Name: Solutions

Student ID:

Write your name and Student ID number above. Show your work, and take care to explain what you are doing; partial credit will be given for incomplete answers that demonstrate some conceptual understanding. Cross out or erase parts of the problem you wish the grader to ignore. If you are stuck on a question, move on to the next part. The maximum score is 50 Points. There are 5 problems in total.

Potentially Useful Formulas

Spin-1/2 states:

$$|+\boldsymbol{x}\rangle = \frac{1}{\sqrt{2}}|+\boldsymbol{z}\rangle + \frac{1}{\sqrt{2}}|-\boldsymbol{z}\rangle$$
$$|-\boldsymbol{x}\rangle = \frac{1}{\sqrt{2}}|+\boldsymbol{z}\rangle - \frac{1}{\sqrt{2}}|-\boldsymbol{z}\rangle$$

$$|+\boldsymbol{y}\rangle = \frac{1}{\sqrt{2}}|+\boldsymbol{z}\rangle + \frac{i}{\sqrt{2}}|-\boldsymbol{z}\rangle$$
$$|-\boldsymbol{y}\rangle = \frac{1}{\sqrt{2}}|+\boldsymbol{z}\rangle - \frac{i}{\sqrt{2}}|-\boldsymbol{z}\rangle$$

Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

A few common Taylor series (for complex numbers and matrices):

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + ...$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + ...$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + ...$$

1. **Hermitian and Unitary Matrices (9 pts):** For each of the following operators and/or matrices, determine whether it is Hermitian, unitary, both, or neither.

(a)

$$A = \begin{pmatrix} \pi & 1 & 0 \\ 1 & e & 1 \\ 0 & 1 & i \end{pmatrix}$$

The matrix A is not Hermitian because of the i on the diagonal. It is not unitary, because $AA^\dagger \neq I$ (for example, the upper left element of $AA^\dagger$ is $\pi^2 + 1 \neq 1$).

$$\boxed{\text{Neither}}.$$

(b)

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}$$

To check that it is Hermitian, we compute:

$$B^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} = B$$

so B is Hermitian. Now check that is unitary:

$$BB^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So it is unitary.

$$\boxed{\text{Both}}.$$

(c) The following operator acts on the state of a spin-1/2 particle:

$$C = |{+x}\rangle \langle{+z}| + |{-x}\rangle \langle{-z}|$$

The operator is manifestly unitary, because it maps an orthonormal basis ($|{+z}\rangle, |{-z}\rangle$) to another orthonormal basis ($|{+x}\rangle, |{-x}\rangle$) via:

$$C\,|{+z}\rangle = |{+x}\rangle\,, \qquad C\,|{-z}\rangle = |{-x}\rangle\,.$$

It is also Hermitian. Here is one way to see it. I will write the states $|{\pm x}\rangle$ in the z basis to get:

$$C = |{+x}\rangle \langle{+z}| + |{-x}\rangle \langle{-z}| = \frac{1}{\sqrt{2}} \left(|{+z}\rangle \langle{+z}| + |{-z}\rangle \langle{+z}| + |{+z}\rangle \langle{-z}| + |{-z}\rangle \langle{-z}| \right).$$

In this form, it is clear that $C^\dagger = C$.

$$\boxed{\text{Both}}.$$

2. **Properties of Hermitian and unitary operators (9 pts)**

(a) True/False: Unitary transformations preserve inner products, in the sense that $\langle \alpha | \beta \rangle = \langle U\alpha | U\beta \rangle$ for all vectors $|\alpha\rangle, |\beta\rangle$. (There is no need to justify your answer).

$$\boxed{\text{True.}}$$

(b) Let $\hat{A}$ and $\hat{B}$ be Hermitian operators. Prove that $\hat{A}\hat{B}$ is Hermitian if and only if $\hat{A}$ and $\hat{B}$ commute with each other.

proof

Let $\hat{A}$ and $\hat{B}$ be Hermitian operators. Then

$$\begin{aligned}(\hat{A}\hat{B})^\dagger &= \hat{B}^\dagger \hat{A}^\dagger \\ &= \hat{B}\hat{A} \\ &= \hat{A}\hat{B} + [\hat{B}, \hat{A}]\end{aligned}$$

where we dropped the daggers because $\hat{A}$ and $\hat{B}$ are Hermitian. The leftmost and rightmost sides of this equation imply that $(\hat{A}\hat{B})^\dagger = \hat{A}\hat{B}$ if and only if $[\hat{B}, \hat{A}] = 0$, which is what we wanted to show.

(c) Show that for any operator $\hat{A}$, the operator $\hat{A}^\dagger \hat{A}$ is Hermitian, and that all its eigenvalues are *non-negative* real numbers. *Proof*

Let $\hat{A}$ be any operator, and consider the operator $\hat{A}^\dagger \hat{A}$. First, we show that $\hat{A}^\dagger \hat{A}$ is Hermitian:

$$(\hat{A}^\dagger \hat{A})^\dagger = \hat{A}^\dagger \hat{A}^{\dagger\dagger} = \hat{A}^\dagger \hat{A}$$

so it is indeed Hermitian. Because it is Hermitian, is possesses a complete eigenbasis and real eigenvalues. Let $|\lambda\rangle$ be a normalized eigenvector of $\hat{A}^\dagger \hat{A}$ with eigenvalue λ. Then:

$$\lambda = \langle \lambda | \hat{A}^\dagger \hat{A} | \lambda \rangle = \| A | \lambda \rangle \|^2 \geq 0$$

Therefore, the eigenvalues of $\hat{A}^\dagger \hat{A}$ are non-negative (in addition to being real numbers).

3. **Spin-1/2 measurements (12 pts)**

A spin-1/2 particle is in the state:

$$|\psi\rangle = \alpha\,(\,3\,|+\boldsymbol{z}\rangle \,+\, 4i\,|-\boldsymbol{z}\rangle\,)$$

(a) Determine the value of $|\alpha|$ for which this state is normalized.

We normalize $|\psi\rangle$ by demanding that $1 = \langle\psi|\psi\rangle$. This translates into the condition:

$$\begin{aligned} 1 &= \alpha^*\,(\,3\,\langle+\boldsymbol{z}| \,-\, 4i\,\langle-\boldsymbol{z}|\,)\,\alpha\,(\,3\,|+\boldsymbol{z}\rangle \,+\, 4i\,|-\boldsymbol{z}\rangle\,) \\ &= |\alpha|^2(|3|^2 + |4i|^2) \\ &= 25|\alpha|^2 \end{aligned}$$

Solving, we find

$$\implies \boxed{|\alpha|^2 = \frac{1}{5}}.$$

(b) Compute the expectation value of the z-component of the spin, $\langle s_z\rangle$.

The normalized state is:

$$|\psi\rangle = \frac{3}{5}\,|+z\rangle + i\frac{4}{5}\,|-z\rangle\,.$$

Therefore, a measurement of s_z results in outcomes $+\hbar/2$ and $-\hbar/2$ with probabilities $9/25$ and $16/25$, respectively. The expectation value is:

$$\langle s_z\rangle = \frac{\hbar}{2}\frac{9}{25} + \left(-\frac{\hbar}{2}\right)\frac{16}{25} = \boxed{-\frac{7}{25}\frac{\hbar}{2}}.$$

(c) Compute the uncertainty Δs_z.

We compute $\Delta s_z = \sqrt{\langle s_z^2\rangle - \langle s_z\rangle^2}$. We already know $\langle s_z\rangle$ from the previous part, but we need to compute $\langle s_z^2\rangle$. It is:

$$\langle s_z^2\rangle = \left(\frac{\hbar}{2}\right)^2\frac{9}{25} + \left(-\frac{\hbar}{2}\right)^2\frac{16}{25} = \frac{\hbar^2}{4}$$

Compute Δs_z:

$$\begin{aligned} \Delta s_z &= \sqrt{\frac{\hbar^2}{4} - \left(-\frac{7}{25}\frac{\hbar}{2}\right)^2} \\ &= \frac{\hbar}{2}\sqrt{\frac{25^2}{25^2} - \left(\frac{7}{25}\right)^2} \\ &= \boxed{\frac{\hbar}{2}\frac{24}{25}} \end{aligned}$$

since $7^2 + 24^2 = 25^2$.

4. **Sequential Measurements (12 pts)**

An operator $\hat{A}$, representing observable A, has two (normalized) eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$, with eigenvalues a_1 and a_2, respectively. Operator $\hat{B}$, representing observable B, has two (normalized) eigenstates $|\phi_1\rangle$ and $|\phi_2\rangle$, with eigenvalues b_1 and b_2, respectively. The eigenstates are related by

$$|\psi_1\rangle = \frac{1}{2}|\phi_1\rangle + \frac{\sqrt{3}}{2}|\phi_2\rangle, \qquad |\psi_2\rangle = \frac{\sqrt{3}}{2}|\phi_1\rangle - \frac{1}{2}|\phi_2\rangle. \tag{1}$$

(a) Let's say you measure A and observe the value a_1. What is the state of the system (immediately) after this measurement?

The state after the measurement collapses to $\boxed{|\psi_1\rangle}$.

(b) If you now measure B, what are the possible results, and what are their probabilities?

The possible results of the measurement are the values b_1 and b_2. Their probabilities are

$$\mathrm{Prob}(b_1) = |\langle\phi_1|\psi_1\rangle|^2 = \boxed{1/4},$$
$$\mathrm{Prob}(b_2) = |\langle\phi_2|\psi_1\rangle|^2 = \boxed{3/4},$$

respectively.

(c) Right after the measurement of B, you measure A again. For each of the possible results of the measurement of B that you listed in (b), compute the probability of getting a_1 in this final measurement.

After the measurement of B, the state collapses to either $|\phi_1\rangle$ or $|\phi_2\rangle$, depending on which outcome the measurement returns. The probability of subsequently measuring A to be a_1 depends on which outcome of B we get. If the measurement of B gives the value b_1, then the state is $|\phi_1\rangle$ and the probability of subsequently measuring $A = a_1$ is $1/4$. If the measurement of B gives the values b_2, then the state is $|\phi_2\rangle$ and the probability of subsequently measuring $A = a_1$ is $3/4$.

5. **Matrix Exponential (8 pts)**

(a) Compute the matrix exponential e^M, where M is the 2×2 matrix:

$$M = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}.$$

To receive full credit, the answer should not be left in the form of an infinite sum.
Note that $M = i\theta\sigma_y$, and $\sigma_y^2 = I$. Then we can write:

$$e^M = \sum_{n=0}^{\infty} \frac{1}{n!}(i\theta)^n \sigma_y^n$$

We can split the sum into even and odd powers:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!}(i\theta)^n \sigma_y^n &= I \sum_{k=0}^{\infty} \frac{1}{(2k)!}(i\theta)^{2k} + \sigma_y \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(i\theta)^{2k+1} \\ &= I\cos\theta + i\sigma_y \sin\theta \end{aligned}$$

Re-writing this as a matrix, we get:

$$e^M = \boxed{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}$$

(b) Let A be Hermitian. Prove that e^{iA} is unitary.
Let A be Hermitian. Then:

$$(e^{iA})^\dagger = e^{-iA^\dagger} = e^{-iA}.$$

Therefore:

$$(e^{iA})^\dagger e^{iA} = e^{-iA} e^{iA} = e^{i(A-A)} = I.$$

So e^{iA} is necessarily unitary.