

Lecture 1: Rigid Body Configuration and Velocity

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Math Basics

Skew symmetric representation

Special Orthogonal Group

Rigid Body Velocity (Twist)

Definition

Change Reference for Twist

Geometric Aspect of Twist: Screw Motion

Definition

Transformation

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Screw Representation of a Twist

Math Basics

Skew symmetric representation

cross product can be expressed as matrix product

$$a \times b = [a]b$$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \leftrightarrow [a] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

$$[a] = -[a]^T$$

Special Orthogonal Group

$$SO(n) = \{R \in \mathbb{R}^{n \times n} : R^T R = I, \det(R) = 1\}$$

Rigid Body Velocity (Twist)

Pick an arbitrary reference point r , then for any body-fixed point on the body

$$v_p = v_r + \omega \times \vec{rp}$$

v_r is the speed of r w.r.t. rigid body fixed frame.

Definition

Twist, namely spatial velocity

Spatial velocity is a set of **parameters**, not the absolute velocity.

$$\mathcal{V}_r = (\omega_r, v_r)$$

Change Reference for Twist

$${}^A\mathcal{V} = {}^AX_B {}^B\mathcal{V}$$

$$T = (R, p)$$

$${}^AX_B = [Ad_T] \triangleq \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$$

Geometric Aspect of Twis: Screw Motion

Definition

Screw Motion: screw axis q, \hat{s}, h + rotation speed $\dot{\theta}$

- \hat{s} : unit vector in the directrion of the rotation axis
- q : any point on the rotation axis
- h : **screw pitch**: the ratio of the linear velocity along the screw axis to the angular velocity about the screw axis

Theorem (Chasles): **Every rigid body motion can be realized by a screw motion.**

Transformation

Screw Motion to Twist

Fix a reference frame $\{A\}$ with origin o_A

$${}^A\omega = {}^A\hat{s}\dot{\theta}$$

$${}^Av_{o_A} = {}^Av_q + {}^A\omega \times (-{}^Aq) = {}^A\hat{s}(h\dot{\theta}) - {}^A\omega \times {}^Aq$$

Twist to Screw Motion

- $\omega = 0$, pure translation

$$\hat{s} = \frac{v}{\|v\|}, \dot{\theta} = \|v\|, h = \infty, q \text{ can be arbitrary}$$

- $\omega \neq 0$

$$\hat{s} = \frac{\omega}{\|\omega\|}, \dot{\theta} = \|\omega\|, q = \frac{\omega \times v}{\|\omega\|^2}, h = \frac{\omega^T v}{\|\omega\|^2}$$

Here v is the velocity of the reference point, not the velocity along the screw axis.

Screw Representation of a Twist

Twist with a unit speed,

$$\nu = \hat{\mathcal{S}}\dot{\theta}$$

$$\hat{S} \leftrightarrow (\hat{s}, h, q), \dot{\theta} = 1$$