Homework 2 solutions

$\boxed{1}$ Is it a vector space?

(a) {polynomials of degree $\leq N$}.

$\boxed{\text{Yes (dim. } N+1)}$

(b) {polynomials of degree $\leq N$ w/ leading coeff. 1}.

$\boxed{\text{No. Space lacks the zero vector}}$

(c) {polynomials of degree $\leq N$ s.t. $p(0)=0$}.

$\boxed{\text{Yes. Dim} = N}$

(d) {polynomials of degree $\leq N$ s.t. $p(0) = 1$}.

$\boxed{\text{No. No zero vector.}}$

(e) {polynomials of degree $\leq N$ s.t. $\int_0^1 p(x)dx = 0$.

$\boxed{\text{Yes. Dim} = N}$

(f) {polynomials of degree $\leq N$ s.t. $p(0) = p(1)$}

$\boxed{\text{Yes. Dim} = N}$

$\boxed{2}$ (a) Let A and B be Hermitian. Then:

$$\begin{aligned}
\langle u | ABv \rangle &= \langle A^\dagger u | Bv \rangle \\
&= \langle Au | Bv \rangle \\
&= \langle B^\dagger A u | v \rangle \\
&= \langle BAu | v \rangle
\end{aligned}$$

This is equal to $\langle ABu | v \rangle$ if and only if $[A, B] = 0$, in which case $(AB)^\dagger = AB$.

(b) Let A be <u>any</u> operator. Then:

$$\begin{aligned}
\langle u | A^\dagger A v \rangle &= \langle Au | Av \rangle \\
&= \langle A^\dagger A u | v \rangle
\end{aligned}$$

$$\Rightarrow \boxed{A^\dagger A \text{ is Hermitian.}}$$

Now let $|a\rangle$ be an eigenvector such that:

$$(A^\dagger A)|a\rangle = a|a\rangle.$$

Consider:

$$\langle a | (A^\dagger A) | a \rangle = a \langle a | a \rangle$$

1 without loss of generality.

$$\langle a | (A^\dagger A) | a \rangle = \langle Aa | Aa \rangle = \| |Aa\rangle \|^2 \geq 0$$

$\Rightarrow$ $\boxed{a \geq 0}$ non-negative eigenvalues.

$\boxed{3}$ (a) compute e^M.

(i) $M = \begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$.

$$M^2 = \begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M^3 = \begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow e^M = I + M + \tfrac{1}{2}M^2 = \boxed{\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}}$$

(ii) $M = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} = -i\theta\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$M^2 = (-i\theta)^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (-i\theta)^2 I.$$

$$\Rightarrow M^k = (-i\theta)^k \times \begin{cases} I & \text{if } k \text{ even.} \\ \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \text{if } k \text{ odd.} \end{cases}$$

so:

$$e^M = \sum_{k=0,2,4,\ldots} \frac{1}{k!}(-i\theta)^k I + \sum_{k=1,3,5,\ldots} \frac{1}{k!}(-i\theta)^k \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \cos\theta \, I - i\sin\theta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}$$

(b) Let M and N commute. Then

$$e^M e^N = \left(\sum_{k=0}^{\infty} \frac{1}{k!} M^k \right) \left(\sum_{n=0}^{\infty} \frac{1}{n!} N^n \right)$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{k!\,n!} M^k N^n$$

Now:

Binomial theorem ☆ Only works b/c M & N commute

$$\frac{1}{m!}(M+N)^m = \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} M^k N^{m-k}$$

$$= \frac{1}{\cancel{m!}} \sum_{k=0}^{m} \frac{\cancel{m!}}{k!(m-k)!} M^k N^{m-k}$$

$$= \sum_{k=0}^{m} \frac{1}{k!(m-k)!} M^k N^{m-k}$$

Therefore:

$$e^M e^N = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{k!\,n!} M^k N^n$$

$$= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \frac{1}{k!\,(m-k)!} M^k N^{m-k} \right)$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} (M+N)^m$$

$$= \boxed{e^{M+N}}$$

(c) Let H be Hermitian. Then:

$$(e^{iH})^{\dagger} = \left(\sum_{k=0}^{\infty} \frac{1}{k!} (iH)^k \right)^{\dagger}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} (-iH)^k$$

$$= e^{-iH}$$

$$\Rightarrow (e^{iH})^{\dagger} e^{iH} = e^{-iH} e^{iH} = e^{0} = I$$

$$\Rightarrow \boxed{e^{iH} \text{ is unitary}}$$

$\boxed{4}$ (a) $\langle U\alpha | U\beta \rangle = \langle \alpha | \cancel{U^\dagger U}^{\,1} \beta \rangle = \langle \alpha | \beta \rangle$.

(b) Let $U|\lambda\rangle = \lambda|\lambda\rangle$. Then:

$$1 = \langle \lambda | \lambda \rangle = \langle U\lambda | U\lambda \rangle = \lambda^* \lambda \langle \lambda | \lambda \rangle$$
$$= |\lambda|^2$$

$$\Rightarrow \boxed{|\lambda|^2 = 1}$$

(c) Let $|\lambda_1\rangle$ and $|\lambda_2\rangle$ be eigenvectors of U so that:

$$U|\lambda_1\rangle = \lambda_1 |\lambda_1\rangle$$
$$U|\lambda_2\rangle = \lambda_2 |\lambda_2\rangle$$

Then:

$$\langle \lambda_1 | \lambda_2 \rangle = \langle U\lambda_1 | U\lambda_2 \rangle$$
$$= \lambda_1^* \lambda_2 \langle \lambda_1 | \lambda_2 \rangle$$

Note that $\lambda_1^* = 1/\lambda_1$ (since $\lambda_1^* \lambda_1 = 1$). So

$$\Rightarrow \lambda_1 \langle \lambda_1 | \lambda_2 \rangle = \lambda_2 \langle \lambda_1 | \lambda_2 \rangle$$

$$\Rightarrow (\lambda_1 - \lambda_2) \langle \lambda_1 | \lambda_2 \rangle = 0$$

$$\Rightarrow \boxed{\text{either } \lambda_1 = \lambda_2 \quad \text{or} \quad \langle \lambda_1 | \lambda_2 \rangle = 0}$$

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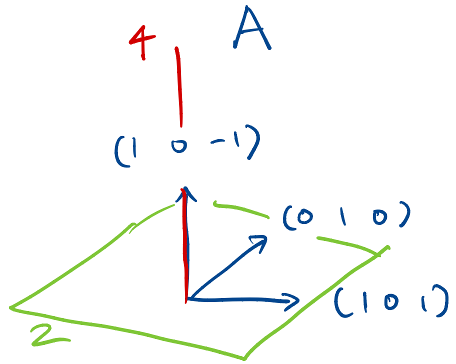

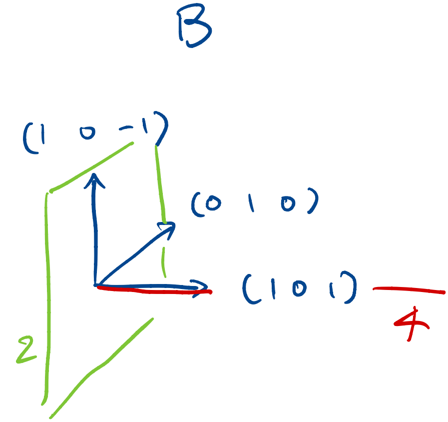

(a) $A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

* Both A and B have eigenvectors:

$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}$ $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\frac{1}{\sqrt{2}}$

with eigenvalues:

$a_1 = 2$ $a_2 = 2$ $a_3 = 4$

$b_1 = 4$ $b_2 = 2$ $b_3 = 2$

(b) Let C be normal such that $[A,C] = [B,C] = 0$.

* Since C is normal it has 3 orthogonal eigenvectors.

* Let $\varphi_1, \varphi_2, \varphi_3$ be e-vectors, and c_1 c_2 c_3 be e-vals of C.

* Since $[A,C]=0$, it is possible to find an ON-basis in which both A and C are diagonal.

* Since $(4, v_3)$ is a non-degenerate eval/evector of A, one of the e-vectors of C must point in the v_3 direction. Without loss of generality, set $\varphi_1 = v_3$

* Similarly, since $[B, C] = 0$, and $(4, v_1)$ is a non-degenerate eval/evector of B, WLOG set $\varphi_2 = v_1$.

* This leaves only φ_2, with is fixed in the direction $\vec{\varphi}_1 \times \vec{\varphi}_2$ so as to be orthonormal.

$\Rightarrow$ C must have eigenvectors v_1 v_2 v_3 and so is diagonal in the (v_1, v_2, v_3) basis. //

(c) No. For example, the matrix

$$F = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 4 & 2 \\ 4 & 8 & 4 \\ 2 & 4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 5 & 3 \\ 5 & 9 & 5 \\ 3 & 5 & 3 \end{pmatrix}$$

commutes with A, but in the (v_1, v_2, v_3)

basis it is:

$$F \longrightarrow \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 3 & 5 & 3 \\ 5 & 9 & 5 \\ 3 & 5 & 3 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 6/\sqrt{2} & 10/\sqrt{2} & 6/\sqrt{2} \\ 5 & 9 & 5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 10/\sqrt{2} & 0 \\ 10/\sqrt{2} & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Not diagonal.