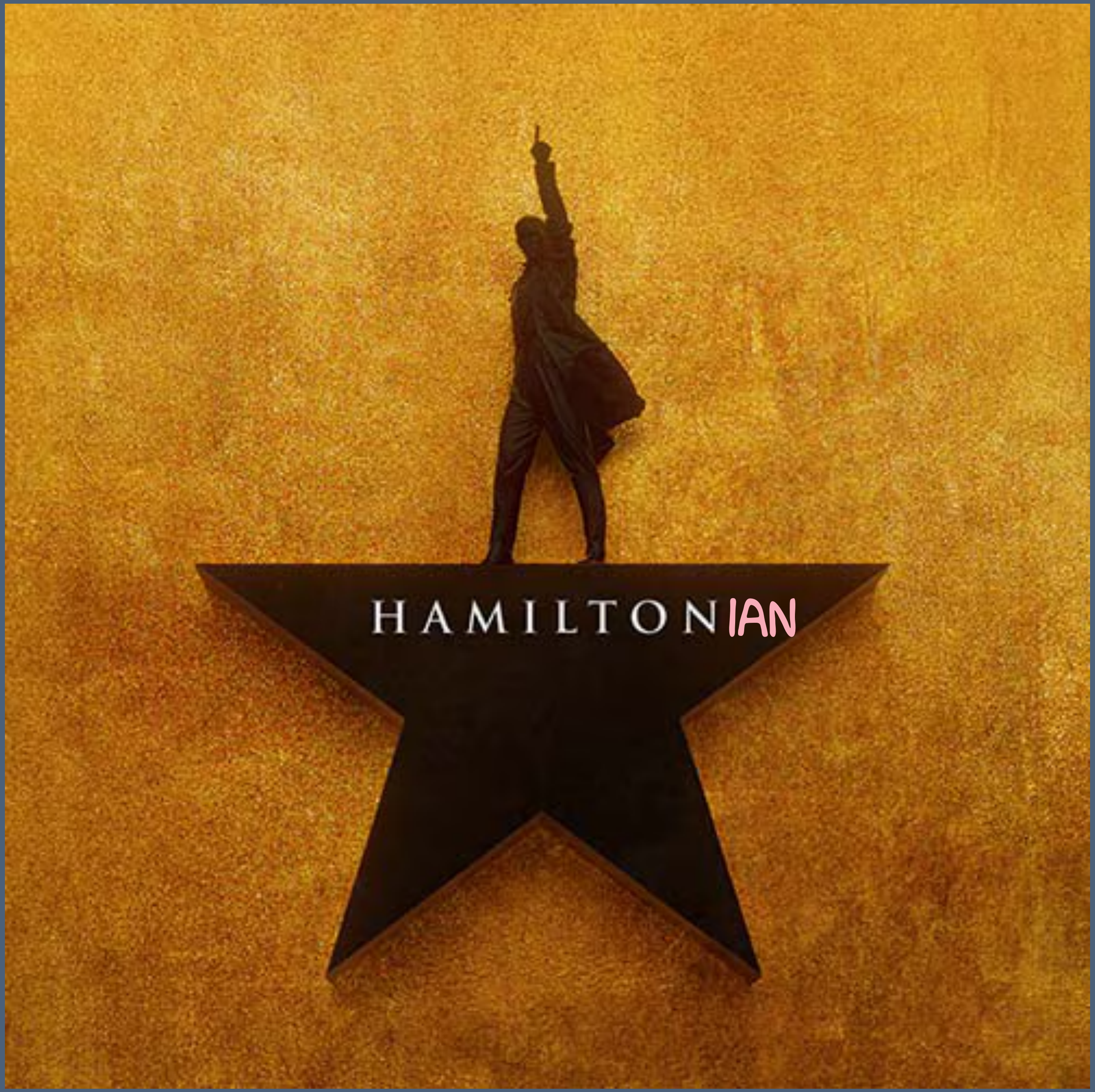
PhysH308

Hamiltonian Mechanics!

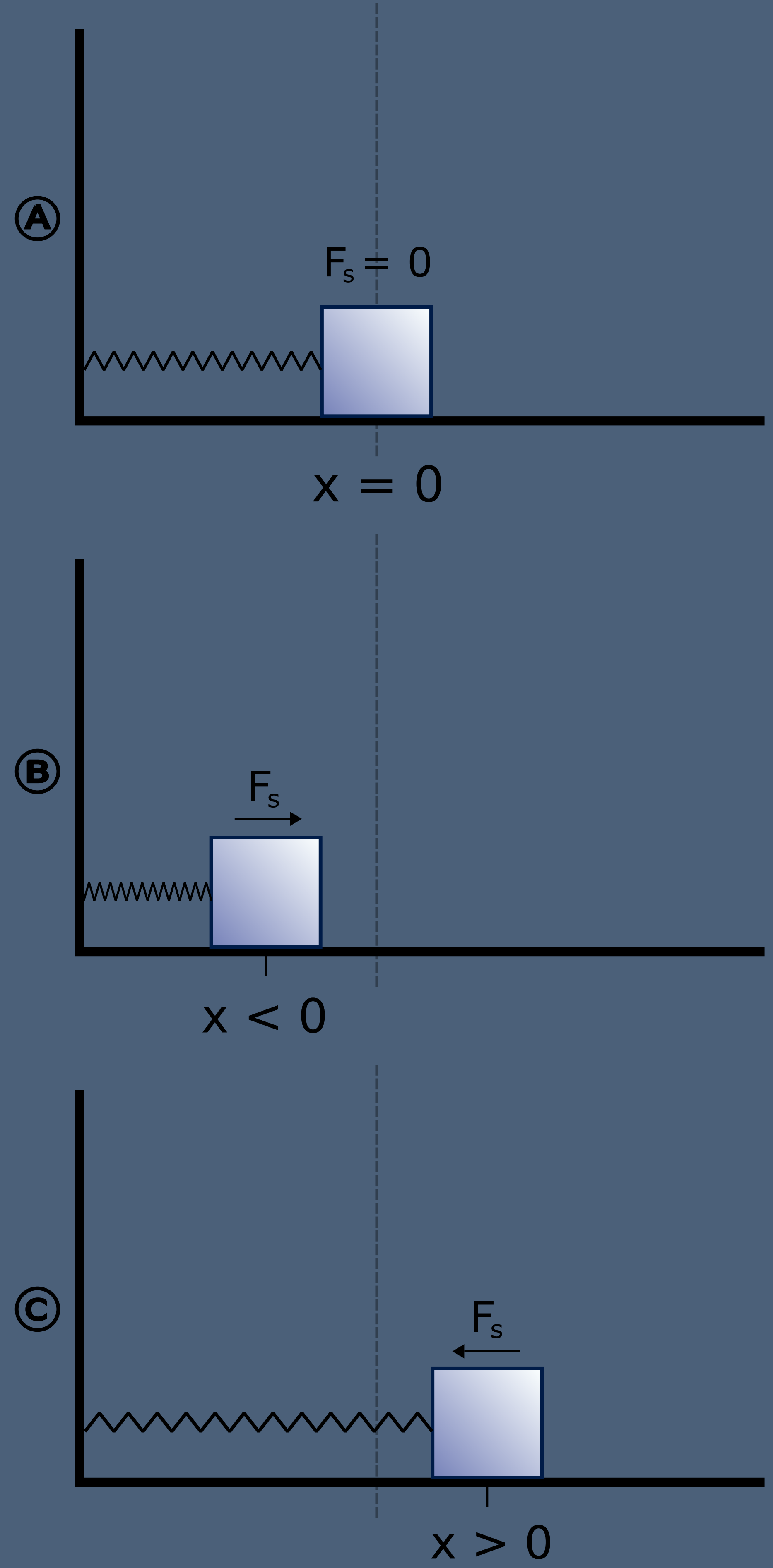


Ted Brzinski, Nov. 7, 2024

Hamiltonian mech

Nearly Lagrangian

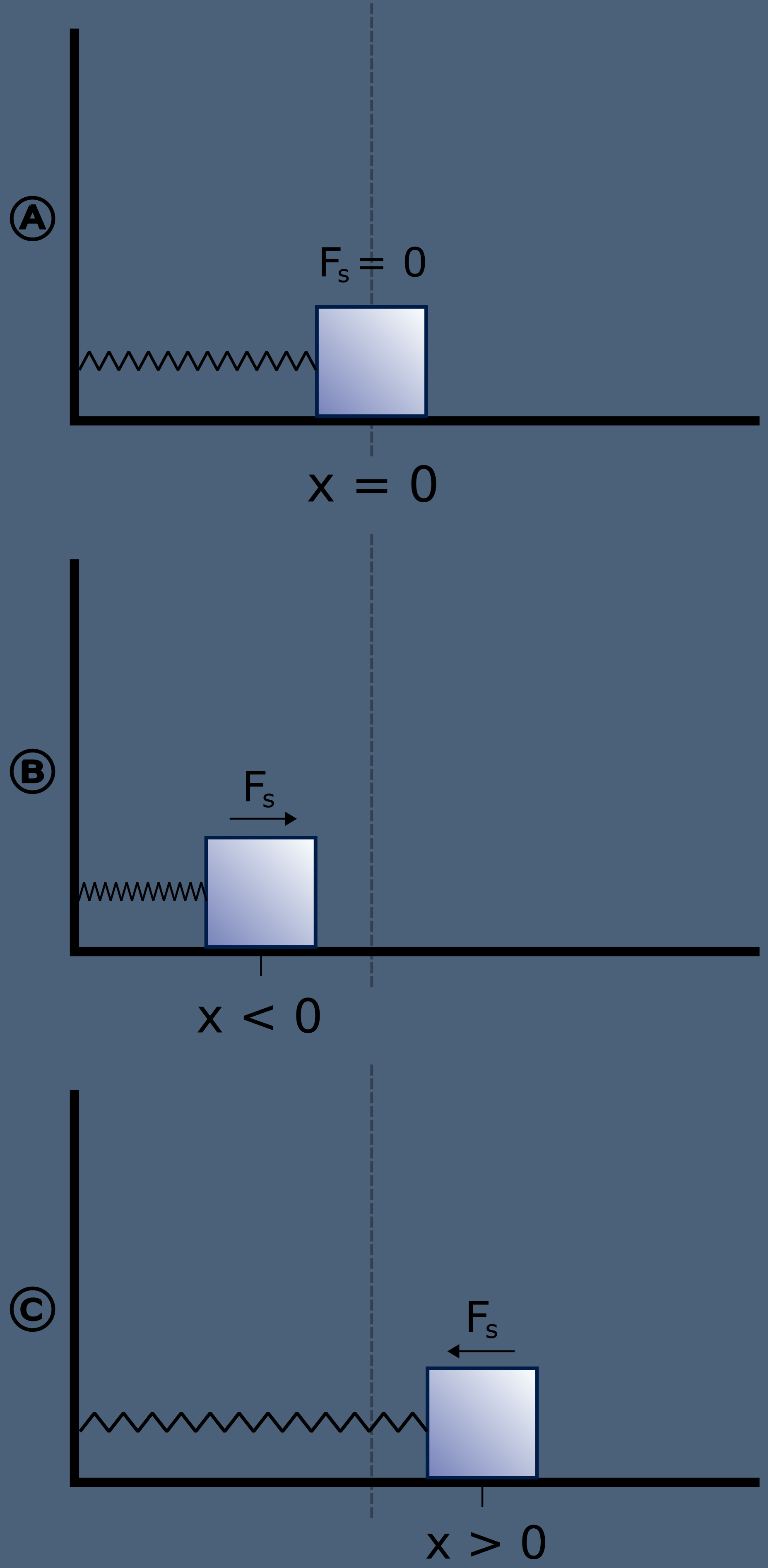
- Consider the system pictured without friction. The spring is ideal and Hookean.
- $T = \frac{1}{2}m\dot{x}^2$
- $U = \frac{1}{2}kx^2$
- $\mathscr{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$



Hamiltonian mech

Nearly Lagrangian

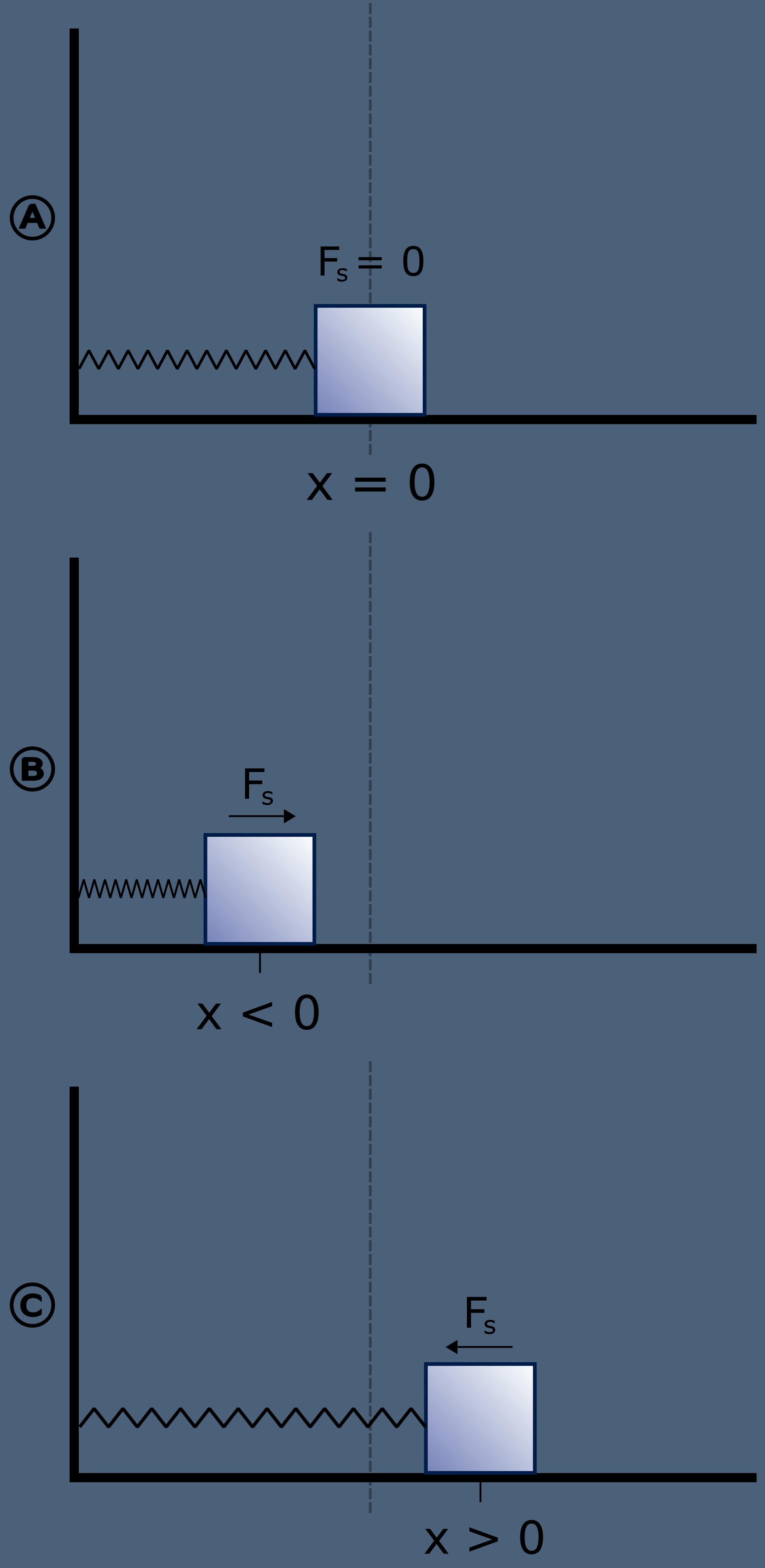
- $\mathscr{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$
- $\frac{\partial \mathscr{L}}{\partial x} = -kx$
  is the *generalized force*
- $\frac{\partial \mathscr{L}}{\partial \dot{x}} = m\dot{x}$
  is the *generalized momentum*
- E-L yields: $\ddot{x} = -\frac{k}{m}x$
  (as we'd expect)



Hamiltonian mech

Nearly Lagrangian

- Let's do this a different way...
- $\mathscr{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$
- $\frac{\partial \mathscr{L}}{\partial \dot{x}} = m\dot{x}$ is the *generalized momentum* p
- We'll define the Hamiltonian: $\mathscr{H} = p\dot{x} - \mathscr{L}$

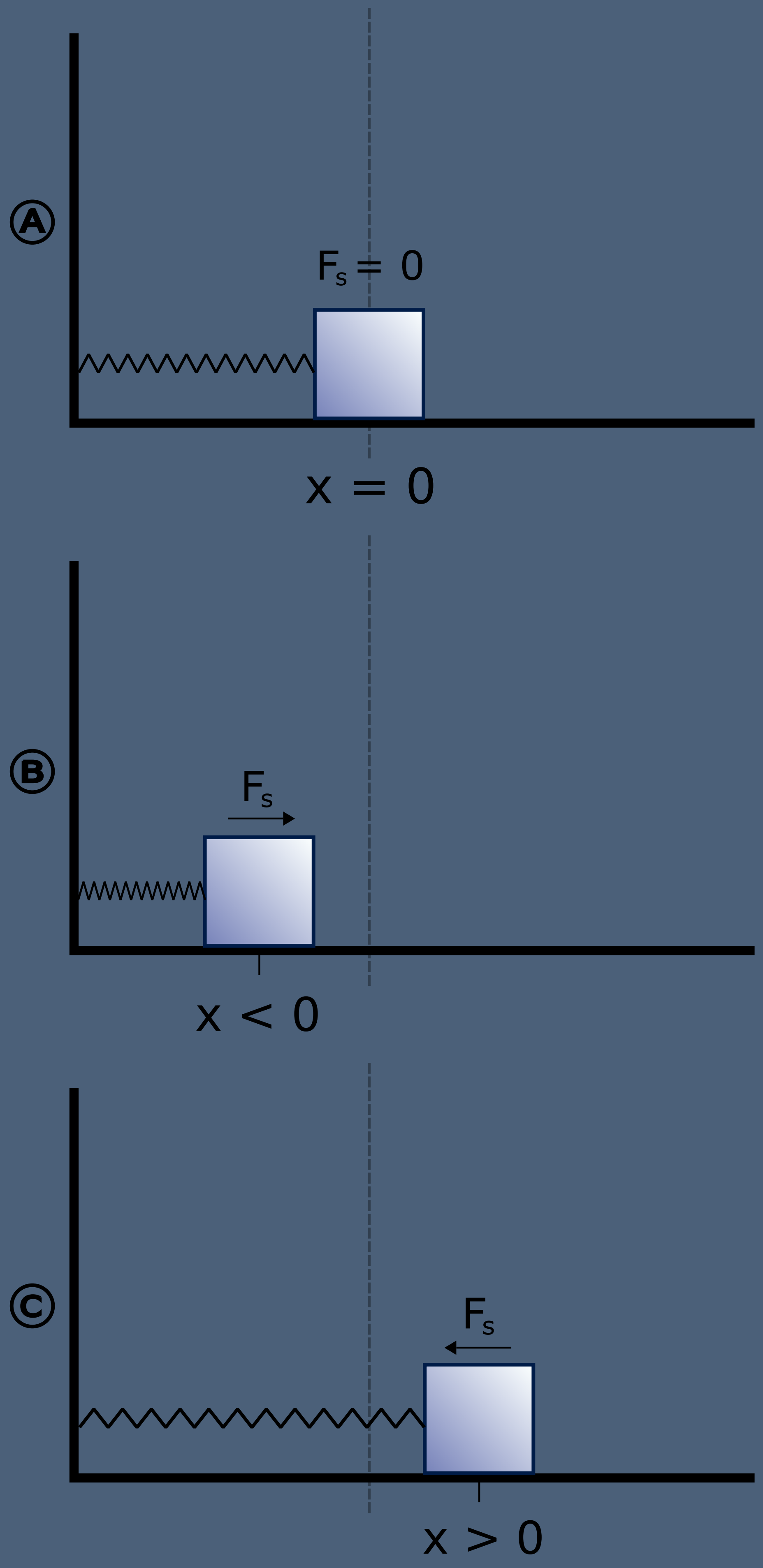


Hamiltonian mech

Nearly Lagrangian

- Let's do this a different way…
- $\mathscr{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$
- $p = m\dot{x}$ is the *generalized momentum*
- We'll define the Hamiltonian:

$$\mathscr{H} = p\dot{x} - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$



Hamiltonian mech

Nearly Lagrangian

- $p = m\dot{x}$
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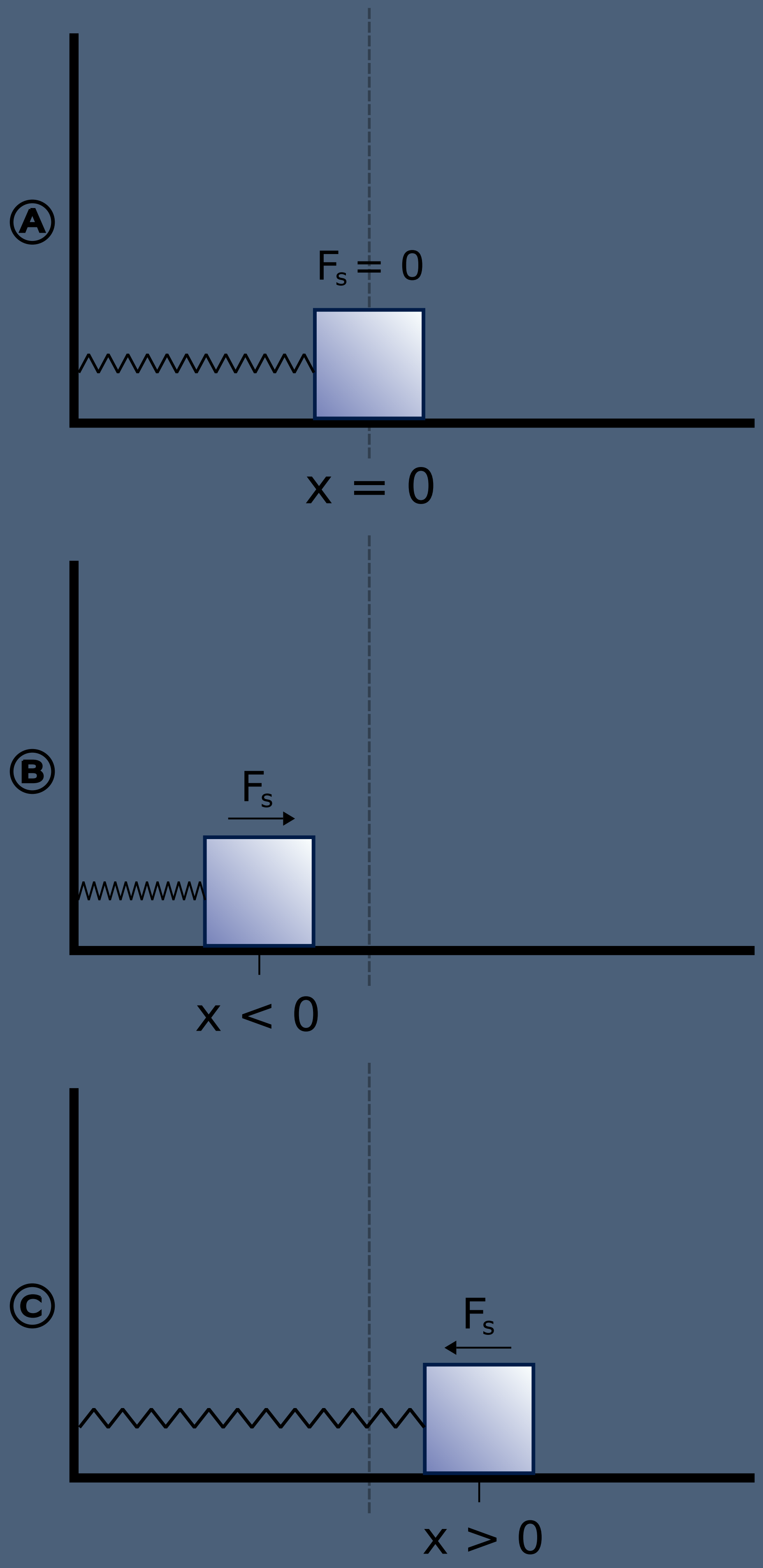


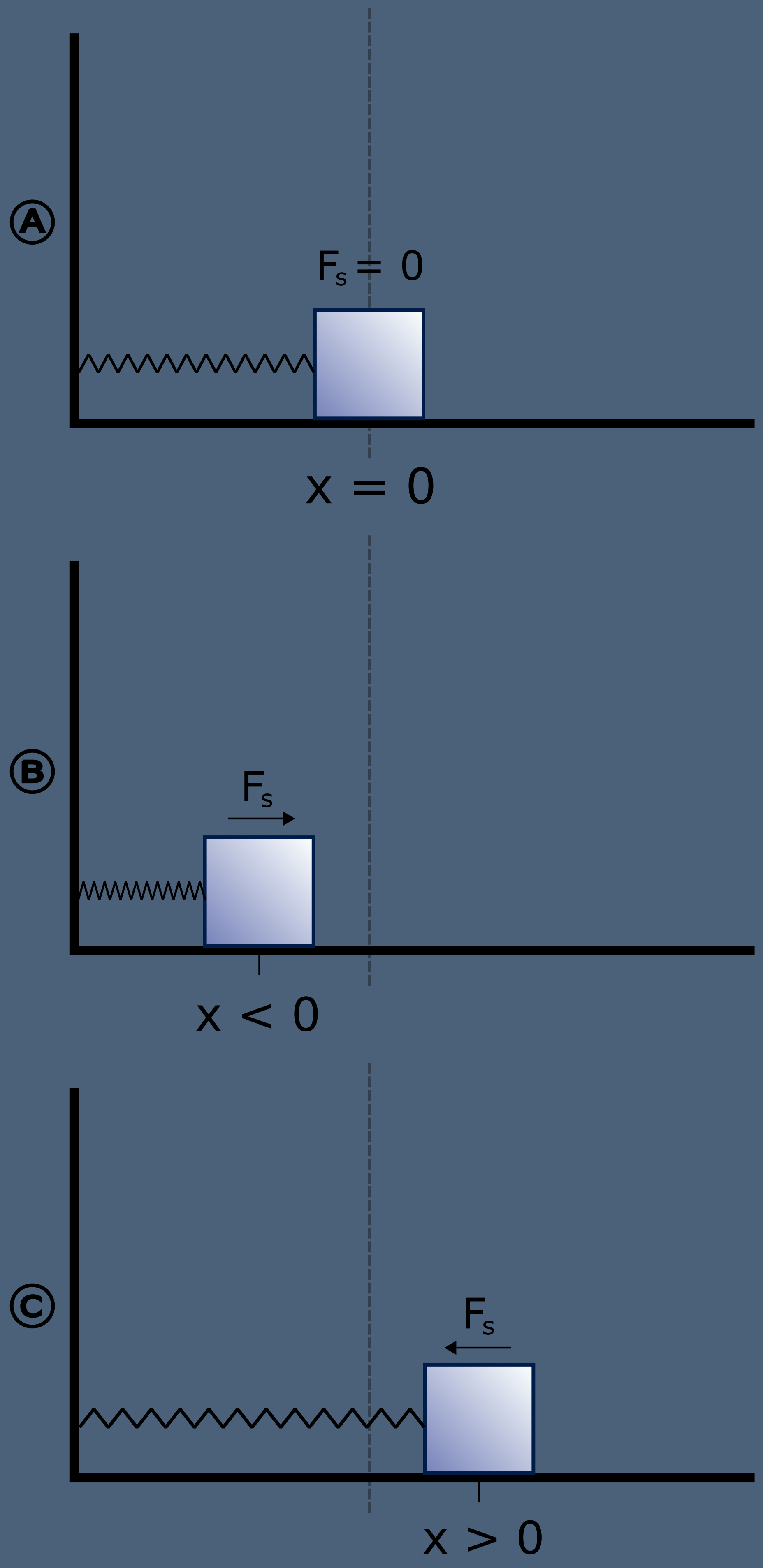
Hamiltonian mech

Nearly Lagrangian

- $p = m\dot{x}$

$$\mathscr{H} = p\dot{x} - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

- $$= m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$



Hamiltonian mech

Nearly Lagrangian

- $p = m\dot{x}$

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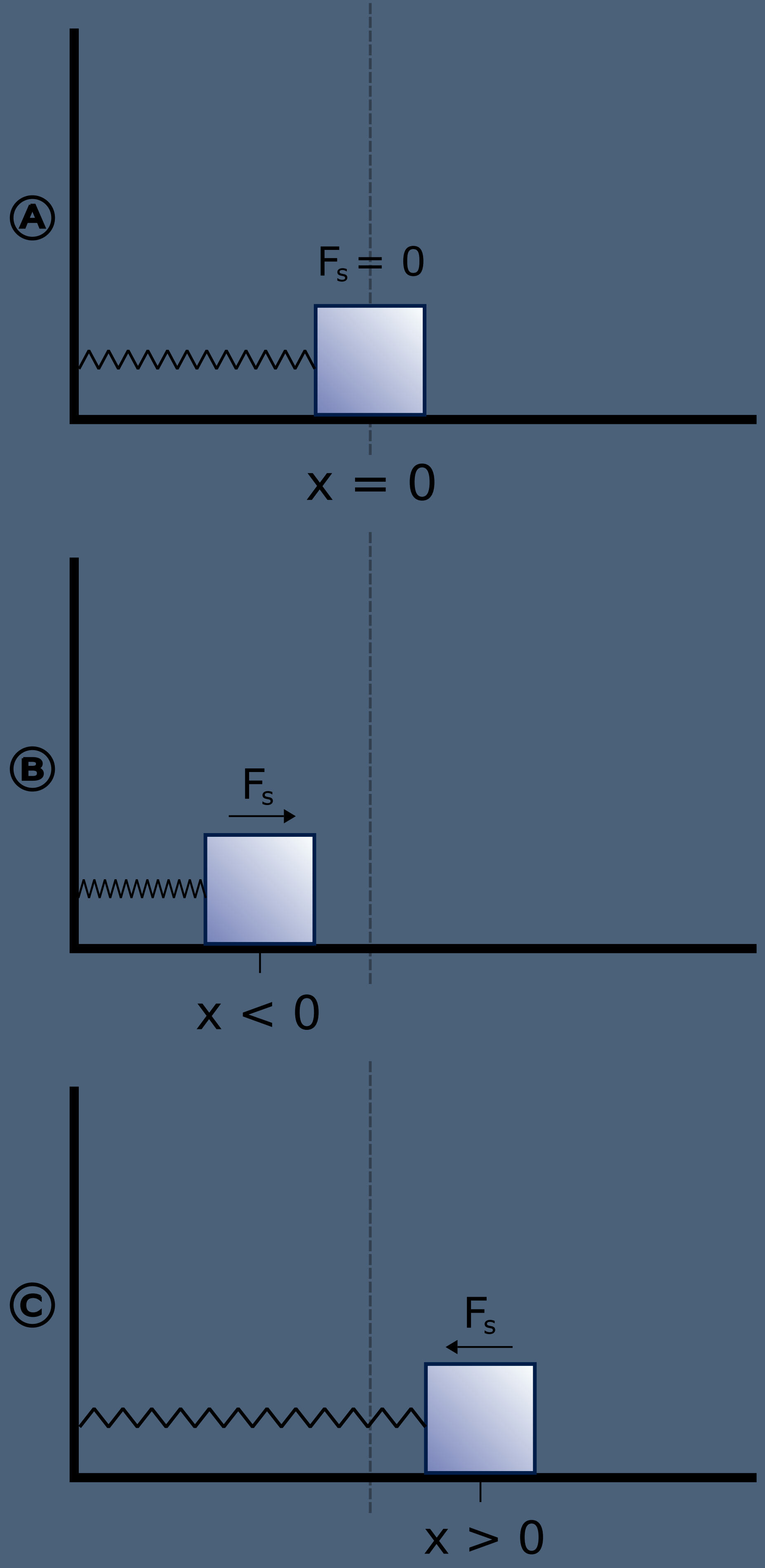
$$= m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

- $$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

- Which looks like the total energy!



Hamiltonian mech

Nearly Lagrangian

But let's go back to here:

- $p = m\dot{x}$
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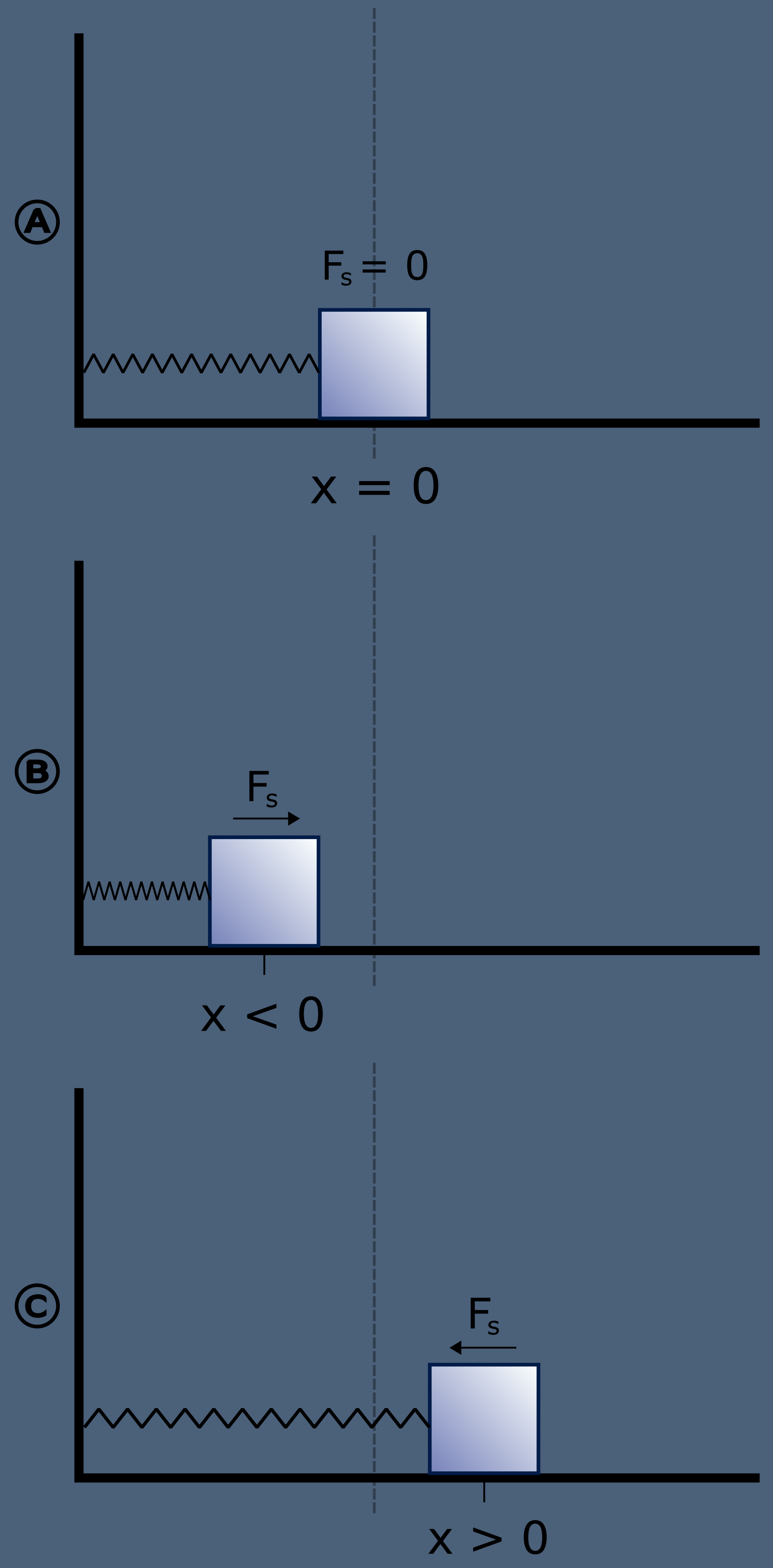
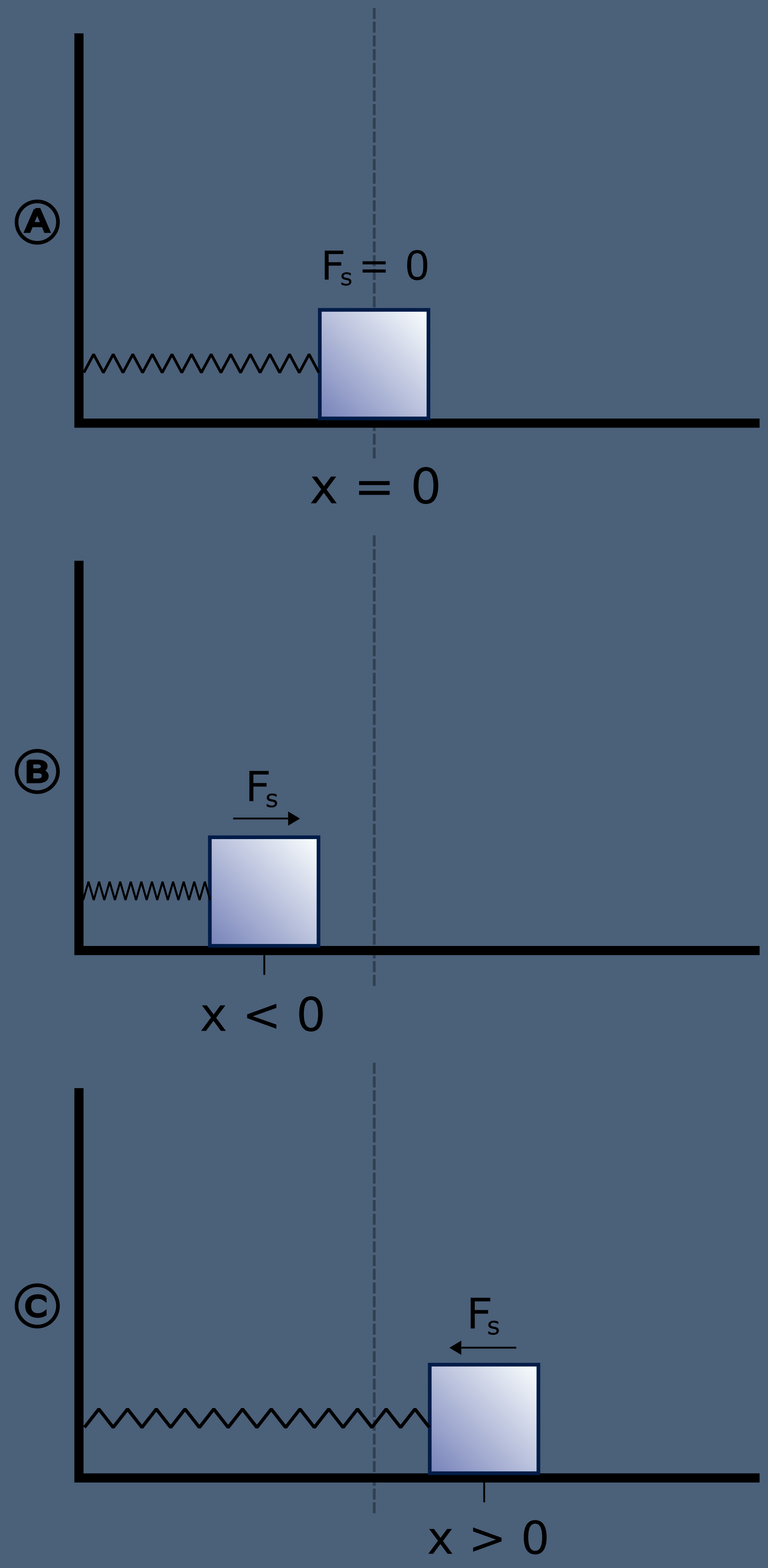


Hamiltonian mech

Nearly Lagrangian

But let's go back to here:

- $\dot{x} = \dfrac{p}{m}$
- $\mathscr{H} = p\dot{x} - \dfrac{1}{2}m\dot{x}^2 + \dfrac{1}{2}kx^2$

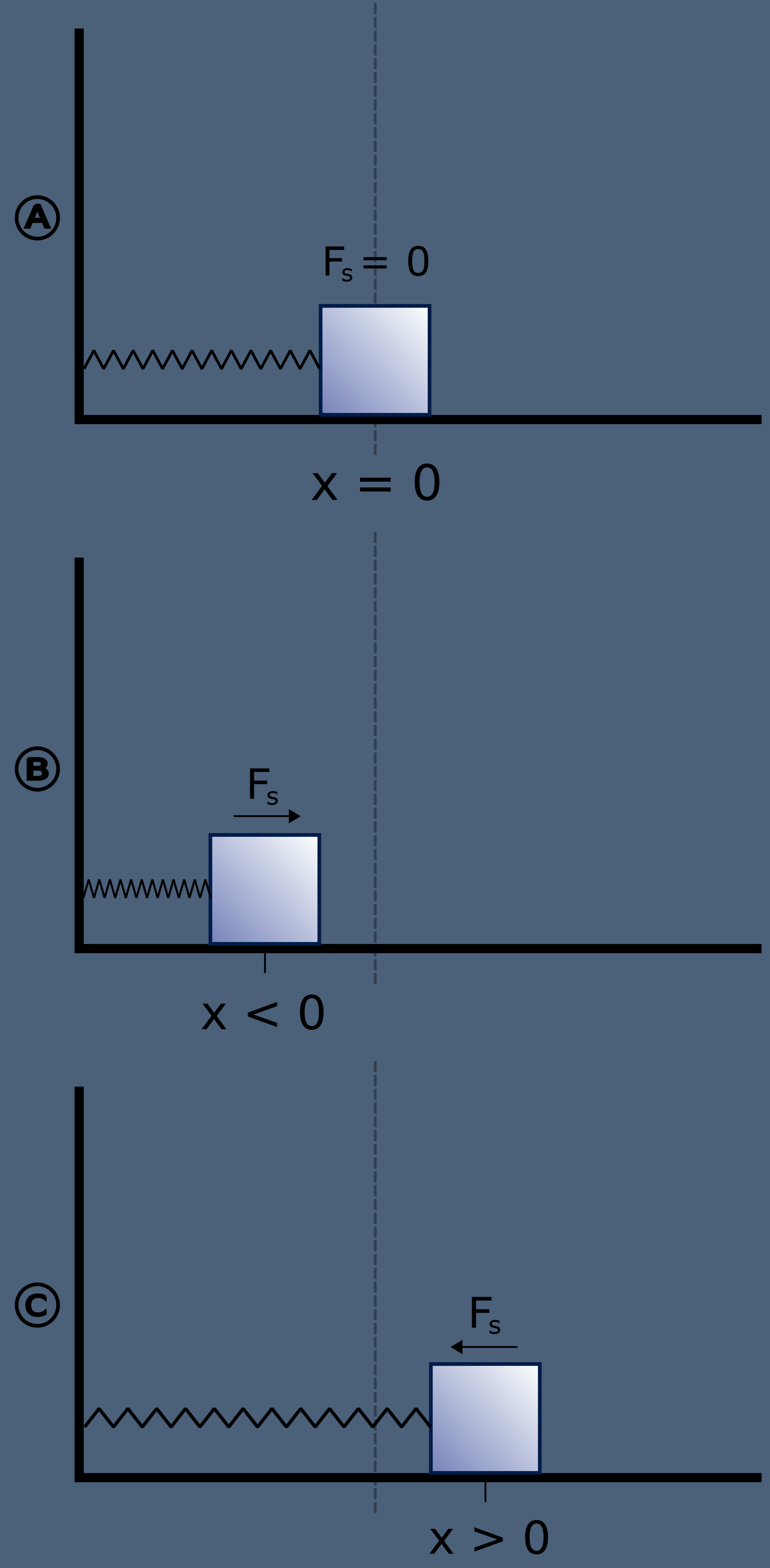


Hamiltonian mech

Nearly Lagrangian

But let's go back to here:

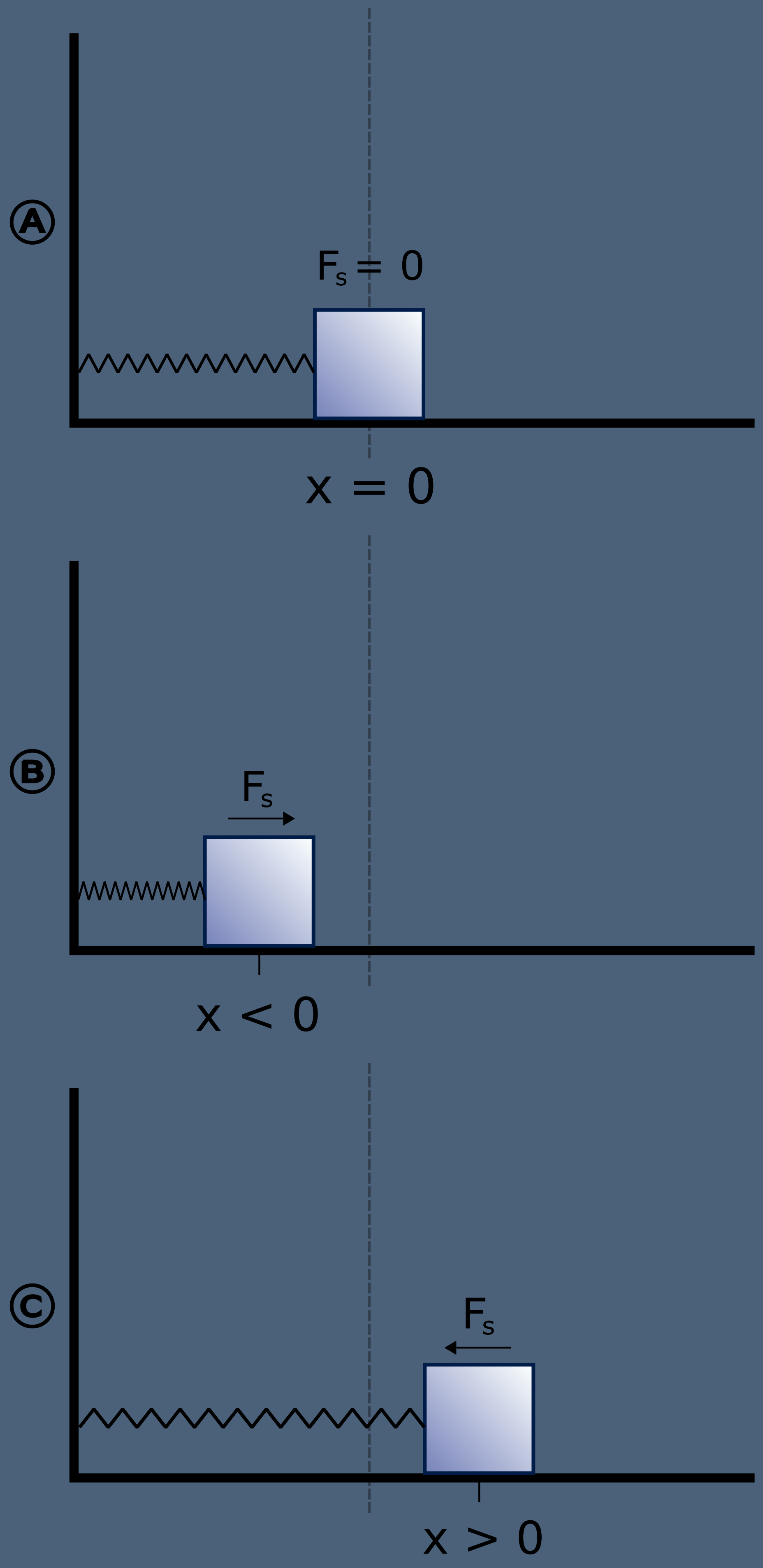
- $\dot{x} = \dfrac{p}{m}$
- $\mathscr{H} = \dfrac{p^2}{2m} + \dfrac{1}{2}kx^2$
- So we've gone from $\mathscr{L}\left(x, \dot{x}, t\right)$ to $\mathscr{H}\left(x, p, t\right)$
- What about E-L?



Hamiltonian mech

Nearly Lagrangian

- What about E-L?
- $\mathscr{H} \equiv p\dot{x} - \mathscr{L}$
- $\dfrac{\partial \mathscr{H}}{\partial x} = p\dfrac{\partial \dot{x}}{\partial x} - \left(\dfrac{\partial \mathscr{L}}{\partial x} + \dfrac{\partial \mathscr{L}}{\partial \dot{x}}\dfrac{\partial \dot{x}}{\partial x}\right)$



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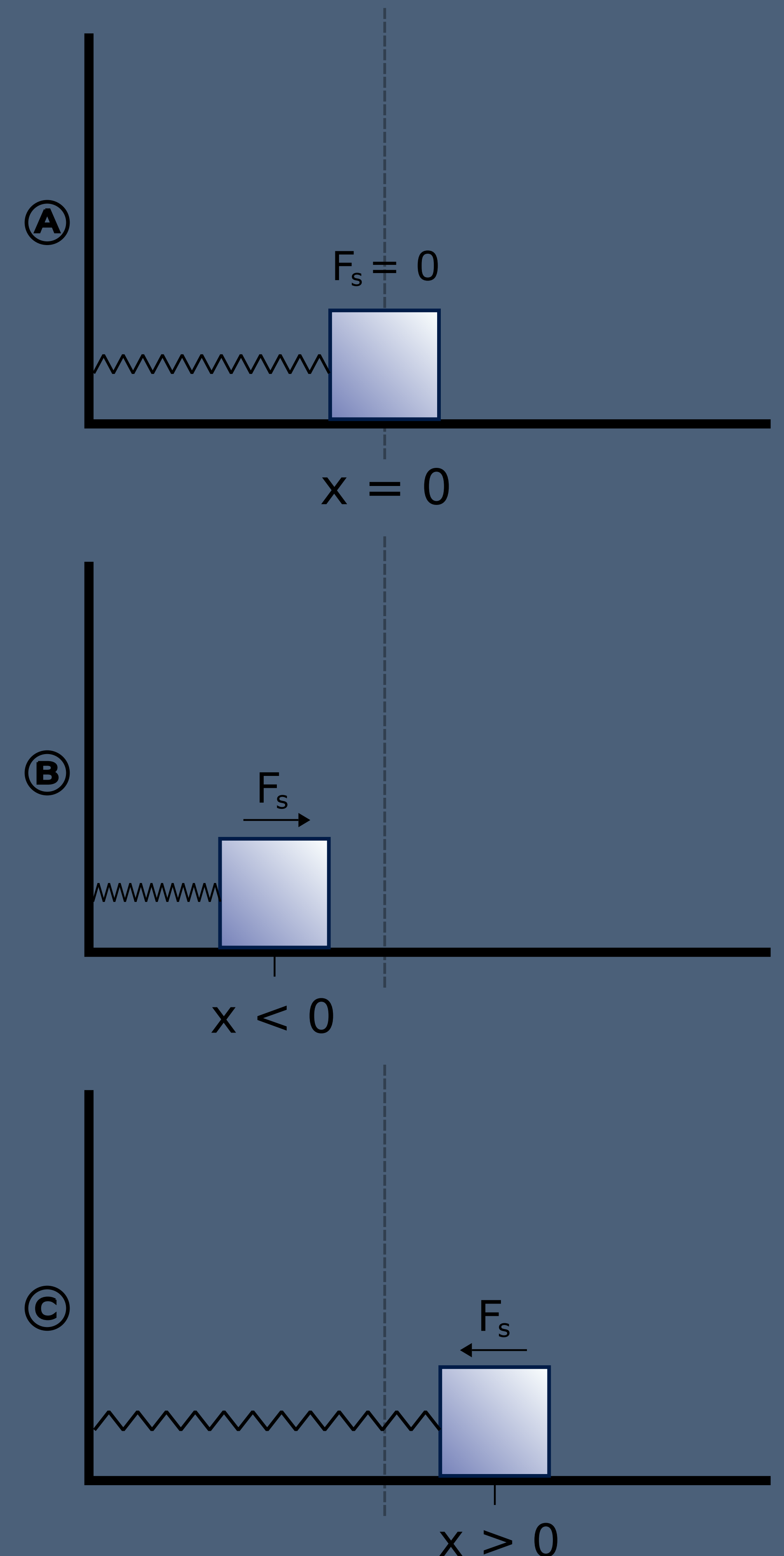


Hamiltonian mech

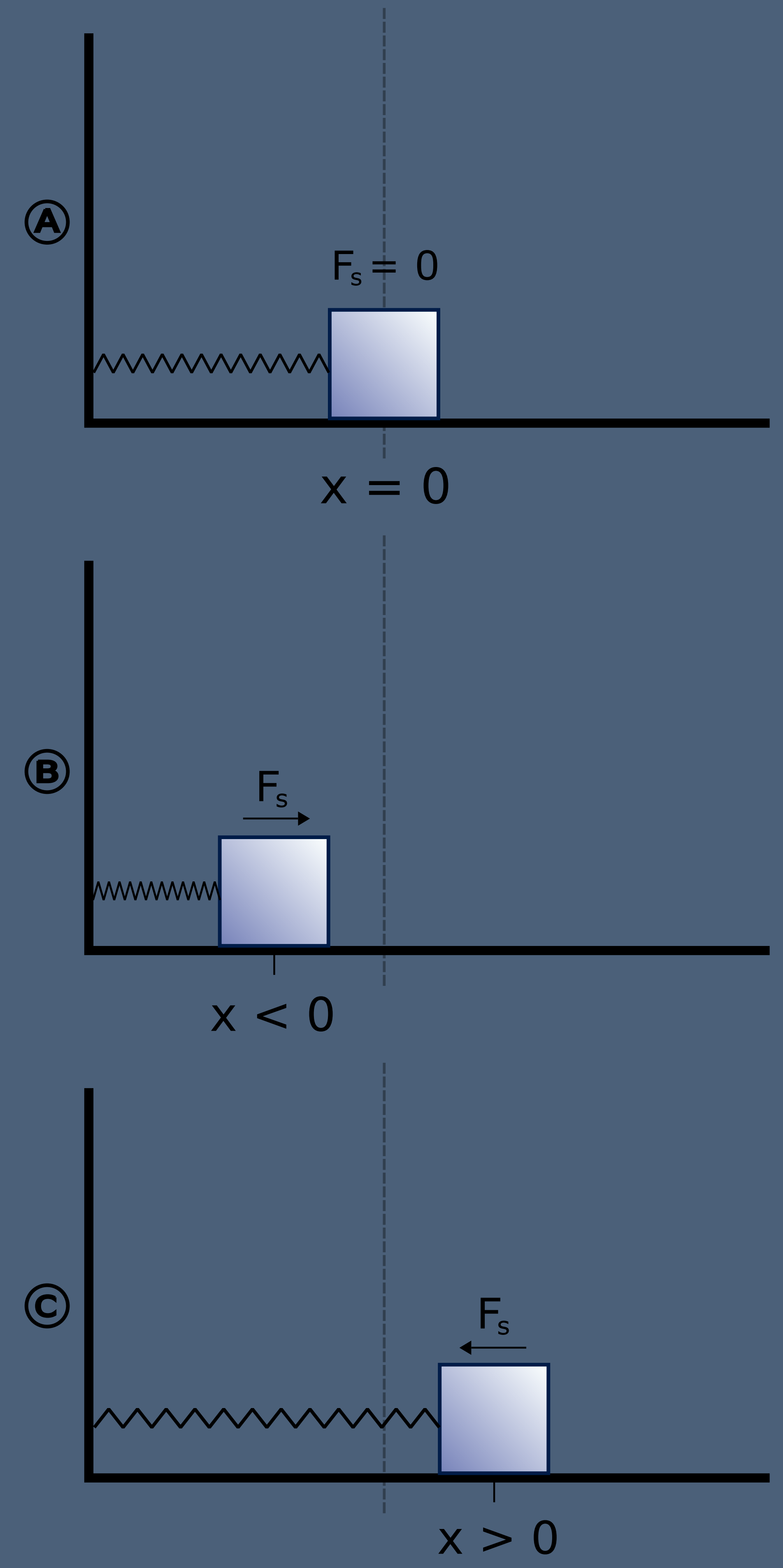
Nearly Lagrangian

- What about E-L?

- $\mathscr{H} \equiv p\dot{x} - \mathscr{L}$

$$\frac{\partial \mathscr{H}}{\partial x} = p\frac{\partial \dot{x}}{\partial x} - \left(\frac{\partial \mathscr{L}}{\partial x} + p\frac{\partial \dot{x}}{\partial x}\right)$$

- $$= -\frac{\partial \mathscr{L}}{\partial x} = -\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}} = -\dot{p}$$



Hamiltonian mech

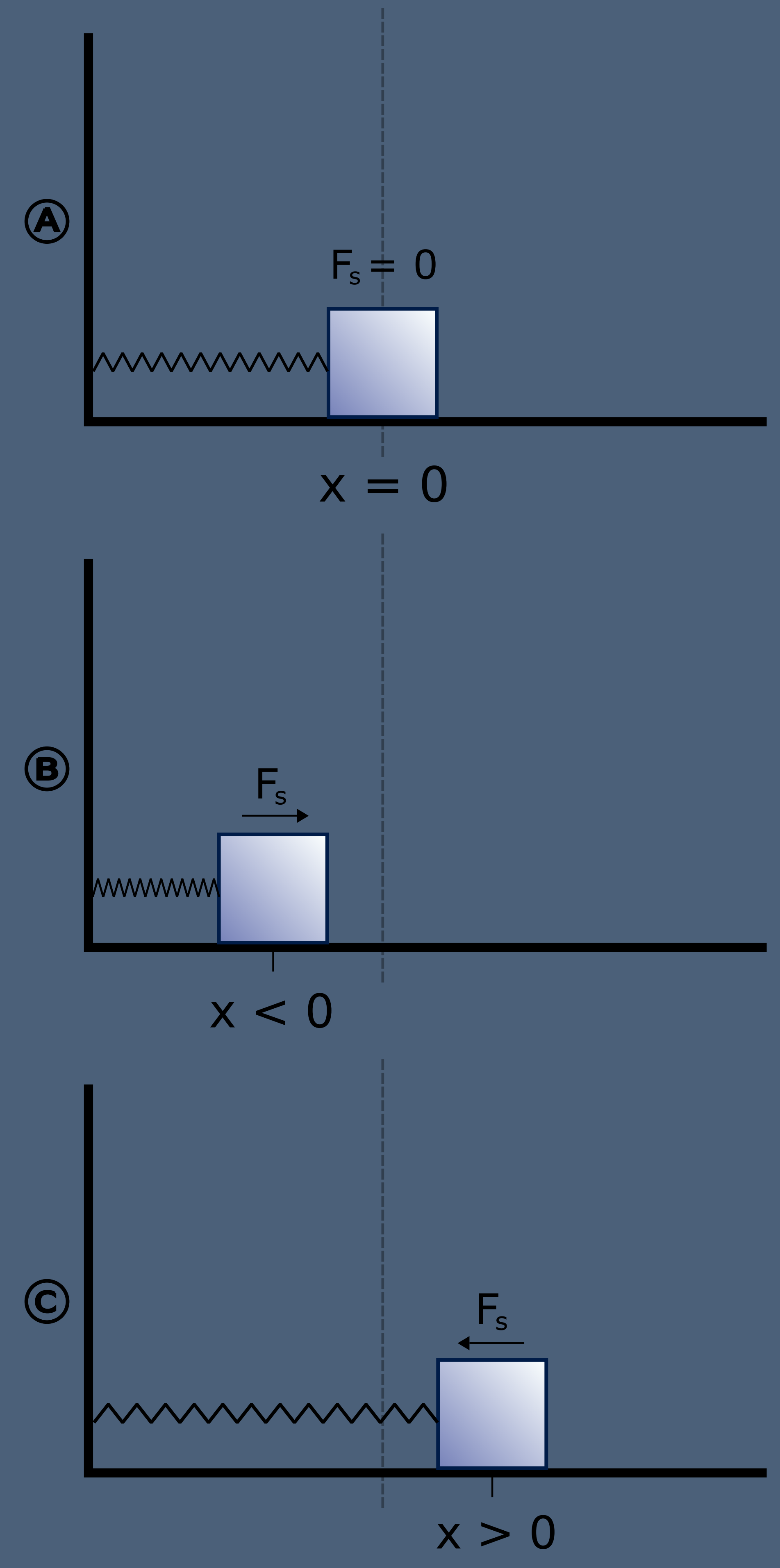
Nearly Lagrangian

- What about E-L?
- $\mathscr{H} \equiv p\dot{x} - \mathscr{L}$
- $\dfrac{\partial \mathscr{H}}{\partial x} = -\dot{p}$

$$\frac{\partial \mathscr{H}}{\partial p} = \dot{x} + p\frac{\partial \dot{x}}{\partial p} - \frac{\partial \mathscr{L}}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial p}$$

- $$= \dot{x} + p\frac{\partial \dot{x}}{\partial p} - p\frac{\partial \dot{x}}{\partial p}$$

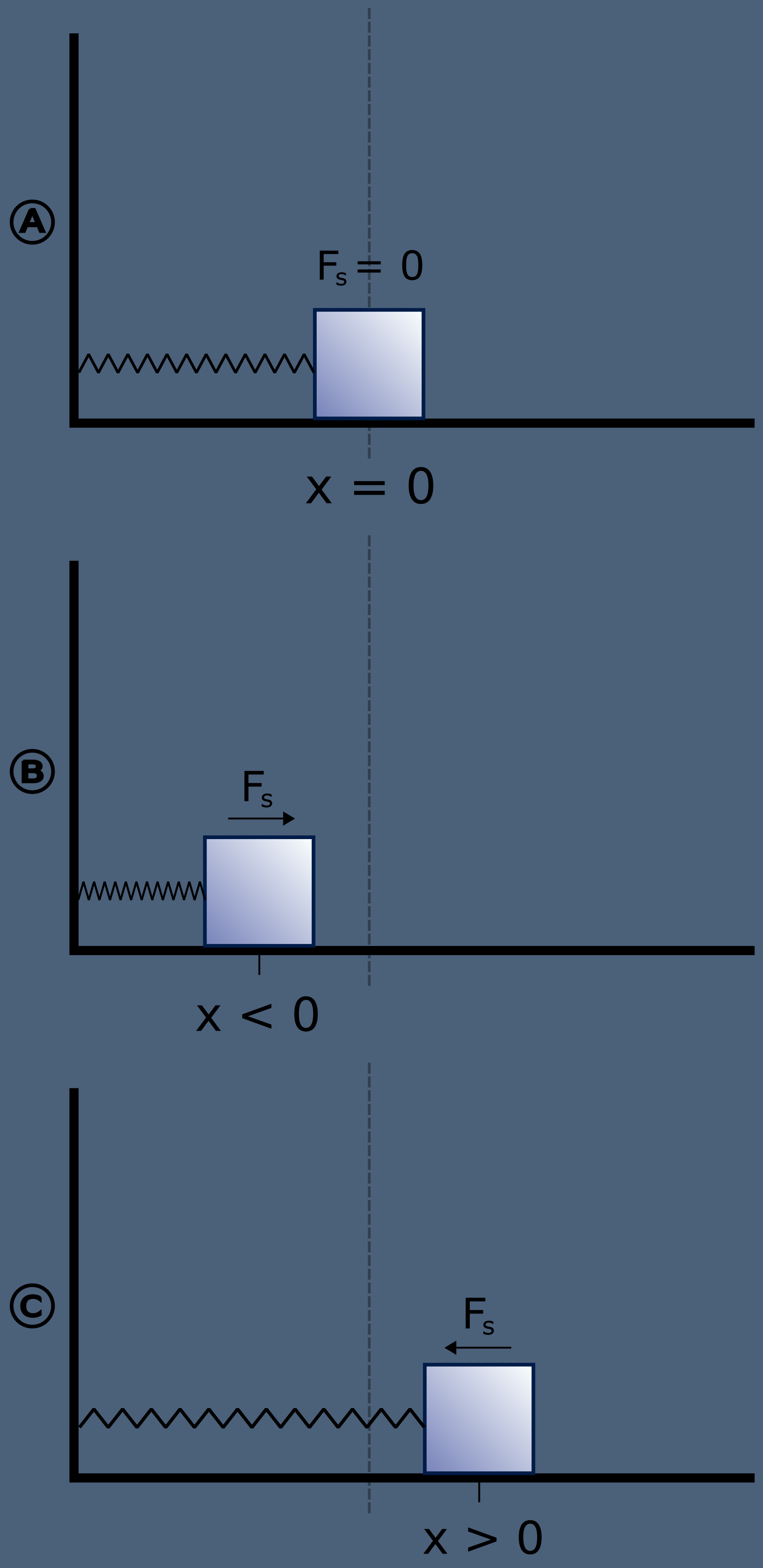
$$= \dot{x}$$



Hamiltonian mech

Nearly Lagrangian

- What about E-L?
- $\mathscr{H} \equiv p\dot{x} - \mathscr{L}$
- $\dfrac{\partial \mathscr{H}}{\partial x} = -\dot{p}$
- $\dfrac{\partial \mathscr{H}}{\partial p} = \dot{x}$
- $\mathscr{L}$ yields a 2nd-order diff EQ,
  $\mathscr{H}$ yields 2x1st-order diff EQs



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- $\mathscr{H} = \dfrac{p^2}{2m} + \dfrac{1}{2}kx^2$
- $\dfrac{\partial \mathscr{H}}{\partial x} = -\dot{p}$
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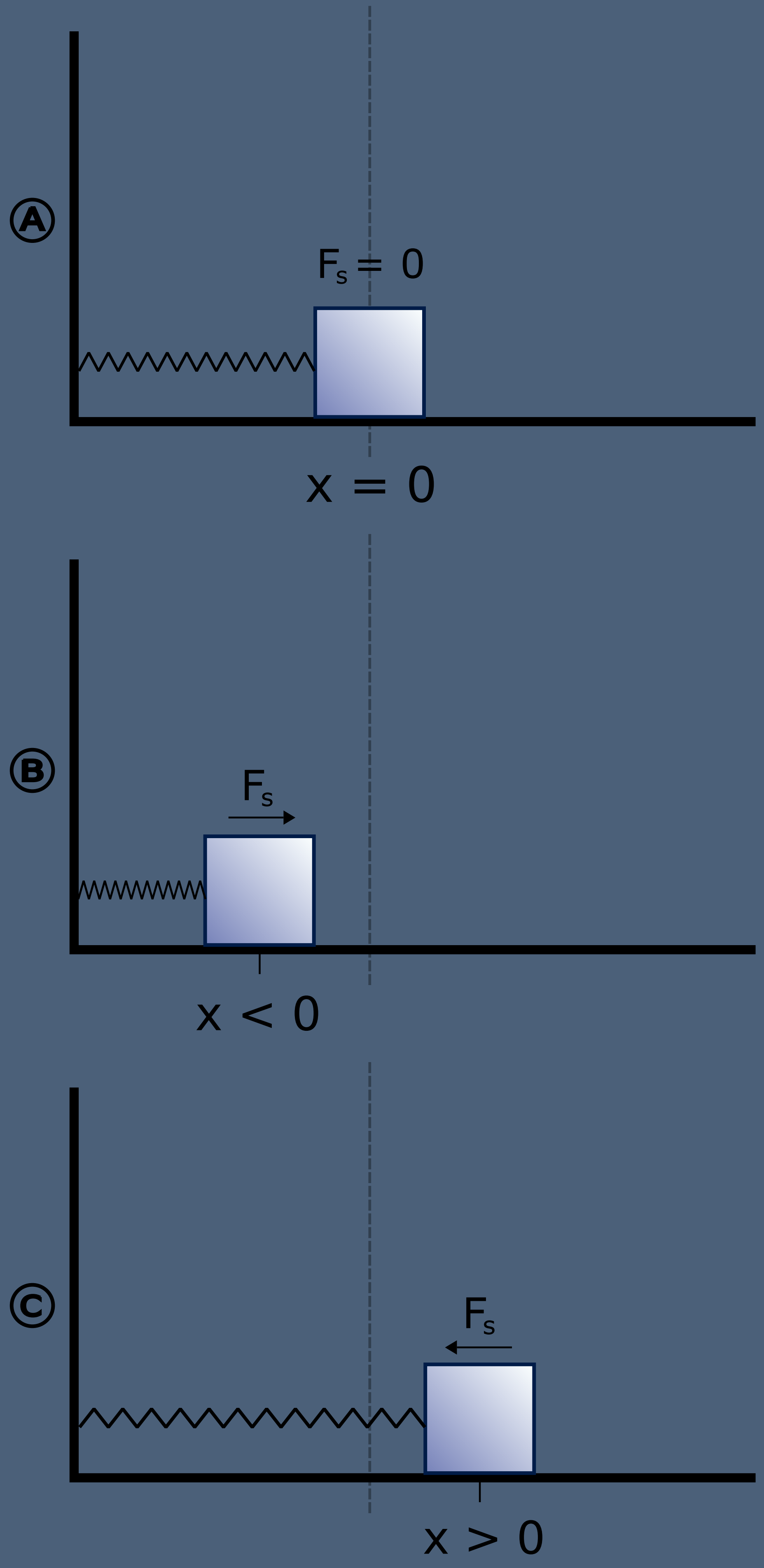
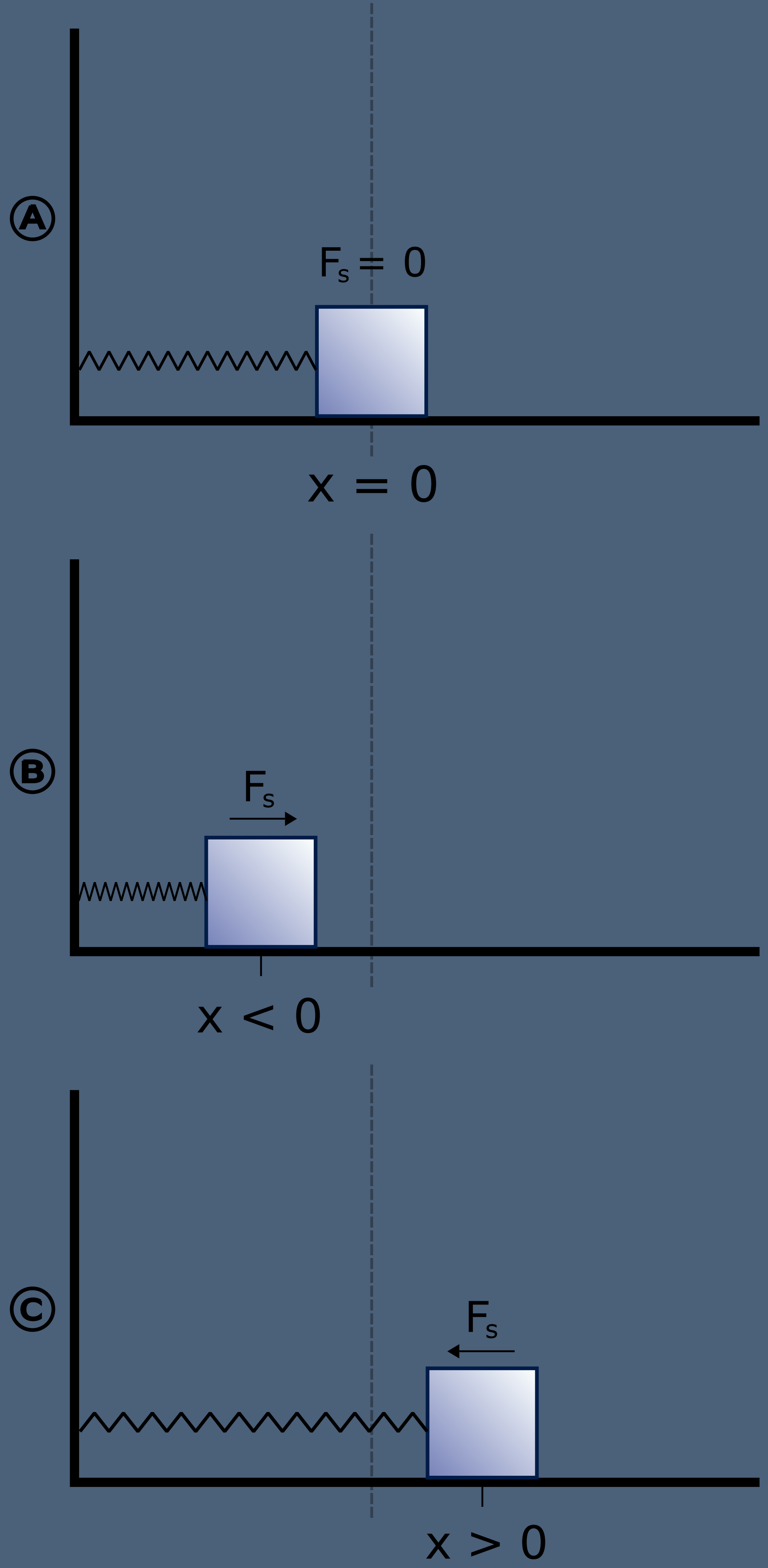


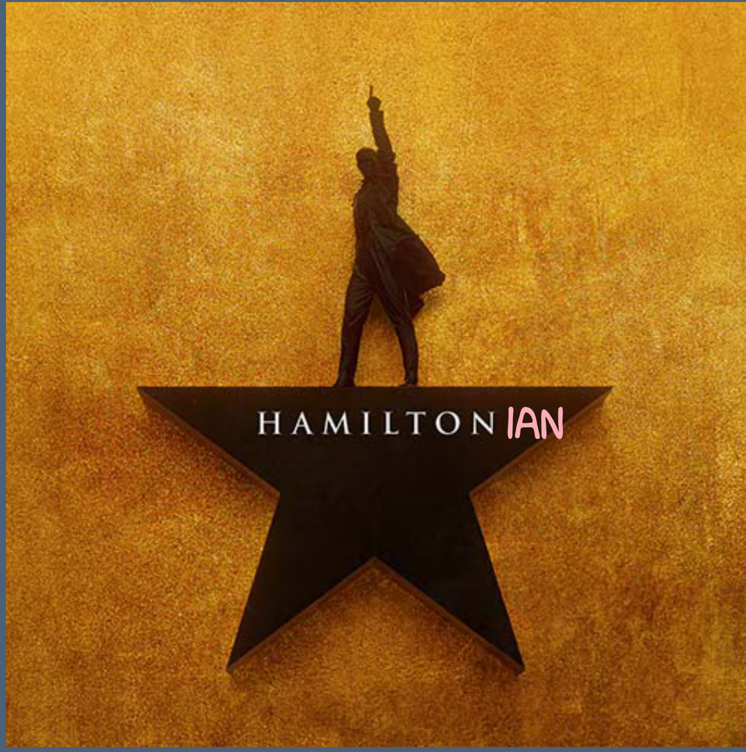
Hamiltonian mech

Nearly Lagrangian

- $\mathscr{H} = \dfrac{p^2}{2m} + \dfrac{1}{2}kx^2$
- $\dfrac{\partial \mathscr{H}}{\partial x} = -\dot{p} = kx$ (i) force
- $\dfrac{\partial \mathscr{H}}{\partial p} = \dot{x} = \dfrac{p}{m}$ (ii) momentum
- $p = m\dot{x} \Rightarrow \dot{p} = m\ddot{x}$ (iii)
- (i) & (iii): $m\ddot{x} = -kx$
- $x = x_0 \cos\left(\sqrt{\dfrac{k}{m}}t + \delta\right)$



Two problems



- 13.4 — It's tempting to treat $\mathscr{H}$ and $E = T + U$ as interchangeable. But they aren't always! Here you'll see when they are.
- 13.3 — Find the motion of a system using the Hamiltonian approach!