Homework 4 Due Monday, September 25th, by midnight

1. *More filters = ...less filtering?* Another important physical example of a two-dimensional Hilbert space is *photon polarization.* A photon travelling in the $+\boldsymbol{z}$ direction has two linear polarizations, corresponding to an electric field that oscillates purely along the x direction or purely along the y direction. I'll denote these states as $|X\rangle$ and $|Y\rangle$, respectively. They satisfy $\langle X|Y\rangle = 0$. Linear polarizations at other angles are real superpositions of these states. In particular, consider linear polarizations along axes x', y' which are related to the x, y directions by a counterclockwise rotation ϕ about the $+z$ direction. These are related to the old polarization states by

$$|X'\rangle = \cos\phi\,|X\rangle + \sin\phi\,|Y\rangle \tag{1}$$

$$|Y'\rangle = -\sin\phi\,|X\rangle + \cos\phi\,|Y\rangle\,. \tag{2}$$

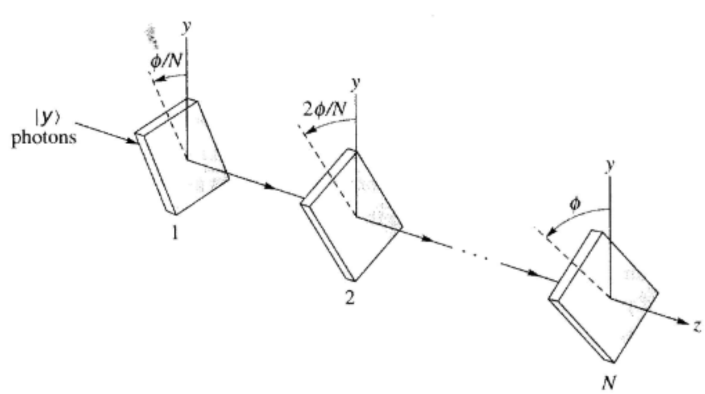
One can measure the (linear) polarization along a certain axis by passing the photon through a (linear) polarizing filter. In this case, photons measured to have their polarization aligned with the filter will pass through it, while photons measured to have their polarizations perpendicular to the filter will be absorbed by the filter.

You might wonder about complex linear combinations. These correspond to circularly polarized light: $|R\rangle = (|X\rangle + i\,|Y\rangle)/\sqrt{2}$ and $|L\rangle = (|X\rangle - i\,|Y\rangle)/\sqrt{2}$ (for right- and left-circularly polzarized light). The most general linear combination is 'elliptical' polarization, which is somewhere between these cases.

   (a) Suppose unpolarized light is passed through a sequence of two filters with their polarization axes arranged in the sequence: x, y. What fraction of the photons will pass through both filters?

   (b) Now suppose unpolarized light is passed through a sequence of three filters with polarization axes y, then $x+y$, then x. (Here $x+y$ represents a direction $+45°$ from the x axis). What fraction of photons will pass through all three filters? For context, and to see this in action, here is a nice video doing this experiment https://www.youtube.com/watch?v=5SIxEiL8ujA.

   Figure 1: A sequence of polarizing filters. Figure reproduced from *Townsend Quantum Mechanics a Modern Approach 2nd*.



   Now, suppose a sequence of N ideal linear polarizers is arranged as shown. The transmission axis of the first polarizer makes an angle of ϕ/N with the y axis. The transmission axis of every other polarizer makes an angle of ϕ/N with respect to the axis of the preceding one. Thus, the transmission axis of the final polarizer makes an angle ϕ with the y axis. A beam of y-polarized photons is incident on the first polarizer.

   (c) What is the probability that an incident photon is transmitted by the array?

   (d) Evaluate the probability of transmission in the limit of large N. This is one way to 'rotate' a photon.

2. *Commutator Identities* For any operators $\hat{A}, \hat{B}, \hat{C}$, show that the following identities hold:

   (a) $[\hat{A}, \hat{B} + \hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$.

   (b) $[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}$. Notice how much this looks similar to the product rule for derivatives. This is an easy way to remember it.

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(c) (Optional, ungraded). $[\hat{A},[\hat{B},\hat{C}]]+[\hat{B},[\hat{C},\hat{A}]]+[\hat{C},[\hat{A},\hat{B}]]=0$. This is called the Jacobi identity.

(d) (Optional, ungraded). Suppose that $[\hat{A},\hat{B}]=c$, where c is just a number (not an operator). Let $f(\hat{B})$ be a function of the operator $\hat{B}$ that can be Taylor expanded. Show that:

$$[\hat{A},f(\hat{B})]=[\hat{A},\hat{B}]\frac{\partial f}{\partial B} \tag{3}$$

Hint: Write $f(\hat{B})$ as a power series and use part b. Notice that this looks just like the chain rule for derivatives.

3. (Griffiths 3.33) *Sequential Measurements*

An operator $\hat{A}$, representing observable A, has two (normalized) eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$, with eigenvalues a_1 and a_2, respectively. Operator $\hat{B}$, representing observable B, has two (normalized) eigenstates $|\phi_1\rangle$ and $|\phi_2\rangle$, with eigenvalues b_1 and b_2, respectively. The eigenstates are related by

$$|\psi_1\rangle=(3|\phi_1\rangle+4|\phi_2\rangle)/5, \qquad |\psi_2\rangle=(4|\phi_1\rangle-3|\phi_2\rangle)/5. \tag{4}$$

(a) Let's say you measure A and observe the value a_1. What is the state of the system (immediately) after this measurement?

(b) If you now measure B, what are the possible results, and what are their probabilities?

(c) Right after the measurement of B, you measure A again. What is the probability of getting a_1? (Note that the answer would be quite different if I had told you the outcome of the B measurement).

4. A spin-1/2 particle is repeatedly prepared in a particular spin state $|\psi\rangle$. It is known that there is a 36% probability of obtaining $\hat{S}_z=\hbar/2$ and therefore a 64% chance of obtaining $\hat{S}_z=-\hbar/2$ if a measurement of $\hat{S}_z$ is carried out. In addition, it is known that the probability of finding the particle with $\hat{S}_x=\hbar/2$, that is in the state $|+\boldsymbol{x}\rangle$, is 50%. Determine the state of the particle as completely as possible from this information.