Discussion 4 137A

Spin-1/2: A Problem Set.

Conventions:

$$\begin{cases} |+x\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle \\ |-x\rangle = \frac{1}{\sqrt{2}}|+z\rangle - \frac{1}{\sqrt{2}}|-z\rangle \end{cases}$$

$$\begin{cases} |+y\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{i}{\sqrt{2}}|-z\rangle \\ |-y\rangle = \frac{1}{\sqrt{2}}|+z\rangle - \frac{i}{\sqrt{2}}|-z\rangle \end{cases}$$

So that the components of the spin-1/2 operator are:

$$S_z \xrightarrow[z]{} \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S_x \xrightarrow[z]{} \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y \xrightarrow[z]{} \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

in the z-basis.

$\boxed{1}$ Write the operator $\hat{S}_x$ in purely braket notation.

$\boxed{2}$ Write the operator $\hat{S}_x$ as a matrix with respect to the x-basis.

$\boxed{3}$ Perform a change-of-basis to express the result of problem 2 in the z basis (you should get $\frac{\hbar}{2}\sigma_x$).

$\boxed{4}$ Use purely braket notation to compute $\langle S_z \rangle$ for the state $|+x\rangle$

$\boxed{5}$ Use "matrix mechanics" to compute $\langle S_z \rangle$ for the state $|+x\rangle$.

$\boxed{6}$ The state $|\psi\rangle$ is given by the column vector

$$|\psi\rangle \xrightarrow[z]{} \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

w.r.t. the z-basis. Compute $\langle S_y \rangle$.

$\boxed{7}$ Use matrix mechanics to compute:

$$[S_z, S_x]$$

You should find they don't commute. We say they are "incompatible". Their uncertainties w.r.t. any state $|\psi\rangle$ satisfy:

$$\Delta S_z \, \Delta S_x \geq \left| \frac{1}{2} \langle [S_z, S_x] \rangle \right|$$

Generalized uncertainty principle.

$\boxed{8}$ For the state $|\psi\rangle \longrightarrow \frac{1}{2}\begin{pmatrix} \sqrt{2} \\ 1+i \end{pmatrix}$

(a) Check its normalization.

(b) Compute $\langle S_x \rangle$, $\langle S_y \rangle$ and $\langle S_z \rangle$.

(c) Compute ΔS_z and ΔS_x and verify that the uncertainty principle holds.

$\boxed{9}$ Is it possible that $\Delta S_z \Delta S_x = 0$?

$\boxed{10}$ Set up and solve the eigenvalue problem for the eigenstates / eigenvalues of the operator:

$$\vec{S}\cdot\hat{n} = \hat{S}_x n_x + \hat{S}_y n_y + \hat{S}_z n_z$$

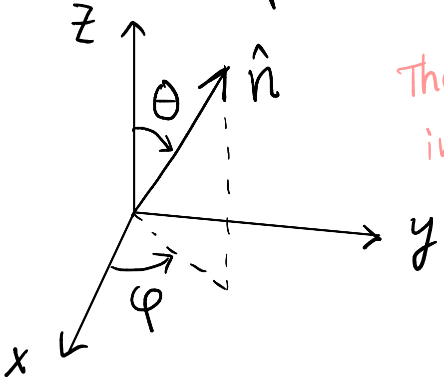
where

$$\hat{n} = (\sin\theta\cos\varphi\ ,\ \sin\theta\sin\varphi\ ,\ \cos\theta)$$

is a unit vector in the direction (θ, φ) in spherical coords. You should find:

$$\begin{cases} |+n\rangle = \cos\frac{\theta}{2}|+z\rangle + e^{i\varphi}\sin\frac{\theta}{2}|-z\rangle \\ |-n\rangle = \sin\frac{\theta}{2}|+z\rangle - e^{i\varphi}\cos\frac{\theta}{2}|-z\rangle \end{cases}$$

up to overall phases.



These are spin eigenstates in the $\pm\hat{n}$ direction.

SOLUTIONS

$\boxed{1}$ $\boxed{\hat{S}_x = \frac{\hbar}{2}|+x\rangle\langle +x| - \frac{\hbar}{2}|-x\rangle\langle -x|}$

$\boxed{2}$ $\hat{S}_x \xrightarrow[x]{} \boxed{\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$

$\boxed{3}$ Change-of-basis matrix:

$$U \equiv \begin{pmatrix} \langle +x|+z\rangle & \langle -x|+z\rangle \\ \langle +x|-z\rangle & \langle -x|-z\rangle \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

Its inverse is:

$$U^{-1} = U^{\dagger} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

Then

$$S_x \xrightarrow[z]{} U^{-1}\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U$$

$$= \frac{\hbar}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{\hbar}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{\hbar}{4}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \boxed{\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \checkmark$$

$\boxed{4}$ State: $|+x\rangle$.

$$\langle S_z \rangle = \langle +x | S_z | +x \rangle$$

$$= \frac{1}{\sqrt{2}}\left(\langle +z| + \langle -z|\right) S_z \frac{1}{\sqrt{2}}\left(|+z\rangle + |-z\rangle\right)$$

$$= \frac{1}{2}\left(\langle +z| + \langle -z|\right)\left(\frac{\hbar}{2}|+z\rangle - \frac{\hbar}{2}|-z\rangle\right)$$

$$= \frac{1}{2}\Big(\frac{\hbar}{2}\underbrace{\langle +z|+z\rangle}_{1} - \frac{\hbar}{2}\underbrace{\langle +z|-z\rangle}_{0}$$

$$+ \frac{\hbar}{2}\underbrace{\langle -z|+z\rangle}_{0} - \frac{\hbar}{2}\underbrace{\langle -z|-z\rangle}_{1}\Big)$$

$$= \frac{1}{2}\left(\cancel{\frac{\hbar}{2}} - \cancel{\frac{\hbar}{2}}\right) = \boxed{0}.$$

$\boxed{5}$

$$\langle S_z \rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{\hbar}{4}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \boxed{0}$$

$\boxed{6}$

$$|\psi\rangle \longrightarrow \frac{1}{\sqrt{3}}\begin{pmatrix}1\\\sqrt{2}\end{pmatrix}$$

$$\langle S_y\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & \sqrt{2}\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{1}{\sqrt{3}}\begin{pmatrix}1\\\sqrt{2}\end{pmatrix}$$

$$= \frac{\hbar}{6}\begin{pmatrix}1 & \sqrt{2}\end{pmatrix}\begin{pmatrix}-i\sqrt{2}\\ i\end{pmatrix}$$

$$= \frac{\hbar}{6}\left(\cancel{-i\sqrt{2}} + \cancel{i\sqrt{2}}\right) = \boxed{0}$$

$\boxed{7}$ $[\hat{S}_z, \hat{S}_x]$

$$\hat{S}_z\hat{S}_x \longrightarrow \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$$

$$\hat{S}_x\hat{S}_z \longrightarrow \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$$

so

$$[S_z, S_x] \longrightarrow \boxed{-i\frac{\hbar}{2}\begin{pmatrix}0 & -i\hbar/2\\ i\hbar/2 & 0\end{pmatrix} \rightarrow -i\frac{\hbar}{2}\hat{S}_y}$$

$\boxed{8}$ For the state $|\psi\rangle \longrightarrow \frac{1}{2}\begin{pmatrix} \sqrt{2} \\ 1+i \end{pmatrix}$

(a) Check its normalization.

(b) Compute $\langle S_x \rangle$, $\langle S_y \rangle$ and $\langle S_z \rangle$.

(c) Compute ΔS_z and ΔS_x and verify that the uncertainty principle holds.

(a) $$\langle \psi | \psi \rangle = \frac{1}{2}\begin{pmatrix} \sqrt{2} & 1-i \end{pmatrix} \frac{1}{2}\begin{pmatrix} \sqrt{2} \\ 1+i \end{pmatrix}$$

$$= \frac{1}{4}\Big(2 + \underbrace{(1-i)(1+i)}_{2}\Big) = \boxed{1} \checkmark$$

(b)

$$\langle S_x \rangle = \frac{1}{2}\begin{pmatrix} \sqrt{2} & 1-i \end{pmatrix}\begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix} \frac{1}{2}\begin{pmatrix} \sqrt{2} \\ 1+i \end{pmatrix}$$

$$= \frac{\hbar}{8}\begin{pmatrix} \sqrt{2} & 1-i \end{pmatrix}\begin{pmatrix} 1+i \\ \sqrt{2} \end{pmatrix} = \frac{\hbar}{8}\left(\sqrt{2} + \cancel{i\sqrt{2}} + \sqrt{2} - \cancel{i\sqrt{2}}\right)$$

$$\boxed{\langle S_x \rangle = \frac{\hbar}{2\sqrt{2}}}$$

$$\langle S_y \rangle = \frac{1}{2}\begin{pmatrix}\sqrt{2} & 1-i\end{pmatrix}\begin{pmatrix} 0 & -i\hbar/2 \\ i\hbar/2 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix}\sqrt{2} \\ 1+i\end{pmatrix}$$

$$= \frac{\hbar}{8}\begin{pmatrix}\sqrt{2} & 1-i\end{pmatrix}\begin{pmatrix}-i+1 \\ \sqrt{2}\, i\end{pmatrix}$$

$$= \frac{\hbar}{8}\left(\sqrt{2} - \cancel{i\sqrt{2}} + \cancel{i\sqrt{2}} + \sqrt{2}\right)$$

$$\boxed{\langle S_y \rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}}$$

$$\langle S_z \rangle = \frac{1}{2}\begin{pmatrix}\sqrt{2} & 1-i\end{pmatrix}\begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}\begin{pmatrix}\sqrt{2} \\ 1+i\end{pmatrix}\frac{1}{2}$$

$$= \frac{\hbar}{8}\begin{pmatrix}\sqrt{2} & 1-i\end{pmatrix}\begin{pmatrix}\sqrt{2} \\ -1-i\end{pmatrix}$$

$$= \frac{\hbar}{8}\left(\cancel{2} - \cancel{(1+i)(1-i)}\right)$$

$$\boxed{\langle S_z \rangle = 0}$$

(c)

$$\langle S_z^2 \rangle = \frac{1}{2}\begin{pmatrix}\sqrt{2} & 1-i\end{pmatrix}\frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}\sqrt{2}\\ 1+i\end{pmatrix}$$

$$= \hbar^2/4$$

$$\langle S_x^2 \rangle = \frac{1}{2}\begin{pmatrix}\sqrt{2} & 1-i\end{pmatrix}\frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}\sqrt{2}\\ 1+i\end{pmatrix}$$

$$= \hbar^2/4$$

$$\Rightarrow \Delta S_z = \sqrt{\frac{\hbar^2}{4} - 0} = \hbar/2$$

$$\Delta S_x = \sqrt{\frac{\hbar^2}{4} - \frac{\hbar^2}{8}} = \frac{\hbar}{2\sqrt{2}}$$

$$\Rightarrow \Delta S_x \Delta S_z = \frac{\hbar^2}{4\sqrt{2}}$$

Meanwhile:

$$\frac{1}{2}\langle [S_z, S_x] \rangle = \frac{1}{2}\langle -i\frac{\hbar}{2}\hat{S}_y \rangle$$

$$= -\frac{i\hbar}{4}\langle S_y \rangle$$

$$= -\frac{i\hbar}{4}\left(\frac{\hbar}{2\sqrt{2}}\right) = -i\frac{\hbar^2}{8\sqrt{2}}$$

so $$\frac{\hbar^2}{4\sqrt{2}} \geq \frac{\hbar^2}{8\sqrt{2}}$$ ✓ yup.

$\boxed{9}$ Yes. For example: $|\psi\rangle = |+z\rangle$.

Then $\Delta S_z = 0$.