PhysH308

Lagrangian Mechanics, cont.

Ted Brzinski, Oct 10, 2024

Noether's (1st) theorem

(the second one is used in gauge theory)

- Noether's Theorem: every *differentiable symmetry* of the Action of a system (without *dissipation*) corresponds to a <u>*conserved quantity.*</u>
  - *Differentiable symmetry*: the Action is unchanged under a transformation (e.g. translation or rotation), OR a change in a continuous variable w.r.t. which the action is differentiable.
  - *Dissipation*: e.g. friction, air resistance, etc

Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918[1]).

Es handelt sich um Variationsprobleme, die eine kontinuierliche Gruppe (im Lieschen Sinne) gestatten; die daraus sich ergebenden Folgerungen für die zugehörigen Differentialgleichungen finden ihren allgemeinsten Ausdruck in den in § 1 formulierten, in den folgenden Paragraphen bewiesenen Sätzen. Über diese aus Variationsproblemen entspringenden Differentialgleichungen lassen sich viel präzisere Aussagen machen als über beliebige, eine Gruppe gestattende Differentialgleichungen, die den Gegenstand der Lieschen Untersuchungen bilden. Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie. Für spezielle Gruppen und Variationsprobleme ist diese Verbindung der Methoden nicht neu; ich erwähne Hamel und Herglotz für spezielle endliche, Lorentz und seine Schüler (z. B. Fokker), Weyl und Klein für spezielle unendliche Gruppen[2]). Insbesondere sind die zweite Kleinsche Note und die vorliegenden Ausführungen gegenseitig durch einander beein-

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1) Die endgiltige Fassung des Manuskriptes wurde erst Ende September eingereicht.

2) Hamel: Math. Ann. Bd. 59 und Zeitschrift f. Math. u. Phys. Bd. 50. Herglotz: Ann. d. Phys. (4) Bd. 36, bes. § 9, S. 511. Fokker, Verslag d. Amsterdamer Akad., 27./1. 1917. Für die weitere Litteratur vergl. die zweite Note von Klein: Göttinger Nachrichten 19. Juli 1918.

In einer eben erschienenen Arbeit von Kneser (Math. Zeitschrift Bd. 2) handelt es sich um Aufstellung von Invarianten nach ähnlicher Methode.

Emmy Noether

1882-1935

- One of 2 women at the time to pursue studies at Erlangen University. Her father (a prof.) facilitated her studies: she was only allowed to audit lectures, yet still passed grad exams in only 3 years (1903).

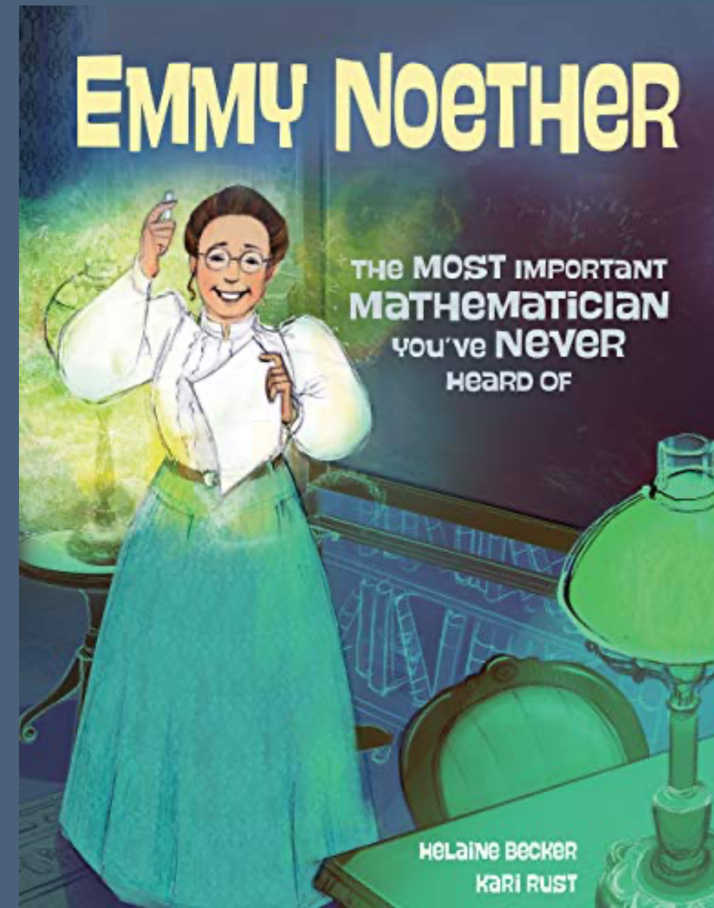
- Went on to graduate studies at Gottingen, but was thrown out within a year when women's enrollment was banned. Completed her PhD work back at Erlangen in 1907.

- Between 1907 and 1923, Noether worked at Erlangen and then Gottingen as an *unpaid* lecturer(!!!)

- Despite prominent lectureships abroad, and numerous important publications, **Noether was never made full professor in Germany, and in 1933 was banned from holding an academic position by the Nazis due to her Jewish identity.**

- Facilitated by colleagues at Princeton (including Einstein), **Noether took a position at Bryn Mawr in 1933**, with an adjunct position at Princeton, where she worked until her sudden death in 1935 due to complications after a surgery.

- Prof. Qinna Shen (BMC German) has been a leader in the study and archival of Noether's legacy at BMC.

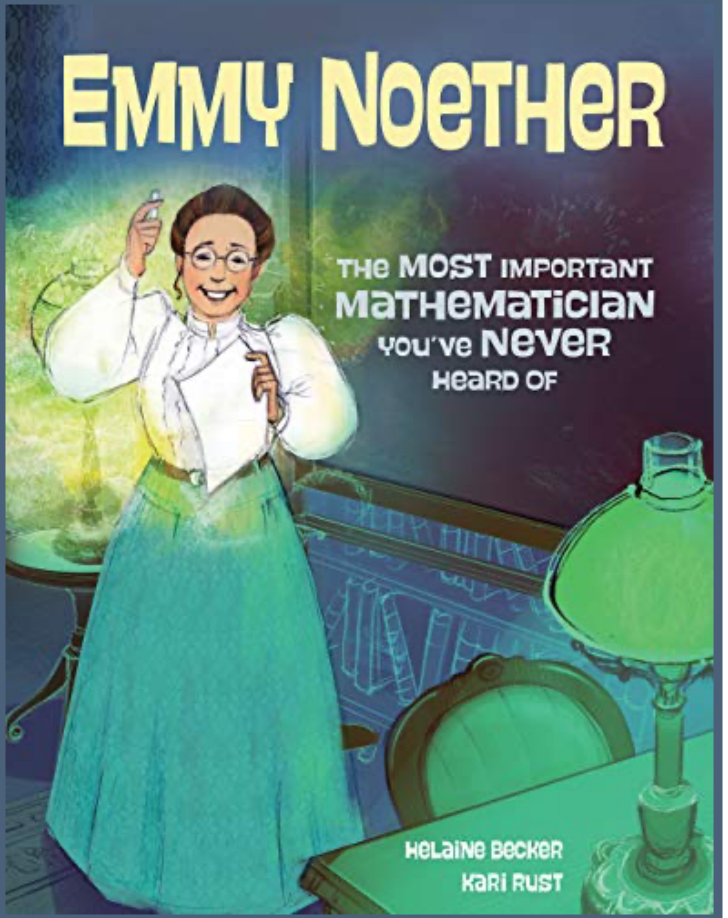


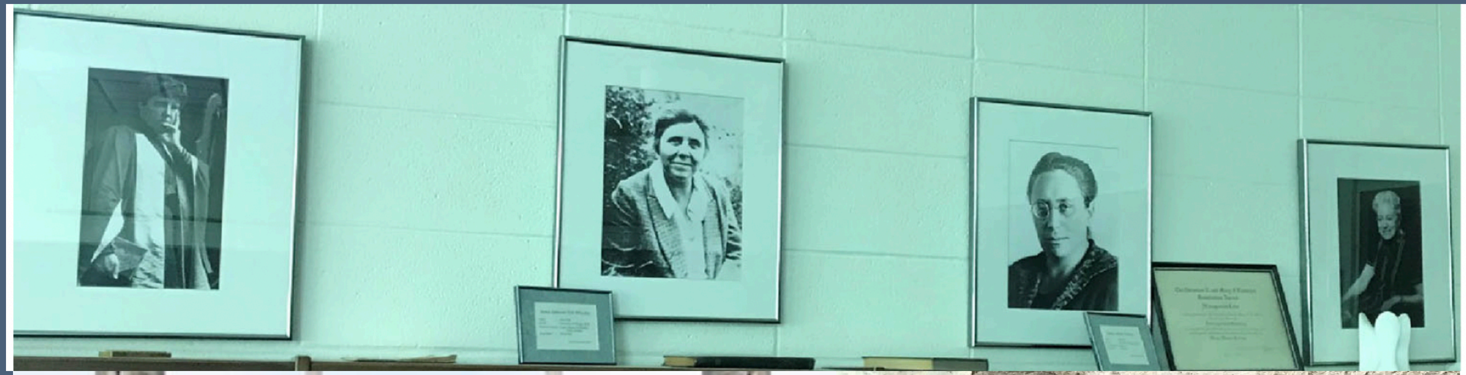
Emmy Noether

1882-1935

- **From Einstein's obituary for Noether:**
In the judgment of the most competent living mathematicians, ***Fräulein Noether was the most significant creative mathematical genius thus far produced since the higher education of women began[…]***

Born in a Jewish family distinguished for the love of learning, Emmy Noether, who, in spite of the efforts of the great Göttingen mathematician, Hilbert, never reached the academic standing due her in her own country, none the less surrounded herself with a group of students and investigators at Göttingen, who have already become distinguished as teachers and investigators. Her unselfish, significant work over a period of many years was rewarded by the new rulers [the Nazi gov't] *of Germany with a dismissal, which cost her the means of maintaining her simple life and the opportunity to carry on her mathematical studies.* ***Farsighted friends of science in this country were fortunately able to make such arrangements at Bryn Mawr College and at Princeton that she found in America up to the day of her death not only colleagues who esteemed her friendship but grateful pupils whose enthusiasm made her last years the happiest and perhaps the most fruitful of her entire career.***





Invariance under translation

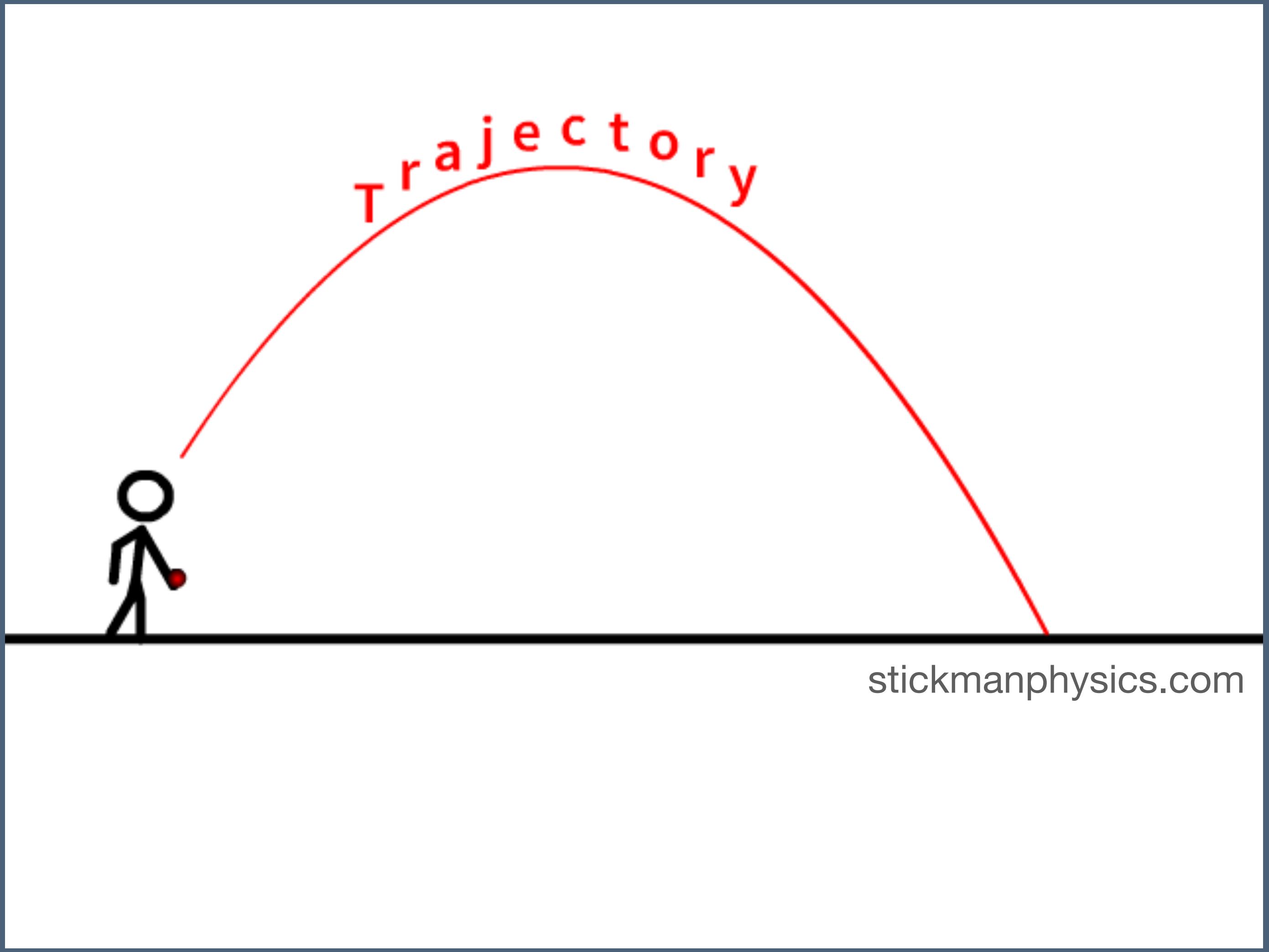
Linear momentum conservation

Example: ballistic motion

- $S = \int_a^b \mathscr{L}(x, y, \dot{x}, \dot{y}, t)\ \mathrm{d}t$
- $\mathscr{L} = T - U = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$
- Translation: $x \to x + \delta x$ OR $y \to y + \delta y$
- S is is unchanged under translation if $\mathscr{L}$ is unchanged under the same translation → only invariant under translation in the x-direction.
- Equivalently (by E-L),

$$\frac{\partial}{\partial x}\mathscr{L} = 0 = \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}} = \frac{d}{dt}(m\dot{x}) = \dot{p}_x$$

So p_x is constant!



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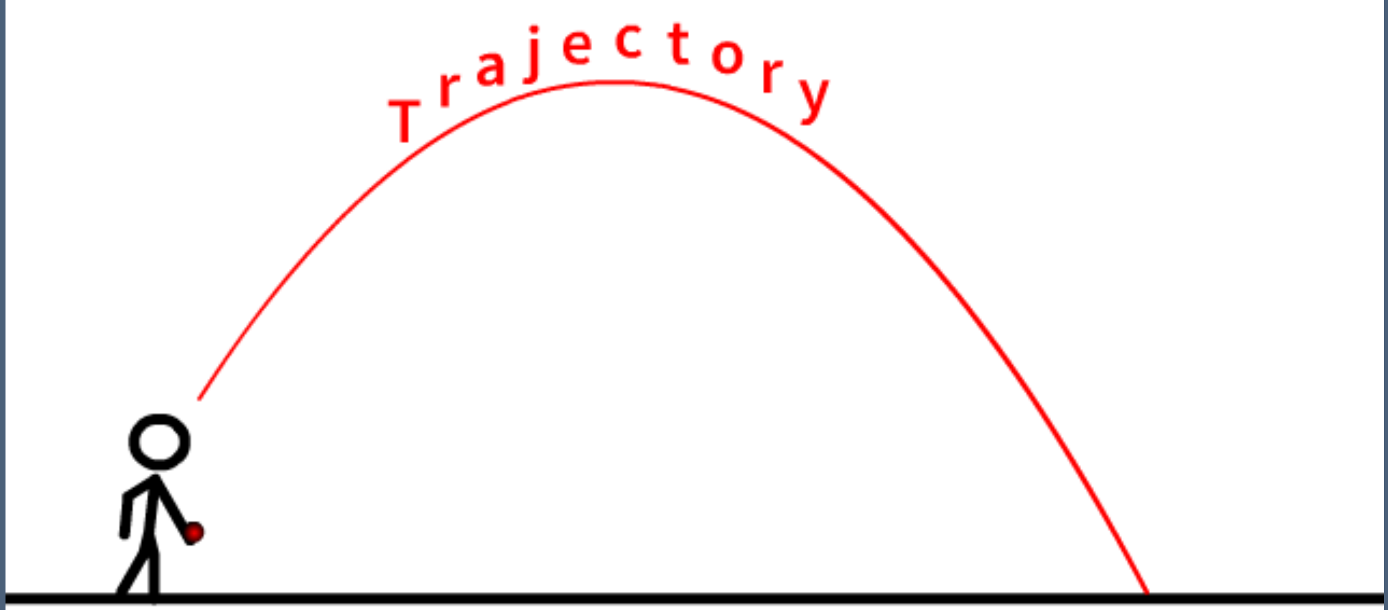


stickmanphysics.com

"Generalized force" and "generalized momentum"

Invariance under rotation

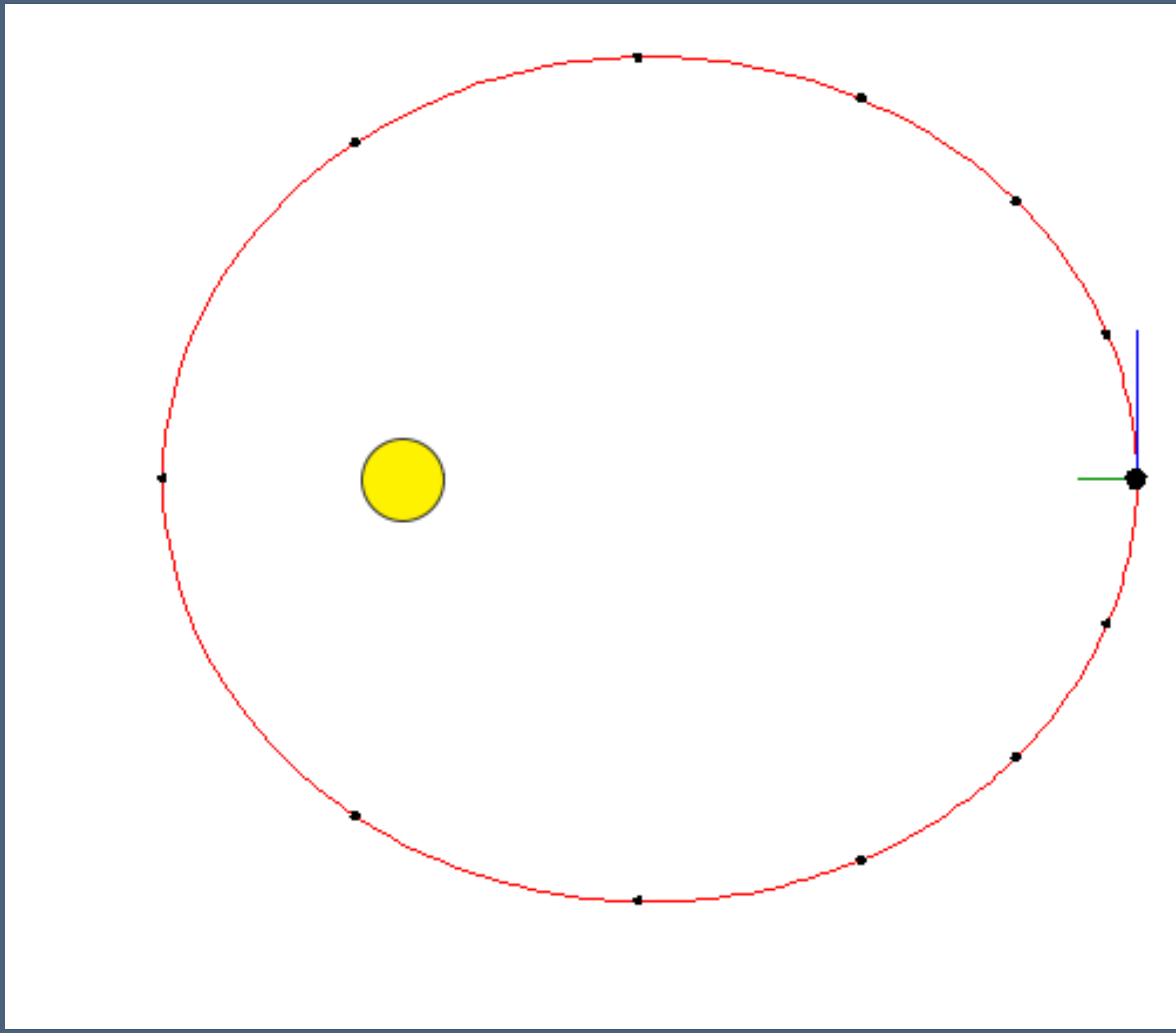
Angular momentum conservation

Example: Keplerian orbit

- $\mathscr{L} = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GMm}{r}$
- Radial translation : $r \to r + \delta r$
  OR
  Rotation : $\theta \to \theta + \delta\theta$
- S is is unchanged under transformation if $\mathscr{L}$ is unchanged under the same tranformation → $\mathscr{L}$ is only invariant under rotation, not radial translation.
- Equivalently (by E-L),

$$\frac{\partial}{\partial\theta}\mathscr{L} = 0 = \frac{d}{dt}\left(mr^2\dot{\theta}\right) = \frac{d}{dt}\left(mrv_\theta\right)$$

So ℓ_z is constant!



Invariance under rotation

Angular momentum conservation

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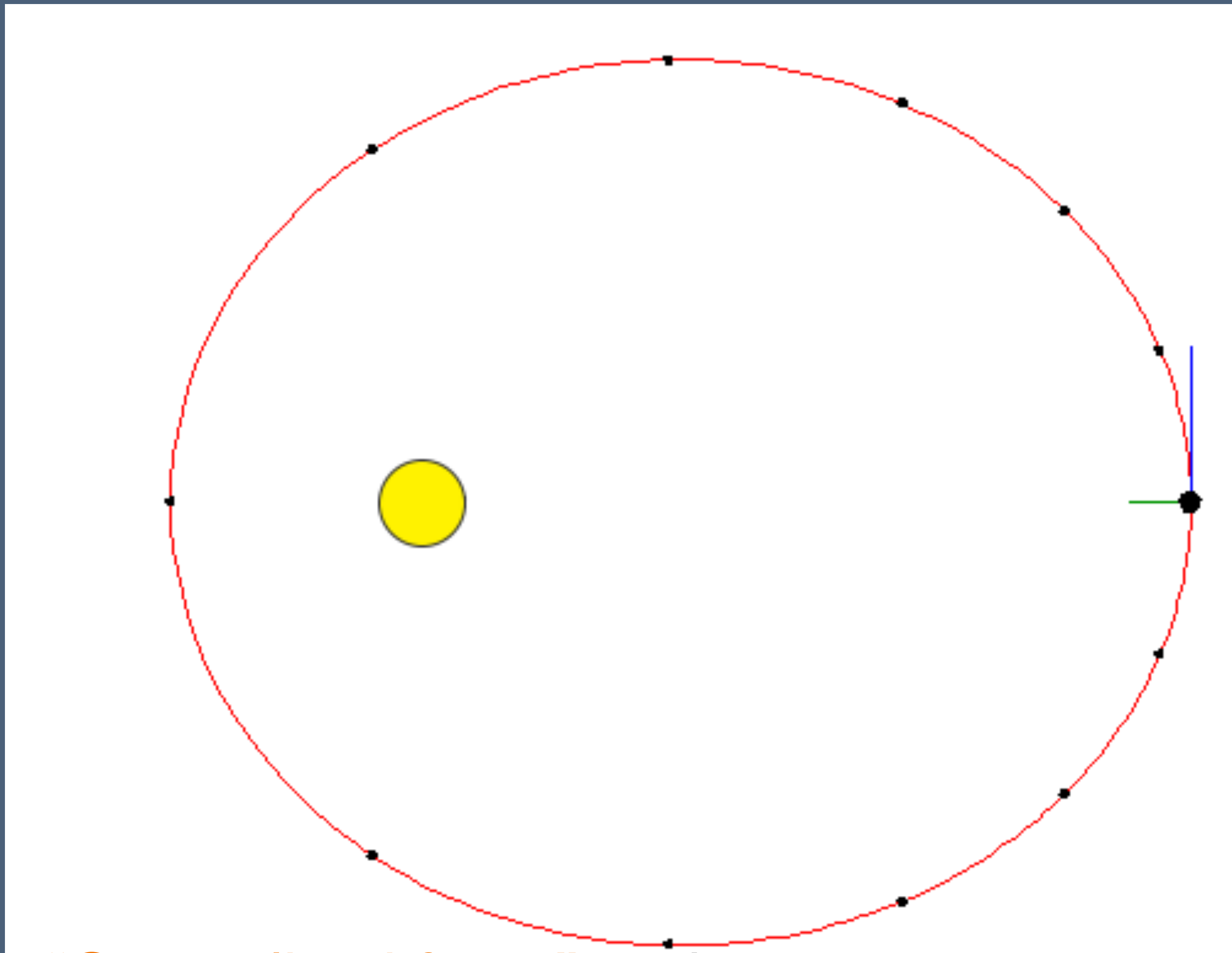

"Generalized force" and "generalized momentum"

Other notes on symmetries and conservation

- (see book) Time translation → energy conservation → origin of the Hamiltonian: $\mathscr{H} = \sum p_i \dot{x}_i - \mathscr{L}$, which for systems in which the Lagrangian has no time-dependence: $\mathscr{H} = 2T - \mathscr{L} = T + U$
- Noether's theorem is inherently classical - based on equations of motion. However the connection Emmy Noether identified between symmetries/ conserved quantities extends further, especially QFT, cosmology (see table).
- The extension of NT to QFT are the Ward-Takahashi Identities

Class	Invariance	Conserved quantity
Proper orthochronous Lorentz symmetry	translation in time (homogeneity)	energy
	translation in space (homogeneity)	linear momentum
	rotation in space (isotropy)	angular momentum
	Lorentz-boost (isotropy)	mass moment **N** = *t***p** − *E***r**
Discrete symmetry	P, coordinate inversion	spatial parity
	C, charge conjugation	charge parity
	T, time reversal	time parity
	CPT	product of parities
Internal symmetry (independent of spacetime coordinates)	U(1) gauge transformation	electric charge
	U(1) gauge transformation	lepton generation number
	U(1) gauge transformation	hypercharge
	$U(1)_Y$ gauge transformation	weak hypercharge
	U(2) [U(1) × SU(2)]	electroweak force
	SU(2) gauge transformation	isospin
	$SU(2)_L$ gauge transformation	weak isospin
	P × SU(2)	G-parity
	SU(3) "winding number"	baryon number
	SU(3) gauge transformation	quark color
	SU(3) (approximate)	quark flavor
	S(U(2) × U(3)) [U(1) × SU(2) × SU(3)]	Standard Model

List from wikipedia: "Symmetry (physics)"

Today's problems

- 7.46 — Here you'll show the two definitions of "differentiable symmetries" are equivalent!
- 7.38 — Calling back the system from the chapter on Energy!