Homework 10 Solutions

1 Commutators

(a) Let's compute $[L_x, L_y]$ using:

$$L_x = yP_z - zP_y$$
$$L_y = zP_x - xP_z$$

We have:

$$[L_x, L_y] = [yP_z - zP_y \,,\, zP_x - xP_z]$$

$$= \overset{①}{[yP_z, zP_x]} - \overset{②}{[yP_z, xP_z]} - \overset{③}{[zP_y, zP_x]} + \overset{④}{[zP_y, xP_z]}$$

There are four commutators:

① $[yP_z, zP_x]$

Only z and P_z don't commute so this is just

$$= yP_x[P_z, z] = -i\hbar\, yP_x$$

② $[yP_z, xP_z] = 0$ All operators commute w/ each other.

③ $[zP_y, zP_x] = 0$ ← All operators commute w/ each other.

④ $[zP_y, xP_z] = P_y x [z, P_z] = i\hbar\, x P_y$

Putting this together:

$$\boxed{[L_x, L_y] = i\hbar(xP_y - yP_x) = i\hbar L_z}$$

(b) Compute $[L^2, L_z]$ where

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

and using:

$$[L_x, L_y] = i\hbar L_z$$
$$[L_y, L_z] = i\hbar L_x$$
$$[L_z, L_x] = i\hbar L_y$$

$$\hookrightarrow [L^2, L_z] = [L_x^2 + L_y^2 + L_z^2, L_z]$$
$$= \underbrace{[L_x^2, L_z]}_{①} + \underbrace{[L_y^2, L_z]}_{②} + \underbrace{\cancel{[L_z^2, L_z]}}_{0}$$

①: $[L_x^2, L_z] = L_x[L_x, L_z] + [L_x, L_z]L_x$
$$= L_x(-i\hbar L_y) + (-i\hbar L_y)L_x$$

$$= -i\hbar(L_x L_y + L_y L_x)$$

②: $[L_y^2, L_z] = L_y[L_y, L_z] + [L_y, L_z]L_y$

$$= L_y(i\hbar L_x) + (i\hbar L_x)L_y$$

$$=$$

Put it all together:

$$[L^2, L_z] = -i\hbar(L_x \cancel{L_y} + L_y \cancel{L_x}) + i\hbar(L_x \cancel{L_y} + L_y \cancel{L_x})$$

$$= \boxed{0}$$

all cancels

(c) $[L_z, L_\pm] = [L_z, L_x \pm iL_y]$

$$= [L_z, L_x] \pm i[L_z, L_y]$$

$$= i\hbar L_y \pm i(-i\hbar L_x)$$

$$= \pm\hbar L_x + i\hbar L_y$$

$$= \pm\hbar(L_x \pm iL_y)$$

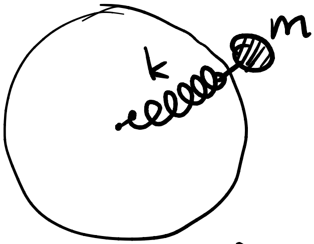
Factor out $\pm\hbar$

$$= \boxed{\pm\hbar L_\pm}$$

(d)

$$
\begin{aligned}
L_\pm L_\mp &= (L_x \pm i L_y)(L_x \mp i L_y) \\
&= L_x^2 + L_y^2 \pm i L_y L_x \mp i L_x L_y \\
&= L_x^2 + L_y^2 \pm i [L_y, L_x] \\
&= L_x^2 + L_y^2 \pm i (-i\hbar) L_z \\
&= \underbrace{L_x^2 + L_y^2}_{L_x^2 + L_y^2 = L^2 - L_z^2} \pm \hbar L_z \\
&= \boxed{L^2 - L_z^2 \pm \hbar L_z}
\end{aligned}
$$

2 3D Harmonic Oscillator



$$V(r) = \frac{1}{2} m\omega^2 r^2$$

The energies of the system are governed by the time-independent Schrodinger equation:

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2)\right] \Psi(x,y,z)$$

$$= E\,\Psi(x,y,z)$$

The trick to solving this is to make a guess:

$$\Psi(x,y,z) = X(x)\,Y(y)\,Z(z)$$

Put this back into the equation:

$$-\frac{\hbar^2}{2m}\left(X''YZ + XY''Z + XYZ''\right) + \frac{1}{2} m\omega^2 (x^2+y^2+z^2) XYZ = EXYZ$$

Divide the equation by $\Psi = XYZ$:

$$\underbrace{\left(-\frac{\hbar^2}{2m}\frac{X''}{X} + \frac{1}{2} m\omega^2 x^2\right)}_{\text{Function of } x \text{ alone}} + \underbrace{\left(-\frac{\hbar^2}{2m}\frac{Y''}{Y} + \frac{1}{2} m\omega^2 y^2\right)}_{\text{Function of } y \text{ alone}} + \underbrace{\left(-\frac{\hbar^2}{2m}\frac{Z''}{Z} + \frac{1}{2} m\omega^2 z^2\right)}_{\text{Function of } z \text{ alone}} = \underbrace{E}_{\text{constant}}$$

The only way for this equation to be true for all x, y, z is if:

$$-\frac{\hbar^2}{2m}\frac{X''}{X} + \frac{1}{2}m\omega^2 x^2 = E_x$$

$$-\frac{\hbar^2}{2m}\frac{Y''}{Y} + \frac{1}{2}m\omega^2 y^2 = E_y$$

$$-\frac{\hbar^2}{2m}\frac{Z''}{Z} + \frac{1}{2}m\omega^2 z^2 = E_z$$

constants

for constants E_x, E_y, E_z such that

$$E = E_x + E_y + E_z$$

So now we have to solve these three differential equations. The first one is:

$$-\frac{\hbar^2}{2m} X''(x) + \frac{1}{2}m\omega^2 x^2 X(x) = E_x X(x) \qquad (*)$$

But this is just the 1D simple harmonic oscillator!
Let $\phi_n(x)$ denote the n^{th} energy eigenstate of the 1D simple harmonic oscillator. Then the solutions to $(*)$ are

$$X_n(x) = \phi_n(x) \qquad n_x = 0, 1, 2, \ldots$$

with energy

$$E_{x,n} = \hbar\omega\left(n_x + \frac{1}{2}\right)$$

* Similarly, the other two equations have solutions:

$$Y_m(y) = \phi_m(y), \qquad E_{y,m} = \hbar\omega\left(m_y + \tfrac{1}{2}\right)$$

$$Z_\ell(z) = \phi_\ell(z), \qquad E_{z,\ell} = \hbar\omega\left(n_z + \tfrac{1}{2}\right)$$

* The full solutions are then

$$\boxed{\Psi_{n_x n_y n_z}(x,y,z) = \phi_{n_x}(x)\,\phi_{n_y}(y)\,\phi_{n_z}(z)}$$

For $n_x, n_y, n_z = 0, 1, 2, \cdots$, with energies:

$$\boxed{E_{n_x n_y n_z} = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right)}$$

(c) I'll make a table:

Energy	Degeneracy	Enumerate n_x, n_y, n_z
$\frac{3}{2}\hbar\omega$	1	$(0,0,0)$
$\frac{5}{2}\hbar\omega$	3	$(1,0,0), (0,1,0), (0,0,1)$
$\frac{7}{2}\hbar\omega$	6	$(2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1)$

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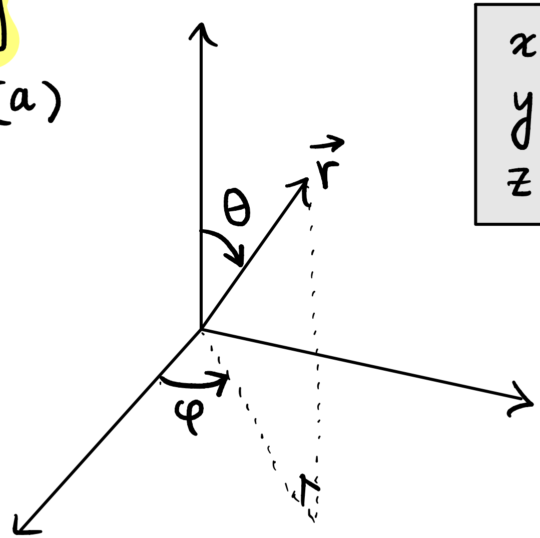
(a)



$$\boxed{\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned}}$$

(b)

$$\boxed{\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \cos^{-1}(z/r) \\ \varphi &= \tan^{-1}(y/x) \end{aligned}}$$

(c)

$$\frac{\partial}{\partial\phi} = \left(\frac{\partial x}{\partial\phi}\right)\frac{\partial}{\partial x} + \left(\frac{\partial y}{\partial\phi}\right)\frac{\partial}{\partial y}$$

$$= -r\sin\theta\sin\varphi\frac{\partial}{\partial x} + r\sin\theta\cos\varphi\frac{\partial}{\partial y}$$

$$= \boxed{-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}}$$

✓