Lecture5: Product of Exponential and Kinematics of Open Chain

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Kinematics

- **Forward Kinematics**

  Given joint variables $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$, derive the configuration of the end-effector frame $T = (R, p)$.

- **Velocity Kinematics**

  Derive the **Jacobian Matrix**: lineariezd map from joint velocities $\dot{\theta}$ to the spatial velocity $\mathcal{V}$ of the end effector.

Product of Exponential Formula Derivations

Definition

Define ${}^0\bar{\mathcal{S}}_i = {}^0\bar{\mathcal{S}}_i(0, \ldots, 0)$: the screw axis of joint i expressed in frame {0} when the robot is at the home position. It is a constant.

The overall forward kinematics:

$$T_{sb}(\theta_1, \theta_2, \ldots, \theta_n) = \underbrace{e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} e^{[{}^0\bar{\mathcal{S}}_2]\theta_2} \ldots e^{[{}^0\bar{\mathcal{S}}_n]\theta_n}}_{PoE} M$$

Proof

For simplicity, assume that $n = 2$

Apply screw motion along ${}^0\bar{\mathcal{S}}_1$ first:

$$T_{sb}(\theta_1, 0) = e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} M \tag{1}$$

Now screw axis for joint 2 has been changed. The new axis:

$${}^0\mathcal{S}_2 = {}^0\mathcal{S}_2(\theta_1, 0) \neq {}^0\bar{\mathcal{S}}_2$$

$$ {}^0\bar{\mathcal{S}}_2 \xrightarrow{\hat{T}_1 = e^{[{}^0\bar{\mathcal{S}}_1]\theta_1}} {}^0\mathcal{S}_2(\theta_1) = \underbrace{[\mathrm{Ad}_{\hat{T}_1}]}_{6\times 6}{}^0\bar{\mathcal{S}}_2 $$

Fact:

$$ \text{If } \mathcal{S}' = [\mathrm{Ad}_T]\mathcal{S} \Leftrightarrow [\mathcal{S}'] = T[\mathcal{S}]T^{-1} $$

Then

$$ \begin{aligned} e^{[{}^0\bar{\mathcal{S}}_2(\theta_1)]\theta_2} &= e^{[[\mathrm{Ad}_{\hat{T}_1}]{}^0\bar{\mathcal{S}}_2]\theta_2} && (2) \\ &= e^{\hat{T}_1[{}^0\bar{\mathcal{S}}_2]\hat{T}_1^{-1}} && (3) \\ &= \hat{T}_1 e^{[{}^0\bar{\mathcal{S}}_2]\theta_2}\hat{T}_1^{-1} && (4) \end{aligned} $$

Finally

$$ \begin{aligned} T_{sb}(\theta_1, \theta_2) &= e^{[{}^0\mathcal{S}_2]\theta_2} T_{sb}(\theta_1, 0) && (5) \\ &= \hat{T}_1 e^{[{}^0\bar{\mathcal{S}}_2]\theta_2}\hat{T}_1^{-1} e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} M && (6) \\ &= \hat{T}_1 e^{[{}^0\bar{\mathcal{S}}_2]\theta_2} M && (7) \\ &= e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} e^{[{}^0\bar{\mathcal{S}}_2]\theta_2} M && (8) \end{aligned} $$

> Note that $\hat{T}_1 = e^{[{}^0\bar{\mathcal{S}}_1]\theta_1}$ and $T^{-1}T = I$