Homework 5 Due Wednesday, October 4th, by midnight

1. *The Spin-1/2 Eigenvalue Problem, In General.*

   So far, you have encountered only the operators for the x, y, and z components of the spin of a spin 1/2 particle: s_x, s_y, and s_z. However, in class you were given a formula for the state corresponding to the spin pointing in the general direction (θ, ϕ). We will now derive this formula by computing the eigenstates of the operator corresponding to the component of the spin pointing in that direction.

   We can define the spin operator along the $\hat{\boldsymbol{n}}$ direction as follows. First, we can organize the operators s_x, s_y, and s_z into a 'vector of operators':

   $$\boldsymbol{s} = (s_x, s_y, s_z).$$

   Next, we can parameterize a unit vector $\hat{\boldsymbol{n}}$ by spherical coordinates:

   $$\hat{\boldsymbol{n}} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta).$$

   Finally, we can define the 'spin operator in the $\hat{\boldsymbol{n}}$ direction' to be:

   $$s_{\hat{\boldsymbol{n}}} \equiv \boldsymbol{s}\cdot\hat{\boldsymbol{n}} = s_x n_x + s_y n_y + s_z n_z. \tag{1}$$

   (a) Set up and solve the eigenvalue problem:

   $$s_{\hat{\boldsymbol{n}}} |\psi\rangle = \lambda |\psi\rangle$$

   That is, find the eigenvalues and eigenvectors of the operator $s_{\hat{\boldsymbol{n}}}$. Express the eigenvectors as column vectors in the z-basis. You may pick the $|{+z}\rangle$ component of the vectors to be real.

   (b) Compute the angular direction (θ, ϕ) in which the spin state $|\psi\rangle = a|{+z}\rangle + b|{-z}\rangle$ is pointing, for arbitrary complex a and b.

2. *Rotating a Spin-1/2 Particle*

   Since spin-1/2 states always 'point' in some particular direction $\hat{\boldsymbol{n}}$, it makes sense to talk about *rotating* those states in 3D space. We will represent rotations as operators which map states that point in direction $\hat{\boldsymbol{n}}$ to states that point in a rotated direction $\hat{\boldsymbol{n}}'$.

   The operator enacting a rotation should be *unitary*, since one would hope that rotations preserve all inner products between states. Therefore, it should be possible to write a rotation operator as e^{iA}, where A is a Hermitian operator. Let's focus on an operator $R(\phi\boldsymbol{z})$ which rotates by an angle ϕ about the z-axis. This must be expressible as:

   $$R(\phi\boldsymbol{z}) = e^{i\phi J_z/\hbar}$$

   for some Hermitian operator J_z. The factor of $\hbar$ has been introduced so that J_z will have units of $\hbar$ (units of angular momentum). The angle ϕ must appear in this way because successive rotations should combine via:

   $$R(\phi_1\boldsymbol{z})R(\phi_2\boldsymbol{z}) = R((\phi_1+\phi_2)\boldsymbol{z}).$$

   In addition, when $R(\phi\boldsymbol{z})$ acts on one of the spin-z eigenstates, it should not rotate the state. That is, we should have:

   $$R(\phi\boldsymbol{z})|{+z}\rangle = (\text{some phase})|{+z}\rangle$$
   $$R(\phi\boldsymbol{z})|{-z}\rangle = (\text{some other phase})|{-z}\rangle$$

   (a) By Taylor expanding the expression $e^{iJ_z\phi/\hbar}$, argue that the above two equations can only be satisfied if:

   $$J_z|{+z}\rangle = \hbar(\text{some number})|{+z}\rangle$$
   $$J_z|{-z}\rangle = \hbar(\text{some other number})|{-z}\rangle$$

   That is, the operator J_z must have the states $|{+z}\rangle$ and $|{-z}\rangle$ as its eigenstates.

(b) If we rotate by an angle $\phi = \pi/2$ about the z-axis, physically we expect that this will cause anything that pointed in the $+x$ direction to get rotated to something that points in the $+y$ direction. We therefore demand that:

$$R(\frac{\pi}{2}\boldsymbol{z})\,|+x\rangle = (\text{possible phase})\,|+y\rangle$$
$$R(\frac{\pi}{2}\boldsymbol{z})\,|-x\rangle = (\text{possible other phase})\,|-y\rangle$$

where we allow for overall phases since that doesn't affect which direction the state is pointing in. Show that the *only* way for these expressions to hold is if the eigenvalues of J_z are:

$$J_z\,|+z\rangle = \frac{\hbar}{2}\,|+z\rangle$$
$$J_z\,|-z\rangle = -\frac{\hbar}{2}\,|-z\rangle$$

Therefore, the operator J_z that generates rotations about the z axis is exactly the spin operator s_z.

3. *Photon Polarizations Are 'Spin 1'*

(a) Consider a photon travelling in the $+z$ direction. The circularly polarized photon states

$$|R\rangle = \frac{1}{\sqrt{2}}\,|X\rangle + \frac{i}{\sqrt{2}}\,|Y\rangle$$
$$|L\rangle = \frac{1}{\sqrt{2}}\,|X\rangle - \frac{i}{\sqrt{2}}\,|Y\rangle$$

carry angular momentum $+\hbar$ and $-\hbar$ in the z-direction, respectively. Suppose N photons per second, each in the state

$$|\psi\rangle = a\,|X\rangle + b\,|Y\rangle$$

are incident upon an ideal black surface that totally absorbs all incident light, and which has its surface normal in the z-direction. In terms of a and b, what is magnitude of the torque exerted on the surface by the beam of photons? (This torque was first measured in 1936 by R.A. Beth).

(b) The observable that one is measuring in part (a) is the angular momentum along the z direction, J_z. Evidently,

$$J_z\,|R\rangle = \hbar\,|R\rangle$$
$$J_z\,|L\rangle = -\hbar\,|L\rangle\,.$$

Write the matrix representation of the operator J_z with respect to the R/L basis, and with respect to the X/Y basis.

(c) Find the matrix representation of the operator

$$R(\phi\boldsymbol{z}) = e^{-iJ_z\phi/\hbar}$$

in both the R/L basis and in the X/Y basis. You should find that this operator enacts the rotations described in the previous problem set, namely it rotates the states $|X\rangle\,, |Y\rangle$ into the state $|X'\rangle\,, |Y'\rangle$ which are polarized along rotated axes x', y'.

(d) Show that photons possess sensible behavior under a 2π rotation:

$$R(2\pi\boldsymbol{z})\,|\psi\rangle = |\psi\rangle\,.$$