Lecture3 Exponential Coordinate of Rigid Body

Exponential Coordinate of $SO(3)$

exp: $[\hat{\omega}]\theta \in so(3) \rightarrow R = e^{[\hat{\omega}]\theta} \in SO(3)$

log: $R \in SO(3) \rightarrow [\hat{\omega}]\theta \in so(3)$

the vector $\hat{\omega}\theta$ is called the *exponential coordinate* for the R, or the canonical coordinates of the $SO(3)$.

Taylor expansion:

$$R = e^{[\hat{\omega}]\theta} = I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \ldots$$

Take the first two order components and do some approximation

Rodrigue's Formula

$$e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\sin(\theta) + [\hat{\omega}]^2(1 - \cos(\theta))$$

Logarithm of Rotations

$\mathrm{tr}(R) = 1 + 2\cos(\theta)$

- If $R = I$, then $\theta = 0$, $\hat{\omega}$ is undifined
- If $\mathrm{tr}(R) = -1$, then $\theta = \pi$, $\hat{\omega}$ equal to one of the following

  $$\hat{\omega} \in \{\frac{1}{\sqrt{2(1+r_{33})}}\begin{bmatrix} r_{13} \\ r_{23} \\ 1+r_{33} \end{bmatrix}, \frac{1}{\sqrt{2(1+r_{22})}}\begin{bmatrix} r_{12} \\ r_{22} \\ 1+r_{32} \end{bmatrix}, \frac{1}{\sqrt{2(1+r_{11})}}\begin{bmatrix} r_{11} \\ r_{21} \\ 1+r_{31} \end{bmatrix}, \}$$

  i.e. rotate about one axis (x, y, or z) for 180 degrees.
- Otherwise, $\theta = \cos^{-1}(\frac{1}{2}(\mathrm{tr}(R) - 1)) \in [0, \pi)$ and $[\hat{\omega}] = \frac{1}{2\sin(\theta)}(R - R^T)$

Euler Angles

There is a very intuitive fact:

- Pre-multiply = extrinsic rotation
- Post-multiply = intrinsic rotation

XYZ corresponds to roll-pitch-yaw respectively

Exponential Coordinate of $SE(3)$

Exponential Map of $se(3)$: from Twist to Rigid-Body Motion

W.l.o.g., assume $||\omega|| = 1$

$$T = e^{[\mathcal{V}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}, \text{with } G(\theta) = I\theta + (1 - \cos(\theta))[\omega] + (\theta - \sin(\theta))[\omega]^2$$

Log of $SE(3)$: from Rigid-Body Motion to Twist

screw axis $S = (\omega, v)$

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

$$T = e^{[\mathcal{S}]\theta} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in SE(3)$$

- if $R = I$, then $\omega = 0, v = p/||p||, \theta = ||p||$
- Use logarithm of rotation to determine ω and θ, then

$$v = G^{-1}(\theta)p, \text{where } G^{-1}(\theta) = \frac{1}{\theta}I - \frac{1}{2}[\omega] + (\frac{1}{\theta} - \frac{1}{2}\cos\frac{\theta}{2})[\omega]^2$$