Discussion 6

Problems on spin-1/2 rotations

* For these problems, we'll take it as a given that

$$R(\phi\hat{z}) = e^{-iJ_z\phi/\hbar}$$

rotates states by angle ϕ about the z axis and

$$J_z|+z\rangle = +\frac{\hbar}{2}|+z\rangle$$
$$J_z|-z\rangle = -\frac{\hbar}{2}|-z\rangle$$

$(J_z = S_z)$

so that:

$$R(\phi\hat{z})|+z\rangle = e^{-i\phi/2}|+z\rangle$$
$$R(\phi\hat{z})|-z\rangle = e^{i\phi/2}|-z\rangle$$

<u>example</u> Compute $\hat{R}(\pi\hat{z})|+\hat{x}\rangle$. What should it be?

↳ Expect $|-x\rangle$. Let's see:

$$\hat{R}(\pi\hat{z})|+\hat{x}\rangle \longrightarrow \begin{pmatrix} e^{-i\pi/2} & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\pi/2}/\sqrt{2} \\ e^{i\pi/2}/\sqrt{2} \end{pmatrix} = e^{-i\pi/2}\begin{pmatrix} 1/\sqrt{2} \\ e^{i\pi}/\sqrt{2} \end{pmatrix}$$

$$= e^{-i\pi/2}\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$= e^{-i\pi/2}|-x\rangle \quad \checkmark$$

<u>Problem</u> Confirm that $\hat{R}(\phi\hat{z}) \rightarrow \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}$ by exponentiating $\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ie using $\hat{R}(\phi\hat{z}) = e^{-i\phi\hat{J}_z/\hbar}$.

↳ For a diagonal matrix, $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$,

$$\exp\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} e^{a} & 0 \\ 0 & e^{b} \end{pmatrix}$$

so

$$\exp\left[-i\phi\,\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}/\hbar\right] = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix} \quad \checkmark$$

<u>problem</u> There's nothing special about the z-direction. We should have

$$\hat{R}(\phi\hat{x}) = e^{-i\phi\hat{J}_x/\hbar}$$

for a rotation by angle ϕ about the x-axis, where $\hat{J}_x$ is the generator of rotations about the x-axis (also known as the x-component of the angular momentum operator).

(a) By symmetry, what is the action of $\hat{R}(\phi\hat{x})$ on the states $|+x\rangle$ and $|-x\rangle$? Write it as a 2×2 matrix in the x-basis.

(b) By making a change of basis from x to z, write $\hat{R}(\phi\hat{x})$ as a 2×2 matrix with respect to the z-basis.

(c) For $\phi \ll 1$, taylor expand to linear order in ϕ and compare with the equation

$$\hat{R}(d\phi\hat{x}) = \mathbb{1} - \frac{i}{\hbar}\hat{J}_x\, d\phi$$

to identify the matrix representation of $\hat{J}_x$ (wrt the z-basis). You should find

$$\hat{J}_x \longrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_x$$

(a) Expect

$$\hat{R}(\phi\hat{x})|+x\rangle = e^{-i\phi/2}|+x\rangle$$
$$\hat{R}(\phi\hat{x})|-x\rangle = e^{i\phi/2}|-x\rangle$$

since there's nothing special about the z-direction.

(b) The matrix representation of $\hat{R}(\phi\hat{x})$ in the x basis is:

$$\hat{R}(\phi\hat{x}) \xrightarrow{x} \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}$$

The matrix elements are $\langle \pm x|\hat{R}|\pm x\rangle$ arranged like:

$$\begin{pmatrix} \langle +x|\hat{R}|+x\rangle & \langle +x|\hat{R}|-x\rangle \\ \langle -x|\hat{R}|+x\rangle & \langle -x|\hat{R}|-x\rangle \end{pmatrix}$$

Then ${}_x\langle a|\hat{R}|b\rangle_x$ with $a,b = \pm 1$ represents the whole matrix. Insert 2 resolutions of the identity:

$${}_z\langle a|\hat{R}|b\rangle_z = \sum_{c,d} {}_z\langle a|c\rangle_x \, {}_x\langle c|\hat{R}|d\rangle_x \, {}_x\langle d|b\rangle_z$$

But

$${}_z\langle a|c\rangle_x = \begin{pmatrix} \langle +z|+x\rangle & \langle +z|-x\rangle \\ \langle -z|+x\rangle & \langle -z|-x\rangle \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

so

$${}_z\langle a|\hat{R}|b\rangle_z = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{-i\phi/2} & e^{-i\phi/2} \\ e^{i\phi/2} & -e^{i\phi/2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} e^{i\phi/2} + e^{-i\phi/2} & e^{-i\phi/2} - e^{i\phi/2} \\ -e^{i\phi/2} + e^{-i\phi/2} & e^{-i\phi/2} + e^{i\phi/2} \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} \cos(\phi/2) & -i\sin(\phi/2) \\ -i\sin(\phi/2) & \cos(\phi/2) \end{pmatrix}}$$

(c) For $\phi << 1$, this is

$$\begin{pmatrix} 1 & -i\phi/2 \\ -i\phi/2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\phi \frac{1}{\hbar}\begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix}$$

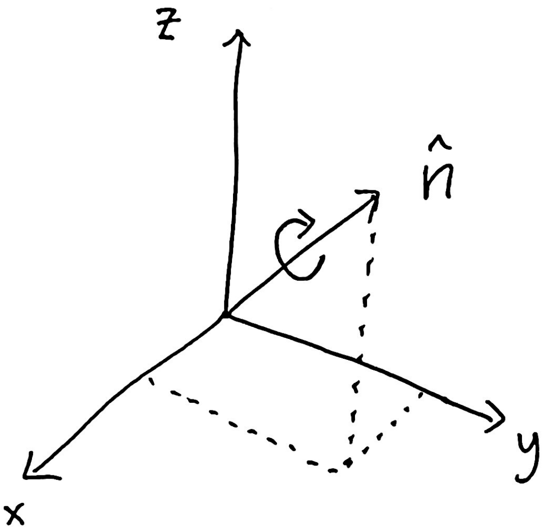
$$\Rightarrow \boxed{J_x \rightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

In the z-basis.

Problem A rotation by angle ϕ about axis $\hat{n}$ is given by

$$\hat{R}(\phi\hat{n}) = e^{-i\hat{\vec{J}}\cdot\hat{n}\phi/\hbar}$$

where $\hat{\vec{J}}\cdot\hat{n} = \hat{J}_x n_x + \hat{J}_y n_y + \hat{J}_z n_z$ and $\hat{n}$ is a unit vector $n_x^2 + n_y^2 + n_z^2 = 1$.



(a) The matrix representation of $\hat{R}(\phi\hat{n})$ in the z-basis is

$$\exp\left[-i\,\vec{\sigma}\cdot\hat{n}\,\phi/2\right]$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. Let's get a closed-form expression for this 2x2 matrix.

(a) Show that $(\vec{\sigma}\cdot\vec{v})^2 = |\vec{v}|^2$. [Hint: use the identity

$$\sigma_j\sigma_k = \delta_{jk}I + i\varepsilon_{jkl}\sigma_l$$

where $I = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ and the indices run over 1,2,3].

(b) Write $\exp\left[-i\vec{\sigma}\cdot\hat{n}\phi/2\right]$ in terms of its Taylor series. Seperate it into even/odd powers and use the result of part (a) to do the sum. Write it as a 2×2 matrix.

(c) Let $\hat{n} = \hat{x}$ and $\hat{n} = \hat{z}$ and check this against the previous problem.

(a) Using $\vec{A}\cdot\vec{B} = \delta_{ij} A_i B_j$, write

$$(\vec{\sigma}\cdot\vec{v})^2 = \delta_{ij}\sigma_i v_j \; \delta_{kl}\sigma_k v_l$$

where all indices are summed and run from 1 to 3. Then everything commutes except the σ's

$$= \delta_{ij}\,\delta_{kl}\, v_j v_l \;\sigma_i \sigma_k$$

Use the identity $\sigma_i\sigma_k = \delta_{ik} I + i\varepsilon_{ikm}\sigma_m$

$$= \delta_{ij}\,\delta_{kl}\, v_j v_l \left[\delta_{ik} I + i\varepsilon_{ikm}\sigma_m\right]$$

$$= \vec{v}\cdot\vec{v}\, I + i(\vec{v}\times\vec{v})\cdot\vec{\sigma}$$

where I used $\delta_{ij}\delta_{ik} = \delta_{jk}$ and $\varepsilon_{ikm} v_i v_k \sigma_m = (\vec{v}\times\vec{v})\cdot\vec{\sigma}$.

But $\vec{v}\times\vec{v} = 0$, so

$$\Rightarrow \quad (\vec{\sigma}\cdot\vec{v})^2 = \vec{v}\cdot\vec{v}\, I \qquad \square$$

(b) Have:

$$\exp\left[\frac{-i\vec{\sigma}\cdot\vec{n}}{2}\phi\right] = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}(\vec{\sigma}\cdot\hat{n})^n\left(\frac{\phi}{2}\right)^n$$

Because $\hat{n}\cdot\hat{n} = 1$, $(\vec{\sigma}\cdot\hat{n})^2 = 1$. This means

$$(\vec{\sigma}\cdot\hat{n})^3 = (\vec{\sigma}\cdot\hat{n}), \qquad (\vec{\sigma}\cdot\hat{n})^4 = I, \qquad (\vec{\sigma}\cdot\hat{n})^5 = (\vec{\sigma}\cdot\hat{n})$$

etc.... So the sum spits into:

$$= \sum_{n=0,2,4}\frac{(-i)^n}{n!}\left(\frac{\phi}{2}\right)^n I + (\vec{\sigma}\cdot\hat{n})\sum_{n=1,3,5}\frac{(-i)^n}{n!}\left(\frac{\phi}{2}\right)^n$$

But these are the Taylor series for sin and cosine so:

$$\boxed{R = I\cos\left(\frac{\phi}{2}\right) - i(\vec{\sigma}\cdot\hat{n})\sin\left(\frac{\phi}{2}\right)}$$

(c) When $\hat{n} = \hat{z}$, $\vec{\sigma}\cdot\hat{n} = \sigma_3$ and we get

$$R = \begin{pmatrix} \cos(\phi/2) & 0 \\ 0 & \cos(\phi/2) \end{pmatrix} - i\begin{pmatrix} \sin(\phi/2) & 0 \\ 0 & -\sin(\phi/2) \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}$$

as expected. Next, when $\hat{n} = \hat{x}$; $\vec{\sigma}\cdot\hat{n} = \sigma_x$

$$R = \begin{pmatrix} \cos(\phi/2) & 0 \\ 0 & \cos(\phi/2) \end{pmatrix} - i\begin{pmatrix} 0 & \sin(\phi/2) \\ \sin(\phi/2) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\phi/2) & -i\sin(\phi/2) \\ -i\sin(\phi/2) & \cos(\phi/2) \end{pmatrix}$$

also as expected.

Problem Show that every spin state is related to every other spin state by a rotation $R(\hat{n}\phi)$ by some angle about some axis (for spin-1/2). That is, given arbitrary $|\chi\rangle$, $|\psi\rangle$,

$$|\chi\rangle = \hat{R}(\hat{n}\theta)|\psi\rangle$$

(Note: This is not true of higher spins, like spin-1).

~~Problem~~

$\boxed{\text{Solution}}$ We need only show that every state can be written as $R(\hat{n}\theta)|+z\rangle$, since we can then use $R(\hat{n}\theta)^\dagger$ to go the other way:

$$|\psi\rangle \xrightarrow{R(\hat{n}\theta)^\dagger} |+z\rangle \xrightarrow{R(\hat{n}'\theta')} |\chi\rangle$$

The composition of two rotations is a rotation, so we get

$$R(\hat{n}'\theta')R^\dagger(\hat{n}\theta)|\psi\rangle = |\chi\rangle$$

$$R(\hat{n}''\theta'')|\psi\rangle = |\chi\rangle$$

where $R(\hat{n}''\theta'')$ is the rotation $R(\hat{n}'\theta')R^\dagger(\hat{n}\theta)$.

So, let's show that every state can be written as

$$R(\hat{n}\theta)|+z\rangle$$

for some $\hat{n}$ and θ.

A general state is $c_1|+z\rangle + c_2|-z\rangle \longrightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, for complex c_1, c_2. Let's ensure that the state is normalized $|c_1|^2 + |c_2|^2 = 1$ and let's pick c_1 to be real (since states are only defined up to an overall phase). Then we can write it as

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi} \sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

for two real numbers, θ and ϕ.

Now, consider the rotation:

$$R(\phi\hat{z})\, R(\theta\hat{y})$$

That is, a rotation by θ about the y-axis followed by a ϕ rotation about the z-axis. In matrices,

$$R(\phi\hat{z}) \longrightarrow \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}$$

and via previous problem

$$R(\theta\hat{y}) \longrightarrow \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

Then their composition is:

$$\begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi/2} & -e^{-i\phi/2} \sin\frac{\theta}{2} \\ e^{i\phi/2} \sin\frac{\theta}{2} & e^{i\phi/2} \cos\frac{\theta}{2} \end{pmatrix}$$

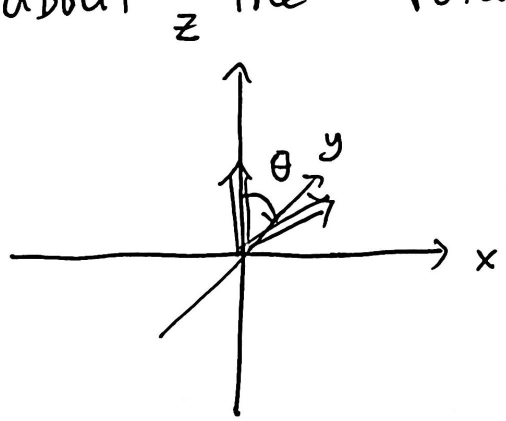
When this acts on $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ it produces

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \xmapsto{R} \begin{pmatrix} e^{-i\phi/2} \cos\frac{\theta}{2} \\ e^{i\phi/2} \sin\frac{\theta}{2} \end{pmatrix}$$

$$= e^{-i\phi/2} \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi} \sin\frac{\theta}{2} \end{pmatrix}$$

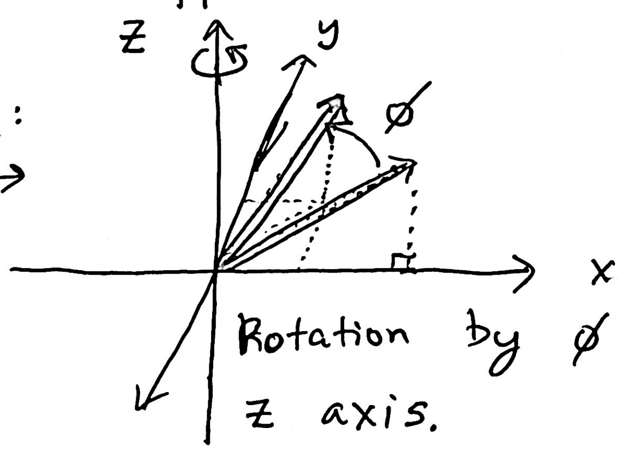
which (up to a phase) is the most general spin state, so indeed every spin state is a rotation away from every other spin state.

In fact the (θ, φ) appearing above are the spherical coordinate angles of the direction $|\chi\rangle \rightarrow \begin{pmatrix} \cos\theta/2 \\ e^{i\phi} \sin\theta/2 \end{pmatrix}$ points. To see this, think about the rotations we applied to $|+z\rangle$:



Rotation by θ about the y axis.

Then:



Rotation by ϕ about z axis.