平均场费米超流中的重杂质相互作用

squid

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1 路径积分推导

无杂质时，相互作用费米场的配分函数利用 Hubbard-Stratonovich 变换可写为

$$\begin{aligned}
Z &= \int \mathcal{D}[\psi^\dagger,\psi] e^{-S[\psi^\dagger_\sigma,\psi_\sigma]} \\
&= \int \mathcal{D}[\psi^\dagger_\sigma,\psi_\sigma] e^{-\int_0^\beta d\tau \int d^3\boldsymbol{r} \left\{\psi^\dagger_\sigma(x)\left(\frac{\partial}{\partial\tau}-\mu-\frac{\nabla^2}{2m}\right)\psi_\sigma(x)+U\psi^\dagger_\uparrow(x)\psi^\dagger_\downarrow(x)\psi_\downarrow(x)\psi_\uparrow(x)\right\}} \\
&= \int \mathcal{D}[\Delta^\dagger_\sigma,\Delta_\sigma] \int \mathcal{D}[\psi^\dagger,\psi] e^{\int dx \frac{|\Delta(x)|^2}{U}} e^{-\int_0^\beta d\tau \int d^3\boldsymbol{r} \left\{\psi^\dagger_\sigma(x)\left(\frac{\partial}{\partial\tau}-\mu-\frac{\nabla^2}{2m}\right)\psi_\sigma(x)-\psi^\dagger_\uparrow(x)\psi^\dagger_\downarrow(x)\Delta(x)-\psi_\downarrow(x)\psi_\uparrow(x)\Delta^*(x)\right\}}
\end{aligned} \tag{1}$$

其中 $x=(\tau,\boldsymbol{r})$。考虑平均场近似 $\Delta(x)=\Delta$，Δ 取为 BCS-BEC crossover 中解出来的值，配分函数简化为

$$\begin{aligned}
Z &= \int \mathcal{D}[\psi^\dagger_\sigma,\psi_\sigma] e^{\int dx \frac{|\Delta|^2}{U}} e^{-\int_0^\beta d\tau \int d^3\boldsymbol{r} \left\{\psi^\dagger_\sigma(x)\left(\frac{\partial}{\partial\tau}-\mu-\frac{\nabla^2}{2m}\right)\psi_\sigma(x)-\psi^\dagger_\uparrow(x)\psi^\dagger_\downarrow(x)\Delta-\psi_\downarrow(x)\psi_\uparrow(x)\Delta\right\}} \\
&= \int \mathcal{D}[\psi^\dagger_\sigma,\psi_\sigma] e^{\int dx \frac{|\Delta|^2}{U}} e^{-S[\psi^\dagger_\sigma,\psi_\sigma]}
\end{aligned} \tag{2}$$

在平均场的基础上加入重杂质（$V_{i\uparrow}=-g\delta(\boldsymbol{r}-\boldsymbol{r}_i)$，只与上自旋作用）的影响，作用量 S 变为

$$\begin{aligned}
S[\psi^\dagger_\sigma,\psi_\sigma] &= \int_0^\beta d\tau \int d^3\boldsymbol{r} \left\{\psi^\dagger_\sigma(x)\left(\frac{\partial}{\partial\tau}-\mu-\frac{\nabla^2}{2m}\right)\psi_\sigma(x)-\psi^\dagger_\uparrow(x)\psi^\dagger_\downarrow(x)\Delta-\psi_\downarrow(x)\psi_\uparrow(x)\Delta\right\} \\
&\quad - g\sum_i \int_0^\beta d\tau \psi^\dagger_\uparrow(\tau,\boldsymbol{r}_i)\psi_\uparrow(\tau,\boldsymbol{r}_i) \\
&= S_0[\psi^\dagger_\sigma,\psi_\sigma] - g\sum_i \int_0^\beta d\tau \psi^\dagger_\uparrow(\tau,\boldsymbol{r}_i)\psi_\uparrow(\tau,\boldsymbol{r}_i)
\end{aligned} \tag{3}$$

其中 i 为杂质指标。利用 δ 函数引入辅助场 $\eta_i(\tau)$，

$$1=\prod_i \int \mathcal{D}[\eta^\dagger_i,\eta_i]\delta(\psi_\uparrow(\tau,\boldsymbol{r}_i)-\eta_i(\tau))\delta(\psi^\dagger_\uparrow(\tau,\boldsymbol{r}_i)-\eta^\dagger_i(\tau)) \tag{4}$$

$$
\begin{aligned}
Z &= \int \mathcal{D}[\psi_\sigma^\dagger,\psi_\sigma] e^{\int dx \frac{|\Delta|^2}{U}} e^{-S_0[\psi_\sigma^\dagger,\psi_\sigma]} e^{g\sum_i \int_0^\beta d\tau \psi_\uparrow^\dagger(\tau,\boldsymbol{r}_i)\psi_\uparrow(\tau,\boldsymbol{r}_i)} \\
&= \prod_i \int \mathcal{D}[\psi_\sigma^\dagger,\psi_\sigma]\mathcal{D}[\eta_i^\dagger,\eta_i] e^{\int dx \frac{|\Delta|^2}{U}} e^{-S_0[\psi_\sigma^\dagger,\psi_\sigma]} e^{g\sum_i \int_0^\beta d\tau \eta_i^\dagger(\tau)\eta_i(\tau)} \delta(\psi_\uparrow(\tau,\boldsymbol{r}_i)-\eta_i(\tau))\delta(\psi_\uparrow^\dagger(\tau,\boldsymbol{r}_i)-\eta_i^\dagger(\tau))
\end{aligned} \tag{5}
$$

再引入辅助场 $\alpha_i(\tau)$ 把 δ 函数变为指数积分的形式，

$$
\delta(\psi_\uparrow(\tau,\boldsymbol{r}_i)-\eta_i(\tau))\delta(\psi_\uparrow^\dagger(\tau,\boldsymbol{r}_i)-\eta_i^\dagger(\tau)) = \int \mathcal{D}[\alpha_i^\dagger,\alpha_i] e^{i\int d\tau \alpha_i^\dagger(\tau)(\psi_\uparrow(\tau,\boldsymbol{r}_i)-\eta_i(\tau)) + i\int d\tau(\psi_\uparrow^\dagger(\tau,\boldsymbol{r}_i)-\eta_i^\dagger(\tau))\alpha_i(\tau)} \tag{6}
$$

此时配分函数写为

$$
Z = \prod_i \int \mathcal{D}[\psi_\sigma^\dagger,\psi_\sigma]\mathcal{D}[\eta_i^\dagger,\eta_i]\mathcal{D}[\alpha_i^\dagger,\alpha_i] e^{\int dx \frac{|\Delta|^2}{U}} e^{-S'[\psi_\sigma^\dagger,\psi_\sigma,\eta_i^\dagger,\eta_i,\alpha_i^\dagger,\alpha_i]} \tag{7}
$$

其中

$$
\begin{aligned}
S'[\psi_\sigma^\dagger,\psi_\sigma,\eta_i^\dagger,\eta_i,\alpha_i^\dagger,\alpha_i] &= \int_0^\beta d\tau \int d^3\boldsymbol{r} \left\{ \psi_\sigma^\dagger(x)\left(\frac{\partial}{\partial\tau}-\mu-\frac{\nabla^2}{2m}\right)\psi_\sigma(x) - \psi_\uparrow^\dagger(x)\psi_\downarrow^\dagger(x)\Delta - \psi_\downarrow(x)\psi_\uparrow(x)\Delta \right\} \\
&\quad - i\sum_i \int d\tau \alpha_i^\dagger(\tau)(\psi_\uparrow(\tau,\boldsymbol{r}_i)-\eta_i(\tau)) - i\sum_i \int d\tau(\psi_\uparrow^\dagger(\tau,\boldsymbol{r}_i)-\eta_i^\dagger(\tau))\alpha_i(\tau) \\
&\quad - g\sum_i \int_0^\beta d\tau \eta_i^\dagger(\tau)\eta_i(\tau)
\end{aligned} \tag{8}
$$

作傅里叶变换

$$
\begin{aligned}
\psi_\sigma(x) = \frac{1}{\sqrt{V\beta}}\sum_{\omega_n,\boldsymbol{k}} \psi_\sigma(i\omega_n,\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega_n\tau}, &\quad \psi_\sigma^\dagger(x) = \frac{1}{\sqrt{V\beta}}\sum_{\omega_n,\boldsymbol{k}} \psi_\sigma^\dagger(i\omega_n,\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}+i\omega_n\tau} \\
\eta(\tau) = \frac{1}{\sqrt{\beta}}\sum_{\omega_n} \eta(i\omega_n) e^{-i\omega_n\tau}, &\quad \eta^\dagger(\tau) = \frac{1}{\sqrt{\beta}}\sum_{\omega_n} \eta^\dagger(i\omega_n) e^{i\omega_n\tau} \\
\alpha(\tau) = \frac{1}{\sqrt{\beta}}\sum_{\omega_n} \alpha(i\omega_n) e^{-i\omega_n\tau}, &\quad \alpha^\dagger(\tau) = \frac{1}{\sqrt{\beta}}\sum_{\omega_n} \alpha^\dagger(i\omega_n) e^{i\omega_n\tau}
\end{aligned} \tag{9}
$$

$S'[\psi_\sigma^\dagger,\psi_\sigma,\eta_i^\dagger,\eta_i,\alpha_i^\dagger,\alpha_i]$ 可以化为

$$
\begin{aligned}
S'[\psi_\sigma^\dagger,\psi_\sigma,\eta_i^\dagger,\eta_i,\alpha_i^\dagger,\alpha_i] &= \sum_{\omega_n,\boldsymbol{k}} \psi_\uparrow^\dagger(i\omega_n,\boldsymbol{k})\left(-i\omega_n+\frac{k^2}{2m}-\mu\right)\psi_\uparrow(i\omega_n,\boldsymbol{k}) + \psi_\downarrow(-i\omega_n,-\boldsymbol{k})\left(-i\omega_n-\frac{k^2}{2m}+\mu\right)\psi_\downarrow^\dagger(-i\omega_n,-\boldsymbol{k}) \\
&\quad - \sum_{\omega_n,\boldsymbol{k}} \Delta\psi_\uparrow^\dagger(i\omega_n,\boldsymbol{k})\psi_\downarrow^\dagger(-i\omega_n,-\boldsymbol{k}) - \sum_{\omega_n,\boldsymbol{k}} \Delta\psi_\downarrow(-i\omega_n,-\boldsymbol{k})\psi_\uparrow(i\omega_n,\boldsymbol{k}) \\
&\quad - i\sum_i\sum_{\omega_n} \alpha_i^\dagger(i\omega_n)\left[\frac{1}{\sqrt{V}}\sum_{\boldsymbol{k}} \psi_\uparrow(i\omega_n,\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i} - \eta_i(i\omega_n)\right] \\
&\quad - i\sum_i\sum_{\omega_n} \left[\frac{1}{\sqrt{V}}\sum_{\boldsymbol{k}} \psi_\uparrow^\dagger(i\omega_n,\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i} - \eta_i^\dagger(i\omega_n)\right]\alpha_i(i\omega_n) \\
&\quad - g\sum_i\sum_{\omega_n} \eta^\dagger(i\omega_n)\eta(i\omega_n)
\end{aligned} \tag{10}
$$

作玻戈留玻夫变换

$$
\begin{aligned}
\gamma_1(i\omega_n,\boldsymbol{k}) = u_{\boldsymbol{k}}\psi_\uparrow(i\omega_n,\boldsymbol{k}) - v_{\boldsymbol{k}}\psi_\downarrow^\dagger(-i\omega_n,-\boldsymbol{k}), &\quad \gamma_1^\dagger(i\omega_n,\boldsymbol{k}) = u_{\boldsymbol{k}}\psi_\uparrow^\dagger(i\omega_n,\boldsymbol{k}) - v_{\boldsymbol{k}}\psi_\downarrow(-i\omega_n,-\boldsymbol{k}) \\
\gamma_2^\dagger(i\omega_n,\boldsymbol{k}) = v_{\boldsymbol{k}}\psi_\uparrow(i\omega_n,\boldsymbol{k}) + u_{\boldsymbol{k}}\psi_\downarrow^\dagger(-i\omega_n,-\boldsymbol{k}), &\quad \gamma_2(i\omega_n,\boldsymbol{k}) = v_{\boldsymbol{k}}\psi_\uparrow^\dagger(i\omega_n,\boldsymbol{k}) + u_{\boldsymbol{k}}\psi_\downarrow(-i\omega_n,-\boldsymbol{k})
\end{aligned} \tag{11}
$$

其中

$$u_{\boldsymbol{k}}^2=\frac{1}{2}\left(1+\frac{\frac{k^2}{2m}-\mu}{\sqrt{\left(\frac{k^2}{2m}-\mu\right)^2+\Delta^2}}\right),\quad v_{\boldsymbol{k}}=\frac{1}{2}\left(1-\frac{\frac{k^2}{2m}-\mu}{\sqrt{\left(\frac{k^2}{2m}-\mu\right)^2+\Delta^2}}\right) \tag{12}$$

代入 $S'[\psi_\sigma^\dagger,\psi_\sigma,\eta_i^\dagger,\eta_i,\alpha_i^\dagger,\alpha_i]$ 可以得到

$$\begin{aligned}
&S'[\gamma_\sigma^\dagger,\gamma_\sigma,\eta_i^\dagger,\eta_i,\alpha_i^\dagger,\alpha_i]\\
&=\sum_{\omega_n,\boldsymbol{k}}\left(-i\omega_n+\sqrt{\left(\frac{k^2}{2m}-\mu\right)^2+\Delta^2}\right)\gamma_1^\dagger(i\omega_n,\boldsymbol{k})\gamma_1(i\omega_n,\boldsymbol{k})+\sum_{\omega_n,\boldsymbol{k}}\left(-i\omega_n-\sqrt{\left(\frac{k^2}{2m}-\mu\right)^2+\Delta^2}\right)\gamma_2(i\omega_n,\boldsymbol{k})\gamma_2^\dagger(i\omega_n,\boldsymbol{k})\\
&\quad-i\sum_i\sum_{\omega_n,\boldsymbol{k}}\alpha_i^\dagger(i\omega_n)\frac{1}{\sqrt{V}}\left(u_{\boldsymbol{k}}\gamma_1(i\omega_n,\boldsymbol{k})+v_{\boldsymbol{k}}\gamma_2^\dagger(i\omega_n,\boldsymbol{k})\right)e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i}\\
&\quad-i\sum_i\sum_{\omega_n,\boldsymbol{k}}\frac{1}{\sqrt{V}}\left(u_{\boldsymbol{k}}\gamma_1^\dagger(i\omega_n,\boldsymbol{k})+v_{\boldsymbol{k}}\gamma_2(i\omega_n,\boldsymbol{k})\right)e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i}\alpha_i(i\omega_n)\\
&\quad-g\sum_i\sum_{\omega_n}\left(\eta^\dagger(i\omega_n)-\frac{i}{g}\alpha_i^\dagger(i\omega_n)\right)\left(\eta(i\omega_n)-\frac{i}{g}\alpha_i(i\omega_n)\right)-\frac{1}{g}\sum_i\sum_{\omega_n}\alpha_i^\dagger(i\omega_n)\alpha_i(i\omega_n)\\
&=\sum_{\omega_n,\boldsymbol{k}}\left(-i\omega_n+\sqrt{\left(\frac{k^2}{2m}-\mu\right)^2+\Delta^2}\right)\left[\gamma_1^\dagger(i\omega_n,\boldsymbol{k})-i\sum_i\alpha_i^\dagger(i\omega_n)\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i}}{\sqrt{V}}u_{\boldsymbol{k}}\right]\left[\gamma_1(i\omega_n,\boldsymbol{k})-i\sum_i\alpha_i(i\omega_n)\frac{e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i}}{\sqrt{V}}u_{\boldsymbol{k}}\right]\\
&\quad+\sum_{\omega_n,\boldsymbol{k}}\left(-i\omega_n-\sqrt{\left(\frac{k^2}{2m}-\mu\right)^2+\Delta^2}\right)\left[\gamma_2(i\omega_n,\boldsymbol{k})-i\sum_i\alpha_i^\dagger(i\omega_n)\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i}}{\sqrt{V}}v_{\boldsymbol{k}}\right]\left[\gamma_2^\dagger(i\omega_n,\boldsymbol{k})-i\sum_i\alpha_i(i\omega_n)\frac{e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i}}{\sqrt{V}}v_{\boldsymbol{k}}\right]\\
&\quad+\sum_{\omega_n}\sum_{i,j}\alpha_i^\dagger(i\omega_n)\frac{1}{V}\sum_{\boldsymbol{k}}\left[\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}}{-i\omega_n+\sqrt{\left(\frac{k^2}{2m}-\mu\right)^2+\Delta^2}}u_{\boldsymbol{k}}^2+\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}}{-i\omega_n-\sqrt{\left(\frac{k^2}{2m}-\mu\right)^2+\Delta^2}}v_{\boldsymbol{k}}^2\right]\alpha_j(i\omega_n)\\
&\quad-g\sum_i\sum_{\omega_n}\left(\eta^\dagger(i\omega_n)-\frac{i}{g}\alpha_i^\dagger(i\omega_n)\right)\left(\eta(i\omega_n)-\frac{i}{g}\alpha_i(i\omega_n)\right)-\sum_{\omega_n}\sum_{i,j}\alpha_i^\dagger(i\omega_n)\frac{\delta_{ij}}{g}\alpha_j(i\omega_n)
\end{aligned} \tag{13}$$

于是 $\gamma_\sigma^\dagger,\gamma_\sigma,\eta_i^\dagger,\eta_i,\alpha_i^\dagger,\alpha_i$ 均构成二次型。积掉 $\gamma_\sigma^\dagger,\gamma_\sigma,\eta_i^\dagger,\eta_i$ 可以得到

$$Z=Z_0\int\mathcal{D}[\alpha^\dagger,\alpha]e^{-S''[\alpha^\dagger,\alpha]} \tag{14}$$

其中 $S''[\alpha^\dagger,\alpha]$ 为

$$\begin{aligned}
S''[\alpha^\dagger,\alpha]&=\sum_{\omega_n}\sum_{i,j}\alpha_i^\dagger(i\omega_n)\left[-\frac{\delta_{ij}}{g}+\frac{1}{V}\sum_{\boldsymbol{k}}\left(\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}}{-i\omega_n+\sqrt{\left(\frac{k^2}{2m}-\mu\right)^2+\Delta^2}}u_{\boldsymbol{k}}^2+\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}}{-i\omega_n-\sqrt{\left(\frac{k^2}{2m}-\mu\right)^2+\Delta^2}}v_{\boldsymbol{k}}^2\right)\right]\alpha_j(i\omega_n)\\
&=\sum_{\omega_n}\sum_{i,j}\alpha_i^\dagger(i\omega_n)\left[-\frac{\delta_{ij}}{g}+\int\frac{d\boldsymbol{k}}{(2\pi)^3}\frac{-i\omega_n-\left(\frac{k^2}{2m}-\mu\right)}{-\omega^2-\left(\frac{k^2}{2m}-\mu\right)^2-\Delta^2}e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}\right]\alpha_j(i\omega_n)\\
&=\sum_{\omega_n}\sum_{i,j}\alpha_i^\dagger(i\omega_n)M_{ij}(i\omega_n)\alpha_j(i\omega_n)
\end{aligned} \tag{15}$$

M 矩阵矩阵元为

$$
\begin{aligned}
M_{ii} &= -\frac{1}{g} + \int \frac{d\boldsymbol{k}}{(2\pi)^3} \frac{-i\omega_n - \left(\frac{k^2}{2m} - \mu\right)}{-\omega^2 - \left(\frac{k^2}{2m} - \mu\right)^2 - \Delta^2} \\
&= \frac{m}{2\pi a} + \int \frac{d\boldsymbol{k}}{(2\pi)^3} \left(\frac{-i\omega_n - \left(\frac{k^2}{2m} - \mu\right)}{-\omega^2 - \left(\frac{k^2}{2m} - \mu\right)^2 - \Delta^2} - \frac{2m}{k^2} \right) \\
&= \cdots \\
&= \frac{m}{2\pi a} + \frac{m}{\pi} \left\{ \frac{S_2 \left[2m\mu(i2m\omega + i2m\sqrt{\omega^2 + \Delta^2}) - 4m^2\omega\sqrt{\omega^2 + \Delta^2} - 4m^2(\omega^2 + \Delta^2) \right]}{8m\sqrt{\omega^2 + \Delta^2}\sqrt{4m^2\mu^2 + 4m^2(\omega^2 + \Delta^2)}} \right. \\
&\qquad \left. + \frac{S_1 \left[2m\mu(i2m\omega - i2m\sqrt{\omega^2 + \Delta^2}) + 4m^2\omega\sqrt{\omega^2 + \Delta^2} - 4m^2(\omega^2 + \Delta^2) \right]}{8m\sqrt{\omega^2 + \Delta^2}\sqrt{4m^2\mu^2 + 4m^2(\omega^2 + \Delta^2)}} \right\}
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
M_{i\neq j} &= \int \frac{d\boldsymbol{k}}{(2\pi)^3} \frac{-i\omega_n - \left(\frac{k^2}{2m} - \mu\right)}{-\omega^2 - \left(\frac{k^2}{2m} - \mu\right)^2 - \Delta^2} e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)} \\
&= \cdots \\
&= \frac{m^2}{\pi|\boldsymbol{r}_i - \boldsymbol{r}_j|} \frac{(\omega + \sqrt{\omega^2 + \Delta^2})e^{iS_1|\boldsymbol{r}_i - \boldsymbol{r}_j|} - (\omega - \sqrt{\omega^2 + \Delta^2})e^{-iS_2|\boldsymbol{r}_i - \boldsymbol{r}_j|}}{4m\sqrt{\omega^2 + \Delta^2}}
\end{aligned}
$$

其中

$$
S_1 = \sqrt{2m\mu + i2m\sqrt{\omega^2 + \Delta^2}}, \quad S_2 = \sqrt{2m\mu - i2m\sqrt{\omega^2 + \Delta^2}} \tag{17}
$$

约定 S_1, S_2 为实部大于零的解。

积掉 $\alpha_i^\dagger, \alpha_i$，得到配分函数以及热力学势

$$
Z = Z_0 e^{\sum_{\omega_n} \ln \det M(i\omega_n)}, \quad \Omega = -\frac{1}{\beta} \sum_{\omega_n} \ln \det M(i\omega_n) + \text{const} \tag{18}
$$

忽略不重要的常数，零温极限下得到

$$
\min\{\langle H - \mu N \rangle\} = \Omega|_{T\to 0} = -\int \frac{d\omega}{2\pi} \ln \det M(i\omega) \tag{19}
$$

于是双杂质相互作用能可以写为

$$
E_{int} = -\int \frac{d\omega}{2\pi} \ln \frac{\det M(i\omega)}{\det M^\infty(i\omega)} \tag{20}
$$

由 M 矩阵矩阵元的表达式可以发现 $M(i\omega) = M^*(-i\omega)$，于是 E_{int} 可以写出实数积分的形式

$$
E_{int} = -\int \frac{d\omega}{2\pi} \mathrm{Re} \left[\ln \frac{\det M(i\omega)}{\det M^\infty(i\omega)} \right] \tag{21}
$$

2 有超流时的标度律

以费米能为单位对具有能量量纲的量进行无量纲化 $\tilde{x} = x/E_F$。重杂质相互作用 E_{int} 的无量纲形式相应写为

$$
\tilde{E}_{int}(d, a_{if}, a_{ff}, k_F) = -\int \frac{d\tilde{\omega}}{2\pi} \mathrm{Re} \left[\ln \frac{\det M(i\tilde{\omega}, \tilde{\mu}, \tilde{\Delta}, k_F a_{if}, k_F d, k_F)}{\det M^\infty(i\tilde{\omega}, \tilde{\mu}, \tilde{\Delta}, k_F a_{if}, k_F))} \right] = -\int \frac{d\tilde{\omega}}{2\pi} f(i\tilde{\omega}, \tilde{\mu}, \tilde{\Delta}, k_F a_{if}, k_F d)
$$

由 M 矩阵矩阵元的形式可以证明 f 是一个只依赖于无量纲数的无量纲数。而平均场 BCS-BEC crossover 的化学势 $\tilde{\mu}$ 和能隙 $\tilde{\Delta}$ 有严格解（Eur. Phys. J. B 1, 151159 (1998)），

$$\tilde{\mu} = \frac{x_0}{(x_0 I_5(x_0) + I_6(x_0))^{2/3}}$$
$$\tilde{\Delta} = \frac{1}{(x_0 I_5(x_0) + I_6(x_0))^{2/3}}$$

其中 x_0 可由 $1/k_F a_{ff}$ 确定，

$$\frac{1}{k_F a_{ff}} = -\frac{4}{\pi} \frac{x_0 I_6(x_0) - I_5(x_0)}{(x_0 I_5(x_0) + I_6(x_0))^{1/3}}$$

$I_5(x_0), I_6(x_0)$ 是与椭圆积分有关的函数，

$$I_5(x_0) = (1 + x_0^2)^{1/4} E(\frac{\pi}{2}, \kappa) - \frac{1}{4x_1^2(1 + x_0^2)} F(\frac{\pi}{2}, \kappa)$$
$$I_6(x_0) = \frac{1}{2(1 + x_0^2)^{1/4}} F(\frac{\pi}{2}, \kappa)$$

其中

$$x_1^2 = \frac{\sqrt{1 + x_0^2} + x_0}{2}, \quad \kappa^2 = \frac{x_1^2}{(1 + x_0^2)^{1/2}}$$

$F(\frac{\pi}{2}, \kappa), E(\frac{\pi}{2}, \kappa)$ 分别为第一类、第二类完全椭圆积分。

于是可以发现，如果作变换 $d \to \lambda^{-1} d, a_{if} \to \lambda^{-1} a_{if}, a_{ff} \to \lambda^{-1} a_{ff}, k_F \to \lambda k_F$，$x_0$ 保持不变，从而 $\tilde{\mu}, \tilde{\Delta}$ 也保持不变。又由于 $k_F a_{if}, k_F d$ 同样保持不变，所以

$$\tilde{E}_{int}(d, a_{if}, a_{ff}, k_F) = \tilde{E}_{int}(\lambda^{-1} d, \lambda^{-1} a_{if}, \lambda^{-1} a_{ff}, \lambda k_F)$$

也就是说

$$E_{int}(\lambda^{-1} d, \lambda^{-1} a_{if}, \lambda^{-1} a_{ff}, \lambda k_F) = \lambda^2 E_{int}(d, a_{if}, a_{ff}, k_F)$$

从而此时重杂质的哈密顿量具有标度对称性

$$H(\lambda^{-1} d, \lambda^{-1} a_{if}, \lambda^{-1} a_{ff}, \lambda k_F) = \lambda^2 H(d, a_{if}, a_{ff}, k_F)$$