universität innsbruck

Advances in the Algebraic Factorization of Overconstrained Closed-Loop Linkages

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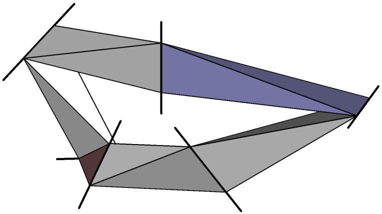
Unit of Geometry and Surveying

6R linkages



Kinematic image space

Via *Study's kinematic mapping*

$$\mathfrak{S} : \mathrm{SE}(3) \hookrightarrow \mathcal{S} \subset \mathbb{P}^7$$

rigid body displacements are associated with points in 7-dimensional projective space $\rightsquigarrow$ dual quaternion representation of SE(3) in our work.

Overconstrained closed-loop linkages:
Their motions are described by curves on the Study quadric $\mathcal{S}$.

Factorization:
The process of computing the axes of a linkage from the curve in the kinematic image space.

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Of particular interest are singular displacements
$\rightsquigarrow$ dual quaternions with norm zero

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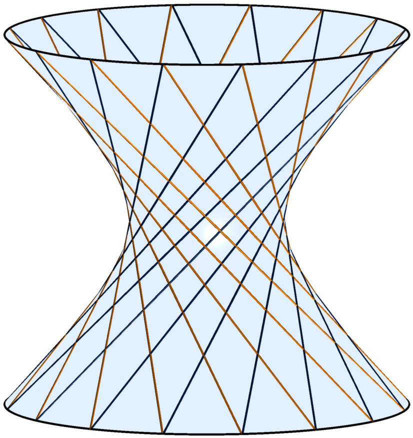

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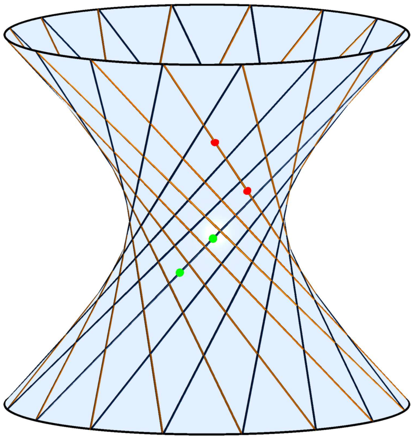

State of the art

As outlined by Li et al. (2025), if some of the null points lie in the same linear subspaces of the null quadric $(n_i)^* n_j = 0$ or $n_i (n_j)^* = 0$, we can compute an axis.

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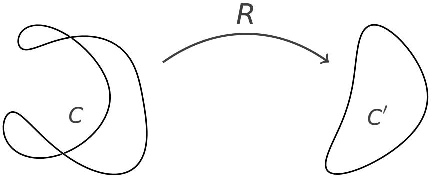
Research question: What are techniques and methods to factor a 6R linkage?

Riemann-Roch maps

Goal: The construction of a map R from a constraint curve C to a curve C' of lower degree.

$$\begin{aligned} R : C &\to C' \\ [c] &\mapsto [(\varrho(c) - h_1)c] \end{aligned}$$



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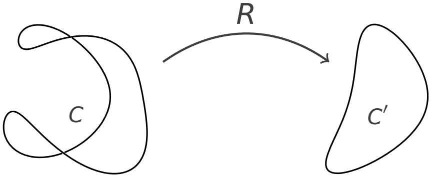

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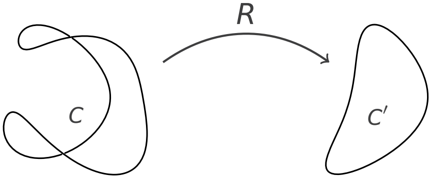

Here h_1 is a known first or last axis in the linkage and

$$\varrho : C \dashrightarrow \mathbb{P}^1$$

is a rational map under certain constraints, namely

- vanish at certain null points,
- be of a given mapping degree,
- ...

Purpose: Eliminate some null points $\rightsquigarrow$ lower degree.

Multiplication trick

Situation: Null points are not suited for the computation of the first and last axes via the standard method $\rightsquigarrow$ Bricard plane-symmetric mechanism

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Result: We get the first and last axes as well as the null points for the middle axis.

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For certain linkage classes – plane-symmetric or line-symmetric Bricard – we have developed a method to factorize the axes of the linkage.

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Conclusion: Factorization can be done algebraically and most information about the mechanisms is contained in the null points.

Thank you for your attention!