Discussion 1 1/20/2021

Today: $\boxed{\text{complex numbers}}$

why?
- Schrodinger's equation has an i in i

$$i\hbar \frac{\partial \psi}{\partial t} = \left[\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi(x,t$$

- Waves conveniently described by

$$e^{ikx} \leftarrow ???$$

- More!

$\boxed{\text{Warm-Up}}$
(1) What is $1/i$
(2) What is $\sqrt{i}$
(3) What is i^i

reveal: (let students answer) (1) $1/i = -i$

(2) $\sqrt{i} = \pm\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$. (3) i^i is a real numbe with several possible values, eg. $e^{-\pi/2}$ mak sense.

<u>Def</u> Let i be a number such that $i^2 = -1$. Call i the "<u>imaginary unit</u>"

<u>Def</u> An "<u>imaginary number</u>" is a multiple of i, e.g. $5i$ is imaginary and $(5i)^2 = -25$.

Def A complex number is any number of the form $a + ib$ where a and b are real.

question: Is 1 a complex number? (yes).

Addition: The real parts and imaginary parts add seperatly e.g. $(1 + 3i) + (3 - i) = 4 + 2i$

Multiplication: The terms distribute intuitively, eg.

$$(1 + 3i)(3 - i) = 3 - i + 9i + 3 = 6 + 8i$$

Division and subtraction are the inverse of these.

question: What is $\frac{2+i}{5+2i}$? Simplify to $a+ib$ form

ans: $\frac{2+i}{5+2i} \cdot \frac{5-2i}{5-2i} = \frac{10 - 4i + 5i + 2}{29} = \frac{12 + i}{29}$

Def "complex conjugate". Let $z = x + iy$. Then

$$\boxed{z^* \equiv x - iy.}$$

Def Let $z = x + iy$. We call x the "real part" of z and y the "imaginary part" of z. We write

$$\begin{cases} Re(z) = x \\ Im(z) = y \end{cases}$$

note: The imaginary part of z is a real number.

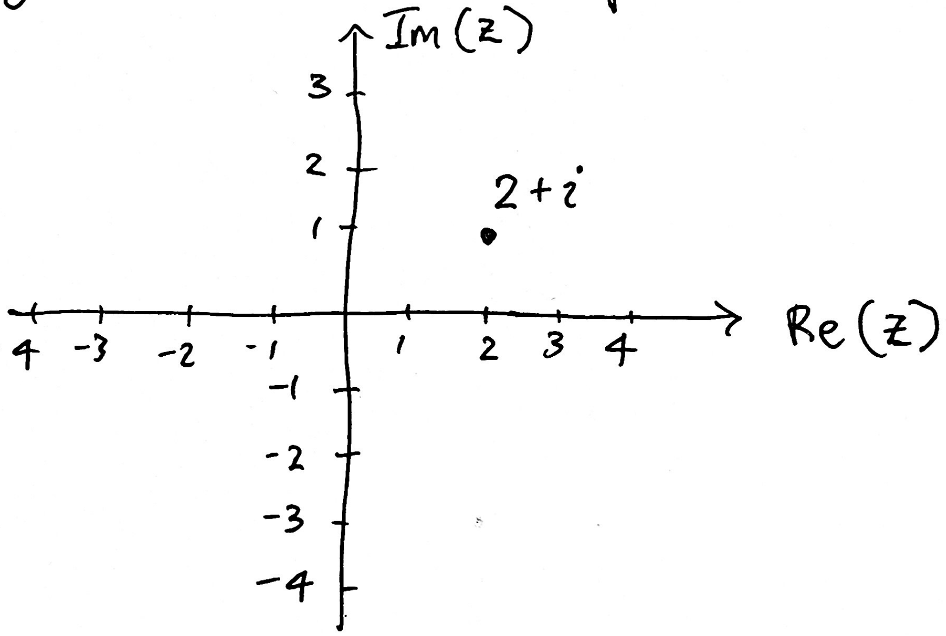
Def The <u>magnitude squared</u> of a complex number $z = x + iy$ is

$$|z|^2 \equiv x^2 + y^2$$

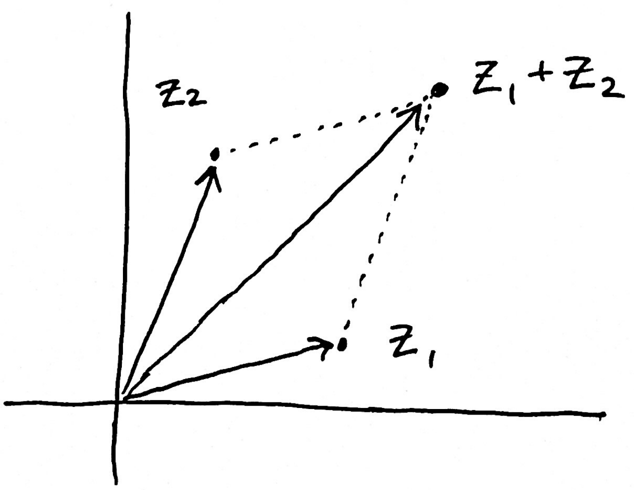
It is always a positive real number (or zero).

<u>question</u>: show that $|z|^2 = z^* z$

<u>Complex plane</u>: The set of all complex numbers is conveniently visualized as a plane



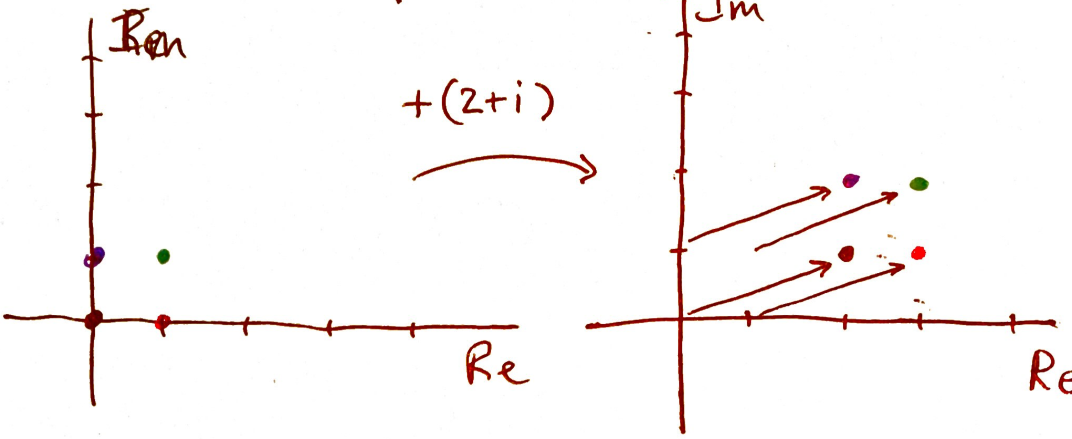
<u>question</u>: What does addition/subtraction of two complex numbers <u>look</u> like on the plane?



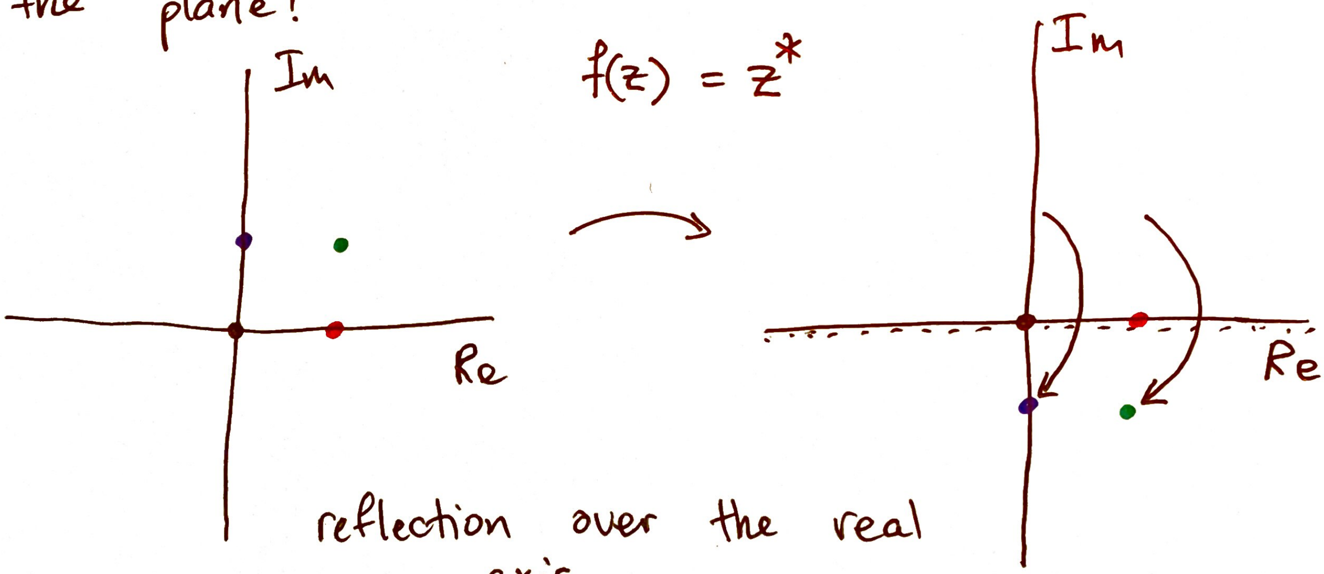
ans: looks like <u>vector</u> <u>addition</u>.

Extend this to speak of the "action" of addition on the complex plane.

eg The function $f(z) = z + (2+i)$ can be visualized by shifting the whole plane two to the right and one up:



question What is the action of complex conjugation on the plane?



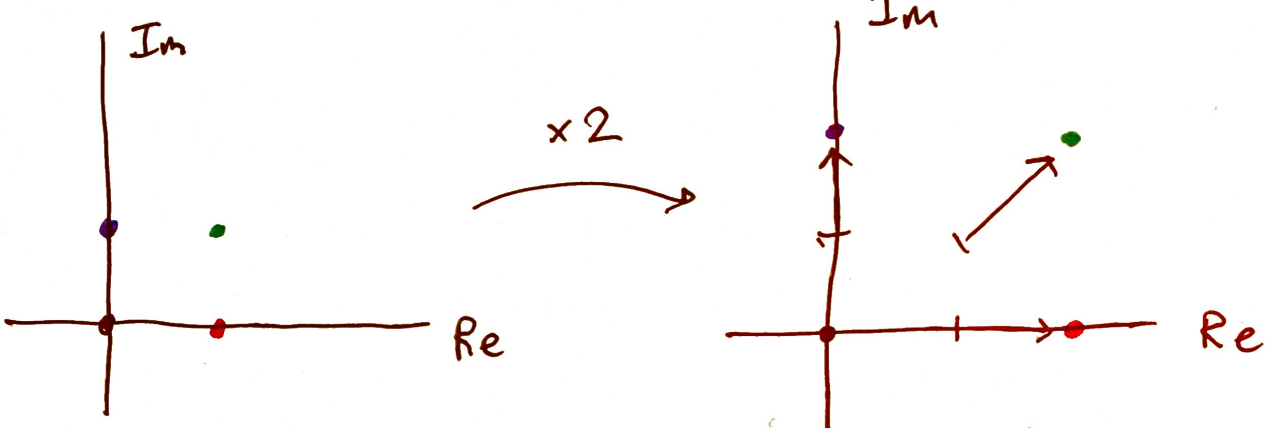
reflection over the real axis.

question: What does multiplication do to the complex plane?

→ A mystery. Looks like a combination of scaling and rotation.

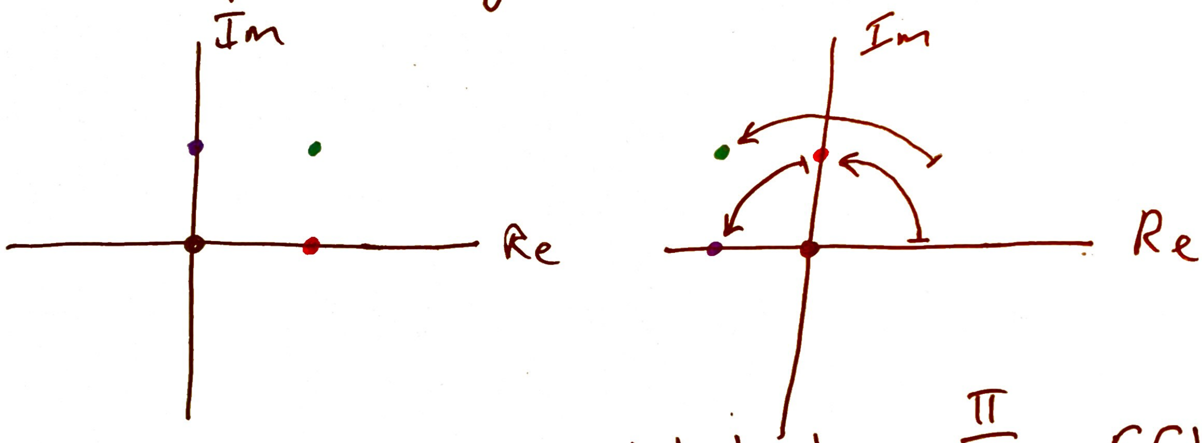
example Multiplication by 2

$$2z = 2(x+iy) = 2x + 2iy$$



All points scaled radially by 2.

example Multiplication by i



All points rotated by $\frac{\pi}{2}$ CCW, about the origin.

Why???

A "tangent" on Taylor series

$$\cos\theta = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + \dots$$

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \frac{1}{7!}\theta^7 + \dots$$

$$e^\theta = 1 + \theta + \frac{1}{2!}\theta^2 + \frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 + \dots$$

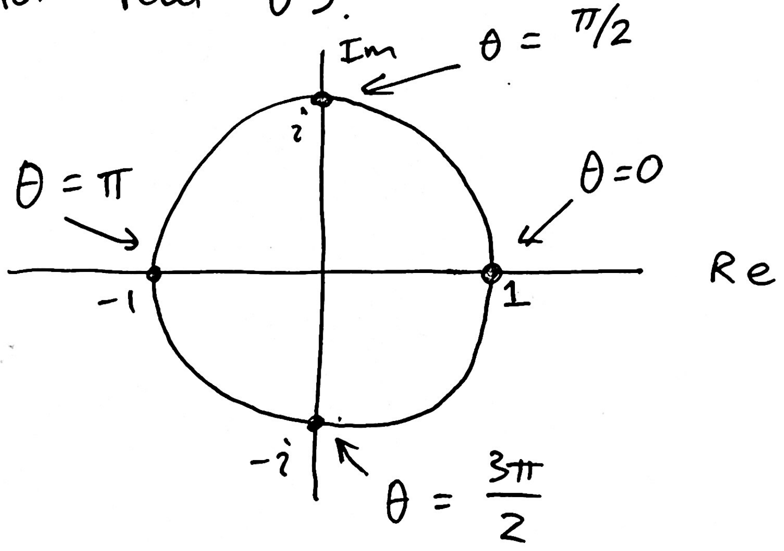
question: Show that the Taylor series for $e^{i\theta}$ agrees with the Taylor series for $\cos\theta + i\sin\theta$

$$e^{i\theta} \equiv 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \dots$$

Euler's Formula

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

question: Plot the image of $e^{i\theta}$ in the complex plane (for real θ).



question Write $\sin\theta$ and $\cos\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$.

→ ans: $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

question Show that $|e^{i\theta}| = 1$ for all θ.

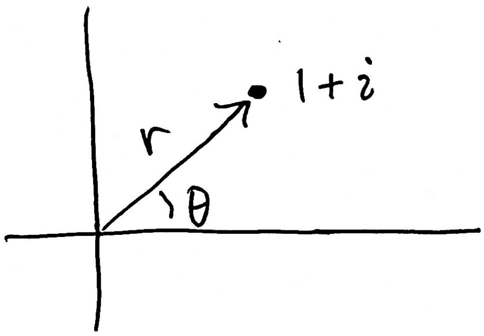
Claim Every complex number $z = x + iy$ can be written as

$$z = re^{i\theta}$$

for real numbers $r > 0$ and $\theta \in [0, 2\pi]$.

These are just polar coordinates!

question Write $1+i$ in $re^{i\theta}$ form.



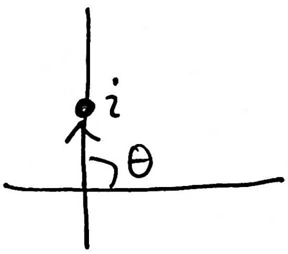
$$r = |z| = \sqrt{1+1} = \sqrt{2}$$

$$\theta = \arctan\left(\frac{1}{1}\right) = \pi/4$$

$$\Rightarrow \boxed{1+i = \sqrt{2}\, e^{i\pi/4}}$$

question Write i in $re^{i\theta}$ form

$r = 1, \quad \theta = \pi/2 \Rightarrow \boxed{i = e^{i\pi/2}}$



multiplication revisited $\quad z_1 = r_1 e^{i\theta_1} \qquad z_2 = r_2 e^{i\theta_2}$

$$\rightarrow \quad z_1 z_2 = r_1 r_2 \, e^{i(\theta_1 + \theta_2)}$$

→ Mystery "solved": $\boxed{\text{angles add, magnitudes multiply}}$

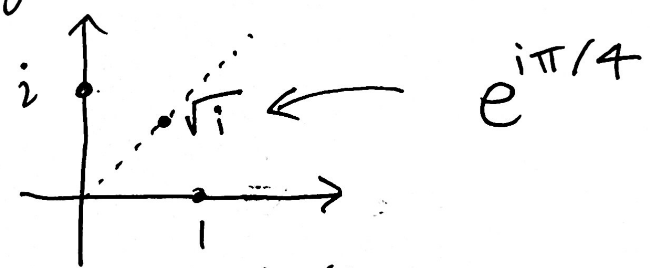
<u>question</u>: What is the action of $15-8i$ on the complex plane under multiplication?

→ Write as $\sqrt{15^2+8^2}\, e^{i\theta} = 17\, e^{-i(0.489)}$

→ scales the plane by 17 radially, rotates by 0.4899 rad CCW.

<u>question</u> Explain $\sqrt{i} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ "geometrically".

- $\sqrt{i}$ is that number where when you square it you get $i = e^{i\pi/2}$
- Since angles add and magnitudes multiply, $\sqrt{i}$ has magnitude 1 and angle $\pi/4$

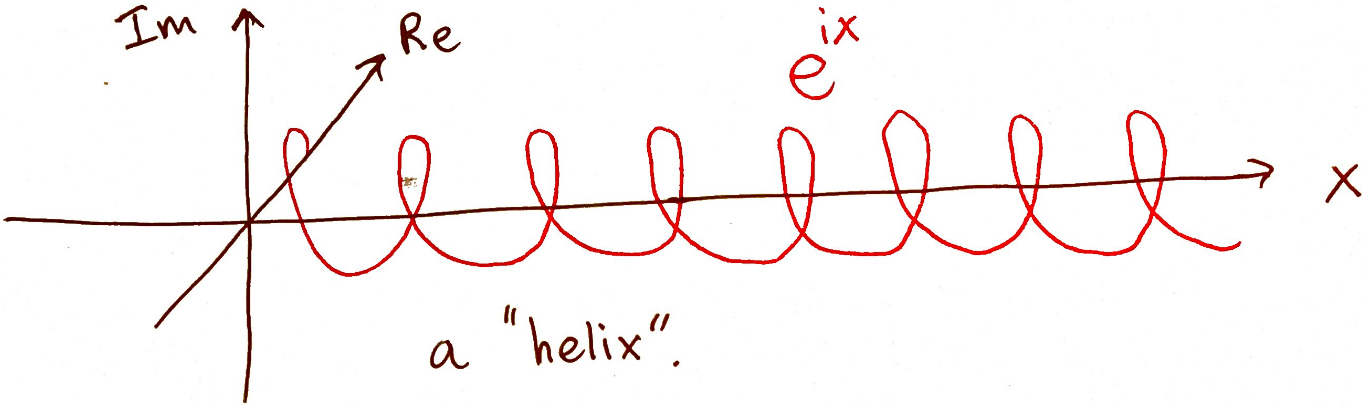


- via trigonometry, $e^{i\pi/4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$.

<u>question</u>: What is i^i?

$i^i = \left(e^{i\pi/2}\right)^i = e^{-\pi/2}$ (or is it?).

Plotting e^{ix} for real x



a "helix".

Problems :

① (True/false)

(a) $|z_1| + |z_2| = |z_1 + z_2|$

(b) $|z_1| \cdot |z_2| = |z_1 z_2|$

(c) $\frac{|z_1|}{|z_2|} = \left|\frac{z_1}{z_2}\right|$

② Plot all the solutions to $z^5 = 1$ in the complex plane & write them in $re^{i\theta}$ form.

③ Is it true that $Re(zw) = (Re z)(Re w)$?

④ Show that (a) $(zw)^* = z^* w^*$

(b) $\begin{cases} Re(z) = \dfrac{z + z^*}{2} \\ Im(z) = \dfrac{z - z^*}{2i} \end{cases}$

Sols

(1) (a) False. Counterexample: $|1| + |-1| = 2$
But $|1+(-1)| = 0$

(b) True. Let $z_1 = r_1 e^{i\theta_1}$ $z_2 = r_2 e^{i\theta_2}$. Then

$$|z_1|\cdot|z_2| = r_1 r_2$$

and

$$|z_1 \cdot z_2| = |r_1 r_2 e^{i\theta_1} e^{i\theta_2}| = r_1 r_2 \quad //$$

(c) True. Let $z_3 = 1/z_2$. Then

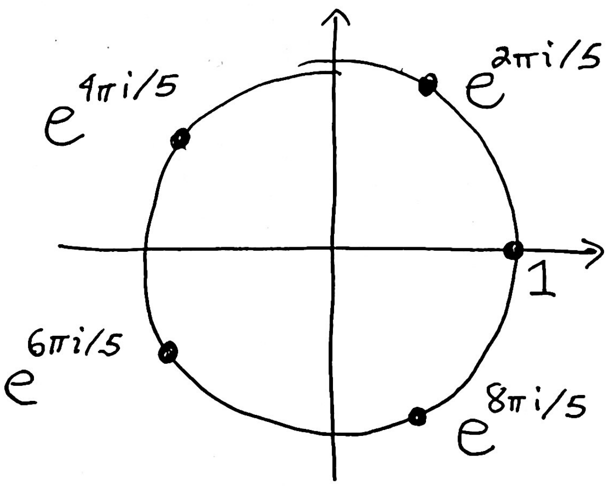
$$\left|\frac{z_1}{z_2}\right| = |z_1 \cdot z_3|$$

$$|z_1|/|z_2| = |z_1|\cdot|z_3|$$

But from (b) these are equal. //

(2) The equation $z^5 = 1$ has five solutions:

$$e^{i\frac{2\pi}{5}},\ e^{i\frac{4\pi}{5}},\ e^{i\frac{6\pi}{5}},\ e^{i\frac{8\pi}{5}},\ e^{i\frac{10\pi}{5}} = 1$$



(3) No, not in general. Let $z = a+ib$, $w = c+id$. then

$zw = (ac-bd) + i(bc+ad)$

so

$$\mathrm{Re}(zw) = ac-bd \quad \text{while} \quad \mathrm{Re}(z)\,\mathrm{Re}(w) = ac.$$

4 (a) Let $z = a + ib$, $w = c + id$. Then

$$(zw)^* = (ac - bd + i(bc + ad))^*$$
$$= (ac - bd) - i(bc + ad)$$

while

$$z^* w^* = (a - ib)(c - id) = (ac - bd) - i(bc + ad)$$

which agree. //

(b) Let $z = a + ib$. Then

$$\frac{z + z^*}{2} = \frac{(a + \cancel{ib}) + (a - \cancel{ib})}{2} = a$$

so $\mathrm{Re}(z) = (z + z^*)/2$.

Similarly

$$\frac{z - z^*}{2i} = \frac{(\cancel{a} + ib) - (\cancel{a} - ib)}{2i} = b$$

so $\mathrm{Im}(z) = (z - z^*)/2i$. //

Example

A particle in an infinite square well $-\frac{a}{2} < x < \frac{a}{2}$ is in a state

$$\Psi(x,0) = \begin{cases} A\cos^3\left(\frac{\pi x}{a}\right) & -\frac{a}{2} < x < \frac{a}{2} \\ 0 & \text{otherwise} \end{cases}$$

at time $t = 0$.

(a) Determine A to normalize Ψ

(b) Write $\Psi(x,0)$ as a linear combination of stationary states $\Psi(x,0) = \sum_{n=1}^{\infty} c_n \varphi_n(x)$ where

$$\varphi_n(x) = \begin{cases} \sqrt{\frac{2}{L}}\cos\left(\frac{n\pi x}{a}\right) & n \text{ odd} \\ -\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{a}\right) & n \text{ even} \end{cases}$$

(c) What is the state of the particle at a later time, $\Psi(x,t)$?

(d) Compute the average value of the particle's position $\langle \hat{x} \rangle$ as a function of time.

soln

(a) $$\int_{-a/2}^{a/2} dx\, |\Psi(x,0)|^2 = |A|^2 \int_{-a/2}^{a/2} dx\, \cos^6\left(\frac{\pi x}{a}\right) = |A|^2 \frac{5a}{16} \overset{!}{=} 1$$

$$\Rightarrow \boxed{A = \frac{4}{\sqrt{5a}}}$$

(b) Wolfram Alpha tells me that $\cos^3(x) = \dfrac{3\cos(x) + \cos(3x)}{4}$

so

$$\Psi(x,0) = \frac{4}{\sqrt{10}}\left(\frac{3}{4}\sqrt{\frac{2}{a}}\cos\left(\frac{\pi x}{a}\right) + \frac{1}{4}\sqrt{\frac{2}{a}}\cos\left(\frac{3\pi x}{a}\right)\right)$$

$$\boxed{\Psi(x,0) = \frac{3}{\sqrt{10}}\varphi_1(x) + \frac{1}{\sqrt{10}}\varphi_3(x)}$$

So $c_1 = 3/\sqrt{10}$, $c_3 = 1/\sqrt{10}$, and all other $c_n = 0$.

To derive this trig identity, note that

$$(e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3 = \cos^3\theta - i\sin^3\theta - 3\cos\theta\sin^2\theta + 3i\cos^2\theta\sin\theta$$

$$\parallel$$

$$e^{i3\theta} = \cos(3\theta) + i\sin(3\theta)$$

Two complex numbers are equal iff both their real and imaginary parts are equal, so

$$\cos^3\theta - 3\cos\theta\sin^2\theta = \cos(3\theta)$$

$$\Rightarrow \cos^3\theta - 3\cos\theta(1-\cos^2\theta) = \cos(3\theta)$$

$$\Rightarrow \cos^3\theta + 3\cos^3\theta - 3\cos\theta = \cos(3\theta)$$

$$\Rightarrow 4\cos^3\theta = \cos(3\theta) + 3\cos\theta$$

$$\Rightarrow \cos^3\theta = \frac{1}{4}\left(\cos(3\theta) + 3\cos\theta\right)$$

//

(c) At time t

$$\boxed{\Psi(x,t) = \frac{3}{\sqrt{10}}\varphi_1(x)\,e^{-iE_1t/\hbar} + \frac{1}{\sqrt{10}}\varphi_3(x)\,e^{-iE_3t/\hbar}}$$

where $E_n = \hbar^2\pi^2n^2/2ma^2$.

(d) First,

$$|\Psi(x,t)|^2 = \frac{9}{10}|\varphi_1(x)|^2 + \frac{1}{10}|\varphi_3(x)|^2 + \frac{6}{10}\varphi_1(x)\varphi_3(x)\cos\left(\frac{E_3-E_1}{\hbar}t\right)$$

where I used $|z|^2 = z^*z$, and $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$.

Next,

$$\langle x\rangle = \int_{-a/2}^{a/2} dx\ x\,|\Psi(x,t)|^2$$

The first term is $\int_{-a/2}^{a/2} dx\ x\ \frac{9}{10}|\varphi_1(x)|^2$. But $\varphi_1(x)$ is even about $x=0$, so this is zero.

The second term $\frac{1}{10}\int_{-a/2}^{a/2} dx\ x\ |\varphi_3(x)|^2$ is also zero, same reason.

The third term is

$$\cos\left(\frac{E_3-E_1}{\hbar}t\right)\frac{6}{10}\int_{-a/2}^{a/2} dx\ x\ \varphi_1(x)\varphi_3(x) = \frac{2}{a}\cos\left(\frac{E_3-E_1}{\hbar}t\right)\frac{6}{10}\underbrace{\int_{-a/2}^{a/2} dx\ x\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{3\pi x}{a}\right)}_{0}$$

$$\Rightarrow \boxed{\langle x\rangle = 0}$$

For all time.

Generically, $\langle x\rangle$ is not constant with time. This is a bit of a fluke.