Lecture12: Value Estimation

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Algorithm: MC estimation, for estimating state values

Notation

For a finite trajectory $\tau = \{s_t, a_t, r_{t+1}\}_{t=0}^{T-1}$

$$G_t = G(s_t) = R(\tau|S_0 = s_t) = r_{t+1} + \gamma r_{t+2} + \cdots + \gamma^{T-1} r_T$$

Corollary:

$$G(s_t) = r_{t+1} + \gamma G(s_{t+1})$$

Process

- Input: policy π to be evaluate
- Output: $V_\pi^*(s)$
- Init:
  - $V(s) \in \mathbb{R},\ \forall s \in \mathbb{S}$
  - an empty list $R(s),\ \forall s \in \mathbb{S}$

Loop for each episode: (termination: $\max V_\pi^{[k+1]}(s_t) - V_\pi^{[k]}(s_t) \leq \Delta$)

generate $\tau = \{s_t, a_t, r_{t+1}\}_{t=0}^{T-1}$ using π

$G_t \leftarrow 0$

Loop for each time step $t = T-1, T-2, \cdots, 0$:

$G_t \leftarrow r_{t+1} + \gamma G_{t+1}$

$R(s_t)$ append G_t

$\hat{V}_\pi(s_t) \leftarrow \text{average}(R(s_t))$

Above is the **"every visit"** condition.

This calculates the value function for all possible states?

Examples

- Frozen Lake
- Example Code

Incremental Implementation

$$V_\pi^{[m+1]}(s_t) = V_\pi^{[m]} + \frac{1}{m+1}\left(G_t^{[m+1]} - V_\pi^{[m]}(s_t)\right)$$

New Estimation $\leftarrow$ Old Estimation + α (New Observation - Old Estimation)

This significantly reduces the storage requirements.

Derivation:

$$\begin{aligned}
V_\pi^{[m+1]}(s_t) &= \frac{1}{m+1}\sum_{i=1}^{m+1} G_t^{[i]} \\
&= \frac{1}{m+1}\left(\sum_{i=1}^{m} G_t^{[i]} + G_t^{[m+1]}\right) \\
&= \frac{1}{m+1}\cdot m \cdot \frac{1}{m}\left(\sum_{i=1}^{m} G_t^{[i]} + G_t^{[m+1]}\right) \\
&= \frac{m}{m+1} V_\pi^{[m]}(s_t) + \frac{1}{m} G_t^{[m+1]} \\
&= \frac{1}{m+1}\left((m+1)V_\pi^{[m]}(s_t) - V_\pi^{[m]}(s_t) + G_t^{[m+1]}\right) \\
&= V_\pi^{[m]}(s_t) + \frac{1}{m+1}\left(G_t^{[m+1]} - V_\pi^{[m]}(s_t)\right)
\end{aligned}$$

From Monte-Carlo (MC) to Temporal Difference (TD)

- Limitation of MC methods: require finished episodes
- TD: $G_t \approx r_{t+1} + \gamma \hat{V}_\pi(s_{t+1})$, applies if some applications have very long episodes

$$V_\pi^{[m+1]}(s_t) = V_\pi^{[m]}(s_t) + \alpha\left(\underbrace{r_{t+1}^{[m+1]} + \gamma \hat{V}_\pi(s_{t+1})}_{\text{TD Target}} - V_\pi^{[m]}(s_t)\right)$$

The right side of α is call **TD Error**

TD with n-step returns

- 1-step TD: $G_t \approx r_{t+1} + \gamma \hat{V}_\pi(s_{t+1})$
- 2-step TD: $G_t \approx r_{t+1} + \gamma r_{t+2} + \gamma^2 \hat{V}_\pi(s_{t+2})$
- n-step TD: $G_t \approx r_{t+1} + \gamma r_{t+2} + \cdots + \gamma^{n-1} r_{t+n} + \gamma^n \hat{V}_\pi(s_{t+n})$
- Monte Carlo methods could be considered as ∞-step TD methods.

Rules for action-values

- MC: $Q_\pi^{[m+1]}(s_t, a_t) = \frac{1}{m+1} \sum_{i=1}^{m+1} G_t^{[i]}$
- MC-incremental: $Q_\pi^{[m+1]}(s_t, a_t) = Q_\pi^{[m]}(s_t, a_t) + \frac{1}{m+1}(A_t^{[m+1]} - Q_\pi^{[m]}(s_t, a_t))$
- n-step TD: $G_t \approx r_{t+1} + \cdots + \gamma^{n-1} r_{t+n} + \gamma^n \hat{Q}_\pi(s_{t+n}, a_{t+n})$

Trade-off between bias and variance:

- MC methods: unbiased, high variance
- TD methods: biased, low variance
- n-step TD methods: the choice of n serves as a **trade-off** between bias and variance in value estimation

λ-return: $\mathrm{TD}(\lambda)$ algorithm

Definition

Infinite Case:

$$G_t^\lambda = (1-\lambda) \sum_{l=1}^{\infty} \lambda^{l-1} G_{t:t+l}$$

Estimating the value functions using the TD method computed with the λ-return.

Another method to trade off between the bias and variance of the estimator. Empirically better than fixed n-step return.

Notation

$G_{t:t+n}$ is equivalent to the n-step TD

$$G_{t:t+n} = r_{t+1} + \gamma r_{t+2} + \cdots + \gamma^{n-1} r_{t+n} + \gamma^n \hat{V}_\pi(s_{t+n})$$

Derivation

$$\begin{aligned}
G_t^\lambda &= (1-\lambda)(G_{t:t+1} + \lambda G_{t:t+2} + \lambda^2 G_{t:t+3} + \cdots) \\
&= (1-\lambda)(r_{t+1} + \gamma \hat{V}_\pi(s_{t+1})) + (1-\lambda)\lambda(r_{t+1} + \gamma r_{t+2} + \gamma^2 \hat{V}_\pi(s_{t+2})) + \cdots \\
&= (1-\lambda)[(1+\lambda+\lambda^2+\cdots) r_{t+1} + \gamma\lambda(1+\lambda+\lambda^2) + \gamma^2\lambda^2(1+\lambda+\lambda^2)] \\
&\quad\ +(1-\lambda)(\gamma \hat{V}_\pi(s_{t+1}) + \gamma^2 \lambda \hat{V}_\pi(s_{t+2}) + \gamma^3 \lambda^2 \hat{V}_\pi(s_{t+3})) \\
&= \sum_{l=1}^{\infty} \left[(\gamma\lambda)^{l-1} r_{t+l} + (1-\lambda)\gamma^l \lambda^{l-1} \hat{V}_\pi(s_{t+l}) \right]
\end{aligned}$$

Intermediate expressions used:

- $1 + \lambda + \lambda^2 + \cdots = \lim_{n\to\infty} \frac{1-\lambda^{n+1}}{1-\lambda} = \frac{1}{1-\lambda}$

Discussion

Observe the form

$$G_t^\lambda = \sum_{l=1}^{\infty}\left[(\gamma\lambda)^{l-1}r_{t+l} + (1-\lambda)\gamma^l\lambda^{l-1}\hat{V}_\pi(s_{t+l})\right]$$

2 Special cases:

- $\lambda = 0 : G_t^0 = t_{t+1} + \gamma\hat{V}_\pi(s_{t+1})$

  This is 1-step return TD
- $\lambda = 1 : G_t^1 = r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \cdots$

  This is Monte Carlo (∞-step return)

When $0 < \lambda < 1$:

- λ-return offers a compromise between bias and variance controlled by λ

Finite Case

assume all rewards after step T are 0

$$G_{t:t+n} = G_{t:t+T},\ \forall n \geq T-t$$

Then the general form of λ-return under finite case is:

$$\begin{aligned} G_t^\lambda &= (1-\lambda)\sum_{l=1}^{T-t-1}\lambda^{l-1}G_{t:t+l} + (1-\lambda)\sum_{l=T-t}^{\infty}(1+\lambda+\lambda^2+\cdots)\lambda^{T-t-1}G_{t:T} \\ &= \sum_{l=1}^{T-t-1}\lambda^{l-1}G_{t:t+l} + \lambda^{T-t-1}G_{t:T} \end{aligned}$$

Summary

$$\hat{V}_\pi^{[m+1]}(s_t) \leftarrow \hat{V}_\pi^{[m]}(s_t) + \frac{1}{m+1}(G_t^{[m+1]} - \hat{V}_\pi^{[m]}(s_t))$$

To estimate $G_t^{[m+1]}$, we can use:

- ∞-step return -> MC
- (n) 1-step return -> TD
- λ-return -> $\mathrm{TD}(\lambda)$

Further Expansion

Value Parameterization

> Q: If there are a large number of or infinitely many states, it is difficult or impossible to storage all state values.

Solution: Using a **function approximator** (e.g. neural network) to estimate the state-value function $\hat{V}_\pi(s; w)$

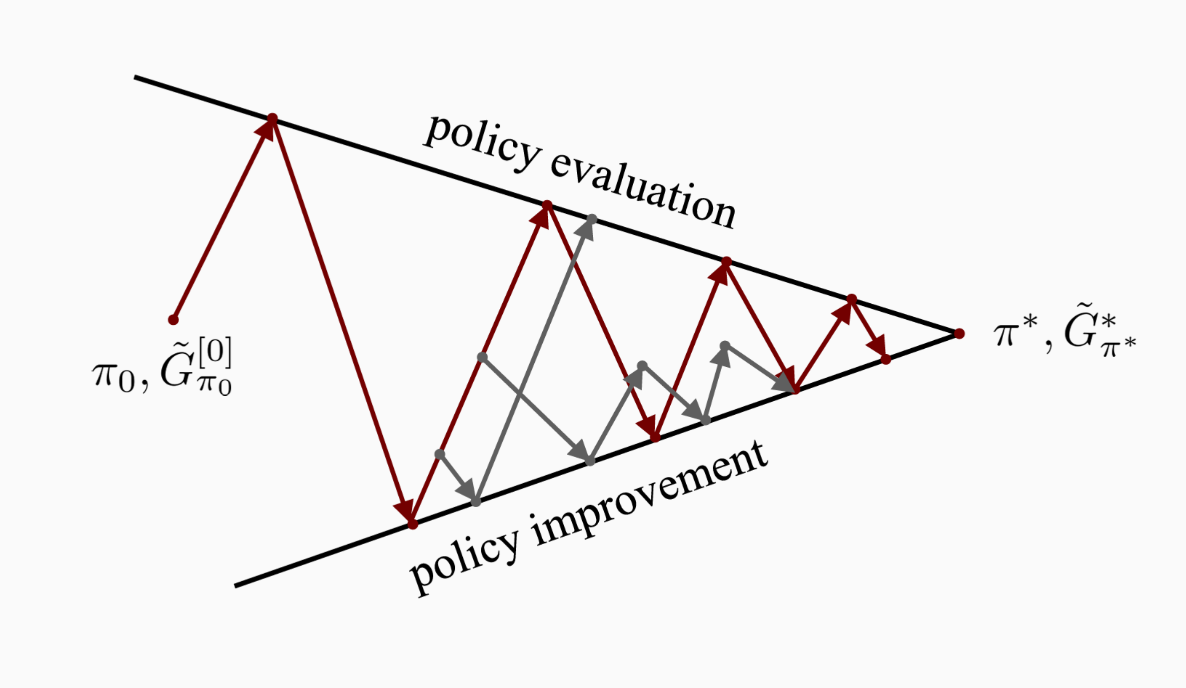
Find w^* s.t. $w^* = \mathrm{argmin}\ E_\pi[(y_t - \hat{V}_\pi(s_t; w))^2]$

- y_t is the true value but may be not accessible
- we can use TD-target $r_{t+1} + \gamma \hat{V}_\pi(s_{t+1}; w)$ to replace y_t

MSE Loss:

$$L(w) = \frac{1}{N} \sum_i (r_{t+1}^{[i]} + \gamma \hat{V}_\pi(s_{t+1}^{[i]}; w) - \hat{V}_\pi(s_t^{[i]}; w))$$

Truncated Policy Evaluation



Truncate early before evaluation converges to improve the policy improvement speed.

Difference between leveraging state-values and action-values

Answers from ChatGPT 4o

> In practice, **state-values** are used when you're interested in evaluating or improving a policy in a model-based way, whereas **action-values** are more commonly used in model-free reinforcement learning algorithms, allowing agents to directly learn the optimal policy by interacting with the environment.