Superconductivity Long Notes

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Contents

1 Superconductivity

1.1 Introduction

Requirements for macroscopic quantum phenomena are

- Sufficiently low temperatures
- Particles must 'notice' that they are indistinguishable
- Non-localisation

Regime for BEC behaviour holds when the Compton wavelength is similar to the interparticle distance.

$$\lambda \geq n^{-1/3} \tag{1}$$

In thermal equilibrium, we find

$$p \sim (mk_B T)^{1/2} \tag{2}$$

and thus we require

$$k_B T \leq \frac{n^{2/3}\hbar^2}{m} \tag{3}$$

where m is the particle mass. Especially for heavy atoms, this means that the temperatures have to be sufficiently low.

Candidate systems include

- some atoms or molecules

- Liquid He-3 or He-4
- Dilute atomic alkali gases
- Electrons in metals
- Quasiparticles, such as excitons, polarisons and magnons
- Photons interacting via coupling to some matter component

1.2 Bosons: BEC Basics

Simple definition A BEC is a system that is in one single macroscopic state. For a wavefunction

$$\psi_s(r_1, r2, \dots r_N) \tag{4}$$

which is symmetric exchange, we get the density matrix

$$\rho(r, r') = N \sum_s p_s \int \mathrm{d}r_2 \mathrm{d}r_3 \dots \int r_N \Psi_s^*(r, r_2, \dots, r_N) \Psi_s(r', r_2, \dots, r_N) \tag{5}$$

This is just the inner product of $\left\langle \psi^\dagger(r)\psi(r') \right\rangle$. where I guess that $\psi(r)$ is a single-particle wavefunction. If we diagonalise, we get

$$\rho(r, r') = \sum_j n_j \chi_j^*(r) \chi_j(r') \tag{6}$$

Once one of the eigenvalues dominate, we say that we are in a macroscopic state.

Infinite and translationally invariant systems gives a single-particle wavefunction

$$\chi_j(r) = e^{i\vec{k}_j \cdot \vec{r}/\sqrt{V}} \tag{7}$$

Macroscopic eigenvalue N_0 for the $\chi_j(r)$ wavefunction given by

$$\lim_{|r-r'| \to \infty} \rho(r, r') = \frac{N_0}{V} \tag{8}$$

Question: If we are dealing with an infinite system, should not $V \to \infty$?

1.3 Non-interacting Bose gas

Statistics for bosons is given by

$$n_k = \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1} \tag{9}$$

where

$$\epsilon_k = \frac{\hbar^2 k^2}{2m} \tag{10}$$

is the energy of a free particle, and where μ is the chemical potential.

Chemical potential often written μ is the amount of energy required to add another paticle to the system.

Total number of particles given by integrating over all energies,

$$N = \int \mathrm{d}\epsilon \frac{g(\epsilon)}{e^{\beta(\epsilon_k - \mu)} - 1} \tag{11}$$

where

$$g(\epsilon) \propto \epsilon^{d/2-1} \tag{12}$$

is the density of states.

Question: I can't recall how the dimensionality of space comes into this.

Answer: We are concerned with the density of states, which necessarily is related to the dimensionality as we ask the question how many states we can fit in per unit volume.

Requirement for well-defined n_k is

$$\mu \leq 0 \tag{13}$$

Maximum number of particles the largest possible value of μ is $\mu = 0$, thus

$$N_{max} = \int \mathrm{d}\epsilon \frac{\sqrt{\epsilon}}{e^{\beta\epsilon_k} - 1} \tag{14}$$

If we add more particles, we cannot describe the statistics by the Bose-Einstein distribution anymore.

Macroscopic occupancy instead, while we are below T_c, any new addition will occupy a single state

$$N = N_0 + N_{max} \tag{15}$$

Question: I thought we were talking about states and not number. However, could this be that below the temperature, we just treat the new particles as one single particle?

1.4 Weakly interacting dilute Bose gas

Full Hamiltonian is given by

$$H = \sum_{i=1}^{N} \left[-\frac{\hbar \nabla_i^2}{2m} + V(r_i) \right] + \sum_{i<j} U\delta(r_i - r_j) \tag{16}$$

Variational Principle is the way of finding the minimum energy. We start with a guess for a ground state, then work out the energy. From there, we look at which parameters go into the wavefunction, the minimise the energy from there.

In the context of superconductivity, the variational principle can be used to find the ground state by minimising the free energy F.

Free energy is given by

$$F = N \int \mathrm{d}^3 r \left[\frac{\hbar^2}{2m} |\nabla \chi|^2 + (V(r) - \mu) |\chi|^2 + \frac{UN(N-1)}{2} |\chi|^4 \right] \tag{17}$$

Derivation of the Gross-Pitaevskii equation We wish to derive the equation of motion for the system with the potential above by minimising the free energy. First, we rescale the wavefunction

$$\chi = \sqrt{N} \psi(r) \tag{18}$$

We obtain

$$F = \int \mathrm{d}^3 r \left[\frac{\hbar^2}{2m} |\nabla^2 \psi^2| + (V(r) - \mu) + \frac{U(N-1)}{2} |\psi|^4 \right] \tag{19}$$

This quantity is at its minimum where the derivative of the quantity inside the integral is zero. Differentiating with respect to ψ gives

$$\frac{\hbar^2}{2m} \nabla^2 \psi + (V(r) - \mu) |\psi|^2 + U(N+1) |\psi|^3 = 0 \tag{20}$$

This is a bit like a non-linear Schrodinger equation.

Chemical potential for translationary invariant systems A translationary invariant system has,

$$\psi = \sqrt{\frac{N}{V}} \tag{21}$$

then solving the Gross-Pitaevskii is given by

$$\mu = \frac{UN}{V} \tag{22}$$

Time dependent Gross-Pitaevskii equation we can combine the above with the TDSE to get

$$i\hbar \frac{d\psi(r,t)}{dt} = \left[-\frac{\hbar^2 \nabla^2}{2m} + V(r) + U|\psi(r,t)|^2 \right] \psi(r,t) \tag{23}$$

BEC ground state is characterised by its average number of particles. It can be either a state with well-defined particle number or well-defined phase.

$$|\Psi_N\rangle = \left(\psi_0^\dagger\right)^N |\Omega\rangle \tag{24}$$

$$|\Psi_\lambda\rangle = e^{\lambda\psi_0^\dagger - |\lambda|^2/2} |\Omega\rangle \tag{25}$$

The first one is a Fock state and the second is a coherent state. The coherent state has lower energy, and thus the ground state does not have well-defined particle number.

1.5 Fragmented BEC

Fragmented BEC for two closely separated states χ_0 and χ_1 can be described by

$$|\Psi\rangle = \left(\psi_0^\dagger\right)^{N-M} \left(\psi_1^\dagger\right)^M |\Omega\rangle \tag{26}$$

Interaction energy difference given by

$$\frac{U}{2V} \left((N-M)(N-M-1) + M(M-1) + 4(N-M)M\right) \simeq \frac{U}{2V} \left[N^2 + 2M(N-M)\right]$$

We have made the approximation of removing one N from the expression above.

1.6 Fluctuations

Guassian state is given by

$$\psi(r) = \frac{1}{\sqrt{V}} \sum_k \psi_k e^{i\vec{k}\cdot\vec{r}} \tag{27}$$

BEC Hamiltonian the Hamiltonian for the weakly interacting gas is, in 2nd quantisation,

$$H - \mu N = \sum_k (\epsilon_k - \mu)\psi_k^\dagger \psi_k + \sum_{k,k',q} \frac{U}{2V} \psi_{k+q}^\dagger \psi_{k'-q}^\dagger \psi_{k'} \psi_k \tag{28}$$

Quasi-particle Hamiltonian the previous Hamiltonian can be diagonalised into

$$H = \sum_k \xi_k b_k^\dagger b_k + \frac{\xi_k - \epsilon_k - \mu}{2} \tag{29}$$

Boboliubov transformation is a unitary transformation of a commutator relation, such as $[a, a^\dagger] = 1$. Note that in addition to the hyperbolic transformation matrix, we can also add a phase $e^{i\theta}$ to the transformation, since this will be cancelled out.

More stuff needed here

1.7 Fermions - BCS-BEC Crossover

BCS theor describes bonding between fermions and can explain most superconductors. BCS works well when

- Transitions temperatures must be a small fraction of the Fermi degeneracy temperature T_F
- Normal state is of a normal metal
- Superconductor transition does not occur close to an additional phase transition
- The symmetry of the Cooper pairs is the same as the crystal lattice or s-wave in the case of amorphous materials
- The principal mechanism of Cooper pairing is the exchange of virtual phonons

Fermi Temperature written T_F is defined as

$$T_F = \frac{E_F}{k_B} \tag{30}$$

where E_F is the Fermi energy. T_F is the temperature at which thermal effects are comparable with quantum efects associated with Fermi statistics.

Virtual phonons are needed to describe static lattice deformations surrounding a perturbation. They are a field/theoretical tool required for the transfer of forces and such.

Cooper problem phonon cause a weak attraction between the electrons.

Cooper pairs are pairs of electrons with momentum with $|k\rangle$ and $-|k\rangle$. Only a few electrons close to the Fermi energy get paired. A Cooper pair is of bosonic nature since it is made up of an even number of fermions.

Each electron has spin-1/2, which when added can have either spin 1 or spin 0.

Pairing instability is based on seeing the superconducting transition as an instabilityof the normal state at some finite temperature. It highlights the potential importance of pairing fluctuations that occur in the normal state even before the critical transition.

BCS instability not entirely sure what this is. I think it is adding an instability to the BEC treatment that we performed before.

1.8 Fermion ground-state wavefunction

Initial assumptions we assume that a number of N electrons in an alkali metal will have $N/2$ be in the spin-up state and $N/2$ be in the spin-down state. The electrons are moving in a free 3D space subject to attractive interactions.

BCS postulate the ground state is a condensate of a macroscopic number of fermion pairs. Any two different species of particles form a singlet.

BCS ground state is given by

$$\Psi_N = \mathcal{A}\left\{\varphi|(\vec{r}_1 - \vec{r}_2|)\chi_{12}\ldots\phi(|\vec{r}_{N-1} - \vec{r}_N|)\chi_{N-1,N}\right\} \tag{31}$$

$\mathcal{A}$ is an operator that anti-symmetrises for any exchange of coordinates or spins for any of the electrons.

BCS ground state in second quantisation is also written

$$|\Psi\rangle_N = \int\prod_i \mathrm{d}^3 r_i \varphi(\vec{r}_1 - \vec{r}_2)\Psi^\dagger_\uparrow(\vec{r}_1)\Psi^\dagger_\downarrow(\vec{r}_2)\ldots\varphi(\vec{r}_{N-1} - \vec{r}_N)\Psi^\dagger_\uparrow(\vec{r}_{N-1})\Psi^\dagger_\downarrow(\vec{r}_N)\,|0\rangle \tag{32}$$

Or, more succinctly,

$$|\Psi\rangle_N = (b^\dagger)^{N/2}\,|0\rangle \tag{33}$$

Field operator lol... is given by

$$\Psi^\dagger_\sigma(\vec{r}) = \sum_k c^\dagger_{k\sigma}\frac{e^{-i\vec{k}\cdot\vec{r}}}{\sqrt{V}} \tag{34}$$

Creation operator for Fermion pair given by

$$b^\dagger = \sum_k \varphi_k v^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \tag{35}$$

1.9 Superfluidity

Superfluid wavefunction is macroscopic and complex, and is given by

$$\psi(\vec{r}) = |\psi(\vec{r})|e^{i\phi(\vec{r})} \tag{36}$$

Current operator given by

$$\vec{J}(r) = \frac{\hbar}{2m*i}\left[\psi^\dagger(r)\vec{\nabla}\psi(r) - \psi(r)\nabla\psi^\dagger(r)\right] \tag{37}$$

Superfluid velocity is the gradient of the phase,

$$\vec{v} = \frac{\vec{J}}{density} = \frac{\hbar}{m*}\vec{\nabla}\phi \tag{38}$$

2 Classical Josephson Junctions

Critical Temperature called T_C, the temperature below which resistivity goes to zero.

Element with highest T_C is niobium at 9K.

Compound with highest T_C is Mercury-Thallium-Barium-Copper-Oxide with 138 K.

Cooper pairs are pairs of electrons which form through a mutually attractive pairing interaction. This arises through electron-phonon interactions.

Spin of electron in a Cooper pair are always opposing because of the Pauli exclusion principle.

Meissner Effect describes the expulsion of magnetic fields from a superconductor. The screening happens through currents appearing in the superconductor, which counter the magnetic field. The superconductor is therefore a perfect diamagnet.

Magnetic susceptibility for a superconductor where we observe the Meissner effect is equal to

$$\frac{dM}{dH} = -1 \tag{39}$$

Here, M is the magnetisation of the material, which is the magnetic dipole moment per unit volume, and H is the magnetic field strength.

Penetration depth is the characteristic length scale for which the magnetic field dies down in the superconductor. The magnetic field decreases exponentially.

Destruction of superconductivity occurs at some critical field strength H_C, where the material will become resistive and allow magnetic fields into its interior.

Critical current called J_C exists, above which the material becomes resistive.

Conditions for superconductivity All quantities must be below T_C, H_C and J_C for superconductivity to occur.

Distance between Cooper pairs can be macroscopic. Therefore, many other Cooper pairs can reside within the same area. This causes the electrons to move coherently.

Wavefunction for electrons in a superconductor can be written down with a single pair wavefunction

$$\tilde{\psi}(\vec{r}) = |\psi(\vec{r})|e^{i\theta(\vec{r})} \tag{40}$$

Origin of zero resistance is the fact that macroscopic electron pairs behave coherently. If an electron pair scatters off a phonon, the other Cooper pairs will move to compensate the phase difference.

8Superconducting coherence length] usually written χ, and is derived from the Cooper pair length. It can also be thought of as the characteristic exponent of the variations of density of the superconducting component. In BCS theory, it is given by

$$\chi = \frac{\hbar v_f}{\pi \Delta} \tag{41}$$

where v_f is the Fermi velocity and Δ is the superconducting energy gap.

Also, it can be found that

$$\xi \propto \frac{1}{T_C} \tag{42}$$

which is bad news for high temperature superconductors.

One consequence of the coherence of the electrons is that we can find many quantum phenomena in superconductors, such as emulating the Young's slit experiment.

Defects in a superconductor do not influence the behaviour as long a they are smaller than the superconducting coherence length ξ.

Number density of superconducting electrons is given by the probability of detecting an electron at position $\vec{r}$, so that

$$n_s(\vec{r}) = |\psi(\vec{r})|^2 \tag{43}$$

Superconducting order parameter is the absolute amplitude of the wavefucntion, $|\psi|$. It provides a measure of 'how strong' the superconductivity in a sample is. It is related to the density of Cooper pairs.

Magnetic flux through superconducting loop is quantised. This follows from the phase coherence. Boundary conditions stipulate that the phase must differ by 2π if you go around the loop once.

Flux quantum the minimal flux that is allowed through a superconducting loop, given by

$$\phi_0 = \frac{h}{2e} \tag{44}$$

2.0.1 Type I and type II superconductors

Type I superconductor reaches its normal state immediately after the external magnetic field H reaches H_C.

Type II superconductor reaches its normal state gradually and not abruptly once the critical field H_C has been applied. For fields below H_C, the magnetic flux creates discrete Abrikosov current vortices that shield the incoming field. Each vortex carries on thread of magnetic field with one flux quantum.

As the external magnetic field increases, more and more vortices appear. Finally, the whole sample is filled with vortices, and it becomes normal (resistive) again.

Abrikosov vortex appears in a Type II superconductor in an external field which is lower than H_C. A vortex can also be called a fluxon. The normal region in the vortex through which magnetic field can penetrate.

Question: Is this just the 'void' that appears in a superfluid?

The circulating part is the current, which screens exactly one ϕ_0.

Size of a vortex the radius is of the scale of the penetration depth λ. We know that magnetic fields are screened on a length scale of λ, and therefore we can think of the vortex as extending over this length and shielding one flux quantum along it.

The normal core of the vortex (where we have no shielding current) is of size ξ. This is because coherence can be maintained over this length scale.

Question: I don't think I have a good, intuitive grasp on this.

Attempt at answer:

Characterisation of Type I or Type II we note that:
- Type I: $\frac{\lambda}{\xi} < \frac{1}{\sqrt{2}}$
- Type II: all other cases.

2.0.2 The Energy Gap and Quasi Particles

Energy required to break Cooper pair is 2Δ.

Temperature dependence we find that
- At $T = 0$ there are no phonons, just Cooper pairs
- At $0 < T < T_C$ there is a small but finite phonon density which can break pairs.
- At $T = T_C$, all Cooper pairs have been broken

Note that we find that the density of Cooper pairs obeys

$$n \sim \frac{1}{T} \tag{45}$$

Breaking a Cooper pair creates two quasi-particles, not electrons. A quasi-particle in a superconductor is a superposition of an electron-like particle and a hole-like particle. Resulting quasi-electrons and quasi-holes have energy Δ more than the rest of the Cooper pairs. There is an energy difference of 2Δ between the 'electron' and the 'hole'.

The states which above T_C were within the energy gap get 'pushed out' to the edge of the gap. The result is a large peak in the quasi-electron and quasi-hole densities of states at the gap edge.

Density of quasi-particle states is given by

$$N_{qp}(E) = N_0 \mathrm{Re}\left(\frac{E}{\sqrt{E^2 - \Delta^2}}\right) \tag{46}$$

where N_0 is the single-spin density of states in the normal state.

Question: Derivation?

Electric field causes the quasi-particles, as they move through it, they cause dissipation. At DC, the quasiparticle current is always shorted by the non-dissipative Cooper pair current.

Question: I don't understand the last sentence.

Two-fluid model a model for describing the conductivity of a superconductor at finite temperatures. Here, current flow is modelled as carried by n_s paired electrons and n_n quasi-particles. For this model, we neglect pair-breaking, by assuming that

$$\frac{\omega}{2\pi} < \frac{2\Delta}{h} \tag{47}$$

The two-fluid model attempts to explain superconductivity in e.g. liquid He below the lambda point. The components are a normal fluid and a superfluid component. Together their sum makes up the total density.

Charge conservation the number of electrons in the normal state (above T_C) is given by

$$n = n_s + n_n \tag{48}$$

Electric field response the Cooper pairs move with velocity $\vec{v}_s$ and do not experience momentum-scattering collisions. The quasi-particles move with velocity $\vec{v}_n$ and experience momentum-scattering collisions at an average rate τ.

We find

$$2m\frac{d\vec{v}_s}{dt} = -2e\vec{E} \tag{49}$$

$$m\frac{d\vec{v}_n}{dt} = -e\vec{E} - m\frac{\vec{v}_n}{\tau} \tag{50}$$

where m is the mass of charge carriers, equal for paired electrons and quasiparticles. Note that the second term of the second equations implies a drag force.

Question: Where does the factor of 2 in the first equation come from?

Current density for Cooper pairs and quasi-particles given by

$$\vec{J} = -n_s e\vec{v}_s - n_n e\vec{v}_n \tag{51}$$

Conductivity we assume that the electric field is harmonic and of the form

$$\vec{E} = \vec{E}_0 e^{i\omega t} \tag{52}$$

then we define real and imaginary conductivities from the current,

$$\vec{J} = (\sigma_1 - i\sigma_2)\vec{E} \tag{53}$$

to obtain

$$\sigma_1 = \frac{n_n e^2 \tau}{(1 + \omega^2\tau^2)m} \tag{54}$$

$$\sigma_2 = \frac{n_s e^2}{m\omega} + \frac{n_n e^2 (\omega\tau)^2}{(1 + \omega^2\tau^2)m\omega} \tag{55}$$

σ_1 describes the dissipative loss when a superconductor is excited by an electric field at frequency ω. It limits the Q factor of superconducting microwave resonators in Josephson qubit devices.

Question: This needs a derivation, will get to it...

2.0.3 Low T_C and high T_C superconductors

Low T_C superconductor are metallic elemental and alloy superconductors.

High T_C superconductors are usually made of compounds.

Differences between low T_Cs and high T_Cs are the following:

- Pairing mechanism in low T_C given by the BCS virtual phonon interaction. In high T_C, it is unknown.
- Properties of T_Cs are anisotropic.
- For copper compounds, electrons that move in parallel to the cuprate planes obtain a band gap that displays four-fold symmetry. Nodes appear in the value of the gap every 90 degrees. This energy gap depends on the momentum of the paired electrons.
- For high T_Cs, the properties can be tuned by doping, usually by increasing the number of oxygen atoms.
- The coherence length is small in high T_Cs. It is limited by the motion of fluxons in the Lorentz force.

Question: I presume then that the properties of low T_Cs are generally isotropic?

2.1 Classical Properties of Josephson Junctions

2.1.1 Basics

Josephson junction consists of two superconducting thin films with a barrier between them. The wavefunctions ψ_1 and ψ_2 in respective section are weakly coupled.

SIS Josephson junction is made of two superconducting thin films and an insulator.

SNS Josephson function is made of two superconducting thin films separated by a normal metal.

Quantum behaviour of the Josephson junction comes from the interference between the two wavefunctions

$$\phi = \phi_1 - \phi_2 \tag{56}$$

Josephson equations are

$$I_S = I_C \sin\phi \tag{57}$$

$$V = \frac{\hbar}{2e}\frac{\partial\phi}{\partial t} \tag{58}$$

Note that I_S is a supercurrent that flows through the device. This is a macroscopic quantum phenomenon.

Static solution can be obtained when

$$I_S < I_C \tag{59}$$

as we can then obtain the phase ϕ from the equations above. This means that ϕ is static, and the voltage V is zero.

Unstable solution if

$$I_S > I_C \tag{60}$$

we can obtain no stable value for ϕ and thus V is finite across the junction.

2.1.2 Resistively-Shunted junction model

Contributions to the current across a Josephson junction given by

- The Josephson supercurrent I_S
- A current from the capacitance induced over a Josephson function

$$I = C\frac{dV}{dt} \tag{61}$$

- A current formed from tunneling of quasi-particles across the junction.

Non-linear differential current equation one can combine the above contributions to the current to write

$$I = I_C \sin\phi + \frac{\hbar}{2eR}\frac{\partial\phi}{\partial t} + \frac{\hbar C}{2e}\frac{\partial^2\phi}{\partial t^2} \tag{62}$$

This can be solved numerically, but see washboard model for physical intuition.

2.1.3 Washboard model

Washboard model provides intuition about the non-linear behaviour of the supercurrent I across the Josephson junction.

Washboard potential from comparison with non-linear equation above, given by

$$U(\phi) = -\frac{\hbar}{2e}\left(I_C \cos\phi + I\phi\right) \tag{63}$$

Rolling ball analogy the behaviour of a ball of mass m that rolls along a periodic washboard describes the dynamics pretty well. Properties:

- Horizontal position of ball = phase difference ϕ
- Viscous drag force on ball = dissipative current

$$F_{drag} = \frac{1}{R}\left(\frac{\hbar}{2e}\right)^2 \frac{\partial\phi}{\partial t} \tag{64}$$

- Mass of ball = proportional to capacitance C of junction

$$m = \left(\frac{\hbar}{2e}\right)^2 C \tag{65}$$

- Velocity of ball = voltage across the junction

Natural oscillation frequency where the ball wiggles around in its local minima is given by

$$\omega_A = \omega_P \left(1 - \left(\frac{I}{I_C}\right)^2\right)^{1/4} \tag{66}$$

where ω_P is the plasma frequency.

Josephson plasma frequency is the oscillation frequency when the net current is zero, given by

$$\omega_P = \left(\frac{2eI_C}{\hbar C}\right)^{1/2} \tag{67}$$

Question: What here is actually oscillating?

Attempt at answer: We said that the horizontal position of the ball is the phase difference. Therefore, it is probably the phase different that oscillates.

2.1.4 Washboard model $T = 0$

Here we assume negligible damping.

At $I = 0$ the ball is trapped in one of the minima, so

$$\frac{d\phi}{dt} = 0 \tag{68}$$

which implies that $V = 0$.

Josephson energy given by

$$E_J = \frac{\hbar I_C}{2e} \tag{69}$$

where $2E_J$ is the height of the potential barrier at $I = 0$.

Non-zero I means that we have a potential step

$$\Delta U \simeq 2E_J \left(1 - \frac{I}{I_C}\right)^{3/2} \tag{70}$$

which means that the ball can escape. With no damping, the ball keeps rolling down the washboard.

At critical current I_C we see that $\Delta U = 0$. This means that

$$\frac{d\phi}{dt} \neq 0 \tag{71}$$

throughout which implies finite V.

Below the critical current I_C the ball can keep rolling. It keeps rolling until the current is reduced to zero.

Hysteretic since the ball keeps moving until $I = 0$ if we are decreasing the current, but doesn't start rolling until $I = I_C$ if increasing the current, we conclude that the system displays hysterecis.

2.1.5 Washboard model: $T > 0$ with negligible damping

Brownian motion the ball will undergo Brownian motion when it sits in the minima. The amplitude of the random walk increases with increasing T. This causes the ball to escape even at $I < I_C$. The average time it takes to escape is

$$\tau_{TA}^{-1} = \frac{\omega_A}{2\pi} e^{-\Delta U/k_B T} \tag{72}$$

Average number of escape attempts per second given by $\omega_A/1\pi$.

Switching current written I_{sw} is the value of the current at which the ball escapes by thermal fluctuation. The same I_{sw} will not be found by repeating the measurement.

2.1.6 Washboard model: $T > 0$ with damping

Behaviour now the ball experiences a drag force and will stop after travelling over a set number of minima.

Quality factor called Q denotes the damping in the system. It is given by

$$Q = \omega_P RC \tag{73}$$

McCumber parameter another way to measure the quality of a Josephson junction. Written

$$\beta_C = Q^2 \tag{74}$$

Hystereic junctions are junctions with

$$Q > 3 \tag{75}$$

These junctions are also known as underdamped.

Non-hysteretic junctions have smaller Q and are called overdamped.

2.2 Radio-Frequency effects in Josephson Junctions

External DC voltage which is constant means we can solve for the phase using one of the Josephson equations,

$$\phi(t) = \frac{2e}{\hbar} V t + \phi_0 \tag{76}$$

Substitute this into the first equation to get the current:

$$I_S = I_C \sin\left(\frac{2e}{\hbar} V t + \phi_0\right) \tag{77}$$

Supercurrent oscillation frequency for an external voltage given by

$$f_J = \frac{1}{2\pi} \frac{2e}{\hbar} V \tag{78}$$

Josephson effect is what we call the fact that the supercurrent flows across the Josephson junction.

2.2.1 Shapiro steps

External RF voltage if the voltage is no longer constant but time-dependent, $Vl(t)$, we integrate the Josephson equation to find

$$I_S(t) = I_C \sin\left(\int_0^t \frac{2e}{\hbar} v(t') \mathrm{d}t' + \phi_0\right) \tag{79}$$

$$= I_C \sin\left(\frac{2eV}{\hbar} t + \frac{2eV_s}{\hbar \omega_s} \sin(\omega_s t) + \phi_0\right) \tag{80}$$

where ω_s is the frequency of the voltage source.

Also note that

$$v(t) = V + V_s \cos \omega_s t \tag{81}$$

Question: What is the V_s voltage?

Attempt at answer: This is maximum amplitude of the oscillating voltage source.

Bessel function are canonical solutions $y(x)$ of Bessel's differential equation:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0 \tag{82}$$

for a complex number α.

Occurrence of Shapiro steps current equation contains a Bessel function can be expanded to give

$$I_S(t) = I_C \sum_{n=-\infty}^{\infty} (-1)^n J_n\left(\frac{2eV_s}{\hbar\omega_s}\right) \sin\left((\omega_J - n\omega_s)t + \phi_0\right) \tag{83}$$

An interesting solution occurs when the source frequency is a multiple of the Josephson frequency $\omega_J = n\omega_s$. Then we see the

Shapiro steps for voltage standard by connecting N Josephson junctions together, we get very accurate Shapiro steps. They are given by

$$V = N\frac{\pi\hbar}{2}f_J \tag{84}$$

The voltage is thus entirely determined by fundamental constants and the f_J, which can be precisely measured.

2.3 Josephson Junctions in a Magnetic Field

Effect of B field on superconducting ring is that we induce a phase change. Can show that

$$\Delta\phi = \frac{2e}{\hbar}\int_A^B \vec{A}\cdot \mathrm{d}\vec{l} \tag{85}$$

That is, the field must be quantised.

Effect of B field on Josephson junction the total phase difference does not both depend on the difference between the two materials and the phase induced by the external magnetic field. We find

$$\phi = \phi_1 - \phi_2 + \frac{2e}{\hbar}\int \vec{A}\cdot \mathrm{d}\vec{l} \tag{86}$$

Note that we integrate from the interior of one of the superconducting films to the interior of the other one.

The effect is that we can very accurately control the junction by applying an external magnetic field.

Spatially varying magnetic field creates changes in the phase $\phi(x)$ that are in turn position dependent. We must rewrite

$$j_S(x) = j_c \sin\phi(x) \tag{87}$$

where $j_s(x)$ is the supercurent density per unit width in the x direction and j_c is the critical current density.

Calculating the total phase we can show that (by integrating over an area inside the superconductor) that

$$\phi(x) - \phi(0) = 2\pi \frac{\Phi(x)}{\Phi_0} \tag{88}$$

where

$$\Phi(x) = \frac{\Phi x}{w} \tag{89}$$

and where w is the width of the junction (I think). Question: Should Φ actually be Φ_0? Question: I do not know what the $\Phi(x)$ term is, unless it is just a dummy variable that we have treated.

Total current under external field is given by integrating the previous equation,

$$I_S = \int_{-w/2}^{w/2} \mathrm{d}x j_c \sin\left(\phi(0) + 2\pi\frac{\Phi(x)}{\Phi_0}\right) \tag{90}$$

$$= \frac{j_c w \Phi_0}{\pi\Phi} \sin\left(\phi(0)\right) \sin\left(\frac{\pi\Phi}{\Phi_0}\right) \tag{91}$$

Maximum superurrent that can flow through the function occurs when

$$\phi(0)0 \pm \frac{\pi}{2} \tag{92}$$

for which we obtain

$$I_C(\Phi) = I_c(0)\left|\frac{\sin\left(\frac{\pi\Phi}{\Phi_0}\right)}{\left(\frac{\pi\Phi}{\Phi_0}\right)}\right| \tag{93}$$

Critical current in zero magnetic field is given by

$$I_C(0) = j_c w \tag{94}$$

where again j_c is the critical current density and w is the width of the side of the junction that runs parallel to the barrier.

Flux required to suppress critical current is equal to one flux quantum ϕ_0.

Question: Is this really correct?

Attempt at answer: Before, we said that certain superconducting properties were the same for every superconductor. However, I would have thought that we require more than one flux quantum to cancel the critical current. Although I might misunderstand. I assume that the critical current is what we whip up in order to cancel out the magnetic field. So something is wrong here.

3 Fabrication and measurement of Josephson junctions

3.1 Fabrication of low T_C Josephson junctions

Double-angle shadow evaporation of aluminium films with aluminium oxide barriers exploits that evaporated Al atoms will travel in a straight line from the source to the substrate. Some of the substrate is protected by a mask through which the deposition is made. The mask is a suspended bi-layer resist mask fabricated using either photolithography or EBL.

The procedure is as follows:

- Apply the mask
- Evaporate the first Al layer onto the substrate
- Allow oxygen into chamber of oxidise the Al
- Rotate sample
- Evaporate other side of Al

'Whole wafer' Niobium Trilayer Junctions Evaporating Nb is impossible due to its high melting point. Therefore, we grow Nb by sputter deposition. Sputter deposition is done by accelerating argon ions towards a Nb target so that Nb atoms are emitted and form a film on a nearby substrate. Nb atoms will not travel in straight line because there is Argon in the chamber. Then, the AlO layer is grown on it without breaking vacuum.

3.2 Measurement of Josephson junctions

Typical measurement consists of controlling an external magnetic field and current while measuring the ingoing and outgoing current. It has to be done in cryogenic temperatures for the junction to remain superconducting.

Shielding the junctions must be shielded from stray electromagnetic fields (earth of mains) and from RF interference.

DC wiring to junction is done with twisted pairs (to cancel picked-up RF on both lines) or with mini coaxial cables. A coaxial cable has an inner conductor surrounded by a tubular insulating layer, surrounded by a tubular conducting shield.

Minimising RF coupling apart from coaxial cables and twisted pairs, we also add low temperature powder filters.

Current bias the junctions are usually current biased because they have very low resistance.

Four terminal configuration is used so that the potential difference across the wiring of the sample and any contact resistance is not measured.

Cryo-CMOS amplifiers don't work at the temperatures where Josephson junctions operate. Therefore, we need to use special heavily doped silicon amplifiers or SQUIDs to amplify the signal.

Magnetic shielding the full sample can be shielded by a superconducting shield or a material with a high permeability (mu metal) which collects the magnetic field lines inside itself.

4 Ed Roman's Lecture - SQUIDS

4.1 Introduction

SQUIDS are ultrasensitive magnetic flux sensors. They can be thought of as flux-to-voltage transducers.

Applications include measuring:

- very small magnetic signals

- measure/amplify small currents or voltages
- measure any quantity that can be related to magnetic flux

Flux quantisation occurs for a superconducting ring. Only an integer n number of flux quanta ϕ_0 are allowed into the ring .This follows from the fact that the phase θ of the wavefunction must be unique. It therefore changes by 2π (goes to its original value) after one turn around the ring. Can show

$$\Delta\theta = \frac{2e}{\hbar}\int_S \vec{B}\cdot \mathrm{d}\vec{S} = \frac{2e}{\hbar}\phi \tag{95}$$

where ϕ is the total magnetic flux.

Flux quantum defined by

$$\phi_0 = \frac{h}{2e} \tag{96}$$

Commercial Josephson junctions are fabricated using conventional photolithography in a cleanroom with the structure defined by etching. We want junctions to have

$$\beta_C < 1 \tag{97}$$

to avoid hysteresis. This is done by first fabricating them with $\beta_c > 1$ and then adding a metal resistor in parallel with the junction. This gives non-hyesteretic behaviour.

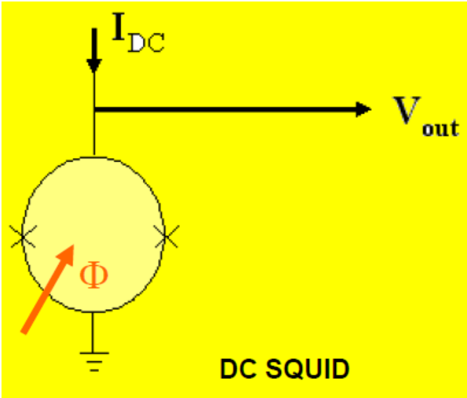
4.2 DC SQUIDS

DC SQUID consists of two Josephson junctions that are connected by a superconducting loop. We interface with the SQUID by inputting a current into the loop or applying a magnetic field perpendicular to the loop crosssection.

Josephson junction analogue One can think of a Josephson junction as a non-linear inductance which accumulates magnetic field energy when a current passes through it. However, the accumulated energy is not expressed in the magnetic field. It is rather thought of as the Josephson energy.

Measurement of DC SQUID is done by applying an external current I_b into the SQUID (see figure). This is also called the measurement current. We measure the voltage over the entire SQUID by comparing the output voltage V_{out} to ground.

Figure 1: DC SQUID.



Measured critical current is $2I_c$, where I_c is the critical current of one single junction. Since we have two junction, and the electrons have two paths to choose from, we can achieve double the critical current before superconductivity breaks down.

Circulating supercurrent will be given by

$$J = \frac{\phi_{ext}}{L} \tag{98}$$

where L is the inductance of the Josephson junction.

Current in the junctions is given by

$$I_{net} = \frac{I_b}{2} \pm J \tag{99}$$

depending on which way the magnetic flux comes in.

I_c flux dependence is as follows: as we increase ϕ, I_c decreases until we hit $\frac{1}{2}\phi_0$. Then it becomes more energetically favourable to change the direction of the supercurrent, after which I_c starts to increase again.

Note: I recall Ed saying that this was a fairly handwavy argument...

The reason for the $\frac{1}{2}\phi_0$ increase is because we gain $\frac{1}{2}\phi_0$ from the supercurrent and $\frac{1}{2}\phi_0$ from the measurement current. At all times, the flux in the loop remains quantised.

Maximum flux sensitivity is achieved by maximising the largest possible modulation of the SQUID current. Requirements:

- Loop inductance L as small as possible
- J as large as possible
- Screening parameter condition

$$\beta_L = 2\frac{LI_c}{\phi_0} \leq 1 \tag{100}$$

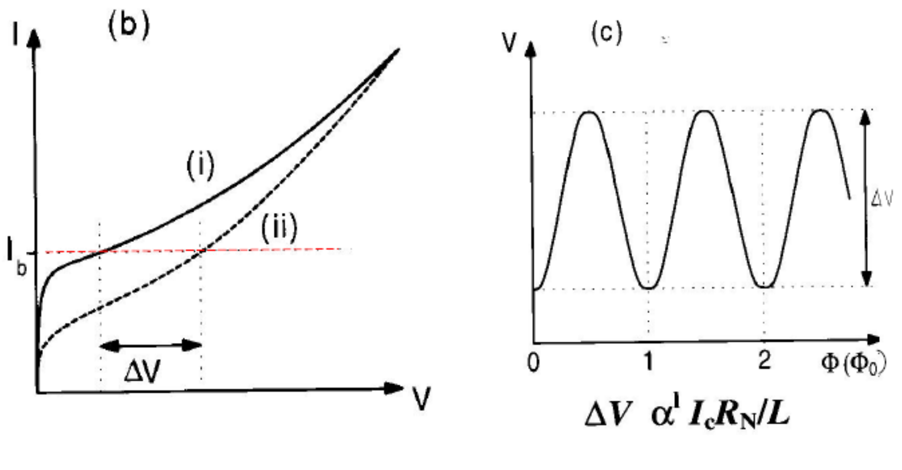
Screening parameter written β_L is related to the hysteresis of the SQUID.

Question: I don't yet know in which way.

Geometric loop inductance L generally scales with the circumference of the loop. Thus we often strive to increase the effective area of the SQUID.

Measuring magnetic flux can be done by fixing the current, usually just above $2I_c$ and measuring the change in voltage V. The voltage will oscillate as a function of ϕ. Note the hyesteretic behaviour in the graph below.

Figure 2: Constant current dependence.



The Voltage amplitude ΔV depends on how we set the external measurement current I_B. We want to hit the largest ΔV so that the oscillations are large. Then we obtain very fine readings for the magnetic flux.

Relative flux sensor Because of the periodicity, the squid cannot sense absolute magnetic field, just changes in magnetic field strength.

Voltage modulation depth ΔV is roughly given by

$$\Delta V \sim \frac{I_c R_N}{1+\beta_L} \tag{101}$$

where R_N is the junction normal resistance. For the ideal ΔV, we require,

$$\beta_L \leq 1 \tag{102}$$

Switching voltage is given by

$$V_{switch} = I_c R_N \tag{103}$$

Its maximum value is $\frac{1}{\Delta}$ where Δ is the energy gap of the superconductor.

Small signal mode is an operational mode of the SQUID where we measure $\phi \ll \phi_0$. It is also known as the voltage state. The procedure is as follows:

- Tune I_b to maximise ΔV
- Add a fixed B field to sit at a steep part of the curve, e.g. $\frac{\phi_0}{4}$
- Measure the transfer function $\delta V = V_\phi \delta\phi$ where

$$V_\phi = \frac{dV}{d\phi}\bigg|max \sim \frac{R_N}{L} \tag{104}$$

- Obtain change $\delta\phi$ in external flux

Question: What exactly is the switching voltage?

Noise in voltage state sets the sensitivity of the SQUID. At most frequencies, the limitation is thermal (Johnson) noise. At lower frequencies, the sensitivity is limited by $1/f$ noise. All frequencies suffer white noise.

Johnson-Nyquist noise is the electric noise generated by the thermal movement of electrons inside a conductor. The derivation of this noise is called the fluctuation-dissipation theorem.

The total amount of thermal noise measured across a resistor depends on the measurement bandwidth Δf of the instrument.

$1/f$ noise also called pink noise where the frequency spectrum has a power spectral density that is inversely proportional to the frequency of the signal.

White noise is a random signal with a constant power spectral density.

Power spectral density is the mean square voltage per unit bandwidth. That is

$$\frac{V^2}{Hz} \tag{105}$$

Theoretical minimum power spectral density for a SQUID has through simulations been found to be

$$S_V(f) \sim 16k_B T R_N \tag{106}$$

at the small signal operating point.

Flux noise the power spectral density of the equivalent flux noise is given by

$$S_\phi(f) \frac{\sim S_V(f)}{V_\phi^2} \simeq \frac{16k_B T L^2}{R_N} \tag{107}$$

Maximising sensitivity of the SQUID is done by minimising $S_\phi(f)$. This is done by

- Low L
- Low T
- High R_N
- Making sure other electronics are not interfering

Note that the increase in R_N will be compensated by the increase in V_ϕ^2.

Flux locked loop shifts the signal frequency to around a high frequency carrier. Instead of trying to measure the absolute frequency, we modulate it by a carrier. This method is useful when we wish to measure small modulations, e.g. a small signal in the background of the Earth's magnetic field.

Measuring small fields We increase sensitivity to flux measurements by increasing the loop area A. However, changes in magnetic field are given by

$$\delta B = \frac{\delta \phi}{A} \tag{108}$$

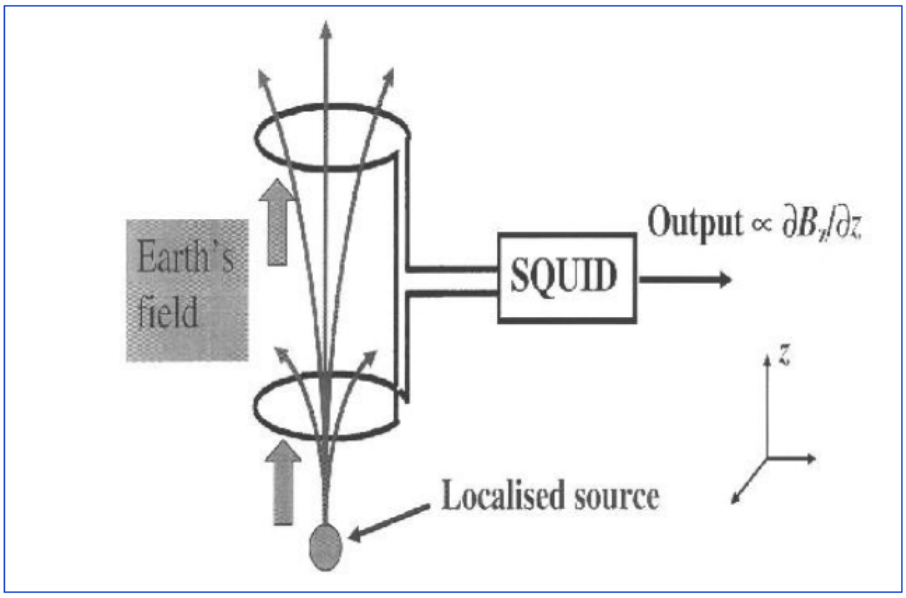
Thus large A will decrease the sensitivity.

Increasing catchment area by coupling the SQUID to a much larger pick-up loop. This is known as a SQUID magnetometer. This way, we keep the area of the loop small while still collecting a lot of magnetic field. For this to work, the entire coupled loop must be superconducting.

Inductively coupled magnetometer is a SQUID inductively coupled to a pickup loop via a multiturn input coil. This matches the inductance of the input coil and the SQUID.

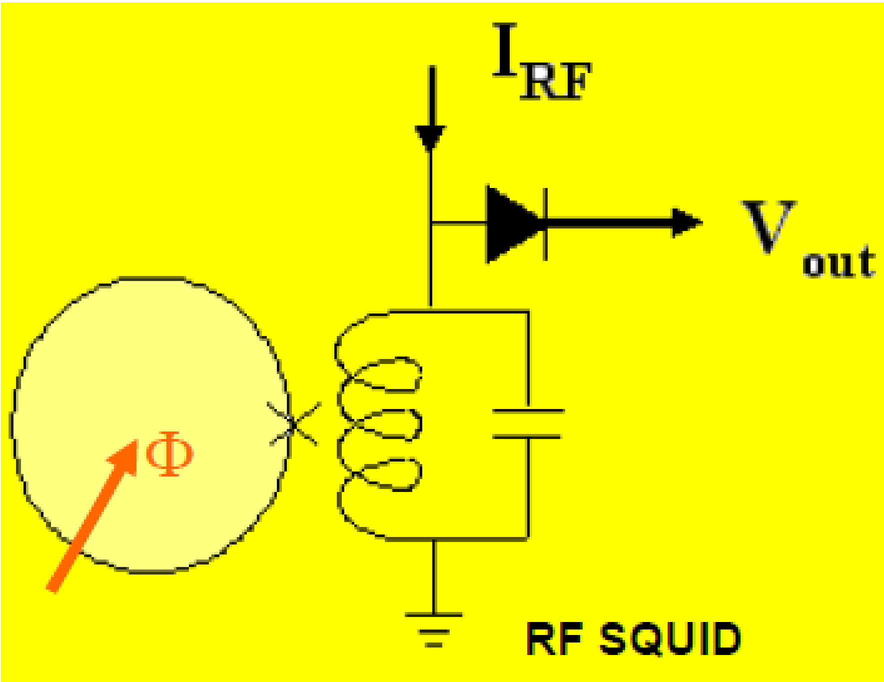
SQUID Gradiometer used to remove any background interference from large fields. Uses two rings where the difference in field lines will be due to their more rapid divergence, which means their source lies closer.

Figure 3: SQUID Gradiometer.



RF SQUID an RF SQUID is less sensitive compared to a DC SQUID but is cheaper and easier to manufacture in smaller quantities. The consist of a single Josephson junction in a loop measured indirectly by an RF antennae.

Figure 4: RF SQUID.



SQUID loop behaviour We have both flux quantisation in the loop and phase change δ around the junction. They are thus related by

$$\delta + 2\pi \frac{\phi_T}{\phi_0} = 2n\pi \tag{109}$$

where ϕ_T is the total flux in the loop.

The total flux ϕ_T is made up of the applied flux ϕ_a and the flux induced by the screening current,

$$J = I_0 \sin\delta \tag{110}$$

This leads to a relation

$$\phi_T = \phi_a - LI_0 \sin\left(2\pi \frac{\phi_T}{\phi_0}\right) \tag{111}$$

which can be either single valued or hysteretic.

Question: How is this derived? Where does the inductance L come into the picture?

Attempt at answer: Probably because we have the relation

$$L = \frac{\phi}{I} \tag{112}$$

That is, the inductance is a measure of how much flux links a circuit when a current flows.

RF SQUID parameter determines whether an RF SQUID is hysteretic or not. It is written

$$\beta_{rf} = \frac{2\pi L I_0}{\phi_0} \tag{113}$$

RF SQUID readout is done as follows:

- Couple the loop with the Josephson junction to an LCR oscillator circuit
- Drive LCR circuit close to its resonance by n RF current
- Measure the voltage oscillation of LCR circuit

Question: I am unsure about the last step - it is not clear from the notes.

Dissipative regime in this mode, the SQUID makes transitions between quantum states. It dissipates energy at a rate that is periodic in ϕ_a. This modulates the Q values of the LCR circuit. As a result, the RF voltage output V_{out} is periodic in ϕ_A.

Dispersive regime (not to be confused with the above!) in this non-hysteretic regime the RF SQUID behaves as a flux sensitive inductor. The inductance is linked to an emf, through

$$E = -\frac{d\phi}{dt} = -L\frac{d\phi}{dt} \tag{114}$$

Question: Something is wrong here. I have never seen Faraday's law with L in it.

Given the Josephson equations,

$$I = I_c \sin \phi \tag{115}$$

$$\frac{d\phi}{dt} = \frac{2e}{\hbar}V \tag{116}$$

We can relate the Josephson current to an inductance

$$L_J = \frac{\phi_0}{2\pi I_c \cos\phi} \tag{117}$$

And we obtain

$$L = \frac{\phi_0}{2\pi I_c \cos\phi} \tag{118}$$

Note that this makes the inductance non-linear.

Question: Derivation?

Dissipative mode readout the RF SQUID is read out by detecting the change in the Josephson junction. This is done by observing the change in Q due to the flux change.

5 John Fenton's Lectures - Three parts

5.1 Quantised Josephson junctions

This is the quantum part about the junctions.

Potentials We can think of the bumps in the washboard model as 1D potentials. Treat the electron as a harmonic oscillator and find energy levels (for a single mode)

$$E_n = \hbar\omega_0 \left(n + \frac{1}{2}\right) \tag{119}$$

Potentials in Josephson junction Potential is not harmonic but anharmonic - energy levels are not spaced evenly. It can be estimated by

$$U(\phi) = -\frac{h}{2e}I\phi - E_J \cos\phi \tag{120}$$

Note a static part (which depends on the external current I) and an oscillating part.

Analogues for the washboard are

- Tilt of washboard = external current bias I

- Bump amplitude = Josephson energy E_J
- Bump periodicity = phase difference $\phi = \phi_1 - \phi_2$

Lifetime of electron in potential is limited due to the electron quantum tunnelling out of the potential.

Voltage state means when the electron escapes the potential, and 'starts rolling' without stopping in an undamped system.

Consequence of tunnelling means that even as we go to 0 K, we see occasional current and voltage spikes once an electron manages to tunnel out.

Irradiation of microwaves increases the tunnel rate.

Question: Exactly which mechanism induces this? There are only graphs in the notes... :(

Josephson junction Hamiltonian for zero current bias given by

$$H = U(\phi) + 4E_c N^2 = U(\phi) - 4E_c \frac{\partial^2}{\partial \phi^2} \tag{121}$$

where

$$N = -i \frac{\partial}{\partial \phi} \tag{122}$$

and

$$E_c = \frac{e^2}{2C} \tag{123}$$

Question: Is this the conductance of the junction, or the combination addition of all capacitances in the circuit?

Attempt at answer: I think it is only the junction. In the notes, the capacitance is used as an analogy of the 'ball's mass'.

Note that ϕ and N are <u>conjugate variables</u> - they are not simultaneously observable.

Question: Why do we have a N^2 term here? I haven't really seen those around before.

$E_J \gg E_c$ regime will see the wavefunction $\psi(\phi)$ sharply peaked around specified ϕ. It implies

- Small $\delta\phi = \phi$ well defined
- Large $\Delta N = N$ not well defined

Question: Which wavefunction are we considering? The total wavefunction of both superconductors, or just one of them?

$E_J \ll E_c$ **regime** wavefunction $\psi(\phi)$ varies slowly over some range.

- Large $\Delta\phi = \phi$ not well defined
- Large $\Delta N = N$ well defined

I believe that these assertions come from when we would average over many turns of the wavefunction.

Deriving the Josephson Equations from the Schrodinger equation can be done by realising that tunnelling changes the number of Cooper pairs on one junction electrode, $N \to N+1$. This warrants the Hamiltonian

$$H_J = -\frac{E_J}{2} \sum_{-\infty}^{\infty} \left(|N\rangle \langle N+1| + |N+1\rangle \langle N| \right) \tag{124}$$

From this we can solve the Schrodinger equation as well as prove the complementarity of N and ϕ

5.2 Qubits form Josephson junctions

Motivation advantages and disadvantages are:

- Good for scaling
- Easy to couple
- Couples too easily to environment
- Short coherence times

Josephson junction qubits come in three varieties:

- Charge qubits
- Flux qubits
- Phase qubits

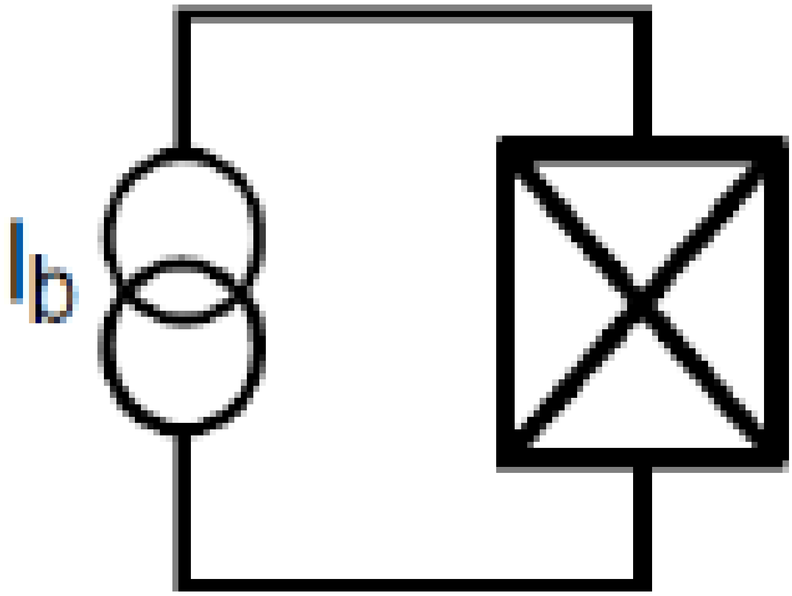
Qubit requirements are tunable separation between energy levels.

Phase qubits is a current-biased Josephson junction, operated in the zero voltage state with a non-zero current bias.

The energy levels are found in local minima of the washboard potential.

We manipulate the qubit by adding microwave pulses which cause Rabi oscillations.

Figure 5: Phase qubit.



Question: What does the box around the function mean? That we are in the zero voltage state?

Flux qubits are Josephson junctions which will have a persistent current flowing continuously as external flux is applied. They are based on RF SQUIDS.

The qubit states are defined by direction of the circulating currents. We make use of the flux quantum to treat it as a qubit. When the external flux reaches $\frac{1}{2}\phi_0$ the two current energy levels are close together, which enables qubit operation.

Potential for RF SQUID is given by

$$U(\phi_T) = -E_J \cos\left(\frac{2\pi\phi_T}{\phi_0}\right) + \frac{(\phi_T - \phi_{ext})^2}{2L} \tag{125}$$

This potential always has two minima, like the wine-bottle potential. We can shift this potential intro three modes:

- $\phi_{ext} < \frac{\phi_0}{2}$ gives low left potential and high right potential
- $\phi_{ext} = \frac{\phi_0}{2}$ gives two equally deep potential
- $\phi_{ext} > \frac{\phi_0}{2}$ gives a high left potential and low right potential

Consequence of double minima there is tunnelling between the two minima, which hybridises the states of the potential wells. This causes them to move apart in energy. We can use this to sit ourselves at $\phi_{ext} = \frac{\phi_0}{2}$, then tunnelling causes there to e a ground state and an excited state. The ground state has a symmetric wavefunction while the excited state has an antisymmetric wavefunction.

Question: Exactly how is the degeneracy broken if we have tunnelling?

Flux qubit readout can be done by inductively couple the qubit to a neighbouring circuit. Then the current flow in the flux can be read out.

Coupling flux qubits can be done by invoking a flux coupling. This means that the state of one qubit will be altered by the state of the second qubit. Through this, we can invoke a CNOT gae.

Question: It is not clear how the coupling is performed.

Charge qubits the basis states of a charge qubit are charge state, representing the presence or absence of Cooper pairs. A charge qubit is formed by a a tiny superconducting island (known as a Cooper-pair box) coupled by a Josephson junction to a superconducting reservoir.

The potential is given by

$$E = -E_J \cos\phi + 4E_c(N - N_g)^2 \tag{126}$$

Central to the setup is a Cooper pair box, it consists of an island the charge states of which it involves a macroscopic number of conduction electrons.

Readout is done by electrostatically coupling the island to an extremely sensitive electrometer.

Coupling between qubits can be done by

- Flux qubits are coupled with an inductive coupling
- Charge qubits are coupled using a capacitive coupling

5.3 Metrological applications of superconducting currents

screw this