# Lecture3 Exponential Coordinate of Rigid Body

#### **Lecture3 Exponential Coordinate of Rigid Body**

Exponential Coordinate of SO(3)

Logarithm of Rotations

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## Exponential Coordinate of SO(3)

exp: 
$$[\hat{\omega}] heta \in so(3) o R = e^{[\hat{\omega}] heta} \in SO(3)$$

log: 
$$R \in SO(3) 
ightarrow [\hat{\omega}] heta \in so(3)$$

the vector  $\hat{\omega}\theta$  is called the *exponential coordinate* for the R, or the canonical coordinates of the SO(3).

Taylor expansion:

$$R=e^{[\hat{\omega}] heta}=I+ heta[\hat{\omega}]+rac{ heta^2}{2!}[\hat{\omega}]^2+rac{ heta^3}{3!}[\hat{\omega}]^3+\ldots$$

Take the first two order components and do some approximation

#### Rodrigue's Formula

$$e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\sin(\theta) + [\hat{\omega}]^2(1 - \cos(\theta))$$

### **Logarithm of Rotations**

$$\operatorname{tr}(R) = 1 + 2\cos(\theta)$$

- If R=I, then  $\theta=0$ ,  $\hat{\omega}$  is undifined
- If  $\mathrm{tr}(R) = -1$ , then  $\theta = \pi$ ,  $\hat{\omega}$  equal to one of the following

$$\hat{\omega} \in \{rac{1}{\sqrt{2(1+r_{33})}}egin{bmatrix} r_{13} \ r_{23} \ 1+r_{33} \end{bmatrix}, rac{1}{\sqrt{2(1+r_{22})}}egin{bmatrix} r_{12} \ r_{22} \ 1+r_{32} \end{bmatrix}, rac{1}{\sqrt{2(1+r_{11})}}egin{bmatrix} r_{11} \ r_{21} \ 1+r_{31} \end{bmatrix}, \}$$

i.e. rotate about one axis (x, y, or z) for 180 degrees.

• Otherwise, 
$$\theta=\cos^{-1}(\frac{1}{2}(\mathrm{tr}(R)-1))\in[0,\pi)$$
 and  $[\hat{\omega}]=\frac{1}{2\sin(\theta)}(R-R^T)$ 

## **Euler Angles**

There is a very intuitive fact:

- Pre-multiply = extrinsic rotation
- Post-multiply = intrinsic rotation

XYZ corresponds to roll-pitch-yaw respectively

# Exponential Coordinate of SE(3)

## Exponential Map of se(3): from Twist to Rigid-Body Motion

W.l.o.g., assume  $\|\omega\|=1$ 

$$T = e^{[\mathcal{V}] heta} = egin{bmatrix} e^{[\omega] heta} & G( heta)v \ 0 & 1 \end{bmatrix}, ext{with } G( heta) = I heta + (1-\cos( heta))[\omega] + ( heta-\sin( heta))[\omega]^2$$

## Log of SE(3): from Rigid-Body Motion to Twist

screw axis  $S=(\omega,v)$ 

$$egin{align} [\mathcal{S}] &= egin{bmatrix} [\omega] & v \ 0 & 0 \end{bmatrix} \in se(3) \ T &= e^{[\mathcal{S}] heta} = egin{bmatrix} R & p \ 0 & 1 \end{bmatrix} \in SE(3) \end{aligned}$$

- if R=I, then  $\omega=0, v=p/\|p\|, \theta=\|p\|$
- Use logarithm of rotation to determine  $\omega$  and  $\theta$ , then

$$v=G^{-1}( heta)p, ext{where } G^{-1}( heta)=rac{1}{ heta}I-rac{1}{2}[\omega]+(rac{1}{ heta}-rac{1}{2}\cosrac{ heta}{2})[\omega]^2$$