Error Correction Lecture 2

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1 Correcting Errors of the Shor Code

Recall that the concatenated Shor code has the following initialisation circuit.

For an error represented by the operator

$$X_1 = XIIIIIIII \tag{1}$$

we get

$$X_1 \left|0\right\rangle_L = (\left|100\right\rangle + \left|011\right\rangle)(\left|000\right\rangle + \left|111\right\rangle)^{\otimes 2} \tag{2}$$

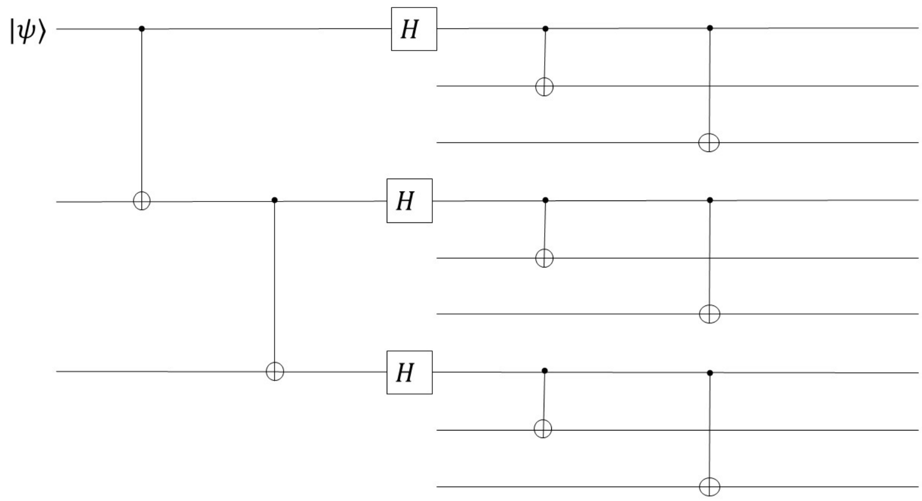
$$X_1 \left|1\right\rangle_L = (\left|100\right\rangle - \left|011\right\rangle)(\left|000\right\rangle - \left|111\right\rangle)^{\otimes 2} \tag{3}$$

This error is detected by the Z_1 and Z_2 parity measurements. In fact, we can write down the following table for error corrections,

$$\begin{array}{ccc} Z_1Z_2 & Z_4Z_5 & Z_7Z_8 \\ Z_2Z_3 & Z_5Z_6 & Z_8Z_9 \\ (Z_3Z_1) & (Z_4Z_6) & (Z_7Z_9) \end{array}$$

We have written the last row in brackets because just as before, it can be constructed from the other two rows and is thus superfluous. Note

Figure 1: Shor code initialisation diagram.



however that we cannot correct for two bitflip errors. The correction would introduce a factor of -1 on the $|1\rangle_L$ state.

However, we know that the operator $X_1X_2X_3 = \bar{Z}$. This error is undetectable. As a consequence, we cannot detect any errors that happen on adjacent bits.

In general, we want

$$\text{error} \cdot \text{error correction} = \mathbb{I} \tag{4}$$

2 Phase errors

Let us introduce the error Z_1. It has the following effect:

$$Z_1 |0\rangle_L = (|000\rangle - |111\rangle)(|000\rangle + |111\rangle)^{\otimes 2} \tag{5}$$

$$Z_1 |1\rangle_L = (|000\rangle + |111\rangle)(|000\rangle - |111\rangle)^{\otimes 2} \tag{6}$$

We can detect this error with the operator $XXXIIIIII$. It has the following effect:

$$XXX(|000\rangle \pm |111\rangle) = \pm(|000\rangle \pm |111\rangle) \tag{7}$$

which enables us to see where the error has introduced a sign error. We get a negative eigenvalue for the error $XXX = \bar{Z}$. But there is a problem. Since this error is a logical operator, its detection would allow us to distinguish between $|0\rangle_L$ and $|1\rangle_L$. We can't allow this, since it would collapse any superposition. So instead, we measure $X_1X_2X_3X_4X_5X_6$. That is, measure two out of three registers. This will check the parity between the first three and the middle three states. So we get the error detection measurements $\begin{array}{c} X_1X_2X_3X_4X_5X_6 \\ X_1X_2X_3X_7X_8X_9 \\ (X_4X_5X_6X_7X_8X_9) \end{array}$ This can detect a single Z error on every single qubit. We find the following syndrome table, using the three measurements above

Z_1	Z_2	Z_3
$-$	$-$	$-$
$-$	$-$	$-$
$+$	$+$	$+$

But notice that the syndrome looks the same for all three measurements! This means that we cannot distinguish between them. Say we detect Z_2, but we then implement the correction Z_1. Notice, however that the two commute. So we have

$$Z_2 \cdot Z_1 |0\rangle_L = |0\rangle_L \tag{8}$$

$$Z_2 \cdot Z_1 |1\rangle_L = |1\rangle_L \tag{9}$$

So it works! This is the same as for the error detection operators for single bitflip errors.

This phenomenon is called **degeneracy**. We have both non-degenerate and degenerate code.

A non-degenerate code means that every possible correctable error has a unique syndrome. This is the case for all classical codes.

It is, understandably, easier to prove things for non-degenerate codes. Thus, some quantum questions are still completely open systems.

3 Slight Generalisation

We shall here look at the properties of Pauli operators and their tensor products. We shall use them extensively throughout the course, and most likely all of their properties.

The Pauli operators are I, X, Y, Z. We include the identity I since some of the properties include it.

The Pauli operators have the following operators.

- **All unitary** we find that for every Pauli operator,

$$U^{-1} = U^\dagger = U \tag{10}$$

- **All Hermitian** we find
$$U^\dagger = U \tag{11}$$

- **All are self-inverse** that is,
$$U = U^{-1} \tag{12}$$

- **Recursive property** we can write
$$Y = iXZ \tag{13}$$
Then,
$$Y\left|\psi\right\rangle = iXZ\left|\psi\right\rangle \tag{14}$$
so that a Y-error can be written in terms of a bitflip and phase error. The i is a global phase which we can ignore.

- **Commute or anti-commute** All Pauli matrices anti commute. That is
$$\{U_i, U_j\} = 0 \text{ if } i \neq j \tag{15}$$

- **All are traceless** except for I.

- **Tensor products of Pauli matrices commute** we find that
$$[X \otimes X, Z \otimes Z] = 0 \tag{16}$$

The last property means that we can always correct for a combination of errors. It means that the following statements are equivalent:

$$X_E Z_E X_C Z_C \tag{17}$$

$$X_E X_C Z_E Z_C \tag{18}$$

where E stands for error and C stands for correction. That is, it doesn't matter in which order we detect and correct the errors since all the operators anti-commute, which just introduces a global phase.

4 Arbitrary errors

Claim: We can correct for arbitrary, unitary errors. Any operator which is a tensor product of Paulis can be written $O_X O_Z$ where O_X contains X and I, and O_Z contains Z and I. For example, we can derive all possible Y errors,

$$YZIX = i(XIIX)(ZZII) \tag{19}$$

In fact, the Pauli operators form a basis for $2^N \times 2^N$ matrices that live in $\mathbb{C}^{2^N \times 2^N}$. Any 2×2 matrix can be written as

$$O = aI + bX + cY + dZ \tag{20}$$

To check whether they actually form a basis, we can make use the Hilbert-Schmidt inner product.

$$\langle A, B \rangle = \mathrm{Tr}\left[AB\right] \tag{21}$$

which can be thought of as an orthogonality condition but for matrices. If you find a set of D operators with zero Hilbert-Schmidt product where D is also the dimension of the space where they live, you have a basis.

5 Measuring Pauli's

We can device a circuit for measuring the Pauli operators. For a number of n qubits, we can imagine a projector

$$P = P_1 \otimes P_2 \otimes P_3 \otimes \ldots \otimes P_n \tag{22}$$

This circuit would simply look like projectors acting on the individual qubits.

We know that all eigenvalues of X, Y, Z are ± 1. If we have

$$P\left|\psi\right\rangle = \left|\psi\right\rangle \tag{23}$$

we measure a $+1$, and if we have

$$P\left|\psi\right\rangle = -\left|\psi\right\rangle \tag{24}$$

we measure -1.