Lecture7: Rigid-Body Dynamics

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Spatial Force (Wrench)

Definition

Consider a rigid body with many forces on it and fix an arbitrary point O in space

The net effect of these forces can be expressed as

- A **net force** f, acting along a line passing through O
- A **moment** n_O about point O

Spatial Force (Wrench) is given by the 6D vector

$$\mathcal{F} = \begin{bmatrix} n_O \\ f \end{bmatrix}$$

Plücker Coordinate Systems

Given a frame {A}, the Plücker coordinate of a spatial force $\mathcal{F}$ is given by

$${}^A\mathcal{F} = \begin{bmatrix} {}^A n_{o_A} \\ {}^A f \end{bmatrix}$$

- Coordinate transform

$${}^A\mathcal{F} = {}^A X_B^* \, {}^B\mathcal{F} \quad \text{where } {}^A X_B^* = {}^B X_A^T$$

Wrench-Twist Pair and Power

Suppose a rigid body has a twist ${}^A\mathcal{V} = ({}^A\omega, {}^A v_{o_A})$ and a wrench ${}^A\mathcal{F} = ({}^A n_{o_A}, {}^A f)$ act on the body. Then the power is

$$P = ({}^A\mathcal{V})^T \ {}^A\mathcal{F}$$

Joint Torque

$$P = \mathcal{V}^T\mathcal{F} = (\hat{\mathcal{S}}^T\mathcal{F})\dot{\theta} \triangleq \tau\dot{\theta}$$

$\tau = \mathcal{S}^{\hat{T}}\mathcal{F} = \hat{\mathcal{S}}\mathcal{F}^T$ is the projection of the wrench onto the screw axis, i.e. the effective part of the wrench.

- τ can be referred to as joint "torque" or **generalized force**

Spatial Momentum

Rotational Inertia

Rotational Inbertia $\bar{I} = \int_V \rho(r)[r][r]^T dr$

- $\rho(\cdot)$ is the density function of the body
- $\bar{I}$ depends on coordinate system (constant if origin coincides with CoM)

Spatial Momentum

- Linear momentum (动量)

  $L \triangleq mv_c$
- Angular momentum about CoM

  $\phi_c = \bar{I}\omega$
- Angular momentum about a point O

  $\phi_o = \sum_i \overrightarrow{Or_i} \times (m_i v_i) = \phi_c + \overrightarrow{OC} \times L$

Spatial Momentum:

$$h \triangleq \begin{bmatrix} \phi_r \\ L \end{bmatrix}$$

n is the reference point.

- Coordinate transform:

$${}^A h = {}^A X_B^* \, {}^B h$$

Spatial Inertia

Spatial inertia $\mathcal{I}$ is given by

$$h = \mathcal{I}\mathcal{V}$$

Let {C} be a frame whose origin coincide with CoM. We have

$$^{C}h = \begin{bmatrix} ^{C}\bar{I}^{C}\omega \\ m^{C}v_c \end{bmatrix} = \begin{bmatrix} ^{C}\bar{I}_c & 0 \\ 0 & mI_3 \end{bmatrix} \begin{bmatrix} ^{C}\omega \\ ^{C}v_c \end{bmatrix}$$

Then

$$^{C}\mathcal{I} = \begin{bmatrix} ^{C}\bar{I}_c & 0 \\ 0 & mI_3 \end{bmatrix}$$

- Coordinate Transform

$$^{A}\mathcal{I} = {^{A}X_{C}^{*}}\,{^{C}\mathcal{I}}\,{^{C}X_{A}}$$