hw 12 sols

1 (Griffiths 4.12)

$\rho \equiv \frac{r}{na}$

*Let's find $R_{30}(r)$. We have in general

$$R_{n\ell}(r) = \frac{1}{r}\rho^{\ell+1} e^{-\rho} v(\rho)$$

For the case of R_{30} we have

$$R_{30}(r) = \frac{1}{\cancel{r}}\frac{\cancel{r}}{3a} e^{-r/3a} v(\rho)$$

with the polynomial $v(\rho)$ determined by the sequence

$$c_{j+1} = \frac{2(j+\ell+1-n)}{(j+1)(j+2\ell+2)} c_j$$

For $n=3$, $\ell=0$, we get the sequence

$$c_0 = 1$$

$$c_1 = \frac{2(0+0+1-3)}{(0+1)(0+0+2)} \cdot 1 = -2$$

$$c_2 = \frac{2(1+1-3)}{(2)(1+2)} \cdot (-2) = \frac{+2}{3}$$

$$c_3 = \frac{2(2+1-3)}{(2+1)(2+2)}\left(-\frac{1}{4}\right) = 0$$

so, up to normalization:

$$v(\rho) = 1 - 2\rho + \frac{2}{3}\rho^2$$

The $n=3$, $\ell=0$ radial wavefunction is:

$$\boxed{R_{30}(r) = \left(1 - 2\frac{r}{3a} + \frac{2}{3}\left(\frac{r}{3a}\right)^2\right) e^{-r/3a}}$$

(not normalized).

* For $R_{31}(r)$, the sequence goes:

$c_0 = 1$

$$c_1 = \frac{2(j+\ell+1-n)}{(j+1)(j+2\ell+2)} = \frac{2(0+1+1-3)}{1\cdot(0+2+2)} = \frac{-2}{4} = -\frac{1}{2}$$

$c_2 = 0$

so

$$\boxed{R_{31}(r) = \left(1 - \frac{1}{2}\left(\frac{r}{3a}\right)\right)\frac{r}{a}\, e^{-r/3a}}$$

Finally, for $R_{32}(r)$, we have:

$c_0 = 1$

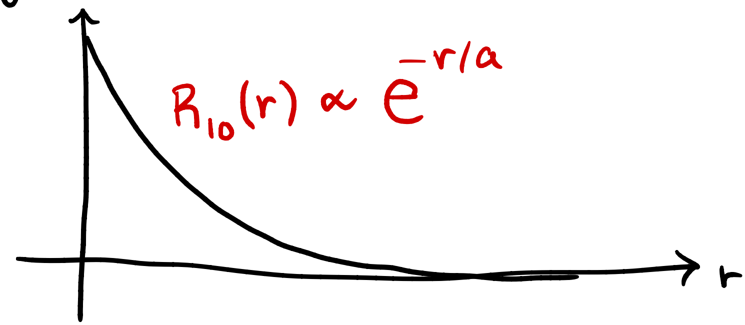
$c_1 = 0$

so

$$\boxed{R_{32}(r) = \left(\frac{r}{a}\right)^2 e^{-r/3a}}$$

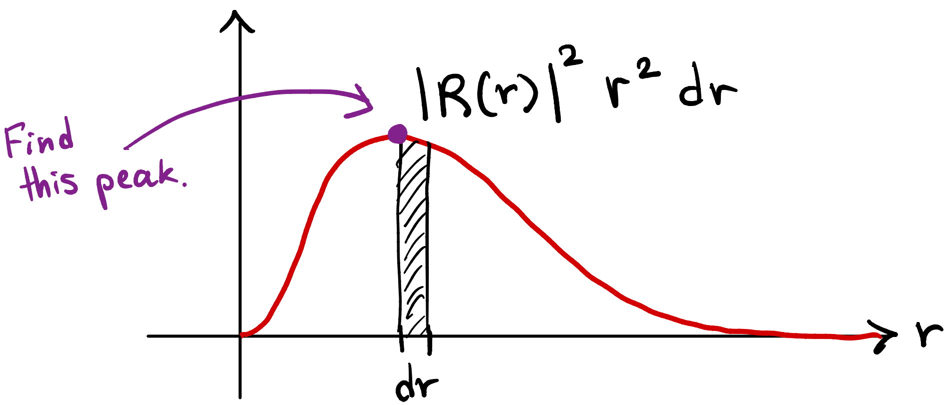
2 (Griffiths 4.16)

The radial wave function of the ground state of hydrogen is:



so naivly we might think that the most probable value of r is $r=0$.

But the probability of finding the particle between r and $r+dr$ is actually



So we need to maximize this function.

$\longrightarrow$

$$\frac{d}{dr}\left(r^2 e^{-r/a}\right) = 2r e^{-r/a} - r^2 \frac{1}{a} e^{-r/a} = 0$$

$$\Rightarrow 2r - \frac{r^2}{a} = 0$$

$$\Rightarrow 2a - r = 0$$

$$\Rightarrow \boxed{r = 2a}$$

Most probable radius.

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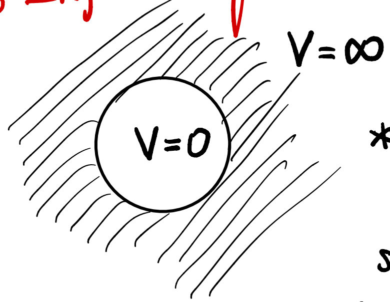
③ Infinite Spherical Well



* Notice that the system is spherically symmetric, so we can find simultaneous energy and angular momentum eigenstates by solving the radial equation:

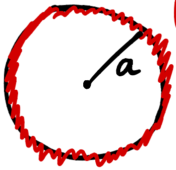
$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2} + V(r)\right]u(r) = E u(r)$$

We will look only for the ground state, which has $\ell = 0$. Then inside the well:

$$-\frac{\hbar^2}{2\mu} u''(r) = E u(r), \quad \left(k \equiv \sqrt{\frac{2mE}{\hbar^2}}\right)$$

$$u(r) = A\sin(kr) + B\cos(kr)$$

* Boundary conditions:



(1) Hard boundary at $r=a$ implies that

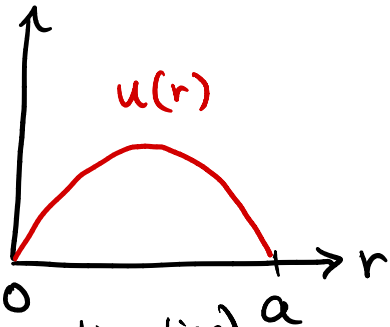
$$u(a) = 0$$

(2) Since $R(r) = u(r)/r$, we need $u(r) \to 0$ at $r \to 0$ so that R does not blow up. This means:

$$u(0) = 0$$

(2) $\Rightarrow$ $u(r) = A\sin(kr)$

(1) $\Rightarrow$ $k = \dfrac{\pi}{a}$



Then the ground state is (up to normalization)

$$\Psi_0(r,\theta,\varphi) = \begin{cases} A\,\dfrac{\sin\left(\frac{\pi r}{a}\right)}{r} & r \leq a \\ 0 & r \geq a \end{cases}$$

with energy:

$$E = \frac{\hbar^2\pi^2}{2ma^2}$$

Ground state wavefunction: