Homework 5 Solutions

Written Assignment

1. Consider a vector with n entries:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Consider the following python function that calculates that computes the average of n entries of a 1D numpy array x:

Code 1:

```
1 def Average_val_1(x):
2     n = x.shape[0]
3     sum = x[0]
4     for i in range(1,x.shape[0]):
5         sum = sum + x[i]
6     sum = sum/x.shape[0]
7 
8     return sum
```

a.) What is the computational complexity of this function, i.e. what is the total number of operations (addition, subtraction, multiplications, and divisions) in this function?

There is one addition operation in line 5, which is executed n-1 times, since i takes on values 1, 2, ..., n-1. There is one division operation occurring outside the loop in line 6. This means that the total number of operations is n-1 + 1 = **n** operations.

b.) What is the asymptotic computational complexity of this function?

The ACC is $\mathcal{O}(n)$.

c.) Consider a slightly different function for calculating the average value of the entries of the 1D array for x:

Code 2:

```
1 def Average_val_2(x):
2     n = x.shape[0]
3     sum = 0
4     for i in range(n):
5         sum = sum + x[i]
6     sum = sum/n
7
8     return sum
```

d.) What is the computational complexity of this function, i.e. what is the total number of operations (addition, subtraction, multiplications, and divisions) in this function?

There is one addition operation in line 5, which is executed n times, since i takes on values 0, 1, ..., n-1. There is one division operation occurring outside the loop in line 6. This means that the total number of operations is **n+1** operations.

e.) What is the asymptotic computational complexity of this function?

The ACC is $\mathcal{O}(n)$.

f.) Finally, consider yet another python function for computing the average value of the entries of the 1D array x:

Code 3:

```
1 def Average_val_3(x):
2     n = x.shape[0]
3     sum = x[0]/n
4     for i in range(1,n):
5         sum = sum + x[i]/n
6     return sum
```

g.) What is the computational complexity of this function, i.e. what is the total number of operations (addition, subtraction, multiplications, and divisions) in this function?

In line 3, there is one division operation. There is one addition operation and one division operation in line 5. These are executed n-1 times, since i takes on values 1, 2, ..., n-1. This means that the total number of operations is 1 + n-1 + n-1 = **2n-1** operations.

h.) What is the asymptotic computational complexity of this function?
The ACC is $\mathcal{O}(n)$.

2. Recall in the coding portion of homework 3 that we created a 2D array of size (12,15) to represent the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{12} & \cdots & \frac{1}{15} \\ 2 & \frac{2}{2} & \frac{2}{3} & \cdots & \frac{2}{12} & \cdots & \frac{2}{15} \\ 3 & \frac{3}{2} & \frac{3}{3} & \cdots & \frac{3}{12} & \cdots & \frac{3}{15} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 12 & \frac{12}{2} & \frac{12}{3} & \cdots & \frac{12}{12} & \cdots & \frac{12}{15} \end{bmatrix}.$$

The following function is one way to accomplish this in python:

Code 4:

```
1 def create_A(n,m):
2     A = np.zeros((n,m))
3     for i in range(n):
4         for j in range(m):
5             A[i,j] = (i+1)/(j+1)
6
7     return A
```

a.) What is the total number of operations (addition, subtraction, multiplications, and divisions) in this function?
In line 5, we have 2 addition operations and one division operation. These are executed m times from the j loop (since j takes on values 0, 1, ..., m-1), and n times from the i loop (since i takes on values 0, 1, ..., n-1). This means that the total number of operations is (2 + 1)nm = **3nm**.

b.) What is the asymptotic computational complexity of this function (if n = m)?
When n = m, the total number of operations is $3n^2$. This means that the ACC is $\mathcal{O}(n^2)$.

c.) If the function above is called, with $n = 12$ and $n = 15$ how many total operations will be made? (use your answer from part a).
The total operations would be $3(12)(15) =$ **540**.

3. Section 8.1 problem 2

a.) We are given matrix A:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 9 & 4 & 0 \\ 5 & 0 & 8 & 10 \end{bmatrix}.$$

We need to fund M such that MA = U, for U an upper triangular matrix. We can accomplish this by building M from three elementary matrices performing operations on A to create an upper-triangular matrix. First, let's look at the first column; we need to eliminate the 5 from row 4 of column 1 of A. To do this, define:

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{M}_1\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 9 & 4 & 0 \\ 0 & 0 & 8 & 0 \end{bmatrix}$$

Now, the second column. We need to eliminate the 9 from row 3, and we will use row 2 to accomplish this.

$$\mathbf{M}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{M}_2\mathbf{M}_1\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 8 & 0 \end{bmatrix}$$

Finally, let's tackle column 3. We will use row 3 to eliminate the element in row 4 of column 3.

$$\mathbf{M}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}, \quad \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{U}.$$

Now that we found an upper triangular matrix U, we can multiply $\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$ to find $\mathbf{M}$ such that $\mathbf{MA} = \mathbf{U}$:

$$\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ -5 & 6 & -2 & 1 \end{bmatrix} = \mathbf{M}.$$

b.) Next, we find the matrix $\mathbf{L}$ such that $\mathbf{A} = \mathbf{LU}$. We can do this by noting that $\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{A} = \mathbf{U}$. Then, multiplying by the inverses of each elementary matrix, we can write: $\mathbf{A} = \mathbf{M}_1^{-1}\mathbf{M}_2^{-1}\mathbf{M}_3^{-1}\mathbf{U}$. This means that $\mathbf{L} = \mathbf{M}_1^{-1}\mathbf{M}_2^{-1}\mathbf{M}_3^{-1}$. Let's compute this using the trick for the inverses of elementary matrices:

$$\mathbf{M}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{M}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Then,

$$\mathbf{M}_1^{-1}\mathbf{M}_2^{-1}\mathbf{M}_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 5 & 0 & 2 & 1 \end{bmatrix} = \mathbf{L}.$$

Thus we have found $\mathbf{L}$, unit lower triangular, so that $\mathbf{A} = \mathbf{LU}$. Furthermore, let's check that $\mathbf{L}$ is indeed the inverse of $\mathbf{M}$. If this is the case, then $\mathbf{LM} = \mathbf{ML} = \mathbf{I}$, the identity matrix.

$$\mathbf{ML} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ -5 & 6 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 5 & 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{I}.$$

And,

$$\mathbf{LM} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 5 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ -5 & 6 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{I}.$$

4. Section 8.1 problem 11(a) We need to solve the following system of equations via forward substitution:

$$\begin{cases} 6x_1 = 12 \\ 6x_2 + 3x_1 = -12 \\ 7x_3 - 2x_2 + 4x_1 = 14 \\ 21x_4 + 9x_3 - 3x_2 + 5x_1 = -2 \end{cases}.$$

First, by inspection, $x_1 = 2$. Second, substitute $x_1 = 2$ into the next equation to get:

$$6x_2 + 3(2) = -12, \quad 6x_2 = -18, \quad x_2 = -3$$

Now substitute into the next equation:

$$7x_3 - 2(-3) + 4(2) = 14, \quad 7x_3 + 14 = 14, \quad x_3 = 0$$

Finally,

$$21x_4 + 9(0) - 3(-3) + 5(2) = -2, \quad 21x_4 + 19 = -2, \quad 21x_4 = -21$$

Therefore, the solution to the system is:

$$\begin{cases} x_1 = 2 \\ x_2 = -3 \\ x_3 = 0 \\ x_4 = -1 \end{cases}.$$

5. Section 8.4 Computer Exercises (pg 423) problem 1: Disregard the books directions for these problems and instead compute the $\mathbf{x}^{(3)}$ using both Richardson iteration and Jacobi iteration for the matrix $\mathbf{A}$ and $\mathbf{b}$ listed in problems 1a and 1b.

   a.) We are given the following:

$$\mathbf{A} = \begin{bmatrix} 5 & -1 \\ -1 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

We need to compute $\mathbf{x}^{(3)}$ using both Richardson iteration and Jacobi iteration. Let's start with Richardson iteration. By definition,

$$x^{(k)} = (\mathbf{I} - \mathbf{A})x^{(k-1)} + \mathbf{b}$$

For this problem,

$$x^{(k)} = \begin{bmatrix} -4 & 1 \\ 1 & -2 \end{bmatrix} x^{(k-1)} + \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

Since the first guess is all zeros, let's go ahead and start computing:

$$x^{(1)} = \begin{bmatrix} -4 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 7 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

$$x^{(2)} = \begin{bmatrix} -4 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \end{bmatrix} + \begin{bmatrix} 7 \\ 4 \end{bmatrix} = \begin{bmatrix} -17 \\ 3 \end{bmatrix}$$

$$x^{(3)} = \begin{bmatrix} -4 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -17 \\ 3 \end{bmatrix} + \begin{bmatrix} 7 \\ 4 \end{bmatrix} = \begin{bmatrix} 78 \\ -19 \end{bmatrix}$$

For Jacobi iteration, we can find $x^{(k)}$ using the following equation:

$$x^{(k)} = -\mathbf{D}^{-1}\mathbf{T}x^{(k-1)} + \mathbf{D}^{-1}\mathbf{b}$$

Where $\mathbf{D}$ is the diagonal elements of the matrix A, and $\mathbf{T}$ is all other elements of A. Thus, for our problem, we have:

$$x^{(k)} = -\begin{bmatrix} 1/5 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} x^{(k-1)} + \begin{bmatrix} 1/5 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

Simplifying,

$$x^{(k)} = -\begin{bmatrix} 0 & -1/5 \\ -1/3 & 0 \end{bmatrix} x^{(k-1)} + \begin{bmatrix} 7/5 \\ 4/3 \end{bmatrix}$$

Now we can start our Jacobi iterations from an initial guess of all zeros.

$$x^{(1)} = -\begin{bmatrix} 0 & -1/5 \\ -1/3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 7/5 \\ 4/3 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 4/3 \end{bmatrix}$$

$$x^{(2)} = -\begin{bmatrix} 0 & -1/5 \\ -1/3 & 0 \end{bmatrix} \begin{bmatrix} 7/5 \\ 4/3 \end{bmatrix} + \begin{bmatrix} 7/5 \\ 4/3 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 9/5 \end{bmatrix}$$

$$x^{(3)} = -\begin{bmatrix} 0 & -1/5 \\ -1/3 & 0 \end{bmatrix} \begin{bmatrix} 5/3 \\ 9/5 \end{bmatrix} + \begin{bmatrix} 7/5 \\ 4/3 \end{bmatrix} = \begin{bmatrix} 44/25 \\ 17/9 \end{bmatrix}$$

Approximately,

$$x^{(3)} \approx \begin{bmatrix} 1.7600 \\ 1.8889 \end{bmatrix}$$

b.) We are given the following:

$$\mathbf{A} = \begin{bmatrix} 5 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ 5 \end{bmatrix}$$

We need to compute $\mathbf{x}^{(3)}$ using both Richardson iteration and Jacobi iteration. Let's start with Richardson iteration. By definition,

$$x^{(k)} = (\mathbf{I} - \mathbf{A})x^{(k-1)} + \mathbf{b}$$

For this problem,

$$x^{(k)} = \begin{bmatrix} -4 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} x^{(k-1)} + \begin{bmatrix} 7 \\ 4 \\ 5 \end{bmatrix}$$

Since the first guess is all zeros, let's go ahead and start computing:

$$x^{(1)} = \begin{bmatrix} -4 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 7 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 5 \end{bmatrix}$$

$$x^{(2)} = \begin{bmatrix} -4 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 7 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -17 \\ 8 \\ 4 \end{bmatrix}$$

$$x^{(3)} = \begin{bmatrix} -4 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} -17 \\ 8 \\ 4 \end{bmatrix} + \begin{bmatrix} 7 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 83 \\ -25 \\ 9 \end{bmatrix}$$

For Jacobi iteration, we can find $x^{(k)}$ using the following equation:

$$x^{(k)} = -\mathbf{D}^{-1}\mathbf{T}x^{(k-1)} + \mathbf{D}^{-1}\mathbf{b}$$

Where $\mathbf{D}$ is the diagonal elements of the matrix A, and $\mathbf{T}$ is all other elements of A. Thus, for our problem, we have:

$$x^{(k)} = -\begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} x^{(k-1)} + \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \\ 5 \end{bmatrix}$$

Simplifying,

$$x^{(k)} = -\begin{bmatrix} 0 & 1/5 & 0 \\ 1/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{bmatrix} x^{(k-1)} + \begin{bmatrix} 7/5 \\ 4/3 \\ 5/2 \end{bmatrix}$$

Now we can start our Jacobi iterations from an initial guess of all zeros.

$$x^{(1)} = -\begin{bmatrix} 0 & 1/5 & 0 \\ 1/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 7/5 \\ 4/3 \\ 5/2 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 4/3 \\ 5/2 \end{bmatrix}$$

$$x^{(2)} = -\begin{bmatrix} 0 & 1/5 & 0 \\ 1/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 7/5 \\ 4/3 \\ 5/2 \end{bmatrix} + \begin{bmatrix} 7/5 \\ 4/3 \\ 5/2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 79/30 \\ 19/6 \end{bmatrix}$$

$$x^{(3)} = -\begin{bmatrix} 0 & 1/5 & 0 \\ 1/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 5/3 \\ 79/30 \\ 19/6 \end{bmatrix} + \begin{bmatrix} 7/5 \\ 4/3 \\ 5/2 \end{bmatrix} = \begin{bmatrix} 289/150 \\ 53/18 \\ 229/60 \end{bmatrix}$$

Approximately,

$$x^{(3)} \approx \begin{bmatrix} 1.9267 \\ 2.9444 \\ 3.8167 \end{bmatrix}$$