Homework 0 ‘Due’ Wednesday, August 30th

We’re all a little rusty after the summer is over. So, this *ungraded* problem set is just here to help remove some of that plaque with everyone’s favorite activity: math review!

1. **Complex Number Arithmetic** Express each of the following in the form $a + bi$, where a and b are real numbers and $i^2 = -1$.
   (a) $(2020 + 3i) + (3 + 2020i)$.
   (b) $(2 + 5i)(3 - 4i)$.
   (c) $(1 + i)/(1 - i)$.
   (d) $(1 + i)^4$.
   (e) $\sqrt{i}$
2. Is it true that $|z|/|w| = |z/w|$ for all complex numbers z and w? Show or provide a counterexample.
3. **Matrix Computations** Given the following two matrices:
$$A = \begin{pmatrix} -1 & 1 & i \\ 2 & 0 & 3 \\ 2i & -2i & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{pmatrix}$$
   compute:
   (a) $A + B$
   (b) AB
   (c) BA
   (d) B^{-1}. Check that $BB^{-1} = I$.
   (e) $\det(A)$. Does A have an inverse?
4. Find the square root and logarithm of the following matrix, if they exist:
$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$$
   [Hint: If only the matrix was diagonal...]
5. **Inner Products**
   (a) Let
$$\boldsymbol{v} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}, \qquad \boldsymbol{w} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \tag{1}$$
   be vectors in ordinary 3D space. Calculate $\boldsymbol{v} \cdot \boldsymbol{w}$ and find the angle between the two vectors.
   (b) Prove the law of cosines $C^2 = A^2 + B^2 - 2AB\cos\theta_{AB}$, where A, B, C are the sides of a triangle and θ_{AB} is the angle between sides A and B. [Hint: Treat the sides of the triangle as displacement vectors. What equation do $\boldsymbol{A}, \boldsymbol{B}$, and $\boldsymbol{C}$ satisfy because they are part of a triangle?]
   (c) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be any vectors. Show that $|\boldsymbol{A} + \boldsymbol{B}| = |\boldsymbol{A} - \boldsymbol{B}|$ if and only if $\boldsymbol{A}$ and $\boldsymbol{B}$ are mutually perpendicular.
   (d) Use vector algebra to prove that the diagonals of an equilateral quadrilateral (i.e., rhombus) must be perpendicular. HINT: use the results from part (c).
6. Let $\boldsymbol{e}_1 = \begin{pmatrix} 1, & 0 \end{pmatrix}$ and $\boldsymbol{e}_2 = \begin{pmatrix} 0, & 1 \end{pmatrix}$ be the unit vectors in the x and y directions. These form a basis for the plane.
   (a) Consider an active rotation of the plane about the origin in the counterclockwise direction by an angle ϕ. Under such a rotation, every vector $\boldsymbol{v}$ gets mapped to a new rotated vector $\boldsymbol{v}'$. Write the rotated vectors $\boldsymbol{e}_1'$ and $\boldsymbol{e}_2'$ in terms of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$.

(b) This rotation is an example of a *linear operator* on a vector space, i.e. a function $R : V \to V$ that maps vectors in V to vectors in V, and which satisfies the linearity property: $R(\alpha \boldsymbol{v} + \beta \boldsymbol{w}) = \alpha R(\boldsymbol{v}) + \beta R(\boldsymbol{w})$. Linear operators can always be expressed as matrices with respect to some basis. Write the matrix of R with respect to the basis $(\boldsymbol{e}_1, \boldsymbol{e}_2)$.

7. **Rotations are always linear transformations** Whatever "rotation" means,

- It is a map of some vector space V.
- Which has a way of measuring "lengths" and "angles" of its vectors, and
- "Rotations" preserve those "lengths" and "angles".

Now, a fancy way to have "lengths" and "angles" in a vector space is to have a dot product in it. So let's assume that V is a euclidian vector space; for instance, a real vector space such as R^n with the standard dot product.

Then, in such a V, "length" means "norm", which is $\|\boldsymbol{v}\| = +\sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$. .

We have to be more careful with "angles" because the standard definition already involves rotations. To avoid a circular argument, we define the (non-oriented) angle determined by two vectors $\boldsymbol{v}, \boldsymbol{w}$, as the *unique* real number $\theta \in [0, \pi]$ such that

$$\cos\theta = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v} \cdot \boldsymbol{w}\|}. \tag{2}$$

So, "rotation" is some kind of map which preserves norms and angles. Since the dot product can be expressed in terms of norms and (cosines of) angles, rotations preserve dot products:

$$R(\boldsymbol{v}) \cdot R(\boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{w}. \tag{3}$$

Now, let's show that a map that preserves the dot product is necessarily linear.

(a) Using the above property, show that $R(\boldsymbol{v} + \boldsymbol{w}) = R(\boldsymbol{v}) + R(\boldsymbol{w})$.

(b) Similarly show that $R(\alpha \boldsymbol{v}) = \alpha R(\boldsymbol{v})$, where α is a number.

These two properties together show that R is always a linear transformation.