Conservation of Momentum

Phys H308, Haverford College

Ted Brzinski, Sept. 7 2022

Conservation of Momentum

- Intro physics - Newton's 3rd:
  - Rockets
  - Collisions

Image Credit: NASA

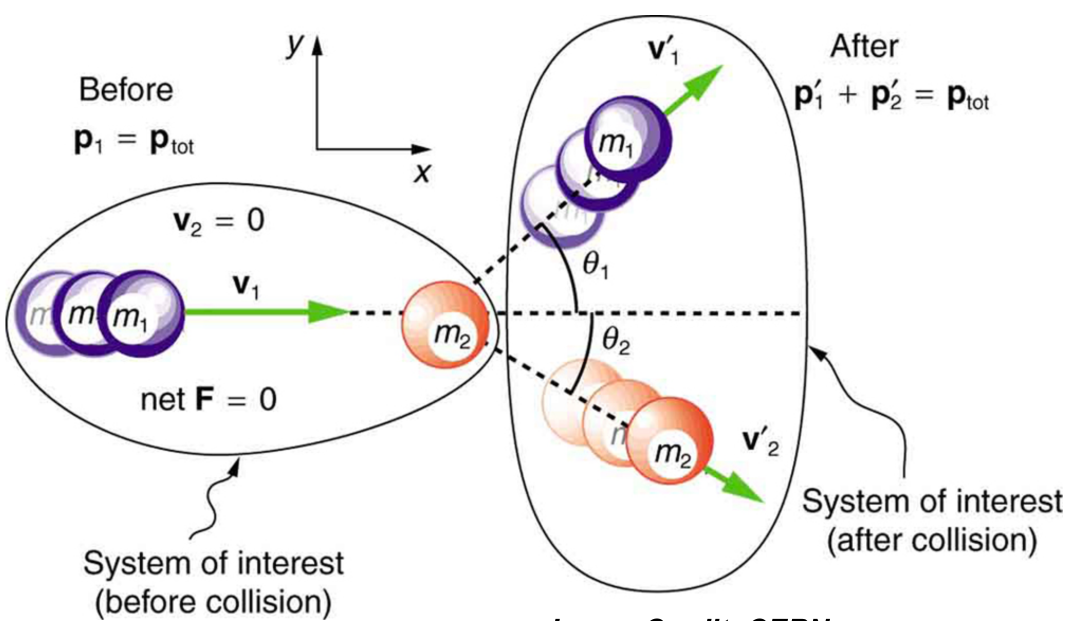


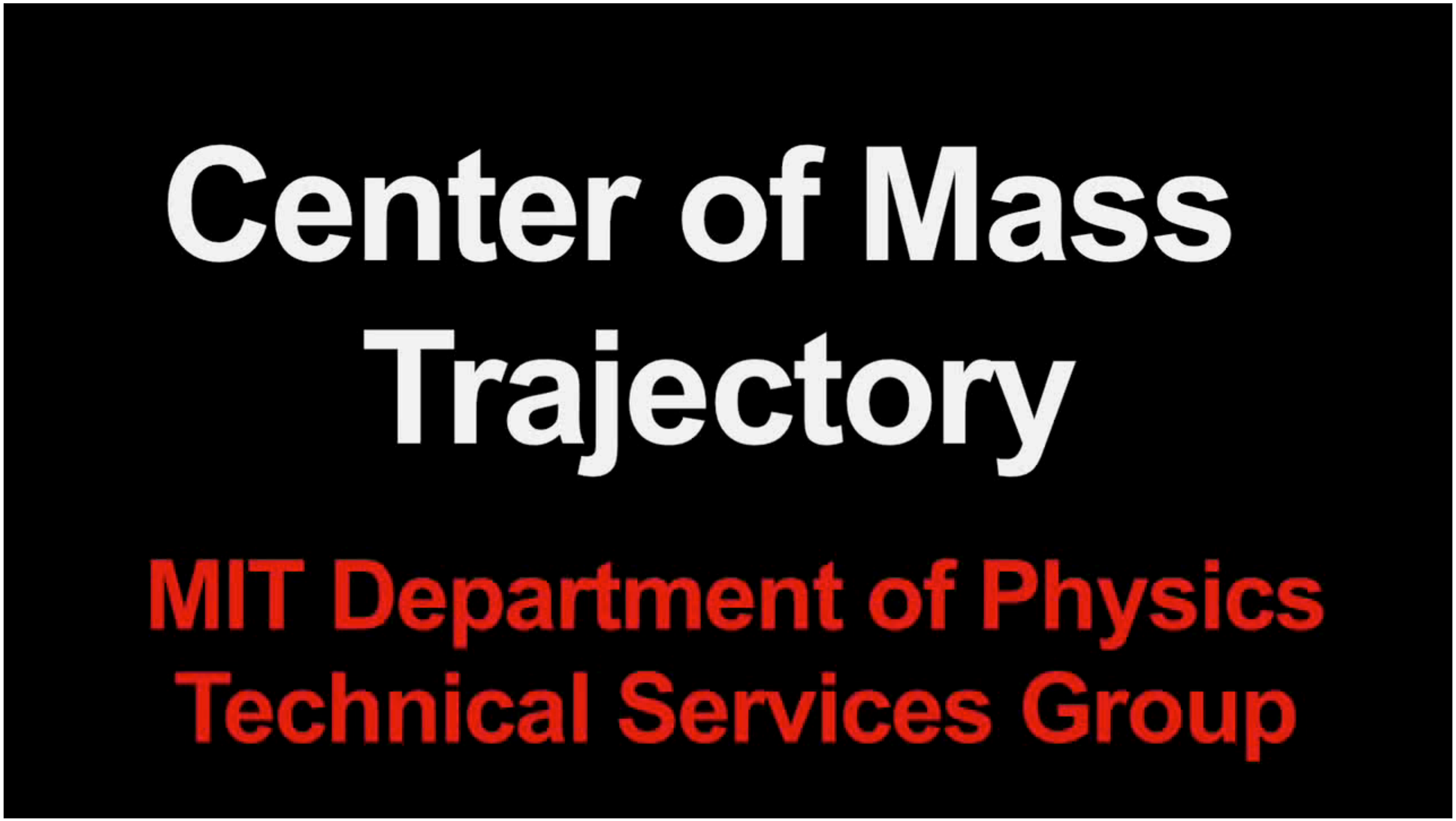
Image Credit: CERN

Conservation of Momentum

- Why we're revisiting:
  - Newton's 3rd Law (recall last week)
  - Rocket motion
    (Diff Eq. Practice in disguise - see $F = m\ddot{\vec{r}}$, basically all eqns of motion)
  - Conservation law! (Next is energy, basis of Lagrangian, Hamiltonian)
  - Center of mass $\mathbf{R_{cm}}$:
    - can simplify problems through choice of reference frame or by considering only $\mathbf{F_{ext}}$, $\mathbf{R_{cm}}$
    - provides nice example of volumetric integration

Center of mass

Simplifies dynamics



Important timestamps:
Throwing stuff - 10 s
Center of mass - 28 s
Geometric center - 49 s

Calculating center of mass

Volumetric integration

EXAMPLE 3.2 The CM of a Solid Cone

Find the CM position for the uniform solid cone shown in Figure 3.4.

It is perhaps obvious by symmetry that the CM lies on the axis of symmetry (the z axis), but this also follows immediately from the integral (3.13). For example, if you consider the x component of that integral, it is easy to see that the contribution from any point (x, y, z) is exactly cancelled by that from the point $(-x, y, z)$. That is, the integral for X is zero. Because the same argument applies to Y, the CM lies on the z axis. To find the height Z of the CM, we must evaluate the integral

$$Z = \frac{1}{M}\int \varrho\, z\, dV = \frac{\varrho}{M}\int z\, dx\, dy\, dz$$

where I could take the factor ϱ outside the integral since ϱ is constant throughout the cone (as long as we understand the integral is limited to the inside of the cone) and I have changed the volume element dV to $dx\, dy\, dz$. For any given z, the integral over x and y runs over a circle of radius $r = Rz/h$, giving a factor of $\pi r^2 = \pi R^2 z^2/h^2$, so that

$$Z = \frac{\varrho\pi R^2}{Mh^2}\int_0^h z^3 dz = \frac{\varrho\pi R^2}{Mh^2}\frac{h^4}{4} = \frac{3}{4}h$$

where in the last step I replaced the mass M by ϱ times the volume or $M = \frac{1}{3}\varrho\pi R^2 h$. We conclude that the CM is on the axis of the cone at a distance $\frac{3}{4}h$ from the vertex (or $\frac{1}{4}h$ from the base).

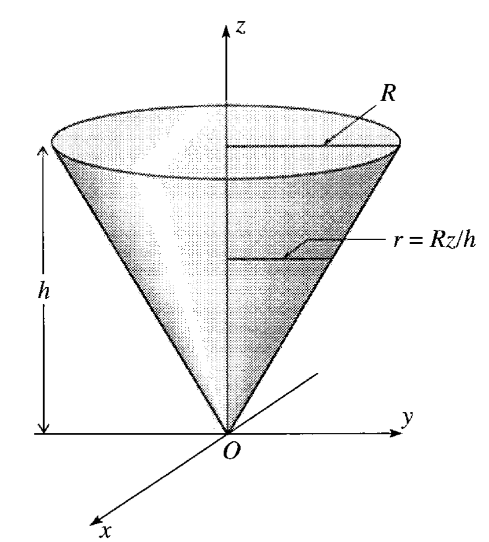
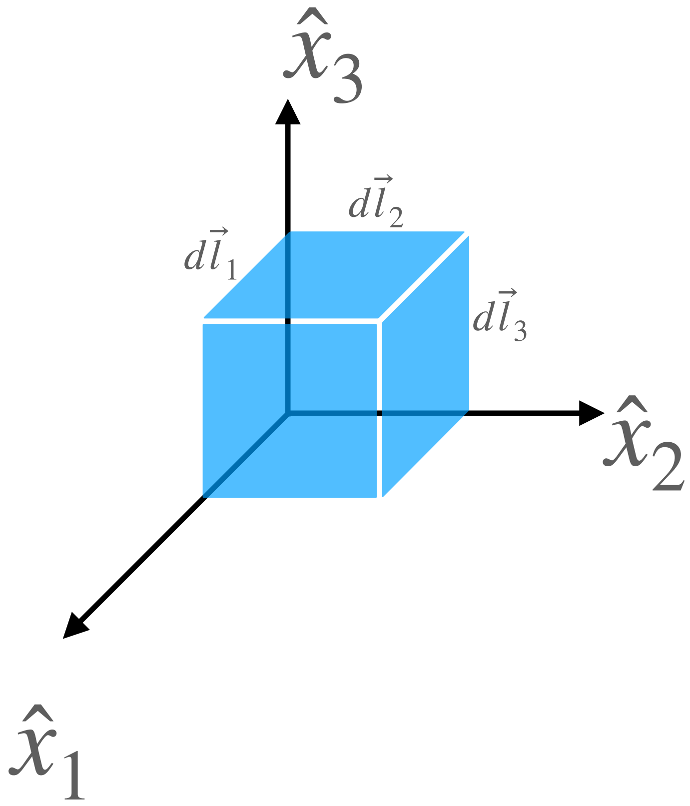


Figure 3.4 A solid cone, centered on the z axis, with vertex at the origin and uniform mass density ϱ. Its height is h and its base has radius R.

Anatomy of a volumetric integral

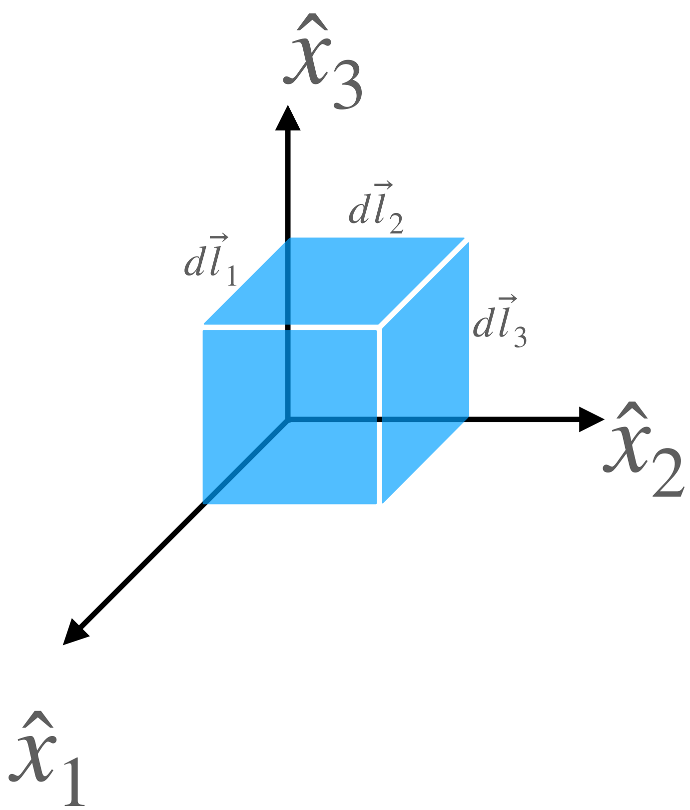


$$V = \int_{V_0} dV = \int_{V_0} dl_1 dl_2 dl_3$$

$$Q_{V_0} = \int_{V_0} q\left(\vec{r}\right) dl_1 dl_2 dl_3$$

Anatomy of a volumetric integral

Cartesian



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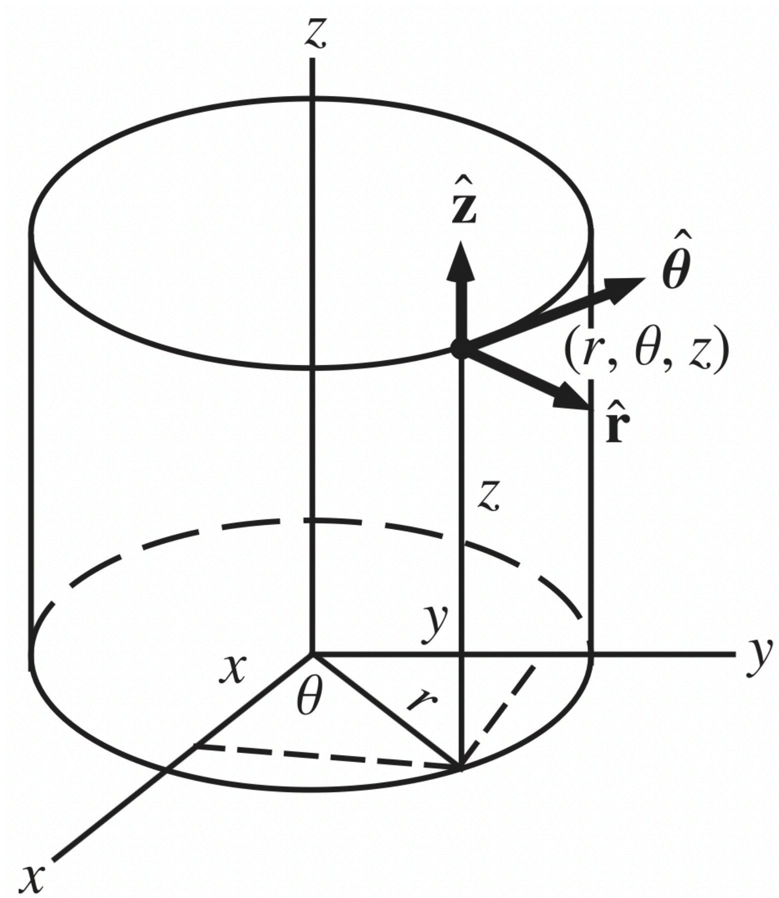
Cartesian: $\hat{x}_i$: $\hat{x}, \hat{y}, \hat{z}$ $\quad dl_1 dl_2 dl_3$: $dz\ dy\ dx$

Anatomy of a volumetric integral

Cylindrical



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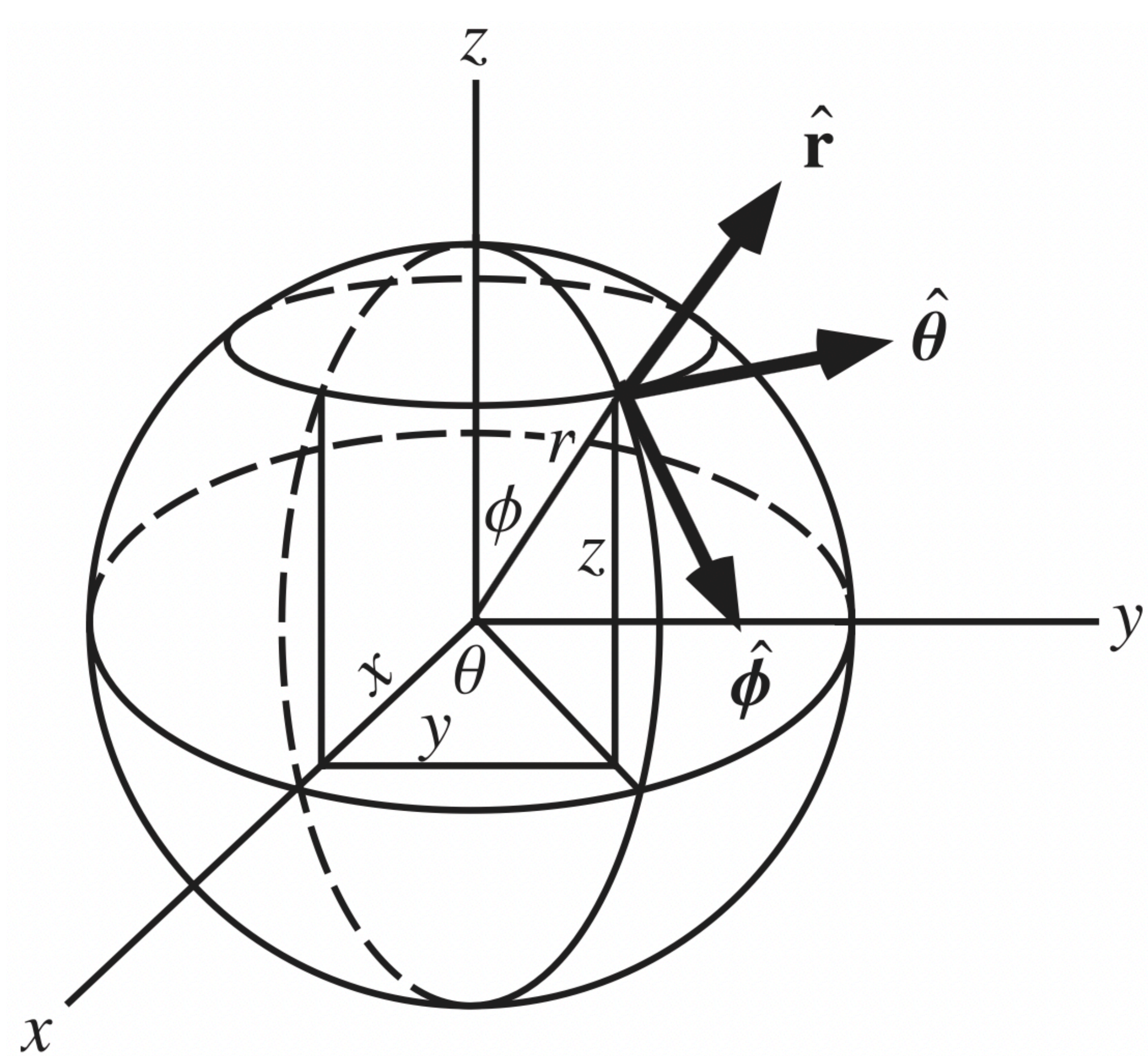
Cylindrical: $\hat{x}_i$: $\hat{r}, \hat{\theta}, \hat{z}$ $\quad dl_1 dl_2 dl_3$: $dr(rd\theta)dz$

Anatomy of a volumetric integral

Spherical

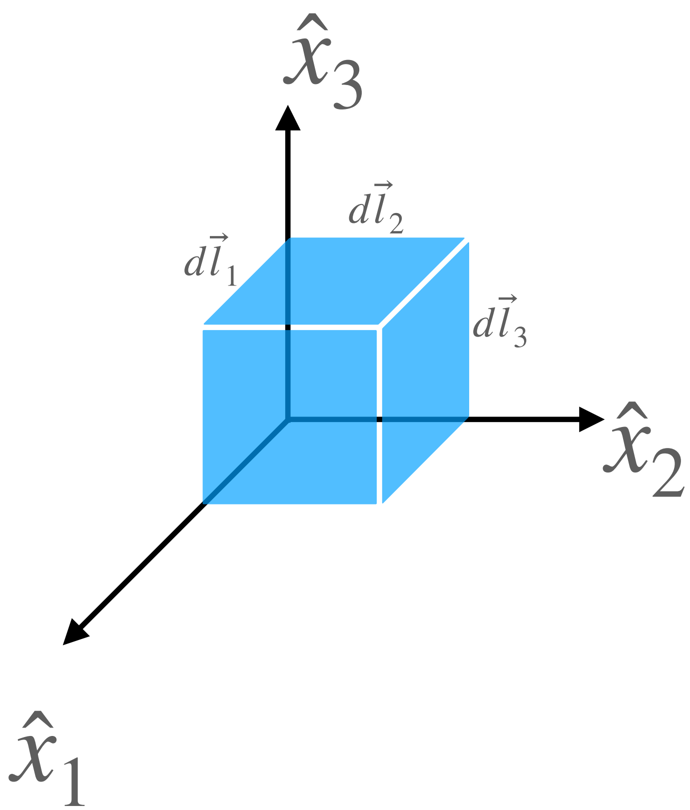


$$V = \int_{V_0} dV = \int_{V_0} dl_1 dl_2 dl_3$$

$$Q_{V_0} = \int_{V_0} q\left(\vec{r}\right) dl_1 dl_2 dl_3$$

Polar: $\hat{x}_i$: $\hat{r}, \hat{\theta}, \hat{\phi}$ $\quad dl_1 dl_2 dl_3$: $dr(rd\phi)(r\sin\phi d\theta)$

Anatomy of a volumetric integral



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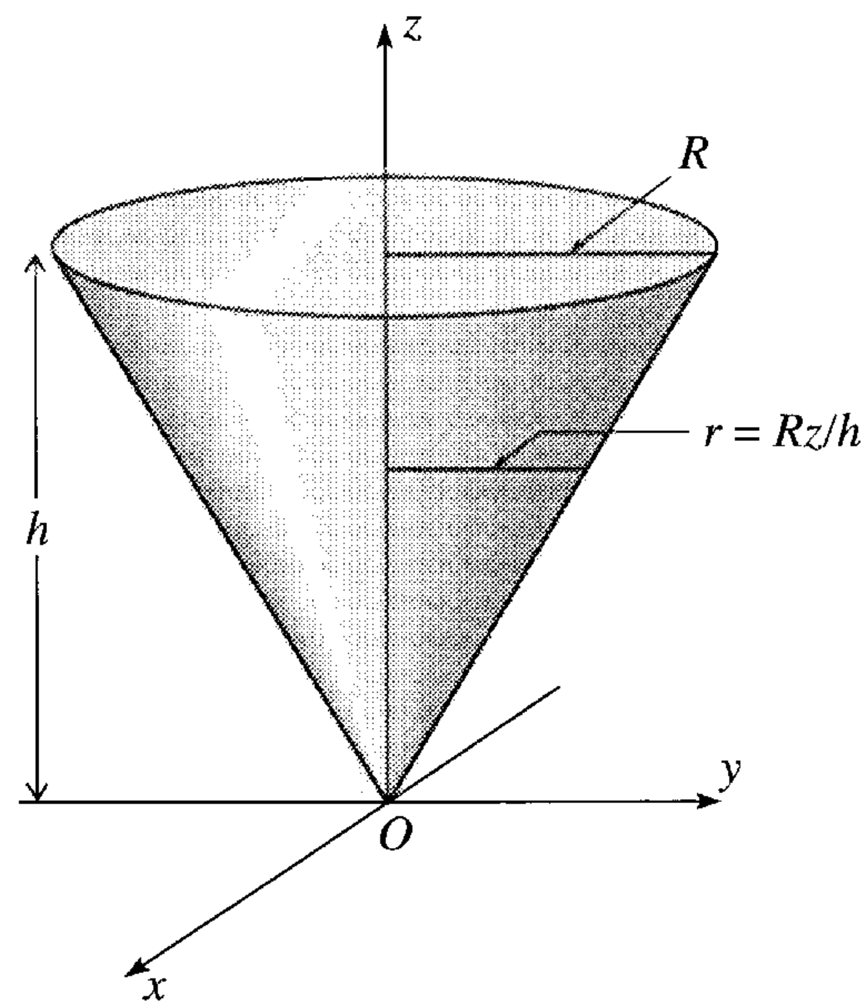
Cartesian: $\hat{x}_i$: $\hat{x}, \hat{y}, \hat{z}$ $\quad dl_1 dl_2 dl_3$: $dz\ dy\ dx$

Cylindrical: $\hat{x}_i$: $\hat{r}, \hat{\theta}, \hat{z}$ $\quad dl_1 dl_2 dl_3$: $dr(rd\theta)dz$

Polar: $\hat{x}_i$: $\hat{r}, \hat{\theta}, \hat{\phi}$ $\quad dl_1 dl_2 dl_3$: $dr(rd\phi)(r \sin\phi d\theta)$

Example



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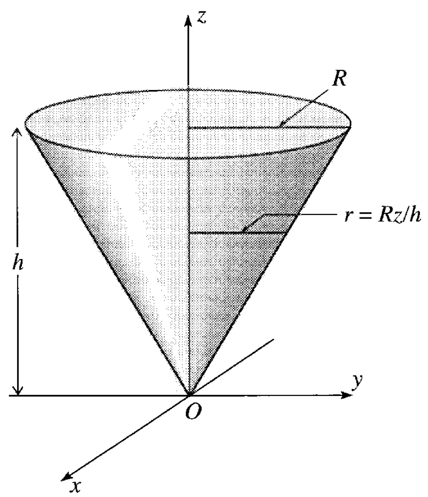

Figure 3.4 A solid cone, centered on the z axis, with vertex at the origin and uniform mass density ϱ. Its height is h and its base has radius R.

$$\rho = \frac{M}{V} = \frac{M}{\frac{1}{3}\pi r^2 h}$$ and, by symmetry, the center of mass lies along the z-axis, so:

$$\mathbf{R} = \frac{1}{M}\int \vec{r} dm = \frac{1}{M}\int \vec{r}\rho\ dV = \frac{\rho\hat{z}}{M}\int r dV$$

Given the symmetry of the system, I would choose to work in cylindrical coordinates:

$$\mathbf{R} = \frac{\rho\hat{z}}{M}\int_{\theta=0}^{2\pi}\int_{z=0}^{h}\int_{r=0}^{Rz/h} r^2 dr\ dz\ d\theta$$