# Lecture6: Velocity Kinematics-Geometric and Analytic Jacobian of Open Chain

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## Velocity Kinematics

Derive the **Jacobian Matrix**: lineariezd map from joint velocities $\dot{\theta}$ to the spatial velocity $\mathcal{V}$ of the end effector (frame {b}).

- Twist representation $\rightarrow$ **Geometric Jacobian**

  $\mathcal{V}_b = J(\theta)\dot{\theta}$
- Local coordinate of $SE(3)$ $\rightarrow$ **Analytic Jacobian**

  Let $x \in \mathbb{R}^p$ be the task space variable

  e.g. some of Cartesain + Euler angle of end-effector frame

$$
\begin{aligned}
x &= g(\theta_1, \theta_2, \ldots \theta_n) \\
\dot{x} &= \frac{\mathrm{d}}{\mathrm{d}t}(g(\theta_1, \ldots \theta_n)) = \underbrace{\frac{\partial g}{\partial \theta}}_{\text{Analytic Jacobian}} \dot{\theta} = J_a(\theta)\dot{\theta}
\end{aligned}
$$

$$
\frac{\partial g}{\partial \theta} \in \mathbb{R}^{n_x \times n}
$$

- Relation between Geometric Jacobian and Analytic Jacobian

  $J_a(\theta) = E(x)J(\theta)$

  where $E(x)$ can be easily found with given parameterization $x$

## Geometric Jacobian Derivations

In general, the end-effector twist $\mathcal{V} = (\omega, v)$

$$
\mathcal{V}_b = \begin{bmatrix} J_1(\theta) & J_2(\theta) & \cdots & J_n(\theta) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}
$$

where $\theta = (\theta_1, \theta_2, \cdots, \theta_n)$,

$J_i(\theta)$ is the screw axis of joint $i$ when all other joints do not move (i.e. $\forall j \neq i, \dot{\theta}_j = 0$)

In coordinate free notation, $J_i(\theta) = \mathcal{S}_i(\theta)$

Using local coordinate

$$^iJ_i = {}^i\mathcal{S}_i, \quad i = 1, 2, \cdots, n$$

In fixed frame {0}

$$^0J_i(\theta) = {}^0X_i(\theta)^iS_i, \quad i = 1, 2, \cdots, n \tag{1}$$

where $^0X_i(\theta)$ is the change of coordinate matrix for special velocities.

Recall the proof we did in Lecture5

$$T_i(\theta) = e^{[^0\bar{\mathcal{S}}_1]\theta_1} e^{[^0\bar{\mathcal{S}}_2]\theta_2} \ldots e^{[^0\bar{\mathcal{S}}_i]\theta_i} M \Rightarrow {}^0X_i(\theta) = [\mathrm{Ad}_{^0T_i(\theta)}] \tag{2}$$

To calculate more efficiently, we then derive a **recursive** Jacobian

$$^0J_i(\theta) = {}^0\mathcal{S}_i(\theta) = [\mathrm{Ad}_{\hat{T}(\theta_1, \cdots, \theta_{i-1})}]^0\bar{\mathcal{S}}_i \tag{3}$$

where $\hat{T}(\theta_1, \cdots, \theta_{i-1}) = e^{[^0\bar{\mathcal{S}}_1]\theta_1} e^{[^0\bar{\mathcal{S}}_2]\theta_2} \ldots e^{[^0\bar{\mathcal{S}}_{i-1}]\theta_{i-1}}$

# Analytic Jacobian Derivations

You should have learned it in traditional robotics course.