勒让德变换

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1 基本内容

1.1 最大值定义

设 $L=L(a_1,a_2,\cdots,a_i,\cdots,a_n)$ 对 a_i 是凸函数。则

$$y=A_ia_i-L(a_1,a_2,\cdots,a_i,\cdots,a_n)$$

是凹函数，在 y 最大值的位置，有

$$\frac{\partial y}{\partial a_i}=A_i-\frac{\partial L}{\partial a_i}=0,\quad A_i=\frac{\partial L}{\partial a_i}$$

这样就有从 a_i 到 $A_i=\frac{\partial L}{\partial a_i}$ 的一个映射。由于 $L=L(a_1,a_2,\cdots,a_i,\cdots,a_n)$ 对 a_i 是凸函数，故存在逆映射 $a_i=a_i(a_1,a_2,\cdots,A_i,\cdots,a_n)$。定义 L 的勒让德变换

$$\begin{aligned}L^*(a_1,a_2,\cdots,A_i,\cdots,a_n)=&a_i(a_1,a_2,\cdots,A_i,\cdots,a_n)A_i\\&-L\left(a_1,a_2,\cdots,a_i(a_1,a_2,\cdots,A_i,\cdots,a_n),\cdots,a_n\right)\end{aligned}$$

从上述表达式可以看出来

$$\begin{aligned}L(a_1,a_2,\cdots,a_i,\cdots,a_n)=&a_iA_i(a_1,a_2,\cdots,a_i,\cdots,a_n)\\&-L^*\left(a_1,a_2,\cdots,A_i(a_1,a_2,\cdots,a_i,\cdots,a_n),\cdots,a_n\right)\end{aligned}$$

从而 $(L^*)^*=L$。

1.2 反函数定义

另一种定义是通过反函数定义，即 $L(a_1,a_2,\cdots,a_i,\cdots,a_n)$ 与 $L^*(a_1,a_2,\cdots,A_i,\cdots,a_n)$ 的一阶导互为反函数

$$\frac{\partial L}{\partial a_i}(x)=\left(\frac{\partial L^*}{\partial A_i}\right)^{-1}(x),\quad \frac{\partial L^*}{\partial A_i}(x)=\left(\frac{\partial L}{\partial a_i}\right)^{-1}(x)$$
$$\Longrightarrow \frac{\partial L}{\partial a_i}=A_i(a_1,a_2,\cdots,a_i,\cdots,a_n),\quad \frac{\partial L^*}{\partial A_i}=a_i(a_1,a_2,\cdots,A_i,\cdots,a_n)$$

L 与 L^* 互为勒让德变换。

由于

$$\frac{\partial}{\partial A_i}(a_iA_i-L(a_1,a_2,\cdots,a_i,\cdots,a_n))=a_i+A_i\frac{\partial a_i}{\partial A_i}-\frac{\partial L}{\partial a_i}\frac{\partial a_i}{\partial A_i}=a_i=\frac{\partial L^*}{\partial A_i}$$

从而可取

$$L^*=A_ia_i-L$$

2 粗糙表达

$$dL = \sum_i \frac{\partial L}{\partial a_i} da_i \overset{def}{=} \sum_i A_i da_i$$
$$A_i = A_i(a_1, a_2, \cdots, a_i, \cdots, a_n), \quad a_i = a_i(a_1, a_2, \cdots, A_i, \cdots, a_n)$$

$$d(A_i a_i) - dL = A_i da_i + a_i dA_i - \sum_i A_i da_i = -\sum_{j \neq i} A_j da_j + a_i dA_i \overset{def}{=} dL^*$$

3 简单推论

$$\frac{\partial^2 L}{\partial a_i \partial a_j} = \frac{\partial^2 L}{\partial a_j \partial a_i} \Longrightarrow \frac{\partial A_j}{\partial a_i} = \frac{\partial A_i}{\partial a_j}$$
$$\frac{\partial^2 L^*}{\partial A_i \partial a_j} = \frac{\partial^2 L^*}{\partial a_j \partial A_i} \Longrightarrow \frac{\partial a_i}{\partial a_j} = -\frac{\partial A_j}{\partial A_i}$$