Discussion 7

Today: Wavefunctions in 1D

* Our particles now have a <u>spatial location</u>, which we describe in terms of a <u>position operator</u> $\hat{x}$ which is an observable so is Hermitian.

$$\hat{x}|a\rangle = a|a\rangle$$

* The eigenstates of $\hat{x}$ are states of definite position — i.e. particles that are located at a certain spot with 100% certainty.
* Position eigenstates at different locations are orthogonal as states:

$$\langle x_1 | x_2 \rangle = 0 \quad \text{for} \quad x_1 \neq x_2$$

* Since position is continuous, the resolution of the identity is:

$$\mathbb{1} = \int_{-\infty}^{\infty} dx' \, |x'\rangle\langle x'|$$

* A general state is a superposition of position states:

$$|\psi\rangle = \int_{-\infty}^{\infty} dx' \, |x'\rangle \langle x'|\psi\rangle$$

$$= \int_{-\infty}^{\infty} dx' \, \psi(x') \, |x'\rangle$$

* The <u>wavefunction</u> $\psi(x)$ describing state $|\psi\rangle$ is the inner product:

$$\psi(x) = \langle x|\psi\rangle$$

amplitude to find a particle in state $|\psi\rangle$ at location x.

* Properly normalized states satisfy:

$$1 = \langle\psi|\psi\rangle = \int dx' \, \langle\psi|x'\rangle\langle x'|\psi\rangle$$

$$= \int dx' \, \psi^*(x')\psi(x')$$

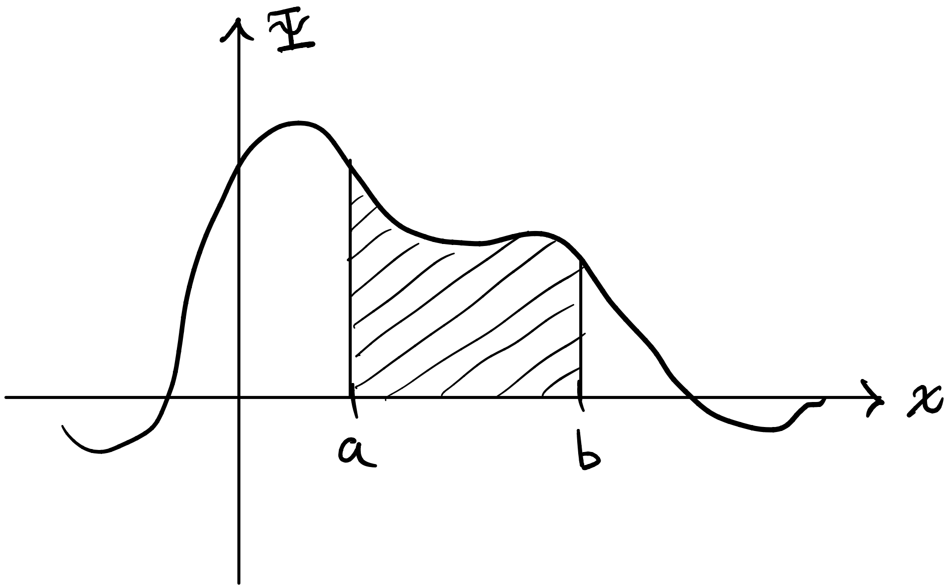
$$= \int dx' \, |\psi(x')|^2$$

* It is natural to identify:

$$dx\, |\psi(x)|^2 = \text{probability of finding particle in the interval between } x \text{ and } x+dx$$

so for example:

$$\int_a^b dx' |\psi(x')|^2 = \text{probability of finding the particle between } x=a \text{ and } x=b$$



$\boxed{1}$ What are the <u>units</u> of the wave function (in 1D) $\psi(x)$?

$\boxed{2}$

(Griffiths 1.4)

At time $t=0$, a particle is in a state $|\Psi\rangle$ represented by the wavefunction:

$$\Psi(x,0) = \begin{cases} A\dfrac{x}{a} & 0 \le x \le a \\ A\dfrac{b-x}{b-a} & a \le x \le b \\ 0 & x > b. \end{cases}$$

where a, b, A are positive constants.

(a) Sketch $\Psi(x,0)$ as a function of x.

(b) Normalize Ψ

(c) What is the probability of finding the particle to the left of $x=a$? Test the limiting cases $b=a$ and $b=2a$.

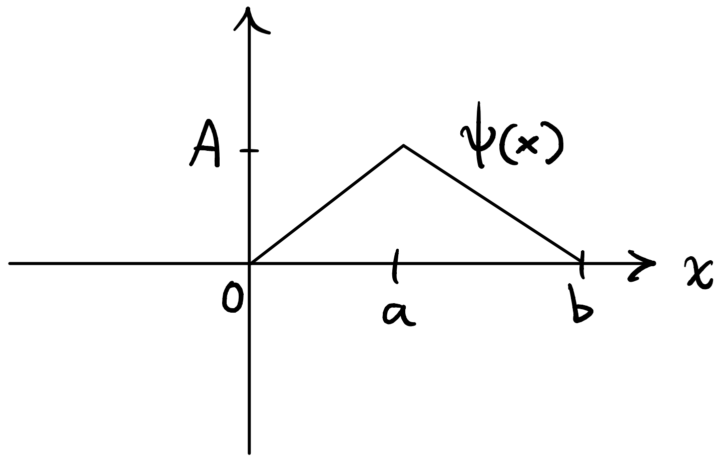
(d) Compute $\langle x \rangle$.

Solutions

1 Since $1 = \int dx' \, |\psi(x)|^2$, $\psi(x)$ must have units of $[\text{length}]^{-1/2}$

2

(a)



(b) $$1 = \int_{-\infty}^{\infty} dx \, |\psi(x)|^2 = \int_0^b dx \, |\psi(x)|^2.$$

Do the integral in 2 pieces:

$$\int_0^a dx \, |\psi(x)|^2 = |A|^2 \int_0^a dx \, \frac{1}{a^2} x^2 = \frac{1}{3} |A|^2 a$$

$$\int_a^b dx \, |\psi(x)|^2 = |A|^2 \int_a^b dx \, \frac{(b-x)^2}{(b-a)^2} = \frac{1}{3} |A|^2 (b-a)$$

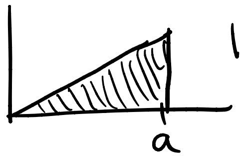
$$\Rightarrow \quad 1 = \frac{1}{3}|A|^2 a + \frac{1}{3}|A|^2 (b-a)$$

$$\Rightarrow \quad 1 = \frac{1}{3}|A|^2 b \rightarrow \boxed{|A| = \sqrt{3/b}}$$

(c) The prob of finding the particle between $x=0$ and $x=a$ is:

$$\int_0^a dx\,|\psi(x)|^2 = \frac{1}{3}|A|^2 a = \frac{1}{\cancel{3}}\frac{\cancel{3}}{b}a = \boxed{\frac{a}{b}}$$

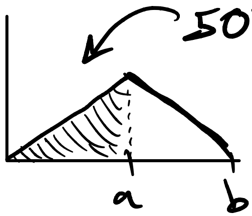
<u>Sanity checks:</u> $b = a$



100% chance ✓

$a/b = 1$

$b = 2a$



50% chance

$a/b = 1/2$ ✓

(d) $\langle x\rangle = \langle\psi|\hat{x}|\psi\rangle$

$$= \int dx' \,\langle\psi|\hat{x}|x'\rangle\langle x'|\psi\rangle$$

$$= \int dx' \,\langle\psi|x'|x'\rangle\langle x'|\psi\rangle$$

$$\boxed{\langle x\rangle = \int dx' \,\psi^*(x')\, x'\, \psi(x')}$$

Generally Useful.

Compute:

$$\langle x \rangle = \int_0^b dx \, |\psi(x)|^2 x$$

Do it in 2 pieces:

$$\int_0^a dx \left(\frac{3}{b}\right)\left(\frac{x}{a}\right)^2 x = \frac{3}{b}\frac{1}{4}a^2$$

$$\int_a^b dx \left(\frac{3}{b}\right)\left(\frac{x-b}{a-b}\right)^2 x = \frac{3}{b}\frac{1}{12}(3a+b)(b-a)$$

So all together

$$\langle x \rangle = \frac{3}{b}\left(\cancel{\frac{a^2}{4}} + \frac{ab}{4} + \frac{b^2}{12} - \cancel{\frac{a^2}{4}}\right)$$

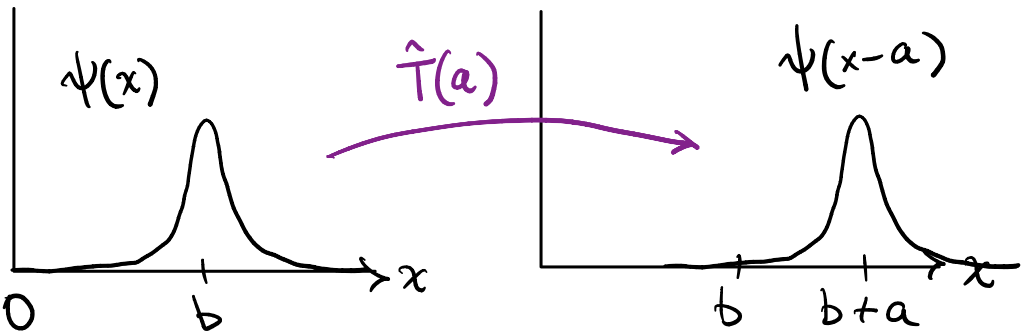
$$= \boxed{\frac{3a+b}{4}}$$

Momentum

* It makes sense to consider spatially translating position states. Invent an operator that does this:

$$\hat{T}(a)|x\rangle = |x+a\rangle$$

* the operator $\hat{T}(a)$ changes the state of a particle in which the particle has position x to one in which the particle has position $x+a$.



* This does what you'd expect to wavefunctions:

$$\hat{T}(a)|\psi\rangle = \int dx' \; \hat{T}(a)|x'\rangle\langle x'|\psi\rangle$$

$$= \int dx' \; |x'+a\rangle\langle x'|\psi\rangle$$

Change dummy variable: $x'' = x'+a$. $dx'' = dx'$. Then:

$$= \int dx'' \; |x''\rangle \underbrace{\langle x''-a|\psi\rangle}_{\text{New wavefunction, } \psi(x''-a).}$$

* Translation operator $\hat{T}(a)$ is <u>unitary</u> because translated states maintain all their inner products:

$$\langle\psi|\phi\rangle = \int dx' \, \psi^*(x')\phi(x')$$

$$\langle\psi|\hat{T}^\dagger(a)\hat{T}(a)|\phi\rangle = \int dx' \, \psi^*(x'-a)\phi(x'-a)$$

$\downarrow x'' = x'-a$

$$= \int dx'' \, \psi^*(x'')\phi(x'')$$

$$\Rightarrow \hat{T}^\dagger(a)\hat{T}(a) = \mathbb{1}.$$

* Consider an infinitesimal translation

$$\hat{T}(dx) = \mathbb{1} - \frac{i}{\hbar}\hat{p}_x dx + \mathcal{O}(dx^2)$$

which acts on position eigenstates by:

$$\hat{T}(dx)|x\rangle = |x+dx\rangle$$

* The operator $\hat{p}_x$ is called the <u>generator of translations</u>, and the $\hbar$ ensures that $\hat{p}_x$ has units of momentum.

* Put a lot of small translations together to get a finite one by distance a:

$$\hat{T}(a) = \lim_{N\to\infty}\left(\mathbb{1} - \frac{i}{\hbar}p_x\left(\frac{a}{N}\right)\right)^N = e^{-ip_x a/\hbar}$$

* The operator $\hat{p}_x$ is Hermitian because $\hat{T}(a)$ is unitary (we've previously seen that e^{iA} is unitary if and only if A is Hermitian) so

$$\hat{p}_x = \hat{p}_x^\dagger$$

* The operator $\hat{p}_x$ does not commute with the position operator $\hat{x}$. To see this, consider a translation by an infinitesimal amount δx:

$$\hat{x}\hat{T}(\delta x)|x_0\rangle = \hat{x}|x_0+\delta x\rangle$$
$$= (x_0+\delta x)|x_0+\delta x\rangle$$

while

$$\hat{T}(\delta x)\hat{x}|x_0\rangle = \hat{T}(\delta x)\, x_0\, |x_0\rangle$$
$$= x_0|x_0+\delta x\rangle$$

so:

$$\left(\hat{x}\hat{T}(\delta x) - \hat{T}(\delta x)\hat{x}\right)|x_0\rangle = \delta x\, |x_0+\delta x\rangle$$

Now expand:

$$\hat{x}\hat{T}(\delta x) - \hat{T}(\delta x)\hat{x} = \hat{x}\left(\mathbb{1} - \frac{i}{\hbar}\hat{p}_x\delta x\right) - \left(\mathbb{1} - \frac{i}{\hbar}\hat{p}_x\delta x\right)\hat{x}$$
$$= -\frac{i}{\hbar}\delta x\,[\hat{x},\hat{p}_x]$$

Set these equal and take $\delta x \to 0$:

$$\Rightarrow -\frac{i}{\hbar}\delta x\,[\hat{x},\hat{p}_x] = \delta x$$

$$\Rightarrow \boxed{[\hat{x}, \hat{p}_x] = i\hbar}$$

"Canonical Commutation Relation"

* When acting on wavefunctions, $\hat{p}_x$ is represented by:

$$\hat{p}_x \xrightarrow{\text{position basis}} -i\hbar \frac{d}{dx}$$

Problems

1. Show that $-i\hbar\frac{d}{dx}$ satisfies the canonical commutation relation by considering

$$[x, -i\hbar\frac{d}{dx}]\ f(x)$$

not an operator, just a function.

2. Show that $\psi(x) = e^{ip_0 x/\hbar}$ is the wave function for a state with a definite value of momentum. Plot it, and relate p_0 to the wavelength.

Solutions

$\boxed{1}$ $[x, -i\hbar \frac{d}{dx}] f(x)$

$$= -i\hbar \left(x \frac{d}{dx} f(x) - \frac{d}{dx}(x f(x)) \right)$$

$$= -i\hbar \left(\cancel{x f'(x)} - f(x) - \cancel{x f'(x)} \right)$$

$$= i\hbar f(x)$$

Since this holds for all functions $f(x)$ we have

$$[x, -i\hbar \frac{d}{dx}] = i\hbar$$

$\boxed{2}$ $$-i\hbar \frac{d}{dx} e^{ip_0 x/\hbar} = p_0 \, e^{ip_0 x/\hbar}$$

$$\Rightarrow \hat{p}_x |\psi\rangle = p_0 |\psi\rangle$$

$\Rightarrow$ Momentum eigenstate w/ eigenvalue p_0

wavenumber: $p_0/\hbar = k$

wavelength: $\lambda = \frac{2\pi}{k} = \frac{2\pi\hbar}{p_0} = \frac{h}{p_0}$

where $2\pi\hbar \equiv h =$ "Planck's constant" (not reduced)

($\hbar$ = "reduced planck's constant").

* The formula

$$\lambda = \frac{h}{p}$$

predates quantum mechanics and is called the <u>de Broglie formula.</u> Here it is a consequence of quantum mechanics.