Superconductivity Long Notes

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Contents

1 Superconductivity

1.1 Introduction

Requirements for macroscopic quantum phenomena are

- Sufficiently low temperatures
- Particles must 'notice' that they are indistinguishable
- Non-localisation

Regime for BEC behaviour holds when the Compton wavelength is similar to the interparticle distance.

$$\lambda \geq n^{-1/3} \tag{1}$$

In thermal equilibrium, we find

$$p \sim (mk_BT)^{1/2} \tag{2}$$

and thus we require

$$k_BT \leq \frac{n^{2/3}\hbar^2}{m} \tag{3}$$

where m is the particle mass. Especially for heavy atoms, this means that the temperatures have to be sufficiently low.

Candidate systems include

- some atoms or molecules
- Liquid He-3 or He-4
- Dilute atomic alkali gases
- Electrons in metals
- Quasiparticles, such as excitons, polarisons and magnons
- Photons interacting via coupling to some matter component

1.2 Bosons: BEC Basics

Simple definition A BEC is a system that is in one single macroscopic state. For a wavefunction

$$\psi_s(r_1, r2, \ldots r_N) \tag{4}$$

which is symmetric exchange, we get the density matrix

$$\rho(r, r') = N \sum_s p_s \int \mathrm{d}r_2 \mathrm{d}r_3 \ldots \int r_N \Psi_s^*(r, r_2, \ldots, r_N)\Psi_s(r', r_2, \ldots, r_N) \tag{5}$$

This is just the inner product of $\left\langle \psi^\dagger(r)\psi(r') \right\rangle$. where I guess that $\psi(r)$ is a single-particle wavefunction. If we diagonalise, we get

$$\rho(r, r') = \sum_j n_j \chi_j^*(r)\chi_j(r') \tag{6}$$

Once one of the eigenvalues dominate, we say that we are in a macroscopic state.

Infinite and translationally invariant systems gives a single-particle wavefunction

$$\chi_j(r) = e^{i\vec{k}_j \cdot \vec{r}/\sqrt{V}} \tag{7}$$

Macroscopic eigenvalue N_0 for the $\chi_j(r)$ wavefunction given by

$$\lim_{|r-r'|\to\infty} \rho(r, r') = \frac{N_0}{V} \tag{8}$$

Question: If we are dealing with an infinite system, should not $V \to \infty$?

1.3 Non-interacting Bose gas

Statistics for bosons is given by

$$n_k = \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1} \tag{9}$$

where

$$\epsilon_k = \frac{\hbar^2 k^2}{2m} \tag{10}$$

is the energy of a free particle, and where μ is the chemical potential.

Chemical potential often written μ is the amount of energy required to add another paticle to the system.

Total number of particles given by integrating over all energies,

$$N = \int \mathrm{d}\epsilon \frac{g(\epsilon)}{e^{\beta(\epsilon_k - \mu)} - 1} \tag{11}$$

where

$$g(\epsilon) \propto \epsilon^{d/2-1} \tag{12}$$

is the density of states.

Question: I can't recall how the dimensionality of space comes into this.

Answer: We are concerned with the density of states, which necessarily is related to the dimensionality as we ask the question how many states we can fit in per unit volume.

Requirement for well-defined n_k is

$$\mu \leq 0 \tag{13}$$

Maximum number of particles the largest possible value of μ is $\mu = 0$, thus

$$N_{max} = \int \mathrm{d}\epsilon \frac{\sqrt{\epsilon}}{e^{\beta\epsilon_k} - 1} \tag{14}$$

If we add more particles, we cannot describe the statistics by the Bose-Einstein distribution anymore.

Macroscopic occupancy instead, while we are below T_c, any new addition will occupy a single state

$$N = N_0 + N_{max} \tag{15}$$

Question: I thought we were talking about states and not number. However, could this be that below the temperature, we just treat the new particles as one single particle?

1.4 Weakly interacting dilute Bose gas

Full Hamiltonian is given by

$$H = \sum_{i=1}^{N} \left[-\frac{\hbar \nabla_i^2}{2m} + V(r_i) \right] + \sum_{i<j} U\delta(r_i - r_j) \tag{16}$$

Variational Principle is the way of finding the minimum energy. We start with a guess for a ground state, then work out the energy. From there, we look at which parameters go into the wavefunction, the minimise the energy from there.

In the context of superconductivity, the variational principle can be used to find the ground state by minimising the free energy F.

Free energy is given by

$$F = N \int \mathrm{d}^3 r \left[\frac{\hbar^2}{2m} |\nabla \chi|^2 + (V(r) - \mu) |\chi|^2 + \frac{UN(N-1)}{2} |\chi|^4 \right] \tag{17}$$

Derivation of the Gross-Pitaevskii equation We wish to derive the equation of motion for the system with the potential above by minimising the free energy. First, we rescale the wavefunction

$$\chi = \sqrt{N} \psi(r) \tag{18}$$

We obtain

$$F = \int \mathrm{d}^3 r \left[\frac{\hbar^2}{2m} |\nabla^2 \psi^2| + (V(r) - \mu) + \frac{U(N-1)}{2} |\psi|^4 \right] \tag{19}$$

This quantity is at its minimum where the derivative of the quantity inside the integral is zero. Differentiating with respect to ψ gives

$$\frac{\hbar^2}{2m} \nabla^2 \psi + (V(r) - \mu) |\psi|^2 + U(N+1)|\psi|^3 = 0 \tag{20}$$

This is a bit like a non-linear Schrodinger equation.

Chemical potential for translationary invariant systems A translationary invariant system has,

$$\psi = \sqrt{\frac{N}{V}} \tag{21}$$

then solving the Gross-Pitaevskii is given by

$$\mu = \frac{UN}{V} \tag{22}$$

Time dependent Gross-Pitaevskii equation we can combine the above with the TDSE to get

$$i\hbar \frac{d\psi(r,t)}{dt} = \left[-\frac{\hbar^2 \nabla^2}{2m} + V(r) + U|\psi(r,t)|^2 \right] \psi(r,t) \tag{23}$$

BEC ground state is characterised by its average number of particles. It can be either a state with well-defined particle number or well-defined phase.

$$|\Psi_N\rangle = \left(\psi_0^\dagger\right)^N |\Omega\rangle \tag{24}$$

$$|\Psi_\lambda\rangle = e^{\lambda \psi_0^\dagger - |\lambda|^2/2} |\Omega\rangle \tag{25}$$

The first one is a Fock state and the second is a coherent state. The coherent state has lower energy, and thus the ground state does not have well-defined particle number.

1.5 Fragmented BEC

Fragmented BEC for two closely separated states χ_0 and χ_1 can be described by

$$|\Psi\rangle = \left(\psi_0^\dagger\right)^{N-M} \left(\psi_1^\dagger\right)^{M} |\Omega\rangle \tag{26}$$

Interaction energy difference given by

$$\frac{U}{2V}\left((N-M)(N-M-1)+M(M-1)+4(N-M)M\right) \simeq \frac{U}{2V}\left[N^2+2M(N-M)\right]$$

We have made the approximation of removing one N from the expression above.

1.6 Fluctuations

Guassian state is given by

$$\psi(r) = \frac{1}{\sqrt{V}}\sum_k \psi_k e^{i\vec{k}\cdot\vec{r}} \tag{27}$$

BEC Hamiltonian the Hamiltonian for the weakly interacting gas is, in 2nd quantisation,

$$H-\mu N = \sum_k (\epsilon_k-\mu)\psi_k^\dagger\psi_k + \sum_{k,k',q}\frac{U}{2V}\psi_{k+q}^\dagger\psi_{k'-q}^\dagger\psi_{k'}\psi_k \tag{28}$$

Quasi-particle Hamiltonian the previous Hamiltonian can be diagonalised into

$$H = \sum_k \xi_k b_k^\dagger b_k + \frac{\xi_k-\epsilon_k-\mu}{2} \tag{29}$$

Boboliubov transformation is a unitary transformation of a commutator relation, such as $[a,a^\dagger]=1$. Note that in addition to the hyperbolic transformation matrix, we can also add a phase $e^{i\theta}$ to the transformation, since this will be cancelled out.

More stuff needed here

1.7 Fermions - BCS-BEC Crossover

BCS theor describes bonding between fermions and can explain most superconductors. BCS works well when

- Transitions temperatures must be a small fraction of the Fermi degeneracy temperature T_F
- Normal state is of a normal metal
- Superconductor transition does not occur close to an additional phase transition
- The symmetry of the Cooper pairs is the same as the crystal lattice or s-wave in the case of amorphous materials
- The principal mechanism of Cooper pairing is the exchange of virtual phonons

Fermi Temperature written T_F is defined as

$$T_F = \frac{E_F}{k_B} \tag{30}$$

where E_F is the Fermi energy. T_F is the temperature at which thermal effects are comparable with quantum efects associated with Fermi statistics.

Virtual phonons are needed to describe static lattice deformations surrounding a perturbation. They are a field/theoretical tool required for the transfer of forces and such.

Cooper problem phonon cause a weak attraction between the electrons.

Cooper pairs are pairs of electrons with momentum with $|k\rangle$ and $-|k\rangle$. Only a few electrons close to the Fermi energy get paired. A Cooper pair is of bosonic nature since it is made up of an even number of fermions.

Each electron has spin-1/2, which when added can have either spin 1 or spin 0.

Pairing instability is based on seeing the superconducting transition as an instabilityof the normal state at some finite temperature. It highlights the potential importance of pairing fluctuations that occur in the normal state even before the critical transition.

BCS instability not entirely sure what this is. I think it is adding an instability to the BEC treatment that we performed before.

1.7.1 Fermion ground-state wavefunction

Initial assumptions we assume that a number of N electrons in an alkali metal will have $N/2$ be in the spin-up state and $N/2$ be in the spin-down state. The electrons are moving in a free 3D space subject to attractive interactions.

BCS postulate the ground state is a condensate of a macroscopic number of fermion pairs. Any two different species of particles form a singlet.

BCS ground state is given by

$$\Psi_N = \mathcal{A}\left\{\varphi|(\vec{r}_1 - \vec{r}_2|)\chi_{12} \dots \phi(|\vec{r}_{N-1} - \vec{r}_N|)\chi_{N-1,N}\right\} \tag{31}$$

$\mathcal{A}$ is an operator that anti-symmetrises for any exchange of coordinates or spins for any of the electrons.

BCS ground state in second quantisation is also written

$$|\Psi\rangle_N = \int \prod_i \mathrm{d}^3 r_i \varphi(\vec{r}_1 - \vec{r}_2)\Psi^\dagger_\uparrow(\vec{r}_1)\Psi^\dagger_\downarrow(\vec{r}_2) \dots \varphi(\vec{r}_{N-1} - \vec{r}_N)\Psi^\dagger_\uparrow(\vec{r}_{N-1})\Psi^\dagger_\downarrow(\vec{r}_N)\left|0\right\rangle \tag{32}$$

Or, more succinctly,

$$|\Psi\rangle_N = (b^\dagger)^{N/2}\left|0\right\rangle \tag{33}$$

Field operator lol... is given by

$$\Psi^\dagger_\sigma(\vec{r}) = \sum_k c^\dagger_{k\sigma}\frac{e^{-i\vec{k}\cdot\vec{r}}}{\sqrt{V}} \tag{34}$$

Creation operator for Fermion pair given by

$$b^\dagger = \sum_k \varphi_k v^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \tag{35}$$

Note the creation of two particles, one with $\vec{k}$ and one with $-\vec{k}$.

Standard modus operandi is to work with the chemical potential instead of the many-party state.

Many-body state which is essentially a coherent state compared to Foxk state, which has of superpositions of states with different number of fermions given by

$$\begin{aligned}
\mathcal{N}\left|\Psi\right\rangle &= \mathcal{N}e^{\sqrt{N_p}b^\dagger}\left|0\right\rangle && (36)\\
&= \mathcal{N}\prod_k e^{\sqrt{N_p}\varphi_k c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}}\left|0\right\rangle && (37)\\
&= \mathcal{N}\prod_k\sum_n \frac{\left(\sqrt{N_p}\varphi_k c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}\right)^n}{n!}\left|0\right\rangle && (38)\\
&= \mathcal{N}\prod_k\left(1+\sqrt{N_p}\varphi_k c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}\right)\left|0\right\rangle && (39)
\end{aligned}$$

where we have used the fact that two fermions cannot be created in the same state, so any

$$\left(c^\dagger_{k\uparrow}\right)^2\left|0\right\rangle = 0 \tag{40}$$

Note that the normalisation constant $\mathcal{N}$ is not included in the equations in the notes, but I think that it should be there.

Normalisation We choose the constant

$$\mathcal{N} = \prod_k\sqrt{1+N_p|\varphi_k|^2} \tag{41}$$

where

$$v_k = \sqrt{N_p}\varphi_k u_k = \sqrt{\frac{N_p\varphi_k^2}{1+N_p\varphi_k^2}} \tag{42}$$

Fermionic BEC state using the normalisation, we have

$$\left|\Psi_{BEC}\right\rangle = \prod_k\left(u_k + v_k c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}\right)\left|0\right\rangle \tag{43}$$

Question: This just seems like we have a superposition of the ground state (normalised) and Cooper pairs. We are describing any excitations with this?

Fermion BEC occupation There are two different cases which determine the occupancy.

For $k < k_F$ we have $u_k = 0, v_k = 1$.

For $k > k_F$ we have $u_k = 1, v_k = 0$.

Fermion to boson crossover Using the wavefunction above, and depending on the form of the normalisation, we can describe both point bosons and large Fermion paris.

I think this means that the wavefunction above is strictly bosonic in both cases.

Question: I still don't see what the point boson part is, unless it is just the first term with u_k.

Question: What determines if I am working with bosons or fermions?

Answer: Since we can describe any wavefunction in terms of annihilation and creation operators, which determine the nature of the particles through their commutator or anti-commutator relations, the wavefunction tells us whether it describes fermions or bosons.

Gap and number equation

In this section, we find the ground-state of the fermionic system by guessing the wavefunction and showing that it minimises the free energy $\mathcal{F}$.

Many-body Hamiltonian for attractive interactions given by

$$H = \sum_{k,\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \frac{U_0}{V} \sum_{k,k',q} c^\dagger_{k+\frac{q}{2}\uparrow} c^\dagger_{-k+\frac{q}{2}\downarrow} c_{k'+\frac{q}{2}\downarrow} c_{-k`+\frac{q}{2}\uparrow} \tag{44}$$

Question: What is the need for q (or interpretation) in this equation?

Attempt at answer: q I think is additional momentum that the particles gain through interaction. This momentum is split between the Cooper pairs. Maybe.

Answer: q is an additional momentum with which the pairs move. They have opposite initial momentum, buta can through interactions gain momentum q.

Free energy is given by the Hamiltonian minus the chemical potential and number.

$$\mathcal{F} = \langle H - \mu N \rangle = \sum_k 2\xi_k v_k^2 + \frac{U_0}{V} \sum_{k,k'} u_k v_k u_{k'} v_{k'} \tag{45}$$

where $\xi_k = \epsilon_k - \mu$.

Minimising the free energy gives

$$v_k^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right) \tag{46}$$

$$u_k^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right) \tag{47}$$

with

$$E_k = \sqrt{\xi_k^2 + \Delta^2} \tag{48}$$

Question: It is not clear to me at all how we minimise the free energy - with respect to what do we minimise it? It implies taking some derivativ? It should be, at least.

Gap equation given by

$$\Delta = -\frac{U_0}{V}\sum_k u_k v_k = -\frac{U_0}{V}\sum_k \frac{\Delta}{2E_k} \tag{49}$$

Question: So... is this a recursive equation? I could just divide by Δ on both sides, the way it is written...

Answer: No, because E_k depends on Δ.

Number equation for the total particle density $n = N/V$ is

$$n = 2\int \frac{\mathrm{d}^3 k}{(2\pi)^3} v_k^2 \tag{50}$$

By solving the gap and number equation simultaneously we can find the energy gap Δ.

1.7.2 Hamiltonian Diagonalisation

Instead of guessing the wavefunction, we here make approximations to the Hamitlonian until we can diagonalise it.

Momentum approximation The fermion pairs with zero total momentum are the dominant part of interactions. We get Hamiltonian

$$H = \sum_{k\sigma} \epsilon_k c_{k\sigma}^\dagger c_{k\sigma} + \frac{U_0}{V}\sum_{k,k'} c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger c_{k'\downarrow} c_{k'\uparrow} \tag{51}$$

Mean-field decoupling Refers to (I think) an approximation which states that there are no interactions, but we have some kind of effective field interaction. That is, individual interactions are replaced with some general term.

$$C_k = \langle c_{k\uparrow} c_{-k\downarrow} \rangle \tag{52}$$

which leads to Hamiltonian

$$H = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} - \Delta \sum_k \left(c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + c_{k\downarrow} c_{-k\uparrow} + \sum_{k'} C_{k'} \right) \tag{53}$$

where

$$\Delta = \frac{U_0}{V} \sum_k C_k \tag{54}$$

Question: Is this approximation based on something similar to high-energy physics, where the low-energy particles that travel through materials actually interact less than particles with high energy?

Diagonalised Hamiltonian Using a Bogoliubov transformation, we get new operators $\gamma_{k\uparrow}$ and $\gamma^\dagger_{-k\downarrow}$

$$H - \mu N = \sum_k E_k \left(\gamma^\dagger_{k\uparrow} \gamma_{k\uparrow} + \gamma^\dagger_{k\downarrow} k_{k\downarrow} \right) + \sum_k \left(\Xi_k - E_k \right) - \frac{\Delta^2}{U_0/V} \tag{55}$$

Ground state is given by the qussi-particle vacuum

$$\gamma_{k\sigma} \left| \Omega \right\rangle = 0 \tag{56}$$

1.7.3 BEC- BCS rossover

BEC limit is the limit of tightly bound pairs. We are not dealing with long-range Cooper pairs

The chemical potential is

$$\mu = -\frac{\hbar^2}{2ma^2} + \frac{\pi \hbar^2 a n}{m} \tag{57}$$

The first time is the binding energy and the second term is the mean-field contribution from attractive interactions.

Question: What is a again? I forget... Is it the average interparticle distance, perhaps?

BEC limit is the limit of weak attractive interactions. Instead of tightly bound molecules of fermions, we get Cooper pairs that are quite spread out. Here

$$\mu \simeq E_F \tag{58}$$

$$\Delta \simeq 2\epsilon_c e^{-\frac{1}{\rho(E_F)U_0}} \tag{59}$$

where ϵ_c is the energy cut-off for interactions.

Changes between regimes All the plots the show the crossover depict the wavevector. Why is this important?

1.8 Superfluidity

Superfluid wavefunction is macroscopic and complex, and is given by

$$\psi(\vec{r}) = |\psi(\vec{r})|e^{i\phi(\vec{r})} \tag{60}$$

Note: This is a solution to the Gross-Pitaevskii equation for a condensate in a macroscopic cylinder.

Current operator given by

$$\vec{J}(r) = \frac{\hbar}{2m * i}\left[\psi^\dagger(r)\vec{\nabla}\psi(r) - \psi(r)\nabla\psi^\dagger(r)\right] \tag{61}$$

Superfluid velocity is the gradient of the phase,

$$\vec{v} = \frac{\vec{J}}{density} = \frac{\hbar}{m*}\vec{\nabla}\phi \tag{62}$$

which we can show if we write out the expression for the current. The requirements for this to hold are: the wavefunction amplitude $|\psi(\vec{r})|$ must (as in this case) be Hermitian, and the phase $\phi(\vec{r})$ must depend on the position of the superfluid. Note that his makes the phase a local phase, and not global.

Question: When I calculate the current, I get exactly $\frac{\hbar}{m*}\vec{\nabla}\phi$. Where does the density come into this?

Attempt at answer: I think that $\vec{J}$ might in the first case actually be the current density.

Consequence of superfluid velocity the curl of a gradient is always zero. Thus

$$\vec{\nabla} \times \vec{v} = \vec{\nabla} \times \frac{\hbar}{m*} \vec{\nabla} \phi = 0 \tag{63}$$

For proof, just write the whole thing out and remember that partial derivatives commute.

Consequence: The superfluid is <u>irrotational</u>.

Mass of fermionic superfluids is $m* = 2m$ because we are dealing with Cooper pairs.

Circulation is the line integral around a closed curve of a velocity field. It is written

$$\Gamma = \oint_C \vec{v} \cdot \mathrm{d}\vec{l} \tag{64}$$

Essentially, it is a useful wauntity in hydrodynamics that helps us deal with various scenarios.

Vortex derivation As we start rotating the cylinder that contains the superfluid, the superfluid can remain stationary until we increase the angular velocity Ω to a certain point. A stationary liquid in a rotating container has high free energy (why?) thus at some point it is beneficial for the superfluid to introduce vortices.

Note that whenever we speak about a vortex, we speak about <u>probability waves</u>!

We know that if we walk around the condensate and return to the same point, the uniqueness (single-valuedness) of the wavefunction requires that the phase ϕ changes by and integer n of 2π. Therefore, if we try and calculate the circulation of the fluid, we find

$$\Gamma = \oint_C \vec{v} \cdot \mathrm{d}\vec{l} \tag{65}$$

Now, the vector $\vec{r}$ is a general displacement vector. If we confine the contour C to a 2D plane, we can limit ourselves to a 2D geometry in polar coordinates. Then, we find

$$\vec{\nabla} = \frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta} \tag{66}$$

where θ is the angle we are walking around. Subsequently, the quantum phase $\phi(\vec{r})$ is now a function of radius r and angle θ in some plane. We also find that

$$\mathrm{d}\vec{l} = r\mathrm{d}\theta\hat{\vec{\theta}} \tag{67}$$

This in turn means that

$$\oint_C \vec{v}\cdot\mathrm{d}\vec{v} = \frac{\hbar}{m*}\oint\left(\frac{\partial\phi(r,\theta)}{\partial r}\hat{\vec{r}} + \frac{1}{r}\frac{\partial\phi(r,\theta)}{\partial\theta}\hat{\vec{\theta}}\right)\cdot r\mathrm{d}\vec{\theta} \tag{68}$$

$$= \frac{\hbar}{m^*}\oint\frac{\partial\phi(r,\theta)}{\partial\theta}\mathrm{d}\theta \tag{69}$$

$$= \frac{\hbar}{m^*}\left(\phi(r,2\pi) - \phi(r,0)\right) \tag{70}$$

$$= \frac{\hbar}{m^*}2\pi n \tag{71}$$

$$= \frac{h}{m^*}n \tag{72}$$

This equation satisfies the situation where we have zero circulation, $\Gamma = 0$ which occurs for $n = 0$. However, we can also have finite circulation, which corresponds to irrotational vertices (with zero curl). <u>The circulation is quantised</u>.

Thus, whereas the overall curl of the liquid $\vec{\nabla}\times\vec{v}$ must be zero, there can still be a local non-zero $\vec{v}$. For $\vec{v}$ to be non-zero, we require a change in ϕ. When we have a vortex, there has been a change in ϕ. In a 2D plane, this occurs at a single point (the axis of the vortex). However, since we can't have a discontinuous flux (which means that $\vec{\nabla}\phi$ would go to infinity, we require that the <u>density</u> at that point also goes to infinity. That is, we know that

$$I = nv \tag{73}$$

The current must remain finite (maybe some kind of current conservation argument?) and therefore the density has to compensate for the velocity diverging.

Irrotational vortex A particle trapped in this vortex will go around the vortex, but will not spin. Therefore, this vortex will have no curl. Any particle trapped within the vortex will follow horizontal flow lines. If we know the circulation of the irrotational vortex, we can infer some other physical quantities.

The tangential component of the probability wave velocity around the vortex is

$$u_\theta = \frac{\Gamma}{2\pi} \tag{74}$$

and the angular momentum per unit mass relative to the vortex axis is therefore constant

$$ru_\theta = \frac{\Gamma}{2\pi} \tag{75}$$

This is how a superfluid can carry angular momentum.

Difference between superfluidity and BEC is that

- BEC is a property of the ground state. Superfluidity is a propery of excited states.
- Ideal BECs are <u>not</u> superfluid
- Not all of e.g. He needs to be in the ground state for superfluidity to occur.

1.8.1 Landau criterion for superfluidity

Landau criterion states that there exists a maximum velocity above which the fluid flow becomes dissipative and loses its superfluid nature.

Landau criterion derivation Consider a superfluid with velocity $\vec{v}$ relative to some container. We create an excitation in the superfluid of momentum $\hbar\vec{k}$. This can be done by introducing an obstacle or similar.

In the lab rest frame, creating this excitation costs the potential and kinetic energy given by

$$E_k + \hbar\vec{k} \cdot \vec{v} \tag{76}$$

The second term is the kinetic energy, since $\hbar\vec{k}$ is a momentum. The expression is minimised for momenta with negative $\vec{k}$ relative to $\vec{v}$. For large enough k, the cost of the excitation is negative, and therefore favourable.

Thus, for any k, the critical velocity is given by

$$v_c = \min_k \frac{E_k}{\hbar k} \tag{77}$$

The nature of k depends on the excitation.

Critical velocity in BCS limit is given by

$$v_{c,BCS} = \frac{\Delta}{\hbar k_F} \tag{78}$$

where Δ is the superconductor gap.

Critical velocity interpretation when an object moves through the superfluid faster than v_c, it will break Cooper pairs. For BECs however, where we deal with tightly bound bosons, we require more energy and so those liquids are usually stabler. Thus for large k

$$v_{c,BEC} = ck \tag{79}$$

where c is the speed of sound.

1.8.2 Superfluid response functions

Redefinition of superfluid The Landau criterion doesn't apply to complex spectra. Thus, we define the superfluid density is defined in terms of linear response theory.

Response function written $\chi_{ij}(q)$ determines what value the current $J_i(q)$ takes from a small perturbation $f(q)$ where q is momentum. That is, the response function maps linear perturbations onto a current. The curent is given by

$$J_i(q) = \sum_k \frac{k_i}{m} \psi^\dagger_{k+q/2} \psi_{k-q/2} \tag{80}$$

General χ_{ij} for isotropic system in the limit of $q \to 0$ is

$$\chi_{ij}(q) = \chi_T \left(\delta_{ij} - \frac{q_i q_j}{q^2} \right) + \chi_L \frac{q_i q_j}{q^2} \tag{81}$$

Superfluid part given by

$$\chi_S = \chi_L - \chi_T \tag{82}$$

1.9 Experimental systems

Ultracold alkali gases Alkali gases can be lasercooled to near 0K. This enables them to form superfluids. Through the hyperfine interaction, we can manipulate the interaction properties of the atoms through Feshback resonance via magnetic fields.

Feshback resonance is achieved when one internal degree of freedom for a many-body system disappears. It occurs when the energy of a bound state of an interacomit potential is equal to the kinetic energy of a colliding pair of atoms.

1.9.1 Excitons and Polaritons

Note on this section: I have not added many details for this section.

Temperature-mass relationship We know that

$$T \propto \frac{1}{m} \tag{83}$$

Therefore, light particles don't need very low temperatures, which makes them good candidates.

Excitons are bound pairs of electrons and holes. They are not stable quasiparticles since they can recombine. This happens when the electron goes back to the conduction band by emitting a photon.

Exciton Hamiltonian given by

$$H = E_g - \frac{\hbar^2 \nabla_e^2}{2m_c} - \frac{\hbar^2 \nabla_h^2}{2m_v} - \frac{e^2}{4\pi\epsilon |r_e - r_h|} \tag{84}$$

The last term is a Coulomb interaction. This has hydrogen-like solutions.

Question: what are m_c and m_v?

Attempt at answer: Could it be that since we are dealing with electrons in either the conduction band or valence band (since we are in a alkali metal which has bands) that the masses refer to the electrons in respective bands? Maybe this is the case, but they don't usually have different mass, do they?

Answer: Yep, electrons in different bands have different effective mass. It all depends on the curvature of the band.

Hamiltonian for many excitons near the band gap is given by

$$H = \sum_k \left(\epsilon_c(k) a_{ck}^\dagger a_{ck} + \epsilon_v(k) a_{vk}^\dagger a_{vk} \right) + \frac{1}{2} \sum_q \left(V_q^{ee} \rho_q^e \rho_{-q}^e + V_q^{hh} \rho_q^h \rho_{-q}^h - 2V_q^{eh} \rho_q^e \rho_{-q}^h \right)$$

where

$$\rho_q^e = \sum_k a_{ck+q}^\dagger a_{ck} \tag{85}$$

$$\rho_q^h = \sum_k a_{vk-q} a^\dagger_{vk} \tag{86}$$

Note that the Hamiltonian incorporates interactions for pairs with q and $-q$ momenta.

The potential is the Coulomb interaction:

$$V(q) = \frac{e^2 4\pi}{\epsilon q^2} \tag{87}$$

Question: Thie doesn't really look like the Coulomb interaction that we are used to.

Attempt at answer: This could just be rewriting the form of the interaction. If q^2 if the wavevector, with

$$q = \frac{2\ pi}{\lambda} \tag{88}$$

we get modifications of the potential. Would need to write this out.

Exciton condensation Electron-hole pairs are similar to Cooper pairs, but they don't have to have opposite spin because the electron and the hole occupy different states anyway.

Exciton creation operator acting on the ground state $|\Omega\rangle$ gives

$$|\Phi\rangle = \frac{1}{\sqrt{V}} \sum_k \phi_{1s}(k) c^\dagger_k h^\dagger_{-k} |\Omega\rangle \tag{89}$$

where $\phi_{1s}(k)$ is the Fourier transform of the 1s wavefunction. That is, we create one hole and one electrom from the vacuum.

Exciton condensate wavefunction is

$$|\Psi\rangle = \mathcal{N} \exp\left[\sum_k \lambda_k a^\dagger_{ck} a_{vk}\right] \tag{90}$$

BEC and BCS crossover can be done with excitons by changing the density. This is good because changing and controlling the interactions (which we might do for cold atoms with the Feshbach resonance) is hard.

Microcavity-polaritons are the normal modes resulting from the strong coupling between quantum well (QW) excitons and cavity photons.

Polariton modes can be found by solving the Schrodinger equation for exciton and photon fields.

$$i\partial_t \begin{pmatrix} \psi_X \\ \psi_C \end{pmatrix} = H_0 \begin{pmatrix} \psi_X \\ \psi_C \end{pmatrix} \tag{91}$$

Polariton Hamiltonian given by

$$H_0 = \begin{pmatrix} \omega_X^0 - i\kappa_X & \Omega_R/2 \\ \Omega_R/2 & \omega_C(-i\nabla) - i\kappa_C \end{pmatrix} \tag{92}$$

Here, κ_X and κ_C are the decay rates of exciton and photon, and Ω_R is the Rabi frequency. Note that we have set $\hbar = 1$.

Finding the eigenstates We can diagonalise the Hamiltonian using a unitary transformation (no need for a Bogoliubov transformation since we are not concerned with preserving any commutator relations) to find the eigenstates. They are made of an equal mixture of light and matter.

We call them <u>upper excitons</u> and <u>lower excitons</u>.

Exciton-Exciton interaction is non-linear, and thus gives rise to some fundamental property of polaritons. It can be written as

$$H_{XX} = \frac{1}{2A} \sum_{k,k',q} V_q \psi^*_{X,k+q} \psi^*_{X,k'-q} \psi_{X,k} \psi_{X,k'} \tag{93}$$

where V_q is the effective interaction potential.

Polariton condensate is best thought of as the steady-state solution for a cavity and excitons. It balances pumping of photons and photon decays (escaping from the cavity).

1.10 Quantum Simulators

1.10.1 Hubbard Model

Confining atoms We can make use of the Stark shift, which is the electric analogue of the Zeeman shift. Using the dipole approximation, the potential is given by

$$V_{dip} = -\frac{1}{2}\vec{d} \cdot \vec{E} \propto \mathrm{Re}(\alpha) E^2 \tag{94}$$

where α is the complex polarisability. The sign also depends on α.

We can control α by detuning to the red or blue sideband. Thus, the laser influences the potential felt by the atoms.

Polarisability is the tendency of a charge distribution to be distorted fom its normal shape by an external electric field.

Question: I haven't seen this used in the dipole approximation before, but then I haven't worked much with the Stark shift....

Creating an optical lattice can be done by using counterpropagating lasers. They will interfere and form a $\sin^2$ interference pattern. This works from 1D to 3D by adding more lasers.

Optical lattice wavefunction is similar to the Bloch wavefunction.

$$\phi_n(\vec{p}) = e^{i\vec{p}\cdot\vec{r}} u_n(\vec{r}) \tag{95}$$

Low energy lattice Hamiltonian away from Feshbach resonance and at low temperatures, we have

$$H = -\sum_{\langle ij \rangle} J \left(b_i^\dagger b_j + b_j^\dagger b_i \right) \tag{96}$$

where the summation is over nearest-neighbour. J is the hoping term which is determined by wavefunction overlap at the lattice sites. Note that this Hamiltonian assumes no interaction between the trapped particles.

We have ngeative term because this is a binding energy imparted by the laser light.

Bose-Hubbard Hamiltonian By adding interactions between atoms on the same lattice site, we get

$$H = -J \sum_{\langle ij \rangle} \left(b_i^\dagger b_j + b_j^\dagger b_i \right) - \sum_{\mu} n_i + \frac{1}{2} \sum_i U n_i (n_i - 1) \tag{97}$$

U is the strength of on-site interactions.

To this Hamiltonian, we could add

- Repulsion between atoms in neighbouring lattice sites

- Forbidden double-occupation
- Limit the number of atoms per site by limiting μ

Bose Hubbard model parameter is U/J, which is the relative strength of on-site repulsion J and kinetic energy U.

Changing the parameter changes the nature of the system. For example, we can induce phase transitions.

Question: I might have gotten this wrong since J should be the attractive on-site potential.

Weak interactions for a small U will cause the system to behave like bosons in free space. The condensate fraction decreases with increasing U.

Strong interaction limit for $U \to \infty$ we have no BEC fraction and all atoms are just localised in their individual sites. This is the Mott-sinsulator state.

Mott insulator is a material which should conductor eletricity, but which is actually an insulator, especially at low temperatures. This is due to electron-electron interactions.

Tuning inter-particle interactions can be done by using Feshbach resonance to tune the scattering length of the atoms. We achieve this by changing the external magnetic field.

Imaging after tuning th eHamiltonian, we wish to measure the positions of the atoms to find out how they behave under the dynamics. This can be done by shining a near-resonant laser onto the atoms so that they flouresce.

Simulating magnetic interactions Starting from some unit filled Mott insulator, applying a potential gradient means shiting neighbouring sites by energy E. This increases the probability of atoms tunnelling between sites. This affects the tunnelling rate of the atom in the target site. This is described by antiferromagnetic Ising type interaction.

1.11 Notes from the slides

1.11.1 BCS Theory

For some reason, the treatment here is different to what there is in the notes. I think what is different is the way we perform the transformation.

Hamiltonian with the approximations is given by

$$H - \epsilon_F N = \sum_{\vec{k}} \begin{pmatrix} c^\dagger_{k\uparrow} & c_{-k\downarrow} \end{pmatrix} \begin{pmatrix} \xi_k & -\Delta \\ -\Delta & -\xi_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} + \sum_k \xi_k - \frac{\Delta^2}{V_0}\Omega \tag{98}$$

where

$$\xi_k = \epsilon_k - \epsilon_F \tag{99}$$

Question: Does here, ϵ_F mean k_F?

Order parameter given by

$$\Delta = -\frac{V_0}{\Omega} \sum_k \langle \psi | c_{-k\downarrow} c_{k\uparrow} | +psi \rangle \tag{100}$$

Mean-field Hamiltonial diagonalisation can be done with

$$c^\dagger_{k\uparrow} = u_k \gamma^\dagger_{k\uparrow} + v_k \gamma_{-k\downarrow} \tag{101}$$

$$c_{-k\downarrow} = -v_k \gamma^\dagger_{k\uparrow} + u_k \gamma_{-k\downarrow} \tag{102}$$

For the anti-commutator to hold, we require

$$v_k^2 + v_k^2 = 1 \tag{103}$$

Proof:

$$\begin{aligned}
\{c_k, c_k^\dagger\} &= \{u_k \gamma_{k\uparrow} + v_k \gamma^\dagger_{-k\downarrow}, u_k \gamma^\dagger_{k\uparrow} + u_k \gamma_{-k\downarrow}\} && (104)\\
&= \{u_k \gamma_{k\uparrow}, u_k \gamma^\dagger_{k\uparrow}\} + \{u_k \gamma_{k\uparrow}, v_k \gamma^\dagger_{-k\downarrow} && (105)\\
&+ \{v_k \gamma_{-k\downarrow}, u_k \gamma_{k\uparrow}\} + \{v_k \gamma_{-k\downarrow}, v_k \gamma^\dagger_{-k\downarrow}\} && (106)\\
&= u_k^2 + v_k^2 && (107)\\
&= 1 && (108)
\end{aligned}$$

because we see that

$$\{\gamma_{k\uparrow}, \gamma^\dagger_{k\uparrow}\} = 1 \tag{109}$$

$$\{\gamma_{k\uparrow}, \gamma^{\dagger}_{-k\downarrow}\} = 0 \tag{110}$$

Question: Do we have to take into account the spin of the fermion when we write down these relations?

Attempt at answer: The spin should really only come in when we are creating states, and many of them. I believe that the only important part of the commutator relations come from the wavevector.

Diagonalisation condition is

$$2\xi_k u_k v_k + \Delta(v_k^2 - u_k^2) = 0 \tag{111}$$

Diagonal Hamiltonian is given by

$$H - \epsilon_F N = \sum_{k\sigma} E_k \gamma^{\dagger}_{k\sigma}\gamma_{k\sigma} + \sum_k (\xi_k - E_k) - \frac{\Delta^2}{V_0}\Omega \tag{112}$$

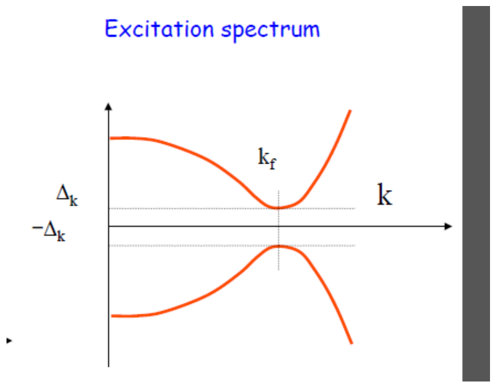
where the first term describes excitations and the second term describes pair-condensate energy.

Quasi-particle excitations given by

$$E_k = \sqrt{\xi_k^2 + \Delta^2} \tag{113}$$

The spectrum looks like the following figure.

Figure 1: Exciton Spectrum.



The energy gap Δ We must calculate the average energy meeded to create a Cooper pair.

$$\Delta = -\frac{V_0}{U} \sum_k \langle\psi| c_{-k\downarrow} c_{k\uparrow} |\psi\rangle \tag{114}$$

Note that in the undiagonalised Hamiltonian, this is exactly the off-diagonal element.

We can rewrite this by expanding into γ_k. We find that

$$\begin{aligned}
|\psi\rangle c_{-k\downarrow} c_{k\uparrow} |\psi\rangle &= u_k v_k \left(-\gamma_{k\uparrow}^\dagger \gamma_{k\uparrow} + \gamma_{-k\downarrow}\gamma_{-k\downarrow}^\dagger\right) &(115)\\
&+ u_k^2 \gamma_{-k\downarrow}\gamma_{k\uparrow} - v_k^2 \gamma_{k\uparrow}^\dagger \gamma_{-k\downarrow}^\dagger |\psi\rangle &(116)\\
&= |\psi\rangle u_k v_k \left(1 - \gamma_{k\uparrow}^\dagger \gamma_{k\uparrow} - \gamma - -k \downarrow^\dagger \gamma_{-k\downarrow}\right) |\psi\rangle &(117)
\end{aligned}$$

We also have that

$$\begin{aligned}
u_k v_k &= \frac{1}{2}\sqrt{\left(1 - \frac{\xi_k}{E_k}\right)\left(1 + \frac{\xi_k}{E_k}\right)} &(118)\\
&= \frac{1}{2}\sqrt{\left(\frac{E_k^2 \xi_k^2}{E_k^2}\right)} &(119)\\
&= \frac{1}{2}\frac{\Delta}{\sqrt{\xi_k^2 + \Delta^2}} &(120)
\end{aligned}$$

Note that this expression is incorrect in the notes!

Therefore, we find that

$$\begin{aligned}
\Delta &= -\frac{V_0}{U} \sum_k \langle\psi| c_{-k\downarrow} c_{k\uparrow} |\psi\rangle &(121)\\
&= -\frac{V_0}{2U} \frac{\Delta}{\sqrt{\xi_k^2 + \Delta^2}} \sum_k \left(1 - |\psi\rangle \sum_\sigma \gamma_{k\sigma}^\dagger \gamma_{k\sigma} |\psi\rangle\right) &(122)
\end{aligned}$$

Note that we have replaced $-k$ with k because it is coming up in the sum anyway.

Question: The only thing I couldn't find was where that factor of -1 comes from. Perhaps it has to do with convention of the transformation.

Physical origin of energy gap Since all Cooper pairs are correlated, It is then claimed that in order to break one Cooper pair, we must change the energy of all the other pairs. Basically, I think they are stabler than other constructions, and therefore there is a larger energy gap that is required in order to get to the next level.

From the Hamiltonian, it is clear that Δ is not dependent on the energy levels, determined by k.

Energy gap for superonductors take the universal value

$$\Delta = 1.764 k_B T_c \tag{123}$$

Quasi-particle expectation value at zero temperature at $T = 0$ we have

$$\langle\psi| \gamma_{k\sigma}^{\dagger} \gamma_{k\sigma} |\psi\rangle = 0 \tag{124}$$

which gives

$$1 = \frac{V_0}{2U} \sum_k \frac{1}{\sqrt{\xi_k^2 + \Delta^2}} \tag{125}$$

which we can solve for

$$\Delta \simeq 2\omega_D e^{-\frac{1}{N(\epsilon_F)V_0}} \tag{126}$$

2 Classical Josephson Junctions

Critical Temperature called T_C, the temperature below which resistivity goes to zero.

Element with highest T_C is niobium at 9K.

Compound with highest T_C is Mercury-Thallium-Barium-Copper-Oxide with 138 K.

Cooper pairs are pairs of electrons which form through a mutually attractive pairing interaction. This arises through electron-phonon interactions.

Spin of electron in a Cooper pair are always opposing because of the Pauli exclusion principle.

Meissner Effect describes the expulsion of magnetic fields from a superconductor. The screening happens through currents appearing in the superconductor, which counter the magnetic field. The superconductor is therefore a perfect diamagnet.

Magnetic susceptibility for a superconductor where we observe the Meissner effect is equal to

$$\frac{dM}{dH} = -1 \tag{127}$$

Here, M is the magnetisation of the material, which is the magnetic dipole moment per unit volume, and H is the magnetic field strength.

Penetration depth is the characteristic length scale for which the magnetic field dies down in the superconductor. The magnetic field decreases exponentially.

Destruction of superconductivity occurs at some critical field strength H_C, where the material will become resistive and allow magnetic fields into its interior.

Critical current called J_C exists, above which the material becomes resistive.

Conditions for superconductivity All quantities must be below T_C, H_C and J_C for superconductivity to occur.

Distance between Cooper pairs can be macroscopic. Therefore, many other Cooper pairs can reside within the same area. This causes the electrons to move coherently.

Wavefunction for electrons in a superconductor can be written down with a single pair wavefunction

$$\tilde{\psi}(\vec{r}) = |\psi(\vec{r})|e^{i\theta(\vec{r})} \tag{128}$$

Origin of zero resistance is the fact that macroscopic electron pairs behave coherently. If an electron pair scatters off a phonon, the other Cooper pairs will move to compensate the phase difference.

Superconducting coherence length usually written ξ, and is derived from the Cooper pair length. It can also be thought of as the characteristic exponent of the variations of density of the superconducting component. In BCS theory, it is given by

$$\xi = \frac{\hbar v_f}{\pi \Delta} \tag{129}$$

where v_f is the Fermi velocity and Δ is the superconducting energy gap.

Also, it can be found that

$$\xi \propto \frac{1}{T_C} \tag{130}$$

which is bad news for high temperature superconductors.

One consequence of the coherence of the electrons is that we can find many quantum phenomena in superconductors, such as emulating the Young's slit experiment.

Defects in a superconductor do not influence the behaviour as long a they are smaller than the superconducting coherence length ξ.

Number density of superconducting electrons is given by the probability of detecting an electron at position $\vec{r}$, so that

$$n_s(\vec{r}) = |\psi(\vec{r})|^2 \tag{131}$$

Superconducting order parameter is the absolute amplitude of the wavefucntion, $|\psi|$. It provides a measure of 'how strong' the superconductivity in a sample is. It is related to the density of Cooper pairs.

Magnetic flux through superconducting loop is quantised. This follows from the phase coherence. Boundary conditions stipulate that the phase must differ by 2π if you go around the loop once.

A more physical explanation is that the wavefunction must be unique, or single-valued everywhere in the material. Thus, going around the loop by 2π should yield the same value. This restricts the magnetic field that can go into the loop.

Flux quantum the minimal flux that is allowed through a superconducting loop, given by

$$\phi_0 = \frac{h}{2e} \tag{132}$$

2.0.1 Type I and type II superconductors

Type I superconductor reaches its normal state immediately after the external magnetic field H reaches H_C.

Type II superconductor reaches its normal state gradually and not abruptly once the critical field H_C has been applied. For fields below H_C, the magnetic flux creates discrete Abrikosov current vortices that shield the incoming field. Each vortex carries on thread of magnetic field with one flux quantum.

As the external magnetic field increases, more and more vortices appear. Finally, the whole sample is filled with vortices, and it becomes normal (resistive) again.

Abrikosov vortex appears in a Type II superconductor in an external field which is lower than H_C. A vortex can also be called a fluxon. The normal region in the vortex through which magnetic field can penetrate. Note that the vortex stretches all the way through the superconductor.

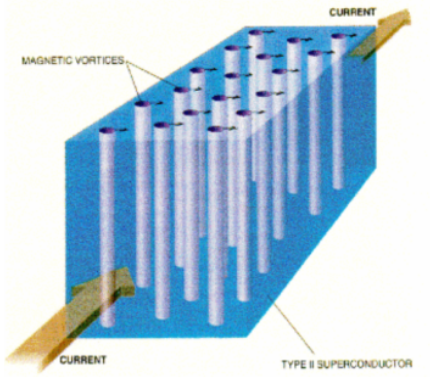
Question: Is this just the 'void' that appears in a superfluid?

The circulating part is the current, which screens exactly one ϕ_0.

Size of a vortex the radius is of the scale of the penetration depth λ. We know that magnetic fields are screened on a length scale of λ, and therefore we can think of the vortex as extending over this length and shielding one flux quantum along it.

The normal core of the vortex (where we have no shielding current) is of size ξ. This is because coherence can be maintained over this length scale.

Figure 2: Magnetic vortices.



Question: I don't think I have a good, intuitive grasp on this.

Attempt at answer:

Characterisation of Type I or Type II we note that:
- Type I: $\frac{\lambda}{\xi} < \frac{1}{\sqrt{2}}$
- Type II: all other cases.

Note that this means that we have no vortices for Type I superconductors, and for Type II the radius of the vortex, λ is larger than the vortex core, ξ, which makes sense.

2.0.2 The Energy Gap and Quasi Particles

Energy required to break Cooper pair is 2Δ. A Cooper pairs have energy lower than the Fermi energy.

Temperature dependence we find that
- At $T = 0$ there are no phonons, just Cooper pairs
- At $0 < T < T_C$ there is a small but finite phonon density which can break pairs.
- At $T = T_C$, all Cooper pairs have been broken

Note that we find that the density of Cooper pairs obeys

$$n \sim \frac{1}{T} \tag{133}$$

Breaking a Cooper pair creates two quasi-particles, not electrons. A quasi-particle in a superconductor is a superposition of an electron-like particle and a hole-like particle. Resulting quasi-electrons and quasi-holes have energy Δ more than the rest of the Cooper pairs. There is an energy difference of 2Δ between the 'electron' and the 'hole'.

The states which above T_C were within the energy gap get 'pushed out' to the edge of the gap. The result is a large peak in the quasi-electron and quasi-hole densities of states at the gap edge.

Density of quasi-particle states is given by

$$N_{qp}(E) = N_0 \mathrm{Re}\left(\frac{E}{\sqrt{E^2 - \Delta^2}}\right) \tag{134}$$

where N_0 is the single-spin density of states in the normal state.

Question: Derivation?

Electric field causes the quasi-particles, as they move through it, they cause dissipation. At DC, the quasiparticle current is always shorted by the non-dissipative Cooper pair current.

Question: I don't understand the last sentence.

Two-fluid model a model for describing the conductivity of a superconductor at finite temperatures. Here, current flow is modelled as carried by n_s paired electrons and n_n quasi-particles. For this model, we neglect pair-breaking, by assuming that

$$\frac{\omega}{2\pi} < \frac{2\Delta}{h} \tag{135}$$

The two-fluid model attempts to explain superconductivity in e.g. liquid He below the lambda point. The components are a normal fluid and a superfluid component. Together their sum makes up the total density.

Charge conservation the number of electrons in the normal state (above T_C) is given by

$$n = n_s + n_n \tag{136}$$

Electric field response the Cooper pairs move with velocity $\vec{v}_s$ and do not experience momentum-scattering collisions. The quasi-particles move with velocity $\vec{v}_n$ and experience momentum-scattering collisions at an average rate τ.

We find

$$2m\frac{d\vec{v}_s}{dt} = -2e\vec{E} \tag{137}$$

$$m\frac{d\vec{v}_n}{dt} = -e\vec{E} - m\frac{\vec{v}_n}{\tau} \tag{138}$$

where m is the mass of charge carriers, equal for paired electrons and quasiparticles. Note that the second term of the second equations implies a drag force.

Question: Where does the factor of 2 in the first equation come from?

Current density for Cooper pairs and quasi-particles given by

$$\vec{J} = -n_s e\vec{v}_s - n_n e\vec{v}_n \tag{139}$$

Conductivity we assume that the electric field is harmonic and of the form

$$\vec{E} = \vec{E}_0 e^{i\omega t} \tag{140}$$

then we define real and imaginary conductivities from the current,

$$\vec{J} = (\sigma_1 - i\sigma_2)\vec{E} \tag{141}$$

to obtain

$$\sigma_1 = \frac{n_n e^2 \tau}{(1+\omega^2\tau^2)m} \tag{142}$$

$$\sigma_2 = \frac{n_s e^2}{m\omega} + \frac{n_n e^2(\omega\tau)^2}{(1+\omega^2\tau^2)m\omega} \tag{143}$$

σ_1 describes the dissipative loss when a superconductor is excited by an electric field at frequency ω. It limits the Q factor of superconducting microwave resonators in Josephson qubit devices.

Question: This needs a derivation, will get to it...

2.0.3 Low T_C and high T_C superconductors

Low T_C superconductor are metallic elemental and alloy superconductors.

High T_C superconductors are usually made of compounds.

Differences between low T_Cs and high T_Cs are the following:

- Pairing mechanism in low T_C given by the BCS virtual phonon interaction. In high T_C, it is unknown.
- Properties of T_Cs are anisotropic.
- For copper compounds, electrons that move in parallel to the cuprate planes obtain a band gap that displays four-fold symmetry. Nodes appear in the value of the gap every 90 degrees. This energy gap depends on the momentum of the paired electrons.
- For high T_Cs, the properties can be tuned by doping, usually by increasing the number of oxygen atoms.
- The coherence length is small in high T_Cs. It is limited by the motion of fluxons in the Lorentz force.

Question: I presume then that the properties of low T_Cs are generally isotropic?

2.1 Classical Properties of Josephson Junctions

2.1.1 Basics

Josephson junction consists of two superconducting thin films with a barrier between them. The wavefunctions ψ_1 and ψ_2 in respective section are weakly coupled.

SIS Josephson junction is made of two superconducting thin films and an insulator.

SNS Josephson function is made of two superconducting thin films separated by a normal metal.

Quantum behaviour of the Josephson junction comes from the interference between the two wavefunctions

$$\phi = \phi_1 - \phi_2 \tag{144}$$

Josephson equations are

$$I_S = I_C \sin\phi \tag{145}$$

$$V = \frac{\hbar}{2e}\frac{\partial\phi}{\partial t} \tag{146}$$

Note that I_S is a supercurrent that flows through the device. This is a macroscopic quantum phenomenon.

Static solution can be obtained when

$$I_S < I_C \tag{147}$$

as we can then obtain the phase ϕ from the equations above. This means that ϕ is static, and the voltage V is zero.

Unstable solution if

$$I_S > I_C \tag{148}$$

we can obtain no stable value for ϕ and thus V is finite across the junction.

2.1.2 Resistively-Shunted junction model

Contributions to the current across a Josephson junction given by

- The Josephson supercurrent I_S
- A current from the capacitance induced over a Josephson function

$$I = C\frac{dV}{dt} \tag{149}$$

- A current formed from tunneling of quasi-particles across the junction.

Non-linear differential current equation one can combine the above contributions to the current to write

$$I = I_C \sin\phi + \frac{\hbar}{2eR}\frac{\partial\phi}{\partial t} + \frac{\hbar C}{2e}\frac{\partial^2\phi}{\partial t^2} \tag{150}$$

This can be solved numerically, but see washboard model for physical intuition.

2.1.3 Washboard model

Washboard model provides intuition about the non-linear behaviour of the supercurrent I across the Josephson junction.

Washboard potential from comparison with non-linear equation above, given by

$$U(\phi) = -\frac{\hbar}{2e}\left(I_C \cos\phi + I\phi\right) \tag{151}$$

Rolling ball analogy the behaviour of a ball of mass m that rolls along a periodic washboard describes the dynamics pretty well. Properties:

- Horizontal position of ball = phase difference ϕ
- Viscous drag force on ball = dissipative current

$$F_{drag} = \frac{1}{R}\left(\frac{\hbar}{2e}\right)^2 \frac{\partial \phi}{\partial t} \tag{152}$$

- Mass of ball = proportional to capacitance C of junction

$$m = \left(\frac{\hbar}{2e}\right)^2 C \tag{153}$$

- Velocity of ball = voltage across the junction

Natural oscillation frequency where the ball wiggles around in its local minima is given by

$$\omega_A = \omega_P \left(1 - \left(\frac{I}{I_C}\right)^2\right)^{1/4} \tag{154}$$

where ω_P is the plasma frequency.

Josephson plasma frequency is the oscillation frequency when the net current is zero, given by

$$\omega_P = \left(\frac{2eI_C}{\hbar C}\right)^{1/2} \tag{155}$$

Question: What here is actually oscillating?

Attempt at answer: We said that the horizontal position of the ball is the phase difference. Therefore, it is probably the phase difference that oscillates. Since the voltage depends on the phase,

$$V \propto \frac{d\phi}{dt} \tag{156}$$

Since this is an oscillator, we can find a pendulum analogy. Here, the kinetic energy depends on the velocity of the pendulum (just like the voltage depends on the phase change). Therefore, the voltage should display sinusoidal behaviour from the harmonic motion. However, it would be very small, and averages to zero because we are not (like in the pendulum case) dealing with a square v^2.

2.1.4 Washboard model $T = 0$

Here we assume negligible damping.

At $I = 0$ the ball is trapped in one of the minima, so

$$\frac{d\phi}{dt} = 0 \tag{157}$$

which implies that $V = 0$.

Josephson energy given by

$$E_J = \frac{\hbar I_C}{2e} \tag{158}$$

where $2E_J$ is the height of the potential barrier at $I = 0$.

Non-zero I means that we have a potential step

$$\Delta U \simeq 2E_J \left(1 - \frac{I}{I_C}\right)^{3/2} \tag{159}$$

which means that the ball can escape. With no damping, the ball keeps rolling down the washboard.

At critical current I_C we see that $\Delta U = 0$. This means that

$$\frac{d\phi}{dt} \neq 0 \tag{160}$$

throughout which implies finite V.

Question: So we cannot just assume that the ball increases in speed, because it feels a continuous acceleration? The analogy does perhaps not work this far.

Below the critical current I_C the ball can keep rolling. It keeps rolling until the current is reduced to zero.

Hysteretic since the ball keeps moving until $I = 0$ if we are decreasing the current, but doesn't start rolling until $I = I_C$ if increasing the current, we conclude that the system displays hysterecis.

2.1.5 Washboard model: $T > 0$ with negligible damping

Brownian motion the ball will undergo Brownian motion when it sits in the minima. The amplitude of the random walk increases with increasing T. This causes the ball to escape even at $I < I_C$. The average time it takes to escape is

$$\tau_{TA}^{-1} = \frac{\omega_A}{2\pi} e^{-\Delta U / k_B T} \tag{161}$$

Average number of escape attempts per second given by $\omega_A / 2\pi$.

Switching current written I_{sw} is the value of the current at which the ball escapes by thermal fluctuation. The same I_{sw} will not be found by repeating the measurement.

2.1.6 Washboard model: $T > 0$ with damping

Behaviour now the ball experiences a drag force and will stop after travelling over a set number of minima.

Quality factor called Q denotes the damping in the system. It is given by

$$Q = \omega_P RC \tag{162}$$

McCumber parameter another way to measure the quality of a Josephson junction. Written

$$\beta_C = Q^2 \tag{163}$$

Hystereic junctions are junctions with

$$Q > 3 \tag{164}$$

These junctions are also known as underdamped.

Non-hysteretic junctions have smaller Q and are called overdamped.

This is because when the junction is damped, decreasing I below I_c will cause the ball to stop because it experiences a drag force. Thus, the voltage goes to zero, and the junction follows a non-hysteretic behaviour.

2.2 Radio-Frequency effects in Josephson Junctions

External DC voltage which is constant means we can solve for the phase using one of the Josephson equations,

$$\phi(t) = \frac{2e}{\hbar}Vt + \phi_0 \tag{165}$$

Substitute this into the first equation to get the current:

$$I_S = I_C \sin\left(\frac{2e}{\hbar}Vt + \phi_0\right) \tag{166}$$

Supercurrent oscillation frequency for an external voltage given by

$$f_J = \frac{1}{2\pi}\frac{2e}{\hbar}V \tag{167}$$

Josephson effect is what we call the fact that the supercurrent flows across the Josephson junction.

2.2.1 Shapiro steps

External RF voltage if the voltage is no longer constant but time-dependent, $Vl(t)$, we integrate the Josephson equation to find

$$I_S(t) = I_C \sin\left(\int_0^t \frac{2e}{\hbar}v(t')\mathrm{d}t' + \phi_0\right) \tag{168}$$

If we choose to apply an external voltage of the following form,

$$v(t) = V + V_s \cos\omega_s t \tag{169}$$

we find that

$$I_S(t) = I_C \sin\left(\frac{2eV}{\hbar}t + \frac{2eV_s}{\hbar\omega_s}\sin(\omega_s t) + \phi_0\right) \tag{170}$$

where ω_s is the frequency of the voltage source.

Question: What is the V_s voltage?

Attempt at answer: This is maximum amplitude of the oscillating voltage source.

Bessel function are canonical solutions $y(x)$ of Bessel's differential equation:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0 \tag{171}$$

for a complex number α.

Occurrence of Shapiro steps current equation contains a Bessel function can be expanded to give

$$I_S(t) = I_C \sum_{n=-\infty}^{\infty} (-1)^n J_n\left(\frac{2eV_s}{\hbar\omega_s}\right) \sin\left((\omega_J - n\omega_s)t + \phi_0\right) \tag{172}$$

An interesting solution occurs when the source frequency is a multiple of the Josephson frequency $\omega_J = n\omega_s$. Then we see the characteristic current steps.

Shapiro steps for voltage standard by connecting N Josephson junctions together, we get very accurate Shapiro steps. They are given by

$$V = N\frac{\pi\hbar}{2} f_J \tag{173}$$

The voltage is thus entirely determined by fundamental constants and the f_J, which can be precisely measured. Note that this measurement has to be done with respect to some external clock reference frame. Thus I guess that this standard also depends on the time standard.

2.3 Josephson Junctions in a Magnetic Field

Effect of B field on superconducting ring is that we induce a phase change. Can show that

$$\Delta\phi = \frac{2e}{\hbar}\int_A^B \vec{A}\cdot \mathrm{d}\vec{l} \tag{174}$$

That is, the field must be quantised.

This can be done by converting the integral into a line integral around the ring, from 0 to 2π. Then, we use Stokes theorem to integrate over the magnetic field, which just gives us the total flux into the loop. Since the wavefunction phase must be unique, this must be a multiple of $n2\pi$.

Question: This explains why the flux is quantised, but not why the phase is affected by the vector potential in the first place.

Effect of B field on Josephson junction the total phase difference does not both depend on the difference between the two materials and the phase induced by the external magnetic field. We find

$$\phi = \phi_1 - \phi_2 + \frac{2e}{\hbar} \int \vec{A} \cdot \mathrm{d}\vec{l} \tag{175}$$

Note that we integrate from the interior of one of the superconducting films to the interior of the other one.

The effect is that we can very accurately control the junction by applying an external magnetic field.

Spatially varying magnetic field creates changes in the phase $\phi(x)$ that are in turn position dependent. We must rewrite

$$j_S(x) = j_c \sin \phi(x) \tag{176}$$

where $j_s(x)$ is the supercurent density per unit width in the x direction and j_c is the critical current density.

Calculating the total phase we can show that (by integrating over an area inside the superconductor) that

$$\phi(x) - \phi(0) = 2\pi \frac{\Phi(x)}{\Phi_0} \tag{177}$$

where

$$\Phi(x) = \frac{\Phi x}{w} \tag{178}$$

and where w is the width of the junction (I think). Question: Should Φ actually be Φ_0? Question: I do not know what the $\Phi(x)$ term is, unless it is just a dummy variable that we have treated.

Total current under external field is given by integrating the previous equation,

$$I_S = \int_{-w/2}^{w/2} \mathrm{d}x j_c \sin\left(\phi(0) + 2\pi \frac{\Phi(x)}{\Phi_0}\right) \tag{179}$$

$$= \frac{j_c w \Phi_0}{\pi \Phi} \sin\left(\phi(0)\right) \sin\left(\frac{\pi \Phi}{\Phi_0}\right) \tag{180}$$

Maximum superurrent that can flow through the function occurs when

$$\phi(0)0 \pm \frac{\pi}{2} \tag{181}$$

for which we obtain

$$I_C(\Phi) = I_c(0)\left|\frac{\sin\left(\frac{\pi\Phi}{\Phi_0}\right)}{\left(\frac{\pi\Phi}{\Phi_0}\right)}\right| \tag{182}$$

Critical current in zero magnetic field is given by

$$I_C(0) = j_c w \tag{183}$$

where again j_c is the critical current density and w is the width of the side of the junction that runs parallel to the barrier.

Flux required to suppress critical current is equal to one flux quantum ϕ_0.

Question: Is this really correct?

Attempt at answer: Before, we said that certain superconducting properties were the same for every superconductor. However, I would have thought that we require more than one flux quantum to cancel the critical current. Although I might misunderstand. I assume that the critical current is what we whip up in order to cancel out the magnetic field. So something is wrong here.

3 Fabrication and measurement of Josephson junctions

3.1 Fabrication of low T_C Josephson junctions

Double-angle shadow evaporation of aluminium films with aluminium oxide barriers exploits that evaporated Al atoms will travel in a straight line from the source to the substrate. Some of the substrate is protected by a mask through which the deposition is made. The mask is a suspended bi-layer resist mask fabricated using either photolithography or EBL.

The procedure is as follows:

- Apply the mask
- Evaporate the first Al layer onto the substrate
- Allow oxygen into chamber of oxidise the Al

- Rotate sample
- Evaporate other side of Al

'Whole wafer' Niobium Trilayer Junctions Evaporating Nb is impossible due to its high melting point. Therefore, we grow Nb by sputter deposition. Sputter deposition is done by accelerating argon ions towards a Nb target so that Nb atoms are emitted and form a film on a nearby substrate. Nb atoms will not travel in straight line because there is Argon in the chamber. Then, the AlO layer is grown on it without breaking vacuum.

3.2 Measurement of Josephson junctions

Typical measurement consists of controlling an external magnetic field and current while measuring the ingoing and outgoing current. It has to be done in cryogenic temperatures for the junction to remain superconducting.

Shielding the junctions must be shielded from stray electromagnetic fields (earth of mains) and from RF interference.

DC wiring to junction is done with twisted pairs (to cancel picked-up RF on both lines) or with mini coaxial cables. A coaxial cable has an inner conductor surrounded by a tubular insulating layer, surrounded by a tubular conducting shield.

Minimising RF coupling apart from coaxial cables and twisted pairs, we also add low temperature powder filters.

Current bias the junctions are usually current biased because they have very low resistance.

Four terminal configuration is used so that the potential difference across the wiring of the sample and any contact resistance is not measured.

Cryo-CMOS amplifiers don't work at the temperatures where Josephson junctions operate. Therefore, we need to use special heavily doped silicon amplifiers or SQUIDs to amplify the signal.

Magnetic shielding the full sample can be shielded by a superconducting shield or a material with a high permeability (mu metal) which collects the magnetic field lines inside itself.

4 Ed Roman's Lecture - SQUIDS

4.1 Introduction

SQUIDS are ultrasensitive magnetic flux sensors. They can be thought of as flux-to-voltage transducers.

Applications include measuring:

- very small magnetic signals
- measure/amplify small currents or voltages
- measure any quantity that can be related to magnetic flux

Flux quantisation occurs for a superconducting ring. Only an integer n number of flux quanta ϕ_0 are allowed into the ring .This follows from the fact that the phase θ of the wavefunction must be unique. It therefore changes by 2π (goes to its original value) after one turn around the ring. Can show

$$\Delta\theta = \frac{2e}{\hbar}\int_S \vec{B}\cdot \mathrm{d}\vec{S} = \frac{2e}{\hbar}\phi \tag{184}$$

where ϕ is the total magnetic flux.

Flux quantum defined by

$$\phi_0 = \frac{h}{2e} \tag{185}$$

Commercial Josephson junctions are fabricated using conventional photolithography in a cleanroom with the structure defined by etching. We want junctions to have

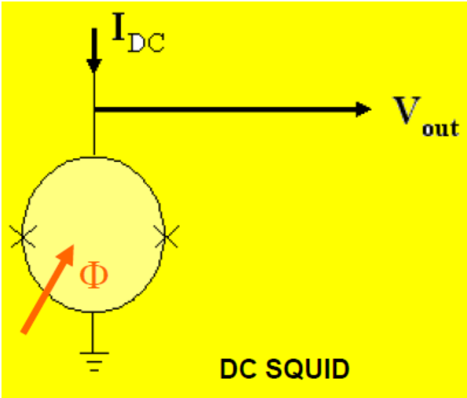
$$\beta_C < 1 \tag{186}$$

to avoid hysteresis. This is done by first fabricating them with $\beta_c > 1$ and then adding a metal resistor in parallel with the junction. This gives non-hyesteretic behaviour.

4.2 DC SQUIDS

DC SQUID consists of two Josephson junctions that are connected by a superconducting loop. We interface with the SQUID by inputting a current into the loop or applying a magnetic field perpendicular to the loop crosssection.

Figure 3: DC SQUID.



Josephson junction analogue One can think of a Josephson junction as a non-linear inductance which accumulates magnetic field energy when a current passes through it. However, the accumulated energy is not expressed in the magnetic field. It is rather thought of as the Josephson energy.

Measurement of DC SQUID is done by applying an external current I_b into the SQUID (see figure). This is also called the measurement current. We measure the voltage over the entire SQUID by comparing the output voltage V_{out} to ground.

Measured critical current is $2I_c$, where I_c is the critical current of one single junction. Since we have two junction, and the electrons have two paths to choose from, we can achieve double the critical current before superconductivity breaks down.

Circulating supercurrent will be given by

$$J = \frac{\phi_{ext}}{L} \tag{187}$$

where L is the inductance of the Josephson junction.

Current in the junctions is given by

$$I_{net} = \frac{I_b}{2} \pm J \tag{188}$$

depending on which way the magnetic flux comes in.

I_c **flux dependence** is as follows: as we increase ϕ, I_c decreases until we hit $\frac{1}{2}\phi_0$. Then it becomes more energetically favourable to change the direction of the supercurrent, after which I_c starts to increase again.

Note: I recall Ed saying that this was a fairly handwavy argument...

The reason for the $\frac{1}{2}\phi_0$ increase is because we gain $\frac{1}{2}\phi_0$ from the supercurrent and $\frac{1}{2}\phi_0$ from the measurement current. At all times, the flux in the loop remains quantised.

I_c **intuition and explanation** Why does the critical current oscillate? After all, the critical current I_c is a material property of the Josephson junction that doesn't really change. It changes when we add external resistor, or change the material in the junction.

However; as we apply flux, I_c changes. Let us analyse the situation. We have a measurement current I_b which is static and goes through both junctions. For flux ϕ_{ext} through the SQUID, there will be a screening current I_s that runs so that generated flux ϕ_s cancels or enhances ϕ_{ext} so that we have an integer number of flux quantua $n\phi_0$ in the superconducting loop.

We have that the current in each junction is

$$I_{junction} = \frac{I_b}{2} \pm I_s \tag{189}$$

We must now ask: what kind of current will we see oscillate in the Josephson junction due to

$$I = I_c' \sin\delta \tag{190}$$

This new I_c' will be given by

$$I_c' = I_c - \left(\frac{I_b}{2} \pm I_s\right) \tag{191}$$

So depending on the strength of the screening current, which depends on ϕ_{ext}, we will see a different I_c' for each junction. That is, in the end we have to keep track of three different currents with three diferent origins.

Maximum flux sensitivity is achieved by maximising the largest possible modulation of the SQUID current. Requirements:

- Loop inductance L as small as possible
- J as large as possible
- Screening parameter condition

$$\beta_L = 2\frac{LI_c}{\phi_0} \leq 1 \tag{192}$$

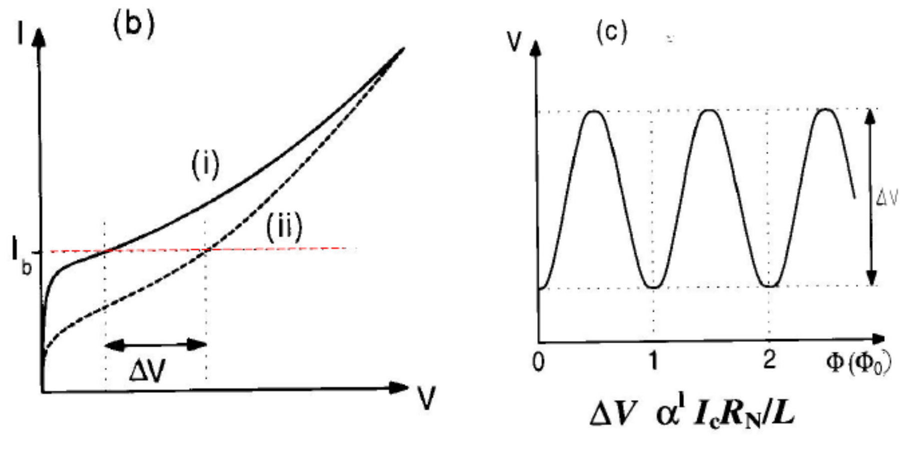
Screening parameter written β_L is related to the hysteresis of the SQUID.

Question: I don't yet know in which way.

Geometric loop inductance L generally scales with the circumference of the loop. Thus we often strive to increase the effective area of the SQUID.

Measuring magnetic flux can be done by fixing the current, usually just above $2I_c$ and measuring the change in voltage V. The voltage will oscillate as a function of ϕ. Note the hyesteretic behaviour in the graph below.

Figure 4: Constant current dependence.



The Voltage amplitude ΔV depends on how we set the external measurement current I_B. We want to hit the largest ΔV so that the oscillations are large. Then we obtain very fine readings for the magnetic flux.

Relative flux sensor Because of the periodicity, the squid cannot sense absolute magnetic field, just changes in magnetic field strength.

Voltage modulation depth ΔV is roughly given by

$$\Delta V \sim \frac{I_c R_N}{1+\beta_L} \tag{193}$$

where R_N is the junction normal resistance. For the ideal ΔV, we require,

$$\beta_L \leq 1 \tag{194}$$

Switching voltage is given by

$$V_{switch} = I_c R_N \tag{195}$$

Its maximum value is $\frac{1}{\Delta}$ where Δ is the energy gap of the superconductor.

Small signal mode is an operational mode of the SQUID where we measure $\phi \ll \phi_0$. It is also known as the voltage state. The procedure is as follows:

- Tune I_b to maximise ΔV
- Add a fixed B field to sit at a steep part of the curve, e.g. $\frac{\phi_0}{4}$
- Measure the transfer function $\delta V = V_\phi \delta\phi$ where

$$V_\phi = \frac{dV}{d\phi}\bigg|max \sim \frac{R_N}{L} \tag{196}$$

- Obtain change $\delta\phi$ in external flux

Question: What exactly is the switching voltage?

Noise in voltage state sets the sensitivity of the SQUID. At most frequencies, the limitation is thermal (Johnson) noise. At lower frequencies, the sensitivity is limited by $1/f$ noise. All frequencies suffer white noise.

Johnson-Nyquist noise is the electric noise generated by the thermal movement of electrons inside a conductor. The derivation of this noise is called the fluctuation-dissipation theorem.

The total amount of thermal noise measured across a resistor depends on the measurement bandwidth Δf of the instrument.

$1/f$ noise also called pink noise where the frequency spectrum has a power spectral density that is inversely proportional to the frequency of the signal.

White noise is a random signal with a constant power spectral density.

Power spectral density is the mean square voltage per unit bandwidth. That is

$$\frac{V^2}{Hz} \tag{197}$$

Theoretical minimum power spectral density for a SQUID has through simulations been found to be

$$S_V(f) \sim 16k_BTR_N \tag{198}$$

at the small signal operating point.

Flux noise the power spectral density of the equivalent flux noise is given by

$$S_\phi(f)\frac{\sim S_V(f)}{V_\phi^2} \simeq \frac{16k_BTL^2}{R_N} \tag{199}$$

Maximising sensitivity of the SQUID is done by minimising $S_\phi(f)$. This is done by

- Low L
- Low T
- High R_N
- Making sure other electronics are not interfering

Note that the increase in R_N will be compensated by the increase in V_ϕ^2.

Flux locked loop shifts the signal frequency to around a high frequency carrier. Instead of trying to measure the absolute frequency, we modulate it by a carrier. This method is useful when we wish to measure small modulations, e.g. a small signal in the background of the Earth's magnetic field.

Measuring small fields We increase sensitivity to flux measurements by increasing the loop area A. However, changes in magnetic field are given by

$$\delta B = \frac{\delta\phi}{A} \tag{200}$$

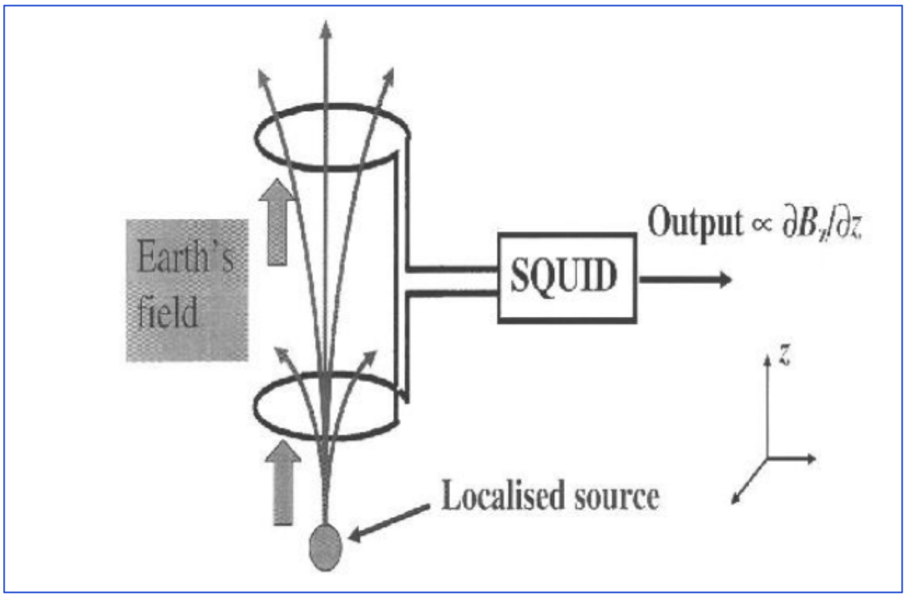
Thus large A will decrease the sensitivity.

Increasing catchment area by coupling the SQUID to a much larger pick-up loop. This is known as a SQUID magnetometer. This way, we keep the area of the loop small while still collecting a lot of magnetic field. For this to work, the entire coupled loop must be superconducting.

Inductively coupled magnetometer is a SQUID inductively coupled to a pickup loop via a multiturn input coil. This matches the inductance of the input coil and the SQUID.

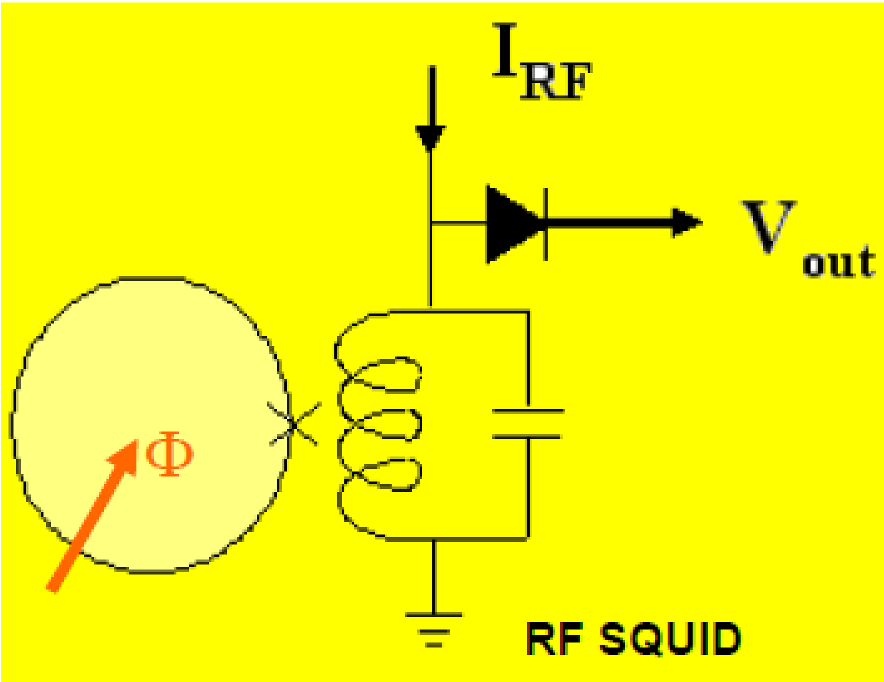
SQUID Gradiometer used to remove any background interference from large fields. Uses two rings where the difference in field lines will be due to their more rapid divergence, which means their source lies closer.

Figure 5: SQUID Gradiometer.



RF SQUID an RF SQUID is less sensitive compared to a DC SQUID but is cheaper and easier to manufacture in smaller quantities. The consist of a single Josephson junction in a loop measured indirectly by an RF antennae.

Figure 6: RF SQUID.



SQUID loop behaviour We have both flux quantisation in the loop and phase change δ around the junction. They are thus related by

$$\delta + 2\pi \frac{\phi_T}{\phi_0} = 2n\pi \tag{201}$$

where ϕ_T is the total flux in the loop.

The total flux ϕ_T is made up of the applied flux ϕ_a and the flux induced by the screening current,

$$J = I_0 \sin\delta \tag{202}$$

This leads to a relation

$$\phi_T = \phi_a - LI_0 \sin\left(2\pi \frac{\phi_T}{\phi_0}\right) \tag{203}$$

which can be either single valued or hysteretic.

Question: How is this derived? Where does the inductance L come into the picture?

Attempt at answer: Probably because we have the relation

$$L = \frac{\phi}{I} \tag{204}$$

That is, the inductance is a measure of how much flux links a circuit when a current flows.

RF SQUID parameter determines whether an RF SQUID is hysteretic or not. It is written

$$\beta_{rf} = \frac{2\pi L I_0}{\phi_0} \tag{205}$$

RF SQUID readout is done as follows:

- Couple the loop with the Josephson junction to an LCR oscillator circuit
- Drive LCR circuit close to its resonance by n RF current
- Measure the voltage oscillation of LCR circuit

Question: I am unsure about the last step - it is not clear from the notes.

Dissipative regime in this mode, the SQUID makes transitions between quantum states. It dissipates energy at a rate that is periodic in ϕ_a. This modulates the Q values of the LCR circuit. As a result, the RF voltage output V_{out} is periodic in ϕ_A.

Dispersive regime (not to be confused with the above!) in this non-hysteretic regime the RF SQUID behaves as a flux sensitive inductor. The inductance is linked to an emf, through

$$E = -\frac{d\phi}{dt} = -L\frac{d\phi}{dt} \tag{206}$$

Question: Something is wrong here. I have never seen Faraday's law with L in it.

Given the Josephson equations,

$$I = I_c \sin\phi \tag{207}$$

$$\frac{d\phi}{dt} = \frac{2e}{\hbar}V \tag{208}$$

We can relate the Josephson current to an inductance

$$L_J = \frac{\phi_0}{2\pi I_c \cos\phi} \tag{209}$$

And we obtain

$$L = \frac{\phi_0}{2\pi I_c \cos\phi} \tag{210}$$

Note that this makes the inductance non-linear.

Question: Derivation?

Dissipative mode readout the RF SQUID is read out by detecting the change in the Josephson junction. This is done by observing the change in Q due to the flux change.

5 John Fenton's Lectures - Three parts

5.1 Quantised Josephson junctions

This is the quantum part about the junctions.

Potentials We can think of the bumps in the washboard model as 1D potentials. Treat the electron as a harmonic oscillator and find energy levels (for a single mode)

$$E_n = \hbar\omega_0 \left(n + \frac{1}{2}\right) \tag{211}$$

Potentials in Josephson junction Potential is not harmonic but anharmonic - energy levels are not spaced evenly. It can be estimated by

$$U(\phi) = -\frac{h}{2e} I\phi - E_J \cos\phi \tag{212}$$

Note a static part (which depends on the external current I) and an oscillating part.

Analogues for the washboard are

- Tilt of washboard = external current bias I
- Bump amplitude = Josephson energy E_J
- Bump periodicity = phase difference $\phi = \phi_1 - \phi_2$

Lifetime of electron in potential is limited due to the electron quantum tunnelling out of the potential.

Voltage state means when the electron escapes the potential, and 'starts rolling' without stopping in an undamped system.

Consequence of tunnelling means that even as we go to 0 K, we see occasional current and voltage spikes once an electron manages to tunnel out.

Irradiation of microwaves increases the tunnel rate.

Question: Exactly which mechanism induces this? There are only graphs in the notes... :(

Josephson junction Hamiltonian for zero current bias given by

$$H = U(\phi) + 4E_c N^2 = U(\phi) - 4E_c \frac{\partial^2}{\partial \phi^2} \tag{213}$$

where

$$N = -i \frac{\partial}{\partial \phi} \tag{214}$$

and

$$E_c = \frac{e^2}{2C} \tag{215}$$

Question: Is this the conductance of the junction, or the combination addition of all capacitances in the circuit?

Attempt at answer: I think it is only the junction. In the notes, the capacitance is used as an analogy of the 'ball's mass'.

Note that ϕ and N are conjugate variables - they are not simultaneously observable.

Question: Why do we have a N^2 term here? I haven't really seen those around before.

$E_J \gg E_c$ regime will see the wavefunction $\psi(\phi)$ sharply peaked around specified ϕ. It implies

- Small $\delta\phi = \phi$ well defined
- Large $\Delta N = N$ not well defined

Question: Which wavefunction are we considering? The total wavefunction of both superconductors, or just one of them?

$E_J \ll E_c$ regime wavefunction $\psi(\phi)$ varies slowly over some range.

- Large $\Delta\phi = \phi$ not well defined
- Large $\Delta N = N$ well defined

I believe that these assertions come from when we would average over many turns of the wavefunction.

Deriving the Josephson Equations from the Schrodinger equation can be done by realising that tunnelling changes the number of Cooper pairs on one junction electrode, $N \to N+1$. This warrants the Hamiltonian

$$H_J = -\frac{E_J}{2} \sum_{-\infty}^{\infty} \left(|N\rangle \langle N+1| + |N+1\rangle \langle N| \right) \tag{216}$$

From this we can solve the Schrodinger equation as well as prove the complementarity of N and ϕ

5.2 Qubits form Josephson junctions

Motivation advantages and disadvantages are:

- Good for scaling
- Easy to couple
- Couples too easily to environment
- Short coherence times

Josephson junction qubits come in three varieties:

- Charge qubits
- Flux qubits
- Phase qubits

Qubit requirements are tunable separation between energy levels.

Phase qubits is a current-biased Josephson junction, operated in the zero voltage state with a non-zero current bias.

The energy levels are found in local minima of the washboard potential.

We manipulate the qubit by adding microwave pulses which cause Rabi oscillations.

Question: What does the box around the function mean? That we are in the zero voltage state?

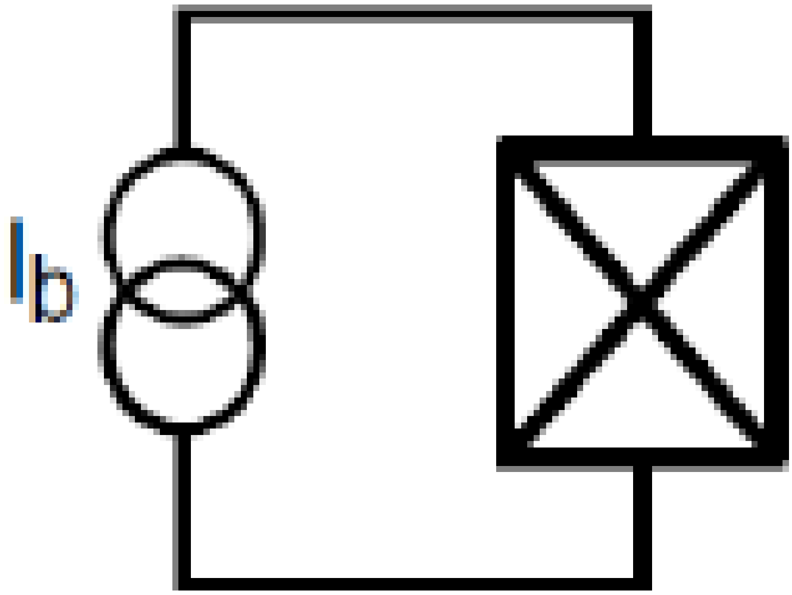
Flux qubits are Josephson junctions which will have a persistent current flowing continuously as external flux is applied. They are based on RF SQUIDS.

The qubit states are defined by direction of the circulating currents. We make use of the flux quantum to treat it as a qubit. When the external flux reaches $\frac{1}{2}\phi_0$ the two current energy levels are close together, which enables qubit operation.

Potential for RF SQUID is given by

$$U(\phi_T) = -E_J \cos\left(\frac{2\pi\phi_T}{\phi_0}\right) + \frac{(\phi_T - \phi_{ext})^2}{2L} \tag{217}$$

Figure 7: Phase qubit.



This potential always has two minima, like the wine-bottle potential. We can shift this potential intro three modes:

- $\phi_{ext} < \frac{\phi_0}{2}$ gives low left potential and high right potential
- $\phi_{ext} = \frac{\phi_0}{2}$ gives two equally deep potential
- $\phi_{ext} > \frac{\phi_0}{2}$ gives a high left potential and low right potential

Consequence of double minima there is tunnelling between the two minima, which hybridises the states of the potential wells. This causes them to move apart in energy. We can use this to sit ourselves at $\phi_{ext} = \frac{\phi_0}{2}$, then tunnelling causes there to e a ground state and an excited state. The ground state has a symmetric wavefunction while the excited state has an antisymmetric wavefunction.

Question: Exactly how is the degeneracy broken if we have tunnelling?

Flux qubit readout can be done by inductively couple the qubit to a neighbouring circuit. Then the current flow in the flux can be read out.

Coupling flux qubits can be done by invoking a flux coupling. This means that the state of one qubit will be altered by the state of the second qubit. Through this, we can invoke a CNOT gae.

Question: It is not clear how the coupling is performed.

Charge qubits the basis states of a charge qubit are charge state, representing the presence or absence of Cooper pairs. A charge qubit is formed by a a tiny superconducting island (known as a Cooper-pair box) coupled by a Josephson junction to a superconducting reservoir.

The potential is given by

$$E = -E_J \cos\phi + 4E_c(N - N_g)^2 \tag{218}$$

Central to the setup is a Cooper pair box, it consists of an island the charge states of which it involves a macroscopic number of conduction electrons.

Readout is done by electrostatically coupling the island to an extremely sensitive electrometer.

Coupling between qubits can be done by

- Flux qubits are coupled with an inductive coupling
- Charge qubits are coupled using a capacitive coupling

5.3 Metrological applications of superconducting currents

5.3.1 Voltage standard

Shapiro steps As already explored previously, the Shapiro steps appear when we apply an externally varying voltage $v(t)$. The resulting equation has Bessel function solutions, which ultimately tells us that the DC supercurrent is nonzero if

$$\omega_J = n\omega_S \tag{219}$$

where I think $\omega_J = \omega_P$ from previous notation (or perhaps ω_A, since we have a non-zero current?), and ω_S is the oscillation of the voltage source.

We find

$$V_n = n\frac{h}{2e}\nu \tag{220}$$

which implies

$$\Delta V = \frac{h}{2e}\nu \tag{221}$$

where ν is some kind of frequency instead of angular frequency, I think it is $2\pi\omega_S$.

External microwaves If we now apply external microwaves, that is, not an external flux ϕ_{ext} from a static magnetic field, but an EM field that varies in time, we see that the Shapiro steps appear as well.

Recall the IV diagram for a single junction, which shows the dicontinuous increase in voltage above I_c. For an applied 70 GHz filed, we see Shapiro steps again.

Question: This mechanism isn't clear to me. The previous treatment told us the Shapiro steps originate from a AC voltage applied to the junction. So why do external microwaves, unless they induce an AC voltage cause

5.3.2 Current Standard

Electron pump based on semiconductor 2D electron gases, is the leading candidate for a current standard.

It works by transferring one electron at a time. The entrance-gate frequency is modulated at a known frequency.

Relationship to frequency From the slides, I think it says that the relationship between a frequency standard and a current standard is not clear.

Conjugate variable relationship N and ϕ are conjugate variables. The derivative of N and ϕ are current and voltage.

I think the question is whether we can relate the voltage standard and get an equally accurate current standard.

Question: If they are conjugate variables, we should have an uncertainty relation between I and V. How would this affect things?

Charge-Flux duality it doesn't really matter in which representation we work in. The Hamiltonia

$$H = E_C(n - n_g)^2 - \left(\frac{E_J}{2} \sum_n |n+1\rangle \langle n| + |n\rangle \langle n+1| \right) \quad (222)$$

This is in the number representation. I don't know what n_g is here. Note that E_J is the coupling energy in this Hamiltonian, and that it couples various number states. These number states represent Cooper paris. I think that the coupling terms represent the tunnelling - a Hamiltonian that on one side of the junction adds another pair.

The same Hamiltonian can be written for an n number of flux eigenstates.

5.3.3 Conduction in nanowires

Connected superconductors can be connected by a nanowire. Recall the order parameter $|\psi|$, which is the phase coherence across the sample. The nanowire acts as a thin pipe between the superconductors.

Fluctuations the order parameter $|\psi|$ can fluctuate to zero in the wire. This requires energy.

Question: Why does this require energy? It is fair enough that you need energy for fluctuations, but what mechanism lies behind this?

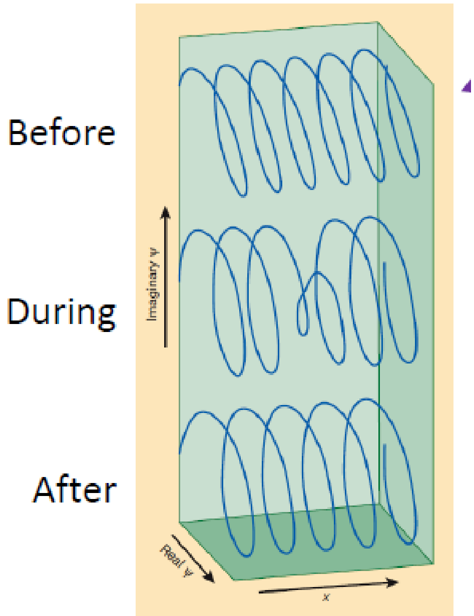
Phase slip can occur in the nanowire. This changes the difference between the phase in the two superconductors by a multiple 2π.

Phase slip junction An alternative to electronn pumps in an attempt to create a current standard. The phase slip junction will behave as a dual of the Josephson junction. I gues instead of Cooper pairs tunnelling through the junction, we have phase quanta changing.

The fluctuation of $|\psi|$ to zero means that the tiny part of the junction where this happens returns to the normal state. This blocks the supercurrent along the axis. This is what changes the phase by $2\pi n$.

Properties of phase-slip junction are exactly the same as the Josephson junction, but with V and I exchanged. This includes the

Figure 8: Phase slip.



IV hysteretic diagrams, Therefore, we should also be able to see Shapiro steps for the current.

Main challenge is to fabricate a nanowire that is good enough to test this. It should have a width of around 10 nm. It must also have resistance

$$R > R_Q = \frac{h}{4e^2} \tag{223}$$

in order to provide damping. We also need to be able to cool it down. People at Surrey are currently thing to do this.

Thermally activated phase slips similar to the thermally activated voltage. They cause short voltage pulses, and the nanowire to become slightly resistive.

5.4 Cryogenic Experimental Techniques

Necessary temperature scale for experiments with Josephson junctions is around mK.

A cooling progression cooling to increasingly low temperatures:

- Cool to low K with liquid N
- Cool with He-4
- Cool with He-3
- Cool by pumping, warmest He removed
- Activated C absorbation

Dilution fridge A dilution fridge works by

- Mixing together He-3 and He-4
- He-3 mixes into He-4
- He-3 evaporates first, lowering the concentration of He-3 in He-4
- New He-3 will mix into the He-4, increasing the entropy and thus decreasing the heat
- He-3 is pumped away

Cryocoolers use gas expansion to obtain cooling to cryogenic temperatures. Work supplied by gas compression at high temperatures. Cooling achieved by gas expansion in the cold part of the system.

We can also have coolers based on various cooling cycles.

5.4.1 Exclusion of EM noise

Sources of EM noise

- Electrical signals
- Magnetic signals
- EM signals

Noise protection done by

- Faraday cage
- Magnetic shielding techniques, sheilds, high mu-metal
- Twisting signal lines
- Filtering

Filtering We can filter signal lines to remove noise at frequencies outisde the range of interest.

Exclusion of EM noise We can maximise the signal-to-noise ratio by

- Low-T amplification. We can use high-electron-mobility-transistor
- Lock-in amplifier can source and detect signals at only a few Hz.

Question: I have no information about the first method. Can't find stuff online.

Exclusion of thermal noise can be done with

- Radiation shields
- Evacuated spaces for conduction/convection noise
- Conduction: thin-walled tubes, use non-metallic materials, wiring up a thermal anchor

Sources of noise in JJ systems We have several kinds of noise.

$1/f$ noise in Josephson junctions arises from two-level fluctuations. Electrons hop between localised trap states.

Consequences of noise are

- I_c fluctuations
- charge fluctuations
- magnetic flux fluctuations
- non-thermal quasi-particles caused by infrared photons
- trapped flux and trapped charge

Noise Hamiltonian for qubits can be written as

$$H = -\frac{1}{2}\left(\epsilon\sigma_z + \Delta\sigma_x\right) \tag{224}$$

Charge qubit noise Charge qubits are sensitive to charge noise. We can minimise the noise by operatoing at the most stable spot, which is closest in terms of the energy crossing

Phase qubit is sensitive to phase noise. It is also sensitive to charge noise and flux noise. It is difficult to reduce the noise, but can do so by tuning I/I_c.

Question: How do we have to tune the current? It is not clear.

5.4.2 Misc stuff

Quantronium tow Josephson junctions connected to a central island, with a larger Josephson junction connected in series with a loop.

Transmon qubits a transmon is a type of superconducting charge qubit. It is designed to have reduced sensitivity to charge noise. This is done by increasing E_J to E_c. Thus resulting energy levels have large spacings that are approximately independent the offset charge.

Fluxonium qubit is an array with a large E_J/E_c ratio. It can be used to provide a large inductance, $L_A > L_J$, where L_A is the inductance of an external coil.

D-Wave is using many many coupled flux qubits.