

Connections on principal bundles

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Notation

These notes compile some general facts about connections on principal bundles, and their relation to connections on vector bundles.

A few notes on notation: We are working in the context of a **principal G -bundle** P over a manifold M . This we denote as $G \hookrightarrow P \xrightarrow{\pi} M$; where $\pi : P \rightarrow M$ is the projection map. The right action of G on P is denoted as $\sigma : P \times G \rightarrow P$. For any $g \in G$, we denote right multiplication by g as $\sigma_g : P \rightarrow P$; and for every $p \in P$, we denote the orbit map as $\sigma_p : G \rightarrow P$.

Given a smooth function $f : M \rightarrow N$ between manifolds, we denote the tangent map at some $x \in M$ as $T_x f : T_x M \rightarrow T_{f(x)} N$. This is to explicitly show the functoriality of T_x .

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I Connections on principal bundles

I.1 Connections as horizontal distributions

Recall that a vector $v \in T_p P$ is called **vertical** if

$$T_p \pi(v) = 0.$$

We denote the subspace of vertical vectors by $V_p P \subset T_p P$. By definition, $V_p P$ is nothing more than the kernel of $T_p \pi$, so we have a short exact sequence

$$0 \longrightarrow V_p P \hookrightarrow T_p P \xrightarrow{T_p \pi} T_{\pi(p)} M \longrightarrow 0.$$

Since this is a sequence of vector spaces, it splits, and thus we have an isomorphism

$$T_p P \cong V_p P \oplus T_{\pi(p)} M.$$

However, the splitting (and thus the isomorphism) is not canonical: it depends on a choice of a subspace $H_p \subset T_p P$ that is complementary to $V_p P$, and an isomorphism $T_{\pi(p)} M \rightarrow H_p$. We call any complementary space to $V_p P$ a **horizontal space** at p , such that:

$$T_p P = V_p P \oplus H_p.$$

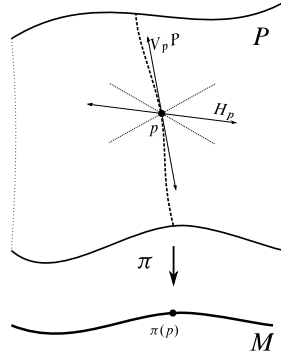


Figure 1: A choice of a horizontal space H_p at $T_p P$. There are many such choices (in dotted lines).

Once we have chosen a single horizontal subspace $H_p \subset T_p P$ at p , we can find horizontal subspaces for all points in the same fiber of p . This follows since the action of G on P , which we denote $\sigma_g(p) = p \cdot g$, is a fiber-preserving diffeomorphism, and thus $T_p \sigma_g$ is an isomorphism of tangent spaces that preserves the vertical subspace. This suggests that $T_p \sigma_g(H_p)$ is a horizontal subspace at $p \cdot g$. Indeed, noting that

$$T_{p \cdot g} \pi \circ T_p \sigma_g = T_p (\pi \circ \sigma_g) = T_p \pi(v),$$

we see that $T_p \pi(V_p P) \subseteq V_{p \cdot g} P$. Similarly, if $u \in V_{p \cdot g} P$, we can write

$$u = T_p \sigma_g(T_{p \cdot g} \sigma_{g^{-1}}(u)) = T_p \sigma_g(\tilde{u}),$$

where by the same argument above $\tilde{u} = T_{p,g}\sigma_{g^{-1}}(u) \in V_p P$ is vertical. Therefore, we obtain that

$$V_{p,g}P = T_p\sigma_g(V_pP).$$

Furthermore, since $T_p\sigma_g : T_pP \rightarrow T_{p,g}P$ is an isomorphism, we obtain that

$$T_{p,g}P = T_p\sigma_g(T_pP) = T_p\sigma_g(V_pP) \oplus T_p\sigma_g(H_p) = V_{p,g}P \oplus T_p\sigma_g(H_p),$$

And so we have proved the following:

Lemma 1.1 (Translation of horizontal subspaces). *If $H_p \subset T_pP$ is horizontal at p , then for all $g \in G$, $T_p\sigma_g(H_p)$ is horizontal at $p \cdot g$.*

So far we have been working at a single point $p \in P$. We can now consider a smooth choice of horizontal spaces above each element of P :

Definition 1.2 ((Principal) Connection). *A connection on P is a distribution H on P such that for all $p \in P$, $H_p \subset T_pP$ is a horizontal subspace. We say that a connection H is **principal** if it is compatible with the group action in the sense that for all $g \in G$ and all $p \in P$,*

$$T_p\sigma_g(H_p) = H_{p \cdot g}.$$

The notion of connection is independent of the group action on the total space P , and indeed it applies to general fiber bundles. The condition for a connection to be principal states that our choice of horizontal subspaces along a single fiber is consistent with the “translation” lemma 1.1.

We think of a connection H as a *preferred* way of relating “neighboring” fibers of the bundle. Once we have $p \in P$, we might think that the preferred way of moving to another fiber is along a “direction” (i.e. tangent vector) in the horizontal space H_p . This gives us a little bit of intuition and (sort of) justifies (kind of) the name *connection*. In practice, however, working with distributions might be cumbersome. Fortunately for us, there are other (equivalent) presentations of connections.

1.2 Connections as 1-forms

Let \mathfrak{g} be the Lie algebra of G . Recall that for all $p \in P$, we have the infinitesimal action of \mathfrak{g} on T_pP , $a_p : \mathfrak{g} \rightarrow T_pP$ given as

$$a_p(X) := \left. \frac{d}{dt} \right|_{t=0} p \cdot \exp(tX).$$

Writing $\sigma_p : G \rightarrow P$ as $\sigma_p(g) = p \cdot g$, we see that the infinitesimal action is simply the differential of σ_p :

$$a_p(X) = T_e\sigma_p(X).$$

This infinitesimal action induces, for each $X \in \mathfrak{g}$, a vector field $X^\#$ called the **fundamental vector field** associated to X given by

$$X_p^\# := a_p(X).$$

We have that σ_p is a diffeomorphism onto the fiber containing p , and thus $a_p = T_e\sigma_p$ induces a linear isomorphism $\mathfrak{g} \xrightarrow{a_p} V_pP$.

Suppose that we have a principal connection H on P . Then in particular, we have a subspace $H_p \subset T_p P$ such that $T_p P = V_p P \oplus H_p$, and so we can construct a map $\omega_p : T_p P \rightarrow \mathfrak{g}$ as

$$\omega_p(v^V + v^H) = a_p^{-1}(v^V),$$

where $v^V \in V_p P$ and $v^H \in H_p$. By construction, we have that

$$\omega_p(a_p(X)) = X$$

for all $X \in \mathfrak{g}$. We can also see how ω_p compares to $\omega_{p \cdot g}$, since we know that our horizontal distribution behaves nicely along the fibers of the action.

For this, first note that for all $g \in G$,

$$\begin{aligned} T_p \sigma_g(a_p(X)) &= \left. \frac{d}{dt} \right|_{t=0} \sigma_g(p \cdot \exp(tX)) \\ &= \left. \frac{d}{dt} \right|_{t=0} p \cdot \exp(tX)g \\ &= \left. \frac{d}{dt} \right|_{t=0} (p \cdot g) \cdot (g^{-1} \exp(tX)g). \end{aligned}$$

Now we ask ourselves, do we know what the tangent vector of $g^{-1} \exp(tX)g$ is? Yes, yes we do:

$$\left. \frac{d}{dt} \right|_{t=0} g^{-1} \exp(tX)g = \left. \frac{d}{dt} \right|_{t=0} \text{Conj}_{g^{-1}}(\exp(tX)) = \text{Ad}_{g^{-1}}(X),$$

where we have written¹ $\text{Conj}_g(h) = ghg^{-1}$, and $\text{Ad}_g = T_e \text{Conj}_g$. Then we have

$$T_p \sigma_g(a_p(X)) = \left. \frac{d}{dt} \right|_{t=0} (p \cdot g) \cdot (g^{-1} \exp(tX)g) = a_{p \cdot g}(\text{Ad}_{g^{-1}}(X)).$$

With this, we can see that for $v \in T_p P$, which we write as $v = v^V + v^H$ with $v^V = a_p(X)$ for some $X \in \mathfrak{g}$:

$$(\sigma_g^* \omega)_p(v^V + v^H) = \omega_{p \cdot g}(T_p \sigma_g(v^V) + T_p \sigma_g(v^H)) = \omega_{p \cdot g}(T_g \sigma_g(a_p(X))) = \text{Ad}_{g^{-1}}(X) = (\text{Ad}_{g^{-1}} \circ \omega_p)(v),$$

and so we conclude that

$$(\sigma_g^* \omega) = \text{Ad}_{g^{-1}} \circ \omega.$$

Then we have proved, modulo the small detail of smoothness², the following:

Proposition 1.3 (1-form induced by principal connection). *Let H be a principal connection on $G \hookrightarrow P \xrightarrow{\pi} M$. Then there exists a (unique) \mathfrak{g} -valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$, such that for all $p \in P$, $g \in G$ and $X \in \mathfrak{g}$:*

1. $\omega_p(a_p(X)) = X$,
2. $\sigma_g^* \omega = \text{Ad}_{g^{-1}} \circ \omega$, and
3. $\ker(\omega_p) = H_p$.

We call any \mathfrak{g} -valued 1-form satisfying these properties a **connection 1-form**:

¹<https://xkcd.com/927/>

²We can handwave it away by saying that it follows from the smoothness of the distribution H .

Definition 1.4 (Connection 1-form). A connection 1-form on P is a \mathfrak{g} -valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ such that for all $p \in P$, $g \in G$ and $X \in \mathfrak{g}$:

1. $\omega_p(a_p(X)) = X$, and
2. $\sigma_g^* \omega = \text{Ad}_{g^{-1}} \circ \omega$.

The converse to proposition 1.3 is also true:

Proposition 1.5. Principal connection induced by connection 1-form Let $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection 1-form. Then the distribution H defined pointwise as

$$H_p = \ker(\omega_p) \subset T_p P$$

is a principal connection on P .

Proof. — First, let's see that indeed $H_p = \ker(\omega_p)$ is horizontal. If $v \in \ker(\omega_p) \cap V_p P$, then $v = a_p(X)$ for some $X \in \mathfrak{g}$, so that

$$0 = \omega_p(v) = \omega_p(a_p(X)) = X,$$

and thus $v = 0$. Therefore $\ker(\omega_p) \cap V_p P = \{0\}$. Now for an arbitrary $v \in T_p P$, set

$$v^V = a_p(\omega_p(v)).$$

Then we have that $T_p \pi(v^V) = 0$, since it is in the image of a_p , and thus $v^V \in V_p P$. Finally, setting $v^H = v - v^V$, we have

$$\omega_p(v^H) = \omega_p(v) - \omega_p(a_p(\omega_p(v))) = \omega_p(v) - \omega_p(v) = 0,$$

and so $v^H \in \ker \omega_p = H_p$. We have then shown that $v = v^V + v^H$, with $v^V \in V_p P$ and $v^H \in H_p$, and so

$$T_p P = V_p P \oplus H_p.$$

Thus H_p is a horizontal subspace. Now to see that H is principal, note that

$$\omega_{p \cdot g}(T_p \sigma_g(v)) = \text{Ad}_{g^{-1}}(\omega_p(v)).$$

Since both $T_p \sigma_g$ and $\text{Ad}_{g^{-1}}$ are isomorphisms, we have that $v \in \ker \omega_p$ if and only if $T_p \sigma_g(v) \in \ker \omega_{p \cdot g}$, and thus

$$T_p \sigma_g(H_p) = H_{p \cdot g}.$$

Finally, smoothness follows from the fact that ω is a smooth form. ■

From now on, if ω is a connection 1-form, we will simply call it a connection. In physics lingo, connections are often called *gauge fields* or *gauge potentials*.

Example 1.6 (Maurer-Cartan connection). Let G be a Lie group, which we interpret as a principal G -bundle over a one-point space $G \hookrightarrow G \xrightarrow{\pi} \{\star\}$. For each $g \in G$, we have a way to map $T_g G$ to $\mathfrak{g} = T_e G$, simply by pushing vectors via one of the multiplications; for instance

$$T_g L_{g^{-1}} : T_g G \rightarrow \mathfrak{g} = T_e G.$$

We then define the **Maurer-Cartan** form of G , denoted by $\Theta \in \Omega^1(G, \mathfrak{g})$, as

$$\Theta_g = T_g L_{g^{-1}}.$$

The heading of the example spoiled the surprise: Θ is a connection on G . Indeed, for $X \in \mathfrak{g} = T_e G$, we have that

$$a_g(X) = \left. \frac{d}{dt} \right|_{t=0} g \exp(tX) = T_e L_g(X),$$

so that

$$\Theta_g(a_g(X)) = T_g(L_{g^{-1}})(T_e L_g(X)) = T_g(L_{g^{-1}} \circ L_g)(X) = X.$$

Now for any $h \in G$, we have

$$(\sigma_g^* \Theta)_h(X) = \Theta_{hg}(T_h \sigma_g(X)) = T_{hg} L_{g^{-1}h^{-1}} T_h \sigma_g(X) = T_h(L_{g^{-1}h^{-1}} \circ \sigma_g)(X).$$

But then, we see that

$$(L_{g^{-1}h^{-1}} \circ \sigma_g)(x) = g^{-1}h^{-1}xg = (\text{Conj}_{g^{-1}} \circ L_{h^{-1}})(x),$$

such that the differential at h is

$$T_h(L_{g^{-1}h^{-1}} \circ \sigma_g) = T_h(\text{Conj}_{g^{-1}} \circ L_{h^{-1}}) = T_e \text{Conj}_{g^{-1}} T_h L_{h^{-1}} = \text{Ad}_{g^{-1}} \circ \Theta_h,$$

and so, indeed

$$(\sigma_g^* \Theta) = \text{Ad}_{g^{-1}} \circ \Theta.$$

With the Maurer-Cartan form, we can construct connections on any principal bundle.

Example 1.7 (Trivial connection on a trivial bundle). Let $P = M \times G$ be a trivial bundle, and $\text{pr}_2 : M \times G \rightarrow G$ be the projection onto G . If Θ is the Maurer-Cartan form of G , then $\text{pr}_2^* \Theta$ is a connection on $M \times G$, and its horizontal distribution is precisely given by $H_{(x,g)} := T_x M \oplus 0 \subset T_{(x,g)}(M \times G)$.

Example 1.8 (Flat connection on a bundle). Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G -bundle, with a trivializing cover $\{(U_j, \Psi_j)\}_{j \in J}$. We write each $\Psi_j : \pi^{-1}(U_j) \rightarrow U_j \times G$ as

$$\Psi_j(p) = (\pi(p), \psi(p)),$$

with $\psi : \pi^{-1}(U_j) \rightarrow G$. Then $\psi_j^* \Theta$ is a connection on the trivial bundle $\pi^{-1}(U_j) \cong U_j \times G$, where Θ is the Maurer-Cartan form of G . Now let $\{f_j\}_{j \in J}$ be a partition of unity subordinate to $\{U_j\}_{j \in J}$. Then the form defined as

$$\omega = \sum_j (f_j \circ \pi) \psi_j^* \Theta$$

is a connection on P .

1.3 Local expressions, or, why physicists did nothing wrong

Consider a trivializing cover $\{(U_j, \Psi_j)\}_{j \in J}$ of the bundle $\pi : P \rightarrow M$, where we write each $\Psi_i : \pi^{-1}(U_i) \rightarrow U_i \times G$ as

$$\Psi_i(p) = (\pi(p), \psi_i(p)),$$

with $\psi_i : U_i \rightarrow G$. We know that each trivialization Ψ_i has an associated section $s_i : U_i \rightarrow P$, given by

$$s_i(x) = \Psi_i^{-1}(x, e)$$

for all $x \in U_i$. These sections are called **local gauges** in the physics literature.

Note that for all $x \in U_i$ and $p \in \pi^{-1}(x)$,

$$\Psi_i(s_i(x) \cdot \psi_i(p)) = (x, \psi_i(s_i(x))\psi_i(p)) = (x, \psi_i(p)) = \Psi_i(p),$$

and therefore we have that

$$p = s_i(x) \cdot \psi_i(p).$$

Now if $x \in U_{ij} = U_i \cap U_j$, for all elements $p \in \pi^{-1}(x)$, we obtain for both sections

$$s_i(x) \cdot \psi_i(p) = p = s_j(x) \cdot \psi_j(p),$$

and thus

$$s_j(x) = s_i(x) \cdot \psi_i(p)\psi_j(p)^{-1}.$$

But now, since the trivializations are G -equivariant, $\psi_i(p \cdot g) = \psi_i(p)g$, the product $\psi_i(p)\psi_j(p)^{-1}$ is G -invariant, and is precisely the transition function $g_{ij} : U_{ij} \rightarrow G$:

$$g_{ij}(x) := \psi_i(p)\psi_j(p)^{-1}.$$

We then conclude:

$$s_j(x) = s_i(x) \cdot g_{ij}(x).$$

See figure 2.

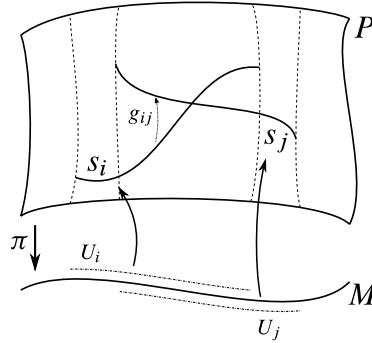


Figure 2: The transition functions g_{ij} relate the sections s_i, s_j induced by the trivializations.

Now let $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection. For each U_i , the pullback of ω by s_i is again a \mathfrak{g} -valued 1-form on U_i . We denote it by

$$\omega_i := s_i^* \omega$$

and call it the **local gauge potential** (in the gauge s_i). How do different local gauges relate to one another?

Proposition 1.9 (Transformation of local potentials). *Let ω be a connection on $G \curvearrowright P \xrightarrow{\pi} M$, and $\{U_i\}_{i \in J}$ a trivializing cover with induced sections $s_i : U_i \rightarrow P$, and transition maps $g_{ij} : U_i \cap U_j \rightarrow G$. Let $\omega_i = s_i^* \omega$ be the local gauge potentials. Then for all $x \in U_{ij} = U_i \cap U_j$,*

$$(\omega_j)_x = \text{Ad}_{g_{ij}(x)^{-1}} \circ (\omega_i)_x + (g_{ij}^* \Theta)_x, \quad (1)$$

where Θ is the Maurer-Cartan form of example 1.6.

Proof. — Let's try to brute-force it first, and see what else we need. we have that

$$(\omega_j)_x = (s_j^* \omega)_x = \omega_{s_j(x)} \circ T_x s_j,$$

so we need to find the expression for $T_x s_j$, preferably in terms of s_i . To do so, let $\sigma : P \times G \rightarrow P$ be the action, i.e. $\sigma(p, g) = p \cdot g$. Then for all $x \in U_{ij}$ we can write $s_j(x)$ as

$$s_j(x) = s_i(x) \cdot g_{ij}(x) = \sigma(s_i(x), g_{ij}(x)) = (\sigma \circ (s_i, g_{ij}))(x),$$

where we have $(s_j, g_{ij}) : U \rightarrow P \times M$ is defined in the natural way. This tells us that

$$T_x s_j = T_x(\sigma \circ (s_i, g_{ij})) = T_{(s_j(x), g_{ij}(x))} \sigma \circ T_x(s_j, g_{ij}) = T_{(s_j(x), g_{ij}(x))} \sigma \circ (T_x s_i, T_x g_{ij}).$$

Now we need to find the expression for $T_{(p,g)} \sigma$. We proceed carefully, in parts, noting that $T_{(p,g)}(P \times G) \cong T_p P \oplus T_g G$. Let $u \in T_p P$, and γ an integral curve of u . Then we have that

$$T_{(p,g)} \sigma(u, 0) = \left. \frac{d}{dt} \right|_{t=0} \sigma(\gamma(t), g) = \left. \frac{d}{dt} \right|_{t=0} \gamma(t) \cdot g = \left. \frac{d}{dt} \right|_{t=0} \sigma_g(\gamma(t)) = T_p \sigma_g(u).$$

On the other hand, let $\xi \in T_g G$. Then we have that $\Theta_g(\xi) := X \in \mathfrak{g} = T_e G$ is the (unique) element of the Lie algebra that satisfies

$$\left. \frac{d}{dt} \right|_{t=0} g \exp(tX) = T_e L_g(X) = \xi,$$

so that $t \mapsto g \exp(t\Theta_g(\xi))$ is an integral curve of ξ . Therefore

$$\begin{aligned} T_{(p,g)} \sigma(0, \xi) &= \left. \frac{d}{dt} \right|_{t=0} \sigma(p, g \exp(t\Theta_g(\xi))) \\ &= \left. \frac{d}{dt} \right|_{t=0} p \cdot g \exp(t\Theta_g(\xi)) \\ &= \left. \frac{d}{dt} \right|_{t=0} (p \cdot g) \cdot \exp(t\Theta_g(\xi)) \\ &= a_{p \cdot g}(\Theta_g(\xi)). \end{aligned}$$

We put these two together, and obtain

$$T_{(p,g)} \sigma(u, \xi) = T_p \sigma_g(u) + a_{p \cdot g}(\Theta_g(\xi)).$$

Substituting in $T_x s_j$, and evaluating at some $v \in T_x U_{ij}$,

$$\begin{aligned} T_x s_j(v) &= T_{(s_i(x), g_{ij}(x))} \sigma(T_x s_i(v), T_x g_{ij}(v)) = T_{s_i(x)} R_{g_{ij}(x)}(T_x s_i(v)) + a_{s_i(x) \cdot g_{ij}(x)}(\Theta_{g_{ij}(x)}(T_x g_{ij}(v))) \\ &= T_{s_i(x)} R_{g_{ij}(x)}(T_x s_i(v)) + a_{s_j(x)}((g_{ij}^* \Theta)_x(v)). \end{aligned}$$

Now we evaluate $\omega_{s_j(x)}$ on $T_x s_j(v)$. By definition, we have

$$\omega_{s_j(x)}(a_{s_i(x)}((g_{ij}^* \Theta)_x(v))) = (g_{ij}^* \Theta)_x(v).$$

We have to do a little bit more work for the other term. We will simply write s_j, g_{ij}, s_j for $s_j(x)$, etc., to avoid the clutter. Then we have

$$\begin{aligned}
\omega_{s_j}(T_x s_j(u)) &= \omega_{s_j}(T_{s_i} R_{g_{ij}}(T_x s_i(u))) \\
&= \omega_{s_i g_{ij}}(T_{s_i} R_{g_{ij}}(T_x s_i(u))) \\
&= (R_{g_{ij}}^* \omega)_{s_i}(T_x s_i(u)) \\
&= \text{Ad}_{g_{ij}^{-1}}(\omega_{s_i}(T_x s_i(u))) \\
&= \text{Ad}_{g_{ij}^{-1}}((s_i^* \omega)_x(u)) \\
&= (\text{Ad}_{g_{ij}(x)^{-1}} \circ (\omega_i)_x)(u).
\end{aligned}$$

Placing these two last results together, we obtain the result. ■

This proposition, in physics, is often called *gauge transformation* of a potential. In physics we mostly work with the local potentials, not with the global connection in the total space P , and we define a gauge potential as *some* object that under a certain set of (local) transformations, transforms as in equation (1). Indeed, the following result tells us that this information is sufficient to reconstruct the global object. The proof is a bit tedious and not particularly enlightening (we did a lot of the work in previous proposition).

Proposition 1.10 (Physicists did nothing wrong). *Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G -bundle, and $\{(U_i, \Psi_i)\}_{i \in J}$ a trivializing cover with induced sections $s_i : U_i \rightarrow P$. Suppose that for each U_i , there is a \mathfrak{g} -valued 1-form $\omega_i \in \Omega^1(U_i, \mathfrak{g})$, such that for all $x \in U_i \cap U_j$,*

$$(\omega_j)_x = \text{Ad}_{g_{ij}(x)^{-1}} \circ (\omega_i)_x + g_{ij}^* \Theta_x.$$

Then there exists a unique connection $\omega \in \Omega^1(P, \mathfrak{g})$ such that for all $i \in J$,

$$s_i^* \omega = \omega_i.$$

1.4 Horizontal lifts

Once we have a connection, we now have a preferred way of *lifting* vectors from TM to TP . Recall that a vector $Y \in T_p P$ is a **lift** of $X \in T_{\pi(p)} M$ if $T_p \pi(Y) = X$. In absence of a connection, there are many different choices of lifts of a vector, and any two choices differ by a vertical vector. That is, if Y, Y' are lifts of X , then $Y - Y'$ is vertical. Once we have a connection, we can define the **horizontal lift** (with respect to a connection H) of $X \in T_x M$ as the horizontal component of *any* lift of X . This definition is, of course, independent of the choice of lift, since any two differ by a vertical vector, whose horizontal component vanishes. Denoting the horizontal component of a vector by Y^H , we have then

$$Y^H = (Y' + (Y - Y'))^H = (Y')^H.$$

Similarly, we can lift vector fields by lifting them in a pointwise fashion.

Definition 1.11 (Horizontal lift of vector fields). *Let $X \in \mathfrak{X}(M)$ be a vector field. We define the **horizontal lift** of X as a vector field $Y \in \mathfrak{X}(P)$, where Y_p is the horizontal lift of $X_{\pi(p)}$.*

If H is a principal connection, then the horizontal lift Y of a vector field X is G -invariant, since $T_p\sigma_g(Y_p)$ is a horizontal vector that projects to $X_{\pi(p)}$. Therefore we have that

$$R_{g*}Y = Y.$$

We also expect a horizontal lift to commute with (some) vertical fields, since, in a sketchy intuitive sense, we define these two directions as independent. Actually, this is true of any G -invariant field.

Lemma 1.12 (G-invariant fields commute with fundamental vector fields). *Let $X^\sharp \in \mathfrak{X}(P)$ be the fundamental vector field associated to $X \in \mathfrak{g}$, and let $Y \in \mathfrak{X}(P)$ be a G -invariant field, i.e. $R_{g*}Y = Y$. Then $[X, Y] = 0$.*

Proof. — Let Φ_t be the flow of X^\sharp . It is straightforward to check that

$$\Phi_t(p) = p \cdot \exp(tX) = R_{g_t}(p),$$

where we denote $g_t = \exp(tX)$. Then

$$[X, Y]_p = \left. \frac{d}{dt} \right|_{t=0} T_{\Phi_t(p)} \Phi_{-t}(Y_{\Phi_t(p)}) = \left. \frac{d}{dt} \right|_{t=0} T_{p \cdot g_t} R_{g_t^{-1}}(Y_{p \cdot g_t}) = \left. \frac{d}{dt} \right|_{t=0} Y_p = 0. \quad \blacksquare$$

2 Curvature

2.1 The curvature 2-form and structure equation

Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G -bundle, and \mathfrak{g} be the Lie algebra of G . For any \mathfrak{g} -valued k -form $\omega \in \Omega^k(P, \mathfrak{g})$, we define $d\omega \in \Omega^{k+1}(P, \mathfrak{g})$ as follows: choose a basis $\{e_1, \dots, e_m\}$ of \mathfrak{g} . Then we can write

$$\omega = \sum_{a=1}^m \omega^a e_a,$$

where each $\omega^a \in \Omega^k(P)$. Then we define

$$d\omega := \sum_{a=1}^m d\omega^a e_a.$$

This definition is independent of the choice of basis of \mathfrak{g} , as can be readily checked.

In order to define curvature, we also need another definition.

Definition 2.1 (Bracket of valued forms). *Let $\alpha \in \Omega^k(P, \mathfrak{g})$ and $\beta \in \Omega^l(P, \mathfrak{g})$. We define a $(k+l)$ -form $[\alpha, \beta] \in \Omega^{k+l}(P, \mathfrak{g})$ in terms of a basis $\{e_1, \dots, e_m\}$ of \mathfrak{g} as*

$$[\alpha, \beta] = \sum_{a,b} \alpha^a \wedge \beta^b [e_a, e_b].$$

This definition is independent of the choice of basis (and in some references it is written as $\alpha \wedge \beta$).

In the case where $\alpha, \beta \in \Omega^1(P, \mathfrak{g})$, the definition becomes

$$\begin{aligned} [\alpha, \beta](X, Y) &= \sum_{a,b} (\alpha^a \wedge \beta^b)(X, Y) [e_a, e_b] \\ &= \sum_{a,b} (\alpha^a(X) \beta^b(Y) - \alpha^a(Y) \beta^b(X)) [e_a, e_b] \\ &= \sum_{a,b} [\alpha^a(X) e_b - \alpha^a(Y) e_b] - [\alpha^a(Y) e_a, \beta^b(X) e_b] \\ &= [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)]. \end{aligned}$$

In general, we have a way to evaluate the bracket of forms:

Lemma 2.2 (Evaluation of bracket). *Let $\alpha \in \Omega^i(P, \mathfrak{g})$ and $\beta \in \Omega^j(P, \mathfrak{g})$. Then for vectors X_1, \dots, X_{i+j} :*

$$[\alpha, \beta](X_1, \dots, X_{i+j}) = \frac{1}{i!j!} \sum_{\sigma \in \mathfrak{S}_{i+j}} \text{sgn}(\sigma) [\alpha(X_{\sigma(1)}, \dots, X_{\sigma(i)}), \beta(X_{\sigma(i+1)}, \dots, X_{\sigma(i+j)})].$$

Proof. — The proof is a straightforward evaluation and application of the definition of the wedge product. ■

Now we define the curvature 2-form of a connection.

Definition 2.3 (Curvature 2-form). *Let ω be a connection on $G \hookrightarrow P \xrightarrow{\pi} M$. The **curvature** of ω is a \mathfrak{g} -valued 2-form $\Omega \in \Omega^2(P, \mathfrak{g})$ defined as*

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]. \quad (2)$$

*Equation (2) is called the **Cartan structure equation**.*

Now let's do an example that comes with a bit of motivation.

Example 2.4 (Curvature of the Maurer-Cartan form). Recall that for a Lie group G (which we see as a G -bundle over a one-point space), we have a canonical connection Θ on G , called the Maurer-Cartan form (see example 1.6), given pointwise as

$$\Theta_g = T_g(L_{g^{-1}}).$$

The Maurer-Cartan form is also left-invariant,

$$(L_g^* \Theta)_h = \Theta_{gh} \circ T_h L_g = T_{gh} L_{(gh)^{-1}} T_h L_g = T_h (L_{(gh)^{-1}} \circ L_g) = T_h L_{h^{-1}} = \Theta_h,$$

and so it is uniquely defined by its value at the identity $e \in G$. Now fix a basis $\{e_1, \dots, e_m\}$ of \mathfrak{g} , so that the Maurer-Cartan form can be written as

$$\Theta = \sum_a \Theta^a e_a,$$

where each component $\Theta^a \in \Omega^1(G)$ is a usual 1-form. If we write ξ_a as the left-invariant field generated by e_a ;

$$\xi_a(g) := T_g L_g(e_a),$$

then at each point $g \in G$, the set $\{\xi_1(g), \dots, \xi_m(g)\}$ are a frame for $T_g G$, and furthermore, we have that

$$\Theta_g(\xi_a(g)) = e_a.$$

But on the other hand, we have

$$\Theta_g(\xi_a(g)) = \sum_b \Theta_g^b(\xi_a(g)) e_b,$$

which implies that for all g , $\Theta_g^b(\xi_a(g)) = \delta_a^b$, and so $\{\Theta^1, \dots, \Theta^m\}$ forms a coframe of $T_g^* G$ that is dual to $\{\xi_1, \dots, \xi_m\}$. Now we have that

$$d\Theta^a(\xi_b, \xi_c) = \xi_b(\Theta^a(\xi_c)) - \xi_c(\Theta^a(\xi_b)) - \Theta^a([\xi_b, \xi_c]) = -\Theta^a([e_b, e_c]) = -[e_b, e_c]^a = -C_{bc}^a,$$

where $[e_b, e_c]^a$ is the a -th component of $[e_b, e_c]$, which is precisely the definition of the structure coefficients C_{bc}^a . Now since the ξ_a vectors form a frame of TG , whose dual coframe is precisely the Θ^a forms, this tells us that

$$d\Theta^a = -\frac{1}{2} \sum_{b,c} C_{bc}^a \Theta^b \wedge \Theta^c,$$

and thus,

$$d\Theta = \sum_a d\Theta^a e_a = -\frac{1}{2} \sum_{a,b,c} C_{bc}^a \Theta^b \wedge \Theta^c e_a = -\frac{1}{2} \sum_{b,c} \Theta^b \wedge \Theta^c [e_b, e_c] = -\frac{1}{2} [\Theta, \Theta].$$

Therefore, conclude that

$$\Omega = d\Theta + \frac{1}{2} [\Theta, \Theta] = 0.$$

From this example, we can see that the curvature of the flat connection of example 1.8 has vanishing curvature as well.

Example 2.5 (Flat connection induced by Maurer-Cartan has vanishing curvature).

Let ω be the flat connection of $G \hookrightarrow P \xrightarrow{\pi} M$, as in example 1.8. Then on each $\pi^{-1}(U_j)$, writing $\omega_j = \omega|_{\pi^{-1}(U_j)}$ we have

$$\Omega|_{\pi^{-1}(U_j)} = d\omega_j + \frac{1}{2} [\omega_j, \omega_j] = d(\psi_j^* \Theta) + \frac{1}{2} [\psi_j^* \Theta, \psi_j^* \Theta] = \psi_j^* (d\Theta + \frac{1}{2} [\Theta, \Theta]) = 0.$$

We now see one of the most (if not the most) important properties of the curvature 2-form:

Proposition 2.6 (Curvature is basic). *Let ω be a connection on $G \hookrightarrow P \xrightarrow{\pi} M$ and Ω its curvature. Then $\Omega \in \Omega_{bas}^2(P, \mathfrak{g})$, that is,*

1. *If X is a vertical field, then $\iota_X \Omega = 0$, i.e. Ω is horizontal; and*
2. *For all $g \in G$, $\sigma_g^* \Omega = \text{Ad}_{g^{-1}} \circ \Omega$, i.e. Ω is pseudotensorial^a of type Ad .*

^a this is sometimes called G -invariance, or G -equivariance, but let's avoid that discussion.

Proof. — First, let's see that Ω is pseudotensorial of type Ad :

$$\sigma_g^* \Omega = \sigma_g^* d\omega + \frac{1}{2} \sigma_g^* [\omega, \omega] = d\sigma_g^* \omega + \frac{1}{2} [\sigma_g^* \omega, \sigma_g^* \omega] = d(\text{Ad}_{g^{-1}} \circ \omega) + \frac{1}{2} [\text{Ad}_{g^{-1}} \circ \omega, \text{Ad}_{g^{-1}} \circ \omega].$$

The occurrences of $\text{Ad}_{g^{-1}}$ in this previous expression may seem like there's some care required with d and the commutator, but by definition, $\text{Ad}_{g^{-1}}$ acts on the element of \mathfrak{g} that ω outputs. We can see this more clearly when we choose a basis $\{e_1, \dots, e_m\}$ of \mathfrak{g} and write $\omega = \sum_a \omega^a e_a$. When we write $\text{Ad}_{g^{-1}} \circ \omega$, this actually stands for

$$\text{Ad}_{g^{-1}} \circ \omega = \sum_a \omega^a \text{Ad}_{g^{-1}}(e_a),$$

so that the terms in the previous expression are

$$d\left(\text{Ad}_{g^{-1}} \circ \sum_a \omega^a e_a\right) = \sum_a d\omega^a \text{Ad}_{g^{-1}}(e_a) = \text{Ad}_{g^{-1}} \circ d\omega.$$

Now we consider the case of the bracket. Since $\text{Ad}_g = T_e C_g$ is the differential of a diffeomorphism, it is a pushforward evaluated at e and thus it distributes into the Lie bracket of vector fields

$$\begin{aligned} \text{Ad}_g[X_e, Y_e] &= (\text{Conj}_{g*}[X, Y])_e \\ &= [\text{Conj}_{g*} X, \text{Conj}_{g*} Y]_e \\ &= [\text{Ad}_g(X_e), \text{Ad}_g(Y_e)]. \end{aligned}$$

Then we have

$$(\sigma_g^* \Omega) = \text{Ad}_{g^{-1}} \left(d\omega + \frac{1}{2}[\omega, \omega] \right) = \text{Ad}_{g^{-1}} \circ \Omega.$$

We now need to show that Ω is horizontal. Since we have a connection, we can decompose any vector $v \in T_p P$ in a vertical and horizontal part, $v = v^V + v^H$. Then the action on Ω on a pair $u, v \in T_p P$ is

$$\Omega_p(u, v) = \Omega_p(u^V + u^H, v^V + v^H) = \Omega_p(u^V, v^V) + \Omega_p(u^V, v^H) + \Omega_p(u^H, v^V) + \Omega_p(u^H, v^H),$$

and thus, it suffices to consider two cases: when both u and v are vertical, or when u is vertical and v is horizontal.

Let's begin with the case where both u and v are vertical, so that $u = a_p(X)$ and $v = a_p(Y)$ for some $X, Y \in \mathfrak{g}$ (namely $X = \omega_p(u)$ and $Y = \omega_p(v)$). If we write X^\sharp, Y^\sharp for the fundamental vector fields associated to X, Y , given by $X_p^\sharp = a_p(X) = \sigma_{p*}(X)$ (and same for Y), we have then that

$$\begin{aligned} \Omega_p(u, v) &= d\omega_p(u, v) + \frac{1}{2}[\omega, \omega](u, v) \\ &= u(\omega(Y^\sharp)) - v(\omega(X^\sharp)) - \omega([u, v]) + [\omega(u), \omega(v)]. \end{aligned}$$

But $\omega(X^\sharp) = X$ and $\omega(Y^\sharp) = Y$ are constant, so

$$\Omega_p(u, v) = -\omega([u, v]) + [X, Y].$$

Finally, we see that

$$[u, v] = [X^\sharp, Y^\sharp]_p = [\sigma_{p*} X, \sigma_{p*} Y]_p = \sigma_{p*}([X, Y]_e) = a_p([X, Y]),$$

so $\omega([u, v]) = [X, Y]$, and thus

$$\Omega_p(u, v) = 0.$$

Now let's consider the case where u is vertical and v is horizontal. Again, let $X = \omega_p(u) \in \mathfrak{g}$, and $X^\#$ be the fundamental vector field associated to X , so that $X_p^\# = u$; and let ν be a horizontal field such that $\nu_p = v$. We then have

$$\begin{aligned}\Omega_p(u, v) &= d\omega_p(u, v) + \frac{1}{2}[\omega, \omega](u, v) \\ &= u(\omega(\nu)) - \nu(\omega(X^\#)) - \omega([u, v]) + [\omega(u), \omega(v)] \\ &= -\omega([u, v]).\end{aligned}$$

Now it suffices to show that $[u, v]$ is horizontal if v is horizontal and u is vertical. First, we have that the flow of the fundamental vector field $X^\#$ is given by

$$\Phi_t(p) = p \cdot \exp(tX),$$

as can be readily checked. Then

$$[u, v] = \left. \frac{d}{dt} \right|_{t=0} (\Phi_{-t*}(\nu))_p = \left. \frac{d}{dt} \right|_{t=0} T_{\Phi_t(p)} \Phi_{-t}(\nu_{\Phi_t(p)})$$

If we write $g_t = \exp(tX)$, then it is clear that $\Phi_t(p) = \sigma_{g_t}(p)$, so

$$[u, v] = \left. \frac{d}{dt} \right|_{t=0} T_{\Phi_t(p)} \Phi_{-t}(\nu_{\Phi_t(p)}) = \left. \frac{d}{dt} \right|_{t=0} T_{p \cdot g_t} \sigma_{g_t}^{-1}(\nu_{p \cdot g_t}).$$

However, we know that $(\sigma_{g*})(v)$ is horizontal for all g if v is horizontal, and thus we obtain that

$$T_{p \cdot g_t} \sigma_{g_t}^{-1}(\nu_{p \cdot g_t}) \in H_p \quad \text{for all } t,$$

and so $[u, v]$ is horizontal as well. Therefore $\omega([u, v]) = 0$, and our result is proved. \blacksquare

Since Ω is horizontal, its values are uniquely determined by the horizontal components of the vectors that it is evaluated at. The following corollary is often given as the definition of the curvature form:

Corollary 2.7. *Let ω be a connection and Ω its curvature. Then for all $u, v \in TP$:*

$$\Omega(u, v) = d\omega(u^H, v^H),$$

where u^H, v^H are the horizontal components of u, v , determined by ω .

2.2 The exterior covariant derivative

From corollary 2.7, we see that the curvature Ω can be defined as the horizontal component of $d\omega$. We can extend this notion, and define the **exterior covariant derivative** $d^\omega : \Omega^k(P, \mathfrak{g}) \rightarrow \Omega^{k+1}(P, \mathfrak{g})$ as the horizontal component of the usual de Rham differential:

$$d^\omega \alpha(X_1, \dots, X_{k+1}) := d\alpha(X_1^H, \dots, X_{k+1}^H).$$

With this definition, we can simply write

$$\Omega = d^\omega \omega.$$

Clearly, by definition, $d^\omega \alpha$ is horizontal for any form $\alpha \in \Omega^k(P, \mathfrak{g})$. We also see that $d^\omega \alpha$ is pseudotensorial of type Ad if α also is. The idea is that σ_g preserves horizontality and the

pullback commutes with d , so in general pulling back by σ_g should behave reasonable well. Indeed, let $\alpha \in \Omega^k(P, \mathfrak{g})$ be pseudotensorial of type Ad . Then

$$\begin{aligned}
(\sigma_g^* d^\omega \alpha)_p(X_1, \dots, X_{k+1}) &= (d^\omega \alpha)_{p \cdot g}(\sigma_{g*} X_1, \dots, \sigma_{g*} X_{k+1}) \\
&= d\alpha_{p \cdot g}((\sigma_{g*} X_1)^H, \dots, (\sigma_{g*} X_{k+1})^H) \\
&= d\alpha_{p \cdot g}(\sigma_{g*}(X_1^H), \dots, \sigma_{g*}(X_{k+1}^H)) \\
&= (\sigma_g^* d\alpha)_p(X_1^H, \dots, X_{k+1}^H) \\
&= d(\sigma_g^* \alpha)_p(X_1^H, \dots, X_{k+1}^H) \\
&= \text{Ad}_{g^{-1}} d\alpha_p(X_1^H, \dots, X_{k+1}^H) \\
&= \text{Ad}_{g^{-1}} d^\omega \alpha_p(X_1, \dots, X_{k+1}).
\end{aligned}$$

He have then shown:

Lemma 2.8 (Exterior covariant derivative preserves basicness). *If $\alpha \in \Omega_{bas}(P, \mathfrak{g})$, then $d^\omega \alpha \in \Omega_{bas}^{k+1}(P, \mathfrak{g})$.*

This result suggests that d^ω is particularly well-behaved on basic forms.

Proposition 2.9 (Expression for exterior covariant derivative on basic forms). *Let $\alpha \in \Omega_{bas}^k(P, \mathfrak{g})$ be a basic form. Then*

$$d^\omega \alpha = d\alpha + [\omega, \alpha].$$

Proof. — Let's consider the right-hand side. Let X_0, \dots, X_k be vectors on $T_p P$. If all of them are horizontal, then the term $[\omega, \alpha]$ vanishes on them because, by definition, ω vanishes on horizontal vectors, and we end up with the definition of the exterior covariant derivative. Recalling the coordinate-free expression for the exterior differential

$$\begin{aligned}
d\alpha(X_0, \dots, X_k) &= \sum_{j=0}^k (-1)^j X_j(\alpha(X_0, \dots, \hat{X}_j, \dots, X_k)) \\
&\quad + \sum_{i < j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k),
\end{aligned}$$

we see that the whole thing vanishes whenever there is more than 1 vertical vector, since we will always end up evaluating α in one of them. Similarly, we can see that in the evaluation of the bracket (following lemma 2.2),

$$[\omega, \alpha](X_0, \dots, X_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_{k+1}} \text{sgn}(\sigma) [\omega(X_{\sigma(0)}), \alpha(X_{\sigma(1)}, \dots, X_{\sigma(k)})],$$

if there is more than one vertical vector, we will always evaluate α in one of them, so everything vanishes. Then, since $d^\omega(\alpha)$ is horizontal, we trivially obtain the result.

The only non-trivial case is the one where we evaluate in exactly one vertical vector. Without loss of generality, suppose X_0 is vertical and X_1, \dots, X_k are horizontal. We still have that

$$d^\omega \alpha(X_0, \dots, X_k) = 0,$$

so we need to show that

$$d\alpha(X_0, \dots, X_k) = -[\omega, \alpha](X_0, \dots, X_k).$$

On the right-hand side, we see that the evaluation of $[\omega, \alpha]$ reduces to the sum of the permutations where we evaluate ω on the vertical vector X_0 , that is,

$$\begin{aligned} [\omega, \alpha](X_0, \dots, X_k) &= \frac{1}{k!} \sum_{\substack{\sigma \in \mathfrak{S}_{k+1} \\ \sigma(0)=0}} \text{sgn}(\sigma) [\omega(X_{\sigma(0)}), \alpha(X_{\sigma(1)}, \dots, X_{\sigma(k)})] \\ &= \frac{1}{k!} \sum_{\sigma' \in \mathfrak{S}_k} \text{sgn}(\sigma') [\omega(X_0), \alpha(X_{\sigma'(1)}, \dots, X_{\sigma'(k)})] \\ &= \frac{1}{k!} \sum_{\sigma' \in \mathfrak{S}_k} \text{sgn}(\sigma')^2 [\omega(X_0), \alpha(X_1, \dots, X_k)] \\ &= [\omega(X_0), \alpha(X_1, \dots, X_k)]. \end{aligned}$$

Here we used the fact that a permutation that fixes 0 can be written as $\sigma(0) = 0; \sigma(i) = \sigma'(i)$ with $\sigma' \in \mathfrak{S}_k$, and these satisfy $\text{sgn}(\sigma) = \text{sgn}(\sigma')$. We have also used the fact that α is antisymmetric.

Now we want to evaluate $d\alpha$, and for such we will use the long coordinate-free expression of the exterior derivative. First, letting $\xi = \omega_p(X_0) \in \mathfrak{g}$, we can extend X_0 to a vertical vector field (which we denote with the same symbol), as $X_0(p) = a_p(\xi)$; i.e. to the fundamental vector field associated to ξ . Second, we can also extend the vectors X_1, \dots, X_k to horizontal vector fields that are furthermore G -invariant. To do so, we extend $T_p\pi(X_j) \in T_{\pi(p)}M$ to a vector field on M , and consider its horizontal lift (see section 1.4), which we denote with the same symbol X_j . With this construction, since horizontal lifts are G -invariant and G -invariant fields commute with fundamental vector fields (lemma 1.12), we have that

$$\alpha([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k) = 0.$$

This follows for $i = 0$, since we evaluate on the bracket of a fundamental vector field and a G -invariant field, which is vanishing. When $i > 0$, we are evaluating α directly on a vertical field, so everything vanishes as well. Then we need only consider

$$d\alpha(X_0, \dots, X_k) = \sum_{j=0}^k (-1)^j X_j(\alpha(X_0, \dots, \hat{X}_j, \dots, X_k)) = X_0(\alpha(X_1, \dots, X_k)).$$

The only term in the sum that does not immediately vanish is the one where we don't evaluate α on X_0 . Now we evaluate at a point p . An integral curve of X_0 at p is $t \mapsto p \cdot \exp(t\xi)$, and we write $g_t = \exp(t\xi)$, so

$$\begin{aligned} d\alpha_p(X_0, \dots, X_k) &= X_0(p)(\alpha(X_1, \dots, X_k)) \\ &= \left. \frac{d}{dt} \right|_{t=0} \alpha_{p \cdot g_t}(X_1(p \cdot g_t), \dots, X_k(p \cdot g_t)) \\ &= \left. \frac{d}{dt} \right|_{t=0} \alpha_{p \cdot g_t}(T_p\sigma_{g_t}(X_1(p)), \dots, T_p\sigma_{g_t}(X_k(p))) \\ &= \left. \frac{d}{dt} \right|_{t=0} (\sigma_{g_t}^* \alpha)_p(X_1(p), \dots, X_k(p)) \\ &= \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{g_t^{-1}} \alpha_p(X_1(p), \dots, X_k(p)) \\ &= \text{ad}(-\xi)(\alpha_p(X_1(p), \dots, X_k(p))) \\ &= -[\xi, \alpha_p(X_1(p), \dots, X_k(p))] \\ &= -[\omega(X_0), \alpha_p(X_1(p), \dots, X_k(p))]. \end{aligned}$$

■

A corollary of this expression is that d^ω is not nilpotent. This means that we cannot (immediately) construct a cohomology theory based on basic forms and the exterior covariant derivative!

Corollary 2.10 (Exterior covariant derivative is not nilpotent). *Let $\varphi \in \Omega_{bas}^0(P, \mathfrak{g})$. Then*

$$(d^\omega \circ d^\omega)\varphi = [\Omega, \varphi].$$

Proof. — We have

$$\begin{aligned} d^\omega(d^\omega\varphi) &= d(d^\omega\varphi) + [\omega, d^\omega\varphi] \\ &= d(d\varphi + [\omega, \varphi]) + [\omega, d\varphi] + [\omega, [\omega, \varphi]] \\ &= d[\omega, \varphi] + [\omega, d\varphi] + [\omega[\omega, \varphi]] \\ &= [d\omega, \varphi] - [\omega, d\varphi] + [\omega, d\varphi] + [\omega[\omega, \varphi]] \\ &= [d\omega, \varphi] + [\omega, [\omega, \varphi]]. \end{aligned}$$

Here we used the fact that for $\alpha \in \Omega^k(P, \mathfrak{g})$ and $\beta \in \Omega^l(P, \mathfrak{g})$:

$$d[\alpha, \beta] = [d\alpha, \beta] + (-1)^k[\alpha, d\beta].$$

This can be readily checked from the definition, and it follows since the bracket is defined in terms of the wedge product.

Now let's evaluate at two vectors $u, v \in TP$:

$$\begin{aligned} [\omega, [\omega, \varphi]](u, v) &= [\omega(u), [\omega, \varphi](v)] - [\omega(v), [\omega, \varphi](u)] \\ &= [\omega(u), [\omega(v), \varphi]] - [\omega(v), [\omega(u), \varphi]] \\ &= -[\omega(u), [\varphi, \omega(v)]] - [\omega(v), [\omega(u), \varphi]] \\ &= [\varphi, [\omega(v), \omega(u)]] \\ &= [[\omega(u), \omega(v)], \varphi] \\ &= \left[\frac{1}{2}[\omega, \omega], \varphi \right](u, v). \end{aligned}$$

Therefore, we obtain

$$d^\omega(d^\omega\varphi) = [d\omega, \varphi] + \frac{1}{2}[[\omega, \omega], \varphi] = [\Omega, \varphi]. \quad \blacksquare$$

3 The relation with connections on vector bundles

3.1 From vector bundles to principal bundles

Let's go back to known waters. Let $\pi_E : E \rightarrow M$ be a vector bundle of rank k over M . Recall that a **connection** ∇ on E is (at least in one of its several flavors) a bilinear map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E),$$

where we denote $\nabla(X)(s) = \nabla_X(s)$, such that for all $X \in \mathfrak{X}(M)$, $s \in \Gamma(E)$ and $f \in C^\infty(M)$:

1. $\nabla_{fX}s = f\nabla_Xs$, and
2. $\nabla_X(fs) = f\nabla_Xs + \mathcal{L}_X(f)s$ (Leibniz rule).

At this point, we know that we have a special principal $GL(k, \mathbb{R})$ -bundle that is directly related to E , namely the frame bundle $\text{Fr}(E)$. Is there any relation between the connection ∇ and possible connections on $\text{Fr}(E)$? Can we find a connection 1-form $\omega_\nabla \in \Omega^1(\text{Fr}(E), \mathfrak{gl}(k, \mathbb{R}))$ that is induced by ∇ ?

Indeed, we can. First, we can rethink this map by fixing $s \in \Gamma(E)$. With s held fixed, we can then write

$$\begin{aligned}\nabla s &: \mathfrak{X}(M) \rightarrow \Gamma(E) \\ X &\mapsto \nabla_X s.\end{aligned}$$

By property (1) above, the map ∇s is $C^\infty(M)$ -linear, and so we can interpret it as an E -valued 1-form on M :

$$\nabla s \in \Omega^1(M, E).$$

If $f \in C^\infty(M)$ is a function, then from the Leibniz rule we obtain that for all $X \in \mathfrak{X}(M)$,

$$\nabla(fs)(X) = \nabla_X(fs) = (\mathcal{L}_X f)s + f\nabla_X s = df(X)s + f\nabla s(X),$$

so we may write

$$\nabla(fs) = df \otimes s + f\nabla s$$

Now let U be a trivializing open set of the bundle, and let $\{e_1, \dots, e_k\}$ be a frame on $E_U := \pi^{-1}(U)$. Of course, each element e_j is a section of E , so we can consider $\nabla e_j \in \Omega^1(U, E_U)$ (why E_U and not just E ?). In particular, we can write ∇e_j as

$$\nabla e_j = \sum_i \Gamma_j^i e_i,$$

where each $\Gamma_j^i \in \Omega^1(U)$ is a 1-form. We can collect all the Γ_j^i in a $\mathfrak{gl}(k, \mathbb{R})$ -valued form, whose entries are called the **connection coefficients** (or in some cases, the Christoffel symbols)

$$\Gamma = \begin{pmatrix} \Gamma_1^1 & \dots & \Gamma_k^1 \\ \vdots & \ddots & \vdots \\ \Gamma_1^k & \dots & \Gamma_k^k \end{pmatrix} \in \Omega^1(U, \mathfrak{gl}(k, \mathbb{R})).$$

What do we have at this point? For each frame $\{e_1, \dots, e_k\}$ of E , which is defined locally on $U \subseteq M$, we have a $\mathfrak{gl}(k, \mathbb{R})$ -valued 1-form Γ . This smells quite a lot like what we're looking for! If we can show that the connection coefficients transform nicely with respect to change of frames, we can invoke the physicists-did-nothing-wrong proposition (proposition 1.10) and construct a connection on $\text{Fr}(E)$.

So let e'_1, \dots, e'_k be another frame, defined on an open $U' \subseteq M$. On $U \cap U'$, each element e'_j can be expressed in terms of the first frame. For each $x \in U \cap U'$ there is a matrix $A(x) \in GL(k, \mathbb{R})$ such that

$$e'_j(x) = \sum_i A(x)^i_j e_i(x),$$

or rather, we have a $GL(k, \mathbb{R})$ -valued function A on $U \cap U'$, which is precisely the transition

function of the trivialization of $\text{Fr}(E)$. Now, when we evaluate the connection on e'_j , we get

$$\begin{aligned}\nabla e'_j &= \sum_i \nabla(A^i_j e_i) \\ &= \sum_i (dA^i_j \otimes e_i + A^i_j \nabla e_i) \\ &= \sum_i \left(dA^i_j \otimes e_i + A^i_j \sum_r \Gamma^r_i e_r \right) \\ &= \sum_i \left(dA^i_j + \sum_r A^r_j \Gamma^i_r \right) \otimes e_i.\end{aligned}$$

On the other hand,

$$\nabla e'_j = \sum_r \Gamma^{jr}_j e'_r = \sum_{i,r} \Gamma^{jr}_j A^i_r e_i.$$

Comparing with the previous result, we obtain

$$\sum_r \Gamma^{jr}_j A^i_r = dA^i_j + \sum_r A^r_j \Gamma^i_r.$$

Noting that the upper index is the column index, we see that the previous equation is for the components of the matrix equation

$$A\Gamma' = dA + \Gamma A,$$

that is

$$\Gamma' = A^{-1}\Gamma A + A^{-1}dA.$$

Indeed, we can now invoke proposition 1.10 and claim:

Theorem 3.1 (Connection induced by connection on vector bundle). *Let ∇ be a connection on a vector bundle $E \rightarrow M$ of rank k . Then there is a unique connection 1-form ω_∇ on the frame bundle $\text{Fr}(E)$ such that, given a local frame $e : U \rightarrow \text{Fr}(E)$, the local gauge potential is given by the connection coefficients:*

$$e^* \omega_\nabla = \Gamma.$$

There's also a direct way to construct ω_∇ given a connection ∇ , that does not require using the physicists-did-nothing-wrong proposition. It can be found in [Bär11, example 2.3.3] and [Cra15, section 2.3.5].

3.2 Interlude: Associated bundles

The converse of theorem 3.1 can be done with a little bit more generality, without any additional complications. We will see that given any connection $\omega \in \Omega^1(P, \mathfrak{g})$, we can find a connection on a wide array of vector bundles that are related to P .

Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G -bundle, V a vector space and $\rho : G \rightarrow \text{GL}(V)$ a representation. We define the **associated bundle** $P \times_\rho V$ as the quotient of $P \times V$ under the action

$$(p, v) \cdot g = (p \cdot g, \rho(g^{-1})v).$$

We will denote $\rho(g)v$ simply as $g \cdot v$ whenever there is no chance for confusion, and the elements of E in terms of representatives, e.g. $[p, v]$. We have a projection map $\pi_E : E \rightarrow M$ a $\pi_E([p, v]) = \pi(p)$. This map is well-defined since $\pi(p \cdot g) = \pi(p)$ for all $p \in P$ and $g \in G$.

Of course, we have just given the definition as a set. We should check that $P \times_\rho V$ is 1. a manifold 2. a vector bundle. Usually we would skip this part but actually the construction of the charts and trivialization on E will give us a better understanding of it, and will tell us how it looks locally.

First, let's look at trivializations. Once we have the trivializations, we can construct a coordinate atlas adapted from the atlas of M , as with all fiber bundles.

For each $p \in P$, we define a map $i_p : V \rightarrow E$ as

$$i_p(v) = [p, v].$$

This map is a bijection from V to the fiber $E_{\pi(p)}$ above $\pi(p)$. Clearly, by construction i_p is surjective. And to see injectivity, suppose that $i_p(v) = i_p(v')$. Then $[p, v] = [p, v']$, so there exists a $g \in G$ such that $(p, v) = (p \cdot g, g^{-1} \cdot v')$. But the action of G is free on P , so necessarily $g = e$, and so $v = v'$. This allows us to endow $E_{\pi(p)}$ with a vector space structure such that i_p is a linear isomorphism; i.e. as $[p, v] + \alpha[p, v'] = [p, v + \alpha v']$ with the *same* p . Of course, this should be the same regardless of the choice of p (in the fiber of π , that is). Indeed, if $p' \in P$ is such that $\pi(p') = \pi(p)$, then there is a $g \in G$ such that $p' = p \cdot g$.

Therefore, we have that

$$i_{p'}(v) = [p', v] = [p \cdot g, v] = [p, g \cdot v] = i_p(g \cdot v) = (i_p \circ \rho(g))(v).$$

Then $i_{p'}$ and i_p are related by an automorphism of V , so the induced vector space structure on $E_{\pi(p)}$ is the same. This last result will be useful again later, so let's put it as a lemma.

Lemma 3.2. *Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G -bundle, V a vector space and $\rho : G \rightarrow \text{GL}(V)$ a representation. Write $E = P \times_\rho V$ for the bundle associated to P via ρ . For $p \in P$, define $i_p : V \rightarrow E_{\pi(p)}$ as $i_p(v) = [p, v]$. Then i_p is a bijection, and for all $g \in G$,*

$$i_{p \cdot g} = i_p \circ \rho(g).$$

Let $\{U_i\}_{i \in I}$ be a cover of M that trivializes P . For each U_i we have a canonical section (or local gauge) $s_i : U_i \rightarrow P_{U_i}$ (see section 1.3). With this canonical section we can construct a trivialization $\Psi_i : U_i \times V \rightarrow E_{U_i}$ as

$$\Psi_i(x, v) = i_{s_i(x)}(v) = [s_i(x), v].$$

It can be shown [see e.g. [Nab10](#), pp. 381] that when we endow E with the quotient topology, E is Hausdorff and each map Ψ_i is a homeomorphism. This is a straightforward (albeit a bit tedious) check.

How do the transition functions look? Consider two trivializing open sets U_i, U_j with their canonical sections s_i, s_j , and let $U_{ij} = U_i \cap U_j$. We have that there is a transition function $g_{ij} : U_{ij} \rightarrow G$ such that for all $x \in U_{ij}$,

$$s_j(x) = s_i(x) \cdot g_{ij}(x).$$

Then, for $(x, v) \in U_{ij} \times V$, we have

$$\Psi_j(x, v) = [s_j(x), v] = [s_i(x) \cdot g_{ij}(x), v] = [s_i(x), g_{ij}(x) \cdot v] = \Psi_i(x, g_{ij}(v)).$$

This implies that

$$(\Psi_i^{-1} \circ \Psi_j)(x, v) = (x, g_{ij}(x) \cdot v) = (x, \rho(g_{ij}(x))(v)). \quad (3)$$

Then the transition functions are of the form $\rho(g_{ij}) \in \text{GL}(V)$, with g_{ij} the gauge transitions of the principal bundle. This tells us that there is a unique smooth structure on E such that the Ψ_i are diffeomorphisms, and such that $\pi_E : E \rightarrow M$ is a smooth surjection. Thus, E is a vector bundle over M with typical fiber V . Let's put it as a proposition.

Proposition 3.3 (Associated bundle is a smooth vector bundle). *Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G -bundle, V a vector space and $\rho : G \rightarrow \text{GL}(V)$ a representation. Write $E = P \times_{\rho} V$ for the bundle associated to P via ρ . Then E is a smooth vector bundle over M with typical fiber V . Furthermore, given a cover $\{U_i\}_{i \in I}$ that trivializes P , with canonical sections $s_i : U_i \rightarrow P$ and transition functions $g_{ij} : U_i \cap U_j \rightarrow G$, the maps $\Psi_i : U_i \times V \rightarrow E$ given as $\Psi_i(x, v) = [s_i(x), v]$ are trivializations of E , with transition functions $\rho(g_{ij}) : U_i \cap U_j \rightarrow \text{GL}(V)$.*

In physics, we usually keep to local gauges. In a local gauge (U_i, s_i) , the bundle $P \times_{\rho} V$ is trivial and “looks like” $U_i \times V$. Equation (3) says that under a change of gauge $s_i \mapsto s_j$, an element $v \in V$ transforms as $v \mapsto \rho(g_{ij})v$.

Example 3.4 (Frame bundles). Let $E \xrightarrow{\pi_E} M$ be a real vector bundle of rank k over a smooth manifold M . We have a principal $\text{GL}(k, \mathbb{R})$ -bundle over M , which is the frame bundle $\text{Fr}(E)$, and the identity representation of $\text{GL}(k, \mathbb{R})$ on \mathbb{R}^k , $\text{id} : \text{GL}(k, \mathbb{R}) \rightarrow \text{GL}(\mathbb{R}^k)$. It is a straightforward check to see that

$$E \cong \text{Fr}(E) \times_{\text{id}} \mathbb{R}^k.$$

Example 3.5 (Adjoint bundle). Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G bundle. We have a natural representation $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$. The vector bundle associated to P via Ad is called the **adjoint bundle** of P , and is denoted $\text{Ad}(P) := P \times_{\text{Ad}} \mathfrak{g}$.

In particular, if $E \xrightarrow{\pi_E} M$ is a vector bundle of rank k , and $\text{Fr}(E)$ is its frame bundle, then we have that

$$\text{Ad}(\text{Fr}(E)) \cong \text{End}(E).$$

How do sections of E look like locally? If we fix a cover $\{U_i\}_{i \in I}$ of M that trivializes P , with canonical sections $s_i : U_i \rightarrow P$, we have that this cover also trivializes E . Suppose that $\psi : M \rightarrow E$ is a section of E . When restricted to U_i , we have that ψ looks locally like

$$(\Psi_i^{-1} \circ \psi)(x) = (x, \psi_i(x)),$$

for some smooth $\psi_i : U_i \rightarrow V$. In fact, given a local gauge s_i , there is a *bijection* between smooth maps $\psi_i : U_i \rightarrow V$ and local sections $\psi : U_i \rightarrow E$. On the overlaps $U_i \cap U_j$, the same argument of before shows that the “trivializations” of the sections transform as

$$\psi_j(x) = g_{ji}(x) \cdot \psi_i(x).$$

Now we see that there is a deeper relation between sections of E (in fact, of E -valued forms) and V -valued forms on P . Again, let $\psi : M \rightarrow E$ be a section. In a local trivialization

U_i with canonical section $s_i : U_i \rightarrow P$, we can write any element $p \in P_{U_i} = \pi^{-1}(U_i)$ uniquely as

$$p = s_i(x) \cdot g_i$$

where $x = \pi(p)$. By the discussion above, there is a function $\psi_i : U_i \rightarrow V$ such that for all $x \in U_i$, ψ looks like

$$\psi(x) = [s_i(x), \psi_i(x)].$$

So in general, we can change the representative of $\psi(x)$ to be of the form $[p, v]$ for any p in the fiber above x ;

$$\psi(x) = [s_i(x), \psi_i(x)] = [s_i(x) \cdot g_i, g_i^{-1} \cdot \psi_i(x)].$$

We can thus define a function $\tilde{\psi}_i : P_{U_i} \rightarrow V$ as

$$\tilde{\psi}_i(s_i(x) \cdot g_i) := g_i^{-1} \cdot \psi_i(x).$$

It is a straightforward check to see that this function is well-defined on P_{U_i} ; and in fact that the collection of $\{\tilde{\psi}_i\}_{i \in I}$ glues together to a map $\tilde{\psi} : P \rightarrow V$ which satisfies that for all $p \in P$ and $g \in G$,

$$\tilde{\psi}(p \cdot g) = g^{-1} \cdot \tilde{\psi}(p).$$

We say that $\tilde{\psi}$ is **G -equivariant** or **pseudotensorial of type ρ** .

This is a general fact: k -forms on M which are valued in E correspond to a certain kind to k -forms on P which are valued in P .

Definition 3.6 (Tensorial form). Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G -bundle. We say that a form $\alpha \in \Omega^k(P, V)$ is **tensorial** of type ρ or **basic** if

1. α is horizontal, i.e. $\iota_X \alpha = 0$ for any vertical vector $X \in TP$; and
2. α is pseudotensorial of type ρ , that is,

$$\sigma_g^* \alpha = \rho(g^{-1}) \circ \alpha.$$

We denote the space of tensorial k -forms of type ρ by $\Omega_\rho^k(P, V)$.

Theorem 3.7 (Lowering of tensorial forms). Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal bundle, V a vector space, $\rho : G \rightarrow \text{GL}(V)$ a representation of G on V and $E = P \times_\rho V$ the associated vector bundle. Then there is a linear isomorphism $h : \Omega_\rho^k(P, V) \rightarrow \Omega^k(M, E)$.

Proof (Sketch). — Define $h : \Omega_\rho^k(P, V) \rightarrow \Omega^k(M, E)$ as follows: given $\bar{\phi} \in \Omega_\rho^k(P, V)$, define

$$h(\bar{\phi})_x(V_1, \dots, V_k) := [p, \bar{\phi}_p(\bar{V}_1, \dots, \bar{V}_k)],$$

where $p \in \pi^{-1}(x)$, $V_1, \dots, V_k \in T_x M$, and the \bar{V}_i are lifts of the V_i ; that is, $T_p \pi(\bar{V}_i) = V_i$ for $i = 1, 2, \dots, k$.

It is straightforward, but a bit long, to check that h is well-defined.

The inverse of h can be given explicitly: given $\psi \in \Omega^k(M, E)$, we define $h^{-1}\psi \in \Omega_\rho^k(P, V)$ as

$$(h^{-1}\psi)_p(\bar{V}_1, \dots, \bar{V}_k) := i_p^{-1}(\pi^* \psi)_x(\bar{V}_1, \dots, \bar{V}_k).$$

Again, it is a straightforward check to see that these maps are well-defined, linear, and inverses of one another. These maps are natural in the sense that they are the obvious choice given the data that we have. ■

The associated bundle is a vector bundle with fiber V , so we now can ask ourselves if, given a connection ω on P , there is an induced connection ∇^ω on E .

3.3 From principal bundles to vector bundles

As above, for any connection ∇ on E , given a section $s \in \Gamma(E)$, we have an E -valued 1-form $\nabla s \in \Omega^1(M, E)$, so we can think of a connection as a map $\nabla : \Gamma(E) \rightarrow \Omega^1(M, E)$. Noting that a section of E is just an E -valued 0-form, and using the isomorphism h of theorem 3.7, we see that the problem is reduced to finding a suggestively-named map

$$d^\omega : \Omega_{\text{bas}}^0(P, V) \rightarrow \Omega_{\text{bas}}^1(P, V),$$

that is *nicely* related to ω and that satisfies the Leibniz rule when we go back to M . Once we have such a map, we can define ∇^ω on E such that the following diagram commutes:

$$\begin{array}{ccc} \Omega_\rho^0(P, V) & \xrightarrow{d^\omega} & \Omega_\rho^1(P, V) \\ \downarrow h & & \downarrow h \\ \Gamma(E) & \xrightarrow{\nabla^\omega} & \Omega^1(M, E) \end{array}$$

But wait a minute... for the case where $\rho = \text{Ad}$ and $V = \mathfrak{g}$, we already have a such a map, namely the exterior covariant derivative d^ω , which acts on basic forms according to proposition 2.9 as

$$d^\omega \alpha = d\alpha + [\omega, \alpha].$$

And now we use the ancient art of reverse-engineering. If α is a 0-form, we can rewrite $[\omega, \alpha]$ in terms of the adjoint representation, precisely as $[\omega, \alpha] = \text{ad}(\omega)(\alpha)$, where $\text{ad} = T_e \text{Ad}$, so that

$$d^\omega \alpha = d\alpha + (T_e \text{Ad} \circ \omega)(\alpha).$$

This suggests that for a general vector space V and representation $\rho : G \rightarrow \text{GL}(V)$, we define

$$d^\omega \alpha := d\alpha + (T_e \rho \circ \omega)(\alpha),$$

on all basic 0-forms. The derivative $T_e \rho : \mathfrak{g} \rightarrow \text{End}(V)$ is called the **infinitesimal action** of \mathfrak{g} on V , induced by the action ρ .

Explicitly, for $p \in P$ and $X \in T_p P$, it is defined as

$$d^\omega \alpha_p(X) = (d\alpha)_p(X) + ((T_e \rho)(\omega_p(X)))(\alpha(p)).$$

What we now need to show is that the map

$$\nabla^\omega := h \circ d^\omega \circ h^{-1} : \Gamma(E) \rightarrow \Omega^1(M, E),$$

satisfies the Leibniz rule,

$$\nabla^\omega(fs) = df \otimes s + f \nabla^\omega s.$$

for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$.

To prove this, we need to get our hands dirty. Let $f \in C^\infty(M)$ and $s \in \Gamma(E)$. The basic 0-form induced on P by fs is

$$h^{-1}(fs)(p) = i_p^{-1}((f(\pi(p))s(\pi(p)))) = (f \circ \pi)(p)i_p^{-1}(s(\pi(p))),$$

and so $h^{-1}(fs) = (f \circ \pi)h^{-1}(s)$. Write $\tilde{f} = f \circ \pi$, and $\tilde{s} = h^{-1}(s)$. Then \tilde{f} is a G -invariant real-valued function and \tilde{s} is a basic V -valued function. Now we apply d^ω :

$$d^\omega(\tilde{f}\tilde{s}) = d(\tilde{f}\tilde{s}) + (T_e\rho \circ \omega)(\tilde{f}\tilde{s}) = d\tilde{f}\tilde{s} + \tilde{f}d\tilde{s} + \tilde{f}(T_e\rho \circ \omega)(\tilde{s}) = d\tilde{f}\tilde{s} + \tilde{f}d^\omega\tilde{s}.$$

Here we have that \tilde{f} comes out of the differential of the representation, because once evaluated at $\omega_p(X)$ for some $p \in P, X \in T_pP$, $(T_e\rho)(\omega_p(X))$ is *linear*. Now we apply h , evaluate at a point $x \in M$ and a vector $X \in T_xM$:

$$\begin{aligned} \nabla^\omega(fs)_x(X) &= h(d^\omega(h^{-1}(fs)))_x(X) \\ &= h(d\tilde{f}\tilde{s} + \tilde{f}d^\omega\tilde{s})_x(X) \\ &= [p, T_p\tilde{f}(\tilde{X})\tilde{s}(p) + \tilde{f}(p)d^\omega\tilde{s}_p(\tilde{X})]. \end{aligned}$$

Now we recall that $\tilde{f} = f \circ \pi$, so $\tilde{f}(p) = f(x)$ and

$$T_p\tilde{f}(\tilde{X}) = T_xfT_p\pi(\tilde{X}) = T_xf(X).$$

Therefore

$$\begin{aligned} \nabla^\omega(fs)_x(X) &= [p, T_xf(X)\tilde{s}(p)] + [p, f(x)d^\omega\tilde{s}_p(\tilde{X})] \\ &= T_xf(X)[p, \tilde{s}(p)] + f(x)[p, d^\omega\tilde{s}_p(\tilde{X})] \\ &= (df \otimes s + f\nabla^\omega s)_p(X). \end{aligned}$$

Then ∇^ω is, indeed, a connection on E . We have then proved

Theorem 3.8 (Connection induced by a connection on principal bundle). *Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G -bundle, V a vector space, $\rho : G \rightarrow \text{GL}(V)$ a representation and $E = P \times_\rho V$ the associated bundle. Given a connection $\omega \in \Gamma^1(P, \mathfrak{g})$, there is an induced connection $\nabla^\omega : \Gamma(E) \rightarrow \Omega^1(M, E)$ such that the following diagram commutes:*

$$\begin{array}{ccc} \Omega_\rho^0(P, V) & \xrightarrow{d^\omega} & \Omega_\rho^1(P, V) \\ \downarrow h & & \downarrow h \\ \Gamma(E) & \xrightarrow{\nabla^\omega} & \Omega^1(M, E) \end{array}.$$

Another way to prove this theorem is to go local, and define the connection in terms of the connection coefficients. This suffices to uniquely define a connection on a vector bundle [see [Niciu8](#), Proposition 3.3.5], if the coefficients transform well enough.

So let's go local, and try to see what the beast of $\nabla^\omega = h \circ d^\omega \circ h^{-1}$ is. As always, let $\{U_i\}_{i \in I}$ be a cover of M that trivializes P , with canonical sections $s_i : U_i \rightarrow P_{U_i}$, and gauge transitions $g_{ij} : U_i \cap U_j \rightarrow G$. As we saw in section 3.2, this trivialization also induces a trivialization of the associated bundle $E = P \times_\rho V$.

We can go further and see that the trivialization of P also makes a further identification in the isomorphism $\Omega_\rho^k(P, V) \cong \Omega^k(M, E)$. Given a form $\alpha \in \Omega_\rho^k(P, V)$, we have that

$$h(\alpha)_x(V_1, \dots, V_k) = [p, \alpha_p(\bar{V}_1, \dots, \bar{V}_k)]$$

for $p \in \pi^{-1}(x)$ and $\bar{V}_1, \dots, \bar{V}_k$ lifts of the V_1, \dots, V_k . But we have a preferred choice of element in the fiber of x , namely $p = s_i(x)$. Similarly, we have a preferred lift of the V_j , namely as $\bar{V}_j = T_x s_i(V_j)$. We then have that

$$h(\alpha)_x(V_1, \dots, V_k) = [s_i(x), (s_i^* \alpha)_x(V_1, \dots, V_k)].$$

Thus, we have that the following diagram commutes:

$$\begin{array}{ccc} \Omega^k_\rho(P_{U_i}, V) & \xrightarrow{s_i^*} & \Omega^k(U_i, E) \\ & \searrow h & \downarrow i_{s_i} \\ & & \Omega^k(U_i, V) \end{array} .$$

This tells us that, given a choice of trivialization (gauge), V -valued tensorial forms on P and E -valued forms on M both reduce to V -valued forms on M .

In particular, given section $\psi : M \rightarrow E$, which looks locally on U_i as

$$\psi(x) = [s_i(x), \psi_i(x)]$$

the above diagram tells us that

$$\psi(x) = (h \circ h^{-1})(\psi(x)) = (i_{s_i(x)} \circ s_i^*)(h^{-1}(\psi(x))) = [s_i(x), s_i^*(h^{-1}\psi)(x)],$$

but since $i_{s_i(x)}$ is an isomorphism, then

$$\psi_i(x) = s_i^*(h^{-1}\psi)(x).$$

Now apply $\nabla^\omega \psi$. By the previous result,

$$\nabla^\omega \psi(x) = h(d^\omega(h^{-1}\psi)(x)) = [s_i(x), s_i^*(d^\omega h^{-1}\psi)(x)].$$

Therefore, it suffices to find $s_i^*(d^\omega h^{-1}\psi)(x)$. For the first term, we have

$$s_i^*(dh^{-1}\psi) = ds_i^*(h^{-1}\psi) = d\psi_i.$$

For the second term we need to be more careful. Let's evaluate at $x \in U_i$ and $V \in T_x M$:

$$s_i^*(T_e \rho \circ \omega(h^{-1}\psi))_x(V) = (T_e \rho(\omega_{s_i(x)}(s_{i*} V)))(h^{-1}\psi(s_i(x))) := (T_e \rho \circ \omega_i)_x(V)(\psi_i(x)).$$

Here we have denoted $\omega_i = s_i^* \omega$ (in consistence with the notation of proposition 1.9). Therefore, the local expression of the connection on E is (dropping the arguments)

$$\nabla^\omega \psi = [s_i, d\psi_i + (T_e \rho \circ \omega_i)(\psi_i)].$$

3.4 In physics language

We can reduce the notation a bit more (and make it a bit more confusing), add coordinates, and obtain the equations of the “covariant derivative on matter fields” that is used by physicists. In physics, a matter field is a (local expression of a) section of the associated bundle E to a principal G -bundle. The group G is called the group of local invariance. A section $s : U \rightarrow P$ is a local gauge, and the local potentials of a connection are denoted by $\mathcal{A} := s^* \omega$.

The infinitesimal action is not denoted explicitly, so we only write $\xi \cdot v$ instead of $T_e \rho(\xi)(v)$, for $\xi \in \mathfrak{g}$ and $v \in V$. Thus, if we choose a basis $\{e_1, \dots, e_k\}$ of V , then we can write the infinitesimal action as a matrix product:

$$\mathcal{A} \cdot e_a = \mathcal{A}^b_a e_b,$$

with $\mathcal{A}^b_a \in \Omega^1(U)$ for all a, b . If we have a section $\Psi : M \rightarrow E$, then in the local gauge s it can be written as $\Psi = [s, \psi]$, with $\psi : U \rightarrow V$. This ψ is called a **matter field**, and in the basis of V , it becomes

$$\psi = \psi^a e_a.$$

If we assume that our trivializing cover is also a coordinate atlas, then on the chart U_i we also have coordinates x^μ . Therefore, we write

$$\mathcal{A}^b_a = \mathcal{A}^b_{a\mu} dx^\mu.$$

Finally, when we apply the connection to Ψ

$$\nabla^\omega \Psi = [s, \mathcal{D}\psi],$$

where

$$\begin{aligned} \mathcal{D}\psi &= d\psi + \mathcal{A} \cdot \psi \\ &= (\partial_\mu \psi^a + \mathcal{A}^a_{b\mu} \psi^b) dx^\mu \otimes e_a \\ &:= \mathcal{D}_\mu \psi^a (dx^\mu \otimes e_a). \end{aligned}$$

Here, the operator \mathcal{D}_μ is called the covariant derivative:

$$\mathcal{D}_\mu = \partial_\mu + \mathcal{A}_\mu.$$

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