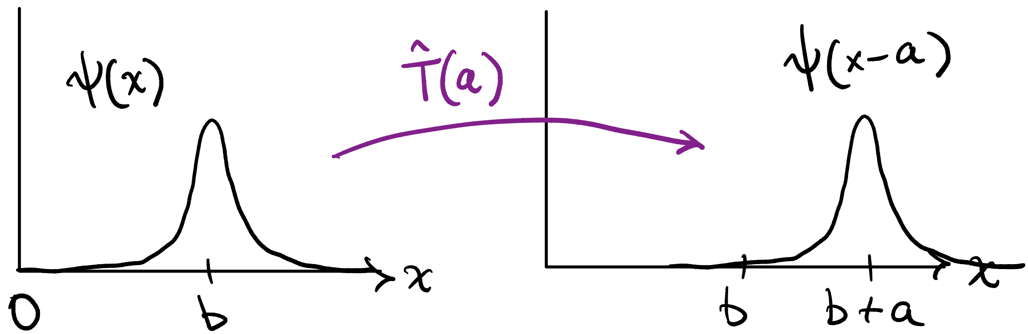
Discussion 8 : Momentum

Momentum

* It makes sense to consider spatially translating position states. Invent an operator that does this:

$$\hat{T}(a)|x\rangle = |x+a\rangle$$

* the operator $\hat{T}(a)$ changes the state of a particle in which the particle has position x to one in which the particle has position $x+a$.



* This does what you'd expect to wavefunctions:

$$\hat{T}(a)|\psi\rangle = \int dx' \, \hat{T}(a)|x'\rangle\langle x'|\psi\rangle$$

$$= \int dx' \, |x'+a\rangle\langle x'|\psi\rangle$$

Change dummy variable: $x'' = x' + a$. $dx'' = dx'$. Then:

$$= \int dx'' \, |x''\rangle \underbrace{\langle x''-a|\psi\rangle}_{\text{New wavefunction, } \psi(x''-a).}$$

* Translation operator $\hat{T}(a)$ is unitary because translated states maintain all their inner products:

$$\langle\psi|\phi\rangle = \int dx'\, \psi^*(x')\,\phi(x)$$

$$\langle\psi|\hat{T}^\dagger(a)\,\hat{T}(a)|\phi\rangle = \int dx'\, \psi^*(x'-a)\,\phi(x'-a)$$

$\downarrow x'' = x'-a$

$$= \int dx''\, \psi^*(x'')\,\phi(x'')$$

$$\Rightarrow \hat{T}^\dagger(a)\hat{T}(a) = \mathbb{1}.$$

* Consider an infinitesimal translation

$$\hat{T}(dx) = \mathbb{1} - \frac{i}{\hbar}\hat{p}_x dx + \mathcal{O}(dx^2)$$

which acts on position eigenstates by:

$$\hat{T}(dx)\,|x\rangle = |x+dx\rangle$$

* The operator $\hat{p}_x$ is called the generator of translations, and the $\hbar$ ensures that $\hat{p}_x$ has units of momentum.

* Put a lot of small translations together to get a finite one by distance a:

$$\hat{T}(a) = \lim_{N\to\infty}\left(\mathbb{1} - \frac{i}{\hbar}\hat{p}_x\left(\frac{a}{N}\right)\right)^N = \boxed{e^{-i\hat{p}_x a/\hbar}}$$

* The operator $\hat{p}_x$ is Hermitian because $\hat{T}(a)$ is unitary (we've previously seen that e^{iA} is unitary if and only if A is Hermitian) so

$$\hat{p}_x = \hat{p}_x^\dagger$$

* The operator $\hat{p}_x$ does not commute with the position operator $\hat{x}$. To see this, consider a translation by an infinitesimal amount δx:

$$\hat{x}\,\hat{T}(\delta x)|x_0\rangle = \hat{x}\,|x_0+\delta x\rangle$$
$$= (x_0+\delta x)\,|x_0+\delta x\rangle$$

while

$$\hat{T}(\delta x)\,\hat{x}\,|x_0\rangle = \hat{T}(\delta x)\,x_0\,|x_0\rangle$$
$$= x_0\,|x_0+\delta x\rangle$$

so:

$$\left(\hat{x}\hat{T}(\delta x) - \hat{T}(\delta x)\hat{x}\right)|x_0\rangle = \delta x\,|x_0+\delta x\rangle$$

Now expand:

$$\hat{x}\,\hat{T}(\delta x) - \hat{T}(\delta x)\hat{x} = \hat{x}\left(\mathbb{1} - \frac{i}{\hbar}\hat{p}_x\delta x\right) - \left(\mathbb{1} - \frac{i}{\hbar}\hat{p}_x\delta x\right)\hat{x}$$
$$= -\frac{i}{\hbar}\delta x\,[\hat{x}, \hat{p}_x]$$

Set these equal and take $\delta x \to 0$:

$$\Rightarrow -\frac{i}{\hbar}\delta x\,[\hat{x}, \hat{p}_x] = \delta x$$

$$\Rightarrow \boxed{[\hat{x}, \hat{p}_x] = i\hbar}$$

"Canonical Commutation Relation"

* When acting on wavefunctions, $\hat{p}_x$ is represented by:

$$\boxed{\hat{p}_x \xrightarrow{\text{position basis}} -i\hbar \frac{d}{dx}}$$

What this means is that:

$$\langle x | \hat{p}_x | \psi \rangle = -i\hbar \frac{d}{dx} \psi(x)$$

Momentum space

* Having introduced the momentum operator, we now introduce a new set of states, momentum eigenstates, satisfying

$$\hat{p}_x |p\rangle = p|p\rangle$$

* Like position, momentum is a continuous variable. The resolution of the identity is:

$$\mathbb{1} = \int_{-\infty}^{\infty} dp\, |p\rangle\langle p|$$

* Momentum eigenstates that have different eigenvalues are orthogonal

$$\langle p' | p \rangle = \delta(p - p')$$

Dirac delta function

* An arbitrary state $|\psi\rangle$ can be expanded as a linear combination of momentum eigenstates:

$$|\psi\rangle = \int dp \, |p\rangle\langle p|\psi\rangle$$

* The condition that states are normalized is:

$$1 = \langle\psi|\psi\rangle = \int_{-\infty}^{\infty} dp \, |\langle p|\psi\rangle|^2$$

* We identify that

$$\boxed{dp \, |\langle p|\psi\rangle|^2 = \text{Probability to find the particle in state } |\psi\rangle \text{ to have momentum in the range between } p \text{ and } p+dp}$$

* Just as we call $\langle x|\psi\rangle \equiv \psi(x)$ the wave function (in position space), we write

$$\langle p|\psi\rangle \equiv \tilde{\psi}(p)$$

and refer to this as the <u>momentum-space wave function</u>.

* We can now determine $\langle x|p\rangle$, which is the position-space wavefunction of a momentum eigenstate, $|p\rangle$. It is generally true that:

$$\langle x|\hat{p}_x|\psi\rangle = -i\hbar\frac{d}{dx}\psi(x)$$

If we specialize to $|\psi\rangle = |p\rangle$, we get:

$$\langle x|\hat{p}_x|p\rangle = -i\hbar\frac{d}{dx}\langle x|p\rangle$$

Let this act on $|p\rangle$

$$p\langle x|p\rangle = -i\hbar\frac{d}{dx}\langle x|p\rangle$$

This is a differential equation for $\psi_p(x) \equiv \langle x|p\rangle$. It is a very simple one: what is the function $\psi_p(x)$ such that taking a derivative is the same as multiplying it by $ip/\hbar$? The answer is:

$$\psi_p(x) = N e^{ipx/\hbar}$$

some constant.

* We determine N by normalizing $|p\rangle$. Namely, we want:

$$\langle p|p'\rangle \overset{!}{=} \delta(p-p')$$

so

$$\langle p|p'\rangle = \int dx\, \langle p|x\rangle\langle x|p'\rangle$$

$$= \int dx\, N^* N\, e^{-ipx/\hbar}\, e^{ip'x/\hbar}$$

$$= |N|^2 \int_{-\infty}^{\infty} dx\, e^{i(p'-p)x/\hbar}$$

From analysis, there is a formula that says:

$$\int_{-\infty}^{\infty} dx\, e^{ikx} = 2\pi\,\delta(k)$$

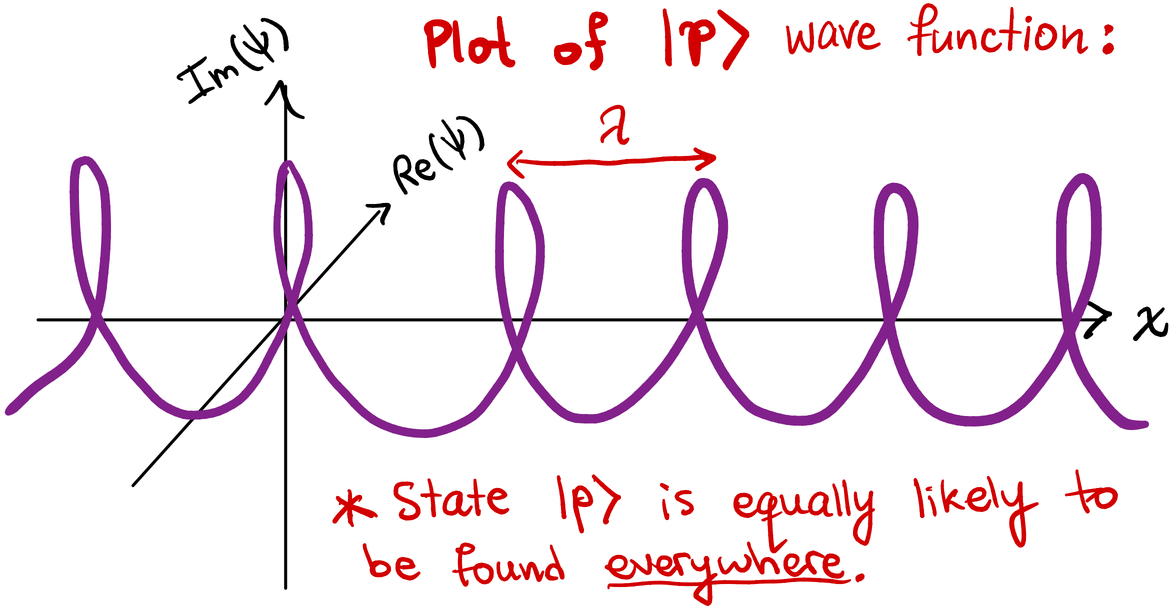
called "the integral representation of the δ-function". We use this to get:

$$\langle p|p'\rangle = |N|^2 (2\pi\hbar)\, \delta(p-p')$$

* To get our desired normalization, we take $N = \frac{1}{\sqrt{2\pi\hbar}}$ so that

$$\boxed{\begin{aligned} \langle x|p\rangle &= \frac{1}{\sqrt{2\pi\hbar}}\, e^{ipx/\hbar} \\ \langle p|x\rangle &= \frac{1}{\sqrt{2\pi\hbar}}\, e^{-ipx/\hbar} \end{aligned}}$$

Plot of $|p\rangle$ wave function:



* State $|p\rangle$ is equally likely to be found <u>everywhere</u>.

* Use right hand rule to determine direction the particle moves (in the drawing it moves <u>right</u>).

* The wavelength of the helix is

$$\lambda = \frac{2\pi}{\text{wavenumber}} = \frac{2\pi}{(p/\hbar)}$$

$$= \frac{2\pi\hbar}{p}$$

De Broglie relation

$$\boxed{\lambda = \frac{h}{p}}$$

where $h \equiv 2\pi\hbar$ is <u>planck's constant</u> with value

$$h = 6.626070 \times 10^{-34}\ \mathrm{J\cdot s}$$

change-of-basis:

Knowledge of $\langle x|p\rangle$ allows us to switch back and forth from position & momentum bases, just like knowledge of amplitudes such as $\langle +x|+z\rangle$ allowed us to switch between the S_x basis and S_z basis.

One simply writes:

$$\boxed{\begin{aligned} \langle p|\psi\rangle &= \int dx\, \langle p|x\rangle\langle x|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\, e^{-ipx/\hbar}\,\psi(x) \\ \langle x|\psi\rangle &= \int dp\, \langle x|p\rangle\langle p|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dp\, e^{ipx/\hbar}\,\tilde{\psi}(p) \end{aligned}}$$

* We see that the position space wave function $\psi(x)$ and the momentum space wave function $\tilde{\psi}(p)$ are the Fourier transform of each other!

Problems

1 Show that $-i\hbar\frac{d}{dx}$ satisfies the canonical commutation relation by considering

$$[x, -i\hbar\frac{d}{dx}]\ f(x)$$

not an operator, just a function.

2 For the Gaussian wave function (a>0 is constant)

$$\psi(x) = \frac{1}{\sqrt{\sqrt{\pi}\,a}}\, e^{-x^2/2a^2}$$

normalization

(a) Compute $\langle p_x \rangle$ via an integral over x (i.e. by working in the position basis).

(b) Find the momentum-space wave function $\tilde{\psi}(p)$ describing the same state

(c) compute $\langle p_x \rangle$ using an integral over p.

(d) Compute $\langle p_x^2 \rangle$ however you choose.

(e) Taking $\langle x \rangle = 0$, $\langle x^2 \rangle = a^2/2$ as given, compute Δx, Δp, and the product $\Delta x \Delta p$.

You should find $\Delta x \Delta p = \frac{\hbar}{2}$, which means that Gaussians are the minimum uncertainty states!

Solutions

$\boxed{1}$ The trick is to let $[x, -i\hbar \frac{d}{dx}]$ act on a "test function" $f(x)$, which we discard later:

$$[x, -i\hbar \frac{d}{dx}] f(x)$$

$$= -i\hbar \left(x \frac{d}{dx} f(x) - \frac{d}{dx}(x f(x)) \right)$$

$$= -i\hbar \left(\cancel{x f'(x)} - f(x) - \cancel{x f'(x)} \right)$$

$$= i\hbar f(x)$$

Since this holds for all functions $f(x)$ we have

$$[x, -i\hbar \frac{d}{dx}] = i\hbar \quad \checkmark$$

$\boxed{2}$

(a) We compute:

$$\langle p_x \rangle = \langle \psi | \hat{p}_x | \psi \rangle = \int dx\, \psi^*(x) \left(\frac{\hbar}{i} \frac{d}{dx} \right) \psi(x)$$

$$= \frac{-i\hbar}{\sqrt{\pi}\, a} \int dx\, e^{-x^2/2a} \left(\frac{d}{dx} e^{-x^2/2a} \right)$$

$$= \frac{-i\hbar}{\sqrt{\pi}\, a} \int dx\, e^{-x^2/2a} e^{-x^2/2a} \cdot \left(-\frac{2x}{2a} \right)$$

$$= \frac{i\hbar}{\sqrt{\pi}\, a^2} \int_{-\infty}^{\infty} dx\, x\, e^{-x^2/a}$$

$= \boxed{0}$ (integral is odd).

(b) The momentum space wave function is:

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx\, e^{-ipx/\hbar}\, \psi(x)$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx\, e^{-ipx/\hbar} \frac{1}{\sqrt{\sqrt{\pi}\, a}} e^{-x^2/2a}$$

General formula:

$$\int_{-\infty}^{\infty} e^{-Ax^2 + Bx + C} = \sqrt{\frac{\pi}{A}}\, e^{\frac{B^2}{4A} + C}$$

$$\boxed{\psi(p) = \sqrt{\frac{a}{\hbar\sqrt{\pi}}}\, e^{-p^2 a^2 / 2\hbar^2}}$$

(c) Now we can do:

$$\langle p_x \rangle = \langle \psi | P_x | \psi \rangle = \int dp\ \tilde{\psi}(p)^* \, p \, \psi(p)$$

$$= \frac{a}{\hbar\sqrt{\pi}} \int dp\ p\, e^{-p^2a^2/\hbar^2}$$

$$= \boxed{0} \quad \text{(odd integral).}$$

(d) Similarly:

$$\langle p_x^2 \rangle = \int_{-\infty}^{\infty} dp\ \tilde{\psi}(p)^* \, p^2 \, \tilde{\psi}(p)$$

$$= \frac{a}{\hbar\sqrt{\pi}} \int_{-\infty}^{\infty} dp\ p^2 e^{-p^2a^2/\hbar^2}$$

Another Gaussian integral

$$= \frac{a}{\hbar\sqrt{\cancel{\pi}}} \frac{1}{2}\sqrt{\cancel{\pi}} \left(\frac{\hbar}{a^2}\right)^{3/2} = \boxed{\frac{\hbar^2}{2a^2}}$$

Then:

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{a^2/2 - 0} = a/\sqrt{2}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\hbar^2/2a^2 - 0} = \frac{\hbar}{\sqrt{2}\,a}$$

so:

$$\Rightarrow \boxed{\Delta x \Delta p = \frac{\cancel{a}}{\sqrt{2}} \frac{\hbar}{\sqrt{2}\,\cancel{a}} = \frac{\hbar}{2}}$$