Homework 6
Solutions

1 (Griffiths 1.3)

(a) Take the probability distribution:

$$\rho(x) = A e^{-\lambda(x-a)^2}$$

where $(A, \lambda > 0)$

* Normalize it by requiring that $\int_{-\infty}^{\infty} dx\, \rho(x) = 1$:

$$1 \overset{!}{=} A \int_{-\infty}^{\infty} dx\, e^{-\lambda(x-a)^2}$$

* We can do this integral with a u-substitution:

$$u = x - a$$
$$du = dx$$

so that

$$\int_{-\infty}^{\infty} dx\, e^{-\lambda(x-a)^2} = \int_{-\infty}^{\infty} du\, e^{-\lambda u^2}$$

* This is a standard Gaussian integral

$$\int_{-\infty}^{\infty} du\, e^{-\lambda u^2} = \sqrt{\frac{\pi}{\lambda}}$$

* The normalization equation is:

$$1 = A\sqrt{\frac{\pi}{\lambda}}$$

so

$$\boxed{A = \sqrt{\frac{\lambda}{\pi}}}$$

(b) Now we compute $\langle x \rangle$. This is given by:

$$\langle x \rangle = \int_{-\infty}^{\infty} dx\; x \sqrt{\frac{\lambda}{\pi}}\, e^{-\lambda(x-a)^2}$$

Again, let

$$\begin{cases} u = x - a \\ du = dx \end{cases}$$

so that

$$\langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} du\, (u+a)\, e^{-\lambda u^2}$$

$$= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} du\; u\, e^{-\lambda u^2} + \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} du\; a\, e^{-\lambda u^2}$$

The first integral has an <u>odd</u> integrand, so the integral is <u>zero</u>.

The second integral is once again a gaussian with

$$a\sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} du\, e^{-\lambda u^2} = a\sqrt{\frac{\cancel{\lambda}}{\cancel{\pi}}}\sqrt{\frac{\cancel{\pi}}{\cancel{\lambda}}} = a$$

so

$$\boxed{\langle x \rangle = a}$$

*Next, we compute $\langle x^2 \rangle$. Following similar steps:

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx\; x^2 \sqrt{\frac{\lambda}{\pi}}\, e^{-\lambda(x-a)^2}$$

Let $u = x - a$. Then:

$$= \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} du\, (u+a)^2 e^{-\lambda u^2}$$

0 (odd)

$$= \sqrt{\frac{\lambda}{\pi}}\left[\underbrace{\int_{-\infty}^{\infty} du\, u^2 e^{-\lambda u^2}}_{\text{This one works out to be } \frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}}} + \int_{-\infty}^{\infty} du\, 2au\, e^{-\lambda u^2} + \underbrace{\int_{-\infty}^{\infty} du\, a^2 e^{-\lambda u^2}}_{a^2\sqrt{\frac{\lambda}{\pi}}}\right]$$

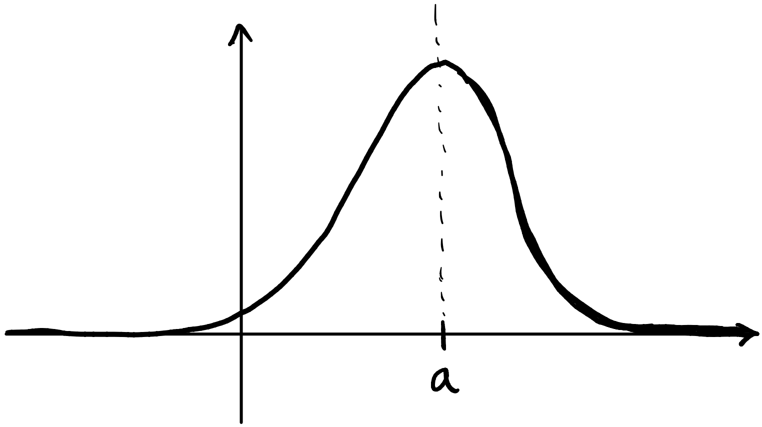
so:

$$\boxed{\langle x^2 \rangle = \frac{1}{2\lambda} + a^2}$$

The standard deviation is:

$$\boxed{\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{1}{\sqrt{2\lambda}}}$$

(c)



$\boxed{2}$ (Griffiths 1.5)

(a)

$$1 = \int_{-\infty}^{\infty} |\Psi(x)|^2 dx = |A|^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|} dx$$

$$= 2|A|^2 \int_0^{\infty} dx \; e^{-2\lambda x}$$

$$= 2|A|^2 \left[\frac{-1}{2\lambda} e^{-2\lambda x} \right]_0^{\infty}$$

$$= \frac{1}{\lambda} |A|^2$$

$$\Rightarrow \boxed{|A| = \sqrt{\lambda}}$$

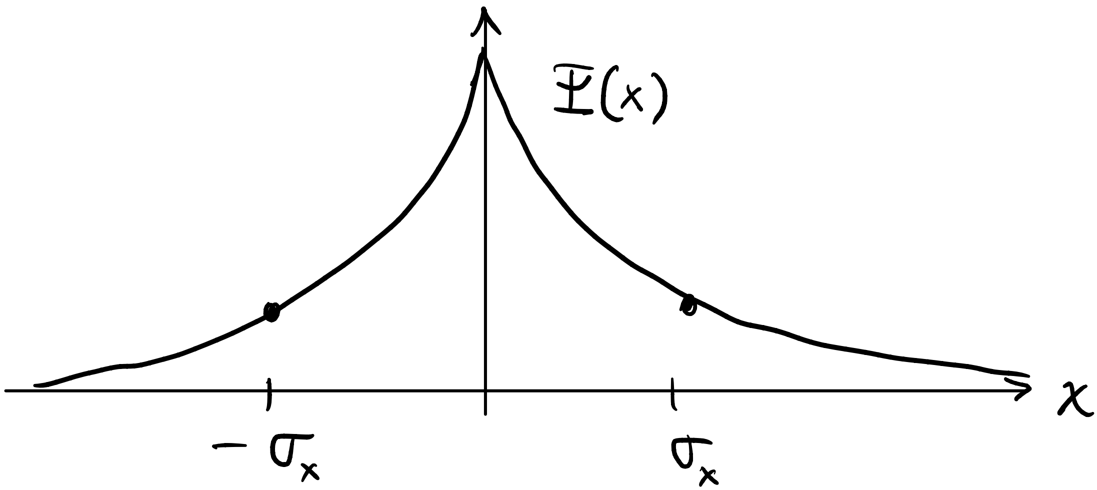
(b) $\langle x \rangle = \lambda \int_{-\infty}^{\infty} dx \underbrace{x\, e^{-2\lambda|x|}}_{\text{Odd.}} = \boxed{0}$

$$\langle x^2 \rangle = \lambda \int_{-\infty}^{\infty} dx\; x^2 e^{-2\lambda|x|} = 2\lambda\left(\frac{2}{(2\pi)^3}\right)$$

$$\Rightarrow \boxed{\langle x^2 \rangle = \frac{1}{2\lambda^2}}$$

(c) Then

$$\sigma_x = \sqrt{\frac{1}{2\lambda^2} - 0} = \boxed{\frac{1}{\sqrt{2}\lambda}}$$



At $x = \pm\sigma_x$, the prob. density is

$$|\Psi(\sigma_x)|^2 = \lambda e^{-2\cancel{\lambda}\left(\frac{1}{\sqrt{2}\cancel{\lambda}}\right)} = e^{-\sqrt{2}}\lambda \approx \boxed{0.24\lambda}$$

The probability of being found with $|x| > \sigma_x$ is

$$2\int_{\sigma_x}^{\infty} dx\, |\Psi|^2 = 2\lambda \int_{\sigma_x}^{\infty} dx\, e^{-2\lambda x}$$

$$= 2\lambda \left[\frac{-1}{2\lambda} e^{-2\lambda x} \right]_{\sigma_x}^{\infty}$$

$$= e^{-2\cancel{\lambda}\left(\frac{1}{\sqrt{2}\cancel{\lambda}}\right)}$$

$$= \boxed{e^{-\sqrt{2}} \simeq 24.3\%}$$

3 (Griffiths 1.16)

(a) $$1 = |A|^2 \int_{-a}^{a} (a^2-x^2)^2 dx = \frac{16}{15} a^5 |A|^2$$

$$\Rightarrow \boxed{|A| = \sqrt{\frac{15}{16a^5}}}$$

(b) $$\langle x \rangle = \int_{-a}^{a} dx \underbrace{x \frac{15}{16a^5} (x^2-a^2)^2}_{\text{odd.}} = \boxed{0}$$

(d) $$\langle x^2 \rangle = \frac{15}{16a^5} \int_{-a}^{a} dx\, x^2 (x^2-a^2)^2 = \frac{15}{16a^5} a^7 \left(\frac{2}{3} - \frac{4}{5} + \frac{2}{7}\right)$$

$$\Rightarrow \boxed{\langle x^2 \rangle = \frac{a^2}{7}}$$

(f) $$\sigma_x = \sqrt{\frac{a^2}{7} - 0} = \boxed{a/\sqrt{7}}$$

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Wavefunctions are only defined up to an overall phase, so we can only determine $|A|$, but never A.