On the Semantic Expressiveness of Recursive Types

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What and Why?

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- clarifies the design of emerging languages
- better understanding of how to answer language expressiveness questions

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- prove that **iso-recursive** and ($\textit{coinductive}$) $\textit{equi-recursive}$ types have the same semantic expressiveness

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- devise a new proof technique for FAC based on approximate cross-**lan**_guage_ LR

Talk Outline

Recursive Types

Comparing Language Expressiveness

Iso is Co-Equi (is Fix)

Recursive Types

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A1: they are **isomorphic** [Gordon *et al.*'79]

A2: they are *equivalent* [Morris '68]

Iso-recursive Types: $\lambda^{\mu}_{\mathrm{I}}$

$$\mathrm{t} ::= \cdots \mid \mathrm{fold}_{\mu\alpha.\tau}\ \mathrm{t} \mid \mathrm{unfold}_{\mu\alpha.\tau}\ \mathrm{t}$$
$$\mathrm{v} ::= \cdots \mid \mathrm{fold}_{\mu\alpha.\tau}\ \mathrm{v}$$

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Nil $\qquad [2, 1, Nil]$ $\qquad LN \stackrel{\text{def}}{=} \mu\alpha.\ Unit \uplus (Nat \times \alpha)$

$$\mathbf{fold}_{\mathrm{LN}}\ \mathbf{inl}\ \mathbf{unit}$$

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Iso-recursive Typing & Semantics: $\lambda^{\mu}_{\mathrm{I}}$

($\lambda^{\mu}_{\mathrm{I}}$-Type-fold)

$$\frac{\Gamma \vdash \mathrm{t} : \tau[\mu\alpha.\,\tau/\alpha]}{\Gamma \vdash \mathrm{fold}_{\mu\alpha.\tau}\ \mathrm{t} : \mu\alpha.\,\tau}$$

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($\lambda^{\mu}_{\mathrm{I}}$-Eval-fold)

$$\frac{}{\mathrm{unfold}_{\mu\alpha.\tau}\ (\mathrm{fold}_{\mu\alpha.\tau}\ \mathrm{v}) \hookrightarrow_{\mathrm{p}} \mathrm{v}}$$

$Equi$-recursive Types: λ^{μ}_{E}

No term-level annotation

$Equi$-recursive Types: λ_E^μ

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(λ_E^μ-Type-eq ($\doteq$))

$$\frac{\Gamma \vdash t : \mu\alpha.\,\tau \qquad \mu\alpha.\,\tau \doteq \sigma}{\Gamma \vdash t : \sigma}$$

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Definitions of $\doteq$:

- inductive — e.g., [Abadi & Fiore '96]
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Coinductive Type Equality: λ^{μ}_{E}

$$(\doteq\text{-prim})\quad \frac{\iota = \mathit{Unit} \lor \mathit{Bool} \lor \alpha}{\iota \doteq \iota}$$

$$(\doteq\text{-bin})\quad \frac{\star \in \{\rightarrow, \times, \uplus\} \quad \tau_1 \doteq \sigma_1 \quad \tau_2 \doteq \sigma_2}{\tau_1 \star \tau_2 \doteq \sigma_1 \star \sigma_2}$$

$$(\doteq\text{-}\mu_l)\quad \frac{\tau[\mu\alpha.\,\tau/\alpha] \doteq \sigma \quad \tau \text{ contractive in } \alpha}{\mu\alpha.\,\tau \doteq \sigma}$$

$$(\doteq\text{-}\mu_r)\quad \frac{\tau \neq \mu\alpha.\,\tau' \quad \tau \doteq \sigma[\mu\alpha.\,\sigma/\alpha] \quad \sigma \text{ contractive in } \alpha}{\tau \doteq \mu\alpha.\,\sigma}$$

Contractiveness: α are used only after a $\star$

Non-contractiveness: (e.g., $\mu\alpha.\,\alpha$) are value-uninhabited

Knowns and Unknowns between $\boldsymbol{\lambda}^{\boldsymbol{\mu}}_{\mathbf{I}}$ & λ^{μ}_{E}

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- Q: termination $\boldsymbol{\lambda}^{\mu}_{\mathbf{I}}$ $\iff$ termination λ^{μ}_{E}?
- Q: (generally) how to compare relative semantic expressiveness of languages?

Comparing Language Expressiveness

History (applied to $\boldsymbol{\lambda}_{\mathrm{I}}^{\mu}$ & λ_{E}^{μ})

Felleisen ’91, Mitchell ’93

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- Q: does $\mathbf{C}[t]$ behave differently from $C[t]$?

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- our $[\![\cdot]\!]$: identity and erase **fold** / **unfold**

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- semantic differentiation: $\Uparrow$ vs $\Downarrow$

Fully Abstract Compilation (FAC) [Abadi (*et al.*) '99]

- Preservation and reflection of contextual equivalence
- $\vdash [\![\cdot]\!] : \mathsf{FAC} \overset{\text{def}}{=} \forall \mathbf{t_1}, \mathbf{t_2}$

  $\mathbf{t_1} \simeq_{\mathbf{ctx}} \mathbf{t_2} \iff [\![\mathbf{t_1}]\!] \simeq_{\mathrm{ctx}} [\![\mathbf{t_2}]\!]$

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or:

$$(\forall \mathbf{C}. \mathbf{C}[\mathbf{t_1}]\Downarrow \iff \mathbf{C}[\mathbf{t_2}]\Downarrow)$$

$$\Updownarrow$$

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$$\Uparrow \text{simple}$$

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$$\Downarrow \text{hard}$$

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Preservation via Step-Idx LR

[Devriese *et al.*'16]

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ctx. eq. preservation

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$\mathbf{t_1} \gtrsim_{\_} [\![\mathbf{t_1}]\!]$

$\langle\!\langle C \rangle\!\rangle_{\mathbf{n}} \gtrsim_n C$

(1)

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$\mathbf{t_1} \gtrsim_{\_} \llbracket \mathbf{t_1} \rrbracket$

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(1) (3)

$\mathbf{t_2} \lesssim_{\_} \llbracket \mathbf{t_2} \rrbracket$

$\langle\!\langle C \rangle\!\rangle_{\mathbf{n}} \lesssim_{\_} C$

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- Q2: what is an approximate backtranslation $\langle\!\langle \cdot \rangle\!\rangle_n : C \to \mathbf{C}$?

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  - define on C's derivation (but $\triangleq$ is coinductive)
  - approximate! $\langle\!\langle\cdot\rangle\!\rangle \to \langle\!\langle\cdot\rangle\!\rangle_n$
- Since $C\,[\,[\![\mathbf{t}]\!]\,]\Downarrow_n$, n-unfolding of recursive types in C suffices to replicate $\Downarrow\_$ in $\boldsymbol{\lambda}^{\mu}_{\mathbf{I}}$

Approximate Backtranslation Type

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$$\mathbf{BtT}_{\mathbf{0};\tau} \overset{\text{def}}{=} \mathbf{Unit}$$

$$\mathbf{BtT}_{\mathbf{n+1};\tau} \overset{\text{def}}{=} \begin{cases} \mathbf{Unit} \uplus \mathbf{Unit} & \text{if } \tau = \mathit{Unit} \\ (\mathbf{BtT}_{\mathbf{n};\boldsymbol{\tau}} \to \mathbf{BtT}_{\mathbf{n};\boldsymbol{\tau}'}) \uplus \mathbf{Unit} & \text{if } \tau = \tau \to \tau' \\ (\mathbf{BtT}_{\mathbf{n};\boldsymbol{\tau}} \uplus \mathbf{BtT}_{\mathbf{n};\boldsymbol{\tau}'}) \uplus \mathbf{Unit} & \text{if } \tau = \tau \uplus \tau' \\ \mathbf{BtT}_{\mathbf{n+1};\boldsymbol{\tau}'[\boldsymbol{\mu\alpha.\tau}'/\boldsymbol{\alpha}]} \uplus \mathbf{Unit} & \text{if } \tau = \mu\alpha.\,\tau' \end{cases}$$

Backtranslation Example and Relation

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Need a special value relation:

$$\mathcal{V}\,[\![\mathbf{BtT}_{\mathbf{n+1};\tau}]\!] \stackrel{\text{def}}{=} \Big\{(\mathbf{v}, v) \mid \text{either } \mathbf{v} = \textbf{inr unit} \\ \text{or } \tau = Unit \text{ and } \exists \mathbf{v}'.\ \mathbf{v} = \textbf{inl } \mathbf{v}' \text{ and} \\ (\mathbf{v}', v) \in \mathcal{V}\,[\![\mathbf{Unit}]\!]\Big\}$$

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\begin{tikzcd}
 & \lambda^{\mathsf{fx}} \arrow[dl, "\llbracket\cdot\rrbracket^{\lambda^{\mathsf{fx}}}_{\boldsymbol{\lambda}^{\mu}_{\mathbf{I}}}"'] & \\
\boldsymbol{\lambda}^{\mu}_{\mathbf{I}} \arrow[ur, dotted, "\langle\!\langle\cdot\rangle\!\rangle^{\boldsymbol{\lambda}^{\mu}_{\mathbf{I}}}_{\lambda^{\mathsf{fx}}}\, n"'] \arrow[rr, "\llbracket\cdot\rrbracket^{\boldsymbol{\lambda}^{\mu}_{\mathbf{I}}}_{\lambda^{\mu}_{E}}"] & & \lambda^{\mu}_{E} \arrow[ll, dotted, bend left, "\langle\!\langle\cdot\rangle\!\rangle^{\lambda^{\mu}_{E}}_{\boldsymbol{\lambda}^{\mu}_{\mathbf{I}}}\, n"]
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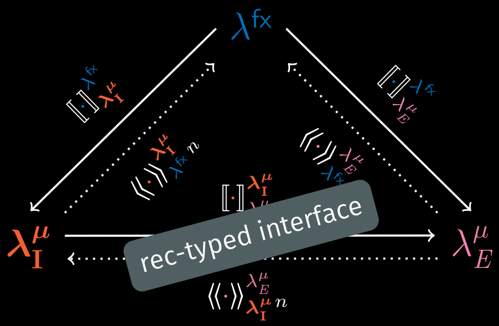

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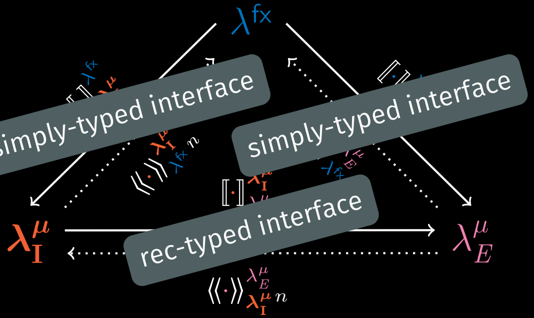

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Questions?

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