Homework 8 Solutions

This homework assignment is about numerical integration, the left-hand rule, the right-hand rule, the midpoint rule, and the trapezoid rule.For the written exercises, you should upload

a scanned PDF to Gradescope and then follow the prompts given by Gradescope to assign certain pages of your PDF document to the correct problems.For the coding exercises, you

will be prompted to upload your python file directly to Grade- scope. Please make sure you submit only one file and that your variables are assigned to the correct variable names.The

course syllabus found on Canvas has information on how homework is graded and how homework should be presented and submitted. Please let me or the TAs know if you have any questions or concerns.This assignment is due on Sunday, March 10 at 11:59pm.

5.1.2

What is the result if we estimate $\int_1^2 x^{-1}dx$ by means of the trapezoid, right-hand, left-hand, and midpoint rule using the partition $P = \{1, 3/2, 2\}$. $f(x) = x^{-1}$. $f(1) = 1$, $f(3/2) = 2/3$, and $f(2) = 1/2$. $\Delta x = 1/2$.

Trapezoidal Rule:

$$\int_1^2 x^{-1}dx \approx \Delta x \left(\frac{f(1) + f(3/2)}{2} + \frac{f(3/2) + f(2)}{2} \right) = \frac{17}{24}$$

Midpoint Rule:

$$\int_1^2 x^{-1}dx \approx \Delta x \left(f(5/4) + f(7/4) \right) = \frac{24}{35}$$

Left-Hand Rule:

$$\int_1^2 x^{-1}dx \approx \Delta x \left(f(1) + f(3/2) \right) = \frac{5}{6}$$

Right-Hand Rule:

$$\int_1^2 x^{-1}dx \approx \Delta x \left(f(3/2) + f(2) \right) = \frac{7}{12}$$

5.1.8

What is the numerical value of the composite trapezoid rule applied to the reciprocal function applied to $f(x) = x^{-1}$ with $P = \{1, 4/3, 2\}$.

Trapezoidal Rule:

$$\int_1^2 x^{-1}dx \approx \frac{1}{3}\frac{f(1) + f(4/3)}{2} + \frac{2}{3}\frac{f(4/3) + f(2)}{2} = \frac{17}{24}$$

Midpoint Rule:

$$\int_1^2 x^{-1}dx \approx \frac{1}{3}f(7/6) + \frac{2}{3}f(5/3) = \frac{24}{35}$$

Left-Hand Rule:

$$\int_1^2 x^{-1}dx \approx \frac{1}{3}f(1) + \frac{2}{3}f(4/3) = \frac{5}{6}$$

Right-Hand Rule:

$$\int_1^2 x^{-1}dx \approx \frac{1}{3}f(4/3) + \frac{2}{3}f(2) = \frac{7}{12}$$

5.1.9

$$
\begin{aligned}
T(f;P) &= \frac{h}{2}[f(0)+f(1)] + hf(\tfrac{1}{2}) \text{ where } f(x) = (x^2+1)^{-1} \text{ and } h = \tfrac{1}{2} \\
&= \frac{1}{4}[1+\frac{1}{2}] + \frac{1}{2}\left(\frac{4}{5}\right) = \frac{31}{40} = 0.775
\end{aligned}
$$

Trapezoid Rule

$$\int_0^1 \frac{dx}{x^2+1} = \arctan x\Big|_0^1 = \frac{\pi}{4} \approx 0.7854$$

$$\text{Error} = -\frac{1}{12}(b-a)h^2 f''(\xi) = -\frac{1}{48}f''(\xi)$$

$$f(x) = (x^2+1)^{-1},\ f'(x) = -(x^2+1)^{-2}(2x),$$

$$
\begin{aligned}
f''(x) &= -2(x^2+1)^{-2} + 2(x^2+1)^{-3}(2x)^2 \\
&= 2(3x^2-1)/(x^2+1)^3 \\
f'''(x) &= [(x^2+1)^3(12x) - 2(3x^2-1)(3)(x^2+1)^2(2x)]/(x^2+1)^6 \\
&= (x^2+1)^2(12x)[x^2+1-3x^2+1]/(x^2+1)^6 \\
&= -24x(x^2-1)(x^2+1)^2/(x^2+1)^6
\end{aligned}
$$

Set $f'''(x) = 0$. The maximum value occurs at $x = 1$ with $f''(1) = 1/2$.

An upper bound of the error term is $-[(b-a)/12]h^2 f''(1) = 1/96$.

Note that $(0.7854 - 0.775) = 0.0104 \leq \dfrac{1}{96} = 0.01042$.

5.1.11

$$|\text{error}| = \frac{1}{12}(b-a)h^2|f''(\xi)|, \;\; f(x) = \sin(x^2), \;\; f'(x) = 2x\cos(x^2),$$
$$f''(x) = 2\cos(x^2) - 4x^2\sin(x^2)$$
$$|f''(\xi)| \le 2|\cos(\xi^2)| + 4\xi^2|\sin(\xi^2)| \le 2 + 4\xi^2 \le 146$$
$$|\text{error}| \le \frac{1}{12}(6)\left(\frac{6}{100}\right)^2(146) = 0.2628$$

5.1.23

$$T(f;P) = \frac{1}{8}[10+5] + \frac{1}{4}[8+7+6] = \frac{15}{8} + \frac{21}{4} = \frac{57}{8} = 7.125$$

5.1.31

Error term $= \dfrac{b-a}{12}h^2 f''(\xi)$ for some $\xi \in (a,b)$. Now $f(x) = e^{-x^2}$, $f'(x) = -2xe^{-x^2}$, $f''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$. Thus, we have

$$\max_{0\le x\le 2} |f''(x)| \le 2 \max_{0\le x\le 2} e^{-x^2}|2x^2-1| = 2$$

and

$$|\text{error}| \le \frac{1}{6}h^2(2) = \frac{1}{3}h^2 = \frac{4}{3n^2} < 10^{-6}$$

since $h = 2/n$. Hence, we obtain $n > (2/\sqrt{3})\ 10^3$ or $n \ge 1155$. Note true solution 0.8820139.