AMATH 301 Exam 1 Version 1 Name: Key

Friday, February 16, 2024 Student ID:

Instructions: Write neatly and show all work. Unless it is stated otherwise, in order to receive full credit, you must show your work and carefully justify your answers. Please circle your final answer. The number in parentheses next to the problem number is the point value of the problem. Besides the one side of a 8.5 by 11 inch note sheet, notes, text-books, or calculators may not be used on this exam. Please verify that this exam consists of 5 problems on the front and back of 5 pages. Any work on scratch paper will not be graded.

This exam is worth 60 points.

Good Luck!

Pledge: By signing your name to this exam you are agreeing to the following statement:

I pledge that the work on this exam is mine only - I received no aid from outside sources nor did I obtain answers in ways that violate the instructions on the exam. I also pledge that I did not provide aid to any other student taking the exam.

Signature:

1. Consider the following python code:

Code 1:

```
import numpy as np

a = np.array([3,-6,5,-2])

b = np.array([4,-9,7,0])
```

State what each of the following code would return if ran in the python console after running the code above: (you do not need to justify/show your work)

(a) (2 points) a[2:4]

[5, -2]

(b) (3 points) b∗(−3∗a)

[4, -9, 7, 0] * [-9, 18, -15, 6]

= [-36, -162, -105, 0]

(c) (2 points) a∗∗2

[16, 81, 49, 0]

2. Consider the following python code:

Code 2:

```python
import numpy as np
n = 3
m = 4
A = np.zeros((n,m))
for i in range(n):
    for j in range(m):
        A[i,j] = 2 + min(i,j)
```

(a) (4 points) If this code is run in python, write out the elements of A. You may write A as a matrix or as a numpy array. (recall the min() function in python. min(1,2) returns 1)

$i=0, 1, 2$

$j=0, 1, 2, 3$

$$A = \begin{array}{cc} & \begin{array}{cccc} j=0 & j=1 & j=2 & j=3 \end{array} \\ \begin{array}{c} i=0 \\ i=1 \\ i=2 \end{array} & \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 3 & 3 & 3 \\ 2 & 3 & 4 & 4 \end{bmatrix} \end{array}$$

(b) (2 points) Write a code that modifies the 2nd column of A to be a column of ones. (you do not need to justify/show your work)

`A[:,1]=1` is one correct answer.

(d) (2 points) Write the code (using array slicing) that extracts the middle two values from a.

a[1:3]

(e) (2 points) Write the code (using array slicing) that extracts the first three values from b.

b[:3]

(f) (3 points) Use numpy functions and numpy constants to write the code that computes $\ln\left(\sqrt{3+\pi^3}\right) * \cos(5)$.

np.log(np.sqrt(3 + np.pi**3))*np.cos(5)

3. Let $\mathbf{A}$ be the following matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 & 1 \\ 0 & 3 & 0 & 10 & 6 \\ 5 & 1 & 0 & 2 & 0 \end{bmatrix}.$$

We would like to solve a system of linear equations, $\mathbf{Ax} = \mathbf{b}$, for a 5x1 vector of unknowns $\mathbf{x}$, given a 5x1 vector of known constant values, $\mathbf{b}$. The following questions concern the setup of the system; you will not be solving the system in this problem.

a.) (3 points) Naive Gaussian elimination will not work on this system. In a single sentence, explain why.

we would get a division by 0 on the first step of G.E. mult $= \frac{a_{ij}}{a_{11}} = \frac{a_{ij}}{0} \times$

b.) (3 points) If two rows of $\mathbf{A}$ are swapped, we will be able to solve this system via either forward or backward substitution. Please identify which two rows must be swapped and write down the matrix resulting from swapping these two rows. Do not swap more than two rows, i.e. three rows must remain unchanged after this operation. You may assume that row numbering begins at 1, thus row 1 is the row containing the numbers $(0, 0, 0, 2, 0)$.

Pivot / ~~Switch~~ Switch row 1 & row 2

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 4 & 0 & 1 \\ 0 & 3 & 0 & 10 & 6 \\ 5 & 1 & 0 & 2 & 0 \end{bmatrix}$$

c.) (3 points) Would the system found by swapping two rows be solvable by forward substitution or backward substitution? Please explain which, and how you know, in a single sentence. Do not solve the system.

This is Backward Sub since we would Be solving for x_5 first then x_4 then x_3 then x_2 then x_1.

(c) (4 points) If the code below is run in python, write out the elements of A. You may write A as a matrix or as a numpy array.

Code 3:

```
import numpy as np
n = 3
m = 4
A = np.zeros((n,m))
for i in range(n):
    for j in range(m):
        A[i,j] = 2 + min(i,j)
for k in range(n):
    if A[k,k] < 3:
        A[k,k] = 1
    elif A[k,k] < 4:
        A[k,k] = 0
    else:
        A[k,k] = -1
```

part a.) $A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 3 & 3 & 3 \\ 2 & 3 & 4 & 4 \end{bmatrix}$

$\to K = 0, 1, 2$

$K=0$ $\quad A[0,0] = 2 < 3 \to A[0,0] = 1$

$K=1$ $\quad A[1,1] = 3 < 4 \to A[1,1] = 0$

$K=2$ $\quad A[2,2] = 4 \to$ else $\to A[2,2] = -1$

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 0 & 3 & 3 \\ 2 & 3 & 4 & -1 \end{bmatrix}$$

4. Consider the system of equations $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{x}$ is a vector of unknowns and

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}.$$

(a) (9 points) Find the LU decomposition for the matrix $\mathbf{A}$. Make sure to list all of your elementary matrices.

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad M_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \\ 0 & 10 & -1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

$$M_2 M_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \\ 0 & 10 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix} \leftarrow \underline{\underline{U}}$$

$$M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$$

$$M_1^{-1} M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} = \underline{\underline{L}}$$

(b) (6 points) Use forward substitution and backward substitution to solve the system $\mathbf{A}\mathbf{x} = \mathbf{b}$.

$$\underline{\underline{A}}\,\underline{x} = \underline{b} \rightarrow \underline{\underline{L}}\,\underline{\underline{U}}\,\underline{x} = \underline{b} \rightarrow \underline{\underline{U}}\,\underline{x} = \underline{z}$$

$$\rightarrow \underline{\underline{L}}\,\underline{z} = \underline{b}$$

F.S. $\underline{\underline{L}}\,\underline{z} = \underline{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$$

$z_1 = 0$, $z_1 + z_2 = -5$

$z_2 = -5 - z_1$

$z_2 = -5$,

$3z_1 - 5z_2 + z_3 = 7 \rightarrow z_3 = 7 + 5z_2 - 3z_1$

$z_3 = 7 - 25 - 0$

$= -18$

B.S. $\underline{\underline{U}}\,\underline{x} = \underline{z}$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -18 \end{bmatrix}$$

$-6x_3 = -18 \rightarrow x_3 = 3$

$-2x_2 - x_3 = -5 \rightarrow -2x_2 = -5 + x_3 \rightarrow -2x_2 = -5 + 3$

$-2x_2 = -2$

$x_2 = 1$

$x_1 - x_2 + 2x_3 = 0 \rightarrow$

$x_1 = x_2 - 2x_3 = 1 - 2 \cdot 3 = 1 - 6 = -5$

$$\boxed{\underline{x} = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}}$$

5. (12 points) Consider the following system of equations, $\mathbf{A}\mathbf{x} = \mathbf{b}$:

$$\mathbf{A} = \begin{bmatrix} 5 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 9 \\ 4 \\ -6 \end{bmatrix}.$$

Using the Jacobi iteration method, compute $\mathbf{x}^{(2)}$. Use the inital guess

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$A = D - C_L - C_u \qquad D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad D^{-1} = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

$$C_L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad C_u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_L + C_u = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad D^{-1}(C_L + C_u) = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/5 & 0 \\ 1/5 & 0 & 1/5 \\ 0 & 1/5 & 0 \end{bmatrix} \qquad D^{-1}b = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/5 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \\ -6 \end{bmatrix} = \begin{bmatrix} 9/5 \\ 4/5 \\ -6/5 \end{bmatrix}$$

$$\underline{x}^{(1)} = \begin{bmatrix} 0 & 1/5 & 0 \\ 1/5 & 0 & 1/5 \\ 0 & 1/5 & 0 \end{bmatrix} \overset{\underline{x}^{(0)}}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} + \begin{bmatrix} 9/5 \\ 4/5 \\ -6/5 \end{bmatrix} = \begin{bmatrix} 9/5 \\ 4/5 \\ -6/5 \end{bmatrix}$$

$$\underline{x}^{(2)} = \begin{bmatrix} 0 & 1/5 & 0 \\ 1/5 & 0 & 1/5 \\ 0 & 1/5 & 0 \end{bmatrix} \begin{bmatrix} 9/5 \\ 4/5 \\ -6/5 \end{bmatrix} + \begin{bmatrix} 9/5 \\ 4/5 \\ -6/5 \end{bmatrix} = \begin{bmatrix} 49/25 \\ 23/25 \\ -26/25 \end{bmatrix}$$