Discussion 5

137A

More... Spin-1/2 problems

$\boxed{1}$ Use matrix mechanics to compute:

$$[S_z, S_x]$$

You should find they don't commute. We say they are "incompatible". Their uncertainties w.r.t. any state $|\psi\rangle$ satisfy:

$$\Delta S_z \, \Delta S_x \geq \left| \frac{1}{2} \langle [S_z, S_x] \rangle \right|$$

Generalized uncertainty principle.

$\boxed{2}$ Is it possible that $\Delta S_z \Delta S_x = 0$?

$\boxed{3}$ Work out the matrix representation of the projection operators $P_+ = |+z\rangle\langle+z|$ and $P_- = |-z\rangle\langle-z|$ using the states $|+y\rangle$ and $|-y\rangle$ as a basis. Check that

$$P_+ + P_- = I$$

Identity

holds as a matrix equation.

Photon Polarization

$\boxed{4}$ A photon polarization for a photon travelling in the $+z$ direction is given by:

$$|\psi\rangle = \sqrt{\frac{2}{3}}\,|X\rangle + \frac{i}{\sqrt{3}}\,|Y\rangle$$

(a) What is the probability that the photon will pass through an ideal linear polarizer with its transmission axis along the y-direction?

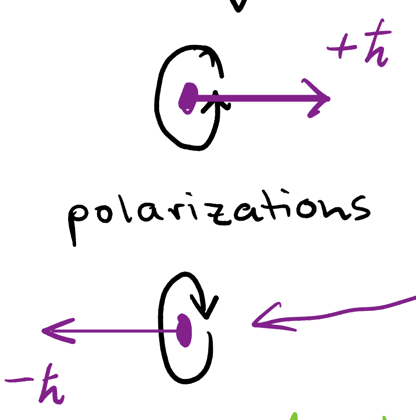
(b) What is the probability that the photon will pass through an ideal linear polarizer with its transmission axis making an angle ϕ with the y-axis?

(c) The states

$$|R\rangle = \frac{1}{\sqrt{2}}|X\rangle + \frac{i}{\sqrt{2}}|Y\rangle$$

$$|L\rangle = \frac{1}{\sqrt{2}}|X\rangle - \frac{i}{\sqrt{2}}|Y\rangle$$

carry angular momentum $+\hbar$ and $-\hbar$, respectively, in the z-direction.

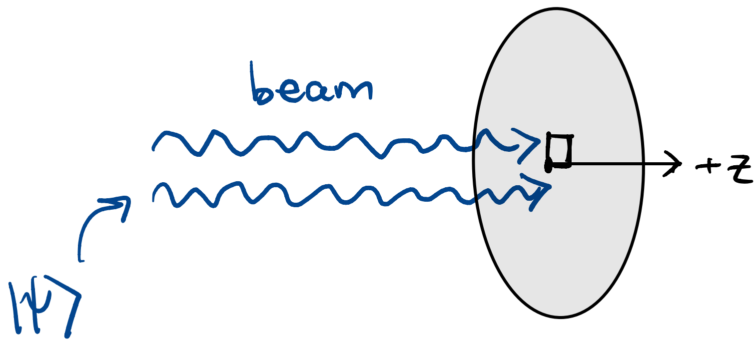


polarizations

"spin" (usually called <u>helicity</u> for photons)

Angular momentum follows right-hand rule.

Suppose a beam carrying N photons per second, each in the state $|\psi\rangle$, is totally absorbed by a black surface with a surface normal in the z direction



How large is the <u>torque</u> exerted on the disk? In which <u>direction</u> does the disk rotate?

→ This expiriment was done in 1936 (R.A. Beth) confirming that photons carry helicity $\pm\hbar$.

$\boxed{5}$ Since photon helicity is an observable, it is represented by an operator. Assuming the photon travels in the $+z$ direction, the relevent operator is the component of the angular momentum along the z-direction:

$$J_z|R\rangle = \hbar|R\rangle$$

$$J_z|L\rangle = -\hbar|L\rangle$$

Write the matrix-representation of J_z in the R-L basis, and in the X-Y basis.

$\boxed{6}$ Compute the matrix representation of the operator

$$R(\phi) = e^{-iJ_z\phi/\hbar}$$

in the R-L basis and in the X-Y basis. This operator performs rotations about the z-axis by angle ϕ.

Solutions

$\boxed{1}$ $[\hat{S}_z, \hat{S}_x]$

$$\hat{S}_z\hat{S}_x \longrightarrow \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\hat{S}_x\hat{S}_z \longrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so

$$[S_z, S_x] \longrightarrow \boxed{-i\frac{\hbar}{2}\begin{pmatrix} 0 & -i\hbar/2 \\ i\hbar/2 & 0 \end{pmatrix} \rightarrow -i\frac{\hbar}{2}\hat{S}_y}$$

$\boxed{2}$ Yes. For example: $|\psi\rangle = |+z\rangle$.

Then $\Delta S_z = 0$.

$\boxed{3}$ $P_+ = |+z\rangle\langle +z|$.

$$P_+ \xrightarrow{\quad y \quad} \begin{pmatrix} \langle +y|P_+|+y\rangle & \langle +y|P_+|-y\rangle \\ \langle -y|P_+|+y\rangle & \langle -y|P_+|-y\rangle \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}}$$

$$P_- = |-z\rangle\langle -z|$$

$$P_- \longrightarrow \begin{pmatrix} \langle +y|P_+|+y\rangle & \langle +y|P_+|-y\rangle \\ \langle -y|P_+|+y\rangle & \langle -y|P_+|-y\rangle \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}}$$

check:

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\boxed{4}$ $|\psi\rangle = \sqrt{\frac{2}{3}}|X\rangle + \frac{i}{\sqrt{3}}|Y\rangle$

(a) $|\langle Y|\psi\rangle|^2 = \boxed{1/3}$

(b) $|Y'\rangle = -\sin\phi\,|X\rangle + \cos\phi\,|Y\rangle$

$$\langle Y'|\psi\rangle = -\sin\phi\sqrt{\frac{2}{3}} + i\cos\phi\frac{1}{\sqrt{3}}$$

$$|\langle Y'|\psi\rangle|^2 = \frac{2}{3}\sin^2\phi + \frac{1}{3}\cos^2\phi$$

$$= \boxed{\frac{1}{3} + \frac{1}{3}\sin^2\phi}$$

(c) Re-write this as a linear combination of $|R\rangle$ and $|L\rangle$:

$$\begin{cases} |X\rangle = \frac{1}{\sqrt{2}}|R\rangle + \frac{1}{\sqrt{2}}|L\rangle \\ |Y\rangle = \frac{1}{\sqrt{2}i}\left(|R\rangle - |L\rangle\right) \end{cases}$$

$$|\psi\rangle = \sqrt{\frac{2}{3}}\,|X\rangle + \frac{i}{\sqrt{3}}|Y\rangle$$

$$= \left(\sqrt{\frac{2}{3}}\frac{1}{\sqrt{2}}|R\rangle + \sqrt{\frac{2}{3}}\frac{1}{\sqrt{2}}|L\rangle\right)$$

$$+ \left(\frac{\cancel{i}}{\sqrt{3}}\frac{1}{\sqrt{2}\,\cancel{i}}|R\rangle - \frac{\cancel{i}}{\sqrt{3}}\frac{1}{\sqrt{2}\,\cancel{i}}|L\rangle\right)$$

$$= \left(\frac{\sqrt{2}}{\sqrt{6}} + \frac{1}{\sqrt{6}}\right)|R\rangle + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\right)|L\rangle$$

$$= \frac{1}{\sqrt{6}}\left(\sqrt{2}+1\right)|R\rangle + \frac{1}{\sqrt{6}}\left(\sqrt{2}-1\right)|L\rangle$$

Then:

$$\langle J_z\rangle = \hbar\left(\frac{5+\sqrt{2}}{6}\right) - \hbar\left(\frac{5-\sqrt{2}}{6}\right)$$

$$= \hbar\sqrt{2}/3$$

$\Rightarrow$ on avg. each photon deposits $\frac{\sqrt{2}}{3}\hbar$ of angular momentum onto the disk.

$$\Rightarrow \boxed{\text{torque} = \frac{\sqrt{2}}{3}\hbar N}$$

$\Rightarrow$ $\boxed{\text{In the CCW direction abt z-axis.}}$

$\boxed{5}$ $J_z = \hbar|R\rangle\langle R| - \hbar|L\rangle\langle L|$ so:

$$\boxed{J_z \xrightarrow{R,L} \hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

Now, since

$$\begin{cases} |R\rangle = \frac{1}{\sqrt{2}}|X\rangle + \frac{i}{\sqrt{2}}|Y\rangle \\ |L\rangle = \frac{1}{\sqrt{2}}|X\rangle - \frac{i}{\sqrt{2}}|Y\rangle \end{cases}$$

We can write:

$$J_z = \hbar\left(\frac{1}{\sqrt{2}}|x\rangle + \frac{i}{\sqrt{2}}|y\rangle\right)\left(\frac{1}{\sqrt{2}}\langle x| - \frac{i}{\sqrt{2}}|Y\rangle\right)$$

$$- \hbar\left(\frac{1}{\sqrt{2}}|X\rangle - \frac{i}{\sqrt{2}}|Y\rangle\right)\left(\frac{1}{\sqrt{2}}\langle X| + \frac{i}{\sqrt{2}}\langle Y|\right)$$

$$= \hbar\left(\frac{1}{2}|x\rangle\langle x| - \frac{i}{2}|X\rangle\langle Y| + \frac{i}{2}|Y\rangle\langle X| + \frac{1}{2}|Y\rangle\langle Y|\right)$$

$$- \hbar\left(\frac{1}{2}|x\rangle\langle x| + \frac{i}{2}|X\rangle\langle Y| - \frac{i}{2}|Y\rangle\langle x| + \frac{1}{2}|Y\rangle\langle Y|\right)$$

$$= \hbar \left(i\,|Y\rangle\langle X| - i\,|X\rangle\langle Y| \right)$$

so:

$$\boxed{J_z \xrightarrow{X,Y} \hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$$

$\boxed{6}$ $$R_z(\phi) = e^{-iJ_z\phi/\hbar}$$

First, in the R/L basis. Then:

$$R_z(\phi)|R\rangle = e^{-iJ_z\phi/\hbar}|R\rangle$$

$$= e^{-i\phi}|R\rangle$$

$J_z|R\rangle = \hbar|R\rangle$

$$R_z(\phi)|L\rangle = e^{-iJ_z\phi/\hbar}|L\rangle$$

$$= e^{i\phi}|L\rangle$$

$$\Rightarrow R_z(\phi) = e^{-i\phi}|R\rangle\langle R| + e^{i\phi}|L\rangle\langle L|$$

$$\rightarrow \boxed{R_z(\phi) \xrightarrow{R/L} \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}}$$

Now just perform a change of basis with

$$U = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}$$

so

$$R_z(\phi) \xrightarrow{X,Y} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^{-i\phi} & ie^{-i\phi} \\ e^{i\phi} & -ie^{i\phi} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} e^{-i\phi} + e^{i\phi} & ie^{-i\phi} - ie^{i\phi} \\ -ie^{-i\phi} + ie^{i\phi} & e^{-i\phi} + e^{i\phi} \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}}$$