Homework 5 Solutions

$\boxed{1}$ The matrix we want to diagonalize is:

$$\vec{S}\cdot\hat{n} \xrightarrow{z} \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\theta\cos\varphi + \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\theta\sin\varphi + \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\theta$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix} \quad //$$

The eigenvalues can be found using the characteristic polynomial:

$$\left|\begin{pmatrix} \frac{\hbar}{2}\cos\theta - \lambda & \frac{\hbar}{2}e^{-i\varphi}\sin\theta \\ \frac{\hbar}{2}e^{i\varphi}\sin\theta & -\frac{\hbar}{2}\cos\theta - \lambda \end{pmatrix}\right| = 0$$

$$\left(-\frac{\hbar}{2}\cos\theta - \lambda\right)\left(\frac{\hbar}{2}\cos\theta - \lambda\right) - \left(\frac{\hbar}{2}\right)^2\sin^2\theta = 0$$

$$-\left(\frac{\hbar}{2}\right)^2\cos^2\theta - \left(\frac{\hbar}{2}\right)^2\sin^2\theta + \lambda^2 = 0$$

$$\Rightarrow \quad \lambda^2 = \left(\frac{\hbar}{2}\right)^2$$

$$\Rightarrow \quad \boxed{\lambda = \pm\hbar/2}$$

eigenvalues.

Now find the eigenvectors:

$\underline{\lambda = \hbar/2:}$

$$\frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \quad x\cos\theta + e^{-i\varphi}\sin\theta\, y = x$$

$$\Rightarrow \quad y = x\left(\frac{1-\cos\theta}{\sin\theta}\right)e^{i\varphi}$$

$$\Rightarrow \quad y = x\,\frac{\sin\theta/2}{\cos\theta/2}\,e^{i\varphi}$$

* pick $x = \cos\theta/2$. Then: $y = \sin\theta/2\, e^{i\varphi}$

So the Normalized eigenvector is:

$$\boxed{|+n\rangle \xrightarrow{z} \begin{pmatrix} \cos\theta/2 \\ e^{i\varphi}\sin\theta/2 \end{pmatrix}}$$

$\underline{\lambda = -\hbar/2:}$

$$\frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{-\hbar}{2}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\cos\theta\cdot x + e^{-i\varphi}\sin\theta\cdot y = -x$$

$$\Rightarrow \quad y = -x\left(\frac{\cos\theta+1}{\sin\theta}\right)e^{i\varphi}$$

$\hookrightarrow \; y = -x \dfrac{\cos\theta/2}{\sin\theta/2} e^{i\varphi}$

* Pick $x = \sin\theta/2$. Then $y = -\cos\frac{\theta}{2} e^{i\varphi}$ and:

$$\boxed{|-n\rangle \longrightarrow \begin{pmatrix} \sin\frac{\theta}{2} \\ -e^{i\varphi}\cos\frac{\theta}{2} \end{pmatrix}}$$

(b) Let

$$|\psi\rangle = a|+z\rangle + b|-z\rangle.$$

Compare with:

$$|+n\rangle = e^{i\delta}\left(\cos\frac{\theta}{2}|+z\rangle + e^{i\phi}\sin\frac{\theta}{2}|-z\rangle\right)$$

possible global phase.

Then we have:

$$|a| = \cos\frac{\theta}{2}$$

$$\Rightarrow \boxed{\theta = 2\cos^{-1}(|a|)}$$

Similarly,

$$\frac{b}{a} = \tan\frac{\theta}{2} e^{i\phi}$$

$$\mathrm{Re}\left(\frac{b}{a}\cot\frac{\theta}{2}\right) = \cos\phi$$

θ known from here.

$$\Rightarrow \boxed{\cos^{-1}\left(\mathrm{Re}\left(\frac{b}{a}\cot\frac{\theta}{2}\right)\right) = \phi}$$

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(a) We start by assuming that:

some real numbers tbd

$$e^{-iJ_z\phi/\hbar}|+z\rangle = e^{ia_+\phi}|+z\rangle$$

$$e^{-iJ_z\phi/\hbar}|-z\rangle = e^{ia_-\phi}|-z\rangle$$

* Taylor expanding in ϕ, this implies:

$$\left(1 - \frac{iJ_z}{\hbar}\phi + O(\phi^2)\right)|+z\rangle = \left(1 + ia_+\phi + O(\phi^2)\right)|+z\rangle$$

$$\left(1 - \frac{iJ_z}{\hbar}\phi + O(\phi^2)\right)|+z\rangle = \left(1 + ia_-\phi + O(\phi^2)\right)|-z\rangle$$

* Different powers of ϕ are linearly independent, so the two sides can <u>only</u> be equal if

$$\begin{cases} -\frac{iJ_z}{\hbar}\phi|+z\rangle = ia_+\phi|+z\rangle \\ -\frac{iJ_z}{\hbar}\phi|-z\rangle = ia_-\phi|-z\rangle \end{cases}$$

or cleaning things up:

$$\boxed{\begin{aligned} J_z|+z\rangle &= -\hbar a_+|+z\rangle \\ J_z|-z\rangle &= -\hbar a_-|-z\rangle \end{aligned}}$$

$\Rightarrow |+z\rangle, |-z\rangle$ are eigenstates of J_z.

(b) Rotate $|+x\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$:

$$e^{-i\hat{J}_z \frac{\pi}{2}/\hbar}|+x\rangle = e^{-i\hat{J}_z \frac{\pi}{2}/\hbar}\left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle\right)$$

$$= \frac{1}{\sqrt{2}} e^{-ia_+\frac{\pi}{2}}|+z\rangle + \frac{1}{\sqrt{2}} e^{-ia_-\frac{\pi}{2}}|-z\rangle$$

$$= e^{-ia_+\frac{\pi}{2}} \underbrace{\left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}} e^{-i(a_- - a_+)\frac{\pi}{2}}|-z\rangle\right)}_{\text{compare this to } |+y\rangle\text{, demand that they are equal.}}$$

$$\Rightarrow i = e^{-i(a_- - a_+)\pi/2}$$

$$\Rightarrow (a_- - a_+)\frac{\pi}{2} = -\frac{\pi}{2}$$

$$\Rightarrow \boxed{a_+ - a_- = 1}$$

There are many choices for a_+, a_-

* We can pick $\boxed{a_+ = \frac{1}{2}, \quad a_- = -\frac{1}{2}}$ in which case

$$\boxed{\begin{aligned} J_z|+z\rangle &= \frac{\hbar}{2}|+z\rangle \\ J_z|-z\rangle &= -\frac{\hbar}{2}|-z\rangle \end{aligned}}$$

$\boxed{3}$

(a) For $|\psi\rangle = a|X\rangle + b|Y\rangle$, the amplitude to be found in the state $|R\rangle$ is:

$$\langle R|\psi\rangle = \left(\frac{1}{\sqrt{2}}\langle X| - \frac{i}{\sqrt{2}}\langle Y|\right)\left(a|X\rangle + b|Y\rangle\right)$$

$$= \frac{a}{\sqrt{2}} - \frac{ib}{\sqrt{2}}$$

The probability to be found in state $|R\rangle$ is then:

$$|\langle R|\psi\rangle|^2 = \left|\frac{a}{\sqrt{2}} - \frac{ib}{\sqrt{2}}\right|^2$$

$$= \left(\frac{a^*}{\sqrt{2}} + \frac{ib^*}{\sqrt{2}}\right)\left(\frac{a}{\sqrt{2}} - \frac{ib}{\sqrt{2}}\right)$$

$$= \frac{|a|^2}{2} + \frac{|b|^2}{2} - \frac{b^*a}{2i} + \frac{a^*b}{2i}$$

$|a|^2 + |b|^2 = 1$

$$= \frac{1}{2} + \mathrm{Im}(a^*b)$$

$\mathrm{Im}(z) = \dfrac{z - z^*}{2i}$

* The probability to be found in state $|L\rangle$ must be 1 minus this:

$$\Rightarrow |\langle L|\psi\rangle|^2 = \frac{1}{2} - \mathrm{Im}(a^*b)$$

* The expected amount of angular momentum exerted per photon is

$$\Delta L = \hbar\left(\frac{1}{2} + \mathrm{Im}(a^*b)\right) - \hbar\left(\frac{1}{2} - \mathrm{Im}(a^*b)\right)$$

$$= \hbar \times 2\,\mathrm{Im}(a^*b)$$

* The net torque is:

$$\boxed{\tau = N\hbar \times 2\,\mathrm{Im}(a^* b)}$$

Note: the maximum value of $\mathrm{Im}(a^* b)$ is $1/2$ (if the state is normalized).

(b) $$J_z \xrightarrow{R/L} \begin{pmatrix} \hbar & 0 \\ 0 & -\hbar \end{pmatrix}$$

$$\begin{aligned}
J_z &= \hbar |R\rangle\langle R| - \hbar |L\rangle\langle L| \\
&= \hbar\left(\frac{1}{\sqrt{2}}|X\rangle + \frac{i}{\sqrt{2}}|Y\rangle\right)\left(\frac{1}{\sqrt{2}}\langle X| - i\frac{1}{\sqrt{2}}\langle Y|\right) \\
&\quad - \hbar\left(\frac{1}{\sqrt{2}}|X\rangle - \frac{i}{\sqrt{2}}|Y\rangle\right)\left(\frac{1}{\sqrt{2}}\langle X| + \frac{i}{\sqrt{2}}\langle Y|\right)
\end{aligned}$$

$$= \hbar\left(\cancel{\frac{1}{2}|X\rangle\langle X|} - \frac{i}{2}|X\rangle\langle Y| + \frac{i}{2}|Y\rangle\langle X| + \cancel{\frac{1}{2}|Y\rangle\langle Y|}\right)$$

$$- \hbar\left(\cancel{\frac{1}{2}|X\rangle\langle X|} + \frac{i}{2}|X\rangle\langle Y| - \frac{i}{2}|Y\rangle\langle X| + \cancel{\frac{1}{2}|Y\rangle\langle Y|}\right)$$

$$= \hbar i |Y\rangle\langle X| - \hbar i |X\rangle\langle Y|$$

Therefore:

$$\Longrightarrow \boxed{J_z \xrightarrow{X/Y} \hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$$

(c) R/L basis is easy:

$$e^{-iJ_z\phi/\hbar} = \exp\begin{pmatrix} -i\phi & 0 \\ 0 & i\phi \end{pmatrix} = \boxed{\begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}}$$

X/Y basis is harder:

$$e^{-iJ_z\phi/\hbar} \xrightarrow[X/Y]{} \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{-i\phi}{\hbar}\begin{pmatrix} 0 & -i\hbar \\ i\hbar & 0 \end{pmatrix}\right]^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\phi^n \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^n$$

Note that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$

so the sum goes like:

$$= 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \phi\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{1}{2!}\phi^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{3!}\phi^3\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \dots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\sum_{k=0,2,4,\dots} \frac{1}{k!}(-1)^{k/2}\phi^k + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\sum_{k=1,3,5,\dots} \frac{1}{k!}(-1)^{\frac{k-1}{2}}\phi^k$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\phi + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\sin\phi$$

so:

$$\boxed{e^{-iJ_z\phi/\hbar} \xrightarrow{x/y} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}}$$

This indeed maps $|x\rangle \to |x'\rangle$, $|y\rangle \to |y'\rangle$:

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{|x\rangle} = \underbrace{\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}}_{|x'\rangle}$$

(d) We have:

$$\begin{cases} e^{-iJ_z(2\pi)/\hbar}|+z\rangle = e^{-2\pi i}|+z\rangle = |+z\rangle \\ e^{-iJ_z(2\pi)/\hbar}|-z\rangle = e^{2\pi i}|-z\rangle = |-z\rangle \end{cases}$$

Therefore:

$$\boxed{R(2\pi\hat{z})|\psi\rangle = |\psi\rangle}$$

For photons.