Euler's method problem

1. Consider the Initial Value Problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = t - x, \quad x(0) = 1.$$

   (a) Verify that the solution to this initial value problem is the function

$$x(t) = 2e^{-t} + t - 1$$

   Solution:
   Left hand side:

$$\frac{\mathrm{d}}{\mathrm{d}t}(2e^{-t} + t - 1) = -2e^{-t} + 1$$

   Right hand side:

$$t - x(t) = t - (2e^{-t} + t - 1) = -2e^{-t} + 1$$

   Thus the function $x(t)$ satisfies the ODE. Now verify that $x(t)$ satisfies the initial condition:

$$x(0) = 2e^{0} + 0 - 1 = 2 - 1 = 1.$$

   The function $x(t)$ satisfies the ODE and the initial condition and thus is the solution to the initial value problem.

   (b) With a step size of $h = \frac{1}{4}$ Approximate $x(\frac{1}{2})$ using Euler's Method.
   Solution: Euler's method states that for the initial value problem $\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x)$, $x(a) = x_0$,

$$x(t + h) \approx x(t) + hf(t, x)$$

   Thus, the first step of Euler's method is:

$$x(0 + \frac{1}{4}) \approx x(0) + \frac{1}{4}\left(0 - x(0)\right)$$
$$x(\frac{1}{4}) \approx 1 + \frac{1}{4}(0 - 1) = \frac{3}{4}.$$

   The second step of Euler's method is:

$$x(0\frac{1}{4} + \frac{1}{4}) \approx x(\frac{1}{4}) + \frac{1}{4}\left(\frac{1}{4} - x(\frac{1}{4})\right)$$
$$x(\frac{1}{2}) \approx \frac{3}{4} + \frac{1}{4}(\frac{1}{4} - \frac{3}{4}) = \frac{5}{8}.$$