Homework 7 Solutions

$\boxed{1}$ (Griffiths 1.16) parts c, e, g, h

For

$$\Psi(x) = \sqrt{\frac{15}{16a^5}}\,(a^2 - x^2)$$

(a) Compute the expectation value of momentum:

$$\langle p \rangle = \frac{\hbar}{i}|A|^2 \int_{-a}^{a} dx\ (a^2 - x^2)\underbrace{\frac{d}{dx}(a^2 - x^2)}_{2x} = \boxed{0}$$

Integrand is odd.

(e)

$$\langle p^2 \rangle = -\hbar^2 |A|^2 \int_{-a}^{a} dx\ (a^2 - x^2)\overbrace{\frac{d^2}{dx^2}(a^2 - x^2)}^{-2}$$

$$= 2\hbar^2 |A|^2 \int_{-a}^{a} dx\ (a^2 - x^2)$$

$$= 4\hbar^2 |A|^2 \left(a^2 x - \frac{1}{3}x^3\right)\Big|_0^a$$

$$= 4\hbar^2 \frac{15}{16a^5}\left(a^3 - \frac{1}{3}a^3\right) = 4\hbar^2 \frac{15}{16}\frac{2}{3}\frac{1}{a^2}$$

$$= \boxed{\frac{5}{2}\left(\frac{\hbar}{a}\right)^2}$$

(g)

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \boxed{\sqrt{\frac{5}{2}}\frac{\hbar}{a}}$$

(h) From last week, $\sigma_x = \frac{1}{\sqrt{7}}a$. So

$$\sigma_x\sigma_p = \frac{1}{\sqrt{7}}a\sqrt{\frac{5}{2}}\frac{\hbar}{a} = \boxed{\sqrt{\frac{10}{7}}\frac{\hbar}{2} > \frac{\hbar}{2}} \checkmark$$

$\boxed{2}$

(a) The four components of $\hat{S}_z$ in the $|\pm x\rangle$ basis are given by

$$\langle +x|S_z|+x\rangle, \quad \langle +x|S_z|-x\rangle, \quad \langle -x|S_z|+x\rangle, \quad \langle -x|S_z|-x\rangle$$

usually arranged into a 2×2 matrix like:

$$S_z \xrightarrow{|\pm x\rangle} \begin{pmatrix} \langle +x|S_z|+x\rangle & \langle +x|S_z|-x\rangle \\ \langle -x|S_z|+x\rangle & \langle -x|S_z|-x\rangle \end{pmatrix}$$

Compute these:

$$\langle +x|S_z|+x\rangle = \langle +x|S_z\left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle\right)$$

$$= \langle +x|\left(\frac{1}{\sqrt{2}}\frac{\hbar}{2}|+z\rangle - \frac{1}{\sqrt{2}}\frac{\hbar}{2}|-z\rangle\right)$$

$$= \frac{\hbar}{2}\langle +x|-x\rangle = \boxed{0}$$

$$\langle -x|S_z|+x\rangle = \frac{\hbar}{2}\langle -x|-x\rangle = \boxed{\hbar/2}$$

$\downarrow$ Hermiticity

$$\langle +x|S_z|-x\rangle = \boxed{\hbar/2}$$

$$\langle -x|S_z|-x\rangle = \langle -x|S_z\left(\frac{1}{\sqrt{2}}|+z\rangle - \frac{1}{\sqrt{2}}|-z\rangle\right)$$

$$= \langle -x|\frac{\hbar}{2}\left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle\right)$$

$$= \frac{\hbar}{2}\langle -x|+x\rangle = \boxed{0}$$

so:

$$\boxed{S_z \xrightarrow{|\pm x\rangle} \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

(b)

$$\langle x'|\hat{x}|x\rangle = x\langle x'|x\rangle$$

$$= x\,\delta(x'-x)$$ ✓

(c)

$$\langle x|\hat{p}|x'\rangle = \int dp\, \langle x|\hat{p}|p\rangle\langle p|x'\rangle$$

$$= \int dp\, p\,\langle x|p\rangle\langle p|x'\rangle$$

$$= \int dp\, p\, \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx'/\hbar}$$

$$= \frac{1}{2\pi\hbar}\int dp\, p\, e^{ip(x-x')/\hbar}$$ ✓

(d) Take the general statement

$$\langle x|\hat{p}|\psi\rangle = -i\hbar \frac{d}{dx}\langle x|\psi\rangle$$

and apply it to $|\psi\rangle = |x'\rangle$. Then:

$$\Rightarrow \langle x|\hat{p}|x'\rangle = -i\hbar \frac{d}{dx}\delta(x-x')$$

(e) Let $f(x)$ be smooth & differentiable. Then

Integrate by parts:

$$\int_{-\infty}^{\infty} dx\, f(x) \frac{d}{dx}\delta(x-x') = \underbrace{\left[f(x)\,\delta(x-x')\right]_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} dx \left(\frac{d}{dx} f(x)\right) \delta(x-x')$$

$$= \boxed{-f'(x')}$$

(f) Use the representation of the delta function as an integral:

$$\delta(x-x') = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp\, e^{ip(x-x')/\hbar}$$

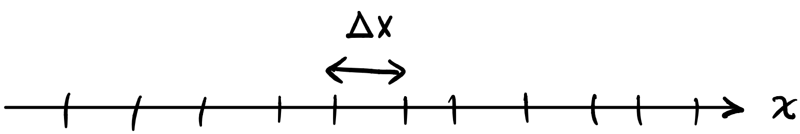
Now differentiate both sides w.r.t. x:

$$\Rightarrow \frac{d}{dx}\delta(x-x') = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp\,(ip/\hbar)\, e^{ip(x-x')/\hbar}$$

so

$$\boxed{-i\hbar\frac{d}{dx}\delta(x-x') = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp\; p\; e^{ip(x-x')/\hbar}}$$

which agrees with the previous expression.



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(a) We need the expression $\sum_{i=-\infty}^{\infty} |i\rangle\langle i| = \mathbb{1}$

to converge to the integral

$$\int_{-\infty}^{\infty} dx\; |x\rangle\langle x| = \mathbb{1}$$

in the limit $\Delta x \longrightarrow 0$. We need a dx, so we have to write:

$$\sum_{i=-\infty}^{\infty} |i\rangle\langle i| = \sum_{i=-\infty}^{\infty} \Delta x \frac{|i\rangle}{\sqrt{\Delta x}}\frac{\langle i|}{\sqrt{\Delta x}}$$

Then

$$\lim_{\Delta x\to 0}\frac{|i\rangle}{\sqrt{\Delta x}} = |x\rangle$$

so that

$$\langle x|\psi\rangle = \psi(x) \simeq \frac{\langle i|\psi\rangle}{\sqrt{\Delta x}}$$

(b) We have

$$\frac{d}{dx}\psi(x) = \lim_{\Delta x \to 0} \frac{\psi(x+\Delta x) - \psi(x-\Delta x)}{2\Delta x}$$

$$\simeq \frac{1}{2\Delta x}\left[\frac{\langle i+1|\psi\rangle}{\sqrt{\Delta x}} - \frac{\langle i-1|\psi\rangle}{\sqrt{\Delta x}} \right]$$

(c) With

$$P_{ij} = \langle i|P|j\rangle = -i\hbar \frac{\langle i+1|j\rangle - \langle i-1|j\rangle}{2\Delta x}$$

$$\boxed{P_{ij} = \frac{-i\hbar}{2\Delta x}\left[\delta_{i+1,j} - \delta_{i-1,j} \right]}$$