Homework 3 Solutions

137A

1 (a)

$$\langle A\rangle_\psi = \sum_i a_i\, p(a_i)$$

$$= \sum_i a_i\, |\langle a_i|\psi\rangle|^2$$

$$= \sum_i a_i\, \langle\psi|a_i\rangle\langle a_i|\psi\rangle$$

$$= \sum_i \langle\psi|a_i|a_i\rangle\langle a_i|\psi\rangle$$

$A|a_i\rangle = a_i|a_i\rangle$

$$= \sum_i \langle\psi|A|a_i\rangle\langle a_i|\psi\rangle$$

$$= \langle\psi|A\underbrace{\left(\sum_i |a_i\rangle\langle a_i|\right)}_{=1}|\psi\rangle$$

$$= \boxed{\langle\psi|A|\psi\rangle.}\quad //$$

(b) $\boxed{\text{True.}}$ $\langle\psi|(A+B)|\psi\rangle = \langle\psi|A|\psi\rangle + \langle\psi|B|\psi\rangle$

$$= \langle A\rangle + \langle B\rangle.$$

(c) $\langle (A-\langle A\rangle)^2\rangle$

$$= \langle A^2 - 2A\langle A\rangle + \langle A\rangle^2\rangle$$

$$= \langle A^2\rangle - 2\langle A\rangle^2 + \langle A^2\rangle$$

$$= \langle A^2\rangle - \langle A\rangle^2$$

$$\Rightarrow \boxed{\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}}.$$

$\boxed{2}$ (i) probabilities don't care about phase:

$$p(a_i) = |\langle a_i|\psi\rangle|^2 = |\langle a_i|e^{i\delta}|\psi\rangle|^2.$$

(ii) expectation values don't care about phase:

$$\langle A\rangle = \langle\psi|A|\psi\rangle = \left(\langle\psi|e^{-i\delta}\right) A \left(e^{i\delta}|\psi\rangle\right)$$

$\boxed{3}$ $|\psi\rangle = \frac{1}{2}|+z\rangle + \frac{i\sqrt{3}}{2}|-z\rangle$

$$\langle S_z\rangle = \begin{pmatrix}\frac{1}{2} & \frac{-i\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}1/2\\ i\sqrt{3}/2\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}\frac{1}{2} & \frac{-i\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}1/2\\ -i\sqrt{3}/2\end{pmatrix}$$

$$= \frac{\hbar}{2}\left(\frac{1}{4} - \frac{3}{4}\right)$$

$$\boxed{\langle S_z\rangle = -\frac{\hbar}{4}}$$

Note that $S_z^2 = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4} I$, so

$$\boxed{\langle S_z^2 \rangle = \hbar^2/4} \quad \text{w.r.t. any state}$$

Then:

$$\Delta S_z = \sqrt{\langle S_z^2 \rangle - \langle S_z \rangle^2}$$

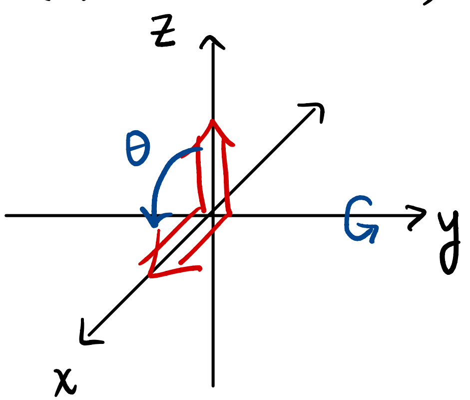
$$= \sqrt{\hbar^2/4 - (-\hbar/4)^2}$$

$$= \sqrt{4\hbar^2/16 - \hbar^2/16}$$

$$\boxed{\Delta S_z = \frac{\sqrt{3}\,\hbar}{4}}$$

$\boxed{4}$ $\quad |+n\rangle = \cos\frac{\theta}{2}|+z\rangle + e^{i\phi}\sin\theta\,|-z\rangle.$

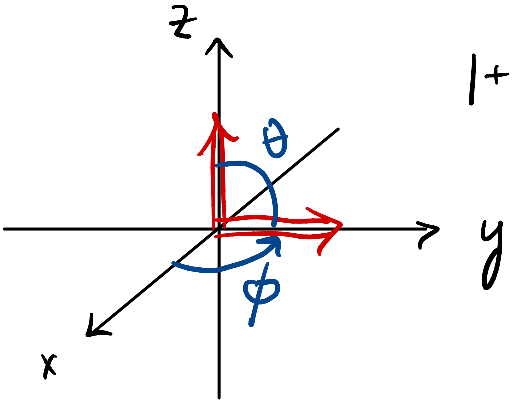
(a) For $\theta = \pi/2$, $\phi = 0$, we get



$$|+n\rangle \to \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$$

$$= |+x\rangle$$

For $\theta = \pi/2$, $\phi = \pi/2$ we get



$$|+n\rangle \to \frac{1}{\sqrt{2}}|+z\rangle + \frac{i}{\sqrt{2}}|-z\rangle$$

$$= |+y\rangle \checkmark$$

(b) $\langle +n|S_z|+n\rangle$

$$= \left(\cos\frac{\theta}{2} \quad e^{-i\phi}\sin\frac{\theta}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$= \frac{\hbar}{2}\left(\cos\frac{\theta}{2} \quad e^{-i\phi}\sin\frac{\theta}{2}\right)\begin{pmatrix} \cos\theta/2 \\ -e^{i\phi}\sin\theta/2 \end{pmatrix}$$

$$= \frac{\hbar}{2}\left[\cos^2\theta/2 - \sin^2\theta/2\right]$$

$$= \boxed{\frac{\hbar}{2}\cos\theta}$$

Half angle identity.

(c) $\langle S_z^2\rangle = \frac{\hbar^2}{4}$ always, so

$$\Delta S_z = \sqrt{\hbar^2/4 - \hbar^2/4\cos^2\theta}$$

$$\Delta S_z = \boxed{\frac{\hbar}{2}\sin\theta}$$