \frac{319}{5} \\ \Rightarrow \quad & 4x_1 + 7x_2 + 19x_3 \geq \left\lceil \frac{319}{5} \right\rceil \\ \Rightarrow \quad & 4x_1 + 7x_2 + 19x_3 \geq 64 \qquad (31) \end{aligned}$$

In a first time we chose the variable that will enter the basis. We select the less penalizing variable *id est* the variable $x_i$ with the lowest ${}^{c_i}/_{a_i}$. It follows that the optimal solution $x^*$ of the Linear relaxation of Problem XXIII is:

$$x^* = \left(0, 0, \frac{64}{19}\right)$$

We generate a Gomory cut to eliminate this point. First we introduce a slack variable $s_1$ in Constraint (31) and rewrite the latter so that the coefficient of variable $x_3$ is equal to 1:

$$\frac{4}{19}x_1 + \frac{7}{19}x_2 + x_3 + \frac{1}{19}s_1 = \frac{64}{19} \qquad (32)$$

The Gomory cut is contructed using nonbasic variables (*id est* $x_1$, $x_2$ and $s_1$) and Equation 32:

$$\begin{aligned} & \frac{4}{19}x_1 + \frac{7}{19}x_2 + \left(-\frac{1}{19} - \left\lfloor -\frac{1}{19} \right\rfloor\right) s_1 \geq \frac{64}{19} - \left\lfloor \frac{64}{19} \right\rfloor \\ \Leftrightarrow \quad & \frac{4}{19}x_1 + \frac{7}{19}x_2 + \frac{18}{19}s_1 \geq \frac{7}{19} \\ \Leftrightarrow \quad & \frac{4}{19}x_1 + \frac{7}{19}x_2 + \frac{18}{19}s_1 - s_2 = \frac{7}{19} \qquad (33) \end{aligned}$$

We now add the cut and reoptimize the linear relaxation:

$$\begin{cases} x_3 = \frac{64}{19} - \frac{4}{19}x_1 - \frac{7}{19}x_2 + \frac{1}{19}s_1 & (34) \\ s_2 = -\frac{7}{19} + \frac{4}{19}x_1 + \frac{7}{19}x_2 + \frac{18}{19}s_1 & (35) \end{cases}$$

Let $x_1$ enter the basis. By Equation (35), we have:

$$x_1 + \frac{7}{4}x_2 + \frac{9}{2}s_1 - \frac{19}{4}s_2 = \frac{7}{4} \tag{36}$$

If we insert Equation (36) into Equation (34), we get:

$$\begin{aligned} & x_3 = \frac{64}{19} - \frac{4}{19}\left(\frac{7}{4} - \frac{7}{4}x_2 - \frac{18}{4}s_1 + \frac{19}{4}s_2\right) - \frac{7}{19}x_2 - \frac{1}{19}s_1 \\ \Leftrightarrow \quad & x_3 = 3 + \frac{7}{19}x_2 + \frac{18}{19}s_1 - s_2 - \frac{7}{19}x_2 - \frac{1}{19}s_1 \\ \Leftrightarrow \quad & x_3 = 3 + s_1 - s_2 \end{aligned} \tag{37}$$

Thus the optimal solution is given by $x^* = \left(\frac{7}{4}, 0, 3\right)$. One can see that $x_1^*$ is not an integer. We can thus derive a Gomory cut by using the Equation (36).

$$\begin{aligned} & \left(\frac{7}{4} - \left\lfloor\frac{7}{4}\right\rfloor\right)x_2 + \left(\frac{9}{2} - \left\lfloor\frac{9}{2}\right\rfloor\right)s_1 + \left(\frac{-19}{4} - \left\lfloor\frac{-19}{4}\right\rfloor\right) \geq \left(\frac{7}{4} - \left\lfloor\frac{7}{4}\right\rfloor\right) \\ \Leftrightarrow \quad & \frac{3}{4}x_2 + \frac{1}{2}s_1 + \frac{1}{4}s_2 \geq \frac{3}{4} \\ \Leftrightarrow \quad & \frac{3}{4}x_2 + \frac{1}{2}s_1 + \frac{1}{4}s_2 - s_3 = \frac{3}{4} \\ \Leftrightarrow \quad & s_3 = -\frac{3}{4} + \frac{3}{4}x_2 + \frac{1}{2}s_1 + \frac{1}{4}s_2 \end{aligned} \tag{38}$$

We add the cut we optimize again:

$$\begin{cases} x_3 = 3 - s_1 + s_2 & (39) \\ x_1 = \frac{7}{4} - \frac{7}{4}x_2 - \frac{9}{2}s_1 + \frac{19}{4}s_2 & (40) \\ s_3 = -\frac{3}{4} + \frac{3}{4}x_2 + \frac{1}{2}s_1 + \frac{1}{4}s_2 & (41) \end{cases}$$

$x_2$ enters the basis:

$$x_2 = 1 - \frac{2}{3}s_1 - \frac{1}{3}s_2 + \frac{4}{3}s_3 \tag{42}$$

We update the other constraints of the system. First Constraint (39) is left unchanged. We insert then Equation (41) into Equation (40):

$$\begin{aligned} & x_1 = \frac{7}{4} - \frac{7}{4}\left(1 - \frac{2}{3}s_1 - \frac{1}{3}s_2 + \frac{4}{3}s_3\right) - \frac{9}{2}s_1 + \frac{19}{4}s_2 \\ \Leftrightarrow \quad & x_1 = 0 + \frac{14}{12}s_1 + \frac{7}{12}s_2 - \frac{28}{12}s_3 - \frac{54}{12}s_1 + \frac{57}{12}s_2 \\ \Leftrightarrow \quad & x_1 = 0 - \frac{40}{12}s_1 + \frac{64}{12}s_2 - \frac{28}{12}s_3 \\ \Leftrightarrow \quad & x_1 = 0 - \frac{10}{3}s_1 + \frac{16}{3}s_2 - \frac{7}{3}s_3 \end{aligned} \tag{43}$$

The optimal solution is now $x^* = (0, 1, 3)$ which is integral and is thus the optimal solution of the Integer Linear Program XXIII.

# TP 6. Strong Valid Inequalities

**Exercise 6.1** In each of the examples below a set $X$ and a point $x^*$ are given. Find a valid inequality for $X$ cutting off the point $x^*$.

1. $X = \{x \in \mathbb{B}^5 \ : \ 9\,x_1 + 8\,x_2 + 6\,x_3 + 6\,x_4 + 5\,x_5 \leq 14\}$ $\qquad x^* = (0,\, \frac{5}{8},\, \frac{3}{4},\, \frac{3}{4},\, 0)$
2. $X = \{x \in \mathbb{B}^5 \ : \ 9\,x_1 + 8\,x_2 + 6\,x_3 + 6\,x_4 + 5\,x_5 \leq 14\}$ $\qquad x^* = (\frac{1}{4},\, \frac{1}{8},\, \frac{3}{4},\, \frac{3}{4},\, 0)$
3. $X = \{x \in \mathbb{B}^5 \ : \ 7\,x_1 + 6\,x_2 + 6\,x_3 + 4\,x_4 + 3\,x_5 \leq 14\}$ $\qquad x^* = (\frac{1}{7},\, 1,\, \frac{1}{2},\, \frac{1}{4},\, 1)$
4. $X = \{x \in \mathbb{B}^5 \ : \ 12\,x_1 - 9\,x_2 + 8\,x_3 + 6\,x_4 - 3\,x_5 \leq 2\}$ $\qquad x^* = (0,\, 0,\, \frac{1}{2},\, \frac{1}{6},\, 1)$

*Correction of Exercise 6.1*

1. $X = \{x \in \mathbb{B}^5 \ : \ 9\,x_1 + 8\,x_2 + 6\,x_3 + 6\,x_4 + 5\,x_5 \leq 14\}$ $\qquad x^* = (0,\, \frac{5}{8},\, \frac{3}{4},\, \frac{3}{4},\, 0)$

Let us first remark that this constraint is a knapsack constraint. Hence we can use some well known valid inequalities: the cover inequalities. Let us recall that a set $C \subseteq N$ is a cover if $\sum_{j \in C} a_i > b$, and that a cover is said to be *minimal* if $\forall j \in C$, $C \setminus \{j\}$ is not a cover anymore.

We can remark that $a_2 + a_3 + a_4 = 8 + 6 + 6 = 20 > 14$, thus $C = \{2, 3, 4\}$ is a minimal cover. We can therefore introduce a cover inequality defined by:

$$\sum_{j \in C} x_j \leq |C|$$

$$\Rightarrow \qquad x_2 + x_3 + x_4 \leq 2 \tag{44}$$

Since $\frac{5}{8} + \frac{3}{4} + \frac{3}{4} = \frac{17}{8} > 2$, the inequality (44) cuts off $x^*$.

2. $X = \{x \in \mathbb{B}^5 \ : \ 9\,x_1 + 8\,x_2 + 6\,x_3 + 6\,x_4 + 5\,x_5 \leq 14\}$ $\qquad x^* = (\frac{1}{4},\, \frac{1}{8},\, \frac{3}{4},\, \frac{3}{4},\, 0)$

Let us remark that $X$ is the same set as the previous one. Hence Inequality (44) is a valid inequality. However, $\frac{1}{8} + \frac{3}{4} + \frac{3}{4} = \frac{13}{8} \leq 2$, the latter does not cut off $x^*$.

Nevertheless, let us remark that $a_1 > a_i,\ \forall i \in C = \{2, 3, 4\}$, thus we can strengthen Inequality (44) by defining an extended cover. Let us recall that, given a minimal cover $C$, the maximal extended cover $E^*(C)$ is defined as:

$$E^*(C) = C \cup \{j \in N : a_j \geq a_i,\ \forall i \in C\}$$

Furthermore every subset $E(C) \subseteq E^*(C)$ such that $E(C) \cap E^*(C) \supseteq C$ is an extended cover. To each extended cover $E(C)$ we can associate an extended cover inequality defined as:

$$\sum_{j \in E(C)} x_j \leq |C|$$

It follows that Inequality (45) is a valid inequality.

$$x_1 + x_2 + x_3 + x_4 \leq 2 \tag{45}$$

This inequality does not cut off the point $x^*$ neither, so we need to strengthen it. We look for the greatest $\alpha \geq 1$ such that Inequality (46) is valid for every integral point of $X = \left\{x \in \mathbb{B}^4 : 9x_1 + 8x_2 + 6x_3 + 6x_4\right\}$.

$$\alpha x_1 + x_2 + x_3 + x_4 \leq 2 \tag{46}$$

If $x_1 = 0$, Inequality (46) is the same as Inequality (45), and is thus valid for every value of $\alpha$.

On the other hand, if $x_1 = 1$, thus Inequality (46) is valid if and only if Inequality (47) is valid for every integral point of $X' = \{x \in \mathbb{B}^3 : 8x_2 + 6x_3 + 6x_4 \leq 14 - 9 = 5\}$.

$$\alpha + x_2 + x_3 + x_4 \leq 2 \tag{47}$$

Inequality (47) is valid for every value of $\alpha$ verifying:

$$\begin{aligned}
& \alpha \leq 2 - \underbrace{\max\{x_2 + x_3 + x_4 : 8x_2 + 6x_3 + 6x_4 \leq 5, x \in \mathbb{B}^3\}}_{\beta} \\
\Leftrightarrow \quad & \alpha \leq 2 - \beta \\
\Leftrightarrow \quad & \alpha \leq 2 \qquad \text{since } \beta = 0 \qquad (48)
\end{aligned}$$

It gives us the following inequality that cuts off $x^*$.

$$2x_1 + x_2 + x_3 + x_4 \leq 2 \tag{49}$$

3. $X = \{x \in \mathbb{B}^5 \; : \; 7\,x_1 + 6\,x_2 + 6\,x_3 + 4\,x_4 + 3\,x_5 \leq 14\} \qquad x^* = (\frac{1}{7},\, 1,\, \frac{1}{2},\, \frac{1}{4},\, 1)$

As previously, this constraint is a knapsack constraint, we can look for a minimal cover. $C = \{2, 3, 5\}$ is one of them. We introduce then the associated inequality:

$$x_2 + x_3 + x_5 \leq 2 \tag{50}$$

Since $1 + \frac{1}{2} + 1 > 2$, Inequality (50) cuts of $x^*$.

4. $X = \{x \in \mathbb{B}^5 \; : \; 12\,x_1 - 9\,x_2 + 8\,x_3 + 6\,x_4 - 3\,x_5 \leq 2\} \qquad x^* = (0,\, 0,\, \frac{1}{2},\, \frac{1}{6},\, 1)$

Note that, we can not introduce cover inequalities, as the constraint is not, as such, a knapsack constraint. Let us rewrite it. To do so, we first introduce the complementary variables for each negatively weighted variable:

$$\begin{cases} \overline{x_2} = 1 - x_2 \\ \overline{x_5} = 1 - x_5 \end{cases}$$

The constraint can then be rewritten as follows:

$$12x_1 + 9\overline{x_2} + 8x_3 + 6x_4 + 3\overline{x_5} \leq 2 + 9 + 3 = 14 \tag{51}$$

We also update $\overline{x^*} = (0, 1, \frac{1}{2}, \frac{1}{6}, 0)$.

We can look for a minimal cover to introduce the associated cover inequality. Let us remark $C = \{2, 3\}$ is a minimal cover, thus Inequality (52) is valid:

$$\overline{x_2} + x_3 \leq 1 \tag{52}$$

Since $1 + \frac{1}{2} > 1$, the latter cuts off $\overline{x^*}$. Thus the Inequality $x_3 - x_2 \leq 0$ cuts off the point $x^*$ (as $x_3^* - x_2^* = \frac{1}{2} - 0 > 0$).

**Exercise 6.2** Consider the following set:

$$
\begin{aligned}
X = \{(x, y) \in \mathbb{R}_+^4 \times \mathbb{B}^4 \ : x_1 + x_2 + x_3 + x_4 & \geq 36 && (53)\\
x_1 & \leq 20y_1, && (54)\\
x_2 & \leq 10y_2, && (55)\\
x_3 & \leq 10y_3, && (56)\\
x_4 & \leq 8y_4 \quad \} && (57)
\end{aligned}
$$

1. Derive a valid inequality that looks like a 0-1 knapsack constraint.
2. Use this to cut off the fractional point $x^* = (20, 10, 0, 6)$, $y^* = (1, 1, 0, \frac{3}{4})$ with an inequality involving only $y$ variables.

*Correction of Exercise 6.2*

We can combine Constraints (54), (55), (56) and (57) to get the following constraint:

$$x_1 + x_2 + x_3 + x_4 \leq 20y_1 + 10y_2 + 10y_3 + 8y_4 \tag{58}$$

Thanks to Constraint (53), we can also derive this inequality:

$$20y_1 + 10y_2 + 10y_3 + 8y_4 \geq 36 \tag{59}$$

From this, we use a separation problem to determine whether there exists a cover inequality cutting off the point $(x^*; y^*)$. As a reminder, a separation problem consists in determining, given a fractional LP solution $x^*$, whether there exists a cover $C \subseteq N$ such that $\sum_{j \in C} \left(1 - x_j^*\right) < 1$.

It can be easily shown that this is equivalent to determine whether the following inequality is verified:

$$\gamma = \min_{C \in N} \left\{ \sum_{j \in C} (1 - x_j) : \sum_{j \in C} a_j > b \right\} < 1 \tag{60}$$

If $\gamma \geq 1$, then the fractional solution satisfies all the cover inequalities, while if $\gamma < 1$ with optimal solution $z^R$ then $\sum_{j \in R} x_j \leq |R| - 1$ is a valid cover inequality cutting off $x^*$.

In our case, we must pay attention to the sense of the Inequality (59). Thus we try to determine whether there exists a cover $C$ such that:

$$\gamma = \max_{j \in C} \left\{ 1 - y^* : \sum_{j \in C} a_j < b \right\} > 1 \tag{61}$$

In other words, we want to solve ILP XXIV.

**Integer Linear Program XXIV.** Separation Problem

$$
\begin{array}{llr}
\text{Maximize} & \displaystyle\sum_{j \in N} (1 - y_j^*) z_j & \text{(XXIV.1)} \\
\text{Subject To} & \displaystyle\sum_{j \in N} a_j z_j \quad \leq b & \text{(XXIV.2)} \\
 & z \in \mathbb{B}^4 & \text{(XXIV.3)}
\end{array}
$$

By considering $N = \{1, 2, 3, 4\}$ and the fractional solution $y = \left(1, 1, 0, \frac{3}{4}\right)$ we need to solve ILP XXV.

**Integer Linear Program XXV.** Separation Problem

$$
\begin{array}{llr}
\text{Maximize} & 0z_1 + 0z_2 + 1z_3 + \dfrac{1}{4} z_4 & \text{(XXV.1)} \\
\text{Subject To} & 20z_1 + 10z_2 + 10z_3 + 8z_4 \quad \leq 36 & \text{(XXV.2)} \\
 & z \in \mathbb{B}^4 & \text{(XXV.3)}
\end{array}
$$

An optimal solution is given by $z = (0, 0, 1, 1)$ with objective value $obj = \frac{5}{4} > 1$. Thus the Inequality (62) is valid and cuts off $x^*$.

$$x_3 + x_4 \geq 1 \tag{62}$$

## Exercise 6.3

Consider the following knapsack set:

$$X = \{x \in \mathbb{B}^6 \ : \ 12\,x_1 + 9\,x_2 + 7\,x_3 + 5\,x_4 + 5\,x_5 + 3\,x_6 \leq 14\}$$

Set $x_1 = x_2 = x_4 = 0$, and consider the cover inequality $x_3 + x_5 + x_6 \leq 2$ that is valid for $X' = X \cap \{x_1 = x_2 = x_4 = 0\}$.

Lift the inequality to obtain a strong valid inequality for $X$.

### *Correction of Exercise 6.3*

We start from the Inequality (63) that is valid for $X' = X \cap \{x_1 = x_2 = x_4 = 0\}$. Note that this Inequality is also valid for $X$ but does not define a strong valid inequality.

$$x_3 + x_5 + x_6 \leq 2 \tag{63}$$

To lift the Inequality (63), we want to *introduce* variables set to zero into the latter, each of them weighted by the greatest $\alpha \geq 0$ such that the Inequality is still a valid one.

Let us begin with variable $x_1$, we look for the greatest $\alpha_1 \geq 0$ such that Inequality (64) is a valid inequality.

$$\alpha_1 x_1 + x_3 + x_5 + x_6 \leq 2 \tag{64}$$

In other words we are looking for the largest $\alpha_1$ that verifies:

$$\alpha_1 \leq 2 - \max\{x_3 + x_5 + x_6 : 7x_3 + 5x_5 + 3x_6 \leq 14 - 12 = 2\} \tag{65}$$

Since $\max\{x_3 + x_5 + x_6 : 7x_3 + 5x_5 + 3x_6 \leq 2\} = 0$, then $\alpha_1 = 2$.
Let us consider variable $x_2$. As previously we look for $\alpha_2 \geq$ such that Inequality (66) is valid.

$$2x_1 + \alpha_2 x_2 + x_3 + x_5 + x_6 \leq 2 \tag{66}$$

Leading to Inequality (67).

$$\alpha_2 \leq 2 - \max\{2x_1 + x_3 + x_5 + x_6 : 12x_1 + 7x_3 + 5x_5 + 3x_6 \leq 14 - 9 = 5\} \tag{67}$$

Since $\max\{2x_1 + x_3 + x_5 + x_6 : 12x_1 + 7x_3 + 5x_5 + 3x_6 \leq 5\} = 1$, then $\alpha_2 = 1$.
In the same way we find that $\alpha_4 = 0$. Thus the following inequality is a strong valid one:

$$2x_1 + x_2 + x_3 + x_5 + x_6 \leq 2 \tag{68}$$

**Exercise 6.4** In each of the examples below a set $X$ and a point $(x^*; y^*)$ are given. Find a valid inequality for $X$ cutting off the point $(x^*; y^*)$.

1.

$$\begin{aligned} X = \{(x, y) \in \mathbb{R}^3_+ \times \mathbb{B}^3 \ : \ & x_1 + x_2 + x_3 \leq 7, && (69)\\ & x_1 \leq 3\, y_1, && (70)\\ & x_2 \leq 5\, y_2, && (71)\\ & x_3 \leq 6\, y_3 \quad \} && (72) \end{aligned}$$

$$(x^*; y^*) = (2,\ 5,\ 0\,;\ \frac{2}{3},\ 1,\ 0)$$

2.

$$\begin{aligned} X = \{(x, y) \in \mathbb{R}^3_+ \times \mathbb{B}^3 \ : \ & 7 \leq x_1 + x_2 + x_3, && (73)\\ & x_1 \leq 3\, y_1, && (74)\\ & x_2 \leq 5\, y_2, && (75)\\ & x_3 \leq 6\, y_3 \quad \} && (76) \end{aligned}$$

$$(x^*; y^*) = (2,\ 5,\ 0\,;\ \frac{2}{3},\ 1,\ 0)$$

3.

$$\begin{aligned} X = \{(x, y) \in \mathbb{R}^6_+ \times \mathbb{B}^6 \ : \ & x_1 + x_2 + x_3 \leq 4 + x_4 + x_5 + x_6, && (77)\\ & x_1 \leq 3\, y_1, && (78)\\ & x_2 \leq 3\, y_2, && (79)\\ & x_3 \leq 6\, y_3, && (80)\\ & x_4 \leq 3\, y_4, && (81)\\ & x_5 \leq 5\, y_5, && (82)\\ & x_6 \leq 1\, y_6 \quad \} && (83) \end{aligned}$$

$$(x^*; y^*) = (3,\ 3,\ 0,\ 0,\ 2,\ 0\,;\ 1,\ 1,\ 0,\ 0,\ \frac{2}{5},\ 0)$$

*Correction of Exercise 6.4*

1.

$$
\begin{aligned}
X = \{(x, y) \in \mathbb{R}^3_+ \times \mathbb{B}^3 \ : & \ x_1 + x_2 + x_3 \leq 7, && (84)\\
& x_1 \leq 3\, y_1, && (85)\\
& x_2 \leq 5\, y_2, && (86)\\
& x_3 \leq 6\, y_3 \qquad \} && (87)
\end{aligned}
$$

$$(x^*; y^*) = (2,\, 5,\, 0\,;\, \frac{2}{3},\, 1,\, 0)$$

Note that, contrary to what we previously did, we cannot use Cover Inequalities as the knapsack Constraint (84) does not contain $y$ variables and we are not able to derive knapsack constraints from the previous inequalities.

However, we can remark that the set is under the following form with $N_1 = \{1, 2, 3\}$ and $N_2 = \emptyset$.

$$X = \left\{ (x, y) \in \mathbb{R}^n_+ \times \mathbb{B}^n_+ : \sum_{j \in N_1} x_j \leq b + \sum_{j \in N_2} x_j,\ x_j \leq a_j y_j \text{ for } j \in N_1 \cup N_2 \right\} \tag{88}$$

Thus, we look for a generalized cover for $X$, *id est* a set $C = C_1 \cup C_2$ with $C_1 \subseteq N_1$ and $C_2 \subseteq N_2$ such that:

$$\sum_{j \in C_1} a_j - \sum_{j \in C_2} a_j = b + \lambda$$

with $\lambda > 0$. We can remark that $C = \{C_1, C_2\} = \{\{1, 2\}\,; \emptyset\}$ is generalized cover with $\lambda = 1$, indeed, by using Constraint (84), we can write:

$$a_1 + a_2 = 3 + 5 = 7 + 1 \tag{89}$$

We can then derive a valid flow cover inequality defined by:

$$\sum_{j \in C_1} x_j + \sum_{j \in C_1} \left(\max\left\{(a_j - \lambda)\,; 0\right\}(1 - y_j)\right) \leq b + \sum_{j \in C_2} a_j + \lambda \sum_{j \in L_2} y_j + \sum_{j \in N_2 \setminus (C_2 \cup L_2)} x_j \tag{90}$$

where $L_2 \subseteq N_2 \setminus C_2$. Applied to our case, $L_2 = \emptyset$, thus we get the following valid Inequality:

$$
\begin{aligned}
& x_1 + x_2 + (3 - 1)(1 - y_1) + (5 - 1)(1 - y_2) \leq 7 \\
\Leftrightarrow \quad & x_1 + x_2 + 2 - 2y_1 + 4 - 4y_2 \leq 7 \\
\Leftrightarrow \quad & x_1 + x_2 - 2y_1 - 4y_2 \leq 1 && (91)
\end{aligned}
$$

We can remark that Constraint (91) cuts off the point $(x^*; y^*) = \left(2, 5, 0; \frac{2}{3}, 1, 0\right)$ as $2 + 5 - \frac{4}{3} - 4 = \frac{5}{3} > 1$.

2.

$$X = \{(x,y) \in \mathbb{R}^3_+ \times \mathbb{B}^3 \ : 7 \leq x_1 + x_2 + x_3, \tag{92}$$

$$x_1 \leq 3\, y_1, \tag{93}$$

$$x_2 \leq 5\, y_2, \tag{94}$$

$$x_3 \leq 6\, y_3 \quad \} \tag{95}$$

$$(x^*; y^*) = (2,\ 5,\ 0\,;\ \frac{2}{3},\ 1,\ 0)$$

Let us first remark that no cover inequality is violated by this solution. Indeed, by using the separation problem as defined in Exercise TP 6., we get the 0/1 seperation problem XXVI:

**Integer Linear Program XXVI.** Separation Problem

| | | | |
|---|---|---|---|
| Maximize | $\frac{1}{3} z_1 + 0 z_2 + 1 z_3$ | | (XXVI.1) |
| Subject To | $3z_1 + 5z_2 + 6z_3$ | $\leq 7$ | (XXVI.2) |
| | $z \in \mathbb{B}^3$ | | (XXVI.3) |

The optimal solution is given by $z = (0, 0, 1)$ with optimal value equal to 1. This proves that no cover constraint is violated by the solution $(x^*; y^*) = (2,\ 5,\ 0\,;\ \frac{2}{3},\ 1,\ 0)$.

We hence look for flow cover inequalities. To find such an inequality, we will use the *Flow Cover Separation Heuristic*. But before let us remark that Constraint (92) can be rewritten as follows:

$$0 \leq -7 + x_1 + x_2 + x_3 \tag{96}$$

It follows that $X$ is under the form defined by (88) with $N_1 = \emptyset$ and $N_2 = \{1, 2, 3\}$. From this point, we could proceed by enumerating every flow cover inequalities for every admissible set $C_2$ and $L_2$, however this would be time consuming. As previously stated, we will use the Flow Cover Separation Heuristic. This heuristic aims at finding a valid cover $C = (C_1, C_2)$ such that, by considering a given set $L_2$, the associated flow inequality is violated.

Note that this is a heuristic, it implies that even if the found set $C$ does not define a violated flow cover inequality, it may nevertheless exist a violated one [2].

If we call $z$ the unknown incidence vector of $C = (C_1, C_2)$, the heuristic solves the 0/1-Program XXVII.

**Integer Linear Program XXVII.** Flow Cover Separation Heuristic

| | | | |
|---|---|---|---|
| Minimize | $\sum_{j \in N_1} \left(1 - y_j^*\right) z_j - \sum_{j \in N_2} y_j^* z_j$ | | (XXVII.1) |
| Subject To | $\sum_{j \in N_1} a_j z_j - \sum_{j \in N_2} a_j z_j$ | $> b$ | (XXVII.2) |
| | $z \in \mathbb{B}^n$ | | (XXVII.3) |

[2] I invite the reader to take a look at *Integer Programming* by Wolser, section 9.4.2

This gives us an optimal solution $z^*$ defining a set $C = \{C_1; C_2\}$. From this, if the Inequality (90) for $L_2 = \left\{j \in N_2 \backslash C_2 : \lambda y_j^* < x_j^*\right\}$, then it defines a violated Flow Cover Inequality.

Applied to our problem, we consider then 0/1-Program XXVIII.

---

**Integer Linear Program XXVIII.** Flow Cover Separation Heuristic

---

$$
\begin{array}{rlrr}
\text{Minimize} & -\dfrac{2}{3}z_1 - z_2 & & \text{(XXVIII.1)} \\
\text{Subject To} & -3z_1 - 5z_2 - 6z_3 & > -7 & \text{(XXVIII.2)} \\
 & z \in \mathbb{B}^3 & & \text{(XXVIII.3)}
\end{array}
$$

---

The optimal solution is given by $z^* = (0, 1, 0)$, defining then $C = \{C_1; C_2\} = \{\emptyset; \{2\}\}$.

We can thus determine $\lambda$ thanks to Constraint (XXVIII.2), since $-5 = -7 + 2$ we get $\lambda = 2$. Thus $L_2 = \left\{j \in \{1, 3\} : 2y_j^* < x_j^*\right\} = \{1\}$.

Since:

$$
b + \sum_{j \in C_2} a_j + \lambda \sum_{j \in L_2} + \sum_{j \in N_2 \backslash (C_2 \cup L_2)} x_j^* = -7 + 5 + \frac{4}{3} < 0 = \sum_{j \in C_1} x_j^* + \sum_{j \in C_1} \max\{0; a_j - \lambda\}\left(1 - y_j^*\right)
$$

We now that the Flow Cover Inequality (97) cuts off the point $(x^*; y^*)$.

$$
-2 + 2y_1 + x_3 > 0 \tag{97}
$$

3.

$$
\begin{array}{rlr}
X = \{(x, y) \in \mathbb{R}_+^6 \times \mathbb{B}^6 & : x_1 + x_2 + x_3 \leq 4 + x_4 + x_5 + x_6, & (98) \\
 & x_1 \leq 3\, y_1, & (99) \\
 & x_2 \leq 3\, y_2, & (100) \\
 & x_3 \leq 6\, y_3, & (101) \\
 & x_4 \leq 3\, y_4, & (102) \\
 & x_5 \leq 5\, y_5, & (103) \\
 & x_6 \leq 1\, y_6 \qquad \} & (104)
\end{array}
$$

$$
(x^*; y^*) = (3,\ 3,\ 0,\ 0,\ 2,\ 0\,;\ 1,\ 1,\ 0,\ 0,\ \frac{2}{5},\ 0)
$$

We proceed as previously since the previous is under form defined by (88) with $N_1 = \{1, 2, 3\}$ and $N_2 = \{4, 5, 6\}$. We use the Flow Cover Separation Heuristic. We want to solve 0/1-Program XXIX.

---

**Integer Linear Program XXIX.** Flow Cover Separation Heuristic

---

$$
\begin{array}{rlrr}
\text{Minimize} & z_3 - \dfrac{2}{5}z_5 & & \text{(XXIX.1)} \\
\text{Subject To} & 3z_1 + 3z_2 + 6z_3 - 3z_4 - 5z_5 - z_6 & > 4 & \text{(XXIX.2)} \\
 & z \in \mathbb{B}^6 & & \text{(XXIX.3)}
\end{array}
$$

---

An optimal solution is given by $z^* = (1,1,0,0,0,0)$ defining $C = \{C_1; C_2\} = \{\{1,2\}; \emptyset\}$. It follows that $\lambda = a_1 + a_2 - b = 2$ and that $L_2 = \{5\}$.
Since $3 + 3 > \frac{24}{5}$, the Inequality (105) cuts off hte point $(x^*; y^*)$.

$$x_1 + x_2 - x_4 - x_6 - y_1 - y_2 - 2y_5 \leq 2 \tag{105}$$

## Exercise 6.5

Consider the set

$$\begin{aligned} X = \{(x_1,\, y_2,\, y_3,\, y_4) \in \mathbb{R}^1_+ \times \mathbb{B}^3 \; : & \; x_1 \leq 1000\, y_2 + 1500\, y_3 + 2200\, y_4, \\ & \; x_1 \leq 2000 \qquad\qquad\qquad\qquad \} \end{aligned}$$

1. Show that it can be written as a mixed 0-1 set

$$\begin{aligned} X = \{(x,y) \in \mathbb{R}^4_+ \times \mathbb{B}^4 \; : & \; x_1 \leq x_2 + x_3 + x_4, \\ & \; x_1 \leq 2000\, y_1 \\ & \; x_2 \leq 1000\, y_2 \\ & \; x_3 \leq 1500\, y_3 \\ & \; x_4 \leq 2200\, y_4 \qquad \} \end{aligned}$$

with the additional constraints

$$\begin{aligned} & y_1 = 1, \\ & x_2 \geq 1000\, y_2, \\ & x_3 \geq 1500\, y_3, \\ & x_4 \geq 2200\, y_4 \end{aligned}$$

2. Use this to find a valid inequality for X cutting off the point

$$(x_1,\, y_2,\, y_3,\, y_4) = (2000,\, 0,\, 1,\, \frac{5}{22})$$

### *Correction of Exercise 6.5*

We consider the two following sets $X$ and $X'$:

$$X = \{(x_1,\, y_2,\, y_3,\, y_4) \in \mathbb{R}^1_+ \times \mathbb{B}^3 \; : \; x_1 \leq 1000\, y_2 + 1500\, y_3 + 2200\, y_4, \tag{106}$$
$$x_1 \leq 2000 \qquad\qquad\qquad\qquad \} \tag{107}$$

$$
\begin{aligned}
X' = \{(x,y) \in \mathbb{R}^4_+ \times \mathbb{B}^4 \ : & \ x_1 \leq x_2 + x_3 + x_4, && (108)\\
& x_1 \leq 2000\, y_1 && (109)\\
& x_2 \leq 1000\, y_2 && (110)\\
& x_2 \geq 1000\, y_2 && (111)\\
& x_3 \leq 1500\, y_3 && (112)\\
& x_3 \geq 1500\, y_3 && (113)\\
& x_4 \leq 2200\, y_4 && (114)\\
& x_4 \geq 2200\, y_4 && (115)\\
& y_1 = 1 \qquad \} && (116)
\end{aligned}
$$

To show that $X$ can be rewritten as $X'$, we show that every point belonging to $X$ can be turned into a point that belongs to $X'$ and inversely.

Let us first consider a point $x = (x_1, y_2, y_3, y_4) \in X$. Obviously, $x$ satisfies Inequalities (106) and (107). First we set $x_2 = 1000y_2$, $x_3 = 1500y_3$ and $x_4 = 2200y_4$.

Since $x$ satisfies Inequality (106), we can write:

$$
\begin{array}{ccccccc}
x_1 & \leq & 1000y_2 & + & 1500y_3 & + & 2200y_4 \\
 & & \shortparallel & & \shortparallel & & \shortparallel \\
x_1 & \leq & x_2 & + & x_3 & + & x_4
\end{array}
$$

We then set $y_1 = 1$, thus since $x_1 \leq 2000$ in $X$, the Inequality (109) is verified in $X'$. Note that variables $x_2$, $x_3$ and $x_4$ have been chosen in such a way that they satisfy Inequalities (110)-(115). Thus every point of $X$ can be turned into a point of $X'$.

Let us now consider a point $x' = (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4)$ belonging to the set $X'$. Thanks to Constraints (108), (110), (112) and (114), we can write:

$$
x_1 \leq x_2 + x_3 + x_4 \leq 1000y_2 + 1500y_3 + 2200y_4 \tag{117}
$$

Furthermore, $x'$ satisfies $x_1 \leq 2000y_1 = 2000$ since $y_1 = 1$. Thus $x'$ can be turned into a point of set $X$.

We now use this reformulation to find a valid inequality cutting off point $x^* = (x_1, y_2, y_3, y_4) = \left(2000, 0, 1, \frac{5}{22}\right)$.

We use as previously a Flow Cover Separation Heuristic with $N_1 = \{1\}$ and $N_2 = \{2, 3, 4\}$.

We solve the 0/1-Problem XXX.

---

**Integer Linear Program XXX.** Flow Cover Separation Heuristic

---

$$
\begin{aligned}
\text{Minimize} \quad & -z_3 - \frac{5}{22} z_4 && \text{(XXX.1)}\\
\text{Subject To} \quad & 2000z_1 - 1000z_2 - 1500z_3 - 2200z_4 \qquad > 0 && \text{(XXX.2)}\\
& z \in \mathbb{B}^4 && \text{(XXX.3)}
\end{aligned}
$$

---

The optimal solution is given by $z^* = (1, 0, 1, 0)$, defining $C = \{C_1; C_2\} = \{\{1\}; \{3\}\}$. Then $\lambda = a_1 + a_3 - b = 500$ and $L_2 = \{4\}$.

Since $2000 - 500\frac{5}{22} > 1500$, the Inequality (118) cuts off the given point.

$$
x_1 - 1000y_2 - 500y_4 \leq 1500 \tag{118}
$$

**Exercise 6.6** Given a graph $G = (V, E)$ with $|V| = n$, consider the set of incidence vectors of the stable sets

$$X = \{x \in \mathbb{B}^n \;:\; x_i + x_j \leq 1 \;\; for \;\; (i, j) \in E\}$$

Show that if $C \subseteq V$ is a maximal clique of $G$, then the inequality

$$\sum_{j \in C} x_j \leq 1$$

1. is valid for $X$
2. defines a facet of $conv(X)$

## *Correction of Exercise 6.6*

Left as homework.