What is a gauge field?
Part 1: Electromagnetism

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The objective of the following posts is to attempt to give an answer to the age-old question: What is a gauge field?

That's a tall order, alright. In order to understand what gauge fields are, I hope to construct a direct *dictionary* between classical gauge fields as physicists know them, and the language of principal bundles that mathematicians use. The parallel between both is striking, but I haven't found an actual dictionary that lets you go straight from one to the other. And well, since I'm a completionist it doesn't just suffice to spell it out, but rather to build it nicely.

This first part does not have too many prerrequisites: only the basics of electromagnetism. I'll try to be as self-contained as possible in the physics part. Without further ado, let's begin.

1 Potentials for the electric and magnetic field

In Gaussian units, the microscopic (or vacuum) Maxwell equations are

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \nabla \cdot \mathbf{E} = 4\pi\rho$$
$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J},$$

where ρ is the electric charge density and $\mathbf{J}$ is the electric current density. Here, the fields $\mathbf{E}, \mathbf{B}$, and the current density $\mathbf{J}$ are time-dependent vector fields on some open subset of $U \subseteq \mathbb{R}^3$,

$$\mathbf{E}, \mathbf{B}, \mathbf{J} : \mathbb{R} \times U \to \mathbb{R}^3,$$

and the charge density ρ is a time-dependent scalar function on U,

$$\rho : \mathbb{R} \times U \to \mathbb{R}.$$

The objective with these equations is to determine the electric and magnetic fields $\mathbf{E}, \mathbf{B}$, given the source functions $\mathbf{J}$ and ρ (and boundary conditions and all that so that the PDE is actually soluble).

Now we have a *trick* to make it easier to find a solution to the equations. The trick is to see that we can automatically satisfy the homogeneous equations (on the left) by a clever rewriting of $\mathbf{E}$ and $\mathbf{B}$. Indeed, the equation $\nabla \cdot \mathbf{B} = 0$ suggests (but does not *imply*[1]) that we write

$$\mathbf{B} = \nabla \times \mathbf{A},$$

for some other vector field $\mathbf{A}$, which we call the **magnetic vector potential**. Once we have this, the other homogeneous equation becomes

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = \nabla \times \left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = 0.$$

Again, this suggests (but not always implies[2]) that we write

$$\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi,$$

or rather

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

for some function ϕ which we call the **electric potential** (the negative sign is a convention[3]). For such choices of $\mathbf{A}$ and ϕ, the homogeneous Maxwell equations are immediately satisfied. Of course the choice of $\mathbf{A}$ and ϕ must be such that the inhomogeneous equations are still satisfied, but this reduces the problem from finding two vector fields $\mathbf{E}$, $\mathbf{B}$ satisfying the full Maxwell equations to finding one scalar field ϕ and a vector field $\mathbf{A}$ that satisfy the (admittedly ugly) equations

$$\begin{aligned} -\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) &= 4\pi\rho \\ \nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\phi) &= \frac{4\pi}{c}\mathbf{J}. \end{aligned}$$

Of course, the choice of $\mathbf{A}$ and ϕ are not *unique*. Once we have a choice of $\mathbf{A}$ and ϕ, then for *any* smooth scalar field Λ, we can change $\mathbf{A}$ as

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda,$$

and of course we will still obtain

$$\nabla \times \mathbf{A}' = \nabla \times \mathbf{A} + \nabla \times (\nabla\Lambda) = \nabla \times \mathbf{A} = \mathbf{B}.$$

Then $\mathbf{A}' = \mathbf{A} + \nabla\Lambda$ is also another magnetic vector potential for $\mathbf{B}$. Under this new magnetic potential, we have for the electric field

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}'}{\partial t} + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\Lambda) = -\nabla\left(\phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}\right) - \frac{1}{c}\frac{\partial \mathbf{A}'}{\partial t},$$

and so if we define a "new" electric potential ϕ' as

$$\phi' = \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t},$$

[1]When does $\nabla \cdot \mathbf{F} = 0$ imply that $\mathbf{F} = \nabla \times \mathbf{G}$ for some other vector field $\mathbf{G}$? It depends solely on the topology of the underlying space, which we have not discussed yet. The functions grad, curl, and div define a cochain complex

$$0 \to C^\infty(U) \overset{\text{grad}}{\to} \mathfrak{X}(U) \overset{\text{curl}}{\to} \mathfrak{X}(U) \overset{\text{div}}{\to} C^\infty(U) \to 0, \tag{1}$$

called the GCD complex, where $U \subseteq \mathbb{R}^3$ is the domain of definition of our vector fields, assumed to be open. The standard metric on $\mathbb{R}^3$ induces an isomorphism between the GCD and the de Rham complex (via the Hodge dual), and so we have that ker(div) = im(curl) if and only if the second cohomology group $H^2(U)$ is trivial. In most applications in physics, the underlying space is convex, so everything works nicely.

[2]Similarly to the previous footnote, this holds if the underlying space is simply-connected, since in this case we would need the first cohomology group $H^1(U)$ to be trivial.

[3]which does have a neat physical interpretation in terms of energy, but which we shall not discuss.

we still can write

$$\mathbf{E} = -\nabla\phi' - \frac{1}{c}\frac{\partial \mathbf{A}'}{\partial t}.$$

This tells us that the pair $\mathbf{A}', \phi'$ is another perfectly good choice of potentials for $\mathbf{E}$ and $\mathbf{B}$.

In summary, we can reduce Maxwell's equations on $\mathbf{E}$ and $\mathbf{B}$ to two (hopefully easier) equations on a pair of potentials $\mathbf{A}, \phi$. Once we have found such potentials $\mathbf{A}, \phi$, we can recover the electric and magnetic fields as

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A}, \\ \mathbf{E} &= -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \end{aligned}$$

The choice of potentials $\mathbf{A}, \phi$ is *not unique*, since for any smooth scalar field Λ, we can define new potentials $\mathbf{A}', \phi'$ as

$$\begin{aligned} \mathbf{A}' &= A + \nabla\Lambda, \\ \phi' &= \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}, \end{aligned}$$

and we still obtain the same $\mathbf{E}, \mathbf{B}$. This change of the potentials is called a **gauge transformation** with **gauge function** Λ, and we say that the electric and magnetic fields $\mathbf{E}, \mathbf{B}$ are **gauge-invariant**.

2 In special-relativistic notation

We can rewrite this more neatly[4] by in the Minkowski spacetime of special relativity. The space we are working in is $M = \mathbb{R} \times U \subseteq \mathbb{R}^4$, with global coordinates

$$x^0 = ct, \qquad x^1 = x, \qquad x^2 = y, \qquad x^3 = z,$$

where c is the speed of light in your favorite units. We also have a metric η given in coordinates as

$$\eta = \mathrm{d}x^0 \otimes \mathrm{d}x^0 - \sum_{i=1}^{3} \mathrm{d}x^i \otimes \mathrm{d}x^i = \eta_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu.$$

Here we used Einstein's notation, and we will follow the usual conventions of raising and lowering indices for the isomorphism $TM \cong T^*M$ induced by the metric[5]. Now we define a 2-form $F \in \Omega^2(M)$, called the **Faraday** or **electromagnetic tensor** whose components with respect to these coordinates are

$$[F_{\mu\nu}] = \begin{pmatrix} 0 & \mathbf{E}^1 & \mathbf{E}^2 & \mathbf{E}^3 \\ -\mathbf{E}^1 & 0 & -\mathbf{B}^3 & \mathbf{B}^2 \\ -\mathbf{E}^2 & \mathbf{B}^3 & 0 & -\mathbf{B}^1 \\ -\mathbf{E}^3 & -\mathbf{B}^2 & \mathbf{B}^1 & 0 \end{pmatrix}.$$

A note on notation: Let's think of the bold symbols as overriding Einstein's notation. This equation should be seen literally, component-wise, e.g. $F_{01} = \mathbf{E}^1$, and of course the Einstein notation doesn't add up here. It doesn't matter too much at this point[6].

[4]For our purposes, this rewriting is simply for the sake of making everything clearer. What we are really doing is rewriting the equations of electromagnetism in the language of special relativity. It is no coincidence that this amounts just to a *rewriting* without any modifications to the equations of electromagnetism: special relativity was essentially *made to work* with classical electromagnetism. See the end of Jackson (or any half-decent book on relativity) for more details on special-relativistic electromagnetism.

[5]For more details on this check any three-quarters decent book on relativity, for example Carroll, D'Inverno or Schutz.

[6]This can be fixed by introducing new 4-vectors E, B with $E^0 = 0, E^i = \mathbf{E}^i$ (same for B), and then *lowering* the index and *defining* $F_{0i} := -E_i = E^i = \mathbf{E}^i$ (and equivalently for B) but that's too much work and potentially more confusing.

A key feature of he electromagnetic tensor is that it is a *closed* 2-form, that is, its deRham differential vanishes. The computation is a bit tedious but we'll give a few components just so that this is not completely blind faith:

$$(\mathrm{d}F)_{012} = \frac{\partial F_{12}}{\partial x^0} - \frac{\partial F_{02}}{\partial x^1} + \frac{\partial F_{01}}{\partial x^2} = -\frac{1}{c}\frac{\partial \mathbf{B}^3}{\partial t} - \frac{\partial \mathbf{E}^2}{\partial x^1} + \frac{\partial \mathbf{E}^1}{\partial x^2} = -\left(\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E}\right)^3,$$

(that is, the third component of the equation, not the equation cubed) and similarly for the components $(\mathrm{d}F)_{013}$ and $(\mathrm{d}F)_{023}$. For the last component,

$$(\mathrm{d}F)_{123} = \frac{\partial F_{23}}{\partial x^1} - \frac{\partial F_{13}}{\partial x^2} + \frac{\partial F_{12}}{\partial x^3} = -\frac{\partial \mathbf{B}^1}{\partial x^1} - \frac{\partial \mathbf{B}^2}{\partial x^2} - \frac{\partial \mathbf{B}^3}{\partial x^3} = -\nabla \cdot \mathbf{B}.$$

Thus, we have that $\mathrm{d}F = 0$ if and only if the homogeneous Maxwell equations hold for $\mathbf{E}, \mathbf{B}$. In this case, F is a closed 2-form, and this suggests[7] we write

$$F = \mathrm{d}A$$

for some 1-form $A \in \Omega^1(M)$, called the **electromagnetic potential**. In components, this is

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu},$$

where $A = A_\mu \mathrm{d}x^\mu$. This electromagnetic potential corresponds to the electric and magnetic potentials $\phi, \mathbf{A}$ as

$$A_0 = \phi \qquad A_i = -\mathbf{A}^i.$$

The annoying sign for the spacial indices tell us that A should be more naturally thought of as a *vector field* instead of a 1-form. Raise that index![8]

$$A^0 = \phi \qquad A^i = \mathbf{A}^i$$

Ah, much better. Indeed, we have for $i \geq 1$,

$$F_{0i} = \mathbf{E}^i = \frac{\partial A_i}{\partial x^0} - \frac{\partial A_0}{\partial x^i} = \left(-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi\right)^i,$$

and for instance,

$$F_{21} = \mathbf{B}^3 = \frac{\partial A_1}{\partial x^2} - \frac{\partial A_2}{\partial x^1} = (\nabla \times \mathbf{A})^3.$$

This tells us that the choice of a primitive A for F such that $\mathrm{d}A = F$ is exactly the same as choosing potentials $\phi, \mathbf{A}$ for the electric and magnetic fields $\mathbf{E}, \mathbf{B}$ as in the previous section.

Once again, we have that the choice of electromagnetic potential is not unique, since we can add to A any closed 1-form $\mathrm{d}\Lambda$ (for a function $\Lambda \in C^\infty(M)$) and still obtain the same electromagnetic tensor F. If $A' = A - \mathrm{d}\Lambda$, then

$$\mathrm{d}A' = \mathrm{d}(A - \mathrm{d}\Lambda) = \mathrm{d}A + \mathrm{d}^2\Lambda = \mathrm{d}A = F.$$

In components, A' looks like

$$A'_0 = \phi' = A_0 - \frac{\partial \Lambda}{\partial x^0} = \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t},$$
$$A'_i = -\mathbf{A}'^i = A_i - \frac{\partial \Lambda}{\partial x^i} = -(\mathbf{A} + \nabla\Lambda)^i.$$

[7]But does not imply! Again, this depends on $H^2(M) \cong H^2(U)$ being trivial.

[8]Index gymnastics with the Minkowski metric is quite easy: In my convention (the one true convention, fight me) with positive time and negative space, the time index remains the same while the space indices gain a negative sign whenever they are raised or lowered (as can be easily checked).

Thus we recover the same equations for a gauge transformation:

$$A' = A - \mathrm{d}\Lambda \qquad \Leftrightarrow \qquad \begin{aligned} \mathbf{A}' &= \mathbf{A} + \nabla\Lambda, \\ \phi' &= \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t} \end{aligned}$$

Again, to summarize, we can put together the electric and magnetic fields into a 2-form F, the electromagnetic tensor, which satisfies

$$\mathrm{d}F = 0.$$

This equation is automatically satisfied if there is a 1-form A such that $F = \mathrm{d}A$. In this case we call A an electromagnetic potential for F. The choice of potential is not unique, for we can add any closed 1-form $\mathrm{d}\Lambda$ to A and obtain the same electromagnetic tensor. The new electromagnetic potential is, then

$$A' = A - \mathrm{d}\Lambda.$$

This is called a **gauge transformation**, and the ability to change the potentials is called a **gauge freedom** (or symmetry).

What about the inhomogeneous Maxwell equations? Well, that's a little bit more tricky. Let's write all the equations again:

$$\begin{aligned} \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned} \qquad\qquad \begin{aligned} \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{J} \\ \nabla \cdot \mathbf{E} &= 4\pi\rho. \end{aligned}$$

Note that in the inhomogeneous equations, the roles of $\mathbf{E}$ and $\mathbf{B}$ seem to be reversed, except for a sneaky negative sign. What is this inhomogeneous equation in terms of the electromagnetic tensor F?

If you've already seen this then: 1. why are you even reading this post and 2. you already know that the inhomogeneous equations, *in components*, take the form

$$\frac{\partial F^{\mu\nu}}{\partial x^{\mu}} = 4\pi J^{\nu},$$

where J is a vector field whose components are

$$J^0 = \rho; \qquad J^i = \frac{1}{c}\mathbf{J}^i, \ \text{ for } i \geq 1.$$

Okay this is good and all, but we want a coordinate-free way to write this. Let's try to reverse-engineer the equation. We have that F is a 2-form, and we want to relate it via some sort of "divergence" with a *vector field*. Instead of that, we can simply convert the vector field J into a 1-form using the metric, but still we need to take a derivative of F. The problem is that the de Rham differential d will annihilate F, and even if it didn't, it would turn F into a 3-form. No bueno!

Instead we want to find a way to switch the roles of $\mathbf{E}$ and $\mathbf{B}$ in the Faraday tensor, so that the resulting tensor does not vanish when we apply the exterior differential. We would then have a three-form, which we want to somehow relate to the current one-form.

If this sounds familiar to you, then you've probably heard of the **Hodge dual** or Hodge star operator. Briefly, if you have a metric g on a manifold M then there is an isomorphism $\star : \Omega^k(M) \to \Omega^{n-k}(M)$, such that for all k-forms ω, ν

$$\alpha \wedge \star\beta = g(\alpha, \beta)\mathrm{vol},$$

where vol is the volume form associated to the metric and the metric evaluated on k-forms is defined as

$$g(\alpha, \beta) := \frac{1}{k!}\alpha^{\mu_1 \ldots \mu_k}\beta_{\mu_1 \ldots \mu_k}.$$

We've discussed the Hodge star in depth in a previous post. It can be shown that, in coordinates, the components of the Hodge star of a k-form β are

$$(\star\beta)_{\lambda_1\ldots\lambda_{n-k}} = \pm\frac{\sqrt{|\det(g)|}}{k!}\beta^{\rho_1\ldots\rho_k}\epsilon_{\rho_1\ldots\rho_k\lambda_1\ldots\lambda_{n-k}},$$

where ϵ is the Levi-Civita symbol. In particular, we will care about the stars of wedges of the basis one-forms $\mathrm{d}x^\mu$. In the previous post we showed that if $\{e^1,\ldots,e^n\}$ is an orthonormal basis then

$$\star(e^{\rho_1}\wedge\cdots\wedge e^{\rho_k}) = g^{\rho_1\rho_1}\ldots g^{\rho_k\rho_k}\epsilon^{\rho_1\ldots\rho_k\nu_1\ldots\nu_{n-k}}e^{\nu_1}\wedge\cdots\wedge e^{\nu_{n-k}} \qquad \text{(no Einstein sum)},$$

where $\{\nu_1\ldots\nu_{n-k}\}$ is the complement of $\{\rho_1,\ldots,\rho_k\}$ in $\{0,\ldots,n-1\}$. In our case, $k=2$ and $n=4$ and the one-forms $\mathrm{d}x^\mu$ are orthonormal, so that

$$\star(\mathrm{d}x^0\wedge\mathrm{d}x^1) = \eta^{00}\eta^{11}\epsilon^{0123}\mathrm{d}x^2\wedge\mathrm{d}x^3 = -\mathrm{d}x^2\wedge\mathrm{d}x^3.$$

In a similar fashion, we can show that

$$\begin{aligned}
\star(\mathrm{d}x^0\wedge\mathrm{d}x^2) &= \mathrm{d}x^1\wedge\mathrm{d}x^3\\
\star(\mathrm{d}x^0\wedge\mathrm{d}x^3) &= -\mathrm{d}x^1\wedge\mathrm{d}x^2\\
\star(\mathrm{d}x^1\wedge\mathrm{d}x^2) &= \mathrm{d}x^0\wedge\mathrm{d}x^3\\
\star(\mathrm{d}x^2\wedge\mathrm{d}x^3) &= \mathrm{d}x^0\wedge\mathrm{d}x^1\\
\star(\mathrm{d}x^3\wedge\mathrm{d}x^1) &= \mathrm{d}x^0\wedge\mathrm{d}x^2.
\end{aligned}$$

Thus, if we rewrite the Faraday tensor as

$$\begin{aligned}
F &= \mathbf{E}^1\mathrm{d}x^0\wedge\mathrm{d}x^1 + \mathbf{E}^2\mathrm{d}x^0\wedge\mathrm{d}x^2 + \mathbf{E}^3\mathrm{d}x^0\wedge\mathrm{d}x^3\\
&\quad - \mathbf{B}^1\mathrm{d}x^2\wedge\mathrm{d}x^3 - \mathbf{B}^2\mathrm{d}x^3\wedge\mathrm{d}x^1 - \mathbf{B}^3\mathrm{d}x^1\wedge\mathrm{d}x^2,
\end{aligned}$$

we obtain

$$\begin{aligned}
\star F &= -\mathbf{E}^1\mathrm{d}x^2\wedge\mathrm{d}x^3 + \mathbf{E}^2\mathrm{d}x^1\wedge\mathrm{d}x^3 - \mathbf{E}^3\mathrm{d}x^1\wedge\mathrm{d}x^2\\
&\quad - \mathbf{B}^1\mathrm{d}x^0\wedge\mathrm{d}x^1 - \mathbf{B}^2\mathrm{d}x^0\wedge\mathrm{d}x^2 - \mathbf{B}^3\mathrm{d}x^0\wedge\mathrm{d}x^3,
\end{aligned}$$

which in matrix form is

$$[(\star F)_{\mu\nu}] = \begin{pmatrix} 0 & -\mathbf{B}^1 & -\mathbf{B}^2 & -\mathbf{B}^3\\ \mathbf{B}^1 & 0 & -\mathbf{E}^3 & \mathbf{E}^2\\ \mathbf{B}^2 & \mathbf{E}^3 & 0 & -\mathbf{E}^1\\ \mathbf{B}^3 & -\mathbf{E}^2 & \mathbf{E}^1 & 0\end{pmatrix}.$$

Thus, we have that the roles of $\mathbf{B}$ and $\mathbf{E}$ in $\star F$ are reversed from those in F, with a sneaky negative sign. Morally, applying the $\star$ operator gives

$$\begin{aligned}
F &\overset{\star}{\mapsto} \star F\\
\mathbf{B} &\mapsto \mathbf{E}\\
\mathbf{E} &\mapsto -\mathbf{B}.
\end{aligned}$$

Now we have that $\mathrm{d}(\star F)$ is a *three*-form, some of whose components are

$$(\mathrm{d}\star F)_{012} = \frac{\partial}{\partial x^0}(\star F)_{12} - \frac{\partial}{\partial x^1}(\star F)_{02} + \frac{\partial}{\partial x^2}(\star F)_{01} = -\frac{1}{c}\frac{\partial\mathbf{E}^3}{\partial t} + \frac{\partial\mathbf{B}^2}{\partial x^1} - \frac{\partial\mathbf{B}^1}{\partial x^2} = \left(-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \nabla\times\mathbf{B}\right)^3,$$

which is

$$(\mathrm{d} \star F)_{012} = \left(-\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B}\right)^3 = \frac{4\pi}{c}\mathbf{J}^3.$$

Similarly,

$$(\mathrm{d} \star F)_{123} = \frac{\partial}{\partial x^1}(\star F)_{23} - \frac{\partial}{\partial x^2}(\star F)_{13} + \frac{\partial}{\partial x^3}(\star F)_{12} = -\frac{\partial \mathbf{E}^1}{\partial x^1} - \frac{\partial \mathbf{E}^2}{\partial x^2} - \frac{\partial \mathbf{E}^3}{\partial x^3} = -\nabla \cdot \mathbf{E} = -4\pi\rho.$$

If we put it all together, we obtain

$$\begin{aligned}
\mathrm{d} \star F &= \left(-\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B}\right)^3 \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \left(-\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B}\right)^1 \mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
&\quad + \left(-\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B}\right)^2 \mathrm{d}x^0 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^1 - (\nabla \cdot \mathbf{E})\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
&= \frac{4\pi}{c}\left(\mathbf{J}^3 \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathbf{J}^2 \mathrm{d}x^0 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^1 + \mathbf{J}^1 \mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\right) - 4\pi\rho \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3
\end{aligned}$$

We see that on the right-hand side we have the components of the 4-current J, but as a *three*-form. If we let j be the one-form with components J_μ (i.e., with the lowered index), we have

$$j = \rho \mathrm{d}x^0 - \frac{1}{c}\left(\mathbf{J}^1 \mathrm{d}x^1 + \mathbf{J}^2 \mathrm{d}x^2 + \mathbf{J}^3 \mathrm{d}x^3\right),$$

so that

$$\star j = \rho \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 - \frac{1}{c}\left(\mathbf{J}^3 \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathbf{J}^2 \mathrm{d}x^0 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^1 + \mathbf{J}^1 \mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\right).$$

Thus, we identify:

$$\mathrm{d} \star F = -4\pi \star j.$$

This can also be written as

$$\star \mathrm{d} \star F = 4\pi j.$$

Finally, we obtain the Maxwell equations written in a coordinate-free way in terms of differential forms:

$$\mathrm{d}F = 0, \qquad\qquad \star \mathrm{d} \star F = 4\pi j.$$

These are the *field equations* of the electromagnetic field. The homogeneous equation $\mathrm{d}F = 0$ talks about *conservation of charge*, and the inhomogeneous equation $\star \mathrm{d} \star F = 4\pi j$ tells how the electromagnetic field F responds to the presence of charges and currents (represented by j). The form of these equations is typical of a gauge field, as we shall see in future posts.

3 Takeaway and future

We saw that the inhomogeneous Maxwell equations allow (in some cases, depending on the topology of the underlying space) us to write the electric and magnetic fields $\mathbf{E}, \mathbf{B}$ in terms of simple, auxiliary potentials $\phi, \mathbf{A}$. The choice of these potentials is not unique, and the fields $\mathbf{E}, \mathbf{B}$ remain invariant under certain transformations of the potentials, called *gauge transformations*. At this point, the potentials are no more than auxiliary mathematical objects that help us solve Maxwell's equations, but we shall see that they can be attributed a physical interpretation in the quantum case.

We also saw a unifying description of the electric and magnetic fields into an electromagnetic tensor in $4D$ spacetime, and we rewrote Maxwell's equations in a neater form that is typical of gauge fields (as we shall see in the future).

What comes next is adding *matter* to this whole issue: introducing objects that can interact with the electromagnetic field. We will also see a Lagrangian description of the field equations, which again will be typical of gauge fields.

4 References

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