Homework 2 Due Monday, September 11th by midnight

1. For each of the following subsets, specify whether or not the subset constitutes a vector space (under ordinary rules for adding polynomials together). If not briefly offer one reason why not. If so, specify the dimension of the space. (Take N to be a fixed non negative integer):
   (a) All polynomials of degree no greater than N.
   (b) All polynomials of degree no greater than N whose leading coefficient is 1.
   (c) All polynomials of degree no greater than N which vanish at $z = 0$.
   (d) All polynomials of degree no greater than N which assume the value 1 at $z = 0$.
   (e) All polynomials of degree no greater than N whose integral from $z = 0$ to $z = 1$ vanishes.
   (f) All polynomials of degree no greater than N for which the value of the polynomial at $z = 0$ is the same as the value at $z = 1$.

2. (a) Show that if $\hat{A}$ and $\hat{B}$ are Hermitian, then $\hat{A}\hat{B}$ is Hermitian provided that $\hat{A}$ and $\hat{B}$ commute with each other.
   (b) Show that for any operator $\hat{A}$, the operator $\hat{A}^\dagger \hat{A}$ is Hermitian, and that all its eigenvalues are *positive* real numbers.

3. (Griffiths A.31) Functions of matrices are typically defined by their Taylor series expansions. For example, we define

$$e^M = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \frac{1}{4!}M^4 + \dots \tag{1}$$

   One can show that this series always converges, so the exponential of a matrix is always defined.
   (a) Find $\exp(M)$, if

$$(i) \quad M = \begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (ii) \quad M = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \tag{2}$$

   (b) Show that if the matrices M and N commute, then $e^M e^N = e^{M+N}$. (If they do not commute, this is not true, in general).
   (c) If H is hermitian, show that e^{iH} is unitary.

4. (Griffiths A.30: unitary operators on a Hilbert space)
   (a) Show that unitary transformations preserve inner products, in the sense that $\langle \alpha|\beta\rangle = \langle U\alpha|U\beta\rangle$ for all vectors $|\alpha\rangle, |\beta\rangle$.
   (b) Show that the eigenvalues of a unitary transformation have absolute value 1.
   (c) Show that the eigenvectors of a unitary transformation with different eigenvalues are orthogonal.

5. Consider the Hermitian matrices

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix} \tag{3}$$

   representing operators with respect to some chosen orthonormal basis.
   (a) Find a common eigenbasis for A and B. (This is possible because A and B commute).
   (b) Let C be any Hermitian operator which commutes with both A and B. Show that C must be diagonal in the same eigenbasis.
   (c) Suppose a Hermitian operator F is known to commute with the operator A, but nothing is known about its commutator with B. Is F necessarily diagonal in the basis you found in part (a) above? If yes, explain. If false, find a counter-example.