Midterm Practice Problems

$\boxed{1}$ True/False

(a) $A + A^\dagger$ is Hermitian for all A.

(b) $[A, B]$ is Hermitian provided A and B are Hermitian.

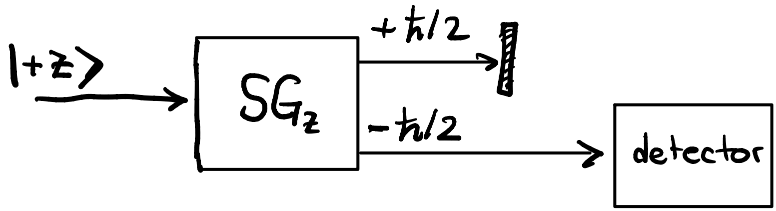
(c) Let $\{|e_i\rangle\}$ and $\{|a_i\rangle\}$ be two orthonormal bases. Then

$$\mathcal{O} = \sum_i |e_i\rangle\langle a_i|$$

is unitary.

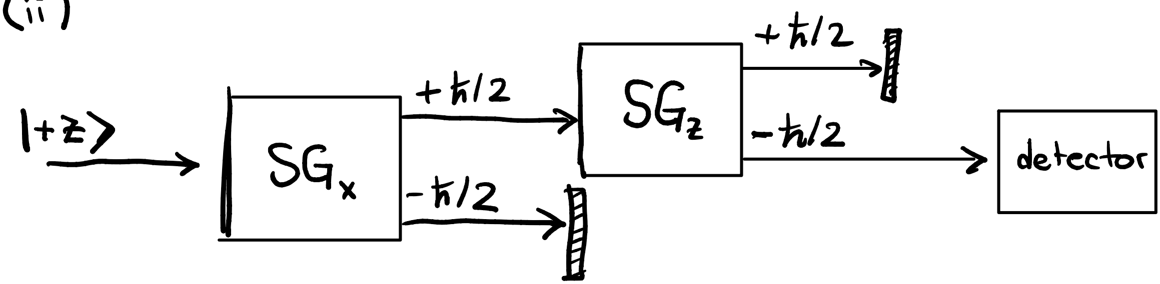
2 Consider the following sequences of Stern-Gerlach devices:

(i)



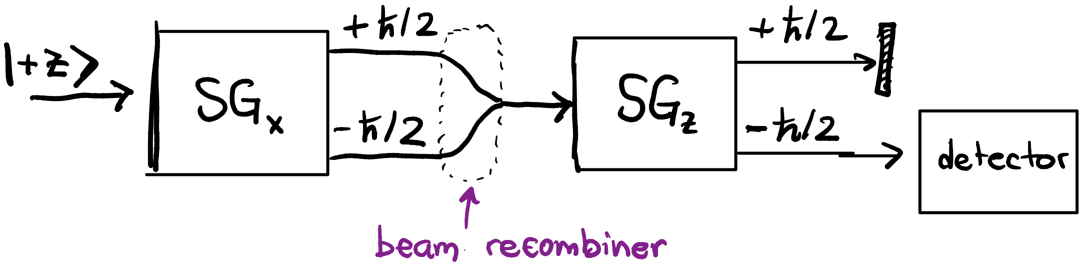
What is the probability that the particle (which begins in the state $|+z\rangle$) is found in the detector?

(ii)



What is the probability that the particle (which begins in the state $|+z\rangle$) is found in the detector?

(iii)



What is the probability that the particle (which begins in the state $|+z\rangle$) is found in the detector?

$\boxed{3}$ The state

$$|\psi\rangle = \frac{1}{2}|+z\rangle + \frac{i\sqrt{3}}{2}|-z\rangle$$

is prepared repeatedly.

(a) Compare this state to

$$|+n\rangle = \cos\frac{\theta}{2}|+z\rangle + e^{i\phi}\sin\frac{\theta}{2}|-z\rangle$$

in order to determine the direction $\hat{n}$ in which $|\psi\rangle$ is spin-up.

(b) Compute $\langle S_x\rangle, \langle S_y\rangle, \langle S_z\rangle$ and show that:

$$\left(\langle S_x\rangle, \langle S_y\rangle, \langle S_z\rangle\right) = \frac{\hbar}{2}\left(n_x, n_y, n_z\right)$$

$\boxed{4}$ Let us assume a particle can only be in states $|\psi_1\rangle$ and $|\psi_2\rangle$, with energies E_1 and E_2, and superpositions thereof. Suppose that application of a laser pulse change the state according to the following rules:

$$|\psi_1\rangle \rightarrow \frac{1}{\sqrt{2}}\left(|\psi_1\rangle + |\psi_2\rangle\right)$$

$$|\psi_2\rangle \rightarrow \frac{1}{\sqrt{2}}\left(|\psi_1\rangle - |\psi_2\rangle\right)$$

(a) Consider the following expiriment: The system is prepared in the state $|\psi_1\rangle$ and the laser is applied twice. If this expiriment is repeated many times, how often, on average, will an energy measurement return a value of E_1?

(b) Consider a similar expiriment: The system is prepared in the state $|\psi_1\rangle$, a pulse is applied, the energy of the system is measured, another pulse is applied, and the energy is measured again. If this expiriment is repeated many times, how often, on average, will the second energy measurement return a value of E_1?

solutions

$\boxed{1}$

(a) True. $(A+A^\dagger)^\dagger = A^\dagger + A$ ✓

(b) False. $[A,B]^\dagger = (AB)^\dagger - (BA)^\dagger$

$$= B^\dagger A^\dagger - A^\dagger B^\dagger$$

$$= BA - AB$$

$$= -[A,B] \quad \times$$

(c) True.

$$\mathcal{O}^\dagger = \sum_i \left(|e_i\rangle\langle a_i| \right)^\dagger$$

$$= \sum_i |a_i\rangle\langle e_i|$$

Now:

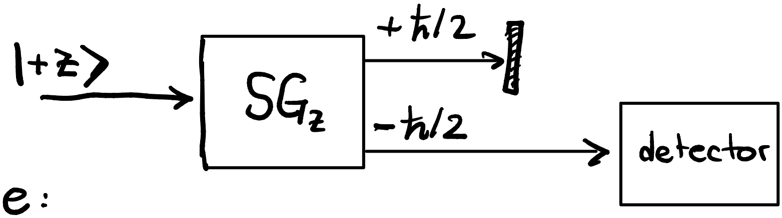
$$\mathcal{O}^\dagger \mathcal{O} = \left(\sum_i |a_i\rangle\langle e_i|\right)\left(\sum_j |e_j\rangle\langle a_j|\right)$$

$$= \sum_i \sum_j |a_i\rangle \underbrace{\langle e_i | e_j\rangle}_{=0 \text{ unless } i=j} \langle a_j|$$

$$= \sum_i |a_i\rangle\langle a_i| = \boxed{\mathbb{1}} \quad ✓$$

$\boxed{2}$

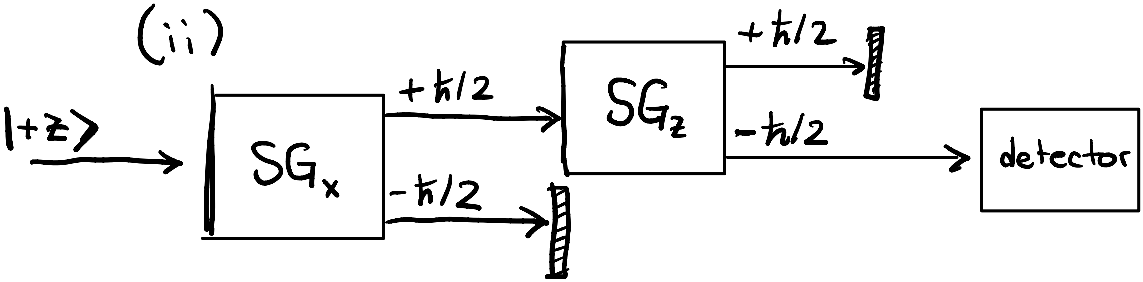
(i)



compute:

$$|\langle -z | +z \rangle|^2 = \boxed{0}$$

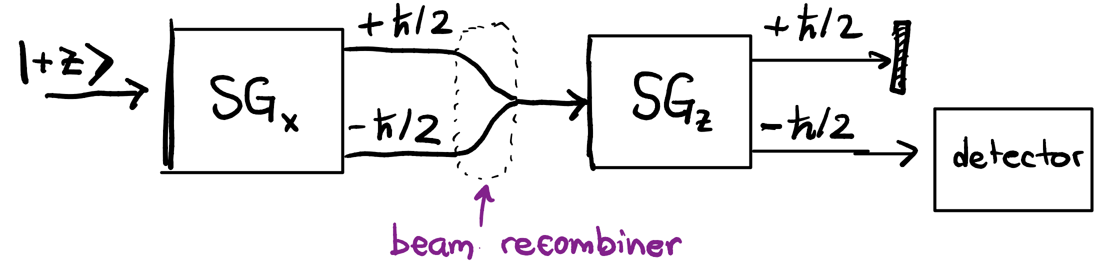
(ii)



compute:

$$|\langle -z | +x \rangle \langle +x | +z \rangle|^2 = \frac{1}{2} \cdot \frac{1}{2} = \boxed{\frac{1}{4}}$$

(iii)



For this one, have to include both paths:

$$\left| \langle -z|+x\rangle\langle +x|+z\rangle + \langle -z|-x\rangle\langle -x|+z\rangle \right|^2$$

$$= \left| \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}\frac{1}{\sqrt{2}} \right|^2 = \boxed{0}$$

$\boxed{3}$

$$|\psi\rangle = \frac{1}{2}|+z\rangle + \frac{i\sqrt{3}}{2}|-z\rangle$$

(a) $$\cos\frac{\theta}{2} = \frac{1}{2} \Rightarrow \frac{\theta}{2} = \frac{\pi}{3}$$

$$\Rightarrow \boxed{\theta = 2\pi/3}$$

$$e^{i\phi} = i \Rightarrow \boxed{\phi = \pi/2}$$

$$\cos\theta = -1/2, \quad \sin\theta = \sqrt{3}/2$$
$$\cos\phi = 0 \quad \sin\phi = 1$$

$$\hat{n} = (\cos\phi\sin\theta, \ \sin\phi\sin\theta, \ \cos\theta)$$

$$\Rightarrow \boxed{\hat{n} = \left(0, \ \frac{\sqrt{3}}{2}, \ -1/2\right)}$$

(b)
$$\langle S_x\rangle = \left(\tfrac{1}{2} \ \ \tfrac{-i\sqrt{3}}{2}\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}1/2\\ i\sqrt{3}/2\end{pmatrix}$$

$$= \frac{1}{4}\left(1 \ \ -i\sqrt{3}\right)\begin{pmatrix}i\sqrt{3}\\ 1\end{pmatrix}\frac{\hbar}{2}$$

$$= \frac{\hbar}{8}\left(\cancel{i\sqrt{3}} - \cancel{i\sqrt{3}}\right) = \boxed{0}$$

$$\langle S_y \rangle = \begin{pmatrix} \frac{1}{2} & \frac{-i\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1/2 \\ i\sqrt{3}/2 \end{pmatrix}$$

$$= \frac{1}{4} \frac{\hbar}{2} \begin{pmatrix} 1 & -i\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} \\ i \end{pmatrix}$$

$$= \frac{\hbar}{8} \left(\sqrt{3} + \sqrt{3} \right) = \boxed{\frac{\hbar}{2} \frac{\sqrt{3}}{2}}$$

$$\langle S_z \rangle = \frac{\hbar}{2} \left| \frac{1}{2} \right|^2 - \frac{\hbar}{2} \left| \frac{\sqrt{3}}{2} \right|^2$$

$$= \frac{\hbar}{2} \left(\frac{1}{4} - \frac{3}{4} \right) = \boxed{-\frac{\hbar}{2} \frac{1}{2}}$$

$$\Rightarrow \boxed{\langle \vec{S} \rangle = \left(0, \frac{\hbar}{2} \frac{\sqrt{3}}{2}, -\frac{\hbar}{2} \frac{1}{2} \right)}$$

So indeed

$$\langle \vec{S} \rangle = \frac{\hbar}{2} \hat{n}$$

$\boxed{4}$

(a) $|\psi_1\rangle \xrightarrow{\text{pulse}} \frac{1}{\sqrt{2}}|\psi_1\rangle + \frac{1}{\sqrt{2}}|\psi_2\rangle$

Do it again:

$$\frac{1}{\sqrt{2}}|\psi_1\rangle + \frac{1}{\sqrt{2}}|\psi_2\rangle \xrightarrow{\text{pulse}} \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|\psi_1\rangle + \frac{1}{\sqrt{2}}|\psi_2\rangle\right)$$

$$+ \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|\psi_1\rangle - \frac{1}{\sqrt{2}}|\psi_2\rangle\right)$$

simplify:

$$= \frac{1}{2}|\psi_1\rangle + \cancel{\frac{1}{2}|\psi_2\rangle}$$

$$+ \frac{1}{2}|\psi_1\rangle - \cancel{\frac{1}{2}|\psi_2\rangle} = \underline{\underline{|\psi_1\rangle}}.$$

→ $\boxed{\text{energy is } \underline{\text{always}} \ E_1 \ 100\%.}$

(b) $|\psi_1\rangle \xrightarrow{\text{pulse}} \frac{1}{\sqrt{2}}|\psi_1\rangle + \frac{1}{\sqrt{2}}|\psi_2\rangle$

<u>measure</u>

E_1 with 50% probability. (case 1).

E_2 with 50% probability. (case 2).

<u>case 1</u> Now the pulse does

$$|\psi_1\rangle \xrightarrow[\text{pulse}]{} \frac{1}{\sqrt{2}}|\psi_1\rangle + \frac{1}{\sqrt{2}}|\psi_2\rangle$$

The second measurement has a 50% chance of giving E_1.

<u>case 2</u> The pulse does:

$$|\psi_2\rangle \xrightarrow[\text{pulse}]{} \frac{1}{\sqrt{2}}|\psi_1\rangle - \frac{1}{\sqrt{2}}|\psi_2\rangle$$

The second measurement has a 50% chance of giving E_1.

$\Rightarrow$ In all cases, the 2^{nd} measurement gives a value of E_1 $\boxed{50\%}$ of the time