# Final Review Problems: Solutions

$\boxed{1}$

1. **Enlarged Square Well**. A particle of mass $m$ sits in the ground state of a 1D infinite square well of width $a$. At time $t = 0$, the right wall of the box is suddenly moved to $x = 2a$, thereby instantly doubling the width of the box. You may assume that the expansion happens so quickly that the wavefunction has no time to change.

   (a) What is the probability of measuring the energy to be $E_2' = E_1 = 4\hbar^2\pi^2/[2m(2a)^2]$ (the first excited state of the new potential) at $t > 0$?

   (b) Find an expression for the time-dependent wavefunction $\Psi(x, t)$ for $t > 0$.

   (c) Find the expectation value of the energy $\langle E\rangle$ for $t > 0$. You need not do computations if you can give a detailed justification of your answer.

(a) $E_2'$ corresponds non-degenerately to the eigenstate $\psi_2'(x)$, so the probability to measure the energy to be $E_2'$ is

$$|\langle\psi_2'|\psi(x,0)\rangle|^2 = \left|\int_0^a dx\,\sqrt{\frac{2}{2a}}\sin\left(\frac{2\pi x}{2a}\right)\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}\right|^2 = \boxed{1/2}$$

(b) Let $\psi_n'(x) = \sqrt{\frac{1}{a}}\sin\left(\frac{n\pi x}{2a}\right)$ denote the eigenfunctions of the enlarged well for $t \geq 0$, and let $E_n' = \hbar^2\pi^2 n^2/2m(2a)^2$ their energies.

Let $c_n = \langle \psi_n' | \psi(x,0) \rangle$

$$= \int_0^a dx \sqrt{\frac{1}{a}} \sin\left(\frac{n\pi x}{2a}\right) \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$$

- For $n=2$, we found $c_2 = 1/\sqrt{2}$.
- For $n =$ even, the integral is zero.
- For $n =$ odd, we have:

$$c_n = \frac{4\sqrt{2}}{\pi(n^2-4)}(-1)^{\frac{n+1}{2}}$$

so:

$$\boxed{\psi(x,t) = \sum_{n=1}^{\infty} c_n e^{-iE_n' t/\hbar} \psi_n'(x)}$$

where

$$\boxed{c_n = \begin{cases} 1/\sqrt{2} & n = 2 \\ 0 & n \text{ even}, \ n \neq 2 \\ \frac{4\sqrt{2}}{\pi^2(n^2-4)}(-1)^{(n+1)/2} & n \text{ odd} \end{cases}}$$

(c) $\langle E \rangle$ is same before and after $t=0$, since

$$\langle E \rangle = \int_0^a dx \; \psi(x,0)^* \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \cancelto{0}{V(x)} \right) \psi(x,0)$$

is the same just before/after. So

$$\boxed{\langle E \rangle = E_1 = 4E_1'}$$

2. At time $t = 0$, a particle is placed in a one-dimensional infinite square well of width $L$ in the following superposition of the $n = 1$ and $n = 2$ states:

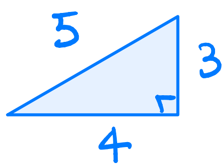
$$\Psi(x, 0) = \frac{3}{5}\psi_1(x) + A\psi_2(x)$$

where $A$ is an unknown constant.

(a) Determine the constant $A$ as fully as possible from normalization.

(b) Write an expression for the wavefunction at a later time, $t$.

(c) The system is repeatedly prepared in the aforementioned state, and a measurement of the position of the particle at time $T = \pi m L^2/(3\pi^2\hbar)$ later yields an average result of $x = L/2$ (the center of the box). Use this information to determine the constant $A$ more fully.




(a) $|A| = \frac{4}{5}$

(b) For $t > 0$,

$$\boxed{\Psi(x,t) = \frac{3}{5} e^{-iE_1 t/\hbar} \psi_1(x) + A e^{-iE_2 t/\hbar} \psi_2(x)}$$

(c) Compute

$$\langle x\rangle = \int_0^a dx\ \Psi^*(x,t)\ x\ \Psi(x,t)$$

$$= \left|\frac{3}{5}\right|^2 \underbrace{\int_0^a dx\ |\psi_1(x)|^2 x}_{L/2} + \left|\frac{4}{5}\right|^2 \underbrace{\int_0^a dx\ |\psi_2(x)|^2 x}_{L/2}$$

$$+\left(\frac{3}{5}\frac{4}{5}e^{i\varphi+i\omega t} + \frac{3}{5}\frac{4}{5}e^{-i\varphi-i\omega t}\right)\underbrace{\int_0^a dx\ \psi_1\psi_2 x}_{-\frac{16}{9\pi^2}L}$$

$$= \frac{L}{2} - \frac{12}{25}\cos(\omega t+\varphi)\frac{16}{9\pi^2}L$$

Set $t = T = \pi m L^2/(3\pi^2\hbar)$:

$$\omega T = \frac{E_2-E_1}{\hbar}T = \frac{\cancel{3\pi^2\hbar}}{2\cancel{mL^2}}\frac{\pi \cancel{mL^2}}{\cancel{3\pi^2\hbar}} = \frac{\pi}{2}$$

so at $t=T$,

$$\langle x\rangle = \frac{L}{2} - \underbrace{\frac{12}{25}\cos\left(\frac{\pi}{2}+\varphi\right)\frac{16}{9\pi^2}L}_{0}$$

$$\Rightarrow \varphi = 0 \quad \text{or} \quad \pi$$

$$\Rightarrow \boxed{A = \pm \frac{4}{5}}$$

4. A particle of mass $m$ is placed in a finite spherical well in three dimensions:

$$V(r) = \begin{cases} -V_0 & \text{if } r < a \\ 0 & \text{otherwise} \end{cases}.$$

Find the ground state, by solving the radial equation with $\ell = 0$. What is the minimum value of $V_0$ for which a bound state exists?

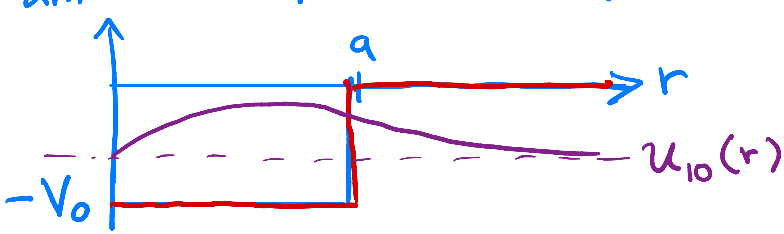
The energies/eigenstates are governed by the radial equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\ell(\ell+1)\hbar^2}{2mr^2} + V(r)\right]u(r) = Eu$$

For $\ell = 0$, we solve:

$$-\frac{\hbar^2}{2m}u'' - V_0 = -|E|u \quad \text{(inside)}$$

$$-\frac{\hbar^2}{2m}u'' = -|E|u \quad \text{(outside)}$$



- The most general solutions in each region are:

$$u = \begin{cases} A\sin(kr) + \cancel{B\cos(kr)} & r<a \quad \text{Physically unacceptable} \\ C e^{-\kappa r} + \cancel{D e^{\kappa r}} & r>a \quad \text{Physically unacceptable} \end{cases}$$

where:

$$\begin{cases} k = \dfrac{\sqrt{2m(V_0 - |E|)}}{\hbar} \\ \kappa = \dfrac{\sqrt{2m|E|}}{\hbar} \end{cases}$$

- Continuity at $r=a$ implies:

$$A\sin(ka) = C e^{-\kappa a}$$

- Continuity of $u'$ at $r=a$ implies:

$$kA\cos(ka) = -\kappa C e^{-\kappa a}$$

- Dividing these:

$$\Rightarrow \frac{1}{k}\tan(ka) = -\frac{1}{\kappa}$$

Notice that

$$k^2 + \kappa^2 = \frac{2m(V_0 - |E|)}{\hbar^2} + \frac{2m|E|}{\hbar^2} = \frac{2mV_0}{\hbar^2}$$

so

$$\kappa = \sqrt{\frac{2mV_0}{\hbar^2} - k^2}$$

and

$$\rightarrow \boxed{\sqrt{z_0^2 - z^2}\,\tan(z) = -z}$$

where I've defined dimensionless variables:

$$z = ka$$

$$z_0 = \frac{\sqrt{2mV_0}}{\hbar}\,a$$

This is a transcendental equation, so we can't solve it analytically. The ground state is

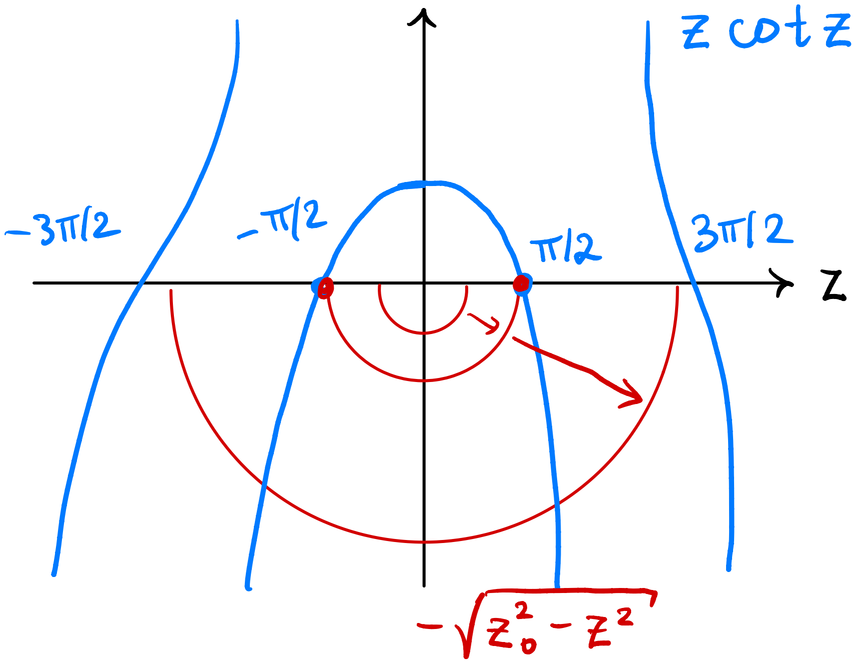
$$u = \begin{cases} A\sin(kr) & r < a \\ C\,e^{-\kappa r} & r > a \end{cases}$$

with $A, C, k$ determined by the transcendental equation.

The equation can be written as:

$$-\sqrt{z_0^2 - z^2} = z\cot(z)$$

Circle of radius $z_0$




A solution first begins to exist when

$$z_0 = \frac{\pi}{2}$$

$$\Rightarrow \frac{\sqrt{2mV_0}}{\hbar} a = \frac{\pi}{2}$$

$$\Rightarrow \boxed{V_0 = \frac{\pi^2\hbar^2}{8ma^2}}$$

Minimum $V_0$.

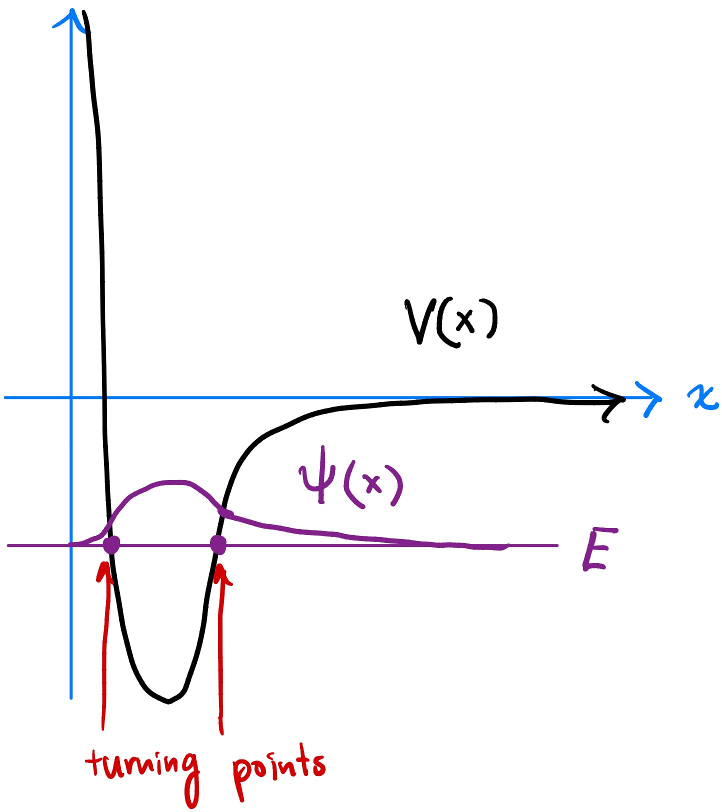
3. A particle of mass $m$ is subject to a potential

$$V(x) = \begin{cases} -\frac{A}{x^4} + \frac{B}{x^8} & x > 0 \\ \infty & x < 0 \end{cases}$$

(a) Sketch the potential. Assuming the potential supports at least one bound state, sketch the corresponding energy eigenfunction, marking any classical turning points.

(b) Make a very rough estimate of the ground state energy. There are several strategies for doing this, so just pick one and explain.

(a)




(b) 1 Dimensional analysis

↳ The only combination of A and B with units of energy is $A^2/B$ so

$$\Rightarrow \boxed{E_1 \cong -A^2/B}$$

## 2 Graphical

↳ From the picture, we know

$$V_{min} < E < 0$$

and we can compute:

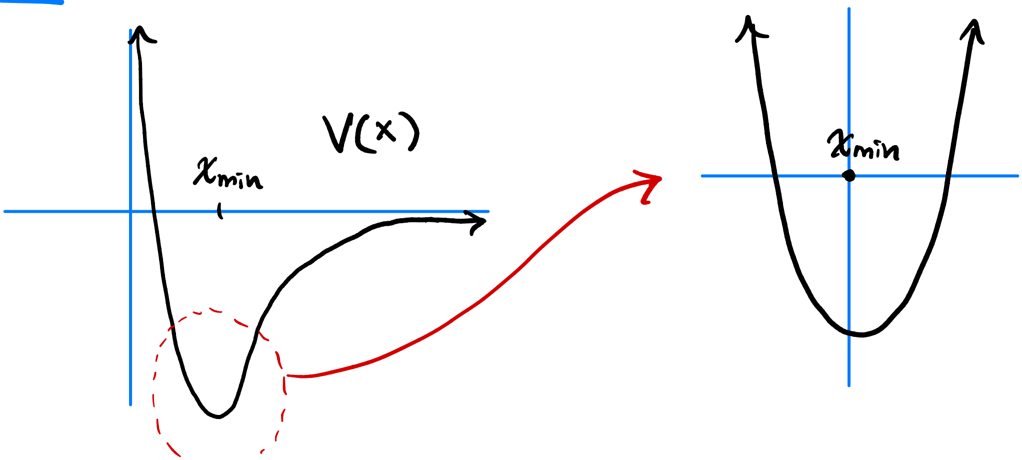
$$\frac{dV}{dx} = 0 \Rightarrow x = \left(\frac{2B}{A}\right)^{1/4}$$

$$\rightarrow V_{min} = \frac{-A^2}{4B^2}$$

so

$$\boxed{E \cong -\frac{A^2}{4B}}$$

## 3 Harmonic Oscillator Approximation




- Near $x_{min}$ it looks like a harmonic oscillator.

• The curvature $\frac{1}{2}V''(x_{min})$ tells us $\omega$. We have

$$V'' = -\frac{20A}{x^6} + \frac{72B}{x^{10}}\Bigg|_{x_{min}}$$

$$= \left(\frac{72}{2^{5/2}} - \frac{20}{2^{3/2}}\right)\frac{A^{5/2}}{B^{3/2}} \stackrel{!}{=} m\omega^2$$

$$\Rightarrow \omega \cong \sqrt{\frac{1}{m}\frac{A^{5/2}}{B^{3/2}}\,5.65}$$

so

$$\Rightarrow \boxed{E_1 \simeq -\frac{A^2}{4B} + \frac{\hbar}{2}\sqrt{\frac{5.65}{m}}\frac{A^{5/4}}{B^{3/4}} + \ldots}$$

overkill...

5. Find $\langle r \rangle$ for an electron in the ground state of hydrogen. Express your answer in terms of the Bohr radius. The normalized ground state is:

$$R_{10}(r) = \frac{2}{a^{3/2}}e^{-r/a}$$

$$\langle r \rangle = 4a^{-3}\int_0^\infty dr\, r^3 e^{-2r/a} = 4a^{-3}\frac{3}{8}a^4 = \boxed{\frac{3}{2}a}$$

6. (a) Given the two angular momenta $j_1 = 1$ and $j_2 = 3/2$, make a table containing two columns. The first column should list all possible uncoupled states $|j_1 m_1 j_2 m_2\rangle$ in order of descending $m_1 + m_2$ (so start with the highest possible value). The second column should give all possible coupled states $|(j_1 j_2) jm\rangle$, ordered in terms of descending $m$. How many states of each kind are there?

(b) Express the state $|(j_1 = 1 j_2 = 3/2) j = 5/2 m = 3/2\rangle$ as a sum of uncoupled states.

(c) What are the possible values of $j_{2z}$ for the state constructed in b)? If you do a measurement of $j_{2z}$ , with what probability would you get these answers?

(a)

| uncoupled | coupled | $m_1+m_2$ |
|---|---|---|
| $\|1 1, \frac{3}{2} \frac{3}{2}\rangle$ | $\|\frac{5}{2}, \frac{5}{2}\rangle$ | $\frac{5}{2}$ |
| $\|1 0, \frac{3}{2} \frac{3}{2}\rangle$ | $\|\frac{5}{2}, \frac{3}{2}\rangle$ | $\frac{3}{2}$ |
| $\|1 1, \frac{3}{2}, \frac{1}{2}\rangle$ | $\|\frac{3}{2}, \frac{3}{2}\rangle$ | |
| $\|1 {-1}, \frac{3}{2} \frac{3}{2}\rangle$ | $\|\frac{5}{2}, \frac{1}{2}\rangle$ | $\frac{1}{2}$ |
| $\|1 0, \frac{3}{2} \frac{1}{2}\rangle$ | $\|\frac{3}{2}, \frac{1}{2}\rangle$ | |
| $\|1 1, \frac{3}{2} \frac{-1}{2}\rangle$ | $\|\frac{1}{2}, \frac{1}{2}\rangle$ | |
| $\|1 {-1}, \frac{3}{2} \frac{1}{2}\rangle$ | $\|\frac{5}{2}, \frac{-1}{2}\rangle$ | $-\frac{1}{2}$ |
| $\|1 0, \frac{3}{2} \frac{-1}{2}\rangle$ | $\|\frac{3}{2}, -\frac{1}{2}\rangle$ | |
| $\|1 1, \frac{3}{2} \frac{-3}{2}\rangle$ | $\|\frac{1}{2}, \frac{-1}{2}\rangle$ | |

| | | |
|---|---|---|
| $\lvert 1\,-1,\ \tfrac{3}{2}\ \tfrac{-1}{2}\rangle$<br>$\lvert 1\,0,\ \tfrac{3}{2}\ \tfrac{-3}{2}\rangle$ | $\lvert \tfrac{5}{2}\ \tfrac{-3}{2}\rangle$<br>$\lvert \tfrac{3}{2}\ \tfrac{-3}{2}\rangle$ | $-3/2$ |
| $\lvert 1\,-1,\ \tfrac{3}{2}\ \tfrac{-3}{2}\rangle$ | $\lvert \tfrac{5}{2}\ \tfrac{-5}{2}\rangle$ | $-5/2$ |

(b) Take the state $\lvert \tfrac{5}{2}\ \tfrac{5}{2}\rangle$ and lower it:

$$S_-\lvert \tfrac{5}{2}\ \tfrac{5}{2}\rangle = \sqrt{\tfrac{5}{2}\tfrac{7}{2} - \tfrac{5}{2}\tfrac{3}{2}}\ \lvert \tfrac{5}{2}\ \tfrac{3}{2}\rangle$$

$$= \sqrt{5}\ \lvert \tfrac{5}{2}\ \tfrac{3}{2}\rangle$$

On the other hand

$$S_-\lvert \tfrac{5}{2}\ \tfrac{5}{2}\rangle = (S_{1-} + S_{2-})\lvert \tfrac{3}{2}\ \tfrac{3}{2}\rangle\lvert 1,1\rangle$$

$$= \sqrt{\tfrac{3}{2}\tfrac{5}{2} - \tfrac{3}{2}\tfrac{1}{2}}\ \lvert \tfrac{3}{2}\ \tfrac{1}{2}\rangle\lvert 1,1\rangle$$

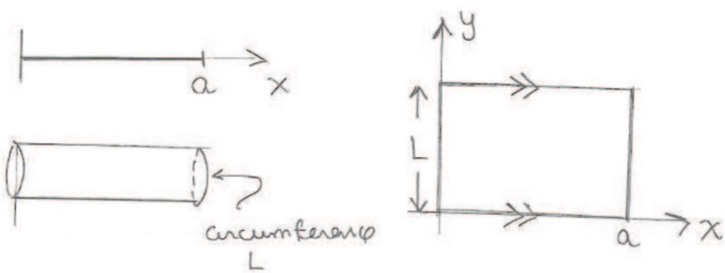
$$+ \sqrt{2-0}\ \lvert \tfrac{3}{2}\tfrac{3}{2}\rangle\lvert 1,0\rangle$$

$$= \sqrt{3}\ \lvert \tfrac{3}{2}\tfrac{1}{2}\rangle\lvert 1,1\rangle + \sqrt{2}\ \lvert \tfrac{3}{2}\tfrac{3}{2}\rangle\lvert 1,0\rangle$$

$$\rightarrow \boxed{\left|\tfrac{5}{2}\,\tfrac{3}{2}\right\rangle = \sqrt{\tfrac{2}{5}}\left|\tfrac{3}{2}\,\tfrac{3}{2}\right\rangle|1,0\rangle + \sqrt{\tfrac{3}{5}}\left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle|1,1\rangle}$$

7. **Infinite square well with extra dimension = a truncated cylinder**. Some models for physics beyond the standard model introduce *extra dimensions* which have not been observed yet. In this problem, we explore how it is, in principle, possible for such extra dimensions to escape detection if they are small enough. For simplicity, we assume we live in a 1-D world and have one extra dimension.

To model this situation, consider a particle that is free to move on the surface of a long and thin *cylinder* of length $a$ and circumference $L$ (but that cannot leave that surface). The surface can thought of as a rectangular region of the $xy$ plane with coordinates $0 \leq x \leq a$ and $0 \leq y \leq L$. The system is described by the two-dimensional Schrodinger




equation with a potential that vanishes in the rectangle $\{(x, y) : 0 < x < a, 0 \leq y \leq L\}$, and is infinite on the vertical edges at $x = 0$ and $x = a$.

(a) **4 points.** Perform separation of variables in the Schrödinger equation and give the two equations that help determine the energy eigenstates. State the boundary conditions that apply.

1 a) In 2D, the time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\left(\partial^2/\partial x^2 + \partial^2/\partial y^2\right)\Psi(x,y) = E\,\Psi(x,y).$$

The separation of variables ansatz, $\Psi(x,y) = \mathcal{X}(x)\mathcal{Y}(y)$ gives

$$-\frac{\hbar^2}{2m}\left(\mathcal{X}''\mathcal{Y} + \mathcal{Y}''\mathcal{X}\right) = E\,\mathcal{X}\mathcal{Y}$$

$$-\frac{\hbar^2}{2m}\left(\frac{\mathcal{X}''}{\mathcal{X}} + \frac{\mathcal{Y}''}{\mathcal{Y}}\right) = E$$

For this to hold for all $x \in (0,a)$, $y \in [0,L]$,

$$\boxed{-\frac{\hbar^2}{2m}\mathcal{X}'' = E_n\mathcal{X}} \quad \text{and} \quad \boxed{-\frac{\hbar}{2m}\mathcal{Y}'' = E_\ell\mathcal{Y}}$$

so that $E = E_{n\ell} = E_n + E_\ell$ is constant.

Boundary conditions: $\mathcal{X}(0) = 0 = \mathcal{X}(a)$
and for any real $\alpha$,
$\mathcal{Y}(\alpha) = \mathcal{Y}(\alpha + L)$.

b) $k^2 \equiv \dfrac{2mE_n}{\hbar^2}$ and $K^2 \equiv \dfrac{2mE_\ell}{\hbar^2}$ so that
for constants A, B, C, and D,

$\mathcal{X} = A\sin kx + B\cos kx$ and $\mathcal{Y} = C\sin Ky + D\cos Ky$.

$\mathcal{X}(0) = B = 0$

$\mathcal{X}(a) = A\sin ka = 0$

$\Downarrow$

$k_n = \dfrac{n\pi}{a}$ for $\boxed{n = 1, 2, 3, \ldots}$

Note $n \neq 0$ because if $\mathcal{X}$ is identically zero, $\Psi(x,y)$ isn't normalizable.

$\Downarrow$

$$\boxed{E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2}}$$

$\mathcal{Y}$ is L-periodic

$\Downarrow$

$\dfrac{2\pi}{K} = \dfrac{L}{\ell}$

$\Downarrow$

$K = \dfrac{2\pi\ell}{L}$

for $\boxed{\ell = 0, 1, 2, \ldots}$

Note $\ell$ can equal 0 because $\mathcal{Y} = D \neq 0$ is L-periodic.

$\Downarrow$

$$E_\ell = \frac{\hbar^2 K_\ell^2}{2m} = \frac{4\ell^2\pi^2\hbar^2}{2mL^2}$$

$$\boxed{E_\ell = \frac{2\ell^2\pi^2\hbar^2}{mL^2}}$$

$$\boxed{E_{n\ell} = E_n + E_\ell}$$

$\Psi_{n\ell} = A\sin(k_n x)(C\sin(K_\ell y) + D\cos(K_\ell y))$

may be rewritten as

$$R\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{2\pi\ell y}{L}+\varphi\right) \quad \text{where } R \equiv AC,\ \varphi \in \mathbb{R}.$$

To normalize,

$$1 = \int_0^a\int_0^L R^2 \underbrace{\sin^2\left(\frac{n\pi x}{a}\right)} \underbrace{\sin^2\left(\frac{2\pi\ell y}{L}+\varphi\right)} dy\,dx$$

$$\frac{1}{R^2} = \left(\frac{a}{2}\right)\left(\frac{L}{2}\right) \quad \Rightarrow \quad R = \frac{2}{\sqrt{aL}}$$

$$\boxed{\Psi_{n\ell} = \frac{2}{\sqrt{a\ell}}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{2\pi\ell y}{L}+\varphi\right) \text{ for arbitrary } \varphi}$$

c) From b), $E_{n\ell} = \frac{\pi^2\hbar^2}{m}\left(\frac{1}{2}\left(\frac{n}{a}\right)^2 + 2\left(\frac{\ell}{L}\right)^2\right)$.

Therefore, whenever $\ell = 0$, $E_{n\ell}$ is the $n^{th}$ energy level of the 1D infinite square well. Setting $\ell \neq 0$, minimizing $E_{n\ell}$ requires $n=1, \ell=1$:

$$\Rightarrow \boxed{\frac{\pi^2\hbar^2}{m}\left(\frac{1}{2a^2}+\frac{2}{L^2}\right)}$$

d) From c) the minimum $E_{n\ell}$ for which the extra dimension "matters" is

$$\frac{\pi^2\hbar^2}{m}\left(\frac{1}{2a^2}+\frac{2}{L^2}\right). \qquad L \approx \frac{a}{1000}$$

Substituting gives $\frac{\pi^2\hbar^2}{m}\left(\frac{1}{2a^2}+\frac{2\cdot 1000^2}{a^2}\right)$

$$= (\pi^2)\left(\frac{\hbar^2}{2ma^2}\right)(2)\left(\frac{1}{2}+2\times10^6\right)$$

$$= \boxed{4{,}000{,}001\,\pi^2 \text{ eV}}$$

or approximately $\boxed{39.5\text{MeV}}$

as the experimenter's required minimum.