PhysH308

Spinning *with* things!

Ted Brzinski, Nov. 7, 2024

Let’s move the independent problem deadline to Monday?

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Pre-registration ends this week!

There are a lot of upper-level options available this Spring!

Physics courses:

1. PHYSH302: Advanced Quantum, with Walter Smith
2. Bryn Mawr is offering PHYSB309 Advanced E&M
3. PHYSH304: Computational Physics with our new faculty member Vijay Singh (whose research is in computational biological physics).
4. PHYSH353: Topics in Soft Matter Physics, a special topics course with visiting faculty member Vianney Gimenez-Pinto.
5. Bryn Mawr is offering PHYSB331: Advanced Experimental Physics

Not listed as physics, but physics, Clyde Daly in Chemistry is teaching CHEMH350: Topics in Computational Chemistry (time TBD), which can use PHYSH214 as a prereq.

Astrophysics/astronomy courses:

6. ASTR344: Topics in Astrophysics: Gravitational Waves with Andrea Lommen.

Problem 10.37

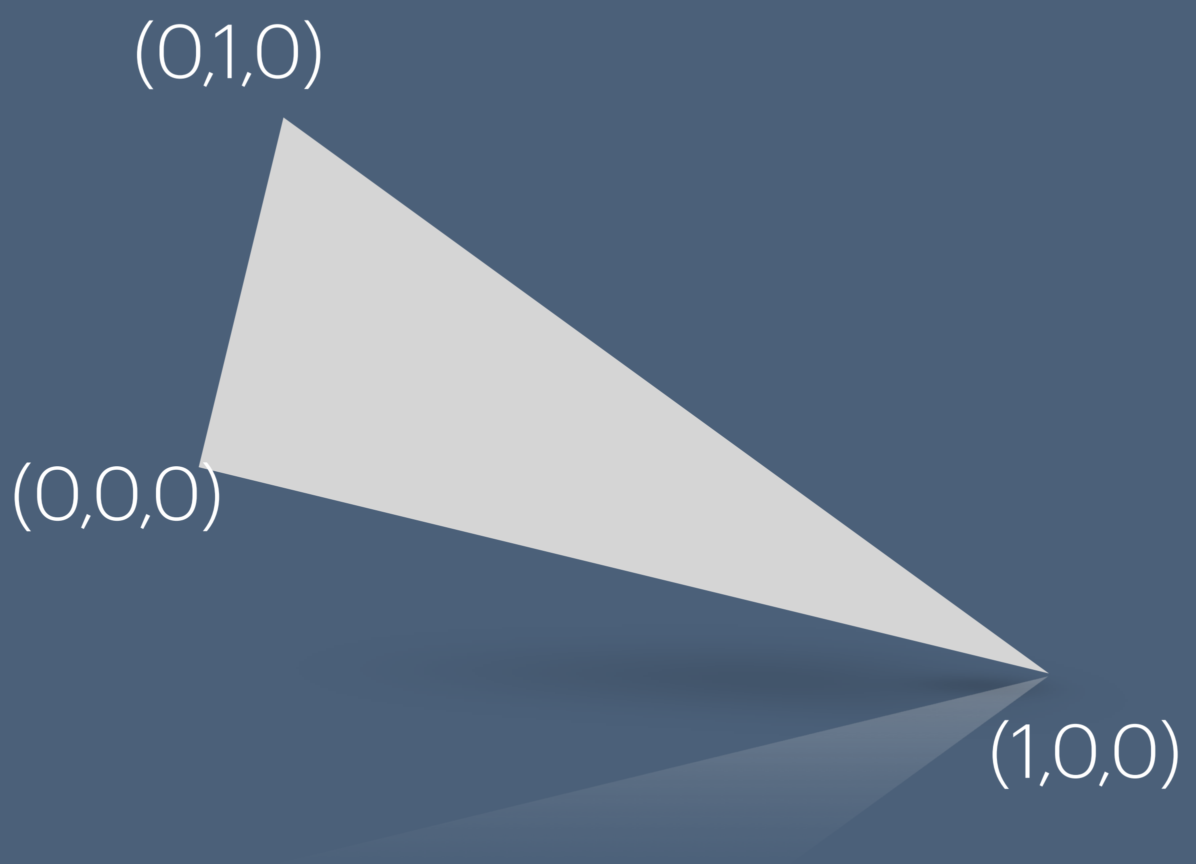
Let's just do it — no need to turn it in!

- Given a shape, find $\mathbf{I}$, diagonalize it, and find the principal axes

  See that from $\mathbf{I}$ in any coordinates you can find the principal axes

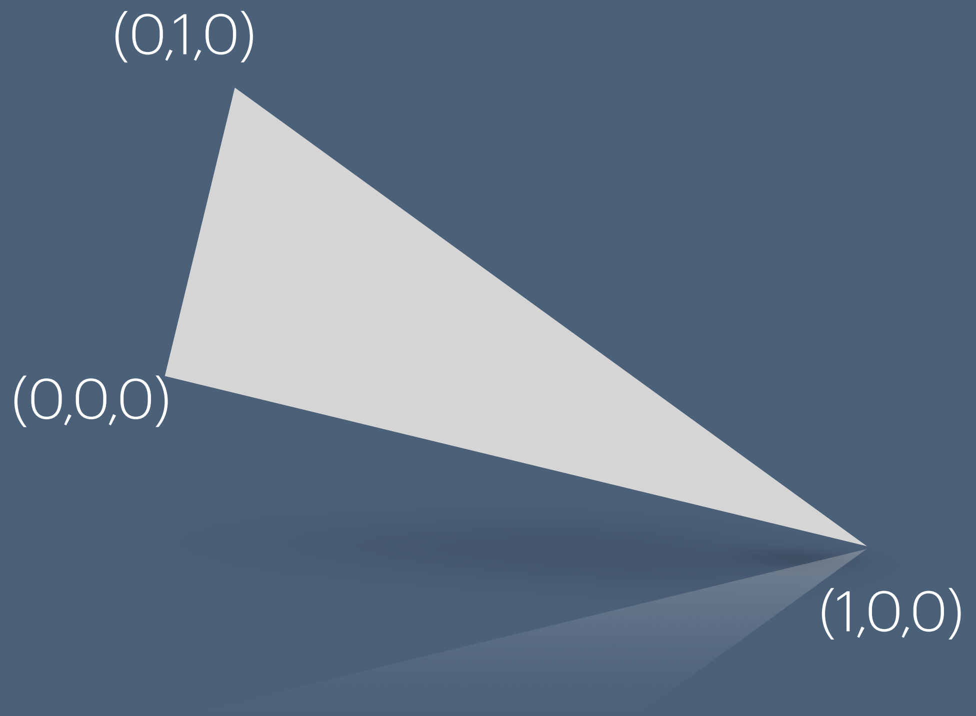
$$I_{xx} = m \sum \left(y^2 + z^2\right)$$

$$I_{xy} = I_{yx} = -m \sum xy$$



Problem 10.37

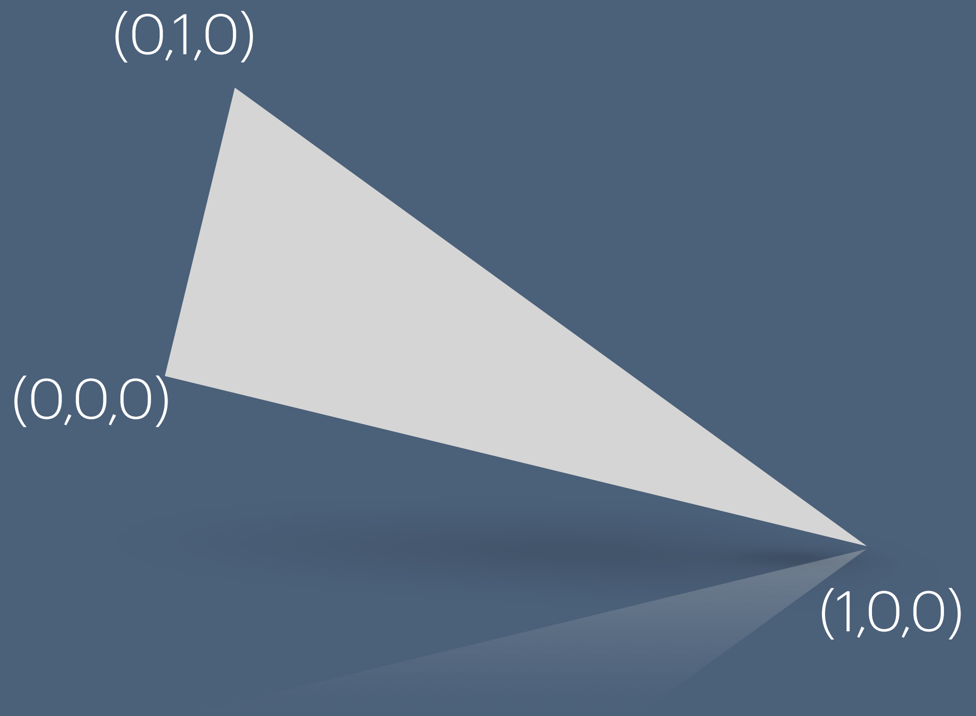
Moments of inertia



$$I_{xx} = \int dm\,\left(y^2 + z^2\right)$$

Problem 10.37

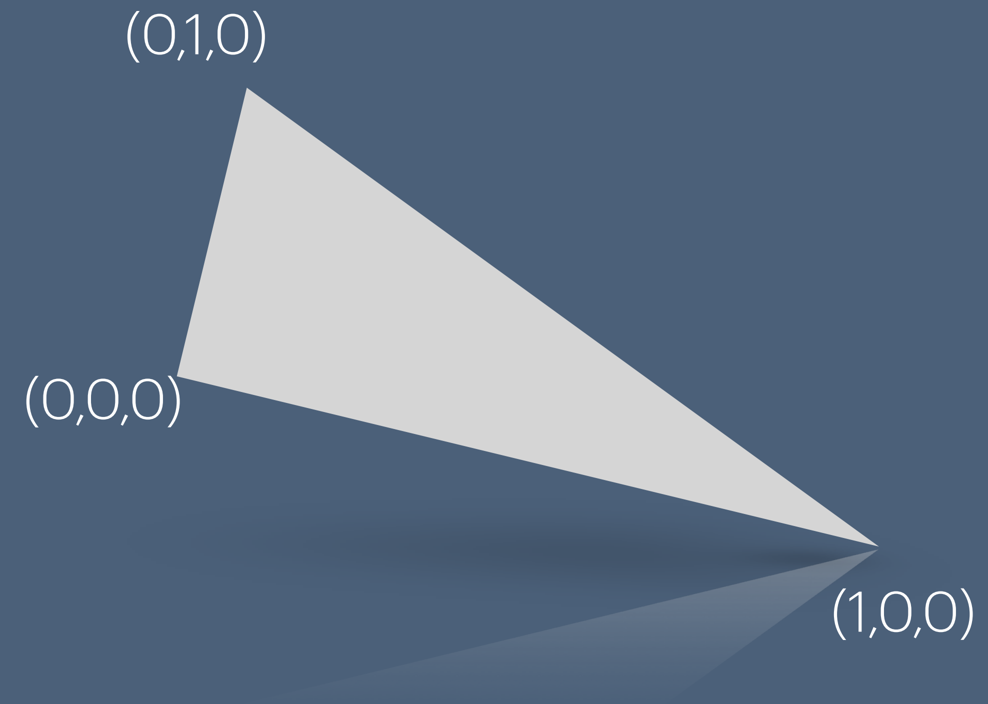
Moments of inertia



$$I_{xx} = \int dm\left(y^2 + z^2\right) = \sigma\int dA\ y^2$$

Problem 10.37

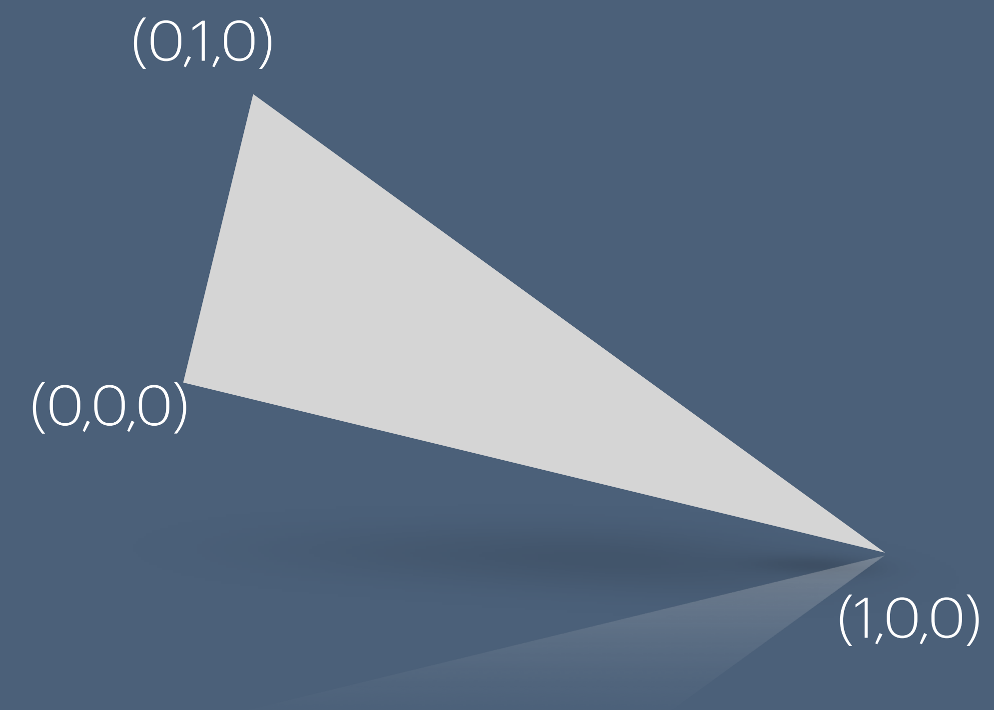
Moments of inertia



$$I_{xx} = \int dm\left(y^2 + z^2\right) = \sigma \int dA\ y^2 = \rho \int_0^1 dx \int_0^{1-x} dy\ y^2$$

Problem 10.37

Moments of inertia



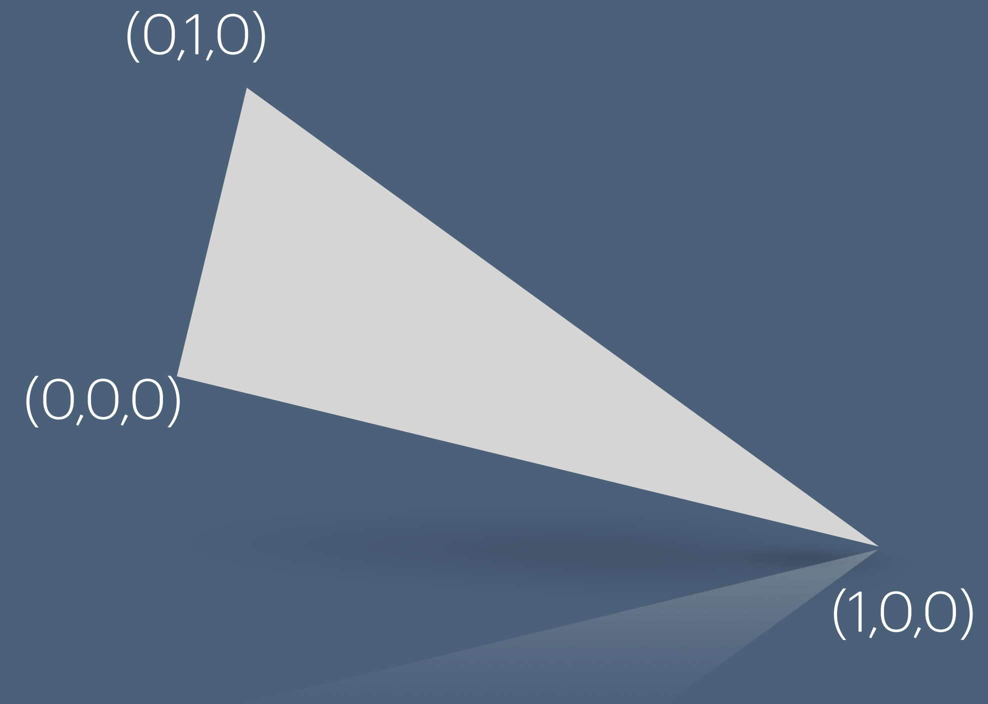
$$I_{xx} = \int dm\,(y^2 + z^2) = \sigma \int dA\; y^2 = \rho \int_0^1 dx \int_0^{1-x} dy\; y^2 = \sigma/12 = 2$$

$$I_{yy} = \int dm\,(x^2 + z^2) = \sigma \int dA\; x^2 = \rho \int_0^1 dy \int_0^{1-y} dx\; x^2 = \sigma/12 = 2$$

$$I_{zz} = \int dm\,(x^2 + y^2) = \int dm\,(x^2 + 2z^2 + y^2) = I_{xx} + I_{yy} = \sigma/6 = 4$$

Problem 10.37

Products of inertia



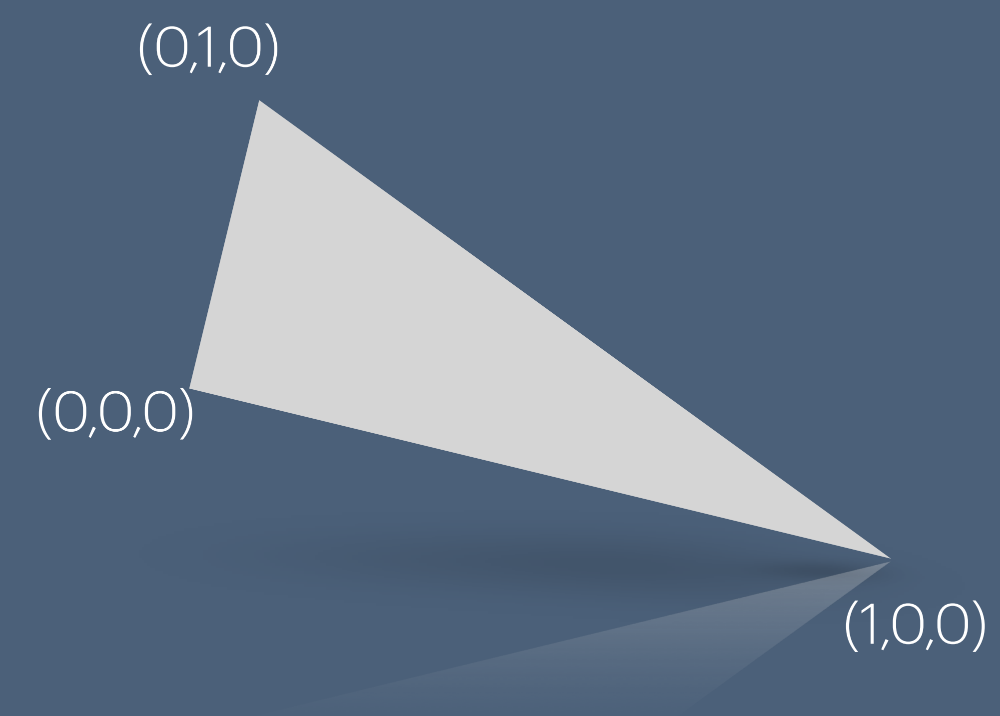
$$I_{xz} = -\sigma \int dA\ xz = 0 = I_{yz}$$

$$I_{xy} = -\sigma \int dA\ xy = -\sigma \int_0^1 dx\ x \int_0^{1-x} dy\ y = -1$$

So, $\mathbf{I} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Problem 10.37

Products of inertia



$$\mathbf{I} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\det\left(\mathbf{I} - \lambda\mathbf{1}\right) = 12 - 19\lambda + 8\lambda^2 - \lambda^3 = (1-\lambda)(3-\lambda)(4-\lambda) = 0$$

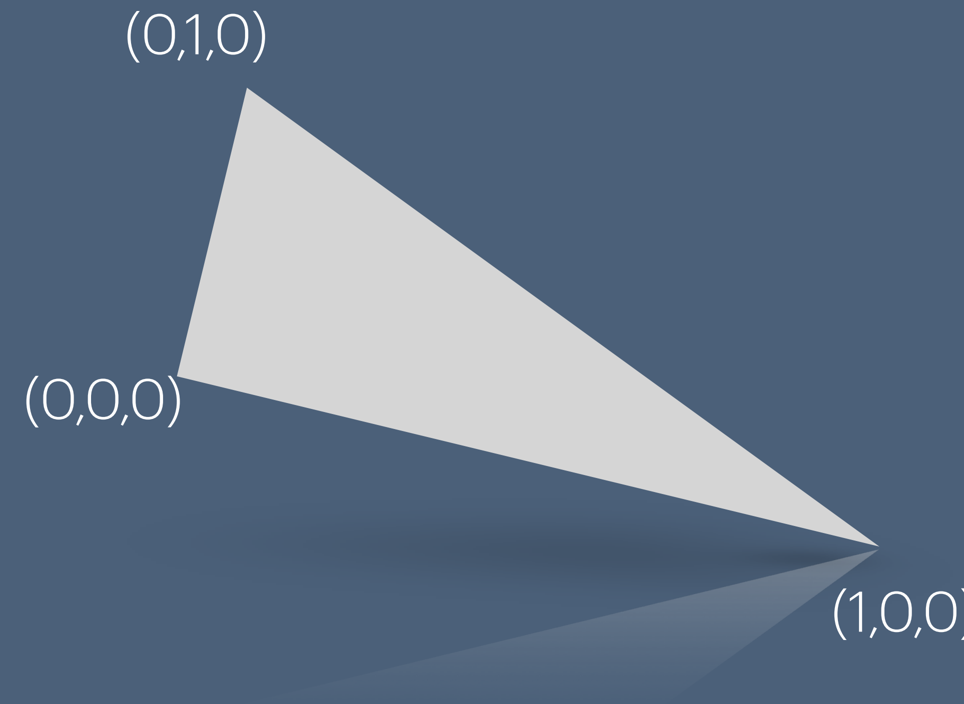
$$\lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = 4$$

Solve $\left(\mathbf{I} - \lambda_i\mathbf{1}\right)\hat{e}_i = 0$ *to find*

$$\hat{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ \hat{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},\ \hat{e}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Problem 10.37

Products of inertia



$$\mathbf{I} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\det\left(\mathbf{I} - \lambda \mathbf{1}\right) = 12 - 19\lambda + 8\lambda^2 - \lambda^3 = (1-\lambda)(3-\lambda)(4-\lambda) = 0$$

$$\lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = 4$$

This procedure can help with 10.42 if you struggle to calculate $\mathbf{I}$ at CoM

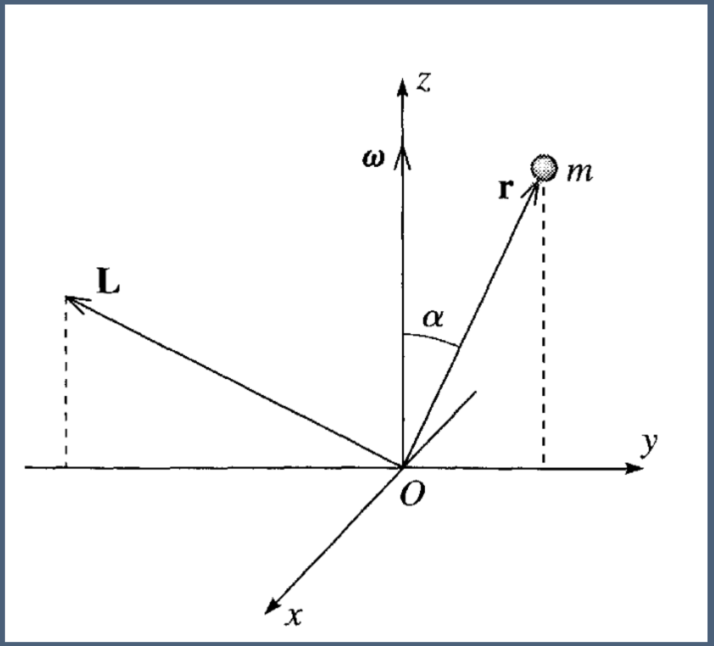
Solve $\left(\mathbf{I} - \lambda_i \mathbf{1}\right)\hat{e}_i = 0$ *to find*

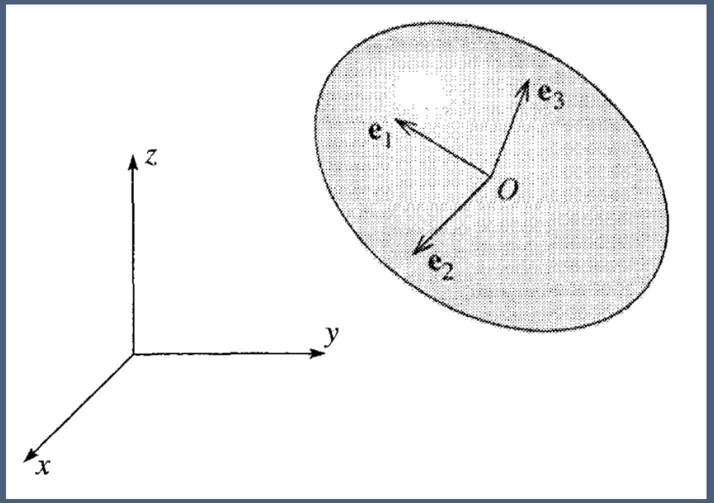
$$\hat{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ \hat{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},\ \hat{e}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

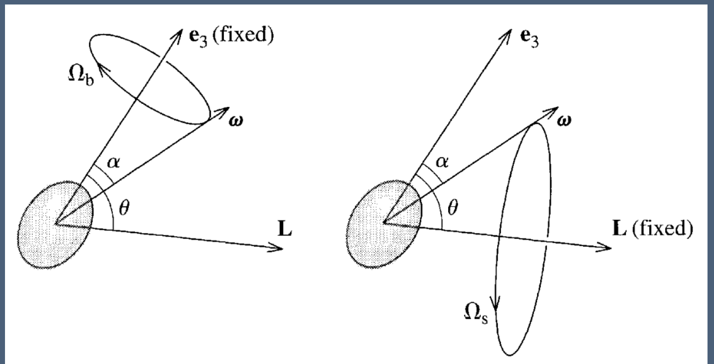
Euler's Angles

Return to the space frame

- We started with a fixed rotation in the space frame
- The inertia tensor allows us to understand the evolution of $\overrightarrow{L}$ *for fixed rotation*
- Euler's Equations allowed us to solve general rotation *in the body frame.*
- Now we want to solve for general rotation *in the space frame*







(a) Body frame
(b) Space frame

Euler's Angles

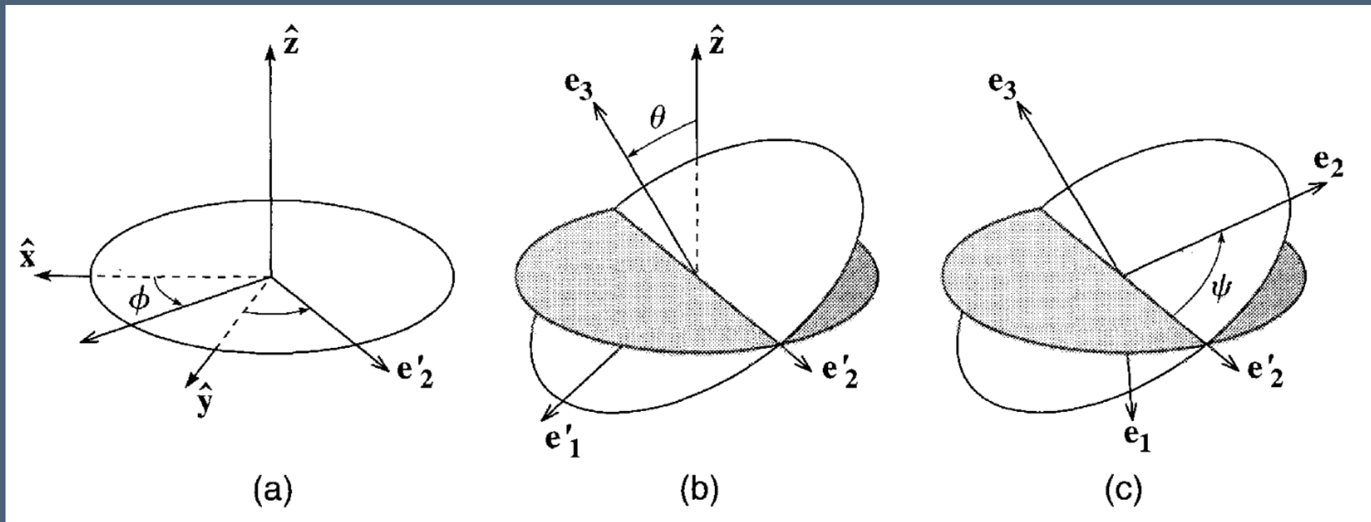
Return to the space frame

- We can describe the *orientation* of the body frame principal axes by considering rotation from space frame:

    1. Rotate by ϕ about $\hat{z}$ *to align* $\hat{e}_1$ *with* θ

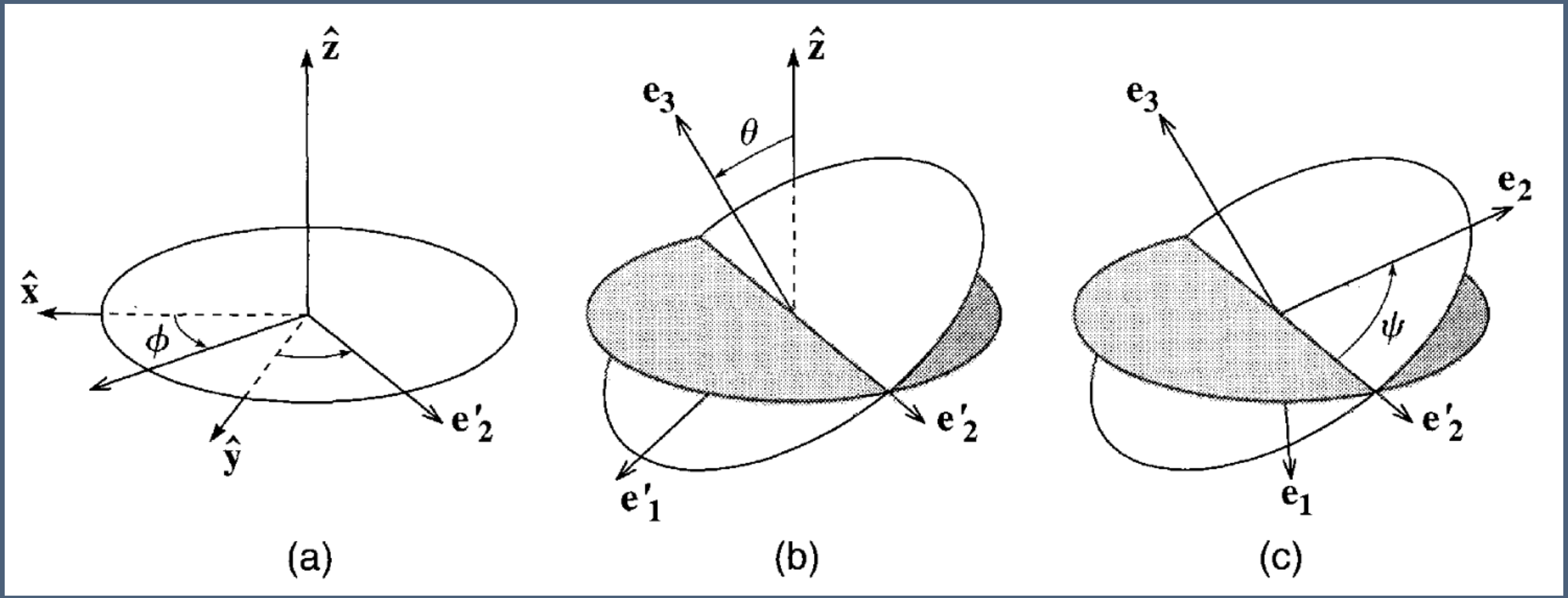
    2. Rotate by θ *around* $\hat{e}_2$ *to put* $\hat{e}_3$ *in its final orientation*

    3. *Rotate by* ψ *about* $\hat{e}_3$ *to put* $\hat{e}_1, \hat{e}_2$ *in their final orientations*



Euler's Angles

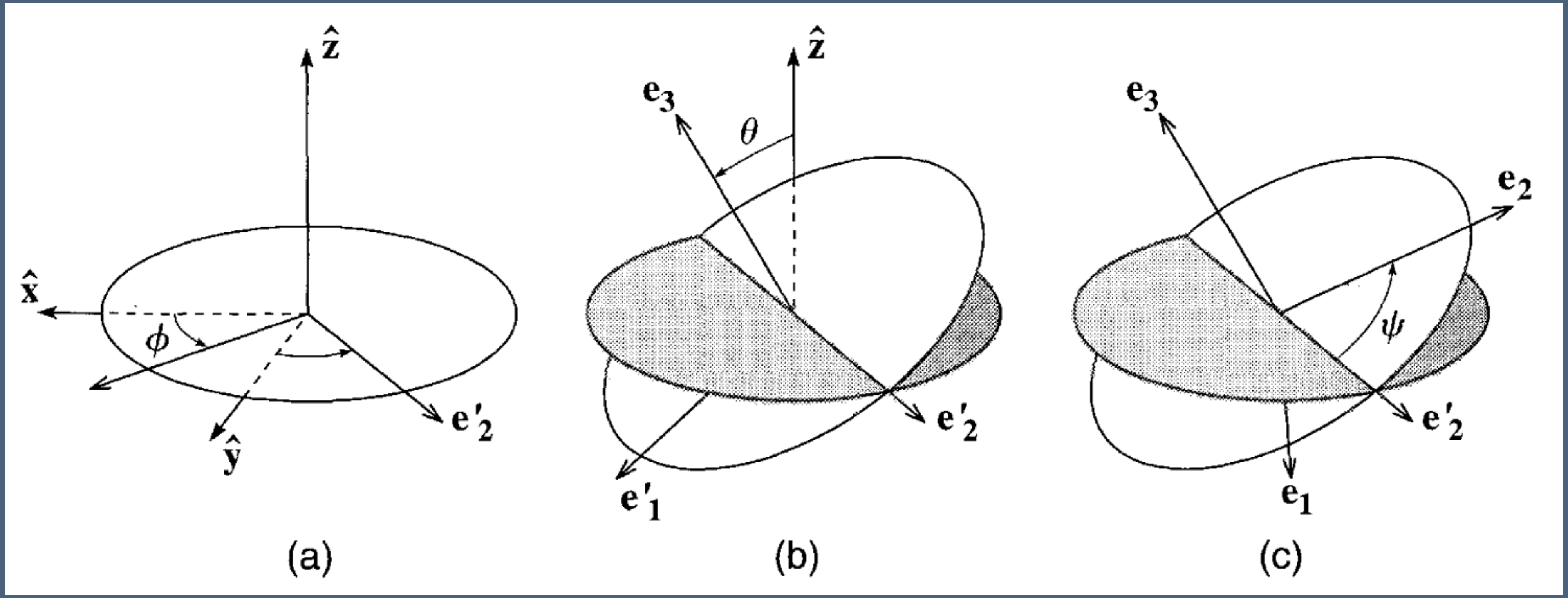
Return to the space frame



- Now we can rewrite $\overrightarrow{\omega}, \overrightarrow{L},$ and thus T (see book for details; in this class we'll always assume $\lambda_1 = \lambda_2$, which lets us disregard the last rotation for $\hat{e}_i$) in a body frame explicitly in terms of it's relation to the space frame!
- $\overrightarrow{\omega} = \left(-\dot{\phi}\sin\theta\right)\hat{e}'_1 + \dot{\theta}\hat{e}'_2 + \left(\dot{\psi} + \dot{\phi}\cos\theta\right)\hat{e}_3$

Euler's Angles

Return to the space frame

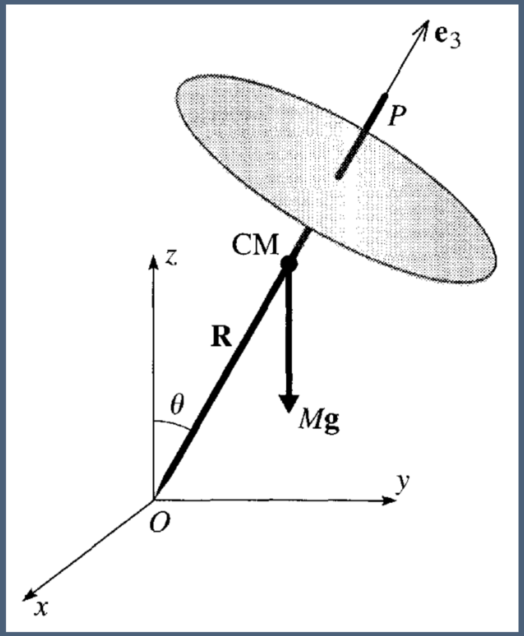


- $\vec{\omega} = \left(-\dot{\phi}\sin\theta\right)\hat{e}'_1 + \dot{\theta}\hat{e}'_2 + \left(\dot{\psi} + \dot{\phi}\cos\theta\right)\hat{e}_3$
- $\vec{L} = \left(-\lambda_1\dot{\phi}\sin\theta\right)\hat{e}'_1 + \lambda_1\dot{\theta}\hat{e}'_2 + \lambda_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\hat{e}_3$
- $T = \frac{1}{2}\lambda_i\omega_i^2 = \frac{1}{2}\lambda_1\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}\lambda_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2$

Euler's Angles

Return to the space frame

- We have $T = \frac{1}{2}\lambda_1\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}\lambda_3\left(\dot{\psi} + \lambda_3\dot{\phi}\cos\theta\right)^2$

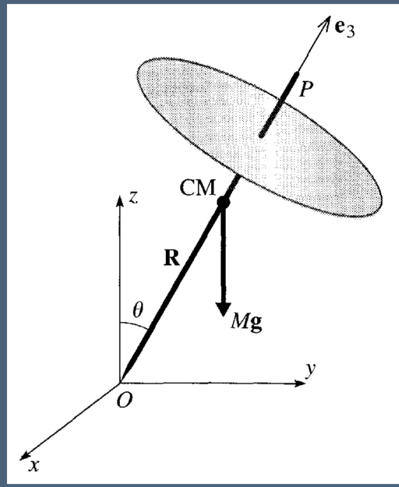


- *For the top,* $U = MgR_{cm}\cos\theta$
- *If we have* T *and* U*, we can use....*

Euler's Angles

Return to the space frame

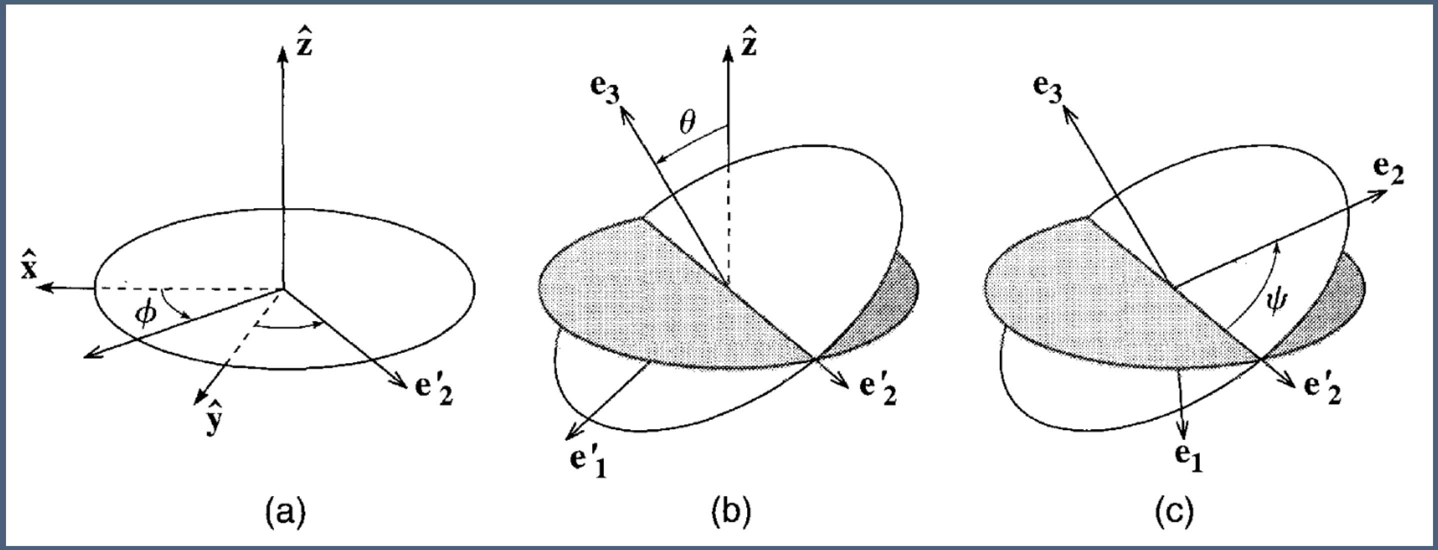
- We have $T = \frac{1}{2}\lambda_1\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}\lambda_3\left(\dot{\psi} + \lambda_3\dot{\phi}\cos\theta\right)^2$
- *For the top,* $U = MgR_{cm}\cos\theta$
- *If we have* T *and* U*, we can use....*



$$\mathscr{L} = \frac{1}{2}\lambda_1\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}\lambda_3\left(\dot{\psi} + \lambda_3\dot{\phi}\cos\theta\right)^2 - MgR_{cm}\cos\theta$$

Euler's Angles

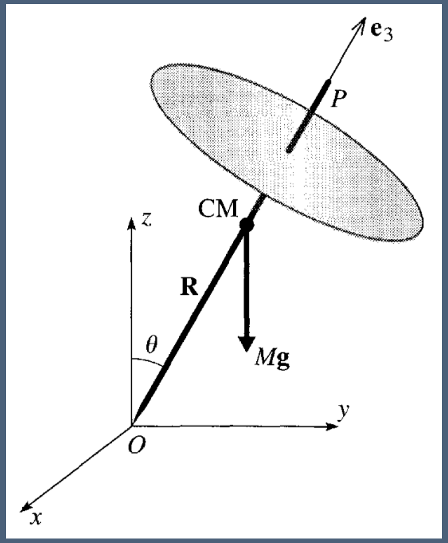
Return to the space frame



- $\mathscr{L} = \frac{1}{2}\lambda_1\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}\lambda_3\left(\dot{\psi} + \lambda_3\dot{\phi}\cos\theta\right)^2 - MgR_{cm}\cos\theta$

- *By inspection,* $\frac{\partial \mathscr{L}}{\partial \psi} = \frac{\partial \mathscr{L}}{\partial \phi} = 0$, *so conserved momenta!*

  (angular momenta about $\hat{z}, \hat{e}_3$*: spin+precession)*

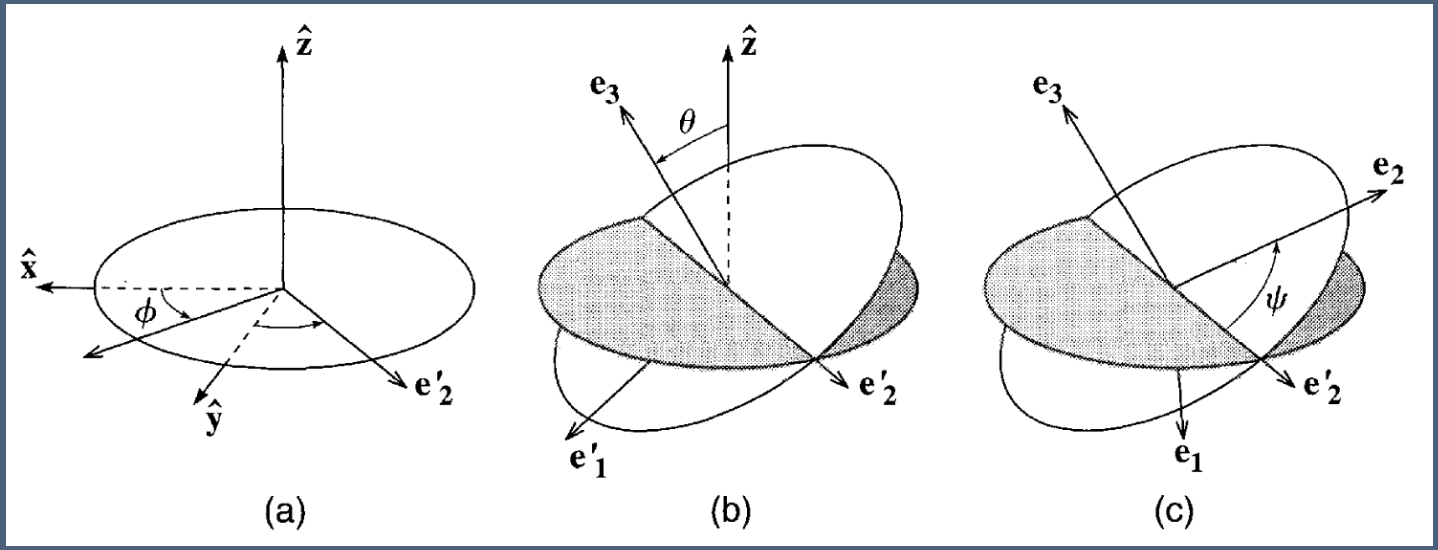


- *Solving for precession rate:*

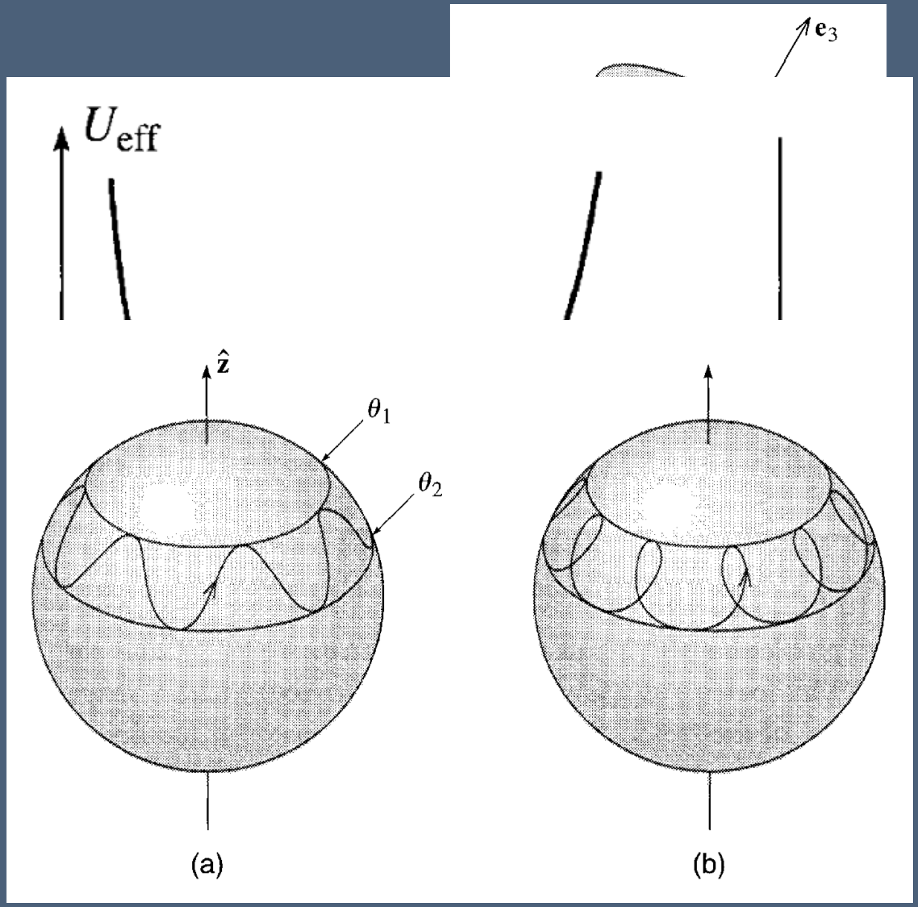
$$\dot{\phi} = \text{const} = \Omega = \begin{cases} \dfrac{MgR}{\lambda_3\omega_3} & \text{Torque due to gravity!} \\[2ex] \dfrac{\lambda_3\omega_3}{\lambda_1\cos\theta} & \text{Like the wobbling book!} \end{cases}$$

Euler's Angles

Return to the space frame

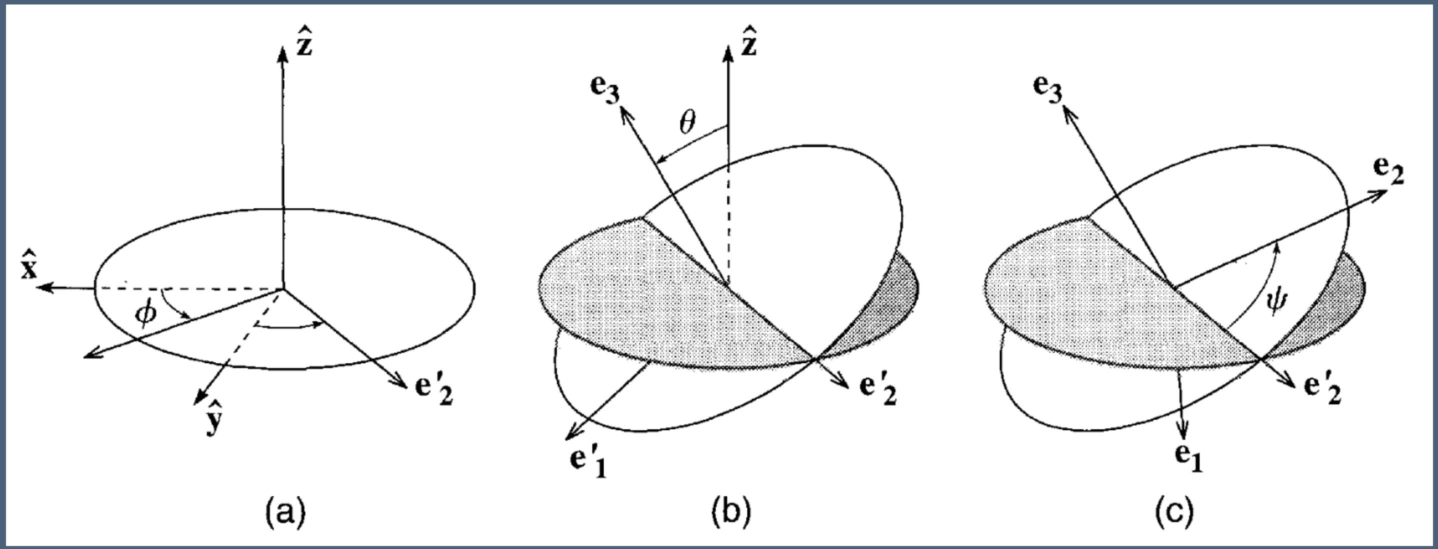


- $\mathscr{L} = \frac{1}{2}\lambda_1\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}\lambda_3\left(\dot{\psi} + \lambda_3\dot{\phi}\cos\theta\right)^2 - MgR_{cm}\cos\theta$
- *What about* $\frac{\partial\mathscr{L}}{\partial\theta} = \frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{\theta}}$*?*
- *Here, it's actually easier to consider* $U_{eff}(\theta)$
- *If* θ *varies, so does* $\dot{\phi} = \frac{L_z - L_3\cos\theta}{\lambda_1\sin^2\theta}$
- *In 10.51, you'll find* $U_{eff}(\theta)$ *and see this relation directly!*



Problem 10.51

Return to the space frame



- *It's actually easier to consider* $U_{eff}(\theta)$
- *If* θ *varies, so does* $\dot{\phi} = \dfrac{L_z - L_3\cos\theta}{\lambda_1 \sin^2\theta}$
- *In* ***10.51****, you'll find* $U_{eff}(\theta)$ *and see this relation directly!*