Assignment 1

Bhupendra Kumar Agarwal (13115036)

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Contents

1 Merge Sort

1.1 Algorithm

```
# To sort A call MergeSort(A,0,len(A)-1)
def MergeSort(A,p,r):
    if p<r:
        q=(p+r)/2
        MergeSort(A,p,q)
        MergeSort(A,q+1,r)
        Merge(A,p,q,r)
```

```
# function to merge A[p:q] and A[q+1:r]
def Merge(A,p,q,r):
    n1 = q-p+1
    n2 = r-q
    L = [None]*(n1+1)
    R = [None]*(n2+1)
```

```
14        for i in range(0,n1,1):
15            L[i]=A[p+i]
16        for j in range(0,n2,1):
17            R[j]=A[q+j+1]
18        L[n1] = float("inf")
19        R[n2] = float("inf")
20        i=0
21        j=0
22        for k in range(p,r+1,1):
23            if L[i] <= R[j]:
24                A[k]=L[i]
25                i=i+1
26            else:
27                A[k]=R[j]
28                j=j+1
```

1.2 Correctness

Defining the following loop invariant:
At the start of each iteration of the for loop of lines 22-28,

1. the subarray $A[p..k-1]$ contains the $k-p$ smallest elements of $L[0..n1]$ and $R[0..n2]$, in sorted order.
2. Moreover, $L[i]$ and $R[j]$ are the smallest elements of their arrays that have not been copied back into A.

Initialization: 1) $k = p$, A[p..k-1] is empty: k-p=0 smallest elements. 2)$i = j = 1$, L[1] and R[1] are the smallest elements that have not copied back to A.

Induction: Suppose L[i]≤ R[j], then L[i] is the smallest element not yet copied back in A and line 24 do that. As subarray A[p..k-1] contained the k-p smallest element initially now it has k-p+1 smallest elements and then k is incremented to hold the 1st point of loop invariant. i is also incremented which maintain 2nd point of loop invariant. Similarly is L[i]>R[i] appropriate actions are taken to maintain the loop invariant.

Termination: k=r+1, at this point A[p..r] is sorted and contain r+1-p smallest element of L and R which are of A. So now we have sorted array A.

1.3 Analyzing Merge Sort

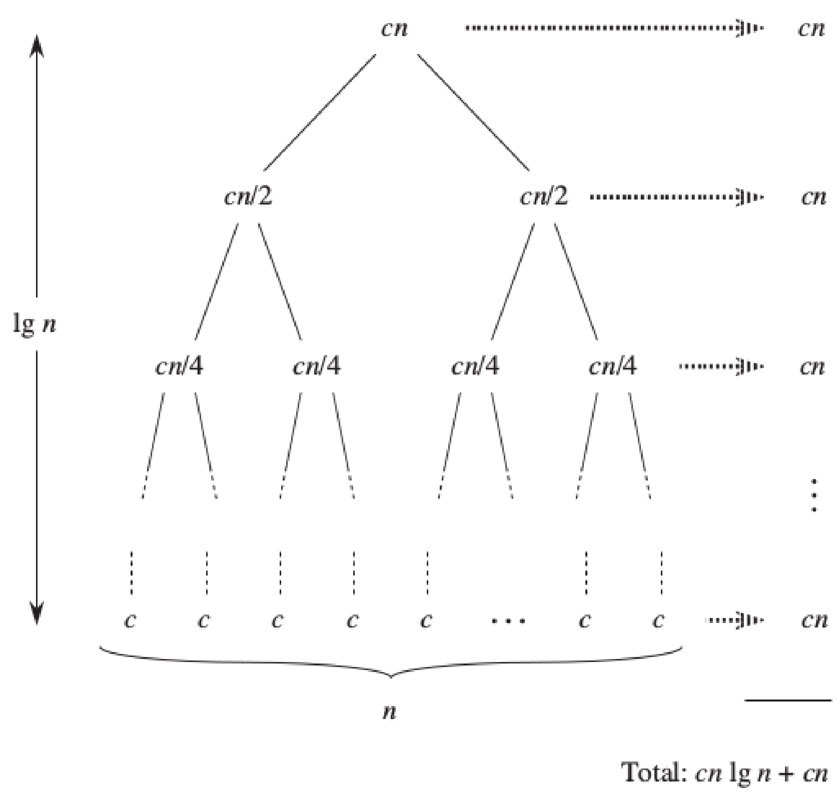
There are three steps:

1. Divide: Ø(1)
2. Conquer: $2T(n/2), solving two sub-problems of n/2 sizes.
3. Combine: Ø(n), In Merge for loop from lines 22 to 28 runs n times.

which gives

$$T(n) = \begin{cases} \theta(1), & \text{if } n = 1. \\ 2\theta(n/2) + \theta(n), & \text{if } n > 1. \end{cases} \tag{1}$$

using recursion tree



thus T(n) = O(n log n).

2 Quick Sort

2.1 Algorithm

```
1  # To sort A call (A,0,len(A)-1)
2  def QuickSort(A,p,r):
3      if p<r:
4          q=Partion(A,p,r)
5          QuickSort(A,p,q-1)
6          QuickSort(A,q+1,r)
```

```
 7  # function to partion A and place pivot at an appropriate position
 8  def Partion(A,p,r):
 9      x=A[r]
10      i=p-1
11      for j in range(p,r):
12          if A[j]<=x:
13              i=i+1
14              A[i],A[j]=A[j],A[i]
15      A[i+1],A[r] = A[r],A[i+1]
16      return i+1
```

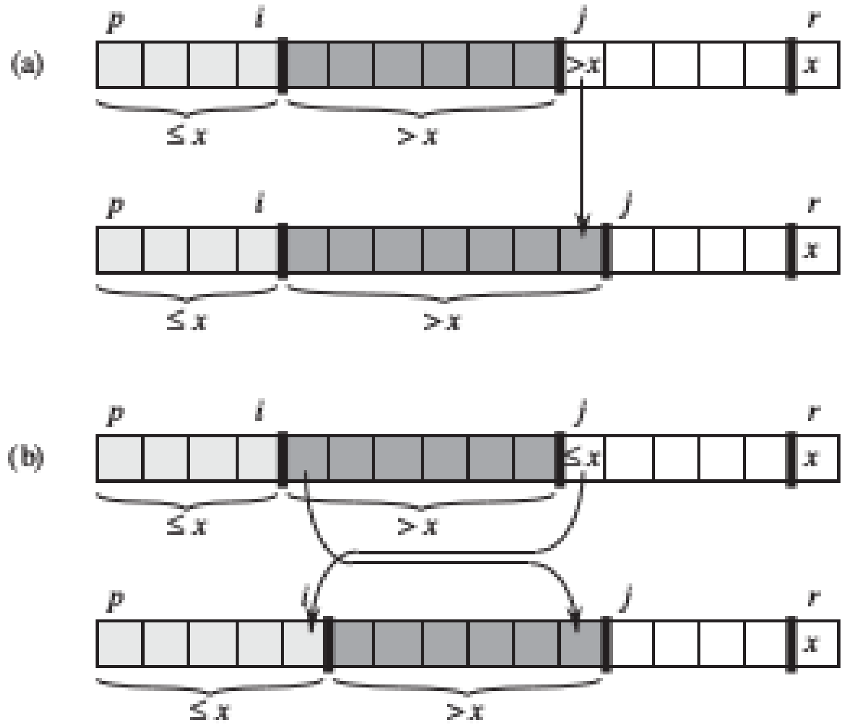
2.2 Correctness

loop invariant for line 11-15, let k be an index $0 \leq k \leq n$ then

1. if $p \leq k \leq i$ then $A[k] \leq x$
2. if $i+1 \leq k \leq j-1$ then $A[k] > x$
3. if $k=r$ then $A[k] = x$

Initialization: initially i=p-1 and j=p then 1) no values between p and i implies $A[k] \leq x$, similarly between i+1 and j-1 no values implies $A[k]>x$, Also $A[r] = x$.

Induction:



a) Figure (a) shows what happens when A[j] > x;the only action in the loop is to increment j . After j is incremented, condition 2 holds for A[j-1] and all other entries remain unchanged.
b) Figure (b) shows what happens when A[j] $\leq$ x; the loop increments i, swaps A[i] and A[j], and then increments j . Because of the swap, we now have that A[i] $\leq$ x, and condition 1 is satisfied. Similarly, we also have that A[j]> x, since the item that was swapped into A[j-1]is, by the loop invariant, greater than x.

Termination: when j=r and at that point. 1) A[p. . . i] $\leq$ x. 2) $A[i+1...r-1] > x$. 3) A[r]=x.At last we exchange pivot with leftmost element greater then x and move it to its correct position.

2.3 Analyzing Quick Sort

Running time of Partition(A,p,r) = θ (n) as for loop runs for n=r-p+1 times.
1 Worst case partitioning : Sub-problems have 0 and (n-1) size.

$$T(n) = T(n-1) + \theta(n)$$

whose solution is

$$T(n) = \theta(n^2) \tag{2}$$

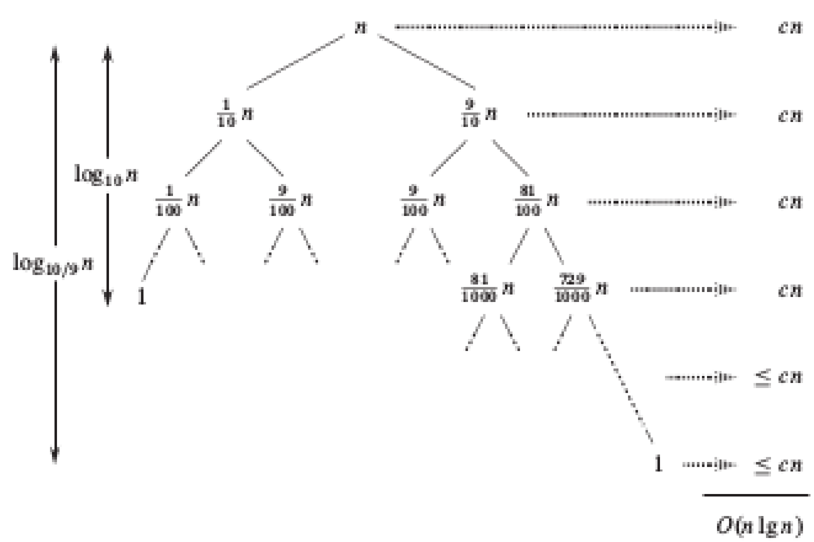
2 Best Case Partitioning: when both problems have size of n/2 then

$$T(n) = 2T(n/2) + \theta(n)$$

whose solution by master theorem is

$$T(n) = (n \log n) \tag{3}$$

3 Average Case: if there is some partitioning lets us say in 9/10 and 1/10 then



which gives

$$T(n) = (n \log n) \tag{4}$$