# Homework 9 Solutions

## $\boxed{1}$ (Griffiths 1.7)

### Problem 1.7

From Eq. 1.33, $\frac{d\langle p\rangle}{dt} = -i\hbar \int \frac{\partial}{\partial t}\left(\Psi^* \frac{\partial \Psi}{\partial x}\right) dx$. But, noting that $\frac{\partial^2 \Psi}{\partial x \partial t} = \frac{\partial^2 \Psi}{\partial t \partial x}$ and using Eqs. 1.23-1.24:

$$\frac{\partial}{\partial t}\left(\Psi^* \frac{\partial \Psi}{\partial x}\right) = \frac{\partial \Psi^*}{\partial t}\frac{\partial \Psi}{\partial x} + \Psi^* \frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial t}\right) = \left[-\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*\right]\frac{\partial \Psi}{\partial x} + \Psi^* \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right]$$

$$= \frac{i\hbar}{2m}\left[\Psi^* \frac{\partial^3 \Psi}{\partial x^3} - \frac{\partial^2 \Psi^*}{\partial x^2}\frac{\partial \Psi}{\partial x}\right] + \frac{i}{\hbar}\left[V\Psi^* \frac{\partial \Psi}{\partial x} - \Psi^* \frac{\partial}{\partial x}(V\Psi)\right]$$

The first term integrates to zero, using integration by parts twice, and the second term can be simplified to $V\Psi^* \frac{\partial \Psi}{\partial x} - \Psi^* V \frac{\partial \Psi}{\partial x} - \Psi^* \frac{\partial V}{\partial x}\Psi = -|\Psi|^2 \frac{\partial V}{\partial x}$. So

$$\frac{d\langle p\rangle}{dt} = -i\hbar\left(\frac{i}{\hbar}\right)\int -|\Psi|^2 \frac{\partial V}{\partial x} dx = \langle -\frac{\partial V}{\partial x}\rangle. \qquad \text{QED}$$

## Problem 1.14

**(a)** $P_{ab}(t) = \int_a^b |\Psi(x,t)|^2 dx$, so $\frac{dP_{ab}}{dt} = \int_a^b \frac{\partial}{\partial t}|\Psi|^2 dx$. But (Eq. 1.25):

$$\frac{\partial |\Psi|^2}{\partial t} = \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)\right] = -\frac{\partial}{\partial x}J(x,t).$$

$$\therefore \frac{dP_{ab}}{dt} = -\int_a^b \frac{\partial}{\partial x}J(x,t)dx = -\left[J(x,t)\right]\Big|_a^b = J(a,t) - J(b,t). \qquad \text{QED}$$

Probability is dimensionless, so $J$ has the dimensions 1/time, and units $\boxed{\text{seconds}^{-1}.}$

**(b)** Here $\Psi(x,t) = f(x)e^{-iat}$, where $f(x) \equiv Ae^{-amx^2/\hbar}$, so $\Psi \frac{\partial \Psi^*}{\partial x} = fe^{-iat}\frac{df}{dx}e^{iat} = f\frac{df}{dx}$, and $\Psi^* \frac{\partial \Psi}{\partial x} = f\frac{df}{dx}$ too, so $\boxed{J(x,t) = 0.}$

## Problem 2.14

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2}, \text{ so } P = 2\sqrt{\frac{m\omega}{\pi\hbar}} \int_{x_0}^{\infty} e^{-\xi^2} dx = 2\sqrt{\frac{m\omega}{\pi\hbar}} \sqrt{\frac{\hbar}{m\omega}} \int_{\xi_0}^{\infty} e^{-\xi^2} d\xi.$$

Classically allowed region extends out to: $\frac{1}{2}m\omega^2 x_0^2 = E_0 = \frac{1}{2}\hbar\omega$, or $x_0 = \sqrt{\frac{\hbar}{m\omega}}$, so $\xi_0 = 1$.

$$P = \frac{2}{\sqrt{\pi}} \int_1^{\infty} e^{-\xi^2} d\xi = 2(1 - F(\sqrt{2})) \text{ (in notation of CRC Table)} = \boxed{0.157.}$$

## Problem 2.18

Equation 2.95 says $\Psi = Ae^{i(kx - \frac{\hbar k^2}{2m}t)}$, so

$$J = \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right) = \frac{i\hbar}{2m}|A|^2\left[e^{i(kx-\frac{\hbar k^2}{2m}t)}(-ik)e^{-i(kx-\frac{\hbar k^2}{2m}t)} - e^{-i(kx-\frac{\hbar k^2}{2m}t)}(ik)e^{i(kx-\frac{\hbar k^2}{2m}t)}\right]$$

$$= \frac{i\hbar}{2m}|A|^2(-2ik) = \boxed{\frac{\hbar k}{m}|A|^2.}$$

It flows in the positive ($x$) direction (as you would expect).

# $\boxed{2}$ (Particle on a circle)

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$ (where $x$ is measured around the circumference), or $\frac{d^2\psi}{dx^2} = -k^2\psi$, with $k \equiv \frac{\sqrt{2mE}}{\hbar}$, so

$$\psi(x) = Ae^{ikx} + Be^{-ikx}.$$

But $\psi(x+L) = \psi(x)$, since $x+L$ is the same point as $x$, so

$$Ae^{ikx}e^{ikL} + Be^{-ikx}e^{-ikL} = Ae^{ikx} + Be^{-ikx},$$

and this is true for *all* $x$. In particular, for $x = 0$ :

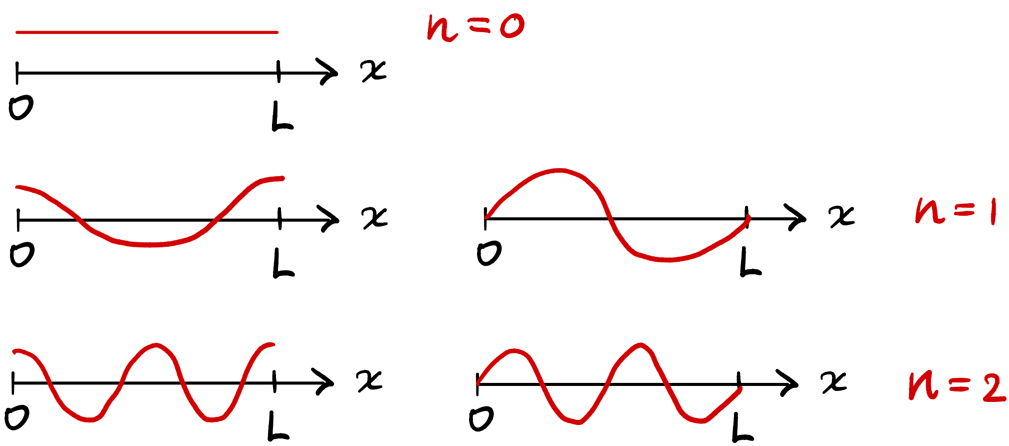
(1) $\quad Ae^{ikL} + Be^{-ikL} = A + B.\quad$ And for $x = \frac{\pi}{2k}$ :

$Ae^{i\pi/2}e^{ikL} + Be^{-i\pi/2}e^{-ikL} = Ae^{i\pi/2} + Be^{-i\pi/2}$, or $iAe^{ikL} - iBe^{-ikL} = iA - iB$, so

(2) $\quad Ae^{ikL} - Be^{-ikL} = A - B.\quad$ Add (1) and (2): $2Ae^{ikL} = 2A$.

Either $A = 0$, or else $e^{ikL} = 1$, in which case $kL = 2n\pi$ $(n = 0, \pm1, \pm2, \ldots)$. But if $A = 0$, then $Be^{-ikL} = B$, leading to the same conclusion. So for every positive $n$ there are *two* solutions: $\psi_n^+(x) = Ae^{i(2n\pi x/L)}$ and $\psi_n^-(x) = Be^{-i(2n\pi x/L)}$ ($n = 0$ is ok too, but in that case there is just *one* solution). Normalizing: $\int_0^L |\psi_\pm|^2 dx = 1 \Rightarrow A = B = 1/\sqrt{L}$. Any *other* solution (with the same energy) is a linear combination of these.

$$\boxed{\psi_n^\pm(x) = \frac{1}{\sqrt{L}}e^{\pm i(2n\pi x/L)};\ E_n = \frac{2n^2\pi^2\hbar^2}{mL^2}\quad (n = 0, 1, 2, 3, \ldots).}$$




etc...