电动力学条目

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目录

1 一些数学

梯度、散度、旋度：

$$d\phi \triangleq \nabla\phi \cdot d\vec{\boldsymbol{l}}$$

$$\nabla \cdot \vec{\boldsymbol{F}} \triangleq \lim_{V\to 0} \frac{\oint_{\partial V} d\vec{\boldsymbol{\sigma}} \cdot \vec{\boldsymbol{F}}}{V}$$

$$\hat{\boldsymbol{n}} \cdot (\nabla \times \vec{\boldsymbol{F}}) \triangleq \lim_{\Sigma\to 0} \frac{\oint_{\partial\Sigma} d\vec{\boldsymbol{l}} \cdot \vec{\boldsymbol{F}}}{\Sigma}$$

迷向张量：

$$\epsilon_{ijk} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix}$$

$$det(A) = \epsilon_{ijk} A_{1i} A_{2j} A_{3k}$$

$$\epsilon_{lmn} det(A) = \epsilon_{ijk} A_{li} A_{mj} A_{nk}$$

$$\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

$$\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = \epsilon_{ijk} a_i b_j c_k$$

张量：

$$T'_{i_1\cdots i_n} = \lambda_{i_1 j_1} \cdots \lambda_{i_n j_n} T_{j_1 \cdots j_n}$$

$$I_{ij} = \delta_{ij} = I'_{ij}$$

$$T_{ij} = \frac{T_{ij} + T_{ji}}{2} + \frac{T_{ij} - T_{ji}}{2}$$

$$T_{ij} = T_{ji}, S_{ij} = -S_{ji} \Longrightarrow T_{ij} S_{ij} = 0$$

$$det(\overleftrightarrow{\boldsymbol{I}} + \vec{\boldsymbol{f}}\vec{\boldsymbol{g}}) = 1 + \vec{\boldsymbol{f}} \cdot \vec{\boldsymbol{g}}$$

$$\overleftrightarrow{\boldsymbol{T}} : \overleftrightarrow{\boldsymbol{S}} = T_{ij} S_{ji} = tr(TS) = \overleftrightarrow{\boldsymbol{S}} : \overleftrightarrow{\boldsymbol{T}}$$

$$\overleftrightarrow{\boldsymbol{T}} : \overleftrightarrow{\boldsymbol{I}} = tr(T)$$

$$\overleftrightarrow{\boldsymbol{I}} \times \vec{\boldsymbol{f}} = \vec{\boldsymbol{f}} \times \overleftrightarrow{\boldsymbol{I}}$$

$$T_{ij} = T_{ji} \Longrightarrow (\nabla \cdot \overleftrightarrow{\boldsymbol{T}}) \times \vec{\boldsymbol{r}} = \nabla \cdot (\overleftrightarrow{\boldsymbol{T}} \times \vec{\boldsymbol{r}})$$

$$S_{ij} = S_{ji} \Rightarrow tr(S) = 0$$

泰勒展开：

$$\vec{\boldsymbol{f}}(\vec{\boldsymbol{r}} + \vec{\boldsymbol{\epsilon}}) = e^{\vec{\boldsymbol{\epsilon}} \cdot \nabla} \vec{\boldsymbol{f}}(\vec{\boldsymbol{r}})$$

积分公式：

$$\int_V dV \nabla = \oint_{\partial_V} d\vec{\boldsymbol{\sigma}}$$

$$\int_\Sigma (d\vec{\boldsymbol{\sigma}} \times \nabla) = \oint_{\partial_\Sigma} d\vec{\boldsymbol{l}}$$

正交曲线坐标系：

$$h_a \triangleq |\frac{\partial \overrightarrow{\boldsymbol{r}}}{\partial u_a}|, H = h_1 h_2 h_3$$

$$\begin{aligned}
d\overrightarrow{r} &= \hat{\boldsymbol{r}}dr + r\hat{\boldsymbol{\theta}}d\theta + r\sin\theta\hat{\boldsymbol{\phi}}d\phi \\
&(h_r = 1, h_\theta = r, h_\phi = r\sin\theta, H = r^2\sin\theta) \\
&= \hat{\boldsymbol{s}}ds + s\hat{\boldsymbol{\phi}}d\phi + \hat{\boldsymbol{z}}dz \\
&(h_s = 1, h_\phi = s, h_z = 1, H = s)
\end{aligned}$$

$$\nabla\phi = \sum_a \frac{\hat{\boldsymbol{u}}_a}{h_a}\frac{\partial\phi}{\partial u_a}$$

$$\begin{cases}
\nabla u_a = \dfrac{\hat{\boldsymbol{u}}_a}{h_a} \\
\nabla \times \dfrac{\hat{\boldsymbol{u}}_a}{h_a} = 0 \\
\nabla \cdot \dfrac{h_a\hat{\boldsymbol{u}}_a}{H} = 0
\end{cases}$$

$$\nabla^2\phi = \nabla\cdot\nabla\phi = \sum_a \frac{1}{H}\frac{\partial}{\partial u_a}(\frac{H}{h_a^2}\frac{\partial\phi}{\partial u_a})$$

$$\nabla\cdot\overrightarrow{\boldsymbol{f}} = \frac{1}{H}\Big[\frac{\partial}{\partial u_1}(h_2h_3f_1) + \frac{\partial}{\partial u_2}(h_1h_3f_2) + \frac{1}{H}(h_1h_2f_3)\Big]$$

$$\nabla\times\overrightarrow{\boldsymbol{f}} = \frac{1}{H}\begin{vmatrix}
h_1\hat{\boldsymbol{u}}_{\boldsymbol{1}} & h_2\hat{\boldsymbol{u}}_{\boldsymbol{2}} & h_3\hat{\boldsymbol{u}}_{\boldsymbol{3}} \\
\dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\
h_1f_1 & h_2f_2 & h_3f_3
\end{vmatrix}$$

一些求导：

$$\nabla\times\frac{\hat{\boldsymbol{r}}}{r^2} = 0, \nabla\cdot\frac{\hat{\boldsymbol{r}}}{r^2} = 4\pi\delta(\overrightarrow{r})$$

$$\nabla\mathbb{R} = \hat{\mathbb{R}}, \nabla\frac{1}{\mathbb{R}} = -\frac{\hat{\mathbb{R}}}{\mathbb{R}}, \nabla^2\frac{\hat{\mathbb{R}}}{\mathbb{R}^2} = 4\pi\delta(\mathbb{R}), \nabla\cdot\frac{\hat{\mathbb{R}}}{\mathbb{R}} = \frac{1}{\mathbb{R}^2}$$

$$\begin{aligned}
Helmholtz : \overrightarrow{\boldsymbol{F}} &= -\nabla\phi + \nabla\times\overrightarrow{\boldsymbol{A}} \\
&= -\nabla\frac{1}{4\pi}\int dV'\frac{\nabla'\cdot\overrightarrow{\boldsymbol{F}}(\overrightarrow{\boldsymbol{r'}})}{\mathbb{R}} + \nabla\times\frac{1}{4\pi}\int dV'\frac{\nabla'\times\overrightarrow{\boldsymbol{F}}(\overrightarrow{\boldsymbol{r'}})}{\mathbb{R}}
\end{aligned}$$

一些积分：

$$\frac{1}{4\pi}\int_{r'\le a} dV'\frac{\hat{\mathbb{R}}}{\mathbb{R}^2} = \frac{1}{4\pi}\int_{r'\le a} dV\frac{-\hat{\mathbb{R}}}{\mathbb{R}^2} = \begin{cases}
r < a : \dfrac{\overrightarrow{\boldsymbol{r}}}{3} \\
\\
r > a : \dfrac{a^3\overrightarrow{\boldsymbol{r}}}{3r^3}
\end{cases}$$

$$\int \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}d\Omega = \frac{4\pi}{3}\overleftrightarrow{\boldsymbol{I}}$$

$$(\int \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}d\Omega : \hat{\boldsymbol{x}}_i\hat{\boldsymbol{x}}_j = \frac{4\pi}{3}\delta_{ij})$$

$$\int d\Omega\alpha_i\alpha_j = \frac{4\pi}{3}\delta_{ij}$$

$$(\alpha_1 = \sin\theta\cos\phi, \quad \alpha_2 = \sin\theta\sin\phi, \quad \alpha_3 = \cos\theta)$$

$$\int d\Omega\alpha_i\alpha_k\alpha_m\alpha_l = \frac{4\pi}{15}(\delta_{ik}\delta_{ml} + \delta_{im}\delta_{kl} + \delta_{il}\delta_{km})$$

上式按照四个指标相同、三个指标相同、两个指标相同 (另两个不同)、两两相同来验证。

2 电磁场

麦克斯韦方程：

真空中：

$$\begin{cases} \nabla\cdot\overrightarrow{\boldsymbol{E}} = \dfrac{\rho}{\epsilon_0} & (EG) \\ \nabla\cdot\overrightarrow{\boldsymbol{B}} = 0 & (BG) \\ \nabla\times\overrightarrow{\boldsymbol{E}} = -\partial_t\overrightarrow{\boldsymbol{B}} & (F) \\ \nabla\times\overrightarrow{\boldsymbol{B}} = \mu_0\overrightarrow{\boldsymbol{j}} + \mu_0\epsilon_0\partial_t\overrightarrow{\boldsymbol{E}} & (A/M) \end{cases}$$

$$\begin{cases} \hat{\boldsymbol{n}}\cdot(\overrightarrow{\boldsymbol{E}_2} - \overrightarrow{\boldsymbol{E}_1}) = \dfrac{\sigma}{\epsilon_0} \\ \hat{\boldsymbol{n}}\cdot(\overrightarrow{\boldsymbol{B}_2} - \overrightarrow{\boldsymbol{B}_1}) = 0 \\ \hat{\boldsymbol{n}}\times(\overrightarrow{\boldsymbol{E}_2} - \overrightarrow{\boldsymbol{E}_1}) = 0 \\ \hat{\boldsymbol{n}}\times(\overrightarrow{\boldsymbol{B}_2} - \overrightarrow{\boldsymbol{B}_1}) = \mu_0\overrightarrow{\boldsymbol{K}} \end{cases}$$

介质中：

$$\overrightarrow{\boldsymbol{D}} \triangleq \epsilon_0\overrightarrow{\boldsymbol{E}} + \overrightarrow{\boldsymbol{P}}$$

$$\overrightarrow{\boldsymbol{H}} \triangleq \frac{\overrightarrow{\boldsymbol{B}}}{\mu_0} - \overrightarrow{\boldsymbol{M}}$$

$$\begin{cases} \nabla\cdot\overrightarrow{\boldsymbol{D}} = \rho_0 & (EG) \\ \nabla\cdot\overrightarrow{\boldsymbol{B}} = 0 & (BG) \\ \nabla\times\overrightarrow{\boldsymbol{E}} = -\partial_t\overrightarrow{\boldsymbol{B}} & (F) \\ \nabla\times\overrightarrow{\boldsymbol{H}} = \overrightarrow{\boldsymbol{j}_0} + \partial_t\overrightarrow{\boldsymbol{D}} & (A/M) \end{cases}$$

$$\begin{cases} \hat{\boldsymbol{n}}\cdot(\overrightarrow{\boldsymbol{D}_2} - \overrightarrow{\boldsymbol{D}_1}) = \sigma_0 \\ \hat{\boldsymbol{n}}\cdot(\overrightarrow{\boldsymbol{B}_2} - \overrightarrow{\boldsymbol{B}_1}) = 0 \\ \hat{\boldsymbol{n}}\times(\overrightarrow{\boldsymbol{E}_2} - \overrightarrow{\boldsymbol{E}_1}) = 0 \\ \hat{\boldsymbol{n}}\times(\overrightarrow{\boldsymbol{H}_2} - \overrightarrow{\boldsymbol{H}_1}) = \overrightarrow{\boldsymbol{K}_0} \end{cases}$$

电磁势：

$$\overrightarrow{\boldsymbol{E}} = -\nabla\phi - \partial_t\overrightarrow{\boldsymbol{A}}, \overrightarrow{\boldsymbol{B}} = \nabla\times\overrightarrow{A}$$

规范变换：

$$\begin{cases}\phi' = \phi - \partial_t \psi \\ \overrightarrow{\boldsymbol{A}'} = \overrightarrow{\boldsymbol{A}} + \nabla\psi\end{cases}$$

$$\oint_C \overrightarrow{\boldsymbol{A}'} \cdot d\overrightarrow{\boldsymbol{l}} = \oint_C \overrightarrow{\boldsymbol{A}} \cdot d\overrightarrow{\boldsymbol{l}}$$

库伦规范： $\nabla \cdot \overrightarrow{\boldsymbol{A}} = 0$

洛伦兹规范： $\nabla \cdot \overrightarrow{\boldsymbol{A}} + \frac{1}{c^2}\partial_t \phi = 0$

势方程：

$$L \triangleq \nabla \cdot \overrightarrow{\boldsymbol{A}} + \frac{1}{c^2}\partial_t \phi, \quad \Box \triangleq \nabla^2 - \frac{1}{c^2}\partial_t^2 (\text{d'Alembert 算子})$$

$$\begin{cases}\Box\phi + \partial_t L = -\frac{\rho}{\epsilon_0} \\ \Box\overrightarrow{A} - \nabla L = -\mu_0 \overrightarrow{\boldsymbol{j}}\end{cases}$$

真空中的守恒律：

能量守恒：

$$w \triangleq \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2 = \frac{1}{2}\epsilon_0 (E^2 + c^2 B^2)$$

$$\overrightarrow{\boldsymbol{S}} \triangleq \frac{1}{\mu_0}\overrightarrow{\boldsymbol{E}} \times \overrightarrow{\boldsymbol{B}} = c^2 \epsilon_0 \overrightarrow{\boldsymbol{E}} \times \overrightarrow{\boldsymbol{B}}$$

$$\overrightarrow{\boldsymbol{E}} \cdot \overrightarrow{\boldsymbol{j}} = -\partial_t w - \nabla \cdot \overrightarrow{\boldsymbol{S}}$$

动量守恒：

$$\overrightarrow{\boldsymbol{g}} \triangleq \epsilon_0 \overrightarrow{\boldsymbol{E}} \times \overrightarrow{\boldsymbol{B}} = \frac{\overrightarrow{\boldsymbol{S}}}{c^2}$$

$$\overleftrightarrow{\boldsymbol{T}} \triangleq \frac{1}{2}\epsilon_0 (E^2 + c^2 B^2)\overleftrightarrow{\boldsymbol{I}} - \epsilon_0 (\overrightarrow{\boldsymbol{E}}\overrightarrow{\boldsymbol{E}} + c^2 \overrightarrow{\boldsymbol{B}}\overrightarrow{\boldsymbol{B}})$$

$$-\partial_t \overrightarrow{\boldsymbol{g}} = \nabla \cdot \overleftrightarrow{\boldsymbol{T}} + \overrightarrow{\boldsymbol{f}}$$

稳恒时，受应力为 $\overrightarrow{\boldsymbol{f}_n} = -\hat{\boldsymbol{n}} \cdot \overleftrightarrow{\boldsymbol{T}}$。

角动量守恒：

$$\overrightarrow{\boldsymbol{l}}_{em} \triangleq \overrightarrow{\boldsymbol{r}} \times \overrightarrow{\boldsymbol{g}}$$

$$\overrightarrow{\boldsymbol{R}} \triangleq -\overleftrightarrow{\boldsymbol{T}} \times \overrightarrow{\boldsymbol{r}}$$

$$-\partial_t \overrightarrow{\boldsymbol{l}_{em}} = \nabla \cdot \overrightarrow{\boldsymbol{R}} + \overrightarrow{\boldsymbol{r}} \times \overrightarrow{\boldsymbol{f}}$$

物质中的守恒律：

能量守恒 (线性无色散介质)：

$$\vec{D}(\vec{r},t)=\overleftrightarrow{\epsilon}(\vec{r})\cdot\vec{E}(\vec{r},t)$$
$$w\triangleq\frac{1}{2}\vec{D}\cdot\vec{E}+\frac{1}{2}\vec{B}\cdot\vec{H}$$
$$\vec{S}\triangleq\vec{E}\times\vec{H}$$
$$\vec{E}\cdot\vec{j_0}=-\partial_t w-\nabla\cdot\vec{S}$$

动量守恒 (线性均匀介质)：

$$\vec{D}(\vec{r},t)=\overleftrightarrow{\epsilon}(t)\cdot\vec{E}(\vec{r},t)$$
$$\vec{g}\triangleq\vec{D}\times\vec{B}$$
$$\overleftrightarrow{T}\triangleq\frac{1}{2}(\vec{D}\cdot\vec{E}+\vec{B}\cdot\vec{H})\overleftrightarrow{I}-(\vec{D}\vec{E}+\vec{B}\vec{H})$$
$$-\partial_t\vec{g}=\nabla\cdot\overleftrightarrow{T}+\vec{f_0}$$

动量守恒 (线性各向同性介质)：

$$\vec{D}(\vec{r},t)=\epsilon(\vec{r},t)\cdot\vec{E}(\vec{r},t)$$
$$\vec{g}\triangleq\vec{D}\times\vec{B}$$
$$\overleftrightarrow{T}\triangleq\frac{1}{2}(\vec{D}\cdot\vec{E}+\vec{B}\cdot\vec{H})\overleftrightarrow{I}-(\vec{D}\vec{E}+\vec{B}\vec{H})$$
$$-\partial_t\vec{g}-\nabla\cdot\overleftrightarrow{T}=\vec{f_0}-\frac{1}{2}E^2\nabla\epsilon-\frac{1}{2}H^2\nabla\mu\triangleq\vec{f}_M$$

角动量守恒 (线性均匀各向同性)：

$$\vec{D}(\vec{r},t)=\epsilon(t)\cdot\vec{E}(\vec{r},t)$$
$$\vec{l}_{em}\triangleq\vec{r}\times\vec{g}$$
$$\vec{R}\triangleq-\overleftrightarrow{T}\times\vec{r}$$
$$-\partial_t\overrightarrow{l_{em}}=\nabla\cdot\vec{R}+\vec{r}\times\vec{f_0}$$

3 静电场

真空中的基本方程：

$$\begin{cases}\nabla\cdot\vec{E}=\dfrac{\rho}{\epsilon_0}\\ \nabla\times\vec{E}=0\end{cases}$$
$$\begin{cases}\hat{n}\cdot(\vec{E_2}-\vec{E_1})=\dfrac{\sigma}{\epsilon_0}\\ \hat{n}\times(\vec{E_2}-\vec{E_1})=0\end{cases}$$
$$Helmholtz:\vec{E}(\vec{r})=-\nabla\frac{1}{4\pi}\int dV'\frac{\nabla'\cdot\vec{E}}{\mathbb{R}}(=-\nabla\phi)$$
$$=\frac{1}{4\pi\epsilon_0}\int dV'\frac{\rho(\vec{r}')\vec{\mathbb{R}}}{\mathbb{R}^3}$$
$$\phi=\frac{1}{4\pi\epsilon_0}\int dV'\frac{\rho(\vec{r}')}{\mathbb{R}}$$

势方程：

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}, \quad \begin{cases} \dfrac{\partial\phi_1}{\partial n} - \dfrac{\partial\phi_2}{\partial n} = \dfrac{\sigma}{\epsilon_0} \\ \phi_1 = \phi_2 \end{cases}$$

格林互易定理：

$$\int dV'\rho'(\vec{\boldsymbol{r}}')\phi(\vec{\boldsymbol{r}}') = \int dV\rho(\vec{\boldsymbol{r}})\phi'(\vec{\boldsymbol{r}})$$

物质中的基本方程：

$$\begin{cases} \nabla\cdot\vec{\boldsymbol{D}} = \rho_0 \\ \nabla\times\vec{\boldsymbol{E}} = 0 \end{cases}$$

$$\begin{cases} \hat{\boldsymbol{n}}\cdot(\vec{\boldsymbol{D}}_2 - \vec{\boldsymbol{D}}_1) = \sigma_0 \\ \hat{\boldsymbol{n}}\times(\vec{\boldsymbol{E}}_2 - \vec{\boldsymbol{E}}_1) = 0 \end{cases}$$

势方程 (简单介质)：

$$\epsilon = \epsilon(\vec{\boldsymbol{r}})$$

$$\nabla(\epsilon\nabla\phi) = -\rho_0, \quad \begin{cases} \epsilon_1\dfrac{\partial\phi_1}{\partial n} - \epsilon_2\dfrac{\partial\phi_2}{\partial n} = \sigma_0 \\ \phi_1 = \phi_2 \end{cases}$$

静电能：

$$\delta A = \int dV\phi\delta\rho_0 = \cdots = \int dV\vec{\boldsymbol{E}}\cdot\delta\vec{\boldsymbol{D}}$$

简单介质时，$\epsilon = \epsilon(\vec{\boldsymbol{r}})$, 静电能为

$$\delta A = \int dV\delta(\frac{1}{2}\vec{\boldsymbol{D}}\cdot\vec{\boldsymbol{E}})$$

$$\begin{aligned} W &= \frac{1}{2}\int dV\vec{\boldsymbol{D}}\cdot\vec{\boldsymbol{E}} = \int dV\frac{1}{2}\epsilon_0E^2 + \int dV\frac{1}{2}\vec{\boldsymbol{P}}\cdot\vec{\boldsymbol{E}} = \frac{1}{2}\int dV\vec{\boldsymbol{D}}\cdot\vec{\boldsymbol{E}} \\ &= -\frac{1}{2}\int dV\vec{\boldsymbol{D}}\cdot\nabla\phi = \frac{1}{2}\int dV\rho_0\phi = \frac{1}{2}\int dV\vec{\boldsymbol{D}}\cdot\vec{\boldsymbol{E}} = \frac{1}{2}\sum_i Q_i\phi_i \end{aligned}$$

移动简单介质 (从无穷远到给定位置) 做功：

$$\begin{aligned} A &= \frac{1}{2}\int dV(\vec{\boldsymbol{D}}\cdot\vec{\boldsymbol{E}} - \vec{\boldsymbol{D}}_0\cdot\vec{\boldsymbol{E}}_0) \\ &= \frac{1}{2}\int dV[\vec{\boldsymbol{E}}\cdot(\vec{\boldsymbol{D}} - \vec{\boldsymbol{D}}_0) + \vec{\boldsymbol{E}}_0\cdot(\vec{\boldsymbol{D}} - \vec{\boldsymbol{D}}_0)] + \frac{1}{2}\int dV(\vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{D}}_0 - \vec{\boldsymbol{E}}_0\cdot\vec{\boldsymbol{D}}) \\ &= \frac{1}{2}\int dV\big[\nabla\cdot\big(\phi(\vec{\boldsymbol{D}} - \vec{\boldsymbol{D}}_0)\big) - \phi\nabla\cdot(\vec{\boldsymbol{D}} - \vec{\boldsymbol{D}}_0)\big] \\ &\quad + \frac{1}{2}\int dV\big[\nabla\cdot\big(\phi_0(\vec{\boldsymbol{D}} - \vec{\boldsymbol{D}}_0)\big) - \phi_0\nabla\cdot(\vec{\boldsymbol{D}} - \vec{\boldsymbol{D}}_0)\big] \\ &\quad + \frac{1}{2}\int dV\big(\epsilon_0\vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{E}}_0 - \vec{\boldsymbol{E}}_0\cdot(\epsilon_0\vec{\boldsymbol{E}} + \vec{\boldsymbol{P}})\big) \\ &= -\frac{1}{2}\int dV\vec{\boldsymbol{P}}\cdot\vec{\boldsymbol{E}}_0 \end{aligned}$$

故介质会被吸入。

唯一性定理：

$$\phi_1 - \phi_2 = \Phi$$

$$\int dV\epsilon(\nabla\Phi)^2 = \int dV\epsilon[\epsilon(\nabla\Phi)^2 + \Phi\nabla\cdot(\epsilon\nabla\Phi)] = \oint_{\partial V} d\sigma\Phi\epsilon\frac{\partial\Phi}{\partial n}$$

Dirichlet 边界条件：

$$\phi(\vec{\boldsymbol{r}}_s) = f(\vec{\boldsymbol{r}}_s) \Rightarrow unique \quad \vec{\boldsymbol{E}}, \phi$$

Neumann 边界条件：

$$\frac{\partial\phi}{\partial n}|_{\partial V} = g(\vec{\boldsymbol{r}}_s) \Rightarrow unique \quad \vec{\boldsymbol{E}}$$

导体电量：

$$Q_i = C_i \Rightarrow unique \quad \vec{\boldsymbol{E}}$$

拉普拉斯方程的求解 (泊松方程转化为拉普拉斯方程)：

分离变量法：

直角坐标系：

$$[\cdots]$$

柱坐标一般解 ($\phi \in [0, 2\pi]$)：

$$\phi = A_0 + B_0 \ln s + \sum_{m=1}^{\infty}\left(A_m s^m + \frac{B_m}{s^m}\right)(C_m \cos m\phi + D_m \sin m\phi)$$

球坐标一般解 ($\phi \in [0, 2\pi]$):

$$\begin{aligned}\phi &= \sum_{l=0}^{\infty}\sum_{m=0}^{l}\left(A_{lm}r^l + \frac{B_{lm}}{r^{l+1}}\right)P_l^m(\cos\theta)(C_{lm}\cos m\phi + D_{lm}\sin m\phi) \\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r^l + \frac{B_{lm}}{r^{l+1}}\right)Y_{lm}\end{aligned}$$

若有轴对称性，上式简化为

$$\phi = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

本征函数展开：

$$[\cdots]$$

格林函数法：

$$\phi(\vec{r}) = \int_V dV'\rho(\vec{r}')G(\vec{r}';\vec{r}) - \epsilon_0 \oint_{\partial V} d\sigma'\phi(\vec{r}')\frac{\partial G(\vec{r}';\vec{r})}{\partial n'} + \epsilon_0 \oint_{\partial V} d\sigma'\frac{\phi(\vec{r}')}{\partial n'}G(\vec{r}';\vec{r})$$

$G(\Delta G = -\delta)$ 满足齐次边界条件时，有倒易性。$G(\vec{r};\vec{r}') = G(\vec{r}';\vec{r})$

Dirichlet 边界条件：

$$\begin{cases} \nabla^2 G_D(\vec{r};\vec{r}') = -\dfrac{\delta(\vec{r}-\vec{r}')}{\epsilon_0} \\ G_D(\vec{r};\vec{r}') = 0 \end{cases}$$

$$G_D(\vec{r};\vec{r}') = G_D(\vec{r}';\vec{r})$$

$$\phi(\vec{r}) = \int_V dV'\rho(\vec{r}')G_D(\vec{r};\vec{r}') - \epsilon_0 \oint_{\partial V} d\sigma'\phi(\vec{r}')\frac{\partial G_D(\vec{r};\vec{r}')}{\partial n'}$$

Neumann 边界条件：

$$\begin{cases} \nabla^2 G_N(\vec{r};\vec{r}') = -\dfrac{\delta(\vec{r}-\vec{r}')}{\epsilon_0} \\ G_N(\vec{r};\vec{r}') = -\dfrac{1}{\epsilon_0 S} \end{cases}$$

$$\phi(\vec{r}) = \int_V dV'\rho(\vec{r}')G_N(\vec{r}';\vec{r}) + \epsilon_0 \oint_{\partial V} d\sigma'\frac{\partial\phi(\vec{r}')}{\partial n'}G_N(\vec{r}';\vec{r}) + \langle\phi\rangle_S$$

也有其他修正方法，但一般所得 G 没有倒易性。

G 的求法：一般用电像法、保形变换 (某些二维)、类似电像猜解。由本征函数与 δ 函数的关系，G 也可以由本征函数表示出来。

多极展开：

$$\begin{aligned} \frac{1}{\mathbb{R}} &= \frac{1}{|\vec{r}-\vec{r}'|} = e^{-\vec{r}'\cdot\nabla}\frac{1}{r} \\ &= [1 - \vec{r}'\cdot\nabla + \frac{1}{2!}\vec{r}'\vec{r}':\nabla\nabla - \cdots]\frac{1}{r} \\ &= [1 - \vec{r}'\cdot\nabla + \frac{1}{6}(3\vec{r}'\vec{r}' - (r')^2\overleftrightarrow{\boldsymbol{I}}):\nabla\nabla - \cdots]\frac{1}{r} \end{aligned}$$

$$\nabla\frac{1}{r} = -\frac{\vec{r}}{r^3}, \quad \nabla\nabla\frac{1}{r} = \frac{3\hat{\boldsymbol{r}}\hat{\boldsymbol{r}} - \overleftrightarrow{\boldsymbol{I}}}{r^3}$$

多极矩：

$$Q \triangleq \int dq$$

$$\vec{p} \triangleq \int \vec{r}'dq$$

$$\overleftrightarrow{\boldsymbol{D}}' \triangleq \int \vec{r}'\vec{r}'dq$$

$$\overleftrightarrow{\boldsymbol{D}} \triangleq \int (3\hat{\boldsymbol{r}}\hat{\boldsymbol{r}} - \overleftrightarrow{\boldsymbol{I}})dq, \quad (D_{ij} = D_{ji}, tr(D) = 0 = \overleftrightarrow{\boldsymbol{D}}:\overleftrightarrow{\boldsymbol{I}})$$

偶极矩和场的体积分 (积分域为包含全部电荷的球)：

$$\begin{aligned}\int dV\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) &= \frac{1}{4\pi\epsilon_0}\int dV\int dV'\frac{\rho(\vec{\boldsymbol{r}}')\hat{R}}{\mathbb{R}^2} \\ &= -\frac{1}{\epsilon_0}\int dV'\rho(\vec{\boldsymbol{r}}')\frac{1}{4\pi}\int dV\frac{-\hat{\mathbb{R}}}{\mathbb{R}^2} \\ &= -\frac{1}{\epsilon_0}\int dV'\rho(\vec{\boldsymbol{r}}')\frac{\vec{\boldsymbol{r}}'}{3} \\ &= -\frac{\vec{\boldsymbol{p}}}{3\epsilon_0}\end{aligned}$$

展开：

$$\begin{aligned}\phi(\vec{\boldsymbol{r}}) &= \frac{1}{4\pi\epsilon_0}\int\frac{dq}{\mathbb{R}} \\ &= \frac{1}{4\pi\epsilon_0}\Big[\frac{Q}{r}-\vec{\boldsymbol{p}}\cdot\nabla\frac{1}{r}+\frac{1}{6}\overleftrightarrow{\boldsymbol{D}}:\nabla\nabla\frac{1}{r}-\cdots\Big] \\ &= \frac{Q}{4\pi\epsilon_0 r}+\frac{\vec{\boldsymbol{p}}\cdot\hat{r}}{4\pi\epsilon_0 r^2}+\frac{\hat{r}\cdot\overleftrightarrow{\boldsymbol{D}}\cdot\hat{r}}{8\pi\epsilon_0 r^3}+\cdots\end{aligned}$$

外场中的小带电体：

电势能：

$$\begin{aligned}U &= \int\phi_e(\vec{\boldsymbol{r}}+\vec{\boldsymbol{r}}')dq = \int e^{\vec{\boldsymbol{r}}'\cdot\nabla}\phi_e(\vec{\boldsymbol{r}}) \\ &= \int dq\Big[1+\vec{\boldsymbol{r}}'\cdot\nabla+\frac{1}{6}(3\vec{\boldsymbol{r}}'\vec{\boldsymbol{r}}'-(r')^2\overleftrightarrow{I}:\nabla\nabla)+\cdots\Big]\phi_e(\vec{\boldsymbol{r}}) \\ &= Q\phi_e(\vec{\boldsymbol{r}})-\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}})-\frac{1}{6}\overleftrightarrow{\boldsymbol{D}}:\nabla\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}})+\cdots \\ &= U^{(0)}+U^{(1)}+U^{(2)}+\cdots\end{aligned}$$

受力：

$$\begin{aligned}\vec{\boldsymbol{F}} &= \int\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}}+\vec{\boldsymbol{r}}')dq \\ &= \int dq\Big[1+\vec{\boldsymbol{r}}'\cdot\nabla+\frac{1}{6}(3\vec{\boldsymbol{r}}'\vec{\boldsymbol{r}}'-(r')^2\overleftrightarrow{I}:\nabla\nabla)+\cdots\Big]\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}}) \\ &= Q\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}})+\vec{\boldsymbol{p}}\cdot\nabla\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}})+\frac{1}{6}(\overleftrightarrow{\boldsymbol{D}}:\nabla\nabla)\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}})+\cdots\end{aligned}$$

其中，

$$-\nabla U^{(0)} = Q\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}})$$

$$-\nabla U^{(1)} = \vec{\boldsymbol{p}}\cdot\nabla\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}})$$

$$\begin{aligned}-\nabla U^{(2)} &= \frac{1}{6}\nabla(\overleftrightarrow{\boldsymbol{D}}:\nabla\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}})) \\ &= \frac{1}{6}\nabla\nabla\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}}):\overleftrightarrow{\boldsymbol{D}} \\ &= \frac{1}{6}\overleftrightarrow{\boldsymbol{D}}:\nabla\nabla\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}})\end{aligned}$$

能换序因为 ∇E_e 为对称张量 (旋度为 0)：

$$\begin{aligned}\nabla(\overleftrightarrow{\boldsymbol{D}}:\nabla\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}})) &= (\nabla)_a(\nabla)_b(E_e)_c(D)^{bc}\\ &= (\nabla)_b(D)^{bc}(\nabla)_a(E_e)_c = (\nabla)_b(D)^{bc}(\nabla)_c(E_e)_a\\ &= (D)^{bc}(\nabla)_b(\nabla)_c(E_e)_a = \overleftrightarrow{\boldsymbol{D}}:\nabla\nabla\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}})\end{aligned}$$

受力矩：

$$\begin{aligned}\vec{\boldsymbol{\tau}} &= \int[\vec{\boldsymbol{r}}'\times\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}}+\vec{\boldsymbol{r}}')]dq\\ &= \int[\vec{\boldsymbol{r}}'\times\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}})]dq+\int\left[\vec{\boldsymbol{r}}'\times[(\vec{\boldsymbol{r}}'\cdot\nabla)\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}})]\right]dq+\cdots\\ &= \vec{\boldsymbol{p}}\times\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}})+[\int\vec{\boldsymbol{r}}'\vec{\boldsymbol{r}}'dq\cdot\nabla]\times\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}})+\cdots\\ &= \vec{\boldsymbol{p}}\times\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}})+\frac{1}{3}\int(3\vec{\boldsymbol{r}}'\vec{\boldsymbol{r}}'-(r')^2\overleftrightarrow{\boldsymbol{I}})dq\cdot\nabla\times\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}})+\cdots\\ &= \vec{\boldsymbol{p}}\times\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}})+\frac{1}{3}(\overleftrightarrow{\boldsymbol{D}}\cdot\nabla)\times\vec{\boldsymbol{E}}_e(\vec{\boldsymbol{r}})+\cdots\end{aligned}$$

球内球外展开：

$$\mathbb{R}=|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|=\sqrt{r^2-2rr'\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}'+(r')^2}$$

$$\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}'=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')$$

$$\begin{aligned}\frac{1}{\mathbb{R}} &= \frac{1}{\sqrt{r^2-2rr'\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}'+(r')^2}}\\ &= \frac{1}{r_>}\sum_{l=0}^{\infty}(\frac{r_<}{r_>})^l P_l(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}')\\ &= \frac{1}{r_>}\sum_{l=0}^{\infty}(\frac{r_<}{r_>})^l\frac{4\pi}{2l+1}\sum_{m=-l}^{m=l}Y_{lm}^*(\Omega')Y_{lm}(\Omega)\end{aligned}$$

球内展开：

$$\phi(\vec{\boldsymbol{r}})=\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}r^lY_{lm}(\Omega),\quad A_{lm}=\frac{4\pi}{2l+1}\int\frac{Y_{lm}^*(\Omega')}{(r')^{l+1}}dq$$

若 $\phi=\phi(r,\theta)$，上式可以简化：

$$\phi(\vec{\boldsymbol{r}})=\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}A_lr^lP_l(\cos\theta),\quad A_l=\int\frac{P_l(\cos\theta')}{(r')^{l+1}}dq$$

球外展开：

$$\phi(\vec{\boldsymbol{r}})=\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{B_{lm}}{r^{l+1}}Y_{lm}(\Omega),\quad B_{lm}=\frac{4\pi}{2l+1}\int Y_{lm}^*(\Omega')(r')^l dq$$

若 $\phi=\phi(r,\theta)$，上式可以简化：

$$\phi(\vec{\boldsymbol{r}})=\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta),\quad B_l=\int P_l(\cos\theta')(r')^l dq$$

4 静磁场

真空中的基本方程：

$$\begin{cases}\nabla\cdot\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}})=0\\ \nabla\times\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}})=\mu_0\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}})\end{cases}$$

$$\begin{cases}\hat{\boldsymbol{n}}\cdot(\vec{\boldsymbol{B}}_2-\vec{\boldsymbol{B}}_1)=0\\ \hat{\boldsymbol{n}}\times(\vec{\boldsymbol{B}}_2-\vec{\boldsymbol{B}}_1)=\mu_0\vec{\boldsymbol{K}}\end{cases}$$

$$\begin{aligned}Helmholtz:\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}})&=\nabla\times\frac{1}{4\pi}\int dV'\frac{\nabla'\times\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}})}{\mathbb{R}}(=\nabla\times\vec{\boldsymbol{A}})\\&=\frac{\mu_0}{4\pi}\int dV'\frac{\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}}')\times\vec{\mathbb{R}}}{\mathbb{R}^3}\end{aligned}$$

从 $\vec{\boldsymbol{B}}$ 的表达式看出 $\vec{\boldsymbol{B}}$ 是一个轴矢量，$B_i'=det(\lambda)\lambda_{ij}B_j$。(一般矢量应满足 $V_i'=\lambda_{ij}V_j$)

磁矢势：

$$\nabla\times\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}})=\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}),\quad \vec{\boldsymbol{A}}'=\vec{\boldsymbol{A}}+\nabla\psi(\vec{\boldsymbol{r}})$$

库伦规范：

$$\nabla\cdot\vec{\boldsymbol{A}}=0$$

(分别可得切向分量连续和法向分量连续的边界条件。)

一个解：

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}})=\frac{\mu_0}{4\pi}\int dV'\frac{\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}})}{\mathbb{R}},\quad \nabla\cdot\vec{\boldsymbol{A}}=0$$

按照这个解，同样有互易定理：

$$\int dV'\vec{\boldsymbol{j}}'(\vec{\boldsymbol{r}}')\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')=\int dV\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}})\vec{\boldsymbol{A}}'(\vec{\boldsymbol{r}})$$

势方程 ($\nabla\cdot\vec{\boldsymbol{A}}=0$)：

$$\nabla^2\vec{\boldsymbol{A}}=-\mu_0\vec{\boldsymbol{j}},\quad \begin{cases}\vec{\boldsymbol{A}}_1=\vec{\boldsymbol{A}}_2\\ \hat{\boldsymbol{n}}\times(\nabla\times\vec{\boldsymbol{A}}_2-\nabla\times\vec{\boldsymbol{A}}_1)=\mu_0\vec{\boldsymbol{K}}\end{cases}$$

解出后需验证是否满足库伦规范。

物质中的基本方程：

$$\begin{cases}\nabla\cdot\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}})=0\\ \nabla\times\vec{\boldsymbol{H}}(\vec{\boldsymbol{r}})=\vec{\boldsymbol{j}}_0(\vec{\boldsymbol{r}})\end{cases}$$

$$\begin{cases}\hat{\boldsymbol{n}}\cdot(\vec{\boldsymbol{B}}_2-\vec{\boldsymbol{B}}_1)=0\\ \hat{\boldsymbol{n}}\times(\vec{\boldsymbol{H}}_2-\vec{\boldsymbol{H}}_1)=\vec{\boldsymbol{K}}_0\end{cases}$$

势方程 (简单介质;$\nabla\cdot\vec{\boldsymbol{A}}=0$)：

$$\nabla\times\left(\frac{\nabla\times\vec{\boldsymbol{A}}}{\mu(\vec{\boldsymbol{r}})}\right)=\vec{\boldsymbol{j}}_0,\quad \begin{cases}\vec{\boldsymbol{A}}_1=\vec{\boldsymbol{A}}_2\\ \hat{\boldsymbol{n}}\times\left(\dfrac{\nabla\times\vec{\boldsymbol{A}}_2}{\mu_2}-\dfrac{\nabla\times\vec{\boldsymbol{A}}_1}{\mu_1}\right)=\vec{\boldsymbol{K}}_0\end{cases}$$

解出后需验证是否满足库伦规范。

磁能：

$$\begin{aligned}\delta A&=\int -dI_0\cdot\epsilon\cdot\delta t=\int dI_0\cdot\delta\Phi_B\\&=\int\vec{\boldsymbol{j}}_0\cdot d\vec{\boldsymbol{\sigma}}_\perp\int d\vec{\boldsymbol{\sigma}}\cdot\delta\vec{\boldsymbol{B}}=\int\vec{\boldsymbol{j}}_0\cdot d\vec{\boldsymbol{\sigma}}_\perp\oint d\vec{\boldsymbol{l}}\cdot\delta\vec{\boldsymbol{A}}\\&=\int dV\vec{\boldsymbol{j}}_0\cdot\delta\vec{\boldsymbol{A}}\\&=\cdots=\int dV\vec{\boldsymbol{H}}\cdot\delta\vec{\boldsymbol{B}}\end{aligned}$$

简单介质时，$\mu=\mu(\vec{\boldsymbol{r}})$, 磁能为

$$\delta A=\int dV\delta\left(\frac{1}{2}\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{H}}\right)$$

$$\begin{aligned}W&=\frac{1}{2}\int dV\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{H}}=\int dV\frac{B^2}{2\mu_0}-\int dV\frac{1}{2}\vec{\boldsymbol{M}}\cdot\vec{\boldsymbol{B}}\\&=\frac{1}{2}\int dV\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{H}}=\frac{1}{2}\int dV\vec{\boldsymbol{j}}_0\cdot\vec{\boldsymbol{A}}\end{aligned}$$

移动简单介质 (从无穷远到给定位置) 做功：

$$\vec{\boldsymbol{A}}=\frac{1}{2}\int dV[\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{H}}-\vec{\boldsymbol{B}}_0\cdot\vec{\boldsymbol{H}}_0]=\cdots=\frac{1}{2}\int dV\vec{\boldsymbol{M}}\cdot\vec{\boldsymbol{B}}_0$$

互能：

$$\begin{aligned}W[\vec{\boldsymbol{j}}_1+\vec{\boldsymbol{j}}_2]&=\frac{1}{2}\int dV(\vec{\boldsymbol{j}}_1+\vec{\boldsymbol{j}}_2)\cdot(\vec{\boldsymbol{A}}_1+\vec{\boldsymbol{A}}_2)\\&=W[\vec{\boldsymbol{j}}_1]+W[\vec{\boldsymbol{j}}_2]+\frac{1}{2}\int dV\vec{\boldsymbol{j}}_1\cdot\vec{\boldsymbol{A}}_2+\frac{1}{2}\int dV\vec{\boldsymbol{j}}_2\cdot\vec{\boldsymbol{A}}_1\\&=W[\vec{\boldsymbol{j}}_1]+W[\vec{\boldsymbol{j}}_2]+\int dV\vec{\boldsymbol{j}}_1\cdot\vec{\boldsymbol{A}}_2\\&\quad\left(\int dV\vec{\boldsymbol{j}}_1\cdot\vec{\boldsymbol{A}}_2=\int dV\vec{\boldsymbol{j}}_2\cdot\vec{\boldsymbol{A}}_1\right)\end{aligned}$$

(若将 $\vec{\boldsymbol{j}}_2$ 视为外场源 ($\vec{\boldsymbol{A}}_2$ 视为外场)，可以按照下式定义 $\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}})$ 在外场中的磁能：$W_e\overset{\Delta}{=}\int dV\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}})\cdot\vec{\boldsymbol{A}}_e(\vec{\boldsymbol{r}})$，更进一步可以由此定义力学势能 (不考虑物体维持电流所需要的能量)：$U\overset{\Delta}{=}-W_e$)

线圈系统的能量：

$$\begin{aligned}W&=\frac{1}{2}\int dV\vec{\boldsymbol{j}}\cdot\vec{\boldsymbol{A}}=\frac{1}{2}\sum_i\int dV_i\vec{\boldsymbol{j}}_i\cdot\vec{\boldsymbol{A}}\\&=\frac{1}{2}\sum_i I_i\oint_{C_i}d\vec{\boldsymbol{l}}_i\cdot\vec{\boldsymbol{A}}=\frac{1}{2}\sum_i I_i\Phi_i\end{aligned}$$

$$\Phi_i = \oint_{C_i} d\vec{l}_i \cdot \vec{A} = \sum_k I_k \frac{\mu_0}{4\pi} \oint_{C_i} \oint_{C_k} \frac{d\vec{l}_i \cdot \vec{l}_k}{R_{ik}} = \sum_k L_{ik} I_k$$

$$L_{ik} = \frac{\mu_0}{4\pi} \oint_{C_i} \oint_{C_k} \frac{d\vec{l}_i \cdot \vec{l}_k}{R_{ik}} = L_{ki}$$

$$W = \frac{1}{2} \sum_{i,k} L_{ik} I_i I_k$$

（两个线圈时，$W = \frac{1}{2}L_{11}I_1^2 + \frac{1}{2}L_{22}I_2^2 + L_{12}I_1I_2$，其中的系数即为自感和互感。）

磁多极子：

电流密度的积分：

$$\int dV \vec{j} = \int dV\big(\nabla \cdot (\vec{j}\vec{r}) + \frac{\partial \rho}{\partial t}\vec{r}\big) = \dot{\vec{p}}$$

$$\begin{aligned}
\int dV \vec{j}\vec{r} &= \frac{1}{2}\int dV(\vec{j}\vec{r} + \vec{r}\vec{j}) + \frac{1}{2}\int dV(\vec{j}\vec{r} - \vec{r}\vec{j}) \cdot \overleftrightarrow{I} \\
&= \frac{1}{2}\int dV[\nabla \cdot (\vec{j}\vec{r}\vec{r}) + \frac{\partial \rho}{\partial t}\vec{r}\vec{r}] + \frac{1}{2}\int dV(\vec{j}\vec{r} - \vec{r}\vec{j}) \cdot \overleftrightarrow{I} \\
&= \frac{1}{2}\int dV(\vec{r} \times \vec{j}) \times \overleftrightarrow{I} + \frac{1}{6}\dot{\overleftrightarrow{D}} + \frac{1}{6}\frac{d}{dt}\int dV \rho r^2 \overleftrightarrow{I} \\
&= \overleftrightarrow{m} \times \overleftrightarrow{I} + \frac{1}{6}\dot{\overleftrightarrow{D}} + \frac{1}{6}\frac{d}{dt}\int dV \rho r^2 \overleftrightarrow{I}
\end{aligned}$$

磁偶极矩：

$$\overleftrightarrow{m} \triangleq \frac{1}{2}\int dV \vec{r} \times \vec{j}$$

对稳恒电流，$\int dV\vec{j} = 0,\quad \int dV\vec{j}\vec{r} = \overleftrightarrow{m} \times \overleftrightarrow{I}$。对稳恒电流，$\overleftrightarrow{m}$ 与原点的选择无关。

若为线圈上的稳恒电流，

$$\overleftrightarrow{m} = \frac{1}{2}I\oint \vec{r} \times d\vec{l} = -\frac{1}{2}I\int (d\vec{\sigma} \times \nabla) \times \vec{r} = I\int d\vec{\sigma}$$

偶极矩和场的体积分 (积分域为包含全部电流的球)：

$$\begin{aligned}
\int dV \vec{B}(\vec{r}) &= \frac{\mu_0}{4\pi}\int dV \int dV' \frac{\vec{j}(\vec{r}') \times \hat{\mathbb{R}}}{\mathbb{R}^2} \\
&= -\mu_0 \int dV' \vec{j}(\vec{r}') \times \frac{1}{4\pi}\int dV \frac{-\hat{\mathbb{R}}}{\mathbb{R}^2} \\
&= -\mu_0 \int dV' \vec{j}(\vec{r}') \times \frac{\vec{r}'}{3} \\
&= \frac{2\mu_0}{3}\frac{1}{2}\int dV' \vec{r}' \times \vec{j}(\vec{r}') \\
&= \frac{2\mu_0}{3}\vec{m}
\end{aligned}$$

磁矢势的多极展开

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int dV'\frac{\vec{j}(\vec{r})}{\mathbb{R}}$$
$$= \frac{\mu_0}{4\pi}\Big[\frac{1}{r}\int dV'\vec{j}(\vec{r}') + \int dV'\vec{j}(\vec{r}')\cdot\frac{\vec{r}}{r^3} + \cdots\Big]$$

稳恒电流时，磁矢势为

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\overleftrightarrow{m}\times\vec{r}}{r^3}$$

外场中的小载流体：

外场中的磁能：

$$W_e = \int dV'\vec{j}(\vec{r}')\cdot\vec{A}_e(\vec{r}+\vec{r}')$$
$$= \int dV'\vec{j}(\vec{r}')\cdot\vec{A}_e(\vec{r}) + \Big[\int dV'\vec{j}(\vec{r}')\cdot\nabla\Big]\cdot\vec{A}_e(\vec{r})$$
$$= [\vec{m}\times\overleftrightarrow{I}\cdot\nabla]\cdot\vec{A}_e(\vec{r})$$
$$= \vec{m}\cdot\vec{B}_e = -U$$

受力：

$$\vec{F} = \int dV'\vec{j}(\vec{r}')\times\vec{B}_e(\vec{r}+\vec{r}')$$
$$= \cdots = (\vec{m}\times\nabla)\times\vec{B}_e(\vec{r})$$
$$= \nabla\vec{B}_e\cdot\vec{m}(= \nabla W_e = -\nabla U)$$

磁能细究：考虑两个间隔无穷远的线圈，电流为 I_1, I_2，维持各自电流不变的情况下移动 I_2 到指定位置，做功详情为：

$$A_{2mot} = MI_1I_2, A_{2amp} = -MI_1I_2$$
$$A_{1ind} = MI_1I_2$$

故磁力指向磁能增大的方向看似不合理，但这是电源也做功的结果。

受力矩：

$$\vec{\tau} = \int dV'\vec{r}'\times[\vec{j}(\vec{r}')\times\vec{B}_e(\vec{r}+\vec{r}')]$$
$$= \cdots = \int dV'\vec{j}(\vec{r}')\vec{r}'\times\vec{B}_e(\vec{r}) - \int dV'\vec{r}'\cdot\vec{j}(\vec{r}')\vec{B}_e(\vec{r})$$
$$= \vec{m}\times\vec{B}_e(\vec{r}) + \frac{1}{2}\int dV'[(\nabla\cdot\vec{j})r^2 - \nabla\cdot(\vec{j}r^2)]\vec{B}_e(\vec{r})$$
$$= \vec{m}\times\vec{B}_e(\vec{r})$$

磁标势法：

在无传导电流的单连通区域 $\nabla\times\vec{\boldsymbol{H}}=0\Rightarrow\vec{\boldsymbol{H}}=-\nabla\phi$，一般情况下可模仿静电场求解关于 ψ 的方程。真空中传导电流已知时可以直接积分；磁化强度 $\vec{\boldsymbol{M}}$ 已知时可以用磁荷法类似库仑定律积分，磁荷的定义如下：$-\nabla\cdot\vec{\boldsymbol{M}}=\rho^*,-\hat{\boldsymbol{n}}\cdot(\vec{\boldsymbol{M}}_2-\vec{\boldsymbol{M}}_1)=\sigma^*$。

5 电磁波

自由空间中的电磁波：

麦克斯韦方程：

$$\begin{cases}\nabla\cdot\vec{\boldsymbol{E}}=0\\ \nabla\cdot\vec{\boldsymbol{B}}=0\\ \nabla\times\vec{\boldsymbol{E}}=-\partial_t\vec{\boldsymbol{B}} & (*)\\ \nabla\times\vec{\boldsymbol{B}}=\dfrac{1}{c^2}\partial_t\vec{\boldsymbol{E}} & (*)\end{cases}$$

带 $*$ 号的式子可以推出前两个以及相应的两个边界条件 $\begin{cases}\hat{\boldsymbol{n}}\times(\vec{\boldsymbol{E}}_2-\vec{\boldsymbol{E}}_1)=0\\ \hat{\boldsymbol{n}}\times(\vec{\boldsymbol{B}}_2-\vec{\boldsymbol{B}}_1)=0\end{cases}$

$$Maxwell+\{\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}},0),\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}},0)\}\Rightarrow\{\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}},t),\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}},t)\}$$

波动方程：

$$\begin{cases}\nabla^2\vec{\boldsymbol{E}}-\dfrac{1}{c^2}\partial_t^2\vec{\boldsymbol{E}}=0=\Box\vec{\boldsymbol{E}}\\ \\ \nabla^2\vec{\boldsymbol{B}}-\dfrac{1}{c^2}\partial_t^2\vec{\boldsymbol{B}}=0=\Box\vec{\boldsymbol{B}}\end{cases}$$

场的求解：

$$\begin{cases}\begin{cases}\Box\vec{\boldsymbol{E}}=0\\ \nabla\cdot\vec{\boldsymbol{E}}=0\end{cases} & \overset{\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}},0)}{\Longrightarrow}\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}},t)\\ \partial_t\vec{\boldsymbol{B}}=-\nabla\times\vec{\boldsymbol{E}}\overset{\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}},0)}{\Longrightarrow}\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}},t)\end{cases}$$

沿某方向传播的电磁波：

沿 z 方向：

$$\begin{aligned}&\vec{\boldsymbol{E}}=\vec{\boldsymbol{E}}(z,t),\vec{\boldsymbol{B}}=\vec{\boldsymbol{B}}(z,t)\\ \Rightarrow&\begin{cases}\vec{\boldsymbol{E}}=\vec{\boldsymbol{E}}_-(z-ct)+\left(\vec{\boldsymbol{E}}_+(z+ct)+static\ field\right)\\ \vec{\boldsymbol{B}}=\dfrac{1}{c}\hat{\boldsymbol{z}}\times\vec{\boldsymbol{E}}_-+\left(-\dfrac{1}{c}\hat{\boldsymbol{z}}\times\vec{\boldsymbol{E}}_++static\ field\right)\end{cases}\\ &(\vec{\boldsymbol{E}}_-\cdot\hat{\boldsymbol{z}}=0)\end{aligned}$$

沿 $\overrightarrow{\boldsymbol{k}}$ 方向传播：

$$\begin{cases}\overrightarrow{\boldsymbol{E}}(\overrightarrow{\boldsymbol{r}},t)=\overrightarrow{\boldsymbol{E}}_{\perp}(\overrightarrow{\boldsymbol{k}}\cdot\overrightarrow{\boldsymbol{r}}-kct)=\overrightarrow{\boldsymbol{E}}_{\perp}(\phi), \quad \overrightarrow{\boldsymbol{k}}\cdot\overrightarrow{\boldsymbol{E}}=0\\ \overrightarrow{\boldsymbol{B}}(\overrightarrow{\boldsymbol{r}},t)=\dfrac{1}{c}\hat{\boldsymbol{k}}\times\overrightarrow{\boldsymbol{E}}\end{cases}$$

力学性质：

$$w=\frac{1}{2}\epsilon_0(E^2+c^2B^2)=\epsilon_0E^2 \quad (w_e=w_m)$$

$$\overrightarrow{\boldsymbol{S}}=\frac{1}{\mu_0}\overrightarrow{\boldsymbol{E}}\times\overrightarrow{\boldsymbol{B}}=wc\hat{\boldsymbol{k}}$$

$$\overrightarrow{\boldsymbol{g}}=\epsilon_0\overrightarrow{\boldsymbol{E}}\times\overrightarrow{\boldsymbol{B}}=\frac{w}{c}\hat{\boldsymbol{k}}$$

$$\overleftrightarrow{\boldsymbol{T}}=w\overleftrightarrow{\boldsymbol{I}}-\overrightarrow{\boldsymbol{D}}\overrightarrow{\boldsymbol{E}}-\overrightarrow{\boldsymbol{B}}\overrightarrow{\boldsymbol{H}}=w\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}$$

平面单色波：

偏振：

$$\hat{\boldsymbol{e}}_1\times\hat{\boldsymbol{e}}_2=\hat{\boldsymbol{k}}$$

$$\overrightarrow{\boldsymbol{E}}=E_1\hat{\boldsymbol{e}}_1+E_2\hat{\boldsymbol{e}}_2=A_1\cos(\phi+\delta_1)\hat{\boldsymbol{e}}_1+A_2\cos(\phi+\delta_2)\hat{\boldsymbol{e}}_2$$

$$\delta\triangleq\delta_2-\delta_1$$

转动方向与传播方向构成右手系时，称为右旋。(与光学中相反)

复表示：

$$\begin{aligned}\overrightarrow{\boldsymbol{E}}=\overrightarrow{\boldsymbol{E}}_0e^{i(\overrightarrow{\boldsymbol{k}}\cdot\overrightarrow{\boldsymbol{r}}-\omega t)}&=(E_{01}\hat{\boldsymbol{e}}_1+E_{02}\hat{\boldsymbol{e}}_2)e^{i\phi}\\&=(A_1e^{i\phi_1}\hat{\boldsymbol{e}}_1+A_2e^{i\phi_2}\hat{\boldsymbol{e}}_2)e^{i\phi}\end{aligned}$$

$$\begin{cases}\nabla^2\overrightarrow{\boldsymbol{E}}=\dfrac{1}{c^2}\partial_t^2\overrightarrow{\boldsymbol{E}}\Longrightarrow k=\dfrac{\omega}{c}\\ \nabla\cdot\overrightarrow{\boldsymbol{E}}=0\Longrightarrow\overrightarrow{\boldsymbol{k}}\cdot\overrightarrow{\boldsymbol{E}}_0=0\\ \nabla\times\overrightarrow{\boldsymbol{E}}=-\partial_t\overrightarrow{\boldsymbol{B}}\Longrightarrow\overrightarrow{\boldsymbol{B}}=\dfrac{\overrightarrow{\boldsymbol{k}}\times\overrightarrow{\boldsymbol{E}}}{\omega}\end{cases}$$

复表示的实部为真实的物理场。

$$\begin{cases}Re\overrightarrow{\boldsymbol{F}}\leftrightharpoons\overrightarrow{\boldsymbol{F}}\\ Linear\ Operator\ L:L[Re\overrightarrow{\boldsymbol{F}}]=0\Longleftrightarrow L[\overrightarrow{\boldsymbol{F}}]=0\\ \partial_t\longleftrightarrow -i\omega,\nabla\longleftrightarrow i\overrightarrow{\boldsymbol{k}}\\ \langle Ref\cdot Reg\rangle=\left\langle\dfrac{f+f^*}{2}\cdot\dfrac{g+g^*}{2}\right\rangle=\dfrac{1}{2}Re(f^*g)\\ \left\langle Re\overrightarrow{\boldsymbol{f}}\cdot Re\overrightarrow{\boldsymbol{g}}\right\rangle=\dfrac{1}{2}Re\left\langle\overrightarrow{\boldsymbol{f}}^*\cdot\overrightarrow{\boldsymbol{g}}\right\rangle,\left\langle Re\overrightarrow{\boldsymbol{f}}\times Re\overrightarrow{\boldsymbol{g}}\right\rangle=\dfrac{1}{2}Re\left\langle\overrightarrow{\boldsymbol{f}}^*\times\overrightarrow{\boldsymbol{g}}\right\rangle\end{cases}$$

复表示的一些结果：

$$\begin{cases} f_0 e^{iax} = g_0 e^{ibx} \Longleftrightarrow f_0 = g_0, a = b \\ f_0 e^{iax} + g_o e^{ibx} = h_0 e^{icx} \Longleftrightarrow f_0 + g_0 = h_0, a = b = c \\ \dots \end{cases}$$

f_i, a_i 均可为复数。

上式中 $[\cdots]$ 可由恒等式求导，结合系数非零解条件，得到范德蒙德行列式，证得其中两个 a_i, a_j 相等，然后相等的项合并，得到低一阶的范德蒙德行列式，同理证明更多的 a_i 相等，进而指数项都相等，消去，得到各项系数的关系。

偏振度：

$$\tilde{R} \triangleq \frac{E_{02}}{E_{01}} = \frac{A_2}{A_1} e^{i\delta}$$

$$Im\tilde{R} = 0 \Rightarrow LP$$

$$Im\tilde{R} \neq 0, \pm 1 \Rightarrow \begin{cases} Im\tilde{R} > 0, & REP \\ Im\tilde{R} < 0, & LEP \end{cases}$$

$$\tilde{R} = \pm i \Rightarrow \begin{cases} \tilde{R} = i, & RCP \\ \tilde{R} = -i, & LCP \end{cases}$$

一般的 $\vec{\boldsymbol{E}}_0$ 由 $\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2$ 的线性组合表示，也可以由圆偏振基表示：

$$\begin{cases} RCP \quad \vec{\boldsymbol{e}}_+ = \dfrac{\vec{\boldsymbol{e}}_1 + i\vec{\boldsymbol{e}}_2}{\sqrt{2}} \\ LCP \quad \vec{\boldsymbol{e}}_- = \dfrac{\vec{\boldsymbol{e}}_1 - i\vec{\boldsymbol{e}}_2}{\sqrt{2}} \end{cases}$$

绝缘介质中的电磁波：

两个参数：

$$\begin{cases} n \triangleq c\sqrt{\mu\epsilon} \\ Z \triangleq \sqrt{\dfrac{\mu}{\epsilon}} \end{cases}$$

麦克斯韦方程：

$$\begin{cases} \nabla \cdot \vec{\boldsymbol{D}} = 0 \\ \nabla \cdot \vec{\boldsymbol{B}} = 0 \\ \nabla \times \vec{\boldsymbol{E}} = -\partial_t \vec{\boldsymbol{B}} & (*) \\ \nabla \times \vec{\boldsymbol{H}} = \partial_t \vec{\boldsymbol{D}} & (*) \end{cases}$$

带 * 号的式子可以推出前两个以及相应的两个边界条件 $\begin{cases}\hat{\boldsymbol{n}}\times(\vec{\boldsymbol{E}}_2-\vec{\boldsymbol{E}}_1)=0\\ \hat{\boldsymbol{n}}\times(\vec{\boldsymbol{H}}_2-\vec{\boldsymbol{H}}_1)=0\end{cases}$

时谐场 (均匀介质)：

$$\begin{cases}\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}},t)=\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})e^{-i\omega t}\\ \vec{\boldsymbol{H}}(\vec{\boldsymbol{r}},t)=\vec{\boldsymbol{H}}(\vec{\boldsymbol{r}})e^{-i\omega t}\end{cases}$$

$$\vec{\boldsymbol{D}}(\vec{r},t)=\epsilon(\omega)\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}},t),\quad \vec{\boldsymbol{B}}(\vec{r},t)=\mu(\omega)\vec{\boldsymbol{H}}(\vec{\boldsymbol{r}},t)$$

场的求解：

$$\begin{cases}\begin{cases}\nabla^2\vec{\boldsymbol{E}}+\omega^2\mu\epsilon\vec{\boldsymbol{E}}=\nabla^2\vec{\boldsymbol{E}}+k^2\vec{\boldsymbol{E}}=0\\ \nabla\cdot\vec{\boldsymbol{E}}=0\end{cases}\\ \vec{\boldsymbol{H}}=-\dfrac{i}{\omega\mu}\nabla\times\vec{\boldsymbol{E}}\end{cases}$$

$$k=\omega\sqrt{\mu\epsilon}=\frac{\omega}{c}n$$

在无限大均匀介质中的解为

$$\nabla^2\vec{\boldsymbol{E}}+k^2\vec{\boldsymbol{E}}=0\Longrightarrow$$

$$\begin{cases}\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})=\sum\limits_{k',k'',k'''}E_{01}e^{i\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{r}}}\hat{\boldsymbol{x}}_1+E_{02}e^{i\vec{\boldsymbol{k}}''\cdot\vec{\boldsymbol{r}}}\hat{\boldsymbol{x}}_2+E_{03}e^{i\vec{\boldsymbol{k}}'''\cdot\vec{\boldsymbol{r}}}\hat{\boldsymbol{x}}_3\\ (k')^2=(k'')^2=(k''')^2=k^2\end{cases}$$

$$\nabla\cdot\vec{\boldsymbol{E}}=0\Longrightarrow\begin{cases}\vec{\boldsymbol{k}}'=\vec{\boldsymbol{k}}''=\vec{\boldsymbol{k}}'''\triangleq\vec{\boldsymbol{k}}\\ \vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{E}}_0=0\end{cases},\quad \vec{\boldsymbol{E}}(\vec{r})=\vec{\boldsymbol{E}}_0e^{i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}}$$

$$\vec{\boldsymbol{H}}=-\frac{i}{\omega\mu}\nabla\times\vec{\boldsymbol{E}}\Longrightarrow\vec{\boldsymbol{H}}(\vec{\boldsymbol{r}})=\frac{\vec{\boldsymbol{k}}\times\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})}{\omega\mu}$$

其中 $\vec{\boldsymbol{k}}$ 可以为复矢量。(但此时 $k^2=\frac{\omega}{c}n$ 不代表复矢量模，只是矢量分量平方和 $\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{k}}$.) 若 $\vec{\boldsymbol{k}}$ 为实矢量，则解为平面电磁波。（真空中自然也是这样）

单色平面波：

$$\begin{cases}\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}},t)=\vec{\boldsymbol{E}}_0e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)},\quad \vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{E}}_0=0\\ \vec{\boldsymbol{H}}=\dfrac{\vec{\boldsymbol{k}}\times\vec{\boldsymbol{E}}}{\omega\mu}(Z\vec{\boldsymbol{H}}=\hat{\boldsymbol{k}}\times\vec{\boldsymbol{E}})\end{cases}$$

相速度：

$$\vec{\boldsymbol{v}}_p=\frac{\omega}{k}\hat{\boldsymbol{k}}=\frac{c}{n}\hat{\boldsymbol{k}}$$

力学性质：

$$w=\epsilon|Re\vec{\boldsymbol{E}}|^2,\quad \langle w\rangle=\frac{1}{2}\epsilon\vec{\boldsymbol{E}}_0^*\cdot\vec{\boldsymbol{E}}_0$$

$$\vec{\boldsymbol{S}}=Re\vec{\boldsymbol{E}}\times Re\vec{\boldsymbol{H}},\quad \left\langle\vec{\boldsymbol{S}}\right\rangle=\frac{1}{2}Re(\vec{\boldsymbol{E}}_0^*\times\vec{\boldsymbol{H}}_0)=\frac{\vec{\boldsymbol{E}}_0^*\cdot\vec{\boldsymbol{E}}_0}{2Z}\hat{\boldsymbol{k}}=\langle w\rangle\vec{\boldsymbol{v}}_p$$

$$\vec{\boldsymbol{g}}=\epsilon\mu Re\vec{\boldsymbol{E}}\times Re\vec{\boldsymbol{H}}=\frac{\vec{\boldsymbol{S}}}{v_p^2},\quad \left\langle\vec{\boldsymbol{g}}\right\rangle=\frac{\langle w\rangle}{v_p^2}\vec{\boldsymbol{v}}_p$$

$$\overleftrightarrow{\boldsymbol{T}}=w\overleftrightarrow{\boldsymbol{I}}-Re\vec{\boldsymbol{D}}Re\vec{\boldsymbol{E}}-Re\vec{\boldsymbol{B}}Re\vec{\boldsymbol{H}}=w\hat{\boldsymbol{k}}\hat{\boldsymbol{k}},\quad \left\langle\overleftrightarrow{\boldsymbol{T}}\right\rangle=\langle w\rangle\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}$$

单色平面波在介质界面的折射和透射 (非全反射)：

边界条件对波矢的限制：沿界面方向分量相等：入射反射折射波的波矢共面；反射定律，折射定律；p 波 (电矢量平行于入射面偏振)、s 波 (电矢量垂直于入射面偏振) 的反射折射波仍是 p 波、s 波。

边界条件对振幅的限制：菲涅尔公式：

$$\begin{cases}r_p=\dfrac{Z_1\cos\theta_1-Z_2\cos\theta_2}{Z_1\cos\theta_1+Z_2\cos\theta_2}\\ t_p=\dfrac{2Z_2\cos\theta_1}{Z_1\cos\theta_1+Z_2\cos\theta_2}\end{cases},\quad \begin{cases}r_s=\dfrac{Z_2\cos\theta_1-Z_1\cos\theta_2}{Z_2\cos\theta_1+Z_1\cos\theta_2}\\ t_s=\dfrac{2Z_2\cos\theta_1}{Z_2\cos\theta_1+Z_1\cos\theta_2}\end{cases}$$

对于弱磁性材料 $\mu_1=\mu_2=\mu_0$，上式化为

$$\begin{cases}r_p=\dfrac{\tan(\theta_1-\theta_2)}{\tan(\theta_1+\theta_2)}\\ t_p=\dfrac{2\cos\theta_1\sin\theta_2}{\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)}\end{cases}\quad \begin{cases}r_s=-\dfrac{\sin(\theta_1-\theta_2)}{\sin(\theta_1+\theta_2)}\\ t_s=\dfrac{2\cos\theta_1\sin\theta_2}{\sin(\theta_1+\theta_2)}\end{cases}$$

能量输运：

$$\begin{cases}R\triangleq\left|\dfrac{\hat{\boldsymbol{n}}\cdot\left\langle\vec{\boldsymbol{S}}_R\right\rangle}{\hat{\boldsymbol{n}}\cdot\left\langle\vec{\boldsymbol{S}}_I\right\rangle}\right|=\dfrac{|\vec{\boldsymbol{E}}_{0R}|^2}{|\vec{\boldsymbol{E}}_{0I}|^2}\\ T\triangleq\left|\dfrac{\hat{\boldsymbol{n}}\cdot\left\langle\vec{\boldsymbol{S}}_T\right\rangle}{\hat{\boldsymbol{n}}\cdot\left\langle\vec{\boldsymbol{S}}_I\right\rangle}\right|=\dfrac{Z_1\cos\theta_2}{Z_2\cos\theta_1}\dfrac{|\vec{\boldsymbol{E}}_{0T}|^2}{|\vec{\boldsymbol{E}}_{0I}|^2}\end{cases}$$

全反射：

$$\vec{\boldsymbol{E}}_T=\vec{\boldsymbol{E}}_{0T}e^{-\kappa z}e^{i(k_1\sin\theta_1x-\omega t)}$$

$$\vec{\boldsymbol{k}}_T=k_{Tx}\hat{\boldsymbol{x}}+i\kappa\hat{\boldsymbol{z}}$$

$$\kappa=k_1\sqrt{\sin^2\theta_1-n_{21}}$$

倏逝波的 $\vec{\boldsymbol{E}},\vec{\boldsymbol{H}}$ 不同相，且不是横波。

若定义 $\begin{cases}k_{Tx}=k_1\sin\theta_1\overset{\Delta}{=}k_2\sin\theta_2\\ k_{Tz}=\sqrt{k_2^2-k_1^2\sin^2\theta_1}\overset{\Delta}{=}k_2\cos\theta_2\end{cases}$，则菲涅尔公式的反射部分仍成立 (因为无散条件和实矢量横波条件的分量等式是一样的)，只是此时 $\sin\theta_2>1,\cos\theta_2\in C$。且此时 $|r_p|,|r_s|=1$，入射波反射波之间有相位差。

透射系数：

$$
\begin{aligned}
\hat{\boldsymbol{z}}\cdot\left\langle\overrightarrow{S}_T\right\rangle &= \hat{\boldsymbol{z}}\cdot\frac{1}{2}Re[\overrightarrow{\boldsymbol{E}}_T^*\times\overrightarrow{\boldsymbol{H}}_T] \\
&= \hat{\boldsymbol{z}}\cdot\frac{1}{2\omega\mu_2}Re[|\overrightarrow{\boldsymbol{E}}_T|^2\overrightarrow{\boldsymbol{k}}_T-(\overrightarrow{\boldsymbol{k}}_T\cdot\overrightarrow{\boldsymbol{E}}_T^*)\overrightarrow{\boldsymbol{E}}_T] \\
&= \frac{1}{2\omega\mu_2}Re[|\overrightarrow{\boldsymbol{E}}_T|^2 i\kappa-(\overrightarrow{\boldsymbol{k}}_T\cdot\overrightarrow{\boldsymbol{E}}_T^*)E_{Tz}] \\
&= \frac{1}{2\omega\mu_2}Re[|\overrightarrow{\boldsymbol{E}}_T|^2 i\kappa-2i\kappa E_{Tz}^*E_{Tz}]=0 \\
&\quad\left(0=(\overrightarrow{\boldsymbol{k}}_T\cdot\overrightarrow{\boldsymbol{E}}_T)^*=k_{Tx}E_{Tx}^*-i\kappa E_{Tz}^*\right)
\end{aligned}
$$

$$
T\triangleq\left|\frac{\hat{\boldsymbol{n}}\cdot\left\langle\overrightarrow{\boldsymbol{S}}_T\right\rangle}{\hat{\boldsymbol{n}}\cdot\left\langle\overrightarrow{\boldsymbol{S}}_I\right\rangle}\right|=\left|\frac{\hat{\boldsymbol{z}}\cdot\left\langle\overrightarrow{\boldsymbol{S}}_T\right\rangle}{\hat{\boldsymbol{z}}\cdot\left\langle\overrightarrow{\boldsymbol{S}}_I\right\rangle}\right|=0
$$

导体对电磁波的影响：

欧姆型导体 $(\sigma<\infty)$：$\begin{cases} interior: & \overrightarrow{\boldsymbol{j}}_0=\sigma\overrightarrow{\boldsymbol{E}},\rho_0=0 \\ surface: & \overrightarrow{\boldsymbol{K}}_0=0,\sigma_0 \end{cases}$

理想导体 $(\sigma\to\infty)$：$\begin{cases} interior: & \overrightarrow{\boldsymbol{j}}_0=0,\rho_0=0 \\ surface: & \overrightarrow{\boldsymbol{K}}_0,\sigma_0 \end{cases}$

麦克斯韦方程：

$$
\begin{cases}
\nabla\cdot\overrightarrow{\boldsymbol{D}}=\rho_0 & \\
\nabla\cdot\overrightarrow{\boldsymbol{B}}=0 & (*) \\
\nabla\times\overrightarrow{\boldsymbol{E}}=-\partial_t\overrightarrow{\boldsymbol{B}} & (**) \\
\nabla\times\overrightarrow{\boldsymbol{H}}=\sigma\overrightarrow{\boldsymbol{E}}+\partial_t\overrightarrow{\boldsymbol{D}} &
\end{cases}
$$

其中 $(**)$ 式可以推出 $(*)$ 式。认为导体内自由电荷为 0 且导体为均匀导体，后两式也可以推出第一式。

边界条件：

$$
\begin{cases}
\begin{cases}
\hat{\boldsymbol{n}}\times(\overrightarrow{\boldsymbol{E}}_2-\overrightarrow{\boldsymbol{E}}_1)=0 \\
\hat{\boldsymbol{n}}\times(\overrightarrow{\boldsymbol{H}}_2-\overrightarrow{\boldsymbol{H}}_1)=0
\end{cases} \\
\hat{\boldsymbol{n}}\times(\overrightarrow{\boldsymbol{D}}_2-\overrightarrow{\boldsymbol{D}}_1)=\sigma_0
\end{cases}
$$

时谐场 (均匀导体)：

$$
\begin{cases}
\nabla\times\overrightarrow{\boldsymbol{E}}=i\omega\mu\overrightarrow{\boldsymbol{H}} \\
\nabla\times\overrightarrow{\boldsymbol{H}}=-i\omega\tilde{\epsilon}\overrightarrow{\boldsymbol{E}}
\end{cases}
\Longrightarrow
\begin{cases}
\begin{cases}
\nabla^2\overrightarrow{\boldsymbol{E}}+\tilde{k}^2\overrightarrow{\boldsymbol{E}}=0 \\
\nabla\cdot\overrightarrow{\boldsymbol{E}}=0
\end{cases} \\
\overrightarrow{\boldsymbol{H}}=-\dfrac{i}{\omega\mu}\nabla\times\overrightarrow{\boldsymbol{E}}
\end{cases}
$$

$$\tilde{\epsilon} \triangleq \epsilon + i\frac{\sigma}{\omega}, \quad \tilde{k} = \omega\sqrt{\mu\tilde{\epsilon}}$$

$$\tilde{n} \triangleq c\sqrt{\mu\tilde{\epsilon}}, \quad \tilde{z} \triangleq \sqrt{\frac{\mu}{\tilde{\epsilon}}}$$

单色平面波：

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_0 e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)}, \quad \vec{\boldsymbol{H}} = \vec{\boldsymbol{H}}_0 e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)}$$

$$\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{k}} = \tilde{k}^2 = \omega^2\mu\epsilon + i\omega\mu\sigma$$

$$\vec{\boldsymbol{k}} = \vec{\boldsymbol{\beta}} + i\vec{\boldsymbol{\alpha}}, \quad \beta^2 - \alpha^2 = \omega^2\mu\epsilon, \quad 2\vec{\boldsymbol{\alpha}} \cdot \vec{\boldsymbol{\beta}} = \omega\mu\sigma$$

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_0 e^{-\vec{\boldsymbol{\alpha}}\cdot\vec{\boldsymbol{r}}} e^{i(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{r}}-\omega t)}$$

$\vec{\boldsymbol{E}}, \vec{\boldsymbol{H}}$ 不同相，且不是横波。

导体表面的反射和透射：

边界条件对波矢的限制：在界面上的分量相等 $\beta_x = k_0 \sin\theta_1$；入射、反射、透射波的波矢共面；$\vec{\boldsymbol{\alpha}} = \alpha\hat{\boldsymbol{z}}$；

$$\begin{cases} \beta^2 - \alpha^2 = \omega^2\mu\epsilon \\ 2\vec{\boldsymbol{\alpha}} \cdot \vec{\boldsymbol{\beta}} = \omega\mu\sigma \end{cases} \implies \frac{\beta_z^2 - \alpha^2}{\alpha\beta_z} = \frac{2\omega\epsilon}{\sigma}\left(1 - \frac{\sin^2\theta_1}{n^2}\right)$$

$$\xRightarrow{\sigma\to\infty} \alpha \approx \beta_z \approx \sqrt{\frac{\omega\mu\sigma}{2}}, \quad \frac{\beta_x}{\beta_z} = \frac{\sin\theta_1}{n}\sqrt{\frac{2\omega\epsilon}{\sigma}} << 1$$

$$v_p = \frac{\omega}{\beta} \approx \frac{\omega}{\beta_z} = \frac{c}{n}\sqrt{\frac{2\omega\epsilon}{\sigma}} << \frac{c}{n}$$

趋肤效应：$d \triangleq \frac{1}{\alpha} \approx \sqrt{\frac{2}{\omega\mu\sigma}}$

良导体正入射：

$$\begin{cases} \vec{\boldsymbol{E}}_1(\vec{\boldsymbol{r}}, t) = E_{0I} e^{i(k_0 z-\omega t)}\hat{\boldsymbol{y}} + E_{0R} e^{i(k_0 z+\omega t)}\hat{\boldsymbol{y}} \\ \vec{\boldsymbol{E}}_2(\vec{\boldsymbol{r}}, t) = E_{0T} e^{i(\alpha(1+i)z-\omega t)}\hat{\boldsymbol{y}} \end{cases}$$

$$\begin{cases} \vec{\boldsymbol{H}}_1(\vec{\boldsymbol{r}}, t) = \frac{k_0}{\omega\mu_0}[E_{0I} e^{i(k_0 z-\omega t)} + E_{0R} e^{i(k_0 z+\omega t)}]\hat{\boldsymbol{x}} \\ \vec{\boldsymbol{H}}_2(\vec{\boldsymbol{r}}, t) = -\frac{\alpha(1+i)}{\omega\mu} E_{0T} e^{i(\alpha(1+i)z-\omega t)}\hat{\boldsymbol{x}} \end{cases}$$

$\vec{\boldsymbol{H}}_2$ 比 $\vec{\boldsymbol{E}}_2$ 落后 $\frac{\pi}{4}$；$\frac{\langle w_m \rangle}{\langle w_e \rangle} = \frac{\mu|\vec{\boldsymbol{H}}_2|^2}{\epsilon|\vec{\boldsymbol{E}}_2|^2} = \frac{\mu\sigma}{\epsilon\omega\mu} = \frac{\sigma}{\epsilon\omega} >> 1$

电流密度 $\vec{\boldsymbol{j}} = \sigma\vec{\boldsymbol{E}}_2$ 指数递减，良导体情况下近似为表面电流：

$$\vec{\boldsymbol{K}} = \int_0^\infty \vec{\boldsymbol{j}}(z, t)dz = \frac{\sigma E_{0T}}{\sqrt{2}\alpha} e^{-i(\omega t-\pi/4)}\hat{\boldsymbol{y}} = \vec{\boldsymbol{K}}_0 e^{-i(\omega t-\pi/4)}\hat{\boldsymbol{y}}$$

由热效应定义表面电阻 $R_s \triangleq \dfrac{1}{\sigma d} = \dfrac{\alpha}{\sigma}$

由边界条件确定正入射时的振幅关系，可有

$$R = |r|^2 = \frac{(1-k_0 d)^2+1}{(1+k_0 d)^2+1} \approx \frac{1-k_0 d}{1+k_0 d}$$

谐振腔与波导管：

理想导体边界下的时谐场：

$$\begin{cases} \begin{cases} \nabla^2 \vec{\boldsymbol{E}} + k^2 \vec{\boldsymbol{E}} = 0 \\ \nabla \cdot \vec{\boldsymbol{E}} = 0 \end{cases} \\ \vec{\boldsymbol{H}} = -\dfrac{i}{\omega\mu_0} \nabla \times \vec{\boldsymbol{E}} \end{cases} , \quad k = \frac{\omega}{c}$$

边界条件：

$$\begin{cases} \hat{\boldsymbol{n}} \times \vec{\boldsymbol{E}} \Big|_{\vec{S}} = 0 \\ (\nabla \cdot \vec{\boldsymbol{E}})_{\vec{S}} = 0 \begin{cases} \dfrac{\partial E_n}{\partial n}\Big|_{\vec{S}} = 0 \\ \dfrac{1}{E_s}\dfrac{\partial E_s}{\partial s}\Big|_{s=R} = \dfrac{1}{R} \\ \dfrac{1}{E_r}\dfrac{\partial E_r}{\partial r}\Big|_{r=R} = -\dfrac{2}{R} \end{cases} \end{cases}$$

得到解后计算其他物理量：

$$\sigma_0 = \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{D}}|_{\vec{S}}, \quad \vec{\boldsymbol{K}}_0 = \hat{\boldsymbol{n}} \times \vec{\boldsymbol{H}}|_{\vec{S}}$$

谐振腔 (尺寸为 $a \times b \times d$)：

$$\nabla^2 \vec{\boldsymbol{E}} + k^2 \vec{\boldsymbol{E}} = 0 \Longrightarrow \begin{cases} E_x = A_1 \cos k_1 x \sin k_2 y \sin k_3 z e^{-\omega t} \\ E_y = A_2 \sin k_1' x \cos k_2' y \sin k_3' z e^{-\omega t} \\ E_z = A_3 \sin k_1'' x \sin k_2'' y \cos k_3'' z e^{-\omega t} \end{cases}$$

$$\nabla \cdot \vec{\boldsymbol{E}} = 0 \Longrightarrow \begin{cases} \vec{\boldsymbol{k}} = \vec{\boldsymbol{k}}' = \vec{\boldsymbol{k}}'' \\ \vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{A}} = 0 \end{cases}$$

$$k_1 = \frac{m\pi}{a}, \quad k_2 = \frac{n\pi}{b}, \quad k_3 = \frac{l\pi}{d}, \quad k = \sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 + (\frac{l\pi}{d})^2}$$

由电场表达式，m, n, l 中最多一个为 0 才有非 0 解。每组 (m, n, l) 表示一种本征模式。m, n, l 全部非 0 时，每个本征模式对应两个独立偏振模式 $A_1 : A_2 : A_3$。

$$\omega_{mnl} = kc = \pi c \sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2 + (\frac{l}{d})^2}$$

频率只能取分立数值，称为本征频率。能在腔中激发的最低频率称为下截止频率 (m,n,l 中一个为 0)，具体取值由腔的尺寸决定。

矩形波导管 (尺寸 $a\times b$)：

求行波形式的解：

$$\begin{cases}\overrightarrow{\boldsymbol{E}}(\overrightarrow{\boldsymbol{r}},t)=\overrightarrow{\boldsymbol{E}}(x,y)e^{i(k_3z-\omega t)}\\ \overrightarrow{\boldsymbol{H}}(\overrightarrow{\boldsymbol{r}},t)=\overrightarrow{\boldsymbol{H}}(x,y)e^{i(k_3z-\omega t)}\end{cases}$$

$$\nabla^2\overrightarrow{\boldsymbol{E}}+k^2\overrightarrow{\boldsymbol{E}}=0\Longrightarrow\begin{cases}E_x=A_1\cos k_1x\sin k_2ye^{i\phi}\\ E_y=A_2\sin k_1'x\cos k_2'ye^{i\phi}\\ E_z=A_3\sin k_1''x\sin k_2''ye^{i\phi}\end{cases}$$

$$\nabla\cdot\overrightarrow{\boldsymbol{E}}=0\Longrightarrow\begin{cases}\overrightarrow{\boldsymbol{k}}=\overrightarrow{\boldsymbol{k}}'=\overrightarrow{\boldsymbol{k}}''\\ k_1A_1+k_2A_2-ik_3A_3=0\end{cases}$$

$$k_1=\frac{m\pi}{a},\quad k_2=\frac{n\pi}{b},\quad k=\sqrt{(\frac{m\pi}{a})^2+(\frac{n\pi}{b})^2+(k_3)^2}$$

由电场表达式，m,n 中最多一个为 0 才有非 0 解。每组 $(m\neq 0,n\neq 0)$ 对应两个独立偏振模式。一个为 0 时，对应一个偏振模式。

$$\omega=c\sqrt{(\frac{m\pi}{a})^2+(\frac{n\pi}{b})^2+(k_3)^2}=\sqrt{\omega_{c,mn}^2+k_3^2c^2}$$

频率可以连续取值，但存在下截止频率，具体取值由尺寸决定。

磁场：

$$\overrightarrow{\boldsymbol{H}}=-\frac{i}{\omega\mu_0}\nabla\times\overrightarrow{\boldsymbol{E}}=\begin{cases}H_x=-\frac{i}{\omega\mu_0}(k_2A_3-ik_3A_2)\sin k_1x\cos k_2ye^{i\phi}\\ H_y=-\frac{i}{\omega\mu_0}(ik_3A_1-k_1A_3)\cos k_1x\sin k_2ye^{i\phi}\\ H_z=-\frac{i}{\omega\mu_0}(k_1A_2-k_2A_1)\sin k_1x\sin k_2ye^{i\phi}\end{cases}$$

波导管的行波解不存在横电磁波 (TEM)($A_3=0,k_1A_2-k_2A_1=0,k_1A_1+k_2A_2-ik_3A_3=0$ 在 m,n 至多一个为 0 的情况下无解)。当 $m\neq 0,n\neq 0$ 时，存在 TE 波 ($A_3=0,k_1A_1+k_2A_2=0$ 有解) 和 TM 波 ($k_1A_2-k_2A_1=0,k_1A_1+k_2A_2-ik_3A_3=0$ 有解); 当两个数有一个为 0 时，只存在 TE 波 (此时电磁为横波，故磁场不可能为横波)。

相速群速：

$$v_p=\frac{\omega}{k_3}=\frac{\omega c}{\sqrt{\omega^2-\omega_{c,mn}^2}}>c,\quad v_g\frac{d\omega}{dk_3}=\frac{k_3c^2}{\omega}=\frac{k_3c^2}{\sqrt{\omega_{c,mn}+k_3^2c^2<c}}<c$$

$$v_pv_g=c^2$$

一般波导管：

分为垂直 ($\perp$) 和平行方向 ($\hat{\boldsymbol{z}}$)

$$\nabla = \nabla_\perp + \hat{\boldsymbol{z}}\partial_z$$

行波形式的解：

$$\begin{cases} \overrightarrow{\boldsymbol{E}}(\overrightarrow{\boldsymbol{r}}, t) = [\overrightarrow{\boldsymbol{E}}_\perp(\overrightarrow{\boldsymbol{r}}_\perp) + E_z(\overrightarrow{\boldsymbol{r}}_\perp)\hat{\boldsymbol{z}}]e^{i(k_3 z - \omega t)} \\ \overrightarrow{\boldsymbol{H}}(\overrightarrow{\boldsymbol{r}}, t) = [\overrightarrow{\boldsymbol{H}}_\perp(\overrightarrow{\boldsymbol{r}}_\perp) + H_z(\overrightarrow{\boldsymbol{r}}_\perp)\hat{\boldsymbol{z}}]e^{i(k_3 z - \omega t)} \end{cases}$$

$$\nabla = \nabla_\perp + ik_3\hat{\boldsymbol{z}}, \quad \partial_t = -i\omega$$

基本方程：

$$\begin{cases} \nabla \times \overrightarrow{\boldsymbol{E}} = i\omega\mu\overrightarrow{\boldsymbol{H}} \\ \nabla \times \overrightarrow{\boldsymbol{H}} = -i\omega\epsilon\overrightarrow{\boldsymbol{E}} \end{cases}$$

$$\Longrightarrow \begin{cases} \nabla_\perp \times \overrightarrow{\boldsymbol{E}}_\perp = i\omega\mu H_z\hat{\boldsymbol{z}}, \quad k_3\hat{\boldsymbol{z}} \times \overrightarrow{\boldsymbol{E}}_\perp - \omega\mu\overrightarrow{\boldsymbol{H}}_\perp = -i\hat{\boldsymbol{z}} \times \nabla_\perp E_z \\ \nabla_\perp \times \overrightarrow{\boldsymbol{H}}_\perp = -i\omega\epsilon E_z\hat{\boldsymbol{z}}, \quad k_3\hat{\boldsymbol{z}} \times \overrightarrow{\boldsymbol{H}}_\perp + \omega\epsilon\overrightarrow{\boldsymbol{E}}_\perp = -i\hat{\boldsymbol{z}} \times \nabla_\perp H_z \end{cases}$$

不存在 TEM:

若 $E_z = H_z = 0$

$$\begin{cases} \nabla_\perp \times \overrightarrow{\boldsymbol{E}}_\perp = 0, \quad k_3\hat{\boldsymbol{z}} \times \overrightarrow{\boldsymbol{E}}_\perp - \omega\mu\overrightarrow{\boldsymbol{H}}_\perp = 0 \\ \nabla_\perp \times \overrightarrow{\boldsymbol{H}}_\perp = 0, \quad k_3\hat{\boldsymbol{z}} \times \overrightarrow{\boldsymbol{H}}_\perp + \omega\epsilon\overrightarrow{\boldsymbol{E}}_\perp = 0 \end{cases} \Longrightarrow \begin{cases} \nabla_\perp \cdot \overrightarrow{\boldsymbol{E}}_\perp = 0 \\ \nabla_\perp \cdot \overrightarrow{\boldsymbol{H}}_\perp = 0 \end{cases}$$

$$\begin{cases} \begin{cases} \nabla_\perp \cdot \overrightarrow{\boldsymbol{E}}_\perp = 0 \\ \overrightarrow{\boldsymbol{E}}_\perp = -\nabla_\perp\phi_\perp(\overrightarrow{\boldsymbol{r}}_\perp) \end{cases} \Longrightarrow \nabla_\perp^2\phi_\perp = 0 \\ \hat{\boldsymbol{n}} \times \overrightarrow{\boldsymbol{E}}_\perp|_{\overrightarrow{S}} = 0 \Longrightarrow \phi_\perp(\overrightarrow{\boldsymbol{r}})|_{\overrightarrow{S}} = const \end{cases} \Longrightarrow E_\perp = 0$$

$$\cdots \Longrightarrow H_\perp = 0$$

H_z 在截面上的积分为 0：

$$\hat{\boldsymbol{n}} \times (\overrightarrow{\boldsymbol{E}}_2 - \overrightarrow{\boldsymbol{E}}_1) = 0 \Longrightarrow \overrightarrow{\boldsymbol{E}}_\perp|_{\overrightarrow{S}} // \hat{\boldsymbol{n}}$$

$$\Longrightarrow 0 = \oint_C d\overrightarrow{\boldsymbol{l}} \cdot \overrightarrow{\boldsymbol{E}}_\perp = \int_\Sigma d\overrightarrow{\boldsymbol{\sigma}} \cdot (\nabla_\perp \times \overrightarrow{\boldsymbol{E}}_\perp) = i\omega\mu\int_\Sigma d\sigma H_z$$

$$\Longrightarrow \int_\Sigma d\sigma H_z = 0$$

同时可以看出 TE 波的 H_z 不为常数。(否则将存在 TEM)

基本方程的转化：

$$\begin{cases}\nabla_\perp \times \overrightarrow{\boldsymbol{E}}_\perp = i\omega\mu H_z \hat{\boldsymbol{z}} \\ \nabla_\perp \times \overrightarrow{\boldsymbol{H}}_\perp = -i\omega\epsilon E_z \hat{\boldsymbol{z}}\end{cases}, \quad \begin{cases}\gamma^2 \overrightarrow{\boldsymbol{E}}_\perp = ik_3 \nabla_\perp E_z - i\omega\mu \hat{\boldsymbol{z}} \times \nabla_\perp H_z \\ \gamma^2 \overrightarrow{\boldsymbol{H}}_\perp = ik_3 \nabla_\perp H_z + i\omega\epsilon \hat{\boldsymbol{z}} \times \nabla_\perp E_z\end{cases} \quad (*)$$

$$\gamma^2 \triangleq \omega^2\mu\epsilon - k_3^2$$

$$\Longrightarrow \begin{cases}\gamma^2(\nabla_\perp \times \overrightarrow{\boldsymbol{E}}_\perp) = -i\omega\mu\nabla_\perp(\hat{\boldsymbol{z}} \times \nabla_\perp H_z) \Longrightarrow \nabla_\perp^2 E_z = -\gamma^2 E_z \\ \gamma^2(\nabla_\perp \times \overrightarrow{\boldsymbol{H}}_\perp) = i\omega\mu\nabla_\perp(\hat{\boldsymbol{z}} \times \nabla_\perp E_z) \Longrightarrow \nabla_\perp^2 H_z = -\gamma^2 H_z\end{cases} \quad (*1)$$

$$\Longrightarrow \begin{cases}\gamma^2(\nabla_\perp \cdot \overrightarrow{\boldsymbol{E}}_\perp) = ik_3\nabla_\perp^2 E_z = -\gamma^2 \partial_z \hat{\boldsymbol{z}} \cdot E_z \hat{\boldsymbol{z}} \Longrightarrow \nabla \cdot \overrightarrow{\boldsymbol{E}} = 0 \\ \gamma^2(\nabla_\perp \cdot \overrightarrow{\boldsymbol{H}}_\perp) = ik_3\nabla_\perp^2 H_z = -\gamma^2 \partial_z \hat{\boldsymbol{z}} \cdot H_z \hat{\boldsymbol{z}} \Longrightarrow \nabla \cdot \overrightarrow{\boldsymbol{H}} = 0\end{cases} \quad (*2)$$

$(*)$ 可以看出，若 E_z, H_z 均为 0 或常数，则只有 0 解。即 TE 波的 H_z 不为常数，TM 波的 E_z 不为常数。

$(*1)$ 对于非 0 解，E_z, H_z 不能均为 0 或常数，故此时必有 $\gamma^2 \neq 0$

$(*2)$ 由于此时方程未耦合 ($(*)$ 可以化回原来的方程)，故无散条件自动满足是显然的。

”Helmholtz 方程”：

$$\begin{cases}(\nabla_\perp^2 + \gamma^2)E_z = 0 \\ (\nabla_\perp^2 + \gamma^2)H_z = 0\end{cases}$$

” 亥姆霍兹方程” 可以由 $(*1)$ 式得到，也可从 $\nabla^2 E_z + k^2 E_z = 0$ 和 γ^2 的定义直接得到。在矩形波导管中要解三个亥姆霍兹方程加上无散条件的限制，这里只需解两个 (无散条件自动满足)。

方程求解：

$$TM: \begin{cases}\begin{cases}(\nabla_\perp^2 + \gamma^2)E_z = 0 \\ \hat{\boldsymbol{n}} \times \overrightarrow{\boldsymbol{E}}|_{\overrightarrow{S}} = 0 \Longrightarrow E_z = 0|_{\overrightarrow{S}}\end{cases} \\ \overrightarrow{\boldsymbol{E}}_\perp = \dfrac{ik_3}{\gamma^2}\nabla_\perp E_z, \quad \overrightarrow{\boldsymbol{H}} = \overrightarrow{\boldsymbol{H}}_\perp = \dfrac{i\omega\epsilon}{\gamma^2}\hat{\boldsymbol{z}} \times \nabla_\perp E_z\end{cases}$$

$$TE: \begin{cases}\begin{cases}(\nabla_\perp^2 + \gamma^2)H_z = 0 \\ \hat{\boldsymbol{n}} \times \overrightarrow{\boldsymbol{E}}|_{\overrightarrow{S}} = 0 \Longrightarrow \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{z}} \times \nabla_\perp H_z)|_{\overrightarrow{S}} = 0 \Longrightarrow \dfrac{\partial H_z}{\partial n}|_{\overrightarrow{S}} = 0\end{cases} \\ \overrightarrow{\boldsymbol{H}}_\perp = \dfrac{ik_3}{\gamma^2}\nabla_\perp H_z, \quad \overrightarrow{\boldsymbol{E}} = \overrightarrow{\boldsymbol{E}}_\perp = -\dfrac{i\omega\mu}{\gamma^2}\hat{\boldsymbol{z}} \times \nabla_\perp H_z\end{cases}$$

边界条件的取定是因为: 面电流密度未知，不能用于求解；无散条件已被自动保证满足，不能用于求解。

$\gamma^2 > 0$：

$$\psi(\vec{\boldsymbol{r}}_\perp) \triangleq E_z(\vec{\boldsymbol{r}}_\perp) \quad (inTM) \quad or \quad H_z(\vec{\boldsymbol{r}}_\perp) \quad (inTE)$$

$$\nabla_\perp^2\psi + \gamma^2\psi = 0 \Longrightarrow$$

$$\begin{aligned}
\gamma^2 \int dV|\psi|^2 &= \int dV\psi^*\gamma^2\psi = -\int dV\psi^*\nabla_\perp^2\psi \\
&= \cdots = -\oint d\vec{\boldsymbol{\sigma}} \cdot \psi^*\nabla_\perp\psi + \int dV|\nabla_\perp\psi|^2 \\
&= -\oint d\sigma\psi^*\frac{\partial\psi}{\partial n} + \int dV|\nabla_\perp\psi|^2 \\
&= \int dV|\nabla_\perp\psi|^2 > 0 \quad \left(E_z = 0|_{\vec{\boldsymbol{S}}} = 0, \frac{\partial H_z}{\partial n}|_{\vec{\boldsymbol{S}}} = 0\right)
\end{aligned}$$

6 电磁辐射

势方程：

$$L \triangleq \nabla \cdot \vec{\boldsymbol{A}} + \frac{1}{c^2}\partial_t\phi, \quad \Box \triangleq \nabla^2 - \frac{1}{c^2}\partial_t^2 (\text{d'Alembert 算子})$$

$$\begin{cases} \Box\phi + \partial_t L = -\dfrac{\rho}{\epsilon_0} \\ \Box\vec{A} - \nabla L = -\mu_0\vec{\boldsymbol{j}} \end{cases}$$

Coulomb 规范：

$$\begin{cases} \nabla \cdot \vec{\boldsymbol{A}} = 0 & (I_c) \\ \nabla^2\phi = -\dfrac{\rho}{\epsilon_0} & (I_a) \\ \Box\vec{\boldsymbol{A}} = -\mu_0\vec{\boldsymbol{j}} & (I_b) \end{cases}$$

Lorentz 规范：

$$\begin{cases} L = \nabla \cdot \vec{\boldsymbol{A}} + \dfrac{1}{c^2}\partial_t\phi = 0 & (I_c) \\ \Box\phi = -\dfrac{\rho}{\epsilon_0} & (I_a) \\ \Box\vec{\boldsymbol{A}} = -\mu_0\vec{\boldsymbol{j}} & (I_b) \end{cases}$$

自由空间单色平面波解：

$$\phi(\vec{\boldsymbol{r}}, t) = \phi_0 e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}} - \omega t)}, \quad \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}, t) = \vec{\boldsymbol{A}}_0 e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}} - \omega t)}$$

代入 Coulomb 规范：

$$I_a \Longrightarrow \phi = 0, I_b \Longrightarrow \omega = kc, I_c \Longrightarrow \vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{A}} = 0$$

$$\vec{\boldsymbol{E}} = -\partial_t\vec{\boldsymbol{A}} = i\omega\vec{\boldsymbol{A}}, \vec{\boldsymbol{B}} = \nabla \times \vec{\boldsymbol{A}} = \frac{\vec{\boldsymbol{k}} \times \vec{\boldsymbol{E}}}{\omega}$$

代入 Lorentz 规范：

$$I_a, I_b \Longrightarrow \omega = kc, I_c \Longrightarrow \vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{A}} = \frac{\omega}{c^2}\phi$$

$$\vec{\boldsymbol{B}} = \nabla \times \vec{\boldsymbol{A}} = i\vec{\boldsymbol{k}} \times \vec{\boldsymbol{A}}, \vec{\boldsymbol{E}} = -\nabla\phi - \partial_t \vec{\boldsymbol{A}} = i\omega[\vec{\boldsymbol{A}} - (\hat{\boldsymbol{k}} \cdot \vec{\boldsymbol{A}})\hat{\boldsymbol{k}}] = c\vec{\boldsymbol{B}} \times \hat{\boldsymbol{k}}$$

推迟势：

$$\nabla^2\phi - \frac{1}{c^2}\partial_t^2\phi = -\frac{\rho(\vec{\boldsymbol{r}}, t)}{\epsilon_0}$$

Green:

$$\nabla^2\phi' - \frac{1}{c^2}\partial_t^2\phi' = -\frac{\rho(\vec{\boldsymbol{r}}', t)\delta(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}')}{\epsilon_0}, \quad \rho(\vec{\boldsymbol{r}}, t) = \int \rho(\vec{\boldsymbol{r}}', t)\delta(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}')dV'$$

$$\Longrightarrow \begin{cases} \nabla^2\phi' - \dfrac{1}{c^2}\partial_t^2\phi' = 0 \\ \displaystyle\int dV(\nabla^2\phi' - \frac{1}{c^2}\partial_t^2\phi') = -\frac{\rho(\vec{\boldsymbol{r}}', t)}{\epsilon_0} \end{cases}$$

$$\Longrightarrow \begin{cases} \phi'(\mathbb{R}, t) = \dfrac{1}{\mathbb{R}} f\big(t - \dfrac{\mathbb{R}}{c}\big) \\ \displaystyle\int dV(f\nabla^2\frac{1}{\mathbb{R}} + 2\nabla f \cdot \nabla\frac{1}{\mathbb{R}} + \frac{1}{\mathbb{R}}\nabla^2 f - \frac{1}{c^2\mathbb{R}}\frac{\partial^2 f}{\partial t^2}) = -4\pi f(t) = -\frac{\rho(\vec{\boldsymbol{r}}', t)}{\epsilon_0} \end{cases}$$

$$\phi'(\mathbb{R}, t) = \frac{1}{4\pi\epsilon_0}\frac{\rho\big(\vec{\boldsymbol{r}}', t - \frac{\mathbb{R}}{c}\big)}{\mathbb{R}}$$

$$\phi(\vec{\boldsymbol{r}}, t) = \int dV'\phi'(\mathbb{R}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\big(\vec{\boldsymbol{r}}', t - \frac{\mathbb{R}}{c}\big)}{\mathbb{R}}dV'$$

$$\cdots \Longrightarrow \vec{A}(\vec{\boldsymbol{r}}, t) = \frac{\mu_0}{4\pi}\int \frac{\vec{\boldsymbol{j}}\big(\vec{\boldsymbol{r}}', t - \frac{\mathbb{R}}{c}\big)}{\mathbb{R}}dV'$$

满足洛伦兹条件:

$$\begin{aligned} \nabla \cdot \vec{\boldsymbol{A}} &= \frac{\mu_0}{4\pi}\int \nabla \cdot \frac{\vec{\boldsymbol{j}}\big(\vec{\boldsymbol{r}}', t - \frac{\mathbb{R}}{c}\big)}{\mathbb{R}}dV' \\ &= \frac{\mu_0}{4\pi}\int dV'\big[-\nabla' \cdot \frac{\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}}', t_r)}{\mathbb{R}} + \frac{[\nabla \cdot \vec{\boldsymbol{j}}(\vec{\boldsymbol{r}}', t_r)]_{t_r}}{\mathbb{R}}\big] \\ &= -\frac{1}{4\pi\epsilon_0 c^2}\int dV'\frac{\partial\rho(\vec{\boldsymbol{r}}', t_r)}{\mathbb{R}} \\ &= -\frac{1}{c^2}\partial_t\phi \end{aligned}$$

电磁场：

$$\vec{\boldsymbol{B}} = \nabla \times \vec{\boldsymbol{A}} = \cdots, \vec{\boldsymbol{E}} = -\nabla\phi - \partial_t\vec{\boldsymbol{A}} = \cdots$$

在周期变换的电偶极子情况下计算坡印廷矢量可以发现在一个周期内流出去的能量和电偶极子二阶导数有关。

谐振电流的辐射场：

$$\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t)=\vec{\boldsymbol{j}}_0(\vec{\boldsymbol{r}})e^{-i\omega t}$$

$$\frac{dP}{d\Omega}=\lim_{r\to\infty}(\hat{\boldsymbol{r}}\cdot\vec{\boldsymbol{S}})r^2=\frac{r^2}{\mu_0}\hat{\boldsymbol{r}}\cdot(\vec{\boldsymbol{E}}_{rad}\times\vec{\boldsymbol{B}}_{rad})$$

$$-\int_V dV\vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{j}}=\frac{dW}{dt}+\oint_{\partial V}d\vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{S}}\Longrightarrow\langle P\rangle=-\frac{1}{2}Re\int_V dV\vec{\boldsymbol{E}}^*\cdot\vec{\boldsymbol{j}}$$

势与场：

$$\begin{cases}\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t)=\vec{\boldsymbol{j}}_0(\vec{\boldsymbol{r}})e^{-i\omega t}\\ \nabla\cdot\vec{\boldsymbol{j}}=-\partial_t\rho=i\omega\rho\Longrightarrow\rho(\vec{\boldsymbol{r}},t)=\rho_0(\vec{\boldsymbol{r}})e^{-i\omega t}=-\frac{i}{\omega}\nabla\cdot\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t)\end{cases}$$

$$\Longrightarrow\begin{cases}\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t)=\vec{\boldsymbol{A}}_0(\vec{\boldsymbol{r}})e^{-i\omega t}=\frac{\mu_0}{4\pi}\int dV'\frac{\vec{\boldsymbol{j}}_0(\vec{\boldsymbol{r}}')e^{ik\mathbb{R}}}{\mathbb{R}}e^{-i\omega t}\\ \phi(\vec{\boldsymbol{r}},t)=\frac{1}{4\pi\epsilon_0}\int dV'\frac{\rho_0(\vec{\boldsymbol{r}}')e^{ik\mathbb{R}}}{\mathbb{R}}e^{-i\omega t}\end{cases}$$

洛伦兹条件的检验：

$$\begin{aligned}\phi(\vec{\boldsymbol{r}},t)&=(-\frac{ic^2}{\omega})\frac{\mu_0}{4\pi}\int dV'\frac{\nabla'\cdot\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}}',t)}{\mathbb{R}}e^{ik\mathbb{R}}\\&=(-\frac{ic^2}{\omega})\frac{\mu_0}{4\pi}[\oint d\vec{\boldsymbol{\sigma}}'\cdot\frac{\vec{\boldsymbol{j}}e^{ik\mathbb{R}}}{\mathbb{R}}+\nabla\cdot\int dV'\frac{\vec{\boldsymbol{j}}e^{ik\mathbb{R}}}{\mathbb{R}}]\\&=-i\frac{c^2}{\omega}\nabla\cdot\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t)\end{aligned}$$

$$\Longrightarrow\nabla\cdot\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t)+i\omega\frac{1}{c^2}\phi(\vec{\boldsymbol{r}},t)=\nabla\cdot\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t)+\frac{1}{c^2}\partial_t^2\phi(\vec{\boldsymbol{r}},t)=0$$

电磁场的计算：

$$\vec{\boldsymbol{B}}=\nabla\times\vec{\boldsymbol{E}},\vec{\boldsymbol{E}}=i\frac{c^2}{\omega}\nabla\times\vec{\boldsymbol{B}}$$

（第二式是因为考虑场点没有源）

场区划分：

近场区：$\mathbb{R}<<\lambda(k\mathbb{R}<<1)$

感应区：$\mathbb{R}\sim\lambda(k\mathbb{R}\sim 1)$

辐射区：$\mathbb{R}>>\lambda(k\mathbb{R}>>1)$

辐射场的提取：

由于球表面积 $S\sim r^2$，故 $\vec{\boldsymbol{E}},\vec{\boldsymbol{B}}$ 只需保留到 $\frac{1}{r}$，由 $\vec{\boldsymbol{E}},\vec{\boldsymbol{B}}$ 与 $\phi,\vec{\boldsymbol{A}}$ 的关系可知，$\vec{\boldsymbol{A}}$ 只需保留到 $\frac{1}{r}$。$(\frac{d}{dr}\frac{e^{ikr}}{r}=ik\frac{e^{ikr}}{r}-\frac{e^{ikr}}{r^2})$

approx.1:$r' << \mathbb{R}$

$$\frac{1}{\mathbb{R}} \approx \frac{1}{r}$$

$$k\mathbb{R} = ke^{-\vec{r}'\cdot\nabla}r = k[r - \vec{r}'\cdot\hat{r} + \frac{1}{2}\vec{r}'\vec{r}':\frac{\overleftrightarrow{I}}{r} + \cdots]$$

由于 λ, r' 的关系不明确，故 $k\mathbb{R}$ 不能化简。

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi r}\int dV'\vec{j}(\vec{r}',t)e^{ik\mathbb{R}}$$

approx.2:$r' << \mathbb{R}, r' \lesssim \lambda$

$$\frac{1}{\mathbb{R}} \approx \frac{1}{r}$$

$$\begin{aligned} k\mathbb{R} &= ke^{-\vec{r}'\cdot\nabla}r = k[r - \vec{r}'\cdot\hat{r} + \frac{1}{2}\vec{r}'\vec{r}':\frac{\overleftrightarrow{I}}{r} + \cdots] \\ &= k[r - \vec{r}'\cdot\hat{r}] = \vec{k}\cdot\vec{\mathbb{R}} \\ \vec{A}(\vec{r},t) &= \frac{\mu_0}{4\pi r}e^{ikr}\int dV'\vec{j}(\vec{r}',t)e^{-i\vec{k}\cdot\vec{r}'} \end{aligned}$$

此时有

$$\vec{B} = \nabla\times\vec{A} \sim \nabla\frac{e^{ikr}e^{-i\vec{k}\cdot\vec{r}'}}{r}\begin{cases} (\nabla\frac{1}{r})e^{ikr}e^{-i\vec{k}\cdot\vec{r}'} \sim \frac{1}{r^2} & (*1) \\ \frac{e^{-i\vec{k}\cdot\vec{r}'}\nabla(e^{ikr})}{r} \sim \frac{k}{r} & (*2) \\ \frac{e^{ikr}\nabla e^{-i\vec{k}\cdot\vec{r}'}}{r} \sim \frac{kr'}{r^2} & (*3) \end{cases}$$

$$\frac{(*3)}{(*2)} \sim \frac{r'}{r} << 1$$

$$\frac{(*1)}{(*2)} \sim \frac{1}{kr}$$

从而 $(*3)$ 式可忽略，但 r, λ 的关系不明确，故 $(*1), (*2)$ 的取舍不明。

approx.3:$r' << \mathbb{R}, r' \lesssim \lambda, \lambda << r$(辐射区)

$$\frac{1}{\mathbb{R}} \approx \frac{1}{r}$$

$$k\mathbb{R} = \vec{k}\cdot\vec{\mathbb{R}}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi r}e^{ikr}\int dV'\vec{j}(\vec{r}',t)e^{-i\vec{k}\cdot\vec{r}'}$$

而此时 $\frac{(*1)}{(*2)} \sim \frac{1}{kr} << 1$，故

$$\vec{B} = \nabla\times\vec{A} = i\vec{k}\times\vec{A}$$

$$\vec{E} = i\frac{c^2}{\omega}\nabla\times\vec{B} = -\frac{c^2}{\omega}\vec{k}\times\vec{B} = c\vec{B}\times\hat{r}$$

$$\partial_t \leftrightarrow -i\omega, \quad \nabla \leftrightarrow i\vec{k} = i\frac{\omega}{c}\hat{r} \leftrightarrow -\frac{\hat{r}}{c}\partial_t$$

可见此时的辐射场是球面波且为横波。

$$\left\langle \vec{S} \right\rangle = \frac{1}{2\mu_0} Re(\vec{E}^* \times \vec{B}) = \frac{c}{2\mu_0} |\vec{B}|^2 \hat{r}$$

小场源辐射：

$$r' << \mathbb{R}, r' << \lambda, \lambda << r$$

$$\begin{aligned}
\vec{A}(\vec{r}, t) &= \frac{\mu_0}{4\pi r} e^{ikr} \int dV' \vec{j}(\vec{r}', t) e^{-i\vec{k}\cdot\vec{r}'} \\
&= \frac{\mu_0}{4\pi r} e^{ikr} \int dV' \vec{j}(\vec{r}', t)(1 - i\vec{k} \cdot \vec{r}') \\
&= \frac{\mu_0}{4\pi r} e^{ikr} [\dot{\vec{p}}(t) - i\vec{m}(t) \times \vec{k} - i\frac{1}{6}\overleftrightarrow{D}(t) \cdot \vec{k} - i\frac{1}{6}\dot{g}(t)\vec{k}] \\
&= \frac{\mu_0}{4\pi r} e^{ikr} [\dot{\vec{p}}(t) + \frac{1}{c}\dot{\vec{m}}(t) \times \hat{r} + \frac{1}{6c}\hat{r} \cdot \dot{\overleftrightarrow{D}}(t) + \frac{1}{6c}\ddot{g}(t)\hat{r}] \\
&= \frac{\mu_0}{4\pi r} [\dot{\vec{p}}(t - \frac{r}{c}) + \frac{1}{c}\dot{\vec{m}}(t - \frac{r}{c}) \times \hat{r} + \frac{1}{6c}\hat{r} \cdot \dot{\overleftrightarrow{D}}(t - \frac{r}{c}) + \frac{1}{6c}\ddot{g}(t - \frac{r}{c})\hat{r}]
\end{aligned}$$

式中的 g 并不影响，因为 g 和 $\hat{r}$ 一起出现，在叉乘中消掉。也可以通过保洛伦兹规范的规范变换消掉：

$$\vec{A}' = \vec{A} + \nabla\psi = \vec{A} + i\vec{k}\psi = \vec{A} - \frac{\hat{r}}{c}\dot{\psi}$$

$$\frac{\dot{\psi}}{c} = \frac{\mu_0}{4\pi r} \frac{1}{6c} \ddot{g}(t - \frac{r}{c}) \Longrightarrow \psi = \frac{\mu_0}{24\pi c} \frac{\dot{g}(t - \frac{r}{c})}{r}$$

可以看出此时 ψ 是满足达朗贝尔方程的 (保洛伦兹规范)。

电偶极辐射：

$$\vec{A}(\vec{r}, t) = \frac{\mu_0 e^{ikr}}{4\pi r} \dot{\vec{p}}(t)$$

$$\vec{B} = \frac{1}{c}\dot{\vec{A}} \times \hat{r} = \frac{\mu_0 e^{ikr}}{4\pi c r} \ddot{\vec{p}} \times \hat{r}$$

$$\vec{E} = c\vec{B} \times \hat{r} = \frac{\mu_0 e^{ikr}}{4\pi r} (\ddot{\vec{p}} \times \hat{r}) \times \hat{r}$$

$$\left\langle \vec{S} \right\rangle = \frac{c}{2\mu_0} |\vec{B}|^2 \hat{r} = \frac{\mu_0}{32\pi^2 c r^2} |\ddot{\vec{p}} \times \hat{r}|^2 \hat{r}$$

$$\frac{d\langle P\rangle}{d\Omega} = \frac{\mu_0}{32\pi^2 c} |\ddot{\vec{p}} \times \hat{r}|^2$$

$$\begin{aligned}
\langle P\rangle &= \frac{1}{32\pi^2 \epsilon_0 c^3} \int d\Omega (\ddot{\vec{p}}^* \times \hat{r}) \cdot (\ddot{\vec{p}} \times \hat{r}) \\
&= \frac{1}{32\pi^2 \epsilon_0 c^3} \int d\Omega [|\ddot{\vec{p}}|^2 - |\ddot{\vec{p}} \cdot \hat{r}|^2] \\
&= \frac{1}{32\pi^2 \epsilon_0 c^3} (4\pi|\ddot{\vec{p}}|^2 - \int d\Omega (\ddot{p}_i^* \alpha_i)(\ddot{p}_j \alpha_j)) \\
&= \frac{1}{32\pi^2 \epsilon_0 c^3} (4\pi|\ddot{\vec{p}}|^2 - \frac{4\pi}{3} \delta_{ij} \ddot{p}_i^* \ddot{p}_j) \\
&= \frac{|\ddot{\vec{p}}|^2}{12\pi \epsilon_0 c^3}
\end{aligned}$$

匀速圆周运动点电荷：

$$\vec{\boldsymbol{p}} = eR(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})e^{-i\omega t}$$

$$\vec{\boldsymbol{B}} = \frac{\mu_0 e^{ikr}}{4\pi cr}\ddot{\vec{\boldsymbol{p}}} \times \hat{\boldsymbol{r}} = \frac{e\omega^2 R}{4\pi\epsilon_0 c^3}\frac{-i\hat{\boldsymbol{\theta}} + \cos\hat{\boldsymbol{\phi}}}{r}e^{i(kr-\omega t+\phi)}$$

$$\vec{\boldsymbol{E}} = \frac{\mu_0 e^{ikr}}{4\pi r}(\ddot{\vec{\boldsymbol{p}}} \times \hat{\boldsymbol{r}}) \times \hat{\boldsymbol{r}} = \frac{e\omega^2 R}{4\pi\epsilon_0 c^2}\frac{\cos\theta\hat{\boldsymbol{\theta}} + i\hat{\boldsymbol{\phi}}}{r}e^{i(kr-\omega t+\phi)}$$

$$R = \frac{E_\phi}{E_\theta} = \frac{i}{\cos\theta}$$

$$\frac{d\langle P\rangle}{d\Omega} = \frac{\mu_0}{32\pi^2 c}|\ddot{\vec{\boldsymbol{p}}} \times \hat{\boldsymbol{r}}|^2 = \frac{e^2\omega^4 R^2}{32\pi^2\epsilon_0 c^3}(1 + \cos^2\theta)$$

$$\langle P\rangle = \frac{|\ddot{\vec{\boldsymbol{p}}}|^2}{12\pi\epsilon_0 c^3} = \frac{e^2\omega^4 R^2}{6\pi\epsilon_0 c^3}$$

磁偶极辐射：

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}, t) = \frac{\mu_0 e^{ikr}}{4\pi cr}\dot{\vec{\boldsymbol{m}}}(t)$$

$$\vec{\boldsymbol{B}} = \frac{1}{c}\dot{\vec{\boldsymbol{A}}} \times \hat{\boldsymbol{r}} = \frac{\mu_0 e^{ikr}}{4\pi c^2 r}(\ddot{\vec{\boldsymbol{m}}} \times \hat{\boldsymbol{r}}) \times \hat{\boldsymbol{r}}$$

$$\vec{\boldsymbol{E}} = c\vec{\boldsymbol{B}} \times \hat{\boldsymbol{r}} = -\frac{\mu_0 e^{ikr}}{4\pi cr}\ddot{\vec{\boldsymbol{m}}} \times \hat{\boldsymbol{r}}$$

$$\left\langle\vec{\boldsymbol{S}}\right\rangle = \frac{c}{2\mu_0}|\vec{\boldsymbol{B}}|^2\hat{\boldsymbol{r}} = \frac{\mu_0}{32\pi^2 c^3 r^2}|\ddot{\vec{\boldsymbol{m}}} \times \hat{\boldsymbol{r}}|^2\hat{\boldsymbol{r}}$$

$$\frac{d\langle P\rangle}{d\Omega} = \frac{\mu_0}{32\pi^2 c^3}|\ddot{\vec{\boldsymbol{m}}} \times \hat{\boldsymbol{r}}|^2$$

$$\langle P\rangle = \frac{\mu_0|\ddot{\vec{\boldsymbol{m}}}|^2}{12\pi c^3}$$

电四极辐射：

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}, t) = \frac{\mu_0 e^{ikr}}{24\pi cr}\hat{\boldsymbol{r}} \cdot \dot{\overleftrightarrow{D}}(t)$$

$$\vec{\boldsymbol{B}} = \frac{1}{c}\dot{\vec{\boldsymbol{A}}} \times \hat{\boldsymbol{r}} = \frac{\mu_0 e^{ikr}}{24\pi c^2 r}(\hat{\boldsymbol{r}} \cdot \ddot{\overleftrightarrow{D}}) \times \hat{\boldsymbol{r}}$$

$$\vec{\boldsymbol{E}} = c\vec{\boldsymbol{B}} \times \hat{\boldsymbol{r}} = \frac{\mu_0 e^{ikr}}{24\pi cr}[(\hat{\boldsymbol{r}} \cdot \ddot{\overleftrightarrow{D}}) \times \hat{\boldsymbol{r}}] \times \hat{\boldsymbol{r}}$$

$$\left\langle\vec{\boldsymbol{S}}\right\rangle = \frac{c}{2\mu_0}|\vec{\boldsymbol{B}}|^2\hat{\boldsymbol{r}} = \frac{\mu_0\omega^6}{1152\pi^2 c^3 r^2}|(\hat{\boldsymbol{r}} \cdot \ddot{\overleftrightarrow{D}}) \times \hat{\boldsymbol{r}}|^2\hat{\boldsymbol{r}}$$

$$\frac{d\langle P\rangle}{d\Omega} = \frac{\mu_0\omega^6}{1152\pi^2 c^3}|(\hat{\boldsymbol{r}} \cdot \ddot{\overleftrightarrow{D}}) \times \hat{\boldsymbol{r}}|^2$$

$$\begin{aligned}\langle P\rangle &= \frac{\mu_0\omega^6}{1152\pi^2 c^3}\int d\Omega[(\hat{\boldsymbol{r}} \cdot \ddot{\overleftrightarrow{D}}^*) \times \hat{\boldsymbol{r}}] \cdot [(\hat{\boldsymbol{r}} \cdot \ddot{\overleftrightarrow{D}}) \times \hat{\boldsymbol{r}}] \\ &= \frac{\mu_0\omega^6}{1152\pi^2 c^3}\int d\Omega[\alpha_2^2\alpha_3^2(D_{22}^* - D_{33}^*)(D_{22} - D_{33}) \\ &\quad + \alpha_3^2\alpha_1^2(D_{33}^* - D_{11}^*)(D_{33} - D_{11})\end{aligned}$$

$$
\begin{aligned}
&\quad + \alpha_1^2\alpha_2^2(D_{11}^* - D_{22}^*)(D_{11} - D_{22})] \\
&(\alpha_1 = \sin\theta\cos\phi, \quad \alpha_2 = \sin\theta\sin\phi, \quad \alpha_3 = \cos\theta) \\
&= \frac{\mu_0\omega^6}{1152\pi^2c^3}\frac{4\pi}{15}(D_{22}^*D_{22} + D_{33}^*D_{33} - D_{22}^*D_{33} - D_{33}^*D_{22} \\
&\quad + D_{33}^*D_{33} + D_{11}^*D_{11} - D_{33}^*D_{11} - D_{11}^*D_{33} \\
&\quad + D_{11}^*D_{11} + D_{22}^*D_{22} - D_{11}^*D_{22} - D_{22}^*D_{11}) \\
&= \frac{\mu_0\omega^6}{1152\pi^2c^3}\frac{4\pi}{15}\big\{2(D_{11}^*D_{11} + D_{22}^*D_{22} + D_{33}^*D_{33}) \\
&\quad - [(D_{11}^* + D_{22}^* + D_{33}^*)(D_{11} + D_{22} + D_{33}) \\
&\qquad - (D_{11}^*D_{11} + D_{22}^*D_{22} + D_{33}^*D_{33})]\big\} \\
&= \frac{\mu_0\omega^6}{1152\pi^2c^3}\frac{4\pi}{5}\overleftrightarrow{\boldsymbol{D}}^* : \overleftrightarrow{\boldsymbol{D}} \\
&= \frac{\mu_0\omega^6}{1440\pi c^3}\overleftrightarrow{\boldsymbol{D}}^* : \overleftrightarrow{\boldsymbol{D}}
\end{aligned}
$$

其中用了主轴坐标系来简化运算。

天线辐射：

$$
I(z,t) = I_0\sin k(\frac{l}{2} - |z|)e^{-i\omega t}(|z| \le \frac{l}{2})
$$

$$
m \triangleq \frac{l}{\lambda}, \quad I(z,t) = I_0\sin(m\pi - k|z|)e^{-i\omega t}
$$

$$
\begin{aligned}
\overrightarrow{\boldsymbol{A}}(\overrightarrow{\boldsymbol{r}},t) &= \frac{\mu_0}{4\pi r}e^{ikr}\int dV'\overrightarrow{\boldsymbol{j}}(\overrightarrow{\boldsymbol{r}}',t)e^{-i\overrightarrow{\boldsymbol{k}}\cdot\overrightarrow{\boldsymbol{r}}'} \\
&= \hat{\boldsymbol{z}}\frac{\mu_0}{4\pi r}e^{i(kr-\omega t)}\int_{-\frac{l}{2}}^{\frac{l}{2}} dz' I_0\sin(m\pi - k|z'|)e^{-ikz'\cos\theta} \\
&= \hat{\boldsymbol{z}}\frac{\mu_0 I_0}{4\pi kr}e^{i(kr-\omega t)}\int_0^{m\pi} d\xi 2\sin(m\pi - \xi)\cos(\xi\cos\theta) \\
&= \hat{\boldsymbol{z}}\frac{\mu_0 I_0}{2\pi kr}e^{i(kr-\omega t)}\frac{\cos(m\pi\cos\theta) - \cos m\pi}{\sin^2\theta}
\end{aligned}
$$

$$
\overrightarrow{\boldsymbol{B}} = i\overrightarrow{\boldsymbol{k}} \times \overrightarrow{\boldsymbol{A}} = -i\frac{\mu_0 I_0}{2\pi r}e^{i(kr-\omega t)}\frac{\cos(m\pi\cos\theta) - \cos m\pi}{\sin\theta}\hat{\boldsymbol{\phi}} = -i\frac{\mu_0 I_0}{2\pi r}e^{i(kr-\omega t)}g(\theta)\hat{\boldsymbol{\phi}}
$$

$$
\overrightarrow{\boldsymbol{E}} = c\overrightarrow{\boldsymbol{B}} \times \hat{\boldsymbol{r}} = -i\frac{\mu_0 c I_0}{2\pi r}e^{i(kr-\omega t)}\frac{\cos(m\pi\cos\theta) - \cos m\pi}{\sin\theta}\hat{\boldsymbol{\theta}} = -i\frac{\mu_0 c I_0}{2\pi r}e^{i(kr-\omega t)}g(\theta)\hat{\boldsymbol{\theta}}
$$

$$
\frac{d\langle P\rangle}{d\Omega} = \frac{c}{2\mu_0}|\overrightarrow{\boldsymbol{B}}|^2 r^2 = \frac{\mu_0 c I_0^2}{8\pi^2}g^2(\theta)
$$

$$
\langle P\rangle = \frac{\mu_0 c I_0^2}{4\pi}\int_0^{\pi} g^2(\theta)\sin\theta d\theta \triangleq \frac{1}{2}I_0^2 R_0 \triangleq \frac{1}{2}I_{max}^2 R_r
$$

$$
\begin{aligned}
R_0 &\triangleq \frac{\mu_0 c}{2\pi}\int_0^{\pi} g^2(\theta)\sin\theta d\theta \\
&= \frac{\mu_0 c}{2\pi}\int_{-1}^{1}\frac{[\cos(m\pi u) - \cos(m\pi)]^2}{1-u^2}du
\end{aligned}
$$

由于 $I(z,t) = I_0 \sin(m\pi - k|z|)e^{-i\omega t}$，故

$$
\begin{aligned}
m \geq \frac{1}{2} &\Longrightarrow I_{max} = I_0 \Longrightarrow R_r = R_0 \\
m < \frac{1}{2} &\Longrightarrow I_{max} = I_0 \sin m\pi \Longrightarrow R_r = \frac{R_0}{\sin^2 m\pi} \\
m << \frac{1}{2} &\Longrightarrow I_{max} = I_0 m\pi \Longrightarrow R_r = \frac{R_0}{m^2\pi^2}
\end{aligned}
$$

$m << \frac{1}{2}$ 时，$R_r = \frac{R_0}{m^2\pi^2} = \cdots \approx 20m^2\pi^2 \approx 197(\frac{l}{\lambda})\Omega$

7 运动电荷的辐射

$$\overrightarrow{\boldsymbol{r}}(t) = \overrightarrow{\boldsymbol{w}}(t)$$

$$\phi(\overrightarrow{\boldsymbol{r}}, t) = e\delta(\overrightarrow{\boldsymbol{r}} - \overrightarrow{\boldsymbol{w}}(t))$$

$$\overrightarrow{\boldsymbol{j}}(\overrightarrow{\boldsymbol{r}}, t) = e\overrightarrow{\boldsymbol{v}}(t)\delta(\overrightarrow{\boldsymbol{r}} - \overrightarrow{\boldsymbol{w}}(t))$$

$$\phi(\overrightarrow{\boldsymbol{r}}, t) = \frac{1}{4\pi\epsilon_0}\int dV'\frac{\rho(\overrightarrow{\boldsymbol{r}}', t_r)}{\mathbb{R}} = \frac{e}{4\pi\epsilon_0}\int dV'\frac{\delta(\overrightarrow{\boldsymbol{r}}' - \overrightarrow{\boldsymbol{w}}(t_r))}{\mathbb{R}}$$

$$\overrightarrow{\boldsymbol{A}}(\overrightarrow{\boldsymbol{r}}, t) = \frac{\mu_0}{4\pi}\int dV'\frac{\overrightarrow{\boldsymbol{j}}(\overrightarrow{\boldsymbol{r}}', t_r)}{\mathbb{R}} = \frac{e}{4\pi\epsilon_0 c^2}\int dV'\frac{\overrightarrow{(}t_r)\delta(\overrightarrow{\boldsymbol{r}}' - \overrightarrow{\boldsymbol{w}}(t_r))}{\mathbb{R}}$$

坐标变换：

$$\overrightarrow{\boldsymbol{r}}' \to \overrightarrow{\boldsymbol{r}}'' = \overrightarrow{\boldsymbol{r}}' - \overrightarrow{\boldsymbol{w}}(t_r)$$

$$
\begin{aligned}
J \triangleq \frac{\partial(x_1'', x_2'', x_3'')}{\partial(x_1', x_2', x_3')} &= det(\nabla'\overrightarrow{\boldsymbol{r}}'') \\
&= det[\nabla'(\overrightarrow{\boldsymbol{r}}' - \overrightarrow{\boldsymbol{w}}(t_r))] \\
&= det(\overleftrightarrow{\boldsymbol{I}} - \frac{\hat{\mathbb{R}}}{c}\overrightarrow{\boldsymbol{v}}(t_r)) \\
&= 1 - \hat{\mathbb{R}} \cdot \overrightarrow{\boldsymbol{\beta}} > 0
\end{aligned}
$$

Lienard-Wechert 势：

$$
\begin{aligned}
\phi(\overrightarrow{\boldsymbol{r}}, t) &= \frac{1}{4\pi\epsilon_0}\int dV'\frac{\rho(\overrightarrow{\boldsymbol{r}}', t_r)}{\mathbb{R}} = \frac{e}{4\pi\epsilon_0}\int dV''\frac{\delta(\overrightarrow{\boldsymbol{r}}'')}{|J|\mathbb{R}} \\
&= \frac{e}{4\pi\epsilon_0\mathbb{R}}\frac{1}{1 - \hat{\mathbb{R}} \cdot \overrightarrow{\boldsymbol{\beta}}}\Big|_{\overrightarrow{\boldsymbol{r}}''=0} \\
&= \frac{e}{4\pi\epsilon_0\mathbb{R}^*}\frac{1}{1 - \hat{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{\beta}}^*}
\end{aligned}
$$

$$\overrightarrow{\boldsymbol{A}}(\overrightarrow{\boldsymbol{r}}, t) = \frac{\overrightarrow{\boldsymbol{v}}^*}{c^2}\phi(\overrightarrow{\boldsymbol{r}}, t) = \frac{\overrightarrow{\boldsymbol{\beta}}^*}{c}\phi(\overrightarrow{\boldsymbol{r}}, t)$$

$*$ 号满足：

$$|\overrightarrow{\boldsymbol{r}} - \overrightarrow{\boldsymbol{w}}(t^*)| = c(t - t^*) = c\Delta t > 0 \Longrightarrow t^* = t^*(\overrightarrow{\boldsymbol{r}}, t)$$

有时把 Δt 看作未知量取正解比较方便。

$$\overrightarrow{\mathbb{R}}^* = \overrightarrow{\boldsymbol{r}} - \overrightarrow{\boldsymbol{w}}(t^*) = \overrightarrow{\mathbb{R}}^*(\overrightarrow{\boldsymbol{r}}, t)$$

$$\mathbb{R}^* = c(t - t^*) = \mathbb{R}^*(\overrightarrow{\boldsymbol{r}}, t)$$

$\overrightarrow{\boldsymbol{n}}^*$:

$$\overrightarrow{\boldsymbol{n}}^* \triangleq \hat{\mathbb{R}}^* - \overrightarrow{\boldsymbol{\beta}}^*$$

$$\phi(\overrightarrow{\boldsymbol{r}}, t) = \frac{e}{4\pi\epsilon_0}\frac{1}{\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*}$$

$$\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^* = \hat{\mathbb{R}}^* \cdot (\mathbb{R}^*\hat{\mathbb{R}}^* - \mathbb{R}^*\overrightarrow{\boldsymbol{\beta}}^*) = \hat{\mathbb{R}}^* \cdot (c\Delta t\hat{\mathbb{R}}^* - \overrightarrow{\boldsymbol{v}}^*\Delta t)$$

从中可以看出 $\mathbb{R}^*\overrightarrow{\boldsymbol{n}}^*$ 表示这样一个矢量：由推迟时刻做匀速运动在 t 时刻到达的点指向场点。

t^* 只有一个解：

$$\begin{cases} c(t - t_1^*) = \mathbb{R}_1^* \\ c(t - t_2^*) = \mathbb{R}_2^* \end{cases}$$

$$\Longrightarrow c|t_1^* - t_2^*| = |\mathbb{R}_1^* - \mathbb{R}_2^*| \leq |\overrightarrow{\mathbb{R}}_1^* - \overrightarrow{\mathbb{R}}_2^*| \leq \Delta s = \langle v\rangle\,|t_1^* - t_2^*| < c|t_1^* - t_2^*|$$

匀速运动点电荷：

$$\phi(\overrightarrow{\boldsymbol{r}}, t) = \frac{e}{4\pi\epsilon_0 R}\frac{1}{\sqrt{1 - \beta^2\sin^2\theta}}$$

$$\overrightarrow{\boldsymbol{A}}(\overrightarrow{\boldsymbol{r}}, t) = \frac{\overrightarrow{\boldsymbol{\beta}}}{c}\phi(\overrightarrow{\boldsymbol{r}}, t)$$

其中 R 为 t 时刻的相对位矢的模长。

电磁场的计算：

t^* 的导数：

$$\begin{cases} \partial_t(\mathbb{R}^{*2}) = 2\mathbb{R}^*\partial_t\mathbb{R}^* = 2\mathbb{R}^*c(1 - \partial_t t^*) \\ \partial_t(\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\mathbb{R}}^*) = 2\overrightarrow{\mathbb{R}}^* \cdot \partial_t\overrightarrow{\mathbb{R}}^* = -2\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{v}}^*\partial_t t^* \end{cases} \Longrightarrow \partial_t t^* = \frac{\mathbb{R}^*}{\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*}$$

$$\begin{cases} \nabla(\mathbb{R}^{*2}) = \cdots \\ \nabla(\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\mathbb{R}}^*) = \cdots \end{cases} \Longrightarrow \nabla t^* = -\frac{\overrightarrow{\mathbb{R}}^*}{c\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*}$$

$$\nabla t^* = -\frac{\hat{\mathbb{R}}^*}{c}\partial_t t^*$$

$\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*$ 的导数：

$$\partial_t(\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*) = \partial_t(\mathbb{R}^* - \overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{\beta}}^*) = \cdots = c - \big(1 - \beta^{*2} + \frac{\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{a}}^*}{c^2}\big)c\partial_t t^*$$

$$\begin{aligned}\nabla(\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*) = \cdots &= -\overrightarrow{\boldsymbol{\beta}}^* - \big(1 - \beta^{*2} + \frac{\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{a}}^*}{c^2}\big)c\nabla t^* \\ &= -\overrightarrow{\boldsymbol{\beta}}^* + \big(1 - \beta^{*2} + \frac{\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{a}}^*}{c^2}\big)\hat{\mathbb{R}}^*\partial_t t^* \\ &= \overrightarrow{\boldsymbol{n}}^* - \frac{\hat{\mathbb{R}}^*}{c}\partial_t(\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*)\end{aligned}$$

Lienard-Wiechert 场；

$$\begin{aligned}\overrightarrow{\boldsymbol{E}} &= -\nabla\phi - \partial_t\overrightarrow{\boldsymbol{A}} = -\nabla\phi - \frac{\overrightarrow{\boldsymbol{\beta}}^*}{c}\partial_t\phi - \frac{\phi}{c}\partial_t\overrightarrow{\boldsymbol{\beta}}^* \\ &= \cdots = \frac{e}{4\pi\epsilon_0}\frac{1}{(\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*)^2}\big[(1 - \beta^{*2} + \frac{\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{a}}^*}{c^2})\overrightarrow{\boldsymbol{n}}^* - \frac{\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*}{c^2}\overrightarrow{\boldsymbol{a}}^*\big]\partial_t t^* \\ &= \frac{e}{4\pi\epsilon_0}\frac{\mathbb{R}^*}{(\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*)^3}\big[(1 - \beta^{*2})\overrightarrow{\boldsymbol{n}}^* + \frac{\overrightarrow{\mathbb{R}}^* \times (\overrightarrow{\boldsymbol{n}}^* \times \overrightarrow{\boldsymbol{a}}^*)}{c^2}\big]\end{aligned}$$

$$\begin{aligned}c\overrightarrow{\boldsymbol{B}} &= \nabla \times (c\overrightarrow{\boldsymbol{A}}) = \nabla \times (\overrightarrow{\boldsymbol{\beta}}^*\phi) \\ &= \cdots = \frac{e}{4\pi\epsilon_0(\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*)^2}\big[-\nabla(\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*) \times \overrightarrow{\boldsymbol{\beta}}^* + \frac{\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*}{c}\nabla t^* \times \overrightarrow{\boldsymbol{a}}^*\big] \\ &= \hat{\mathbb{R}}^* \times \frac{e}{4\pi\epsilon_0}\frac{1}{(\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*)^2}\big[(1 - \beta^{*2} + \frac{\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{a}}^*}{c^2})(\overrightarrow{\boldsymbol{n}}^* - \hat{\mathbb{R}}^*) - \frac{\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*}{c^2}\overrightarrow{\boldsymbol{a}}^*\big]\partial_t t^* \\ &= \hat{\mathbb{R}}^* \times \frac{e}{4\pi\epsilon_0}\frac{1}{(\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*)^2}\big[(1 - \beta^{*2} + \frac{\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{a}}^*}{c^2})\overrightarrow{\boldsymbol{n}}^* - \frac{\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*}{c^2}\overrightarrow{\boldsymbol{a}}^*\big]\partial_t t^* \\ &= \hat{\mathbb{R}}^* \times \overrightarrow{\boldsymbol{E}}(\overrightarrow{\boldsymbol{r}}, t)\end{aligned}$$

可见 $\overrightarrow{\boldsymbol{E}} \cdot \overrightarrow{\boldsymbol{B}} = 0, \hat{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{B}} = 0$，但一般 $\overrightarrow{\boldsymbol{E}} \cdot \hat{\mathbb{R}}^* \neq 0$。$c|\overrightarrow{\boldsymbol{B}}| \leq \overrightarrow{\boldsymbol{E}}$，从而 $w_B \leq w_E$。

速度场与加速度场：

$$\overrightarrow{\boldsymbol{E}}_v = \frac{e}{4\pi\epsilon_0}\frac{\mathbb{R}^*}{(\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*)^3}(1 - \beta^{*2})\overrightarrow{\boldsymbol{n}}^*$$

$$\overrightarrow{\boldsymbol{E}}_a = \frac{e}{4\pi\epsilon_0}\frac{\mathbb{R}^*}{(\overrightarrow{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*)^3}\frac{\overrightarrow{\mathbb{R}}^* \times (\overrightarrow{\boldsymbol{n}}^* \times \overrightarrow{\boldsymbol{a}}^*)}{c^2}$$

$$c\overrightarrow{\boldsymbol{B}}_v = \hat{\mathbb{R}}^* \times \overrightarrow{\boldsymbol{E}}_v, \quad c\overrightarrow{\boldsymbol{B}}_a = \hat{\mathbb{R}}^* \times \overrightarrow{\boldsymbol{E}}_a$$

可见 $\overrightarrow{\boldsymbol{E}}_v, \overrightarrow{\boldsymbol{B}}_v \sim \frac{1}{\mathbb{R}^{*2}}$，而 $\overrightarrow{\boldsymbol{E}}_a, \overrightarrow{\boldsymbol{B}}_a \sim \frac{1}{\mathbb{R}^*}$，从而辐射场是加速度场。

和之前相比，此时 $\overrightarrow{\boldsymbol{E}}_a \cdot \overrightarrow{\boldsymbol{B}}_a = 0, \hat{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{B}}_a = 0, \hat{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{E}}_a = 0$，同时 $c|\overrightarrow{\boldsymbol{B}}_a| = \overrightarrow{\boldsymbol{E}}_a, w_B = w_E$。

点电荷的辐射：

$$\overrightarrow{E} = \frac{e}{4\pi\epsilon_0 c^2 \mathbb{R}^*} \frac{\hat{\mathbb{R}}^* \times (\overrightarrow{n}^* \times \overrightarrow{a}^*)}{(\hat{\mathbb{R}}^* \cdot \overrightarrow{n}^*)^3}$$

$$c\overrightarrow{B} = \hat{\mathbb{R}} \times \overrightarrow{E}$$

$$w = \epsilon_0 E^2 = \frac{e^2}{16\pi^2\epsilon_0 c^4 \mathbb{R}^{*2}} \frac{|\hat{\mathbb{R}}^* \times (\overrightarrow{n}^* \times \overrightarrow{a}^*)|^2}{(\hat{\mathbb{R}}^* \cdot \overrightarrow{n}^*)^6}$$

$$\overrightarrow{S} = \frac{1}{\mu_0}\overrightarrow{E} \times \overrightarrow{B} = wc\hat{\mathbb{R}}^*$$

功率计算：

接收者：

$$\begin{aligned}\frac{dP}{d\Omega}|_{receiver} &= \frac{dW}{dtd\Omega} = \frac{S\mathbb{R}^{*2} d\Omega dt}{dtd\Omega} \\ &= S\mathbb{R}^{*2} = \frac{e^2}{16\pi^2\epsilon_0 c^3} \frac{|\hat{\mathbb{R}}^* \times (\overrightarrow{n}^* \times \overrightarrow{a}^*)|^2}{(\hat{\mathbb{R}}^* \cdot \overrightarrow{n}^*)^6}\end{aligned}$$

发射者：

$$\begin{aligned}\frac{dP}{d\Omega}|_{launcher} &= \frac{dW}{dt^* d\Omega} = \frac{w\mathbb{R}^{*2} d\Omega(\mathbb{R}^* - \mathbb{R}^{*'} - v^* dt^* \cos\theta)}{dt^* d\Omega} \\ &= \frac{w\mathbb{R}^{*2} d\Omega(dt^* - v^* dt^* \cos\theta)}{dt^* d\Omega} \\ &= \frac{w\mathbb{R}^{*2} d\Omega c dt^* \hat{R}^* \cdot \overrightarrow{n}^*}{dt^* d\Omega} \\ &= wc\mathbb{R}^{*2}(\hat{R}^* \cdot \overrightarrow{n}^*) \\ &= \frac{dP}{d\Omega}|_{receiver}(\hat{R}^* \cdot \overrightarrow{n}^*) = \frac{dt}{dt^*}\frac{dP}{d\Omega}|_{receiver}\end{aligned}$$

低速运动粒子的辐射：

$$\beta << 1$$

$$\overrightarrow{n} = \hat{\mathbb{R}}^* - \overrightarrow{\beta}^* \approx \hat{\mathbb{R}}^*, \hat{\mathbb{R}}^* \cdot \overrightarrow{n}^* \approx 1$$

$$\hat{\mathbb{R}}^* \times (\overrightarrow{n}^* \times \overrightarrow{a}^*) \approx \hat{\mathbb{R}}^* \times (\hat{\mathbb{R}}^* \times \overrightarrow{a}^*) = a^* \sin\theta\hat{\boldsymbol{\theta}}$$

$$(\theta = \angle(\hat{\mathbb{R}}^*, \overrightarrow{a}^*))$$

$$\frac{dP}{d\Omega}|_{launcher} \approx \frac{dP}{d\Omega}|_{receiver} = \frac{e^2 a^{*2}\sin^2\theta}{16\pi^2\epsilon_0 c^3} = \frac{e_S^2 a^{*2}\sin^2\theta}{4\pi c^3}$$

$$P = \int d\Omega\frac{dP}{d\Omega} = \frac{2e_S^2 a^{*2}}{3c^3}$$

相对论性粒子的辐射：

$$\hat{\mathbb{R}}^* \cdot \overrightarrow{n}^* = 1 - \beta^* \cos\theta$$

$$(\theta = \angle(\hat{\mathbb{R}}^*, \overrightarrow{\beta}^*))$$

速度与加速度平行：

$$\hat{\mathbb{R}}^* \times (\overrightarrow{\boldsymbol{n}}^* \times \overrightarrow{\boldsymbol{a}}^*_{//}) = \hat{\mathbb{R}}^* \times (\hat{\mathbb{R}}^* \times \overrightarrow{\boldsymbol{a}}^*_{//}) = a^*_{//} \sin\theta \hat{\boldsymbol{\theta}}$$

$$\frac{dP_{//}}{d\Omega}|_{launcher} = \frac{e^2}{16\pi^2\epsilon_0 c^3} \frac{|\hat{\mathbb{R}}^* \times (\overrightarrow{\boldsymbol{n}}^* \times \overrightarrow{\boldsymbol{a}}^*_{//})|^2}{(\hat{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{n}}^*)^5} = \frac{e_S^2 {a^*_{//}}^2}{4\pi c^3} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$

角度因子 $g_{//}(\theta) = \dfrac{\sin^2\theta}{(1-\beta\cos\theta)^5}$，$\beta \to 1$ 时，辐射主要集中在 θ 较小的方向。

$$P_{//} = \int d\Omega \frac{dP_{//}}{d\Omega} = \cdots = \frac{2e_S^2 {a^*_{//}}^2}{3c^3}\gamma^6$$

速度与加速度垂直：

$$\overrightarrow{\boldsymbol{a}}^*_{\perp} = a^*_{\perp}\hat{\boldsymbol{x}}, \overrightarrow{\boldsymbol{\beta}}^* = \beta^* \hat{\boldsymbol{z}}$$

$$\begin{aligned}
\hat{\mathbb{R}}^* \times (\overrightarrow{\boldsymbol{n}}^* \times \overrightarrow{\boldsymbol{a}}^*_{\perp}) &= (\hat{\mathbb{R}}^* \cdot \overrightarrow{\boldsymbol{a}})\hat{\mathbb{R}}^* - a^*_{\perp}\hat{\boldsymbol{x}} - \beta a^*_{\perp}\hat{\mathbb{R}}^* \times \hat{\boldsymbol{y}} \\
&= \cdots \\
&= (-a^*_{\perp}\cos\theta\cos\phi + \beta^* a^*_{\perp}\cos\phi)\hat{\boldsymbol{\theta}} \\
&\quad + (a^*_{\perp}\sin\phi - \beta^* a^*_{\perp}\cos\theta\sin\phi)\hat{\boldsymbol{\phi}}
\end{aligned}$$

$$|\hat{\mathbb{R}}^* \times (\overrightarrow{\boldsymbol{n}}^* \times \overrightarrow{\boldsymbol{a}}^*_{\perp}|^2 = \cdots = {a^*_{\perp}}^2[(1-\beta^*\cos\theta)^2 - (1-\beta^{*2})\sin^2\theta\cos^2\phi]$$

$$\frac{dP_{\perp}}{d\Omega}|_{launcher} = \frac{e_S^2 {a^*_{\perp}}^2}{4\pi c^3} \frac{(1-\beta^*\cos\theta)^2 - (1-\beta^{*2})\sin^2\theta\cos^2\phi}{(1-\beta^*\cos\theta)^5}$$

角度因子 $g_{\perp}(\theta,\phi) = \dfrac{(1-\beta^*\cos\theta)^2 - (1-\beta^{*2})\sin^2\theta\cos^2\phi}{(1-\beta^*\cos\theta)^5}$，$\theta = 0$ 时辐射最强：

$$g_{\perp}(\theta,\phi) \le g_{\perp}(\theta,\frac{\pi}{2}) = g_{\perp}(\theta,\frac{3\pi}{2}) \le g_{\perp}(0,\phi)$$

$$P_{\perp} = \int d\Omega \frac{dP_{\perp}}{d\Omega} = \cdots = \frac{2e_S^2 {a^*_{\perp}}^2}{3c^3}\gamma^4$$

一般情形的加速度：

$$\begin{aligned}
\hat{\mathbb{R}}^* \times (\overrightarrow{\boldsymbol{n}}^* \times \overrightarrow{\boldsymbol{a}}^*) &= a^*_{//}\sin\theta\hat{\boldsymbol{\theta}} \\
&\quad + (-a^*_{\perp}\cos\theta\cos\phi + \beta^* a^*_{\perp}\cos\phi)\hat{\boldsymbol{\theta}} \\
&\quad + (a^*_{\perp}\sin\phi - \beta^* a^*_{\perp}\cos\theta\sin\phi)\hat{\boldsymbol{\phi}}
\end{aligned}$$

$$\frac{dP}{d\Omega}|_{launcher} = \frac{dP_{//}}{d\Omega}|_{launcher} + \frac{dP_{\perp}}{d\Omega}|_{launcher} + \frac{e^2 a^*_{//} a^*_{\perp}}{2\pi c^3} \frac{(\beta - \cos\theta)\sin\theta\cos\phi}{(1-\beta\cos\theta)^5}$$

交叉项有 $\cos\phi$，积分后为 0：

$$\begin{aligned}
P = \int d\Omega \frac{dP}{d\Omega} &= P_{//} + P_{\perp} = \frac{2e_S^2}{3c^3}\gamma^6[{a^*_{//}}^2 + \frac{{a^*_{\perp}}^2}{\gamma^2}] \\
&= \frac{2e_S^2}{3c^3}\gamma^6\left[a^{*2} - |\overrightarrow{\boldsymbol{\beta}}^* \times \overrightarrow{\boldsymbol{a}}^*|^2\right] \\
&= \frac{2e_S^2 a^{*2}}{3c^3}\gamma^6\left[1 - |\overrightarrow{\boldsymbol{\beta}}^* \times \hat{\boldsymbol{a}}^*|^2\right] \\
&\quad (Lienard\ formula)
\end{aligned}$$

8 狭义相对论

Einstein 假设：

相对性原理：物理定律在所有惯性系中具有相同的形式。

光速普适原理：真空中光速对所有惯性系具有相同的数值。

洛伦兹变换：

4-位矢：

$$x^\alpha \triangleq (x^0, x^1, x^2, x^3) = (ct, x, y, z) = (ct, \overrightarrow{\boldsymbol{r}})$$

洛伦兹度规 $g_{\alpha\beta}$ 及其逆 $g^{\alpha\beta}$：

$$g_{\alpha\beta} = g^{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha{}_\beta$$

$$g_{\alpha\rho} T^{\cdots\rho\cdots} = T^{\cdots}{}_{\alpha}{}^{\cdots}, \quad g^{\alpha\rho} T_{\cdots\rho\cdots} = T_{\cdots}{}^{\alpha}{}_{\cdots}$$

时空间隔：$\Delta s^2 \triangleq g_{\alpha\beta} \Delta x^\alpha \Delta x^\beta$

坐标变换的性质：

时空均匀性 $\Longrightarrow$ 变换是线性的：$x'^\alpha = \Lambda^\alpha{}_\beta x^\beta + a^\alpha$

光速普适; 空间各向同性; 变换回本参考系连续性 $\Longrightarrow$ 时空间隔不变

$$g_{\alpha\beta} \Lambda^\alpha{}_\rho \Lambda^\beta{}_\sigma = g_{\rho\sigma} \quad (\Lambda^T g \Lambda = g)$$

(1) 由于 $g_{\alpha\beta}$ 的对称性，分量方程有 10 个是独立的。$\Lambda^\alpha{}_\beta$ 有 6 个独立参数。

(2)$\Lambda^T g \Lambda = g \Longrightarrow det(\Lambda = \pm 1$

庞加莱变换：$x'^\alpha = \Lambda^\alpha{}_\beta x^\beta + a^\alpha$(10 个参数)

洛伦兹变换：$x'^\alpha = \Lambda^\alpha{}_\beta x^\beta$(6 个参数)

(在洛伦兹变换下，由于时空间隔不变可得到标量积 ($\triangleq g_{\alpha\beta} x^\alpha y^\beta$) 为一不变量。)

几个坐标变换：

$$x'^{\alpha} = \Lambda^{\alpha}{}_{\beta} x^{\beta}, \quad x^{\alpha} = \Lambda_{\beta}{}^{\alpha} x'^{\beta}$$

$$x'_{\alpha} = \Lambda_{\alpha}{}^{\beta} x_{\beta}, \quad x_{\alpha} = \Lambda^{\beta}{}_{\alpha} x'_{\beta}$$

$$(x^{\alpha} = \Lambda_{\beta}{}^{\alpha} x'^{\beta}, \quad ds^2 = ds'^2 \Longrightarrow \Lambda g \Lambda^T = g)$$

(这没有增加独立方程的个数，因为 $\Lambda^T g \Lambda = g$ 已经蕴含了这点：$\Lambda^T g \Lambda = g \Longrightarrow g\Lambda = (\Lambda^T)^{-1} g \Longrightarrow \Lambda g = g(\Lambda^T)^{-1} \Longrightarrow g\Lambda^T = \Lambda^{-1} g \Longrightarrow \Lambda g \Lambda^T = g$)

无穷小洛伦兹变换：

$$\Lambda^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} + \Omega^{\alpha}{}_{\beta}$$

$$g_{\rho\sigma} = g_{\alpha\beta} \Lambda^{\alpha}{}_{\rho} \Lambda^{\beta}{}_{\sigma} = g_{\alpha\beta} (\delta^{\alpha}{}_{\rho} + \Omega^{\alpha}{}_{\rho})(\delta^{\beta}{}_{\sigma} + \Omega^{\beta}{}_{\sigma}) \approx g_{\rho\sigma} + g_{\rho\beta} \Omega^{\beta}{}_{\sigma} + g_{\alpha\sigma} \Omega^{\alpha}{}_{\rho}$$

$$\Longrightarrow \begin{cases} \Omega_{\rho\sigma} = -\Omega_{\sigma\rho} \\ \Omega^T = -g\Omega g \quad (\Omega = \left(\Omega^{\alpha}{}_{\beta}\right)) \end{cases}$$

$$\Lambda^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} + g^{\alpha\rho} \Omega_{\rho\beta}$$

两个推论：

(1)

$$\Omega_{\rho\sigma} = -\Omega_{\sigma\rho} \Longrightarrow \Lambda = \begin{pmatrix} 1 & -\xi_1 & -\xi_2 & -\xi_3 \\ -\xi_1 & 1 & -\theta n_3 & \theta n_2 \\ -\xi_2 & \theta n_3 & 1 & -\theta n_1 \\ -\xi_3 & -\theta n_2 & \theta n_1 & 1 \end{pmatrix}$$

其中 $\xi_1, \xi_2, \xi_3, \theta, n_1, n_2, n_3 (n_1^2 + n_2^2 + n_3^2 = 1)$ 为 6 个参数。

$$\begin{pmatrix} ct' \\ \vec{\boldsymbol{r}}' \end{pmatrix} = \Lambda \begin{pmatrix} ct \\ \vec{\boldsymbol{r}} \end{pmatrix} = \begin{pmatrix} ct - \vec{\boldsymbol{\xi}} \cdot \vec{\boldsymbol{r}} \\ \vec{\boldsymbol{r}} - \vec{\boldsymbol{\xi}} ct + \theta \hat{\boldsymbol{n}} \times \vec{\boldsymbol{r}} \end{pmatrix}$$

$\vec{\boldsymbol{\xi}} = 0$：转动；$\theta = 0$：推动。

(2)

$$\Omega^T = -g\Omega g \Longrightarrow e^{\Omega^T} = e^{-g\Omega g} = \Sigma_{n=0}^{\infty} \frac{(-1)^n}{n!} (g\Omega g)^n = g e^{-\Omega} g$$

$$\Longrightarrow e^{\Omega^T} g e^{\Omega} = g, \quad e^{\Omega} \in \{\Lambda | \Lambda^T g \Lambda = g\}$$

事实上，$\forall\Lambda\in\{\Lambda|\Lambda^T g\Lambda=g\}$，可以写为 e^{Ω} 的形式。粗糙理解：同被 6 个参数描述，存在一个对应，由参数连续性，一一对应。

$$\boldsymbol{\Omega}=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix}\theta\Longrightarrow e^{\Omega}=\begin{pmatrix}1&0&0&0\\0&\cos\theta&\sin\theta&0\\0&-\sin\theta&\cos\theta&0\\0&0&0&1\end{pmatrix}$$

$$\boldsymbol{\Omega}=\begin{pmatrix}0&-1&0&0\\-1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\xi\Longrightarrow e^{\Omega}=\begin{pmatrix}\cosh\xi&-\sinh\xi&0&0\\-\sinh\xi&\cosh\xi&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

沿 x 方向运动的洛伦兹变换及反变换：

$$\begin{pmatrix}ct'\\x'\\y'\\z'\end{pmatrix}=\begin{pmatrix}\gamma&-\beta\gamma&0&0\\-\beta\gamma&\gamma&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}ct\\x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}ct\\x\\y\\z\end{pmatrix}=\begin{pmatrix}\gamma&\beta\gamma&0&0\\\beta\gamma&\gamma&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}ct'\\x'\\y'\\z'\end{pmatrix}$$

写为矢量形式：

$$\begin{pmatrix}ct'\\\overrightarrow{\boldsymbol{r}}'_{//}\\\overrightarrow{\boldsymbol{r}}'_{\perp}\end{pmatrix}=\begin{pmatrix}\gamma(ct-\overrightarrow{\boldsymbol{\beta}}\cdot\overrightarrow{\boldsymbol{r}})\\\gamma(\overrightarrow{\boldsymbol{r}}_{//}-\overrightarrow{\boldsymbol{\beta}}ct)\\\overrightarrow{\boldsymbol{r}}_{\perp}\end{pmatrix}$$

$$\begin{pmatrix}ct\\\overrightarrow{\boldsymbol{r}}_{//}\\\overrightarrow{\boldsymbol{r}}_{\perp}\end{pmatrix}=\begin{pmatrix}\gamma(ct'+\overrightarrow{\boldsymbol{\beta}}\cdot\overrightarrow{\boldsymbol{r}}')\\\gamma(\overrightarrow{\boldsymbol{r}}'_{//}+\overrightarrow{\boldsymbol{\beta}}ct')\\\overrightarrow{\boldsymbol{r}}'_{\perp}\end{pmatrix}$$

速度变换：

$$\begin{cases}\beta_x=\dfrac{dx}{cdt}=\dfrac{\beta'_x+\beta_0}{1+\beta_0\beta'_x}\\[2em]\beta_{y,z}=\dfrac{dy,z}{cdt}=\dfrac{\beta'_{y,z}}{\gamma_0(1+\beta_0\beta'_x)}\end{cases}\qquad\begin{cases}\beta'_x=\dfrac{dx}{cdt}=\dfrac{\beta_x-\beta_0}{1-\beta_0\beta'_x}\\[2em]\beta'_{y,z}=\dfrac{dy,z}{cdt}=\dfrac{\beta_{y,z}}{\gamma_0(1-\beta_0\beta'_x)}\end{cases}$$

加速度变换：

$$\begin{cases}a_x=\dfrac{1}{\gamma_0(1+\beta_0\beta'_x)}\left[\dfrac{a'_x}{1+\beta_0\beta'_x}-\dfrac{(\beta'_x+\beta_0)\beta_0a'_x}{(1+\beta_0\beta'_x)^2}\right]\\[2em]a_{y,z}=\dfrac{1}{\gamma_0^2(1+\beta_0\beta'_x)}\left[\dfrac{a'_y}{1+\beta_0\beta'_x}-\dfrac{\beta'_y\beta_0a'_x}{(1+\beta_0\beta'_x)^2}\right]\end{cases}\qquad\begin{cases}a'_x=\cdots\\[2em]a'_y=\cdots\end{cases}$$

瞬时惯性系 (MCRF)：

在这个参考系中 $\beta' = 0$，有时便于计算一些参考系无关量。加速度变换在 MCRF 里有较简单的形式：

$$a_x = \frac{a'_x}{\gamma_0^3}, \quad a_y = \frac{a'_y}{\gamma_0^2}, \quad a_z = \frac{a'_z}{\gamma_0^2}$$

事件分类：

$\Delta s^2 < 0$：类时间隔，可找到标架使两者同地发生。

$\Delta s^2 = 0$：类光间隔

$\Delta s^2 > 0$：类空间隔，可找到标架使两者同时发生。

原时 $d\tau$：

物体自身系中时钟走时，又称固有时。

$$ds^2 = 0 - (cd\tau)^2, \quad d\tau = \frac{\sqrt{-ds^2}}{c}$$

由时空间隔不变性，质点在时空图上轨迹的长度 $\int \sqrt{-ds^2} = c\Delta\tau$

一些 4-矢量：

导数算符：

$$\partial_\alpha \triangleq (\frac{1}{c}\frac{\partial}{\partial t}, \nabla)$$
$$\partial'_\alpha = \Lambda_\alpha^{\ \rho}\partial_\rho$$
$$\Box = \partial_\alpha\partial^\alpha = -\frac{1}{c^2}\partial^2 t + \nabla^2$$

标量积为 4 不变量，故达朗贝尔算子是 4 维不变的。

4-速度：

$$U^\alpha \triangleq \frac{dx^\alpha}{d\tau} = \gamma(c, \vec{\boldsymbol{u}})$$
$$U'^\alpha = \Lambda^\alpha_{\ \beta}U^\beta$$
$$U^\alpha U_\alpha = -c^2$$

4-加速度：

$$A^\alpha \triangleq \frac{dU^\alpha}{d\tau} = \gamma^4(\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{a}}, \frac{\vec{\boldsymbol{a}}}{\gamma^2} + (\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{a}})\vec{\boldsymbol{\beta}})$$
$$A'^\alpha = \Lambda^\alpha_{\ \beta}A^\alpha$$
$$A^\alpha A_\alpha = A^\alpha A_\alpha \ in \ MCRF = a'^2$$
$$0 = \frac{d(U^\alpha U_\alpha)}{d\tau} = \frac{dU_\alpha}{d\tau}U^\alpha + \frac{dU^\alpha}{d\tau}U_\alpha = 2U_\alpha\frac{dU^\alpha}{d\tau} = 2U_\alpha A^\alpha$$

4-动量：

$$P^\alpha \triangleq mU^\alpha = (\gamma mc, \gamma m\vec{\boldsymbol{u}}) = (\frac{\epsilon}{c}, \vec{\boldsymbol{p}})$$

$$P'^\alpha = \Lambda^\alpha{}_\beta P^\beta$$

$$P^\alpha P_\alpha = p^2 - \frac{\epsilon^2}{c^2} = P^\alpha P_\alpha \ in \ MCRF = -m^2c^2$$

对于无质量粒子，$P^\alpha = \hbar(\frac{\omega}{c}, \vec{\boldsymbol{k}}) = \hbar k^\alpha$。

4-张量：

$$T'^{\alpha_1\cdots\alpha_n}{}_{\beta_1\cdots\beta_n} = \Lambda^{\alpha_1}{}_{\rho_1}\cdots\Lambda^{\alpha_n}{}_{\rho_n}\Lambda_{\beta_1}{}^{\sigma_1}\cdots\Lambda_{\beta_n}{}^{\sigma_n}T^{\rho_1\cdots\rho_n}{}_{\sigma_1\cdots\sigma_n}$$

二阶张量可以写为：

$$\begin{pmatrix} \phi & \vec{\boldsymbol{p}} \\ \vec{\boldsymbol{q}} & \overleftrightarrow{\boldsymbol{T}} \end{pmatrix}$$

Leci-Civita 符号：

$$\epsilon^{\alpha\beta\mu\nu} = \begin{vmatrix} \delta^\alpha{}_0 & \delta^\alpha{}_1 & \delta^\alpha{}_2 & \delta^\alpha{}_3 \\ \delta^\beta{}_0 & \delta^\beta{}_1 & \delta^\beta{}_2 & \delta^\beta{}_3 \\ \delta^\mu{}_0 & \delta^\mu{}_1 & \delta^\mu{}_2 & \delta^\mu{}_3 \\ \delta^\nu{}_0 & \delta^\nu{}_1 & \delta^\nu{}_2 & \delta^\nu{}_3 \end{vmatrix}$$

反对称张量 $A^{\alpha\beta}$ 的对偶张量：

$$A^{*\alpha\beta} \triangleq \frac{1}{2!}\epsilon^{\alpha\beta\mu\nu}A_{\mu\nu}$$

$$\left(A^{\alpha\beta}\right) = \begin{pmatrix} 0 & p_1 & p_2 & p_3 \\ -p_1 & 0 & a_3 & -a_2 \\ -p_2 & -a_3 & 0 & a_1 \\ -p_3 & a_2 & -a_1 & 0 \end{pmatrix} \triangleq \{\vec{\boldsymbol{p}}, \vec{\boldsymbol{a}}\}$$

$$\left(A_{\alpha\beta}\right) = \begin{pmatrix} 0 & -p_1 & -p_2 & -p_3 \\ p_1 & 0 & a_3 & -a_2 \\ p_2 & -a_3 & 0 & a_1 \\ p_3 & a_2 & -a_1 & 0 \end{pmatrix} = \{-\vec{\boldsymbol{p}}, \vec{\boldsymbol{a}}\}$$

$$\left(A^{*\alpha\beta}\right) = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & -p_3 & p_2 \\ -a_2 & p_3 & 0 & -p_1 \\ -a_3 & -p_2 & p_1 & 0 \end{pmatrix} = \{\vec{\boldsymbol{a}}, -\vec{\boldsymbol{p}}\}$$

二阶张量场的导数：

$$(T^{\alpha\beta}) = \begin{pmatrix} \phi & \vec{\boldsymbol{p}} \\ \vec{\boldsymbol{q}} & \overleftrightarrow{\boldsymbol{T}} \end{pmatrix} \Longrightarrow \begin{cases} \partial_\beta T^{0\beta} = \partial_0\phi + \nabla\cdot\vec{\boldsymbol{p}} \\ \partial_\beta T^{i\beta} = \partial_0\phi + \partial_j T^{ij} \end{cases}$$

$$(T^{\alpha\beta}) = (T^{\beta\alpha}) = \begin{pmatrix} \phi & \vec{\boldsymbol{p}} \\ \vec{\boldsymbol{p}} & \overleftrightarrow{\boldsymbol{T}} \end{pmatrix} \Longrightarrow (\partial_\beta T^{\alpha\beta}) = \begin{pmatrix} \partial_0\phi + \nabla\cdot\vec{\boldsymbol{p}} \\ \partial_0\vec{\boldsymbol{p}} + \nabla\cdot\overleftrightarrow{\boldsymbol{T}} \end{pmatrix}$$

$$(T^{\alpha\beta}) = (-T^{\beta\alpha}) = \{\vec{\boldsymbol{p}}, \vec{\boldsymbol{a}}\} \Longrightarrow \partial_\beta T^{\alpha\beta} = \begin{pmatrix} \nabla\cdot\vec{\boldsymbol{p}} \\ -\partial_0\vec{\boldsymbol{p}} + \nabla\times\vec{\boldsymbol{a}} \end{pmatrix}$$

电磁规律的相对论协变性：

4-电流密度：

电荷守恒：$\rho dV = \rho_0 dV_0 \overset{dV_0=\gamma dV}{\Longrightarrow} \begin{cases} \rho = \gamma\rho_0 \\ \vec{\boldsymbol{j}} = \gamma\rho_0\vec{\boldsymbol{u}} \end{cases}$

$$j^\alpha \triangleq \rho_0 U^\alpha = \rho_0\gamma(c, \vec{\boldsymbol{u}}) = (\rho c, \vec{\boldsymbol{j}})$$

连续性方程：$\partial_t\rho + \nabla\cdot\vec{\boldsymbol{j}} = 0 \Longleftrightarrow \partial_\alpha j^\alpha = 0$

麦克斯韦方程组：

电磁场强张量：

$$F^{\alpha\beta} = \begin{pmatrix} 0 & \frac{E_1}{c} & \frac{E_2}{c} & \frac{E_3}{c} \\ -\frac{E_1}{c} & 0 & B_3 & -B_2 \\ -\frac{E_2}{c} & -B_3 & 0 & B_1 \\ -\frac{E_3}{c} & B_2 & -B_1 & 0 \end{pmatrix} = \{\frac{\vec{\boldsymbol{E}}}{c}, \vec{\boldsymbol{B}}\}$$

$$F_{\alpha\beta} = \{-\frac{\vec{\boldsymbol{E}}}{c}, \vec{\boldsymbol{B}}\}$$

$$G^{\alpha\beta} \triangleq F^{*\alpha\beta} = \frac{1}{2!}\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu} = \{\vec{\boldsymbol{B}}, -\frac{\vec{\boldsymbol{E}}}{c}\}$$

场的导数：

$$(\partial_\beta F^{\alpha\beta}) = \begin{pmatrix} \frac{1}{c}\nabla\cdot\vec{\boldsymbol{E}} \\ \nabla\times\vec{\boldsymbol{B}} - \frac{1}{c^2}\partial_t\vec{\boldsymbol{E}} \end{pmatrix}$$

$$(\partial_\beta G^{\alpha\beta}) = \begin{pmatrix} \nabla\cdot\vec{\boldsymbol{B}} \\ -\frac{1}{c}\nabla\times\vec{\boldsymbol{E}} - \frac{1}{c}\partial_t\vec{\boldsymbol{B}} \end{pmatrix}$$

麦克斯韦方程组：

$$\begin{cases}\partial_\beta F^{\alpha\beta} = \mu_0 j^\alpha \\ \partial_\beta G^{\alpha\beta} = 0\end{cases}$$

其中 $\partial_\beta G^{\alpha\beta} = 0 \Longleftrightarrow \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$：

$\Longrightarrow$: $\partial_\beta G^{\alpha\beta} = 0 \Longrightarrow \epsilon^{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} = 0 \Longrightarrow$ 由 $F_{\alpha\beta}$ 的反对称性，可得三个数不同时，成立等式。而三个数有两个相同时，可直接得等式成立。三个数都相同时，由于对角线为 0，故也成立。所以对任意 α, β, γ 均成立。

$\Longleftarrow$: $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0, F_{\alpha\beta} = -F_{\beta\alpha}$

$$\Longrightarrow \epsilon^{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} = 0 \Longrightarrow \partial_\beta G^{\alpha\beta} = 0$$

验证电磁场强张量确实为张量：

$$\partial'_\beta F'^{\alpha\beta} = \mu_0 j'^\alpha$$

$$\Lambda_\beta{}^\nu \partial_\nu F'^{\alpha\beta} = \mu_0 \Lambda^\alpha{}_\gamma j^\gamma$$

$$\Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu \partial_\nu F'^{\alpha\beta} = \mu_0 \Lambda_\alpha{}^\mu \Lambda^\alpha{}_\gamma j^\gamma$$

$$\partial_\nu(\Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu F'^{\alpha\beta}) = \mu_0 \delta^\mu{}_\gamma j^\gamma = \mu_0 j^\mu = \partial_\nu F^{\mu\nu}$$

$$\Longrightarrow \begin{pmatrix} \nabla\cdot\vec{\boldsymbol{Y}} \\ -\frac{1}{c}\partial_t\vec{\boldsymbol{Y}} + \nabla\times\vec{\boldsymbol{Z}}\end{pmatrix} = \begin{pmatrix} \nabla\cdot\vec{\boldsymbol{E}} \\ -\frac{1}{c}\partial_t\vec{\boldsymbol{E}} + \nabla\times\vec{\boldsymbol{B}}\end{pmatrix}$$

其中

$$\vec{\boldsymbol{Y}} \triangleq E'_1\hat{\boldsymbol{x}}_1 + \gamma_0(E'_2 + \beta_0 c B'_3)\hat{\boldsymbol{x}}_2 + \gamma_0(E'_3 - \beta_0 c B'_2)\hat{\boldsymbol{x}}_3$$

$$\vec{\boldsymbol{Z}} \triangleq B'_1\hat{\boldsymbol{x}}_1 + \gamma_0(B'_2 - \beta_0\frac{E'_3}{c})\hat{\boldsymbol{x}}_2 + \gamma_0(B'_2 + \beta_0\frac{E'_3}{c})\hat{\boldsymbol{x}}_3$$

$$\left(\Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu F'^{\alpha\beta}\right) = \{\frac{\vec{\boldsymbol{Y}}}{c}, \vec{\boldsymbol{Z}}\}$$

假设电磁场变换是线性齐次的，则可得 $\Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu F'^{\alpha\beta} = F^{\mu\nu}$。

(在线性齐次的情况下，由散度相等可得 $\vec{\boldsymbol{E}} - \vec{\boldsymbol{Y}} = \vec{\boldsymbol{W}}$。$\vec{\boldsymbol{W}}$ 满足 $\nabla\cdot\vec{\boldsymbol{W}} = 0, \vec{\boldsymbol{W}} = \vec{\boldsymbol{W}}(E'_1, \cdots, B'_3)$，是 $E'_1, \cdots, B'_3$ 的齐次线性函数。代入特殊电磁场 (平面电磁波，叠加无限大场源使得只有某个场分量变化，...) 可以得到 $\vec{\boldsymbol{W}} = 0, \vec{\boldsymbol{E}} = \vec{\boldsymbol{Y}}$。类似可得 $\vec{\boldsymbol{B}} = \vec{\boldsymbol{Z}}$)

(设 $F'^{\mu\nu} = M[F^{\mu\nu}]$，不同参考系中的场应该满足各自参考系的叠加原理 $F'^{\mu\nu} = (F'^{\mu\nu})_1 + (F'^{\mu\nu})_2 = M[(F^{\mu\nu})_1] + M[(F^{\mu\nu})_2] = M[F^{\mu\nu}] = M[(F^{\mu\nu})_1 + (F^{\mu\nu})_2]$，故 $M[F^{\mu\nu}]$ 的泰勒展开中应当只包含电磁场分量的一次项，即变换应为线性齐次变换。)

电磁场的变换：

$$\Lambda_{\alpha}{}^{\mu}\Lambda_{\beta}{}^{\nu}F'^{\alpha\beta} = F^{\mu\nu}$$

$$\begin{cases}\vec{E}'_{//} = \vec{E}_{//}, \quad \vec{E}'_{\perp} = \gamma_0(\vec{E}_{\perp} + \vec{\beta}_0 \times c\vec{B}) \\ \vec{B}'_{//} = \vec{B}_{//}, \quad c\vec{B}'_{\perp} = \gamma_0(c\vec{B}_{\perp} - \vec{\beta}_0 \times \vec{E})\end{cases}$$

$$\begin{cases}\vec{E}_{//} = \vec{E}'_{//}, \quad \vec{E}_{\perp} = \gamma_0(\vec{E}'_{\perp} - \vec{\beta}_0 \times c\vec{B}') \\ \vec{B}_{//} = \vec{B}'_{//}, \quad c\vec{B}_{\perp} = \gamma_0(c\vec{B}'_{\perp} + \vec{\beta}_0 \times \vec{E}')\end{cases}$$

低速时：

$$\vec{E}' = \vec{E} + \vec{v}_0 \times \vec{B}, \quad \vec{B}' = \vec{B} - \frac{\vec{v}_0 \times \vec{E}}{c^2}$$

(保留到 β_0 的一级小量)

张量缩并带来的不变量：

$$\mathcal{L}_0 \triangleq -\frac{1}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta} = \frac{1}{2}\epsilon_0(E^2 - c^2B^2)$$

$$-\frac{c}{4}F_{\alpha\beta}G^{\alpha\beta} = \vec{E}\cdot\vec{B}$$

根据 $\mathcal{L}_0$ 的取值将电磁场分类：$E > cB$：类电；$E < cB$：类磁；$E = cB$：电磁波。若 $\vec{E}\cdot\vec{B} = 0$，且 $E > cB$，则存在标架使得在其中为纯电场 (取 $\vec{\beta}_0 = \dfrac{\vec{E}\times c\vec{B}}{E^2}$；若 $\vec{E}\cdot\vec{B} = 0$，且 $E < cB$，则存在标架使得在其中为纯磁场 (取 $\vec{\beta}_0 = \dfrac{\vec{E}\times c\vec{B}}{(cB)^2}$)。若 $\vec{E}\cdot\vec{B} \neq 0$，则可找到标架使得 $\vec{E}'//\vec{B}'$(取 $\vec{\beta}_0$ 满足 $\dfrac{\vec{\beta}_0}{1+\beta_0^2} = \dfrac{\vec{E}\times c\vec{B}}{E^2 + c^2B^2}$)。

4-波矢：

$$k^{\alpha} = (\frac{\omega}{c}, \vec{k}) = \frac{\omega}{c}(1, \hat{\boldsymbol{k}}) = \frac{2\pi}{c}f(1, \hat{\boldsymbol{k}})$$

k^{α} 是 4-矢量：

单色平面波：

$$\begin{cases}\vec{E}(\vec{r}, t) = \vec{E}_0 e^{ik^{\alpha}x_{\alpha}} \\ \vec{B}(\vec{r}, t) = \vec{B}_0 e^{ik^{\alpha}x_{\alpha}}\end{cases}$$

$k^{\alpha}x_{\alpha} = Const + 2n\pi$ 时电磁场都相等，且 $k^{\alpha}x_{\alpha} = Const + 2n\pi$ 确定一族等间隔“平面”，又由于 $x^{\alpha} \to x'^{\alpha}$ 为齐次线性变换，从而在动系中也对应一族等间隔“平面”，故动系电磁场也应为单色平面波。

$$\vec{E}' = \vec{E}'_0 e^{i(k'^{\alpha}x'_{\alpha})} = [\cdots]e^{i(k^{\alpha}x_{\alpha})}$$

$$\Longrightarrow k'^{\alpha}x'_{\alpha} = k^{\alpha}x_{\alpha}$$

$$\Longrightarrow k'^{\alpha} = \Lambda^{\alpha}{}_{\beta}k^{\beta}$$

光行差效应和多普勒效应：

$$k^{\alpha} = \frac{2\pi}{c} f(1, \hat{\boldsymbol{k}})$$

$$\text{观察者：} k^{\alpha} = \frac{2\pi}{c} f \begin{pmatrix} 1 \\ \cos\theta \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \end{pmatrix}, \quad \text{光源：} k^{\alpha} = \frac{2\pi}{c} f_0 \begin{pmatrix} 1 \\ \cos\theta_0 \\ \sin\theta_0\cos\phi_0 \\ \sin\theta_0\sin\phi_0 \end{pmatrix}$$

$$k'^{\alpha} = \Lambda^{\alpha}{}_{\beta} k^{\beta}$$

$$\Longrightarrow \begin{cases} f = f_0 \dfrac{\sqrt{1-\beta^2}}{1-\beta\cos\theta} \\ \cos\theta_0 = \dfrac{\cos\theta - \beta}{1-\beta\cos\theta} \\ \phi_0 = \phi \end{cases} \qquad \begin{cases} f = f_0 \dfrac{1+\beta\cos\theta_0}{\sqrt{1-\beta^2}} \\ \cos\theta = \dfrac{\cos\theta_0 + \beta}{1+\beta\cos\theta_0} \\ \phi = \phi_0 \end{cases}$$

能动量守恒：

4-力密度：

$$f^{\mu} \triangleq F^{\mu\alpha} j_{\alpha} = (\frac{\vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{j}}}{c}, \rho\vec{\boldsymbol{E}} + \vec{\boldsymbol{j}} \times \vec{\boldsymbol{B}})$$

$$\begin{aligned} f^{\mu} &= \frac{1}{\mu_0} F^{\mu\alpha} \mu_0 j_{\alpha} \\ &= \frac{1}{\mu_0} F^{\mu\alpha} \partial^{\beta} F_{\alpha\beta} \\ &= \frac{1}{\mu_0} \partial^{\beta}(F^{\mu\alpha} F_{\alpha\beta}) - \frac{1}{\mu_0} F_{\alpha\beta} \partial^{\beta} F^{\mu\alpha} \\ &= \frac{1}{\mu_0} \partial^{\beta}(F^{\mu\alpha} F_{\alpha\beta}) + \frac{1}{\mu_0} F_{\alpha\beta} \partial^{\mu} F^{\alpha\beta} + \frac{1}{\mu_0} F_{\alpha\beta} \partial^{\alpha} F^{\beta\mu} \\ &\quad (\partial_{\alpha} F_{\beta\gamma} + \partial_{\beta} F_{\gamma\alpha} + \partial_{\gamma} F_{\alpha\beta} = 0) \\ &= \frac{1}{\mu_0} \partial^{\beta}(F^{\mu\alpha} F_{\alpha\beta}) + 2\partial^{\mu} \mathcal{L}_0 + \frac{1}{\mu_0} F_{\alpha\beta} \partial^{\beta} F^{\mu\alpha} \\ &= \frac{1}{\mu_0} \partial^{\beta}(F^{\mu\alpha} F_{\alpha\beta}) + 2\partial^{\mu} \mathcal{L}_0 + \frac{1}{\mu_0} \partial^{\beta}(F^{\mu\alpha} F_{\alpha\beta}) - f^{\mu} \\ &\Longrightarrow f^{\mu} = \partial_{\nu}(\frac{1}{\mu_0} F^{\mu\alpha} F_{\alpha}{}^{\nu}) - \partial_{\nu}(g^{\mu\nu} \mathcal{L}_0) \triangleq -\partial_{\nu} T^{\mu\nu} \end{aligned}$$

能动量张量：

$$T^{\mu\nu} = g^{\mu\nu} \mathcal{L}_0 - \frac{1}{\mu_0} F^{\mu\alpha} F_{\alpha}{}^{\nu} \Longrightarrow T^{\mu\nu} = T^{\nu\mu}$$

$$(T^{\mu\nu}) = \begin{pmatrix} w & \dfrac{\vec{\boldsymbol{S}}}{c} \\ \dfrac{\vec{\boldsymbol{S}}}{c} & \overleftrightarrow{\boldsymbol{T}} \end{pmatrix}$$

$$(-\partial_{\nu} T^{\mu\nu}) = -\begin{pmatrix} \partial_0 w + \nabla \cdot \dfrac{\vec{\boldsymbol{S}}}{c} \\ \partial_0 \dfrac{\vec{\boldsymbol{S}}}{c} + \nabla \cdot \overleftrightarrow{\boldsymbol{T}} \end{pmatrix} = f^{\mu} = \begin{pmatrix} \dfrac{\vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{j}}}{c} \\ \rho\vec{\boldsymbol{E}} + \vec{\boldsymbol{j}} \times \vec{\boldsymbol{B}} \end{pmatrix}$$

规范势：

$(1)F_{\alpha\beta} = -F_{\beta\alpha} \Longrightarrow F_{ab}$ 是 2-形式

$(2)\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0, F_{\alpha\beta} = -F_{\beta\alpha} \Longrightarrow \partial_{[\alpha} F_{\beta\gamma]} = 0$

$$\Longrightarrow (dF)_{abc} = 0 \Longrightarrow F_{ab} = (dA)_{ab}$$

$(3)F_{ab} = (dA)_{ab}$ 显然也可以推出

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0, F_{\alpha\beta} = -F_{\beta\alpha}$$

从而有如下等价关系：

$$\partial_\beta G^{\alpha\beta} = 0 \Longleftrightarrow F_{ab} = (dA)_{ab}$$

$$\Longleftrightarrow \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0, F_{\alpha\beta} = -F_{\beta\alpha}$$

规范势 A^α：

$$A^\alpha = (\frac{\phi}{c}, \overrightarrow{\boldsymbol{A}})$$

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

$$\begin{cases} \overrightarrow{\boldsymbol{E}} = -\nabla\phi - \partial_t \overrightarrow{\boldsymbol{A}} \\ \overrightarrow{\boldsymbol{B}} = \nabla \times \overrightarrow{\boldsymbol{A}} \end{cases}$$

规范变换：

$$\partial^\alpha A^\beta - \partial^\beta A^\alpha = \partial^\alpha A'^\beta - \partial^\beta A'^\alpha$$

$$\Longrightarrow \partial_\alpha K_\beta - \partial_\beta K_\alpha = 0$$

$$(K^\alpha = A'^\alpha - A^\alpha)$$

$$\Longrightarrow (dK)_{ab} = 0 \Longrightarrow K_a = (d\psi)_a$$

$$A'^\alpha = A^\alpha + \partial^\alpha \psi$$

势方程：

$$\begin{aligned} \mu_0 j^\alpha &= \partial_\beta F^{\alpha\beta} = \partial_\beta(\partial^\alpha A^\beta - \partial^\beta A^\alpha) \\ &= \partial^\alpha(\partial_\beta A^\beta) - \partial_\beta \partial^\beta A^\alpha \\ &= \partial^\alpha L - \Box A^\alpha \\ &\Box A^\alpha - \partial^\alpha L = -\mu_0 j^\alpha \end{aligned}$$

（另外的方程已经被规范势表示自动满足。）

洛伦兹规范：

$$L = 0$$

$$\Box A^\alpha = -\mu_0 j^\alpha$$

介质中的麦克斯韦方程：

$$\nabla\cdot\frac{\vec{\boldsymbol{E}}}{c}=\mu_0\rho c,\nabla\times\vec{\boldsymbol{B}}-\frac{1}{c^2}\partial_t\vec{\boldsymbol{E}}=\mu_0\vec{\boldsymbol{j}}\Longleftrightarrow\partial_\beta F^{\alpha\beta}=\mu_0 j^\alpha$$

$$\nabla\cdot=\rho_0 c,\nabla\times\vec{\boldsymbol{H}}-\partial_t\vec{\boldsymbol{D}}=\vec{\boldsymbol{j}}_0\Longleftrightarrow\partial_\beta H^{\alpha\beta}=j_0^\alpha$$

$$\left(H^{\alpha\beta}\right)=\begin{pmatrix}0&cD_1&cD_2&cD_3\\-cD_1&0&H_3&-H_2\\-cD_2&-H_3&0&H_1\\-cD_3&H_2&-H_1&0\end{pmatrix}=\{c\vec{\boldsymbol{D}},\vec{\boldsymbol{H}}\}$$

$$\begin{cases}\vec{\boldsymbol{D}}=\epsilon_0\vec{\boldsymbol{E}}+\vec{\boldsymbol{P}}\\\vec{\boldsymbol{H}}=\dfrac{\vec{\boldsymbol{B}}}{\mu_0}-\vec{\boldsymbol{M}}\end{cases}$$

$$\left(M^{\alpha\beta}\right)=\left(\frac{1}{\mu_0}F^{\alpha\beta}-H^{\alpha\beta}\right)=\begin{pmatrix}0&-cP_1&-cP_2&-cP_3\\cP_1&0&M_3&-M_2\\cP_2&-M_3&0&M_1\\cP_3&M_2&-M_1&0\end{pmatrix}$$

运动介质 ($\beta_0<<1$) 中的 $\vec{\boldsymbol{D}}=\vec{\boldsymbol{D}}(\vec{\boldsymbol{E}},\vec{\boldsymbol{H}}),\vec{\boldsymbol{B}}=\vec{\boldsymbol{B}}(\vec{\boldsymbol{E}},\vec{\boldsymbol{H}})$：

$$\begin{cases}\vec{\boldsymbol{D}}=\epsilon\vec{\boldsymbol{E}}+(\epsilon\mu-\dfrac{1}{c^2})\vec{\boldsymbol{v}}_0\times\vec{\boldsymbol{H}}\\\vec{\boldsymbol{B}}=\mu\vec{\boldsymbol{H}}-(\epsilon\mu-\dfrac{1}{c^2})\vec{\boldsymbol{v}}_0\times\vec{\boldsymbol{E}}\end{cases}$$

粒子动力学：

$$\epsilon\overset{\Delta}{=}\gamma mc^2=\frac{mc^2}{\sqrt{1-\dfrac{u^2}{c^2}}}$$

$$\vec{\boldsymbol{p}}\overset{\Delta}{=}\gamma m\vec{\boldsymbol{u}}=\frac{m\vec{\boldsymbol{u}}}{\sqrt{1-\dfrac{u^2}{c^2}}}$$

$$\Longrightarrow\vec{\boldsymbol{u}}=\frac{c^2\vec{\boldsymbol{p}}}{\epsilon}$$

$$P^\alpha=(\frac{\epsilon}{c},\vec{\boldsymbol{p}})$$

$$P^\alpha P_\alpha=-mc^2\Longrightarrow\epsilon^2=p^2c^2+m^2c^4$$

动力学方程：

假设 MCRF 中，牛二定律成立：$\vec{\boldsymbol{F}}'=m\vec{\boldsymbol{a}}'$

电磁作用有

$$ma'_x = eE'_x, ma'_{y,z} = eE'_{y,z}$$

$$a'_x = \gamma^3 a_x, a'_{y,z} = \gamma^2 a_{y,z}$$

$$E'_x = E_x, E'_{y,z} = \gamma(\overrightarrow{\boldsymbol{E}} + \overrightarrow{\boldsymbol{\beta}} \times c\overrightarrow{\boldsymbol{B}})_{y,z}$$

$$\Longrightarrow \begin{cases} \gamma^3 ma_x = eE_x \\ \gamma ma_{y,z} = e(\overrightarrow{\boldsymbol{E}} + \overrightarrow{\boldsymbol{u}} \times \overrightarrow{\boldsymbol{B}})_{y,z} \end{cases}$$

$$\Longrightarrow \overrightarrow{\boldsymbol{F}} = e\overrightarrow{\boldsymbol{E}} + e\overrightarrow{\boldsymbol{u}} \times \overrightarrow{\boldsymbol{B}} = \frac{d\overrightarrow{\boldsymbol{p}}}{dt}$$

写为 4-矢量：

$$\frac{dP^\alpha}{d\tau} = K^\alpha, \quad K^\alpha \triangleq \gamma(\overrightarrow{\boldsymbol{F}} \cdot \overrightarrow{\boldsymbol{\beta}}, \overrightarrow{\boldsymbol{F}})$$