Lecture2 Operator View of Rigid-Body

Rotation Operation vis ODE

Consider a rotation with unit angular velocity $\hat{\omega}$

$$\dot{p}(t) = \hat{\omega} \times p(t) = [\hat{\omega}]p(t), \text{with } p(0) = p_0 \tag{1}$$

There is linear ODE solution: $p(t) = e^{[\hat{\omega}]t}p_0$

At $t = \theta$, the point has been rotated by θ degree, $p(\theta) = e^{[\hat{\omega}]\theta}p_0$

Rotation Matrix as a Rotation Operator

(*coordinate free*)

Rotation matrix can be written as $R = \mathrm{Rot}(\hat{\omega}, \theta) \triangleq e^{[\hat{\omega}]\theta}$

i.e. rotation operation about $\hat{\omega}$ by θ

Rotation Matrix Properties

- $RR^T = I \quad$ definition
- $R_1R_2 \in SO(3)$ if $R_1, R_2 \in SO(3)$

  product of rotation matrices is also a rotation matrix
- $||Rq - Rp|| = ||p - q||$

  rotation matrix preserves distance
- $R(v \times \omega) = (Rv) \times (R\omega)$

  rotation matrix preserves orientation
- $R[\omega]R^T = [R\omega]$

Rotation Operator in Different Frames

Consider the same rotation operation $\mathrm{Rot}(\hat{\omega, \theta})$

In frame-{A} and frame-{B}

$${}^A\hat{\omega} = {}^AR_B{}^B\hat{\omega}$$

$${}^A\mathrm{Rot}({}^A\hat{\omega}, \theta) = {}^AR_B{}^B\mathrm{Rot}({}^B\hat{\omega}, \theta){}^BR_A$$

You can think like this:
There is a point in frame {A}, first we transform it to frame {B}, then do rotation, finally transform back to frame {A}. This should be equivalent to rotation in frame {A}.

Rigid-Body Operation via ODE

$$\dot{p}(t) = \omega \times p(t) + v \rightarrow \begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ 1 \end{bmatrix} \tag{2}$$

Solution to equation (2) is

$$\begin{bmatrix} p(t) \\ 1 \end{bmatrix} = \exp(\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} t) \begin{bmatrix} p(0) \\ 1 \end{bmatrix}$$

For any twist $\mathcal{V} = (\omega, v)$

$$[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$$

The above definition also applies to screw axis $\mathcal{S} = (\omega, v)$

With this notation, the solution to (2) is

$$\tilde{p} = e^{[\mathcal{V}]t} \tilde{p}(0) = e^{[\mathcal{S}]\theta} \tilde{p}(0)$$

Define $se(3) = \{([\omega], v) : [\omega] \in so(3), v \in \mathbb{R}^3\}$

Any $T \in SE(3)$ can be written as $T = e^{[\mathcal{S}]\theta}$

Homogeneous Transformation as Rigid-Body Operator

- $\tilde{p}' = T\tilde{p}$: "rotate" p about screw axis $\mathcal{S}$ by θ degree
- TT_A: "rotate" {A} frame about $\mathcal{S}$ by θ degree
- $T \leftrightarrow T_B^{-1} T T_B$

Rigid-Body Operation of Screw Axis

After transformation $T = (R, p)$, screw axis $\mathcal{S}$ becomes

$$\mathcal{S}' = [Ad_T] S$$

$[Ad_T] \triangleq \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$