Homework 1 Due Tuesday, September 5th by midnight

1. (Griffiths A.1) Consider the ordinary vectors in three dimensions $a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$, with complex components.
   (a) Does the subset of all vectors with $a_z = 0$ constitute a vector space? If so, what is its dimension; if not, why not?
   (b) What about the subset of all vectors whose z component is 1? Hint: Would the sum of two such vectors be in the subset? How about the null vector?
   (c) What about the subset of vectors whose components are all equal?

   (a) Yes. To verify that this is a vector subspace we need only check that it is closed under vector addition and scalar multiplication. For vector addition we have: $(a_x, a_y, 0) + (a'_x, a'_y, 0) = (a_x + a'_x, a_y + a'_y, 0)$, which is in the space. And for scalar multiplication we have: $c(a_x, a_y, 0) = (c\, a_x, c\, a_y, 0)$, which is also in the space. This is a vector space of dimensions 2.
   (b) No. It is not closed under vector addition: $(a_x, a_y, 1) + (a'_x, a'_y, 1) = (a_x + a'_x, a_y + a'_y, 2)$, which is no longer in the space. It is also not closed under scalar multiplication.
   (c) Yes. It is closed under addition and multiplication, and it has dimension 1.

2. (Griffiths A.8) Given the following two matrices:

$$A = \begin{pmatrix} -1 & 1 & i \\ 2 & 0 & 3 \\ 2i & -2i & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{pmatrix}$$

   compute the following:
   (a) $A + B$
   (b) AB
   (c) $[A, B]$
   (d) $A^\intercal$. (Note: This stands for the transpose. Griffiths writes this as $\tilde{A}$).
   (e) A^*
   (f) $A^\dagger$
   (g) $\det(B)$
   (h) B^{-1}. Check that $BB^{-1} = I$. Does A have an inverse?

   (a) Compute

$$A + B = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ 3i & -2i + 3 & 4 \end{pmatrix}$$

   (b) Compute

$$AB = \begin{pmatrix} -3 & 1 + 3i & 3i \\ 4 + 3i & 9 & 6 - 2i \\ 6i & 6 - 2i & 6 \end{pmatrix}$$

   (c)

$$[A, B] = \begin{pmatrix} -2 - 6i & -2 + 3i & -3 - 9i \\ -14 - 6i & 9 + 9i & -30 - 2i \\ -12 + 4i & 6 + 4i & -18 + 6i \end{pmatrix}$$

(d)

$$A^{\intercal} = \begin{pmatrix} -1 & 2 & 2i \\ 1 & 0 & -2i \\ i & 3 & 2 \end{pmatrix}$$

(e)

$$A^{*} = \begin{pmatrix} -1 & 1 & -i \\ 2 & 0 & 3 \\ -2i & 2i & 2 \end{pmatrix}$$

(f)

$$A^{\dagger} = \begin{pmatrix} -1 & 2 & -2i \\ 1 & 0 & 2i \\ -i & 3 & 2 \end{pmatrix}$$

(g)

$$\det(B) = \begin{vmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{vmatrix} = 3$$

(h)

$$B^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -3i & i \\ 0 & 3 & 0 \\ -i & -6 & 2 \end{pmatrix}$$

One has

$$BB^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3i & i \\ 0 & 3 & 0 \\ -i & -6 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Also, one can compute that $\det(A) = 0$, so A has no inverse.

3. (Griffiths A.9) Using the square matrices in Problem 2, and the column matrices

$$a = \begin{pmatrix} i \\ 2i \\ 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ (1-i) \\ 0 \end{pmatrix},$$

find:

(a) Aa

(b) $a^{\dagger}b$

(c) $a^{\intercal}Ab$

(d) $ab^{\dagger}$

(a)

$$Aa = \begin{pmatrix} -1 & 1 & i \\ 2 & 0 & 3 \\ 2i & -2i & 2 \end{pmatrix}\begin{pmatrix} i \\ 2i \\ 2 \end{pmatrix} = \begin{pmatrix} 3i \\ 6+2i \\ 6 \end{pmatrix}$$

(b)

$$a^\dagger b = \begin{pmatrix} -i & -2i & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1-i \\ 0 \end{pmatrix} = -2 - 4i$$

(c)

$$a^\intercal A b = \begin{pmatrix} i & 2i & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 & i \\ 2 & 0 & 3 \\ 2i & -2i & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1-i \\ 0 \end{pmatrix} = -3 + 11i$$

(d)

$$ab^\dagger = \begin{pmatrix} i \\ 2i \\ 2 \end{pmatrix} \begin{pmatrix} 2 & 1+i & 0 \end{pmatrix} = \begin{pmatrix} 2i & -1+i & 0 \\ 4i & -2+2i & 0 \\ 4 & 2+2i & 0 \end{pmatrix} \tag{1}$$

4. (Griffiths A.13) Noting that $\det(A) = \det(A^\intercal)$, show that:
   (a) the determinant of a hermitian matrix is real
   (b) the determinant of a unitary matrix has modulus 1 (hence the name)
   (c) the determinant of an orthogonal matrix (footnote 13) is either 1 or -1.

   (a) Let H be a Hermitan matrix. Then

   $$\det(H) = \det\left(H^\dagger\right) = \det((H^*)^\intercal) = \det(H^*) = \det(H)^*$$

   Therefore $\det(H)$ is a real number.

   (b) Let U be a unitary matrix. Because U is unitary, it satisfies $UU^\dagger = I$. Take the determinant of both sides:

   $$\begin{aligned} 1 = \det(I) &= \det\left(UU^\dagger\right) \\ &= \det(U)\det(U)^* \\ &= |\det(U)|^2 \end{aligned}$$

   Therefore, $\det(U)$ is a number with magnitude 1.

   (c) Let R be an orthogonal matrix. Because it is orthogonal, it satisfies $RR^\intercal = I$. Take the determinant of both sides of this equation:

   $$\begin{aligned} 1 = \det(I) &= \det\left(RR^T\right) \\ &= \det(R)\det(R) \\ &= \det(R)^2 \end{aligned}$$

   Then $\det(R)$ is equal to either $+1$ or -1.

5. (Griffiths A.19) Find the eigenvalues and eigenvectors of the following matrix:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Can this matrix be diagonalized?

By inspecting the matrix, we see that $\begin{pmatrix} 1 & 0 \end{pmatrix}^\intercal$ is an eigenvector of the matrix, with eigenvalue 1. The matrix has no other eigenvectors. This can be seen by inspecting the characteristic equation

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0.$$

The only solution is $\lambda = 1$.

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6. (Griffiths A.27) A hermitian linear transformation must satisfy $\langle\alpha|T\beta\rangle = \langle T\alpha|\beta\rangle$ for all vectors $|\alpha\rangle$ and $|\beta\rangle$. Prove that it is (surprisingly) sufficient that $\langle\gamma|T\gamma\rangle = \langle T\gamma|\gamma\rangle$ for all vectors $|\gamma\rangle$. Hint: First let $|\gamma\rangle = |\alpha\rangle + |\beta\rangle$, and then let $|\gamma\rangle = |\alpha\rangle + i\,|\beta\rangle$.

Our goal is to show that the (seemingly weaker) condition $\langle\gamma|T\gamma\rangle = \langle T\gamma|\gamma\rangle$ actually implies the (seemingly stronger) condition $\langle\alpha|T\beta\rangle = \langle T\alpha|\beta\rangle$, and is therefore an equally good definition of a Hermitian operator. Suppose T is an operator that satisfies $\langle\gamma|T\gamma\rangle = \langle T\gamma|\gamma\rangle$ for all vectors $|\gamma\rangle$. Consider $|\gamma\rangle = |\alpha\rangle + |\beta\rangle$. Then

$$\begin{aligned}\langle\gamma|T\gamma\rangle &= (\langle\alpha| + \langle\beta|)(|T\alpha\rangle + |T\beta\rangle)\\ &= \langle\alpha|T\alpha\rangle + \langle\alpha|T\beta\rangle + \langle\beta|T\alpha\rangle + \langle\beta|T\beta\rangle\,.\end{aligned}$$

Meanwhile,

$$\begin{aligned}\langle T\gamma|\gamma\rangle &= (\langle T\alpha| + \langle T\beta|)(|\alpha\rangle + |\beta\rangle)\\ &= \langle T\alpha|\alpha\rangle + \langle T\alpha|\beta\rangle + \langle T\beta|\alpha\rangle + \langle T\beta|\beta\rangle\,.\end{aligned}$$

Since these are equal, we have:

$$\begin{aligned}\cancel{\langle\alpha|T\alpha\rangle} + \langle\alpha|T\beta\rangle + \langle\beta|T\alpha\rangle + \langle\beta|T\beta\rangle &= \cancel{\langle T\alpha|\alpha\rangle} + \langle T\alpha|\beta\rangle + \langle T\beta|\alpha\rangle + \langle T\beta|\beta\rangle\\ \langle\alpha|T\beta\rangle + \langle\beta|T\alpha\rangle + \cancel{\langle\beta|T\beta\rangle} &= \langle T\alpha|\beta\rangle + \langle T\beta|\alpha\rangle + \cancel{\langle T\beta|\beta\rangle}\\ \langle\alpha|T\beta\rangle + \langle\beta|T\alpha\rangle &= \langle T\alpha|\beta\rangle + \langle T\beta|\alpha\rangle\end{aligned}$$

where we have used the property that T satisfies to cancel some terms. Now consider the case when $|\gamma'\rangle = |\alpha\rangle + i\,|\beta\rangle$. Then

$$\begin{aligned}\langle\gamma'|T\gamma'\rangle &= (\langle T\alpha| - i\,\langle T\beta|)(|\alpha\rangle + i\,|\beta\rangle)\\ &= \langle T\alpha|\alpha\rangle + i\,\langle T\alpha|\beta\rangle - i\,\langle T\beta|\alpha\rangle + \langle T\beta|\beta\rangle\,.\end{aligned}$$

Meanwhile,

$$\begin{aligned}\langle T\gamma'|\gamma'\rangle &= (\langle\alpha| - i\,\langle\beta|)(|T\alpha\rangle + i\,|T\beta\rangle)\\ &= \langle\alpha|T\alpha\rangle + i\,\langle\alpha|T\beta\rangle - i\,\langle\beta|T\alpha\rangle + \langle\beta|T\beta\rangle\,.\end{aligned}$$

Since these are equal, we have

$$\begin{aligned}\cancel{\langle T\alpha|\alpha\rangle} + i\,\langle T\alpha|\beta\rangle - i\,\langle T\beta|\alpha\rangle + \cancel{\langle T\beta|\beta\rangle} &= \cancel{\langle\alpha|T\alpha\rangle} + i\,\langle\alpha|T\beta\rangle - i\,\langle\beta|T\alpha\rangle + \cancel{\langle\beta|T\beta\rangle}\\ i\,\langle T\alpha|\beta\rangle - i\,\langle T\beta|\alpha\rangle &= i\,\langle\alpha|T\beta\rangle - i\,\langle\beta|T\alpha\rangle\end{aligned}$$

Divide this equation by i:

$$\langle T\alpha|\beta\rangle - \langle T\beta|\alpha\rangle = \langle\alpha|T\beta\rangle - \langle\beta|T\alpha\rangle\,.$$

Now add this equation to what we found for the case using $|\gamma\rangle$:

$$\begin{array}{rl} & \langle\alpha|T\beta\rangle + \langle\beta|T\alpha\rangle = \langle T\alpha|\beta\rangle + \langle T\beta|\alpha\rangle\\ + & \langle\alpha|T\beta\rangle - \langle\beta|T\alpha\rangle = \langle T\alpha|\beta\rangle - \langle T\beta|\alpha\rangle\\ \hline \end{array}$$

$$\implies \langle\alpha|T\beta\rangle = \langle T\alpha|\beta\rangle$$

which is what we wanted to show. Note that this would not work for a real vector space (complex numbers are required).