Lecture9: probability Review for Reinforcement Learning

> Notes taken by [squarezhong]
> Repo address: [squarezhong/SDM5008-Lecture-Notes]

Probability and Conditional Probability

Probability

Definition

A formal way to quantify the uncertainty of our knowledge about the physical world.

Formalism: Probability Space $(\Omega, \mathcal{F}, P)$

- Ω : **sampling space**: a set of all possible outcomes (maybe infinite)
- $\mathcal{F}$: **event space**: collection of events of interest (event is a subset of Ω)
- $P : \mathcal{F} \to [0, 1]$ **probability measure**: assign event in $\mathcal{F}$ to a real number between 0 and 1.

Axioms of Probability

- $P(A) \geq 0$

- $P(\Omega) = 1$
- $A \cap B = 0 \Rightarrow P(A \cup B) = P(A) + P(B)$

Important consequences

- $P(\emptyset) = 0$
- Law of total probability: $P(B) = \sum_i^n P(B \cap A_i)$ for any partitions $\{A_i\}$ of Ω
  - A collection of sets $A_1, \cdots, A_n$ is called a partition of Ω if
    - $A_i \cap A_j = \emptyset$, for all $i \neq j$
    - $A_1 \cup A_2 \cdots \cup A_n = \Omega$

Conditional Probability

Definition

$$P(A|B) \triangleq \frac{P(A \cap B)}{P(B)}$$

- $(\Omega, \mathcal{F}, P) \rightarrow (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$
- $\tilde{\Omega} = B$
- $\tilde{\mathcal{F}}$ = all subsets of B

Bayes rule

$$P(A|B) = \frac{P(A|B)P(A)}{P(B)}$$

Independent

A and B are called (statistically) independent if

- $P(A|B) = P(A)$ or $P(A \cap B) = P(A)P(B)$

Random Variables and Random Vectors

Deterministic variable and random variable

- z is a **deterministic variable** (single-valued variable), which means z can take only one value that may or may not be known
- z is a **random variable** (multi-valued variable), which means z can take multiple (even infinite) possible values, each value occurs with certain probability

probability measure

- Discrete random variable: **probability mass function** (pmf)
- Continuous random variable: **probability density function** (pdf)

Random vector

- n-dimensional random vector $X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$
- density function $f(x), x \in R^n$
- probability evaluation: $P(X \in A) = \int_A f(x)dx$

Expectation of a random vector $X \in R^n$

- Continuous: $E(X) = \triangleq \int_{R^n} xf(x)dx$
- Discrete: $E(X) = \sum_x x \cdot \mathrm{Prob}(X = x)$
- $E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_n) \end{bmatrix}$

Linearity of Expection:

Expection of AX with deterministic constant $A \in R^{m \times n}$

- $E(AX) = AE(X)$
- $E(AX + BY) = AE(X) + BE(Y)$

Jointly Distributed Random Vectors and Conditional Expectation

Jointly distributed random vectors: $X \in R^n, Y \in R^m$

- Joint density function: $(X, Y) \sim f_{XY}(x, y)$
- Marginal density: $X \sim f_X(x), Y \sim f_Y(y)$

  $f_X(x) = \int_{R^m} f_{XY}(x, y)dy, f_Y(y) = \int_{R^n} f_X Y(x, y)dx$

Conditional probability

$$p_{X|Y}(X = i|Y = j) = \frac{p_{XY}(X = i, Y = j)}{\sum_i p_{XY}(X = i, Y = j)}$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

Law of total probability

- $f_X(x) = \int_{R^m} f_{X|Y}(x|y) f_Y(y) dy$
- $f_Y(y) = \int_{R^n} f_{Y|X}(y|x) f_X(x) dx$

Independent

X is independent to Y, denoted by $X \perp Y$ **iff** $f_{XY}(x, y) = f_X(x) f_Y(y)$

Conditional expectation

- Continuous: $E(X|Y = y) = \int_{R^n} x f_{X|Y}(x|y) dx$
- Discrete: $E(X|Y = y) = \sum_i i \cdot \mathrm{Prob}(X = i|Y = y)$
- $E(X) = \sum_y E(X|Y = y) \cdot p_Y(Y = y)$
- $E(g(X, Y)) = \sum_y E(g(X, Y)|Y = y) \cdot p_Y(Y = y)$