On the Semantic Expressiveness of Recursive Types

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19th November 2020

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- clarifies the design of emerging languages
- better understanding of how to answer language expressiveness questions

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- prove that **iso-recursive** and (*coinductive*) *equi-recursive* types have the same semantic expressiveness

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Talk Outline

Recursive Types

Comparing Language Expressiveness

Iso is Co-Equi (is Fix)

Recursive Types

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A1: they are *equivalent* [Morris '68]

A2: they are **isomorphic** [Gordon *et al.*'79]

Iso-recursive Types: $\lambda_{\mathrm{I}}^{\mu}$

$$\mathrm{t} ::= \cdots \mid \mathrm{fold}_{\mu\alpha.\tau}\ \mathrm{t} \mid \mathrm{unfold}_{\mu\alpha.\tau}\ \mathrm{t}$$

$$\mathrm{v} ::= \cdots \mid \mathrm{fold}_{\mu\alpha.\tau}\ \mathrm{v}$$

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Nil $\quad [\mathit{2}, \mathit{1}, Nil]$ $\quad LN \overset{\text{def}}{=} \mu\alpha.\ Unit \uplus (Nat \times \alpha)$

$$\mathrm{fold}_{\mathrm{LN}}\ \mathrm{inl\ unit}$$

$$\mathrm{fold}_{\mathrm{LN}}\ \mathrm{inr}\ \langle 2, \mathrm{fold}_{\mathrm{LN}}\ \mathrm{inr}\ \langle 1, \mathrm{fold}_{\mathrm{LN}}\ \mathrm{inl\ unit} \rangle\rangle$$

Iso-recursive Typing & Semantics: $\lambda^{\mu}_{\mathrm{I}}$

($\lambda^{\mu}_{\mathrm{I}}$-Type-fold)

$$\frac{\Gamma \vdash \mathrm{t} : \tau[\mu\alpha.\,\tau/\alpha]}{\Gamma \vdash \mathrm{fold}_{\mu\alpha.\tau}\ \mathrm{t} : \mu\alpha.\,\tau}$$

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($\lambda^{\mu}_{\mathrm{I}}$-Eval-fold)

$$\frac{}{\mathrm{unfold}_{\mu\alpha.\tau}\ (\mathrm{fold}_{\mu\alpha.\tau}\ \mathrm{v}) \hookrightarrow_{\mathrm{p}} \mathrm{v}}$$

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No term-level annotation

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(λ^{μ}_{E}-Type-eq ($\circeq$))

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Definitions of $\doteq$:

- inductive — e.g., [Abadi & Fiore '96]
- coinductive — e.g., [Cai *et al.* '16]

  (proven to be strictly stronger than the first one wrt expressing type equality)

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Coinductive Type Equality: λ_E^μ

$$\frac{\iota = \mathit{Unit} \lor \mathit{Bool} \lor \alpha}{\iota \triangleq \iota}\ (\triangleq\text{-prim}) \qquad \frac{\star \in \{\rightarrow, \times, \uplus\} \quad \tau_1 \triangleq \sigma_1 \quad \tau_2 \triangleq \sigma_2}{\tau_1 \star \tau_2 \triangleq \sigma_1 \star \sigma_2}\ (\triangleq\text{-bin})$$

$$\frac{\begin{array}{r}\tau[\mu\alpha.\,\tau/\alpha] \triangleq \sigma \\ \tau \text{ contractive in } \alpha\end{array}}{\mu\alpha.\,\tau \triangleq \sigma}\ (\triangleq\text{-}\mu_l) \qquad \frac{\begin{array}{r}\tau \neq \mu\alpha.\,\tau' \\ \tau \triangleq \sigma[\mu\alpha.\,\sigma/\alpha] \\ \sigma \text{ contractive in } \alpha\end{array}}{\tau \triangleq \mu\alpha.\,\sigma}\ (\triangleq\text{-}\mu_r)$$

Contractiveness: α are used only after a $\star$

Non-contractiveness: (e.g., $\mu\alpha.\,\alpha$) are value-uninhabited

Knowns and Unknowns between $\boldsymbol{\lambda}^{\mu}_{\mathbf{I}}$ & λ^{μ}_{E}

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- Q: termination $\boldsymbol{\lambda}^{\mu}_{\mathbf{I}}$ $\iff$ termination λ^{μ}_{E}?
- Q: (generally) how to compare relative semantic expressiveness of languages?

Comparing Language Expressiveness

History (applied to $\boldsymbol{\lambda}^{\mu}_{\mathrm{I}}$ & λ^{μ}_{E})

Felleisen ’91, Mitchell ’93

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- our $[\![\cdot]\!]$: identity and erase **fold** / **unfold**

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- semantic differentiation: $\Uparrow$ vs $\Downarrow$

Fully Abstract Compilation (FAC) [Abadi (*et al.*) '99]

- Preservation and reflection of
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or:

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$$\Updownarrow$$

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FAC for Language Expressiveness

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FAC for Language Expressiveness

$$(\forall \mathbf{C}.\mathbf{C}[\mathbf{t_1}]\Downarrow \iff \mathbf{C}[\mathbf{t_2}]\Downarrow)$$

$$\Uparrow \text{simple}$$

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$$\Downarrow \text{hard}$$

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Preservation via Step-Idx LR

[Devriese *et al.*'16]

$$\mathbf{t}_1 \simeq_{\mathrm{ctx}} \mathbf{t}_2$$

ctx. eq. preservation

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$$C\Big[[\![\mathbf{t_1}]\!]\Big]\Downarrow_n \quad \stackrel{?}{\Rightarrow} \quad C\Big[[\![\mathbf{t_2}]\!]\Big]\Downarrow_{\_}$$

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$$? \gtrsim_n C$$

(1)

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$\mathbf{t} \gtrsim_{\_}$ and $\lesssim_{\_} [\![\mathbf{t}]\!]$

$\langle\!\langle C \rangle\!\rangle_{\mathbf{n}} \gtrsim_{\_}$ and $\lesssim_{\_} C$

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$$\langle\!\langle C \rangle\!\rangle_{\mathbf{n}}\left[\mathbf{t_1}\right] \Downarrow_{\_} \underset{(2)}{\Rightarrow} \langle\!\langle C \rangle\!\rangle_{\mathbf{n}}\left[\mathbf{t_2}\right] \Downarrow_{\_}$$

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$$\begin{tikzcd} \langle\!\langle C \rangle\!\rangle_{\mathbf{n}}[\mathbf{t_1}] \Downarrow_{\_} \arrow[r, Rightarrow, "(2)"'] & \langle\!\langle C \rangle\!\rangle_{\mathbf{n}}[\mathbf{t_2}] \Downarrow_{\_} \arrow[d, "(3)"] \\ C\big[[\![\mathbf{t_1}]\!]\big] \Downarrow_{n} \arrow[u, "(1)"] \arrow[r, Rightarrow, "?"] & C\big[[\![\mathbf{t_2}]\!]\big] \Downarrow_{\_} \end{tikzcd}$$

$\mathbf{t_1} \gtrsim_{\_} [\![\mathbf{t_1}]\!]$
$\langle\!\langle C \rangle\!\rangle_{\mathbf{n}} \gtrsim_{n} C$

$\mathbf{t_2} \lesssim_{\_} [\![\mathbf{t_2}]\!]$
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$$[\![\mathbf{t_1}]\!] \overset{?}{\simeq}_{\mathrm{ctx}} [\![\mathbf{t_2}]\!]$$

ctx. eq. preservation

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- Since $C\,[\,[\![\mathbf{t}]\!]\,] \Downarrow_n$, n-unfolding of recursive types in C suffices to replicate $\Downarrow\_$ in $\boldsymbol{\lambda}^{\mu}_{\mathbf{I}}$

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$$\mathbf{BtT}_{\mathbf{0};\tau} \stackrel{\text{def}}{=} \mathbf{Unit}$$

$$\mathbf{BtT}_{\mathbf{n+1};\tau} \stackrel{\text{def}}{=} \begin{cases} \mathbf{Unit} \uplus \mathbf{Unit} & \text{if } \tau = \mathit{Unit} \\ (\mathbf{BtT}_{\mathbf{n};\boldsymbol{\tau}} \to \mathbf{BtT}_{\mathbf{n};\boldsymbol{\tau}'}) \uplus \mathbf{Unit} & \text{if } \tau = \tau \to \tau' \\ (\mathbf{BtT}_{\mathbf{n};\boldsymbol{\tau}} \uplus \mathbf{BtT}_{\mathbf{n};\boldsymbol{\tau}'}) \uplus \mathbf{Unit} & \text{if } \tau = \tau \uplus \tau' \\ \mathbf{BtT}_{\mathbf{n+1};\boldsymbol{\tau}'[\boldsymbol{\mu\alpha}.\boldsymbol{\tau}'/\boldsymbol{\alpha}]} \uplus \mathbf{Unit} & \text{if } \tau = \mu\alpha.\,\tau' \end{cases}$$

Backtranslation Example and Relation

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Need a special value relation:

$$\mathcal{V}\left[\!\left[\mathbf{BtT}_{\mathbf{n+1};\tau}\right]\!\right] \stackrel{\text{def}}{=} \Big\{(\mathbf{v}, v) \mid \text{either } \mathbf{v} = \mathbf{inr\ unit} \text{ or } \tau = Unit \text{ and } \exists \mathbf{v}'.\ \mathbf{v} = \mathbf{inl\ v}' \text{ and } (\mathbf{v}', v) \in \mathcal{V}[\![\mathbf{Unit}]\!]\Big\}$$

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Our Contributions, Visually

$$\boldsymbol{\lambda}^{\mu}_{\mathbf{I}} \underset{\langle\!\langle \cdot \rangle\!\rangle^{\lambda^{\mu}_{E}}_{\boldsymbol{\lambda}^{\mu}_{\mathbf{I}}} n}{\overset{[\![\cdot]\!]^{\boldsymbol{\lambda}^{\mu}_{\mathbf{I}}}_{\lambda^{\mu}_{E}}}{\rightleftarrows}} \lambda^{\mu}_{E}$$

$[\![\cdot]\!]^{\boldsymbol{\lambda}^{\mu}_{\mathbf{I}}}_{\lambda^{\mu}_{E}}$ erases $\mathbf{fold}$/$\mathbf{unfold}$, $\langle\!\langle \cdot \rangle\!\rangle^{\lambda^{\mu}_{E}}_{\boldsymbol{\lambda}^{\mu}_{\mathbf{I}}} n$ is approximate

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 & \lambda^{\mathsf{fx}} \arrow[dl, "{[\![\cdot]\!]^{\lambda^{\mathsf{fx}}}_{\boldsymbol{\lambda}^{\mu}_{\mathbf{I}}}}"'] & \\
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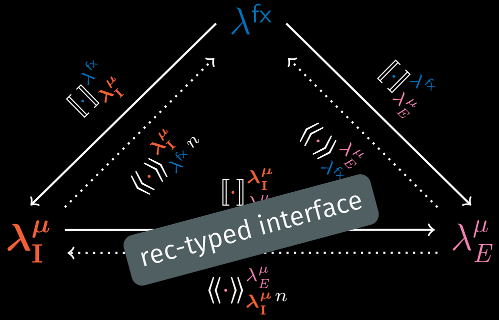

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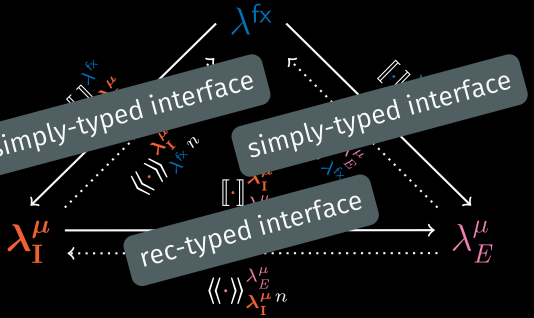

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