Lecture 1: Rigid Body Configuration and Velocity

Math Basics

Skew symmetric representation

cross product can be expressed as matrix product

$a \times b = [a]b$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \leftrightarrow [a] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

$[a] = -[a]^T$

Special Orthogonal Group

$$SO(n) = \{R \in \mathbb{R}^{n \times n} : R^T R = I, det(R) = 1\}$$

Rigid Body Velocity (Twist)

Pick an arbitrary reference point r, then for any body-fixed point on the body

$v_p = v_r + \omega \times \overrightarrow{rp}$

v_r is the speed of r w.r.t. rigid body fixed frame.

Definition

Twist, namely spatial velocity

Spatial velocity is a set of **parameters**, not the absolute velocity.

$\mathcal{V}_r = (\omega_r, v_r)$

Change Reference for Twist

${}^A\mathcal{V} = {}^A X_B {}^B\mathcal{V}$

$T = (R, p)$

${}^A X_B = [Ad_T] \triangleq \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$

Geometric Aspect of Twist: Screw Motion

Definition

Screw Motion: screw axis $q, \hat{s}, h$ + rotation speed $\dot{\theta}$

- $\hat{s}$: unit vector in the directrion of the rotation axis
- q: any point on the rotation axis
- h: **screw pitch**: the ratio of the linear velocity along the screw axis to the angular velocity about the screw axis

Theorem (Chasles): **Every rigid body motion can be realized by a screw motion.**

Transformation

Screw Motion to Twist

Fix a reference frame {A} with origin o_A

${}^A\omega = {}^A\hat{s}\dot{\theta}$

${}^A v_{o_A} = {}^A v_q + {}^A\omega \times (-{}^A q) = {}^A\hat{s}(h\dot{\theta}) - {}^A\omega \times {}^A q$

Twist to Screw Motion

- $\omega = 0$, pure translation

  $\hat{s} = \dfrac{v}{\|v\|},\ \dot{\theta} = \|v\|,\ h = \infty$, q can bre arbitrary

- $\omega \neq 0$

  $\hat{s} = \dfrac{\omega}{\|\omega\|},\ \dot{\theta} = \|\omega\|,\ q = \dfrac{\omega \times v}{\|\omega\|^2},\ h = \dfrac{\omega^T v}{\|\omega\|^2}$

Here v is the velocity of the reference point, not the velocity along the screw axis.

Screw Representation of a Twist

Twist with a unit speed,

$$\mathcal{V} = \hat{\mathcal{S}}\dot{\theta}$$

$$\hat{S} \leftrightarrow (\hat{s}, h, q), \dot{\theta} = 1$$