1. *Parity Operator*

   This is a fancy way of saying "mirror." We define $\hat{\Pi}$ by how it acts on each element of the basis $\{|x\rangle\}$:

   $$\hat{\Pi}\,|x\rangle = |-x\rangle \ . \tag{1}$$

   Each of the following subproblems should not take more than a line (or at most two) to solve.

   a. Prove that $\hat{\Pi}^2 = \mathbb{I}$

   b. Use this to find all possible eigenvalues of $\hat{\Pi}$.

   c. Is $\hat{\Pi}$ Hermitian? Why/why not?

   d. Is $\hat{\Pi}$ Unitary? Why/why not?

   e. How does $\hat{\Pi}$ act on a wavefunction $\psi(x)$? (In other words, what is the new wavefunction after we have acted with $\hat{\Pi}$?)

   f. How does $\hat{\Pi}$ act on the momentum eigenstate $|p\rangle$?

   g. How does $\hat{\Pi}$ act on the momentum-space wavefunction $\tilde{\psi}(p)$?

   h. Prove that $\langle x\rangle = 0$ and $\langle p\rangle = 0$ for any parity eigenstate.

   i. For the 1D Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2\mu} + V(\hat{x})$, show that $[\hat{H}, \hat{\Pi}] = 0$ if and only if the potential satisfies $V(x) = V(-x)$. *Remark:* The final result implies that when $V(x) = V(-x)$, we can simultaneously diagonalize $\hat{\Pi}$ and $\hat{H}$. In other words, we can find a basis of energy eigenstates all of which are also parity eigenstates. Both the harmonic oscillator and the infinite square well (if we shift its position by $-a/2$) satisfy $V(x) = V(-x)$.

   j. Prove that the energy eigenstates we found for the infinite square well (after shifting them by $-a/2$) are already parity eigenstates. What is the parity eigenvalue of the n-th state? (The answer turns out to be a general pattern that holds for any symmetric 1D potential.)

   k. Now let's turn to the harmonic oscillator, where we have not written down an explicit formula for all the energy eigenstates. Without evaluating $\psi_n(x)$, in fact without doing any algebra at all, give a one-line argument why the energy eigenstates must all be parity eigenstates.

Remark In light of the fact that $\langle x \rangle = \langle p \rangle = 0$ for parity eigenstates, and that all the energy eigenstates we found in the I.S.W. and in the H.O. are parity eigenstates, we seem to have a very strange situation: there is no sense in which the particle is ever *not* at the origin, or ever changing direction, at least on average. All these states, in this average sense, are just like the completely trivial classical solution in which the particle sits at $x = 0$ and does not move at all. More generally, we already showed that all expectation values are time-independent in energy eigenstates (they might just not be zero). But let's not forget that energy eigenstates are not the most general states; they just form a convenient basis. The point of the next problem is to see that the expectation value of x will be time-dependent as soon as we sum over states with different parity eigenvalues.

2. *Seeing some movement*

   Griffiths 3rd ed., Problem 2.5: subproblems a, b, c only.

3. *Explicit form of the energy eigenstates of the harmonic oscillator*

   In Griffiths 3rd ed., study Page 62 (which picks up exactly where we just left off in class) and carefully study Example 2.4. Then do Problem 2.10.