Lab Assignment 1

Numerical approximations to the solutions of first order initial value problems

April Dawn Kester
AMS 20 • UCSC • January 12, 2014

PART I

a. Write implementation of Euler's method

Exhibit 1.1 - euler.m file

PART II

a. Sketch direction field for equation:

$$y' = 0.05y(y-5)$$

Exhibit 2.1 - See attached screen shot of solutions.m file for direction field

b. Explicit solutions:

$y(o) = 4 \qquad \varphi(t) = \dfrac{20}{4+e^{0.25t}}$

$y(o) = 6 \qquad \psi(t) = \dfrac{-30}{e^{0.25t}-6}$

Exhibit 2.1 - See attached screen shot of solutions.m file for plot of two solutions

c. Using Euler's method with step-size h=0.1

* denotes true values of φ (t) and ψ (t) at points t = 0.5, 1, and 7 using explicit solutions

φ (0.5)	3.8970	φ * (0.5)	3.8962	0.020%
φ (1.0)	3.7866	φ * (1.0)	3.7849	0.044%
φ (7.0)	2.0566	φ * (7.0)	2.0503	0.30%
ψ (0.5)	6.1610	ψ * (0.5)	6.1641	0.050%
ψ (1.0)	6.3536	ψ * (1.0)	6.3613	0.12%
ψ (7.0)	47.1916	ψ * (7.0)	122.251	61.39%

Exhibit 2.2 & 2.3 - See attached screen shot of direction_field.m file for plot of approximation

d. Using Euler's method with step-size h=0.01

φ (0.5)	3.8963	φ * (0.5)	3.8962	0.0025%
φ (1.0)	3.7852	φ * (1.0)	3.7849	0.0079%
φ (7.0)	2.0509	φ * (7.0)	2.0503	0.029%
ψ (0.5)	6.1638	ψ * (0.5)	6.1641	0.0048%
ψ (1.0)	6.3606	ψ * (1.0)	6.3613	0.011%
ψ (7.0)	100.6985	ψ * (7.0)	122.251	17.62%

Exhibit 2.2 & 2.3 - See attached screen shot of direction_field.m file for plot of approximation

e. Using ode45

φ (0.5)	3.8962	φ * (0.5)	3.8962	0%
φ (1.0)	3.7850	φ * (1.0)	3.7849	0.0026%
φ (7.0)	2.0503	φ * (7.0)	2.0503	0%
ψ (0.5)	6.1642	ψ * (0.5)	6.1641	0.0016%
ψ (1.0)	6.3614	ψ * (1.0)	6.3613	0.0015%
ψ (7.0)	122.31	ψ * (7.0)	122.251	0.048%

Exhibit 2.4 - See attached print out of direction_field.m file for code

f. Summarize results

Euler's method, also known as the tangent line method, is used to approximate the solutions of a differential equation that we may not be able to solve analytically. The purpose of this lab was to illustrate the inherent issues with Euler's method. Euler's approximation is dependent on the previous step which is only an approximation of the step before. The only point that is known is at the initial point (t_0, y_0) because we know that a unique solution in some interval around the initial point satisfies the initial condition. Because Euler's method is using a nearby solution and not an exact solution for each step, the accuracy of the approximation depends on the behavior of the set of solutions.

You will notice in part II of this lab that $\varphi(t)$ has a converging set of solutions and $\psi(t)$ has a diverging set of solutions. As $t \to \infty$ a converging set of solutions get closer together, so using Euler's method, $\varphi(t)$ stays reasonably close to the exact solution. However, for large values of t in a diverging set of solutions, nearby solutions become further apart and Euler's method becomes less accurate. This can be seen clearly in part c when t = 7, φ has an error of 0.30% but ψ has an error of 61.39%. The farther apart solutions are for nearby values of C the harder they become to approximate. A way to improve the accuracy is to limit the approximation to a small interval away from the initial point. If we were to evaluate ψ at an interval of $0 \le t \le 2$ our approximate solution would be more accurate.

Another way to improve the accuracy of the Euler approximation is to decrease the step-size. A smaller step-size increases the number of computational steps and the approximate values become more accurate. We can see the difference illustrated in parts c and d, for $\varphi(0.5)$ the error at h=0.1 is 0.020% but at h=0.01 the error is only 0.0025%. It is important to note however, that increasing the step-size can cause the error to increase more rapidly in diverging set of solutions because with each computation you get even further away from where you started.

ode45 is an improvement on Euler's method and approximates a much more accurate result especially for large values of t.

PART III

a. Sketch direction field for equation:

$$y' = y^2 + t^2$$

Let be $\vartheta(t)$ the solution for $y(0) = 1$

Exhibit 3.1 - See attached screen shot of problem_3.m file for direction field

b. Using Euler's method with step-size h=0.1

ϑ (0.8)	3.5078
ϑ (0.9)	4.8023
ϑ (1.0)	7.1895

c. Using Euler's method with step-size h=0.01

ϑ (0.8)	5.3428
ϑ (0.9)	10.8041
ϑ (1.0)	90.7555

Observations: As discussed in part II, Euler's method becomes more accurate as we decrease the step-size especially for large values of t. We can assume that at h=0.01 these values are closer to the exact value (if we could find it) than in part b where h = 0.1.

d. Using Euler's method with step-size h=0.001

ϑ (0.8)	5.7905
ϑ (0.9)	13.8022
ϑ (1.0)	∞

Observations: As discussed in part 11, Euler's method becomes more accurate as we decrease the step-size especially for large values of t. We can assume that at h=0.001 these values are closer to the exact value (if we could find it) than in part c where h = 0.01.

e. Explain what is happening

This differential equation solution for this initial value problem has a vertical asymptote at t=1. The $\lim_{t \to 1} f(t) = \infty$

f. Use Euler's method or ode45 to find approximate point t* between 0.9 and 1, where ϑ (t) ceases to be defined

ϑ (0.96)	71.61
ϑ (0.97)	202.22
ϑ (0.975)	849.23

It is difficult to approximate exactly where it becomes undefined. If you zoom in on the graph we can approximate 0.965.

Exhibit 3.1 - See attached screen shot of problem_3.m file for zoom plot

PART IV

Stefan-Boltzman law:

$$\frac{dH}{dt} = -\alpha(H^4 - T^4)$$

Initial Conditions:

$$H_0 = 600^\circ K$$
$$T = 300^\circ K$$
$$\alpha = 5 * 10^{-11\circ} K^{-3} / s$$

Equation:

$$\frac{dH}{dt} = (-5 * 10^{-11})(H^4 - 300^4)$$

a. Use ode45 to determine the temperature of the body after 10 seconds

H(10)=549.8186$^\circ K$

Exhibit 4.1 - See attached screen shot of problem_4.m file for direction field and plot of approximate solution

b. Find the approximate time it takes for the temperature of the body to reach $400^\circ K$

H(t)= $400^\circ K$

t=90.56 seconds

Exhibit 4.2 - See attached screen shot of problem_4.m file for zoom plot

c. What is the limiting temperature of the body? Explain. How long does it take to come within $1^\circ K$ of the limiting temperature.

It makes sense that the temperature of the body can never go below the ambient temperature. In more precise terms the $\lim_{t \to \infty}$ of H is $300^\circ K$ and is a stable equilibrium solution. For values of H greater than $300^\circ K$ the slope will decrease and for values below $300^\circ K$ the slope will increase. Following the theorem of existence and uniqueness these solutions will approach the equilibrium solution but never cross. It takes approximately 863.7 seconds for the body to come within $1^\circ K$ of the ambient temperature.

Exhibit 4.2 - See attached screen shot of problem_4.m file for zoom plot