Homework 8 Solutions

1 $\Pi|x\rangle = |-x\rangle$

(a) • $\hat{\Pi}^2|x\rangle = \hat{\Pi}|-x\rangle = |x\rangle$ for all x

• since the $\{|x\rangle\}$ form a basis, this implies that

$$\hat{\Pi}^2|\psi\rangle = |\psi\rangle \quad \text{for all } |\psi\rangle$$

$$\Rightarrow \boxed{\hat{\Pi}^2 = 1}$$

(b) Let $|\lambda\rangle$ be an eigenvector of $\hat{\Pi}$, such that

$$\hat{\Pi}|\lambda\rangle = \lambda|\lambda\rangle$$

Then:

$$|\lambda\rangle = \hat{\Pi}^2|\lambda\rangle = \lambda^2|\lambda\rangle$$

$$\Rightarrow \lambda^2 = 1$$

$$\Rightarrow \boxed{\lambda = \pm 1}$$

(c) Yes, since: $\langle x'|\hat{\Pi}x\rangle = \delta(x'+x)$

$$\langle \hat{\Pi}x'|x\rangle = \delta(x'+x)$$

equal

Since $\{|x\rangle\}$ is a basis, this implies that

$$\langle\phi|\hat{\Pi}\psi\rangle = \langle\hat{\Pi}\phi|\psi\rangle \quad \text{for all } |\phi\rangle, |\psi\rangle.$$

which is the definition of $\boxed{\text{Hermitian.}}$

(d) Since $\hat{\Pi}^\dagger = \hat{\Pi}$, we have

$$\hat{\Pi}\hat{\Pi}^\dagger = \hat{\Pi}^2 = 1$$

So $\boxed{\hat{\Pi} \text{ is unitary}}$

(e) We have:

$$(\Pi\psi)(x) = \langle x|\hat{\Pi}|\psi\rangle = \langle -x|\psi\rangle = \boxed{\psi(-x)}$$

(f)

$$\hat{\Pi}|p\rangle = \int dx\, |x\rangle\langle x|\hat{\Pi}|p\rangle$$

$$= \int dx\, |x\rangle\langle -x|p\rangle \qquad \langle -x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}$$

$$= \int dx\, |x\rangle\langle x|-p\rangle \qquad \langle x|-p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}$$

$$= \boxed{|-p\rangle}$$

(g)

$$(\pi\tilde{\psi})(p) = \langle p|\hat{\pi}|\psi\rangle = \langle -p|\psi\rangle = \boxed{\tilde{\psi}(-p)}$$

(h) Since

$$(\hat{\Pi}\psi)(x) = \psi(-x)$$

and the eigenvalues of $\hat{\Pi}$ are ± 1, we learn that parity eigenstates satisfy either:

$$\underbrace{\psi(-x) = \psi(x)}_{\text{Eigenvalue } +1} \quad \text{or} \quad \underbrace{\psi(-x) = -\psi(x)}_{\text{Eigenvalue } -1}$$

So we learn that:

★ Parity eigenstates are even/odd functions

Therefore:

$$\star\ |\psi(x)|^2 \text{ is even}$$

Then:

$$\langle x \rangle = \int_{-\infty}^{\infty} dx\, |\psi(x)|^2 x = \boxed{0} \quad \text{(odd integrand)}$$

For the same reason,

$$\langle p \rangle = \int_{-\infty}^{\infty} dp\, |\tilde{\psi}(p)|^2 = \boxed{0} \quad \text{(odd integrand)}$$

(i) $[\hat{H}, \hat{\pi}] = 0 \iff \hat{\pi}\hat{H}\hat{\pi} = \hat{H}$

These are equivalent, so let's show the latter.

First, note that:

$$\hat{\pi}\hat{x}\hat{\pi} = -\hat{x}$$
$$\hat{\pi}\hat{p}\hat{\pi} = -\hat{p}$$

Therefore:

$$\begin{aligned} \hat{\pi}\hat{p}^2\hat{\pi} &= \hat{\pi}\hat{p}\hat{\pi}\hat{\pi}\hat{p}\hat{\pi} \\ &= (-\hat{p})(-\hat{p}) \\ &= \hat{p}^2 \end{aligned}$$

So the kinetic term always commutes with $\hat{\pi}$.

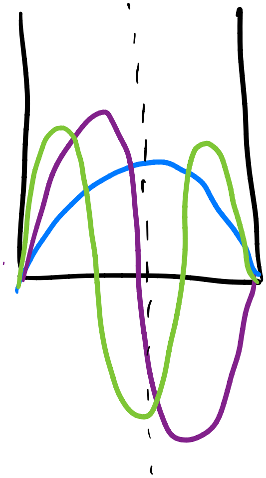
Next, we have

$$\hat{\pi}V(\hat{x})\hat{\pi} = V(-\hat{x})$$

Now, this is equal to $V(\hat{x})$ i.f.f V is an <u>even</u> function. So we learn that:

$$\boxed{\hat{\pi}\hat{H}\hat{\pi} = \hat{H} \quad \text{iff} \quad V(\hat{x}) \text{ is even.}}$$

(j)



$$\psi_n(x) = \sqrt{\frac{2}{a}} \begin{cases} \sin\left(\frac{n\pi x}{a}\right), & n \text{ even} \\ \cos\left(\frac{n\pi x}{a}\right), & n \text{ odd} \end{cases}$$

Clearly all even/odd.

(k) The potential is: $\frac{1}{2}m\omega^2 x^2$ ← even

- So $[H, \Pi] = 0$.
- Therefore H and Π are simultaneously diagonalizable
- But since $E_n = \hbar\omega(n + \frac{1}{2})$ is non-degenerate, we learn that the energy eigenstates $|n\rangle$ <u>must</u> be parity eigenstates!

$\boxed{2}$ (Griffiths 2.5)

(a) $\boxed{|A| = 1/\sqrt{2}}$

(b)
$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_1(x)e^{-iE_1t/\hbar} + \psi_2(x)e^{-iE_2t/\hbar}\right]$$
$$= \frac{1}{\sqrt{2}}e^{-i\omega t}\left[\psi_1(x) + \psi_2(x)e^{-3i\omega t}\right]$$

where
$$\omega \equiv \frac{E_1}{\hbar} = \frac{\hbar\pi^2}{2ma^2}$$

$$\Rightarrow |\Psi(x,t)|^2 = \frac{1}{\sqrt{2}}e^{+i\omega t}\left[\psi_1^*(x) + \psi_2^*(x)e^{3i\omega t}\right]$$
$$\times \frac{1}{\sqrt{2}}e^{-i\omega t}\left[\psi_1(x) + \psi_2(x)e^{-i3\omega t}\right]$$

$$\boxed{|\Psi(x,t)|^2 = \frac{1}{2}\left[|\psi_1(x)|^2 + |\psi_2(x)|^2 + \psi_1(x)\psi_2(x)\,2\cos(3\omega t)\right]}$$

(c)
$$\langle x\rangle = \int_{-\infty}^{\infty} dx\, |\Psi(x,t)|^2 x$$
$$= \frac{1}{2}\underset{1}{\int_{-\infty}^{\infty} dx\, |\psi_1(x)|^2 x} + \frac{1}{2}\underset{2}{\int_{-\infty}^{\infty} dx\, |\psi_2(x)|^2 x} +$$

$$\cos(3\omega t)\int_{-\infty}^{\infty} dx \; \psi_1(x)\psi_2(x)\,x$$

3

Integral 1 is $\frac{a}{2}$ because the integrand is odd about $a/2$.

Integral 2 is $\frac{a}{2}$ for the same reason.

Integral 3 is :

$$\int_{-\infty}^{\infty} dx \; x\,\psi_1(x)\psi_2(x) = \frac{2}{a}\int_0^a dx \, \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)x$$

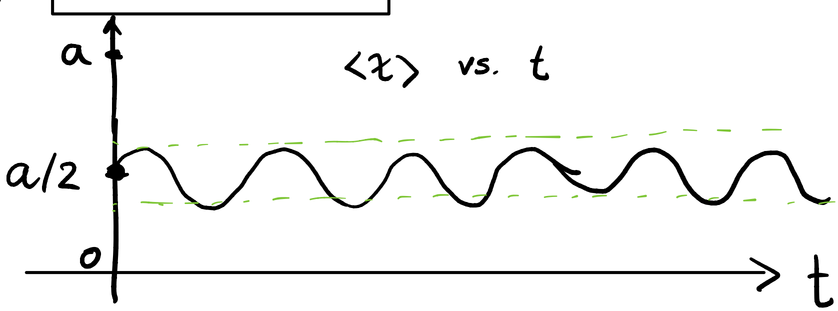
$$= \frac{2}{a}\left(-\frac{8a^2}{9\pi^2}\right)$$

so

$$\Rightarrow \boxed{\langle x\rangle = \frac{a}{2} - \frac{16a}{9\pi^2}\cos(3\omega t)}$$

Amplitude: $\boxed{\frac{16}{9\pi^2}a = (0.3603)\frac{a}{2}}$

Frequency: $\boxed{3\omega = \frac{3\pi^2\hbar}{2ma^2}}$



$\boxed{3}$ From example 2.4 in Griffiths, we know that the first excited state of the harmonic oscillator is:

$$\psi_1(x) = \left(\frac{m\omega}{m\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\, x\, e^{-m\omega x^2/2\hbar}$$

We want to find $\psi_2(x)$. We have the equation

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

so:

$$a^\dagger|1\rangle = \sqrt{2}\,|2\rangle$$

Then $\psi_2(x)$ is given by:

$$\psi_2(x) = \frac{1}{\sqrt{2}}\hat{a}^\dagger\psi_1(x) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x})\psi_1(x)$$

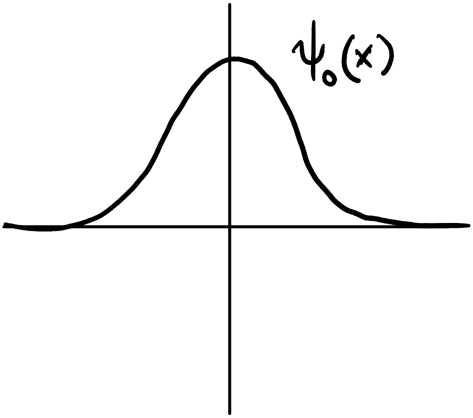
Compute:

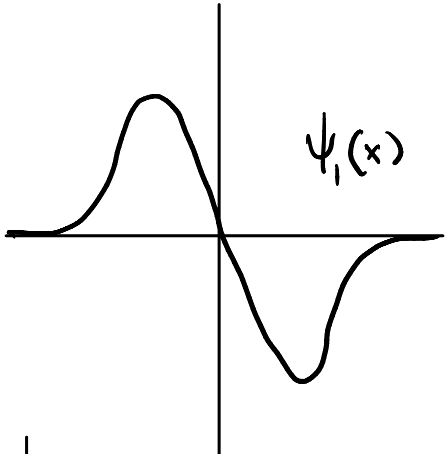
$$\psi_2(x) = \frac{1}{2}\frac{1}{\sqrt{\hbar\cancel{m\omega}}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2\cancel{m\omega}}{\hbar}}\left(-\hbar\frac{d}{dx} + m\omega x\right)x\, e^{-m\omega x^2/2\hbar}$$

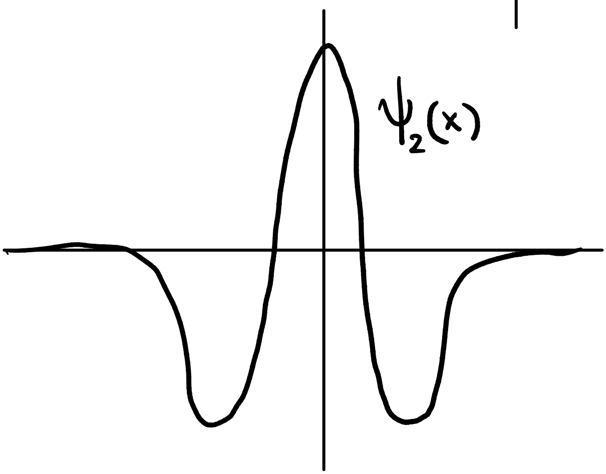
$$= \frac{1}{2\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(-\hbar + \cancel{\hbar}x\left(+\frac{m\omega}{\cancel{2\hbar}}\cancel{2}x\right) + m\omega x^2\right)e^{-m\omega x^2/2\hbar}$$

$$= \frac{1}{2\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(-\hbar + 2m\omega x^2\right)e^{-m\omega x^2/2\hbar}$$

$$\psi_2(x) = \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)e^{-m\omega x^2/2\hbar}$$







(c) Since $\psi_0(x)$ and $\psi_2(x)$ are even, they are automatically orthogonal to $\psi_1(x)$. So we only have:

$$\int_{-\infty}^{\infty} dx\, \psi_0(x)\psi_2(x) \propto \int_{-\infty}^{\infty} dx \left(\frac{2m\omega}{\hbar}x^2 - 1\right)e^{-m\omega x^2/\hbar} = \boxed{0}$$

So indeed, they are all mutually orthogonal.