Homework 11 Solutions

1. Let $|\alpha\rangle$ be an eigenstate of a such that

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

(a)

$$\langle x\rangle = \langle\alpha|\hat{x}|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle\alpha|(a+a^\dagger)|\alpha\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}(\alpha+\alpha^*)$$

$$\boxed{\langle x\rangle = \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}(\alpha)}$$

$$\langle x^2\rangle = \langle\alpha|\hat{x}^2|\alpha\rangle = \frac{\hbar}{2m\omega}\langle\alpha|(a+a^\dagger)^2|\alpha\rangle$$

commute

$$= \frac{\hbar}{2m\omega}\langle\alpha|(aa+aa^\dagger+a^\dagger a+a^\dagger a^\dagger)|\alpha\rangle$$

$$aa^\dagger = a^\dagger a + 1$$

$$= \frac{\hbar}{2m\omega}\langle\alpha|(aa+2a^\dagger a+1+a^\dagger a^\dagger)|\alpha\rangle$$

$$= \frac{\hbar}{2m\omega}(\alpha^2+\alpha^{*2}+2\alpha^*\alpha+1)$$

$$= \frac{\hbar}{2m\omega}\left[(\alpha+\alpha^*)^2+1\right]$$

$$\boxed{\langle x^2 \rangle = \frac{\hbar}{2m\omega}\left(1 + 4\,\mathrm{Re}(\alpha)^2\right)}$$

$$\langle p \rangle = \langle \alpha | \hat{p} | \alpha \rangle = i\sqrt{\frac{m\omega\hbar}{2}}\,\langle \alpha | (a^\dagger - a) | \alpha \rangle$$

$$= i\sqrt{\frac{m\omega\hbar}{2}}\,(\alpha^* - \alpha)$$

$$= -2i \cdot i\sqrt{\frac{m\omega\hbar}{2}}\,\mathrm{Im}(\alpha)$$

$$\boxed{\langle p \rangle = \sqrt{2m\omega\hbar}\,\mathrm{Im}(\alpha)}$$

$$\langle p^2 \rangle = -\frac{m\omega\hbar}{2}\langle \alpha | (a^\dagger - a)^2 | \alpha \rangle$$

$$= -\frac{m\omega\hbar}{2}\langle \alpha | (a^\dagger a^\dagger - a^\dagger a - a a^\dagger + a a) | \alpha \rangle$$

$$= -\frac{m\omega\hbar}{2}\langle \alpha | (a^\dagger a^\dagger - 2a^\dagger a - 1 + a a) | \alpha \rangle$$

$$= -\frac{m\omega\hbar}{2}\left(\alpha^*\alpha^* - 2\alpha^*\alpha - 1 + \alpha\alpha\right)$$

$$= \frac{m\omega\hbar}{2}\left[1 + 4\left(\frac{\alpha - \alpha^*}{2i}\right)^2\right]$$

$$\boxed{\langle p^2 \rangle = \frac{m\omega\hbar}{2}\left[1 + 4\,\mathrm{Im}(\alpha)^2\right]}$$

(b) $$\sigma_x = \left[\langle x^2\rangle - \langle x\rangle^2\right]^{1/2}$$

$$= \sqrt{\hbar/2m\omega}$$

$$\sigma_p = \left[\langle p^2\rangle - \langle p\rangle^2\right]^{1/2}$$

$$= \sqrt{m\omega\hbar/2}$$

$$\Rightarrow \boxed{\sigma_x \sigma_p = \frac{\hbar}{2}}$$

(c) Write:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

Because $|\alpha\rangle$ is an eigenstate of a, we have

$$\hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} c_n \hat{a}|n\rangle = \sum_{n=1}^{\infty} c_n \sqrt{n}\, |n-1\rangle$$

$$= \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1}\, |n\rangle$$

This is equal to:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle = \sum_{n=0}^{\infty} \alpha c_n |n\rangle$$

Dotting $\langle m|$ into both of these equations:

$$c_{m+1}\sqrt{m+1} = \alpha c_m$$

$$c_{n+1} = \frac{\alpha}{\sqrt{n+1}} c_n$$ recursion relation.

The sequence is

$$c_0, \quad \alpha c_0, \quad \frac{\alpha^2}{\sqrt{2}} c_0, \quad \frac{\alpha^3}{\sqrt{3\cdot 2}} c_0, \quad \frac{\alpha^4}{\sqrt{4\cdot 3\cdot 2}} c_0, \quad \ldots$$

$$\Longrightarrow \boxed{c_n = \frac{\alpha^n}{\sqrt{n!}} c_0}$$

(d) The state is:

$$|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Normalize the state:

$$1 = \langle\alpha|\alpha\rangle = |c_0|^2 \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\alpha^{*n}\alpha^m}{\sqrt{n!\,m!}} \underbrace{\langle m|n\rangle}_{1 \text{ if } m=n}$$

$$= |c_0|^2 \sum_{n=0}^{\infty} \frac{(|\alpha|^2)^n}{n!}$$

$$1 = |c_0|^2 e^{|\alpha|^2}$$

$$\Rightarrow \boxed{c_0 = e^{-|\alpha|^2/2}}$$

(e) Let the state be $|\alpha\rangle$ at $t=0$. Then

$$|\alpha(t)\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i\omega n t} e^{-i\omega t/2} |n\rangle$$

$$= e^{-i\omega t/2} e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle$$

$$\boxed{|\alpha(t)\rangle = e^{-i\omega t/2} |\alpha e^{-i\omega t}\rangle}$$

still a coherent state, but with $\alpha \to \alpha e^{-i\omega t}$

(f) The state at time zero is $|\alpha\rangle$ where

$$\alpha = C\sqrt{\frac{m\omega}{2\hbar}} e^{i\phi}$$

Then at time t the state is $|\alpha(t)\rangle$ with

$$\alpha(t) = C\sqrt{\frac{m\omega}{2\hbar}} e^{i(\phi-\omega t)}$$

From part (a), we have

$$\langle x\rangle = \sqrt{\frac{2\hbar}{m\omega}} \operatorname{Re}(\alpha) = \boxed{C\cos(\phi-\omega t)}$$

and

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega}\left(1 + 4\,\mathrm{Re}(\alpha)^2\right)$$

$$\langle x^2 \rangle = \boxed{\frac{\hbar}{2m\omega} + C^2\cos(\phi - \omega t)^2}$$

(g) <u>Yes!</u> $a|0\rangle = 0, \implies \boxed{\alpha = 0}$

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2. $$\begin{cases} |11\rangle = |\uparrow\uparrow\rangle \\ |10\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ |1{-1}\rangle = |\downarrow\downarrow\rangle \end{cases}$$

$$\left\{ |00\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)\right.$$

(a) Apply $S_- = S_{1-} + S_{2-}$ to $|10\rangle$:

$$S_-|10\rangle = (S_{1-} + S_{2-})\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\Big(S_{1-}|\uparrow\downarrow\rangle + S_{2-}|\uparrow\downarrow\rangle + S_{1-}|\downarrow\uparrow\rangle + S_{2-}|\downarrow\uparrow\rangle\Big)$$

$$= \frac{\hbar}{\sqrt{2}}\left(|\downarrow\downarrow\rangle + 0 + 0 + |\downarrow\downarrow\rangle\right)$$

$$= \sqrt{2}\hbar\,|\downarrow\downarrow\rangle$$

$$= \sqrt{2}\hbar\,|1\,{-1}\rangle \quad \checkmark$$

(b) $$S_+|00\rangle = (S_{1+}+S_{2+})\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$= \frac{\hbar}{\sqrt{2}}\left(\cancel{|\uparrow\uparrow\rangle} + 0 + 0 - \cancel{|\uparrow\uparrow\rangle}\right) = \boxed{0}$$

$$S_-|00\rangle = (S_{1-}+S_{2-})\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$= \frac{\hbar}{\sqrt{2}}\left(\cancel{|\downarrow\downarrow\rangle} + 0 + 0 - \cancel{|\downarrow\downarrow\rangle}\right) = \boxed{0}$$

(c) $$S^2|11\rangle = \left(S_1^2 + S_2^2 + 2\vec{S}_1\cdot\vec{S}_2\right)|\uparrow\uparrow\rangle$$

$$= \left(\hbar^2\frac{3}{4} + \hbar^2\frac{3}{4} + \vec{S}_1\cdot\vec{S}_2\right)|\uparrow\uparrow\rangle$$

Notice that:

$$S_{1+}S_{2-} + S_{1-}S_{2+} = (S_{1x}+iS_{1y})(S_{2x}-iS_{2y}) + (S_{1x}-iS_{1y})(S_{2x}+iS_{2y})$$

$$= S_{1x}S_{2x} - \cancel{iS_{1x}S_{2y}} + \cancel{iS_{1y}S_{2x}} + S_{1y}S_{2y} + S_{1x}S_{2x} + \cancel{iS_{1x}S_{2y}} - \cancel{iS_{2x}S_{1y}} + S_{1y}S_{2y}$$

$$= 2(S_{1x}S_{2x} + S_{1y}S_{2y})$$

So

$$\Longrightarrow \vec{S}_1 \cdot \vec{S}_2 = S_{1z} S_{2z} + \frac{1}{2}\left(S_{1+} S_{2-} + S_{1-} S_{2+}\right)$$

Compute:

$$\vec{S}_1 \cdot \vec{S}_2 \,|\uparrow\uparrow\rangle = \left[S_{1z} S_{2z} + \frac{1}{2}\left(S_{1+} S_{2-} + S_{1-} S_{2+}\right)\right]|\uparrow\uparrow\rangle$$

$$= \left[\frac{\hbar}{2}\frac{\hbar}{2} + 0 + 0\right]|\uparrow\uparrow\rangle$$

$$= \frac{\hbar^2}{4}|\uparrow\uparrow\rangle$$

So

$$S^2|\uparrow\uparrow\rangle = \hbar^2\left(\frac{3}{4} + \frac{3}{4} + \frac{2}{4}\right)|\uparrow\uparrow\rangle = 2\hbar^2|10\rangle \checkmark$$

Expect $\ell(\ell+1)\hbar^2$, with $\ell = 1$ ✓

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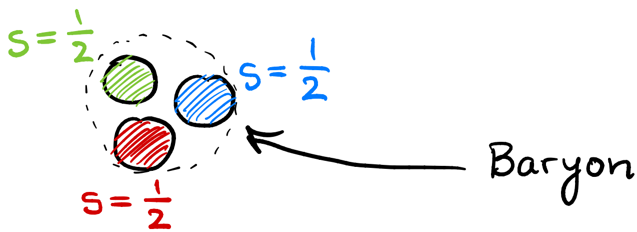
3. 

* First, the total spin of quarks 1&2 can take on possible values $s=0$ or $s=1$

* Next, add the third quark. We have:

$$\frac{1}{2} \otimes 0 = \frac{1}{2}$$

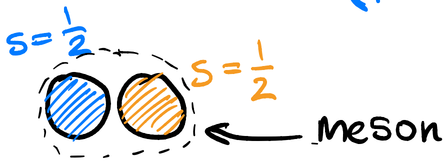
$$\frac{1}{2} \otimes 1 = \frac{3}{2} \oplus \frac{1}{2}$$

So the possible spins are:

$$\boxed{s = \frac{3}{2},\ \frac{1}{2},\ \frac{1}{2}}$$

(two different spin 1/2's)

(b)



Possible spins are $\boxed{s = 0, 1}$