Basic Collisions

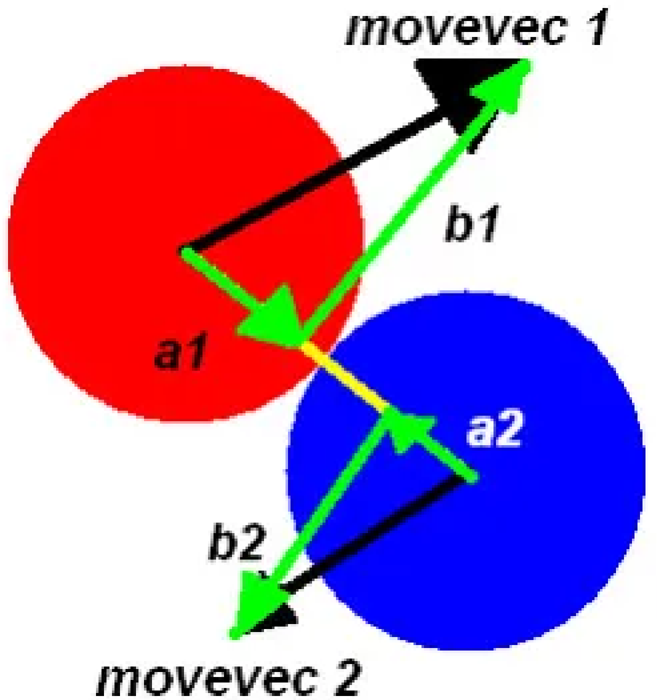

Figure 1: Collision of two objects $a1$ and $a2$

Conservation of Momentum

Let $\mathbf{v_1}$ represent the velocity of an object with mass m_1, and $\mathbf{v_2}$ represent velocity of an object with mass m_2. The conservation of momentum states that

$$m_1\mathbf{v_1} + m_2\mathbf{v_2} = m_1\mathbf{v_1'} + m_2\mathbf{v_2'}, \tag{1}$$

where the primed velocities denote the velocity of the objects after the collision has been resolved.

Rearranging our equation and solving for the final moments, we obtain

$$m_1\mathbf{v_1'} = m_1\mathbf{v_1} - m_2\Delta\mathbf{v_2} \tag{2}$$

$$m_2\mathbf{v_2'} = m_2\mathbf{v_2} - m_1\Delta\mathbf{v_1} \tag{3}$$

We note that physically, any momentum lost by the first object is subsequently gained by the second. In this simplified simulation, we can also assume that any transference of momentum only happens along the direction of the center line of contact of the two objects. Letting $\hat{\mathbf{r}}$ denote the unit vector that points in the direction along the center line of contact of the two objects and letting $\hat{\mathbf{n}}$ denote the direction perpendicular to that, we can write the change in momentum as $\mathbf{P} = P\hat{\mathbf{r}}$. This allows us to rewrite Equations (2) and (3) as

$$m_1\mathbf{v_1'} = m_1\mathbf{v_1} - P\hat{\mathbf{r}} \iff \mathbf{v_1'} = \mathbf{v_1} - \frac{P}{m_1}\hat{\mathbf{r}} \tag{4}$$

$$\mathbf{v_2'} = \mathbf{v_2} + \frac{P}{m_2}\hat{\mathbf{r}}, \tag{5}$$

where P denotes the magnitude of the change in momentum.

We can also rewrite the velocity vectors as follows

$$\mathbf{v_1} = a_1\hat{\mathbf{r}} + b_1\hat{\mathbf{n}} \tag{6}$$
$$\mathbf{v_1'} = a_1'\hat{\mathbf{r}} + b_1'\hat{\mathbf{n}} \tag{7}$$
$$\mathbf{v_2} = a_2\hat{\mathbf{r}} + b_2\hat{\mathbf{n}} \tag{8}$$
$$\mathbf{v_2'} = a_2'\hat{\mathbf{r}} + b_2'\hat{\mathbf{n}} \tag{9}$$

for some $a_1, b_1, a_2, b_2, a_1', b_1', a_2', b_2' \in \mathbb{R}$.

Substituting $\mathbf{v_1}$ from Equation (6) into Equation (4) and $\mathbf{v_2}$ from Equation (8) into Equation (5), we obtain

$$\mathbf{v_1'} = a_1\hat{\mathbf{r}} + b_1\hat{\mathbf{n}} - \frac{P}{m_1}\hat{\mathbf{r}} = \left(a_1 - \frac{P}{m_1}\right)\hat{\mathbf{r}} + b_1\hat{\mathbf{n}} \tag{10}$$
$$\mathbf{v_2'} = \left(a_2 + \frac{P}{m_2}\right)\hat{\mathbf{r}} + b_2\hat{\mathbf{n}} \tag{11}$$

These imply the following relations

$$a_1' = a_1 - \frac{P}{m_1} \tag{12}$$
$$b_1' = b_1 \tag{13}$$
$$a_2' = a_2 + \frac{P}{m_2} \tag{14}$$
$$b_2' = b_2 \tag{15}$$

Conservation of Energy

The conservation of energy states that

$$m_1\mathbf{v_1}^2 + m_2\mathbf{v_2}^2 = m_1\mathbf{v_1'}^2 + m_2\mathbf{v_2'}^2 \tag{16}$$

Breaking the vectors into their components, and substituting using Equations (12) to (15), we obtain

$$\begin{aligned} m_1(a_1^2 + b_1^2) + m_2(a_2^2 + b_2^2) &= m_1\left(\left(a_1 - \frac{P}{m_1}\right)^2 + b_1^2\right) + m_2\left(\left(a_2 + \frac{P}{m_2}\right)^2 + b_2^2\right) \\ 0 &= -2Pa_1 + \frac{P^2}{m_1} + 2Pa_2 + \frac{P^2}{m_2} \\ 0 &= P^2\left(\frac{1}{m_1} + \frac{1}{m_2}\right) + 2P(a_2 - a_1) \\ 0 &= P\left(P\left(\frac{1}{m_1} + \frac{1}{m_2}\right) + 2(a_2 - a_1)\right) \end{aligned}$$

Using the fact that

$$\frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1 + m_2}{m_1 m_2},$$

we have that either $P = 0$ or

$$P = \frac{2(a_1 - a_2)m_1 m_2}{m_1 + m_2} \tag{17}$$

Substituting this back into Equations (4) and (5) we obtain

$$\mathbf{v_1'} = \mathbf{v_1} - \frac{2(a_1 - a_2)}{m_1 + m_2} \cdot m_2 \hat{\mathbf{r}} \tag{18}$$

$$\mathbf{v_2'} = \mathbf{v_2} + \frac{2(a_1 - a_2)}{m_1 + m_2} \cdot m_1 \hat{\mathbf{r}} \tag{19}$$

We note that

$$\begin{aligned} a_1 - a_2 &= \mathbf{v_1} \cdot \hat{\mathbf{r}} - \mathbf{v_2} \cdot \hat{\mathbf{r}} \\ &= (\mathbf{v_1} - \mathbf{v_2}) \cdot \hat{\mathbf{r}} \\ &= (\mathbf{v_1} - \mathbf{v_2}) \cdot \frac{(\mathbf{r_2} - \mathbf{r_1})}{|\mathbf{r_2} - \mathbf{r_1}|^2} \end{aligned}$$

and so we can rewrite Equations (18) and (19) as

$$\mathbf{v_1'} = \mathbf{v_1} - 2 \cdot \frac{(\mathbf{v_1} - \mathbf{v_2}) \cdot (\mathbf{r_2} - \mathbf{r_1})}{|\mathbf{r_2} - \mathbf{r_1}|^2} \cdot \left(\frac{m_2}{m_1 + m_2} \right) \cdot \hat{\mathbf{r}} \tag{20}$$

$$\mathbf{v_2'} = \mathbf{v_2} - 2 \cdot \frac{(\mathbf{v_1} - \mathbf{v_2}) \cdot (\mathbf{r_2} - \mathbf{r_1})}{|\mathbf{r_2} - \mathbf{r_1}|^2} \cdot \left(\frac{m_1}{m_1 + m_2} \right) \cdot \hat{\mathbf{r}} \tag{21}$$

Collision Between Particle and Stationary Object

We now consider a collision between a particle and a stationary object. Since the second object is stationary, we can model it as a "particle" with infinite mass, and zero velocity (i.e., $a_2 = 0$). Letting m_2 tend towards infinity, we obtain

$$\begin{aligned} \lim_{m_2 \to \infty} \mathbf{v_1'} &= \lim_{m_2 \to \infty} \left(\mathbf{v_1} - \frac{2(a_1 - a_2)}{m_1 + m_2} \cdot m_2 \hat{\mathbf{r}} \right) \\ &= \mathbf{v_1} - 2a_1 \, \hat{\mathbf{r}} \\ &= \mathbf{v_1} - 2(\mathbf{v_1} \cdot \hat{\mathbf{r}}) \, \hat{\mathbf{r}} \\ \lim_{m_2 \to \infty} \mathbf{v_2'} &= \lim_{m_2 \to \infty} \left(\mathbf{v_2} + \frac{2(a_1 - a_2)}{m_1 + m_2} \cdot m_1 \hat{\mathbf{r}} \right) \\ &= \mathbf{v_2} \\ &= \mathbf{0} \end{aligned}$$

Thus, we see that for a collision with a stationary object, the final velocity of the particle is given by

$$\mathbf{v_1'} = \mathbf{v_1} - 2(\mathbf{v_1} \cdot \hat{\mathbf{r}}) \, \hat{\mathbf{r}} \tag{22}$$