Homework 3 Due Monday, September 18th, by midnight

1. *Expectation values.*

   Let's use what we've learned about hermitian operators to find a useful expression for the expectation value. From the postulates of quantum mechanics, for any given observable represented by the hermitian operator $\hat{A}$, a measurement of the state $|\psi\rangle$ will result in the value a_i with probability $P(a_i) = |\langle a_i|\psi\rangle|^2$, where a_i and $|a_i\rangle$ are the eigenvalue and corresponding eigenvector of $\hat{A}$. Let us say we would like to calculate the *expectation value* (or mean) of $\hat{A}$. As in probability and statistics, this is defined as:

$$\langle \hat{A} \rangle_\psi = \sum_i a_i P(a_i) = \sum_i a_i |\langle a_i|\psi\rangle|^2$$

   (a) Show that the above expression for $\langle A \rangle_\psi$ can be equivalently written as $\langle\psi|\, \hat{A}\, |\psi\rangle$. This is often more useful for performing calculations.

   (b) True/False: For two observables $\hat{A}$ and $\hat{B}$, $\langle(\hat{A}+\hat{B})\rangle = \langle\hat{A}\rangle + \langle\hat{B}\rangle$.

   (c) Suppose the system is prepared in state $|\psi\rangle$ repeatedly and a measurement of $\hat{A}$ is made each time. The *standard deviation* of these outcomes ΔA (also sometimes called the *uncertainty*) is given by $\Delta\hat{A} = \sqrt{\langle\, (A - \langle A \rangle)^2\, \rangle}$. First, convince yourself that this agrees with the usual notion of standard deviation from statistics. Then, show that it is equal to $\sqrt{\langle A^2\rangle - \langle A\rangle^2}$ (in the limit of infinite repetitions of the experiment).

2. Show that neither the probability of obtaining the result a_i nor the expectation value $\langle\hat{A}\rangle$ is affected by $|\psi\rangle \to e^{i\delta}\,|\psi\rangle$, that is, by an overall phase change for the state $|\psi\rangle$.

3. A spin-1/2 particle is in the state

$$|\psi\rangle = \frac{1}{2}\,|+\boldsymbol{z}\rangle + \frac{i\sqrt{3}}{2}\,|-\boldsymbol{z}\rangle \tag{1}$$

   What are the expectation value $\langle\hat{S}_z\rangle$ and the uncertainty ΔS_z for this state? You might find the results of problem 1 useful.

4. We will see that the state of a spin-1/2 particle that is spin-up along the axis whose direction is specified by the unit vector

$$\boldsymbol{n} = \sin\theta\cos\phi\boldsymbol{i} + \sin\theta\sin\phi\boldsymbol{j} + \cos\theta\boldsymbol{k}$$

   with (θ, ϕ) being standard spherical coordinates, is given by:

$$|+\boldsymbol{n}\rangle = \cos\frac{\theta}{2}\,|+\boldsymbol{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|-\boldsymbol{z}\rangle\ . \tag{2}$$

   (a) Verify that the state $|+\boldsymbol{n}\rangle$ reduces to the states

$$|+\boldsymbol{x}\rangle = \frac{1}{\sqrt{2}}\,|+\boldsymbol{z}\rangle + \frac{1}{\sqrt{2}}\,|-\boldsymbol{z}\rangle\,, \tag{3}$$

$$|+\boldsymbol{y}\rangle = \frac{1}{\sqrt{2}}\,|+\boldsymbol{z}\rangle + \frac{i}{\sqrt{2}}\,|-\boldsymbol{z}\rangle \tag{4}$$

   for the appropriate choice of the angles θ and ϕ.

   (b) Suppose that a measurement of $\hat{S}_z$ is carried out on a particle in the state $|+\boldsymbol{n}\rangle$. What is the probability that the measurement yields (i) $\hbar/2$? (ii) $-\hbar/2$?

   (c) Determine the uncertainty ΔS_z of the measurement described in part (b).