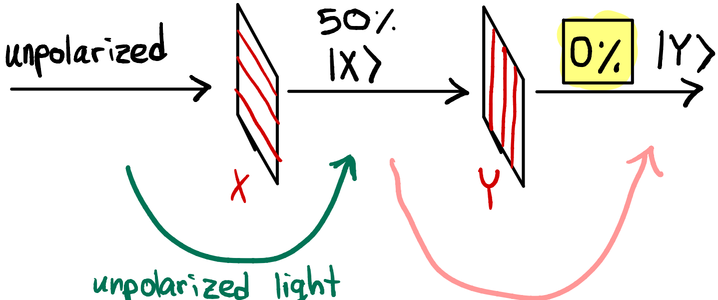
PHY 137A

HW4 Solutions

1

a

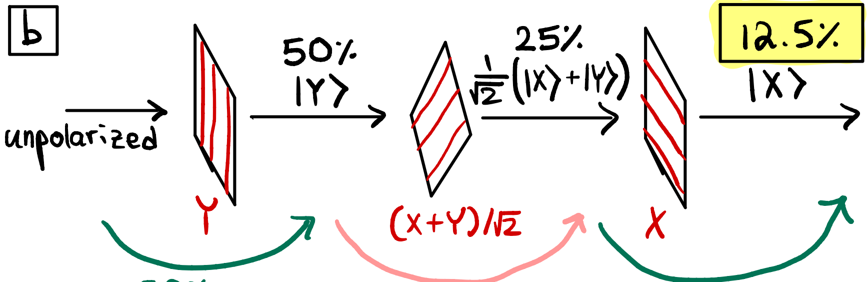


unpolarized light is 50% transmitted by all filters

$\langle X | Y \rangle = 0.$

All light is blocked.

b



50% transmitted

Let:

$$|D\rangle = \frac{1}{\sqrt{2}}\left(|X\rangle + |Y\rangle\right)$$

Then

$$|\langle D|Y\rangle|^2 = \underline{\underline{1/2}}$$

$$|\langle X|D\rangle|^2 = \underline{\underline{\frac{1}{2}}}$$

c

The first filter allows the state:

$$|Y'\rangle = -\sin\left(\frac{\phi}{N}\right)|X\rangle + \cos\left(\frac{\phi}{N}\right)|Y\rangle$$

to pass through. Then:

$$|\langle Y'|Y\rangle|^2 = \left|\cos\frac{\phi}{N}\right|^2 = \cos^2\frac{\phi}{N}$$

is the fraction transmitted.

This is the same for each filter in the sequence, so:

$$\boxed{\text{transmission probability} = \left(\cos\left(\frac{\phi}{N}\right)\right)^{2N}}$$

(d) $$\lim_{N\to\infty}\left(\cos\left(\frac{\phi}{N}\right)\right)^{2N} = \lim_{N\to\infty}\left(1-\frac{1}{2}\frac{\phi^2}{N^2}+\cdots\right)^{2N}$$

$$= \lim_{N\to\infty}\left(1-\cancel{2N}\cdot\frac{1}{\cancel{2}}\frac{\phi^2}{N^{\cancel{2}}}+\cdots\right)$$

$$= \lim_{N\to\infty}\left(1-\frac{1}{N}\phi^2+\cdots\right)$$

$$= \boxed{1}$$

$\boxed{2}$ $\boxed{a}$ $$[A,B+C] = A(B+C)-(B+C)A$$

$$= AB+AC-BA-CA$$

$$= [A,B]+[A,C] \quad //$$

$\boxed{b}$

$$B[A,C]+[A,B]C = \cancel{BAC}-BCA+ABC-\cancel{BAC}$$

$$= [A,BC] \quad //$$

$\boxed{C}$ $\underbrace{[A,[B,C]]}_{(1)} + \underbrace{[B,[C,A]]}_{(2)} + \underbrace{[C,[A,B]]}_{(3)}$

$$(1) = [A, BC] - [A, CB]$$
$$= ABC - BCA - ACB + CBA$$

$$(2) = [B, CA] - [B, AC]$$
$$= BCA - CAB - BAC + ACB$$

$$(3) = [C, AB] - [C, BA]$$
$$= CAB - ABC - CBA + BAC$$

$$\Rightarrow (1) + (2) + (3) = \cancel{ABC} - \cancel{BCA} - \cancel{ACB} + \cancel{CBA}$$
$$+ \cancel{BCA} - \cancel{CAB} - \cancel{BAC} + \cancel{ACB}$$
$$+ \cancel{CAB} - \cancel{ABC} - \cancel{CBA} + \cancel{BAC}$$
$$= \boxed{0}$$

$\boxed{D}$ Let $f(\hat{B}) = \sum_{n=0}^{\infty} f_n B^n$ (Taylor series).

Then: $$\frac{\partial f}{\partial B} = \sum_{n=0}^{\infty} n f_n B^{n-1}$$ ← term-by-term differentiation.

Therefore we only need to show that

$$\boxed{[A, B^n] \stackrel{?}{=} c\, n B^{n-1}}$$ ** want to show.

when $[A,B] = c$ a number.

proof by induction:

* base case: $[A, B^0] = [A, I] = 0$ ✓

  and $c \cdot 0 \cdot B^{-1} = 0$ ✓

* inductive step:

Fix n. Suppose that it is true that:

$$[A, B^{n-1}] = c\,(n-1) B^{n-2}$$

Then:

$$[A, B^n] = [A, B B^{n-1}]$$ via part (b).

$$= B[A, B^{n-1}] + [A,B] B^{n-1}$$ via the inductive hypothesis

$$= B\, c\,(n-1) B^{n-2} + c B^{n-1}$$

$$= c\,(n-1) B^{n-1} + c B^{n-1}$$

$$= c\, n B^{n-1}$$ ✓

By induction, we therefore conclude that

$$[A, B^n] = c\, n B^{n-1}$$

is true for all n.

$$\Longrightarrow \boxed{[A, f(B)] = c \frac{\partial f}{\partial B}}$$

$\boxed{3}$ $\boxed{a}$ $\boxed{|\psi_1\rangle}$

$\boxed{b}$ Possible outcomes: $\boxed{b_1, b_2}$

Probabilities:

$$|\langle \phi_1 | \psi_1 \rangle|^2 = \boxed{9/25}$$

$$|\langle \phi_2 | \psi_1 \rangle|^2 = \boxed{16/25}$$

$\boxed{c}$

$$\begin{tikzcd}
 & \overset{9/25}{|\phi_1\rangle} \arrow[dl, "9/25"'] \arrow[dr, "16/25"] & & & \overset{16/25}{|\phi_2\rangle} \arrow[dl, "16/25"'] \arrow[dr, "9/25"] & \\
\underset{\frac{81}{625}}{|\psi_1\rangle} & & \underset{\frac{144}{625}}{|\psi_2\rangle} & \underset{\frac{256}{625}}{|\psi_1\rangle} & & \underset{\frac{144}{625}}{|\psi_2\rangle}
\end{tikzcd}$$

tot. probability to get $|\psi_1\rangle$: $\boxed{\frac{337}{625}} = 53.92\%$

tot. probability to get $|\psi_2\rangle$: $\boxed{\frac{288}{625}} = 46.08\%$

$\boxed{4}$ Write $|\psi\rangle = a|+z\rangle + b|-z\rangle$.

Since we get $S_z = \hbar/2$ 36% of the time, we have

$$|\langle +z|\psi\rangle|^2 = |a|^2 = 0.36 = (0.6)^2$$

and so

$$|b|^2 = 0.64 = (0.8)^2$$

Next, since $S_x = \hbar/2$ 50% of the time, we have:

$$|\langle +x|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}a \\ b\end{pmatrix}\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}(a+b)\right|^2$$

$$= \frac{1}{2}(a^* + b^*)(a+b)$$

$$= \frac{1}{2}\left(|a|^2 + |b|^2 + a^*b + b^*a\right)$$

Pick a to be real so that $a = a^*$. Then:

$$50\% = \frac{1}{2}\Big(\underset{0.36}{\underset{\uparrow}{|a|^2}} + \underset{0.64}{\underset{\uparrow}{|b|^2}} + a(b+b^*)\Big)$$

$$\cancel{0.5} = \frac{1}{2}\left(\cancel{1} + a(b+b^*)\right)$$

$$\Rightarrow \quad 0 = a(b+b^*)$$

$$\Rightarrow \quad 0 = b+b^*$$

$a \neq 0$

$$\Rightarrow \quad \boxed{Re(b) = 0}$$

Then:

$$\boxed{|\psi\rangle = \frac{3}{5}|+z\rangle \pm i\,\frac{4}{5}|-z\rangle}$$

not enough information to determine sign