

**Instructions:** Write neatly and show all work. Unless it is stated otherwise, in order to receive full credit, you must show your work and carefully justify your answers. Please circle your final answer. The number in parentheses next to the problem number is the point value of the problem. Besides the one side of a 8.5 by 11 inch note sheet, notes, textbooks, or calculators may not be used on this exam. Please verify that this exam consists of 4 problems and 1 extra credit problem, on the front and back of 4 pages. Any work on scratch paper will not be graded.

This exam is worth 40 points.

Good Luck!

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**Pledge:** By signing your name to this exam you are agreeing to the following statement:

*I pledge that the work on this exam is mine only - I received no aid from outside sources nor did I obtain answers in ways that violate the instructions on the exam. I also pledge that I did not provide aid to any other student taking the exam.*

Signature: \_\_\_\_\_

1. Consider the function

$$f(x) = x^3 - 3x^2 + 1.$$

It is possible to use Newton's method and the bisection method to approximate the roots of this function.

- (a) (3 points) If the initial guess is  $x_0 = \frac{1}{2}$ , calculate  $x_1$  using Newton's method.

- (b) (3 points) With an interval of  $[0, 1]$  and an initial midpoint of  $x_0 = \frac{1}{2}(0 + 1) = \frac{1}{2}$ , calculate  $x_1$  using the bisection method.

- (c) (4 points) Determine how many steps of the bisection method from part b, are needed to determine the root with an error of at most  $(1/2) * 10^{-5}$ .

*hint:*  $\frac{\ln(10^{-5})}{\ln(2)} = -16.6096$  rounded to 4 decimal places and recall that  $\ln(1) = 0$ .

2. (a) (8 points) Consider the following table of data

$x$	4	9	16
$f(x)$	2	3	4

Find the polynomial of degree 2 or less that interpolates the data above. You can find this polynomial using either Lagrange or Newton form, and you do not need to simplify your answer.

- (b) (2 points) Now consider the table of data

$x$	0	1
$f(x)$	1	2

The polynomials  $p(x) = x + 1$ ,  $q(x) = -(x - 1)^2 + 2$ , and  $r(x) = x^2 + 1$  interpolate the data (you do not need to verify this). Explain why this does not violate the uniqueness part of the theorem on existence of polynomial interpolation.

3. Consider the data table:

$x$	-1	0	1	2	3
$f(x)$	$\frac{1}{4}$	1	4	16	64

- (a) (3 points) This code below uses the forward difference formula to approximate the  $f'(-1)$ ,  $f'(0)$ ,  $f'(1)$ , and  $f'(2)$  and assigns these approximations to the first four entries of the variable `forward_diff`. Assuming this code is run in python, write the variable `forward_diff` as python output or as a vector.

Code 1:

```
1 import numpy as np
2 x = np.array([-1,0,1,2,3.0])
3 y = np.array([1/4,1,4,16,64])
4 n = x.shape[0]
5 forward_diff = np.zeros(n)
6 for i in range(n-1):
7     forward_diff[i] = (y[i+1] - y[i])/(x[i+1] - x[i])
```

- (b) (3 points) Consider the code below and add to it the code that saves the backward difference approximations for  $f'(0)$ ,  $f'(1)$ ,  $f'(2)$ , and  $f'(3)$  to the last four elements of the variable `backward_diff`

**Code 2:**

```
1 import numpy as np
2 x = np.array([-1,0,1,2,3.0])
3 y = np.array([1/4,1,4,16,64])
4 n = x.shape[0]
5 forward_diff = np.zeros(n)
6 for i in range(n-1):
7     forward_diff[i] = (y[i+1] - y[i])/(x[i+1] - x[i])
8 backward_diff = np.zeros(n)
```

Write your code here:

- (c) (2 points) Using array slicing, write the code that assigns the middle three elements of `forward_diff` to the variable `dx1` and assigns the middle three elements of `backward_diff` to the variable `dx2`.

Write your code here:

- (d) (2 points) For the given data, one way to approximate  $f'(0)$ ,  $f'(1)$ , and  $f'(2)$  using the central difference formula is to average the forward and backward difference approximations for  $f'(0)$ ,  $f'(1)$ , and  $f'(2)$ . Write the code that assigns the average of `dx1` and `dx2` (from part c) to the variable `central_diff`.

Write your code here:

4. (a) (6 points) Approximate the integral

$$\int_0^2 x^3 dx,$$

with the left-hand rule, right-hand rule, and the midpoint rule using the partition

$$P = \{0, 1, 2\}.$$

You do not need to compute the trapezoid rule approximation.

- (b) (4 points) Determine the number of sub-intervals  $n$  that guarantees an error no larger than  $\epsilon = 10^{-3}$  when the trapezoid rule is used to approximate the integral from part a.

*hint:*  $\sqrt{8 \times 10^3} = 89.443$  rounded to three decimal places

5. Consider the Initial Value Problem

$$\frac{dx}{dt} = -2tx, \quad x(0) = 2.$$

- (a) (2 points extra credit) Verify that the solution to this initial value problem is the function

$$x(t) = 2e^{-t^2}.$$

- (b) (2 points extra credit) With a step size of  $h = \frac{1}{4}$  Approximate  $x(\frac{1}{2})$  using Euler's Method.