PhysH308

Fluid Mechanics!

Ted Brzinski, Nov. 7, 2024

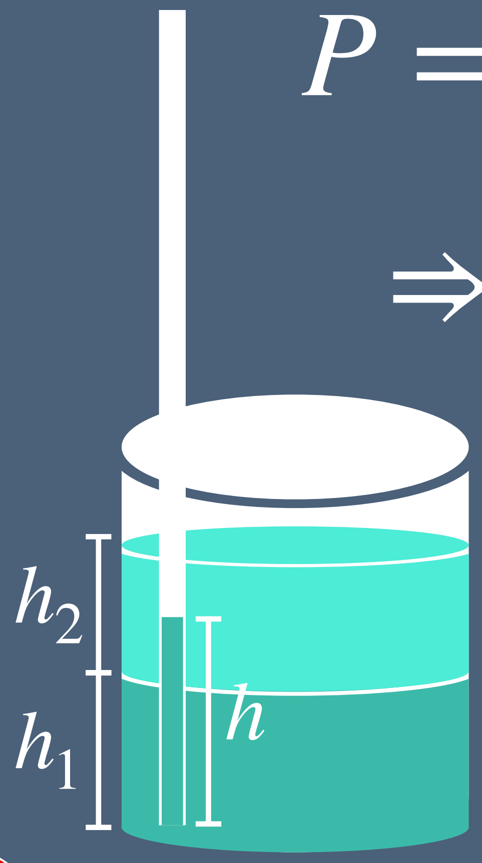
Last time!

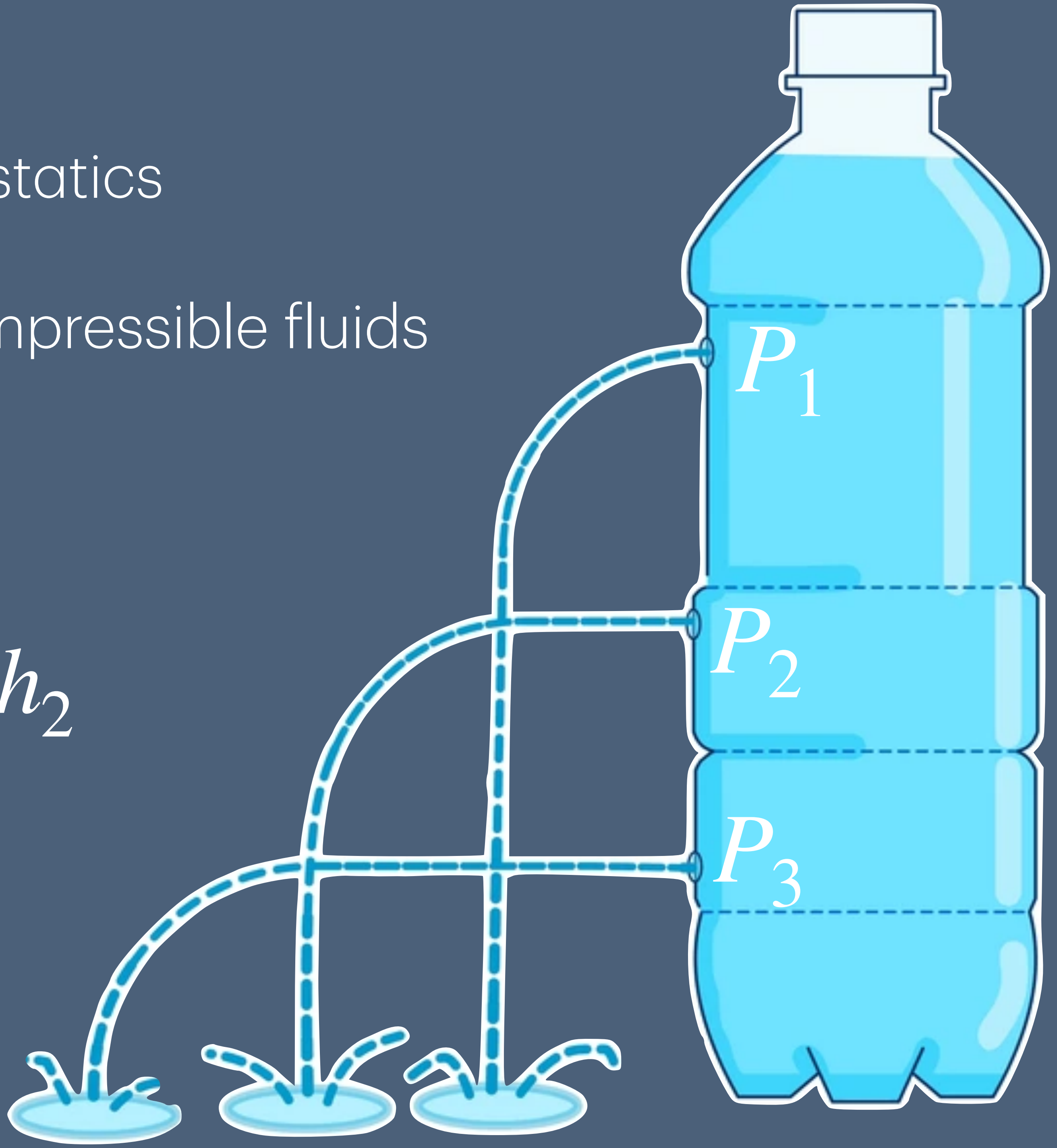
Lautrup 2.5

- Pressure, stress/strain, hydrostatics
- Hydrostatic pressure in incompressible fluids

$$P = P_0 + \rho g z$$

$$\Rightarrow h = h_1 + \frac{\rho_2}{\rho_1} h_2$$

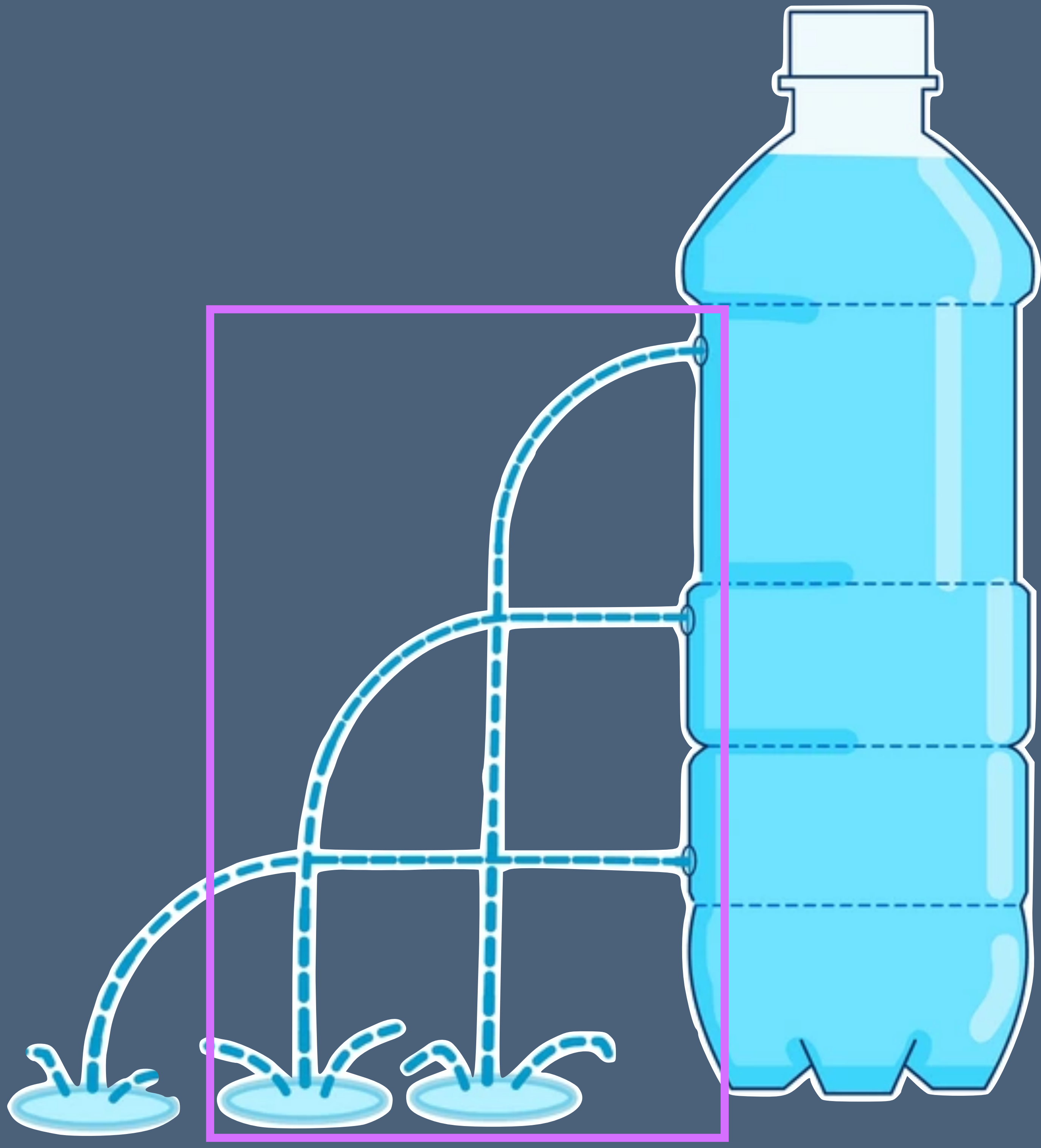




This time:

Ideal Flow! (No viscosity)

- Discreet particles: $\overrightarrow{F} = \dot{\vec{p}}$,
  $\vec{p} = m\vec{v}$, $T = \dfrac{1}{2m}\vec{p}^2$,
  $U = -\int \overrightarrow{F} \cdot d\vec{r}$ and
  $\mathscr{H} = T + U$
- All of this is still true for fluid particles! Only now the particle properties become fields....



This time:

Ideal Flow! (No viscosity)

- All of this is still true for fluid particles! Only now the particle properties become fields....

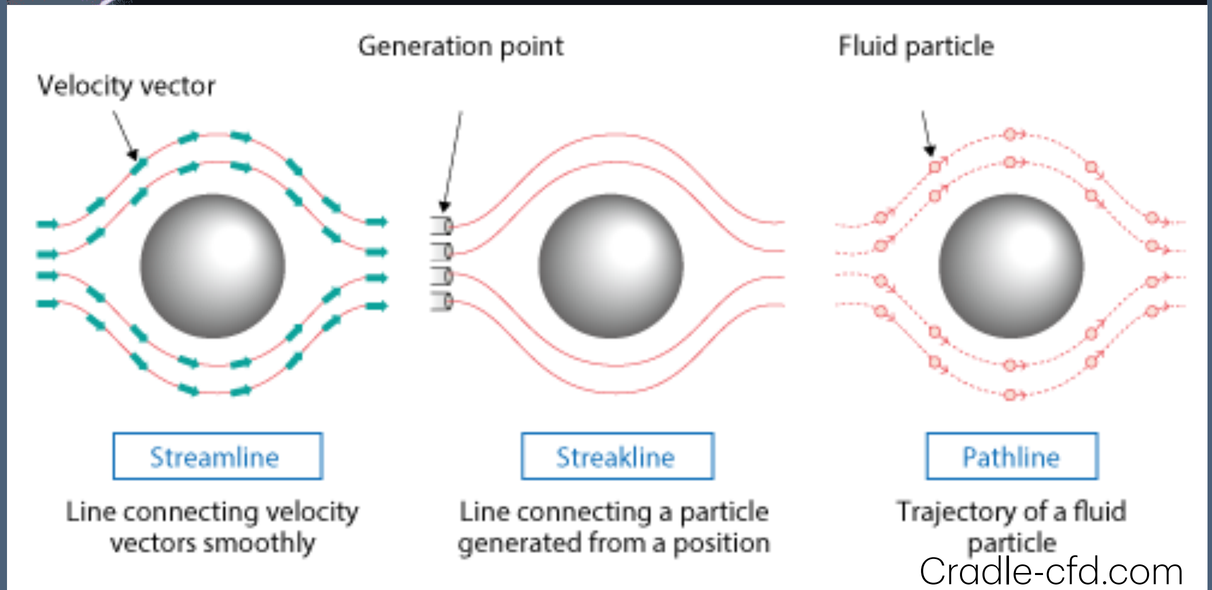
- $\vec{p} = m\vec{v}$ becomes
  $d\overrightarrow{\mathscr{P}} = \vec{v}\left(\vec{x}, t\right) dM = \rho\left(\vec{x}, t\right) \vec{v}\left(\vec{x}, t\right) dV$

- $\vec{v}\left(\vec{x}, t\right)$ is a field, as is $\vec{a}\left(\vec{x}, t\right)$
  (acceleration is $\overrightarrow{w}$, according to Lautrup)

- Newton's 2nd law thus becomes
  $d\overrightarrow{\mathscr{F}} = \vec{a}\left(\vec{x}, t\right) dM$

Visualizing flow

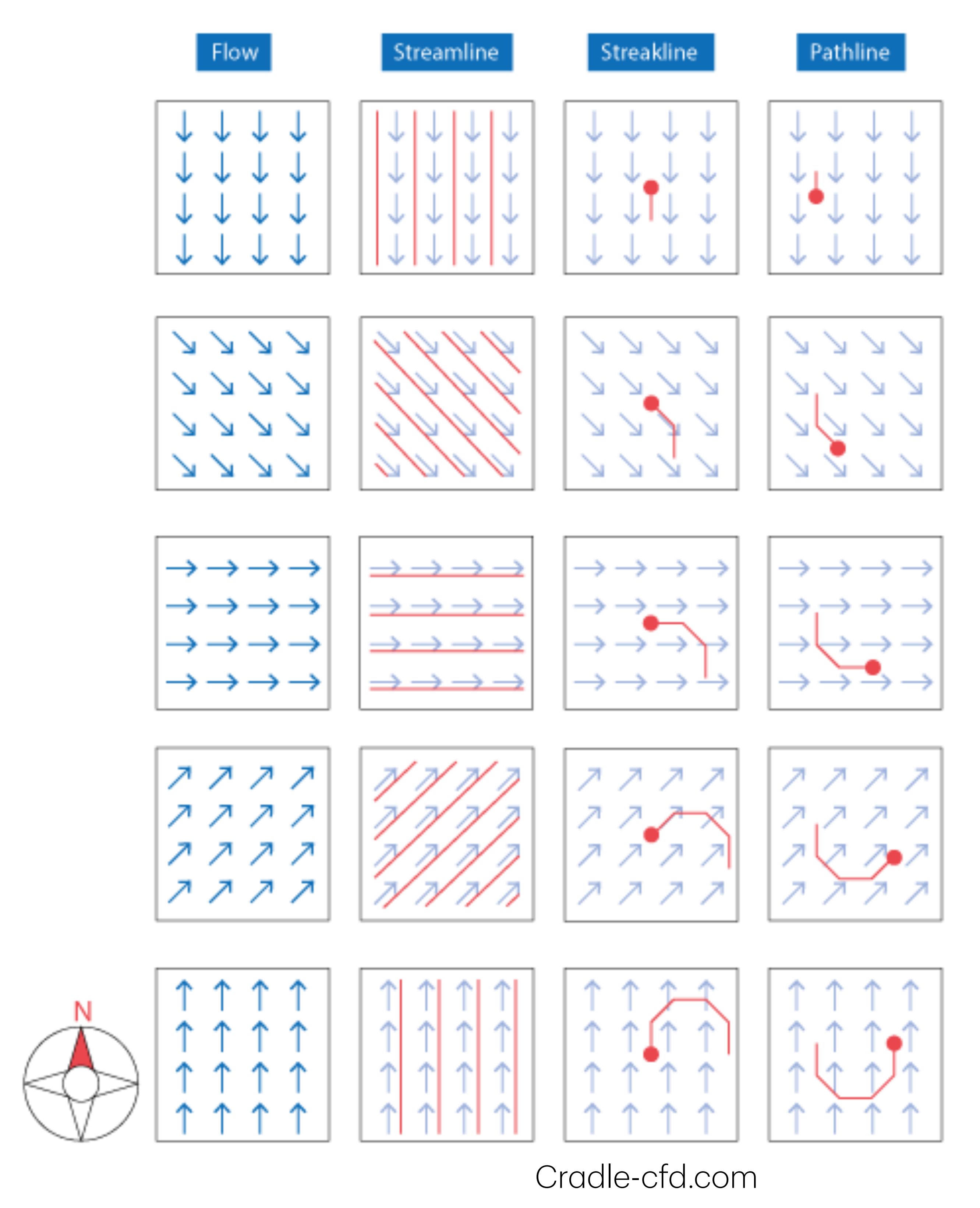
- $\vec{v}\left(\vec{x}, t\right)$ is the field we usually want to visualize:
    - Particle Orbits (individual particle paths)
    - Streaklines (lines of particles from a shared origin)
    - Streamlines (tangent to flow)



Visualizing flow

- Particle Orbits (individual particle paths)
- Streaklines (lines of particles from a shared origin)
- Streamlines (tangent to flow)

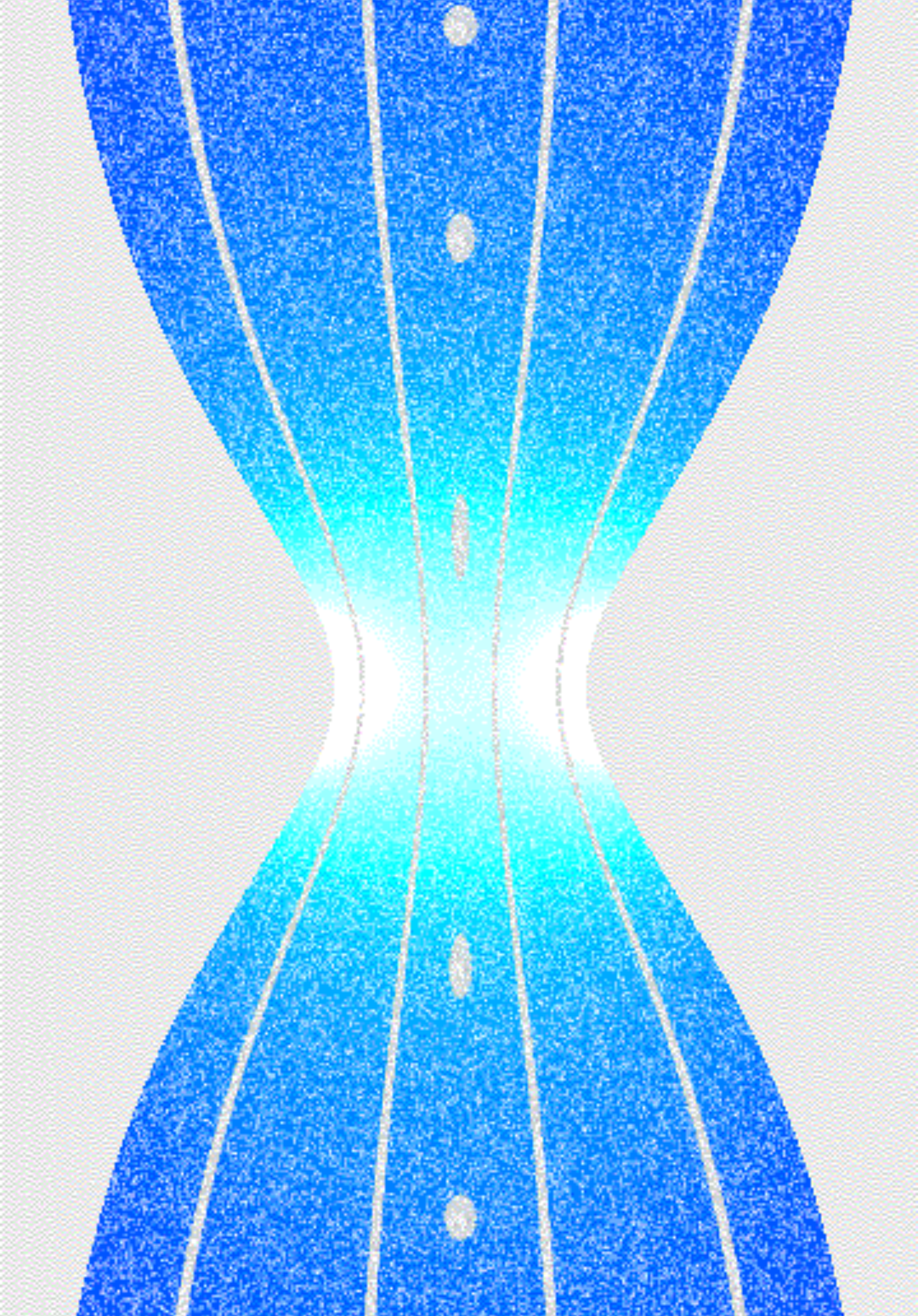
Same for steady flow, but not time-dependent flows!



Material derivatives

Time dependence plus continuity

- $\vec{v}\left(\vec{x}, t\right)$ is now a field, as is $\vec{a}\left(\vec{x}, t\right)$
- Only now $\vec{a}\left(\vec{x}, t\right) \neq \dfrac{d\vec{v}}{dt}$
- To see why, consider steady flow through a constriction. Conservation of mass requires flow to accelerate at the constriction.
- The velocity field is constant in time, but not space, which requires the same to be true of the acceleration!


Credit: Thierry Dugnolle

Material derivatives

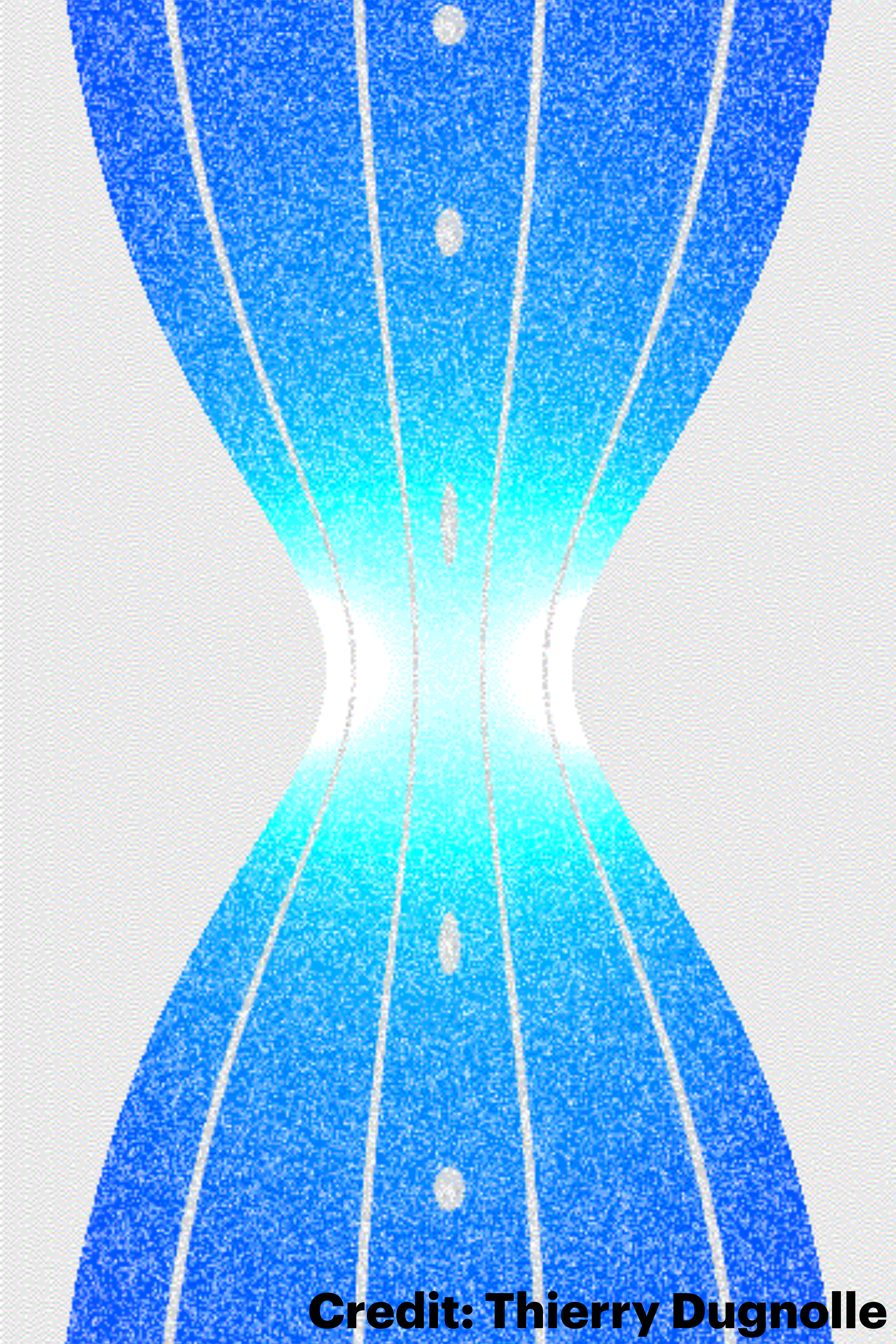
Time dependence plus continuity

- We define a "material derivative" or a "comoving derivative":

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$$

- Thus

$$\vec{a} = \frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \nabla\right) \vec{v}$$

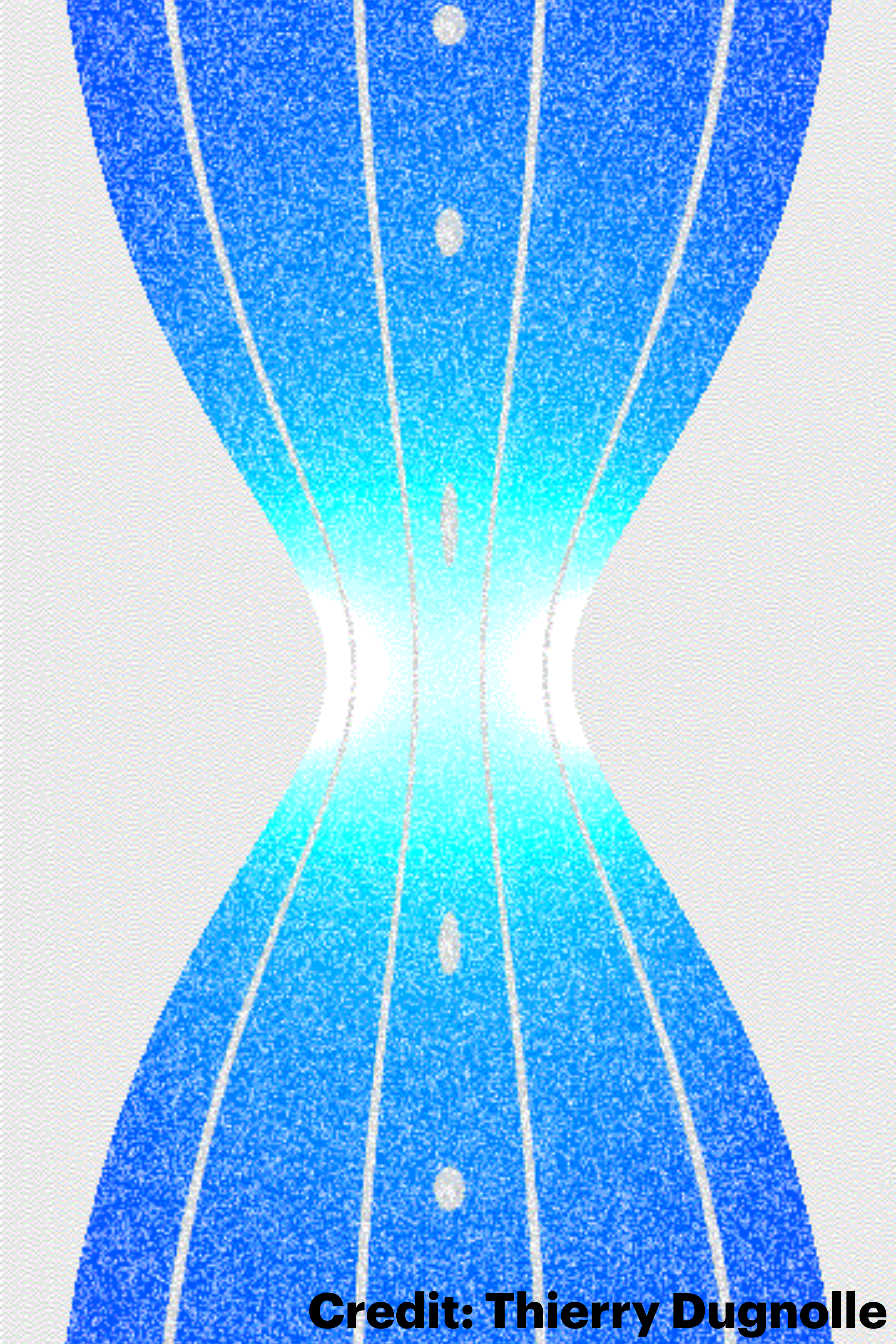


Material derivatives

Time dependence plus continuity

- Now, let's return to Newton's 2nd law:
  $d\overrightarrow{\mathscr{F}} = \vec{a}dM = \rho\vec{a}dV$

- $f^* \equiv \dfrac{d\overrightarrow{\mathscr{F}}}{dV} = \rho\vec{a} = \rho\dfrac{D\vec{v}}{Dt}$
  (we call f^* a "force density)


Credit: Thierry Dugnolle

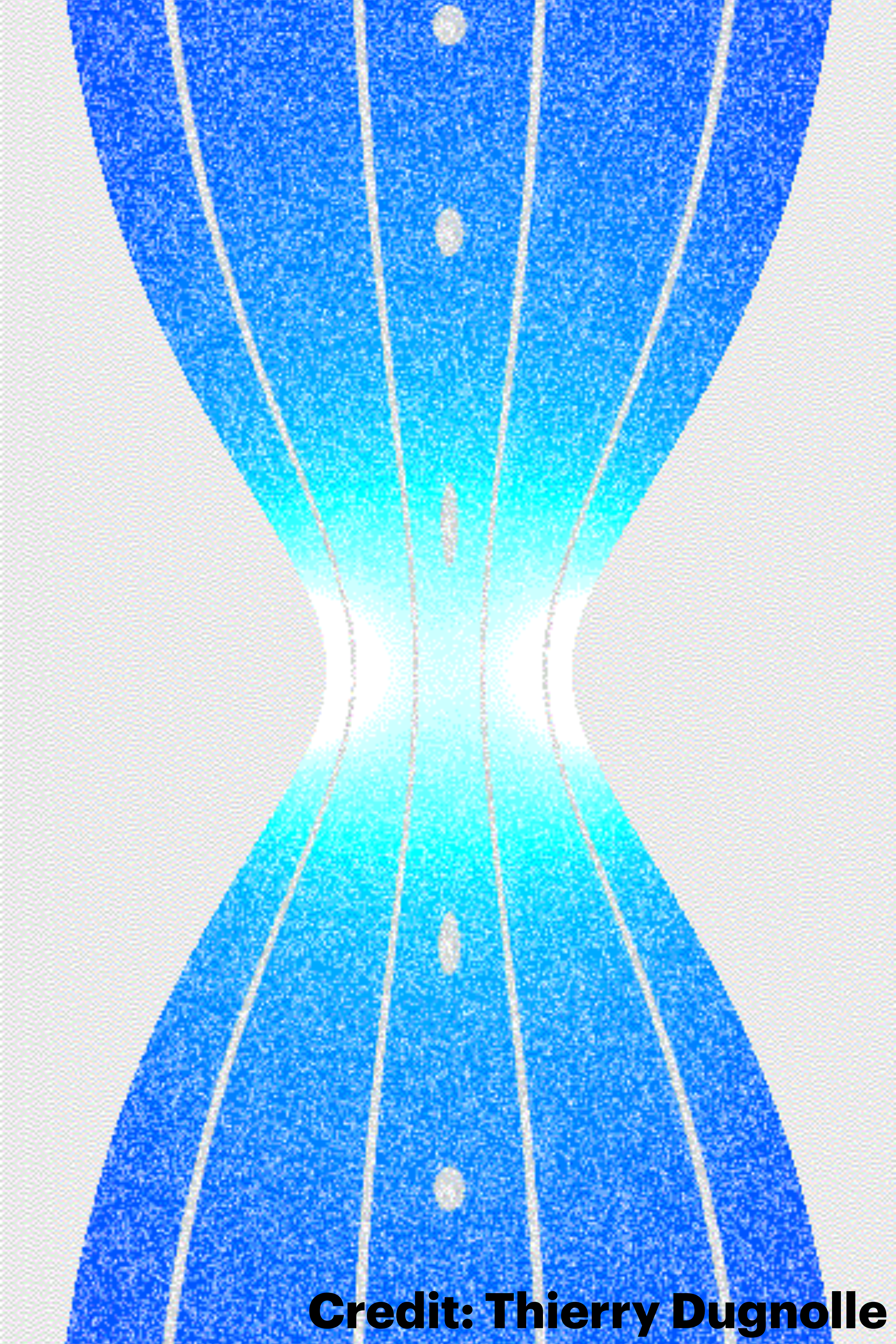
Newton's 2nd Law

For continuous matter

- $f^* \equiv \dfrac{d\overrightarrow{\mathcal{F}}}{dV} = \rho\vec{a} = \rho\dfrac{D\vec{v}}{Dt}$

$$= \rho\left(\frac{\partial\vec{v}}{\partial t} + \left(\vec{v}\cdot\nabla\right)\vec{v}\right)$$

- This is "Cauchy's Equation", and is the final form of Newton's 2nd for all continuous matter!



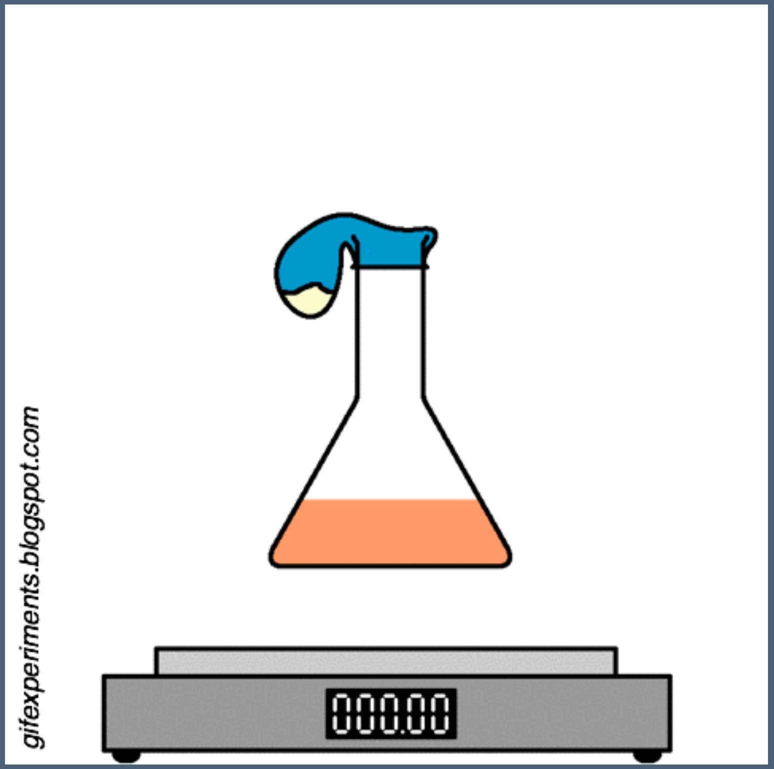
Credit: Thierry Dugnolle

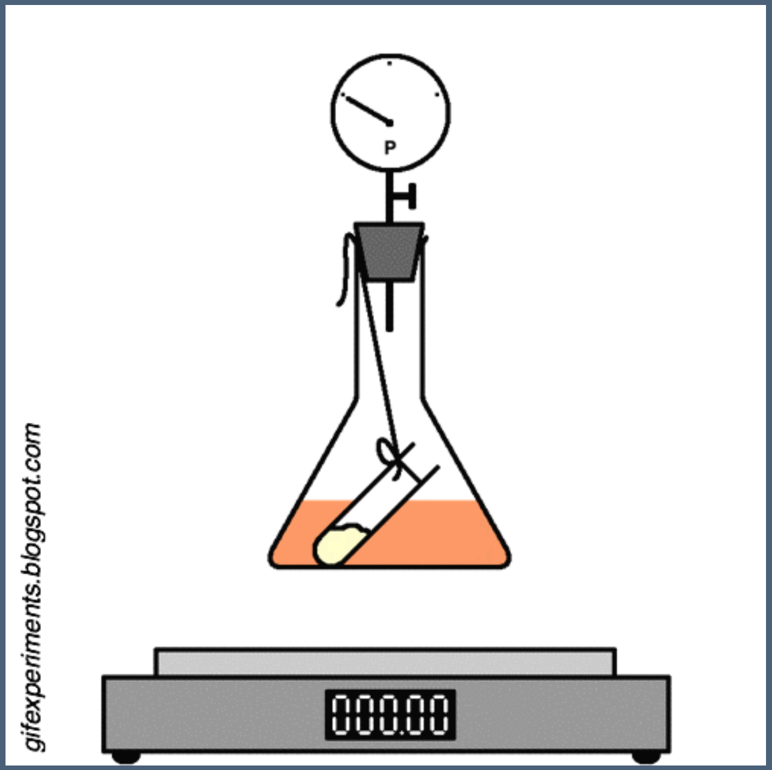
Conservation of mass

A 2nd constraint!

- Regardless of the density and pressure, the mass of a closed system must remain constant.
- $$\frac{dM}{dt} = \boxed{\frac{d}{dt}\int_V \rho \, dV = -\oint_S \rho\vec{v}\cdot d\vec{S}}$$

The change in mass is due to the mass flux



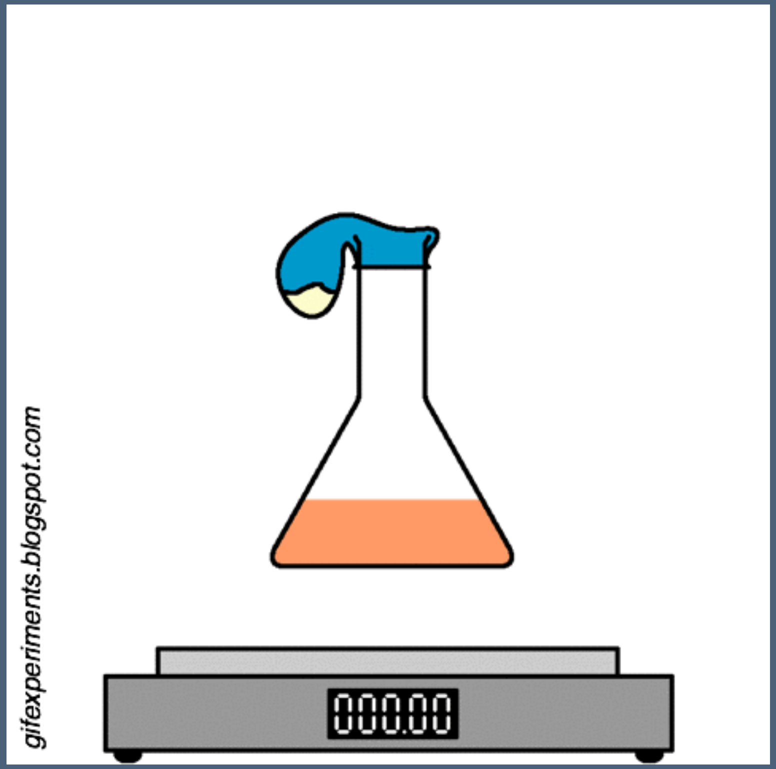


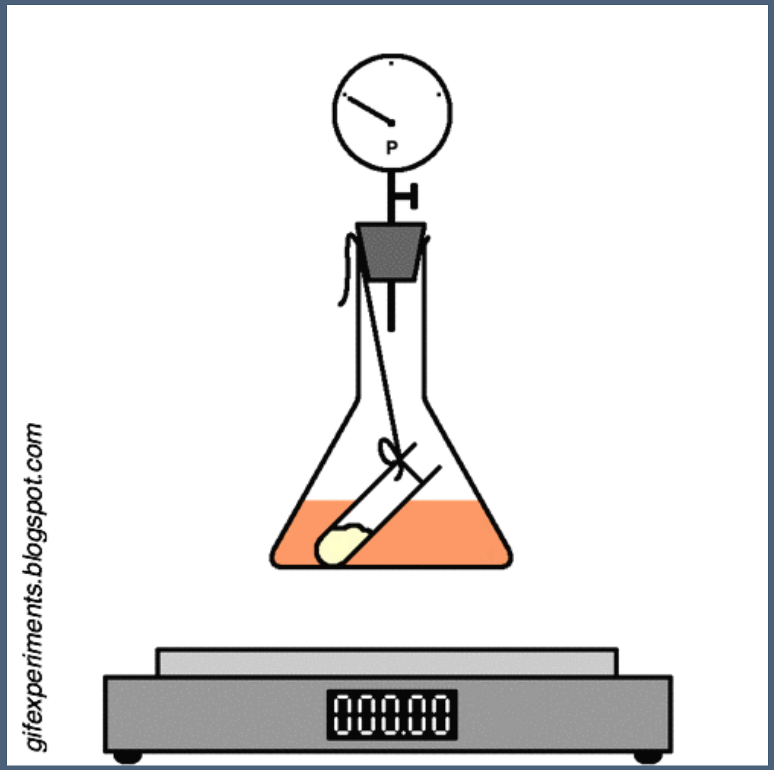
Conservation of mass

A 2nd constraint!

- $$\frac{dM}{dt} = \frac{d}{dt}\int_V \rho \ dV = -\oint_S \rho\vec{v}\cdot d\vec{S}$$
- $$= \int_V \frac{d\rho}{dt} \ dV = -\int_V \nabla\cdot\rho\vec{v} \ dV$$
- $$\Rightarrow \boxed{\frac{d\rho}{dt} + \nabla\cdot\rho\vec{v} = 0}$$

Mass is conserved everywhere!

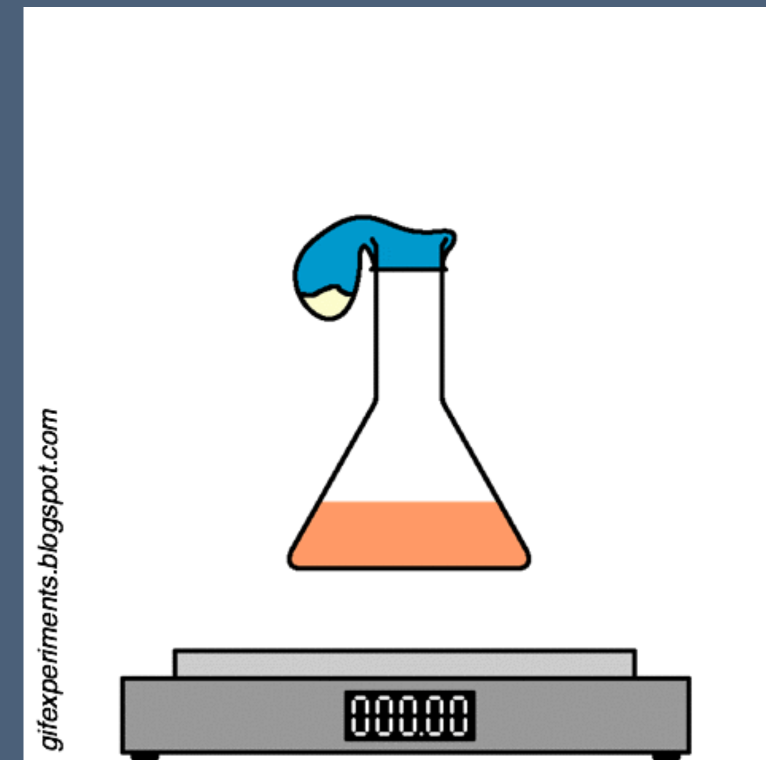


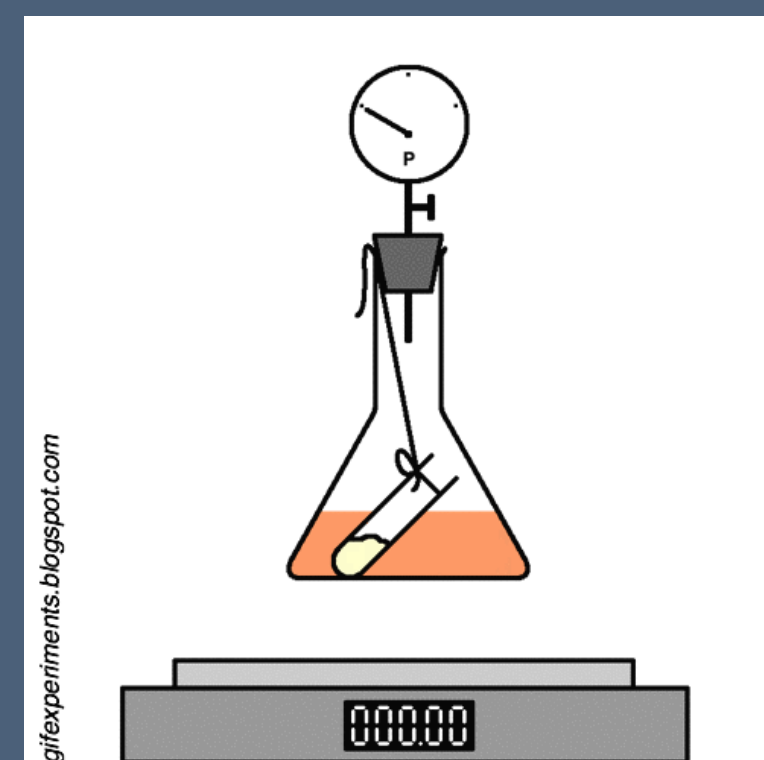


Conservation of mass

A 2nd constraint!

- $$\Rightarrow \frac{d\rho}{dt} + \nabla \cdot \rho\vec{v} = 0$$
- Now consider
$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \left(\vec{v} \cdot \nabla\right)\rho$$
- $$= \frac{\partial\rho}{\partial t} + \nabla \cdot \left(\rho\vec{v}\right) - \rho\nabla \cdot \vec{v}$$
- So $\dfrac{D\rho}{Dt} = -\rho\nabla \cdot \vec{v}$
- We see that in incompressible flow,
$\dfrac{D\rho}{Dt} = 0$, then we have $\nabla \cdot \vec{v} = 0$





Equations of motion

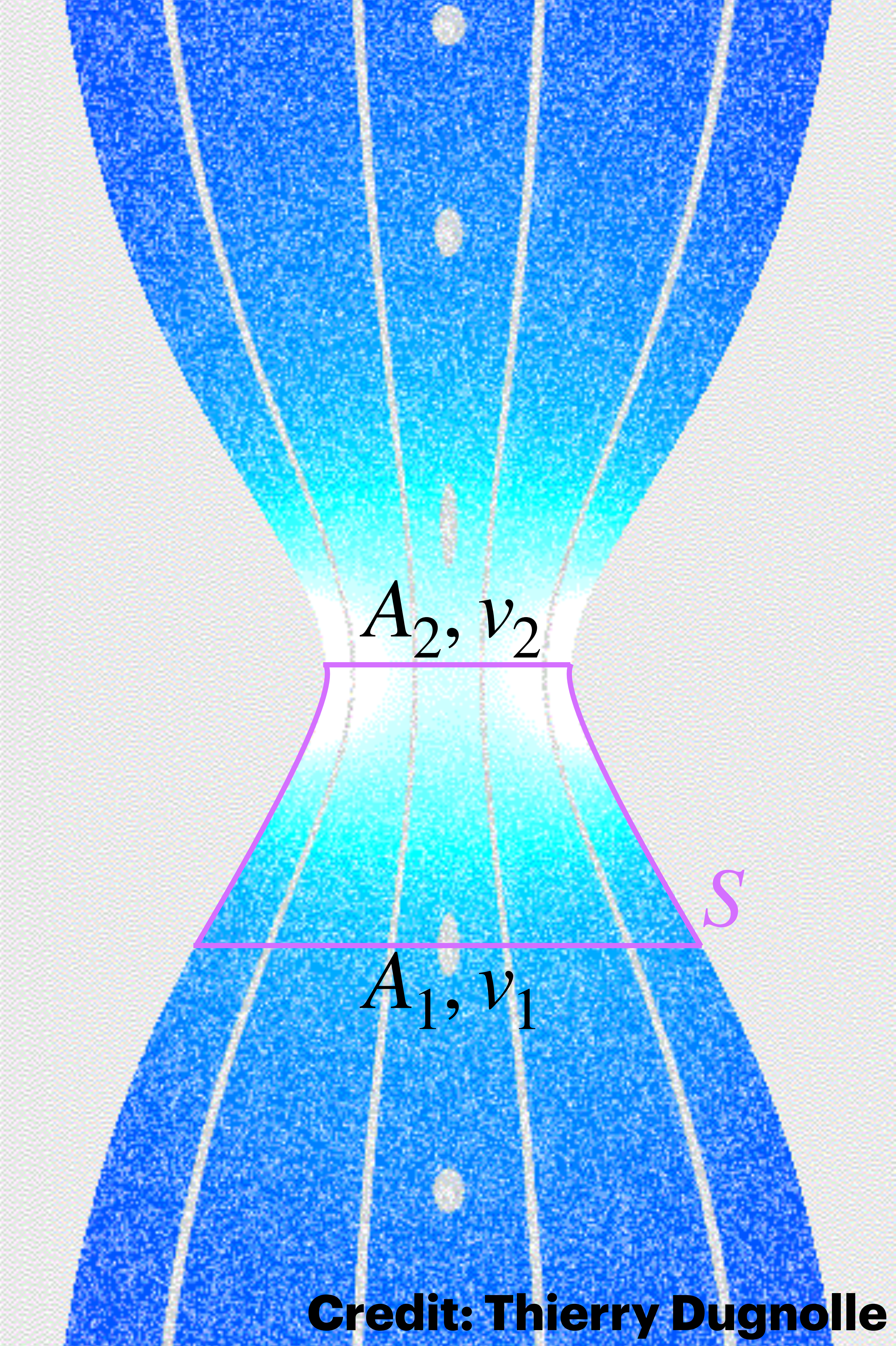
- Cauchy's equation gives us $\frac{\partial v}{\partial t} = -\left(\vec{v} \cdot \nabla\right)\vec{v} + \rho^{-1} f^*$
- Conservation of mass gives $\frac{\partial \rho}{\partial t} = -\left(\vec{v} \cdot \nabla\right)\vec{v} - \rho \nabla \cdot \vec{v}$
- All we need is $f^* = f^*\left[\rho, \vec{v}\right]\left(\vec{x}, t\right)$ to solve the system of equations!

(Gross, but you've developed these skills already!)

A simple case

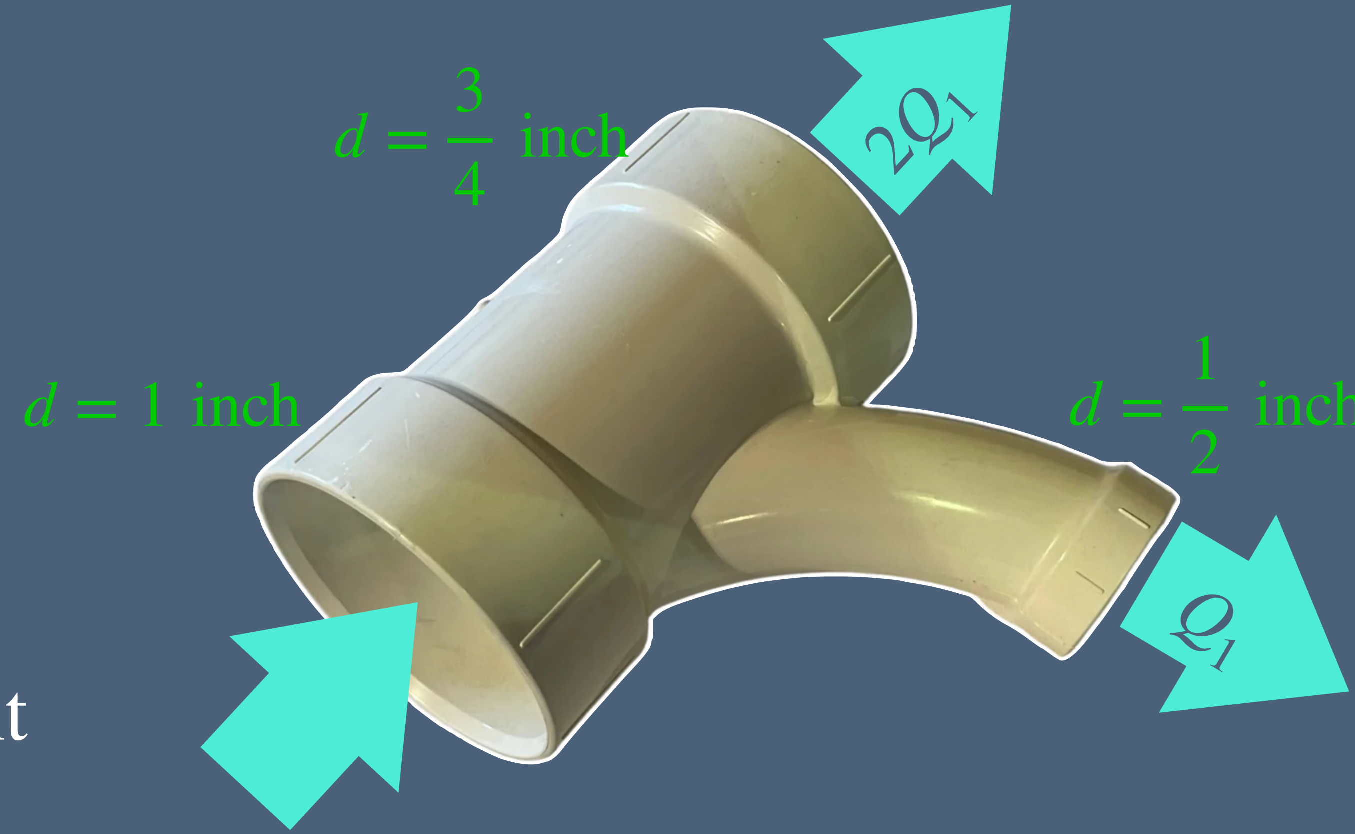
Incompressible fluid, steady flow

- For an incompressible fluid, we found $\nabla \cdot \vec{v} = 0$
- By Gauss' Law, this is equivalent to $\oint_S \vec{v} \cdot d\vec{S} = 0$
- $\oint_S \vec{v} \cdot d\vec{S} = -A_1 v_1 + A_2 v_2 = 0$
- Leonardo's Law (const. flux): $Q = \overrightarrow{A} \cdot \vec{v} = \text{constant}$



Problems

Ideal, incompressible flow



- Leonardo's Law: $Q = Av = \text{constant}$

12.4 A water pipe with diameter 1 inch branches into two pipes with diameters $3/4$ inch and $1/2$ inch. Water is tapped from the largest branch at double the rate as from the other. What is the ratio of velocities in the pipes?

- **2nd problem (conceptual):** A pipe reduces diameter by 50% at a conical constriction. (Hint, recall the holey bottle demo from Tuesday)
  - How does the velocity change across the constriction?
  - How does the pressure change? In particular, on which side of the constriction is the pressure higher? Why? (This is a preview of Bernoulli's effect!)