雅可比行列式

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1 行列式与体积

假设矢量 $\vec{a}_1, \vec{a}_2, \cdot, \vec{a}_n$ 张成体 Ω，也就是

$$\Omega = \{\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \cdots + \alpha_n\vec{a}_n | \alpha_i \in [0,1]\}$$

体积满足这样的性质，将张成 Ω 的矢量中的一个 $\vec{a}_i$ 加上其中另一个矢量 $\vec{a}_j$ 的任意倍数，新的体积等于旧的体积，即

$$\Omega' = \{\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \cdots + \alpha_i(\vec{a}_i + k\vec{a}_j) + \cdots + \alpha_n\vec{a}_n | \alpha_i \in [0,1]\},\ k \in \mathbb{R}$$

$$V(\Omega') = V(\Omega)$$

体积还满足

$$\Omega' = \{\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \cdots + \alpha_i k\vec{a}_i + \cdots + \alpha_n\vec{a}_n | \alpha_i \in [0,1]\},\ k \in \mathbb{R}$$

$$V(\Omega') = kV(\Omega)$$

从而在某个坐标下，将各基矢表达为列矢量，则体积和对应行列式的绝对值 (取绝对值是因为体积没有反对称性) 满足相同的关系

$$V(\Omega) \sim |det(\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n)|$$

故通过“定标”之后，行列式可以用来计算体积。一般来说选取的坐标都是正交归一的，在这种情况下，由于单位阵的行列式为 1，故

$$V(\Omega) = |det(\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n)|$$

2 雅可比行列式

雅可比行列式在变量代换时出现。单变量函数的导数可以看作是线元的比值，雅可比行列式 (的绝对值) 则是变量代换前后参数空间体元的比值。或者说是测度的变化。

考虑坐标变换

$$(x_1, x_2, x_3, \cdots, x_n) \to (u_1, u_2, u_3, \cdots, u_n)$$

雅可比行列式为

$$\frac{\partial(u_1, u_2, \cdots, u_n)}{\partial(x_1, x_2, \cdots, x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

等式右边对应的矩阵显然描述这样一种关系

$$\begin{pmatrix} du_1 \\ du_2 \\ \vdots \\ du_n \end{pmatrix} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{pmatrix}$$

考虑更多的微矢量，则

$$\begin{pmatrix} (du_1)^{(1)} & (du_1)^{(2)} & \cdots & (du_1)^{(n)} \\ (du_2)^{(1)} & (du_2)^{(2)} & \cdots & (du_2)^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ (du_n)^{(1)} & (du_n)^{(2)} & \cdots & (du_n)^{(n)} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} (dx_1)^{(1)} & (dx_1)^{(2)} & \cdots & (dx_1)^{(n)} \\ (dx_2)^{(1)} & (dx_2)^{(2)} & \cdots & (dx_2)^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ (dx_n)^{(1)} & (dx_n)^{(2)} & \cdots & (dx_n)^{(n)} \end{pmatrix}$$

取行列式即得

$$dV_u = \left| \frac{\partial(u_1, u_2, \cdots, u_n)}{\partial(x_1, x_2, \cdots, x_n)} \right| dV_x$$

3 常用关系

逆变换的雅可比行列式 ：

由雅可比矩阵描述的关系容易看出

$$\begin{pmatrix} du_1 \\ du_2 \\ \vdots \\ du_n \end{pmatrix} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{pmatrix} \begin{pmatrix} du_1 \\ du_2 \\ \vdots \\ du_n \end{pmatrix}$$

$$\frac{\partial(u_1, u_2, \cdots, u_n)}{\partial(x_1, x_2, \cdots, x_n)} \frac{\partial(x_1, x_2, \cdots, x_n)}{\partial(u_1, u_2, \cdots, u_n)} = 1$$

链式法则 ：

由雅可比矩阵描述的关系也容易得到

$$\frac{\partial(u_1, u_2, \cdots, u_n)}{\partial(x_1, x_2, \cdots, x_n)} \frac{\partial(x_1, x_2, \cdots, x_n)}{\partial(y_1, y_2, \cdots, y_n)} = \frac{\partial(u_1, u_2, \cdots, u_n)}{\partial(y_1, y_2, \cdots, y_n)}$$

部分冗余变量 ：

$$\frac{\partial(u_1,u_2,\cdots,u_n,y_1,y_2,\cdots,y_m)}{\partial(x_1,x_2,\cdots,x_n,y_1,y_2,\cdots,y_m)}=\begin{vmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} & \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} & \cdots & \frac{\partial u_1}{\partial y_m}\\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} & \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} & \cdots & \frac{\partial u_2}{\partial y_m}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} & \frac{\partial u_n}{\partial y_1} & \frac{\partial u_n}{\partial y_2} & \cdots & \frac{\partial u_n}{\partial y_m}\\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1\end{vmatrix}$$

$$=\frac{\partial(u_1,u_2,\cdots,u_n)}{\partial(x_1,x_2,\cdots,x_n)}$$