Quantitative Macroeconomics
Part 2
Project 1

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1 Problem 1: Baseline Consumption-Savings Model: Derivation of Dynamic Programming Solution

1.1

Derivation of the dynamic programming solution backwards in t up to period $T-2$.

1.1.1 In period T.

$$V_T(w_T) = \max_{c_T, w_{T+1}} \Bigg[u(c_T) + \beta V_{T+1}(w_{t+1}) \Bigg] \tag{1}$$

And because we know that in the last period everything will be consumed:

$$c_T = w_T \tag{2}$$

Hence:

$$w_{T+1} = (w_T - c_T)R = 0 \tag{3}$$

So our value function will be:

$$V_T(w_T) = \frac{c_T^{1-\theta}}{1-\theta} = \frac{w_T^{1-\theta}}{1-\theta} \tag{4}$$

1.1.2 In period $T-1$.

$$V_{T-1}(w_T) = \max_{c_{T-1}, w_T} \Bigg[u(c_{T-1}) + \beta V_T(w_t) \Bigg] \tag{5}$$

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We insert $V_T(w_t)$ from 1.1.1 into this maximization problem and receive:

$$V_{T-1}(w_T) = \max_{c_{T-1},w_T} \left[u(c_{T-1}) + \beta \frac{w_T^{1-\theta}}{1-\theta} \right] \tag{6}$$

We know that

$$w_T = (w_{T-1} - c_{T-1})R \tag{7}$$

After inserting above equation into value function we receive:

$$V_{T-1}(w_T) = \max_{c_{T-1},w_T} \left[\frac{c_{T-1}^{1-\theta}}{1-\theta} + \beta \frac{((w_{T-1} - c_{T-1})R)^{1-\theta}}{1-\theta} \right] \tag{8}$$

Now we maximize with respect to c_{T-1} and receive following first-order condition:

$$c_{T-1}^{-\theta} - \beta R^{1-\theta}((w_{T-1} - c_{T-1})^{-\theta} = 0 \tag{9}$$

Hence the optimal consumption in period $T-1$ is given by:

$$c_{T-1} = \frac{\beta^{\frac{-1}{\theta}} R^{\frac{\theta-1}{\theta}} w_{T-1}}{1 + \beta^{\frac{-1}{\theta}} R^{\frac{\theta-1}{\theta}}} = m_{T-1} w_{T-1} \tag{10}$$

where

$$m_{T-1} = \frac{\beta^{\frac{-1}{\theta}} R^{\frac{\theta-1}{\theta}}}{1 + \beta^{\frac{-1}{\theta}} R^{\frac{\theta-1}{\theta}}} \tag{11}$$

Finally the value function in this period is given by:

$$V_{T-1}(w_T) = \frac{1}{1-\theta} w_{T-1}^{1-\theta} (m_{T-1}^{1-\theta} + \beta((1 - m_{T-1})R)^{1-\theta} \tag{12}$$

To make it simpler we introduce γ.

$$V_{T-1}(w_T) = \frac{w_{T-1}^{1-\theta} \gamma_{T-1}}{1-\theta} \tag{13}$$

Where:

$$\gamma_{T-1} = (m_{T-1}^{1-\theta} + \beta((1 - m_{T-1})R)^{1-\theta} \tag{14}$$

1.1.3 In period $T-2$.

$$V_{T-2}(w_{T-1}) = \max_{c_{T-2},w_{T-1}} \left[\frac{c_{T-2}^{1-\theta}}{1-\theta} + \beta V_{T-1}(w_T) \right] \tag{15}$$

After inserting $V_{T-1}(w_T)$ from previous subpoint we receive:

$$V_{T-2}(w_{T-1}) = \max_{c_{T-2},w_{T-1}} \left[\frac{c_{T-2}^{1-\theta}}{1-\theta} + \frac{w_{T-1}^{1-\theta} \gamma_{T-1}}{1-\theta} \right] \tag{16}$$

After simplifications:

$$V_{T-2}(w_{T-1}) = \max_{c_{T-2}, w_{T-1}} \frac{1}{1-\theta}\left[c_{T-2}^{1-\theta} + \beta((w_{T-2} - c_{T-2})R)^{1-\theta}\gamma_{T-1}\right] \tag{17}$$

FOC:

$$c_{T-2}^{-\theta} - \beta R^{1-\theta}(w_{T-2} - c_{T-2})^{-\theta}\gamma_{T-1} = 0 \tag{18}$$

Hence the optimal consumption in period $T-2$ is:

$$c_{T-2} = \frac{(w_{T-2})R^{\frac{\theta-1}{\theta}}(\beta\gamma_{T-1})^{\frac{-1}{\theta}}}{1 + R^{\frac{\theta-1}{\theta}}(\beta\gamma_{T-1})^{\frac{-1}{\theta}}} = m_{T-2}w_{T-2} \tag{19}$$

Where $m_{T-2} = \frac{R^{\frac{\theta-1}{\theta}}(\beta\gamma_{T-1})^{\frac{-1}{\theta}}}{1+R^{\frac{\theta-1}{\theta}}(\beta\gamma_{T-1})^{\frac{-1}{\theta}}}$.

Finally the value function is:

$$V_{T-2}(w_{T-1}) = \frac{1}{1-\theta}w_{T-2}^{1-\theta}\left[m_{T-2}^{1-\theta} + \beta((1-m_{T-2})R)^{1-\theta}(\beta\gamma_{T-1})^{\frac{-1}{\theta}}\right] \tag{20}$$

1.2 Plots

We plotted our functions for standard values of parameters that is $\beta = 0.98$, $R = 1.05$ and $\theta = 0.5$, but we can also analyse them qualitatively. Plotted in Python with the use of matplotlib package.

1.2.1 Consumption policy functions

Plot:

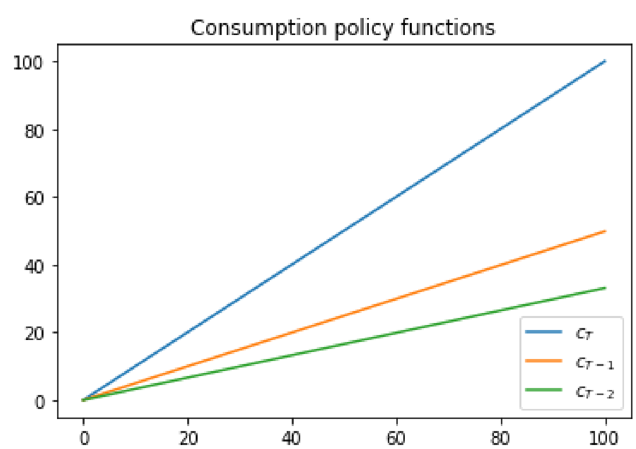


Figure 1: Consumption policy functions

1.2.2 Value functions

Plot:

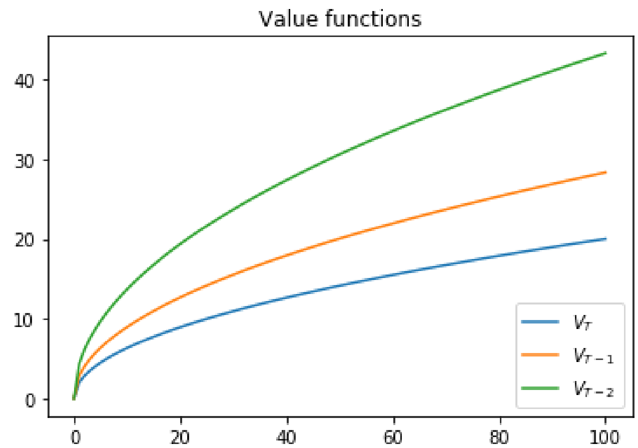


Figure 2: Value functions

2 Problem 2: Baseline Consumption-Savings Model: Numerical Solution

In this exercise we use pure life-cycle model to understand how to implement this macroeconomic model into the MATLAB software.

2.1

The code used in *hhmodel.m* file consists of two methods, defined by the value of parameter *solmeth*.

In solving method 1 human capital was solved in backward induction part and was described by equation:
$h(i) = \frac{h(i+1)+w(i+1)}{1+R}$,
where w(i) represents wage equal to 1 if individual is working (i.e. his age is below 65) plus capital income represented by return on assets. Rest of the variables, i.e. marginal propensity to consume and parameter b representing the discount factor was solved in the forward solving process:

- $b = \frac{1}{1+R}(\frac{1+R}{1+\rho})^{1/\theta}$ is based on the parameters R and ρ

- $c_0(i) = \frac{1+R}{1+\rho}^{\frac{i-1}{\theta}} \frac{1-b}{1-b^{MA}}$, where MA represents maximum age of an individual (here valued at 80 representing 100 yo persons, because individual start their life with age = 20)

- $C(i) = c_0(i)(a(1) + w(1) + h(1))$
- $c(i) = \frac{C(i)}{a(i)+w(i)+h(i)}$

On the other hand, in solving method 2, parameter b, marginal propensity to consume and human capital were calculated in the backward induction step.:

- $b = (\beta(1+R)^{1-\theta})^{\frac{1}{\theta}}$ is based on the parameters R and beta
- $c(i) = \frac{\frac{c(i+1}{b}}{1+\frac{c(i+1)}{b}}$
- $h(i) = \frac{h(i+1)+w(i+1)}{1+R}$

In the forward solving step, C and $c_0 are solved$:
$C(i) = c(i)(a(i) + w(i) + h(i))$

$c_0(i) = \frac{C(i)}{a(i)+inc(i)+h(i)}$
Therefore two solving methods differ one from another - first one involes more forward solving, consumption is calcutated in a different way as well as the parameter b, while the second method is baed on backward iteration method.

Pseudocode for each method:
0. Define parameters of the model. Most important parameters are already defined in the beginning of the document.
1. We define the sets of variables (marginal propensity to consume, human capital, comsumption and c_0) as described in **2.1**:
define VariableSet1Backward
define VariableSet1Forward
define VariableSet2Backward
define VariableSet2Forward

2. We define a vector of income with respect to the age of individual and his retirement age: def IncomeVector as zeros[min_age, max_age, step = 1]
if i in IncomeVector ¡ ret_age+1:
IncomeVector(i) = 1
else:
IncomeVector(i+1) = rr*IncomeVector(i),
where rr represents replacement level after retirement. In this particular case rr = 0.

3. We solve backwards for part of variables as defined in Point 1. here: for i in [max_age, min_age, step = -1]:
if SolMeth == 1:
VariableBackward = VariableSet1Backward
else if SolMeth == 2:

VariableBackward = VariableSet1Backward

4. We solve forwards for other part of variables as defined in Point 1. here:
for i in [min_age, max_age, step = 1]:
if SolMecth == 1:
VariableForward = VariableSet1Forward
else if SolMeth == 2:
VariableForward = VariableSet2Forward

5. Finally, we solve the model for consumption, assets and savings:
wealth(i) = a(i) + w(i) + h(i)
x(i) = a(i) + w(i)
mpx(i) = C(i)/x(i)
A(i) = x(i) - w(i) = a(i)
S(i) = x(i) - C(i)
if i ¡ max_age: w(i+1) = (w(i) - C(i))*(1+R)

2.2

In this part, we check what would be the effects of a decrease in rate of return on capital. So, we decrease R from its base value of 0.04 to (0.02, my proposition). The differences between these two scenarios can be seen on these two plots:

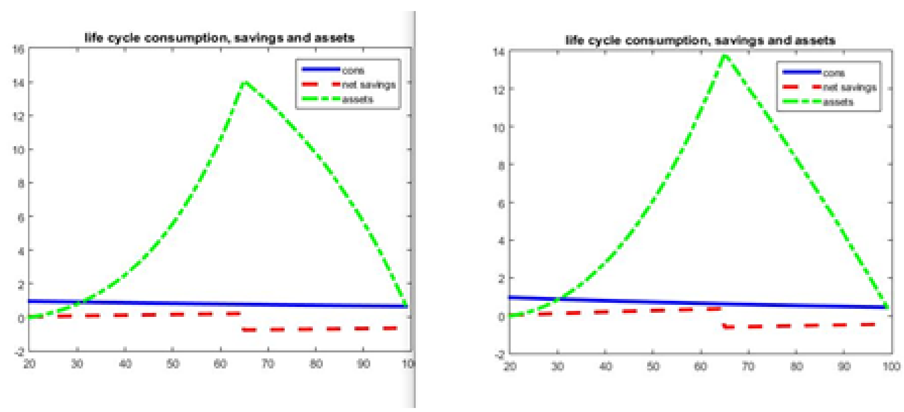


Figure 3: Comparison of 2 scenarios - R = 0.04 (left) and R = 0.02 (right)

As we can see, the difference between scenarios is not very large. Even though, it is visible that the distance between consumption and net savings becomes smaller over time, smaller in the case with lower rate of return. Additionally, the peak of assets that falls on the end of professional career is a bit higher in the left scenario.

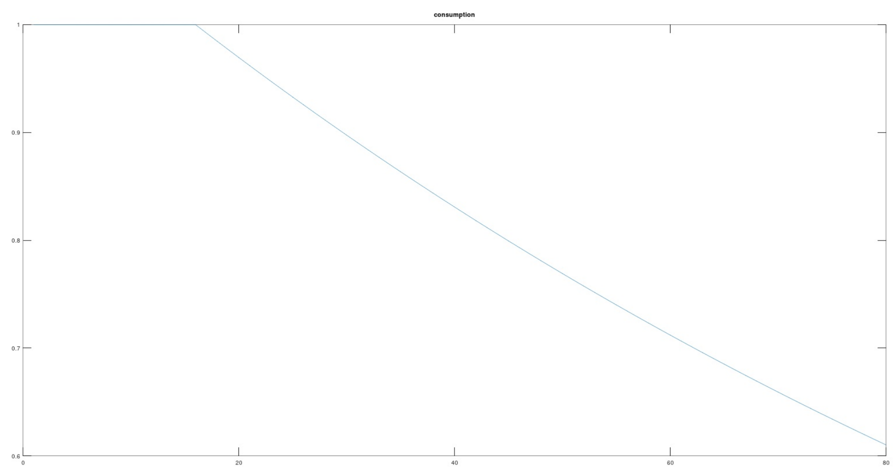


Figure 4: Policy function in the OLG model with borrowing constraint

3 Problem 3: Extension I: Borrowing Constraints

3.1 Non-differentiable policy functions

In a model with borrowing constraints policy functions might not be continuously differentiable objects. This is due to the presence of borrowing constraint itself: once an agent hits the constraint, his consumption becomes a corner solution. Since such a corner solution is optimal, it must be given by the policy function - but one cannot obtain a corner solution from differentiation.

Actually this is also visible in the Figure 4, which presents our policy function in the model with borrowing constraint. Due to existence of that constraint we observe a kink in the 16th period (that is, for a 36 years old individual). Obviously, a function with kinks (that is, not smooth) cannot be continuously differentiable.

3.2 Solution method

3.2.1 Solution algorithm - general discussion

Generally, OLG model with borrowing constraint may be solved with standard sequential problem solving method. One needs to iterate forward starting at $t = 0$, using budget constraint at the initial period. Should we at any point hit the borrowing constraint, it must be imposed at the consumption (that won't be strictly optimal anymore), and from that point on, we can continue forward iteration by equating present value of future stream of consumption to the present value of the future stream of income.

3.2.2 Solution algorithm - *pseudo*-code

Pseudo-code based on the provided Matlab algorithm is presented below.

1. Compute a vector of relative[1] consumption for the whole life-cycle $relcons$:

   (a) Compute a relative consumption growth as if there was no borrowing constraint:
   $$consgr\_nobc = (\beta \cdot R)^{\frac{1}{\theta}}$$

   (b) Fill the $relcons$ vector:
   $$relcons[i] = consgr\_nobc \cdot relcons[i-1]$$

2. Obtain present value of the vectors of relative consumption ($relcons$) and incomes (inc):

   (a)
   $$pv\_relcons = \sum_i R^{-i} relcons[i]$$

   (b)
   $$pv\_inc = \sum_i R^{-i} inc[i]$$

3. Calculate initial consumption:
   $$cons[1] = \frac{pv\_inc}{pv\_relcons}$$

4. Compute vector of consumption (it will be updated later):
   $$cons[i > 1] = consgr\_nobc \cdot cons[i-1]$$

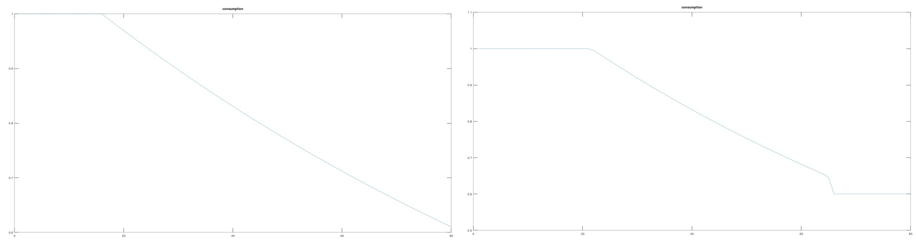
5. Introduce borrowing constraint and update consumption:

   (a) Define and compute *cash-on-hand*:
   $$coh[1] = inc[1]$$
   $$coh[i > 1] = R \cdot (coh[i-1] - cons[i-1]) + inc[i]$$

   (b) Check the borrowing constraint, i.e.:
   $$coh[i] - cons[i] \geq 0$$

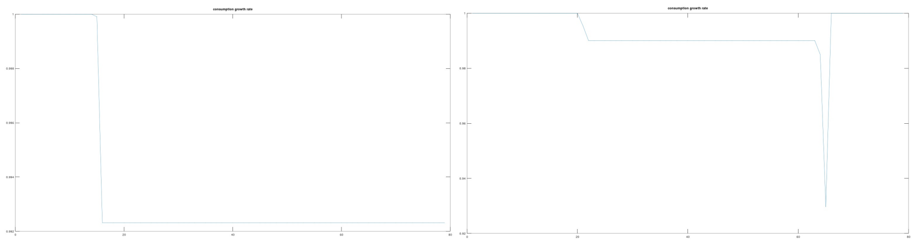
   (c) If borrowing constraint is satisfied, go to the next period and repeat. If not, go to the following points:

[1] Relative to the initial consumption



(a) Consumption with $R = 1\%$ (b) Consumption with $R = 0.1\%$

Figure 5: Consumption under different interest rate regimes



(a) Consumption growth with $R = 1\%$ (b) Consumption growth with $R = 0.1\%$

Figure 6: Consumption growth paths under different interest rate regimes

i. Satisfy the borrowing constraint:

$$cons[i] = coh[i]$$

ii. Calculate present value of the remaining stream of income and relative consumption:

$$pv\_inc = (pv\_inc - inc[i]) \cdot R$$

$$pv\_relcons = (pv\_relcons - 1) \cdot \frac{R}{consgr\_nobc}$$

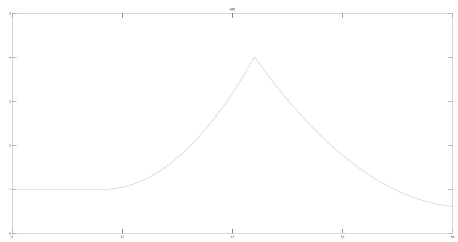
iii. Update the remaining stream of consumption:

$$cons[i+1] = \frac{pv\_inc}{pv\_relcons}$$

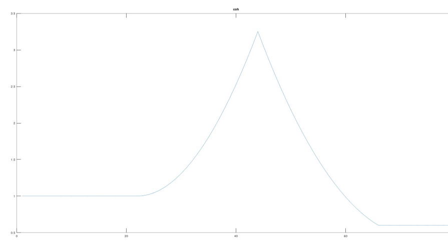
$$cons[j > i+1] = cons[i+1] \cdot relcons[j]$$

3.3 Interest rate cuts

Figures 5 to 8 present paths of consumption, consumption growth, *cash-on-hand* and assets under two interest rate regimes: first, $R = 1\%$; second, $R = 0.1\%$.

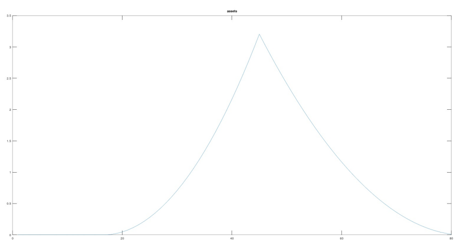


(a) *Cash-on-hand* with $R = 1\%$

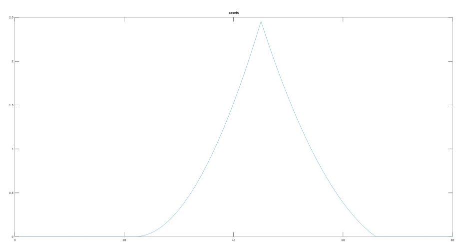


(b) *Cash-on-hand* with $R = 0.1\%$

Figure 7: *Cash-on-hand* under different interest rate regimes



(a) Assets with $R = 1\%$



(b) Assets with $R = 0.1\%$

Figure 8: Asset under different interest rate regimes

As we can see, different policy regarding interest rate bears significant consequences to the variables of interest. Those differences are discussed below in detail.

1. **Consumption**: with lower interest rate, decrease in consumption begins later and is characterised by presence of multiple kinks. After 60 period (that is, after 80 year of life) it stabilises at the level 0.6, which means simply eating each period one's old age pension. On the contrary, with higher interest rate, consumption decreases till death, but it never hits the lower bound of 0.6. This is understandable, though - when interest rate is high, savings yield higher return, which in addition makes agents more willing to save; hence elderly agents dispose of more wealth they can gradually eat up.

2. **Consumption growth**: this variable shows 1:1 relation to consumption itself; hence more kinky[2] consumption path under low rate regime translates into more complex shape of the consumption growth plot. Eventually consumption growth stabilises, as consumption reaches steady-state value of 0.6.

3. ***Cash-on-hand***: here it is more convenient to start from the low interest

[2]No innuendo here!

rate policy. One might note that constant consumption produces constant value of cash-on-hand. Then, under both policies, periods of consumption decrease is a period when *cash-on-hand* first grows (in a convex manner) and later falls (again, in a convex way). Such a behaviour of this variable is explicable in a following way: decreasing consumption with fixed income is equivalent to rising amount of cash disposable, which eventually begins to shrink as individual becomes retired at the period 45 (that is, being 65 years old).

4. **Assets**: Assets behaviour resembles that of *cash-on-hand*.

4 Problem 4: Extension II: Bequest Motive

In this problem we analyze household model with surival risk and bequest motive. The value function households maximize has a form:

$$V_j(a(i)) = \max_{c,a'}\{u(c) + sV_{j+1}(a(i+1)) + (1-s)W_{j+1}(a(i+1))\}$$

subject to:

$$a(i+1) = a(i)R + y - c$$
$$W(a) = \frac{\phi}{1-\theta}(\psi + a)^{1-\theta}$$

4.1

In this subpoint we are supposed to interpret the code as well as provide the pseudo-code for it. The best approach here is to do both at the same time.

1. Define the parameters of the model.
   Most important parameters are already defined in the beginning of the document.

   $\phi_L = 10$
   $\phi_H = 40$

   ϕ is a parameter representing the strength of the bequest motive. Households have utility from dying with positive assets that are going to be inherited by the future generations. When the parameter ϕ increases, bequest motive utility increases too.

2. 1. Define the vector of income:
   income[age_min; age_ret] = 1
   income[age_ret+1; age_max] = 0.4

3. 2. Define the function of assets and bequest motive:
   Assets: $a(i) = \frac{a(i+1)+c(i)-y(i)}{R}$
   Bequest motive: $W(i) = \phi(\psi + a_0)^{-\theta}$

4. 3. Define inverse function for consumption:
   Consumption: $c(i) = marg\_c^{-1/\theta}$

5. 4. Define function solving the problem:
   4. a) The solution for the problem is recursive. So, first, we define the size of the vectors:
   a = zeros[max_age+1; 1]
   c = zeros[max_age; 1]
   U = zeros[max_age; 1]

   b) Next, we have to initialize in the last period:
   a(max_age+1) = 1
   $marg\_c = \frac{W(a(i+1))}{R}$
   $c(max\_age) = marg\_c^{\frac{-1}{\theta}}$
   $a(max\_age) = \frac{1+c(max\_age)+w(max\_age}{R}$

   c) Then, we loop backwards to calculate the values of our variables from (max_age - 1) to (1):
   $W(a(i+1)) = \phi(\psi + a(i+1))^{-\theta}$
   $V(a(i)) = \beta(s(i)V(a(i+1)) + (1 - s(i)) * W(a(i+1)))R$
   $marg\_c = \frac{V(a(i))}{R}$
   $c(i) = marg\_(c)^{\frac{-1}{\theta}}$
   $a(i) = \frac{a(i+1)+c(i)-y(i)}{R}$

6. 5.Finally, we solve the households' problem:
   options = optimize(MaxIter, Tolerance)
   solution = root(a(max+1), options) (fzero - finding root of nonlinear function)
   if (solution(MaxIter)-var) ¿ Tolerance; print ('No convergence') (iterations and result) (var is the matrix of variables)
   x[min_age; max_age] = a[min_age; max_age]R + w[min_age; max_age]

4.2

Now we interpret what is the differents between two scenarios:
1) When $\phi = \phi_L = 10$
2) When $\phi = \phi_H = 40$

(czerwona przerywana - fi wysokie, niebieska ciagła - fi niskie)

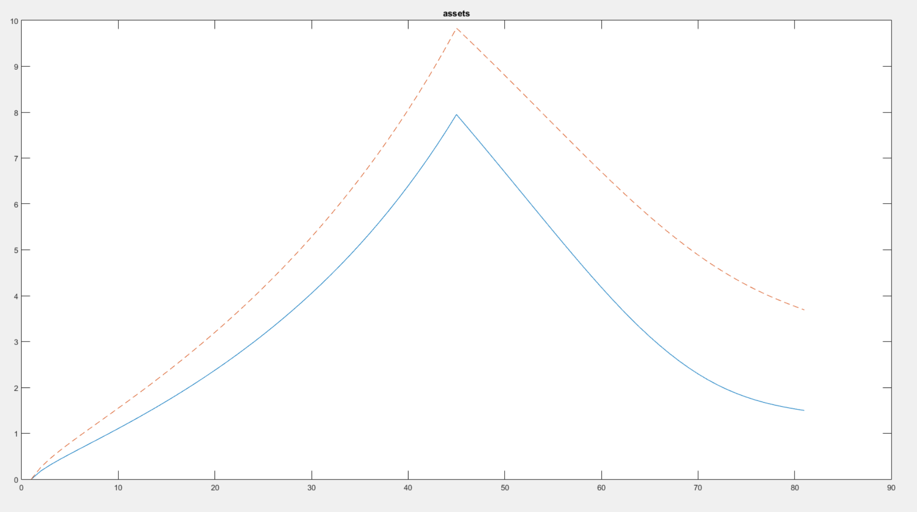


Figure 9: Assets with different values of ϕ

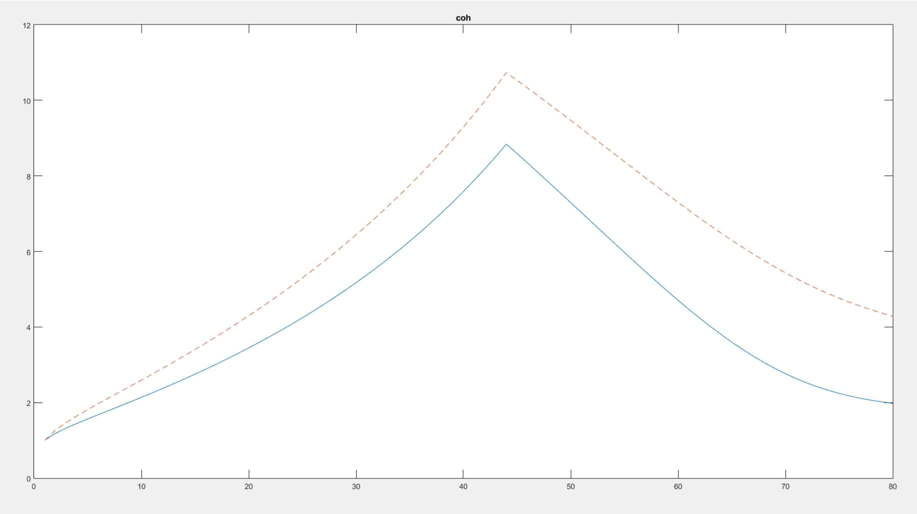


Figure 10: Cash on hand with different values of ϕ

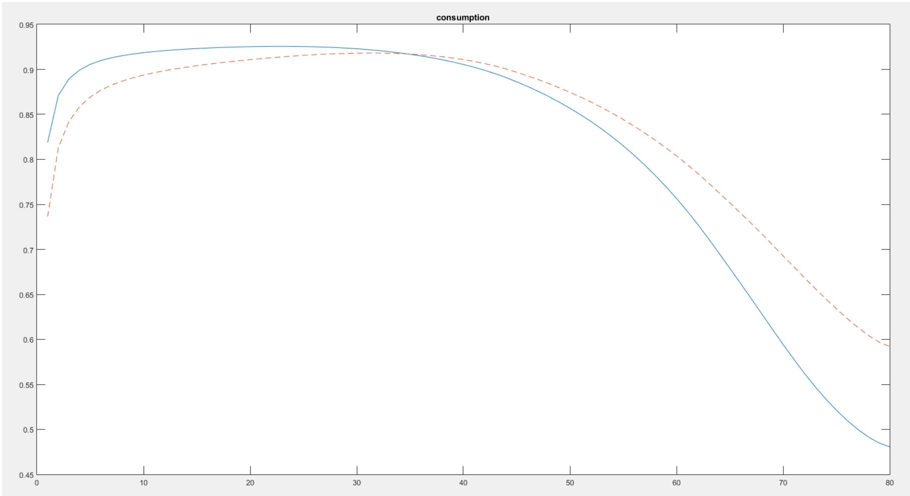


Figure 11: Consumption with different values of ϕ

The red dotted line represents ϕ_H, while the blue continuous one is for ϕ_L Therefore, higher utility parameter from bequest motive gives higher assets and greater value of cash at hand. For ϕ_H the consumption is smaller for earlier age of the individual and is greater for individual when he/she gets older.

4.3

In this scenario we assume that $\psi = 0$.

In this case bequest motive constraint has a following form:
$W(a(i+1)) = \phi * (a(i+1))^{-\theta}$

Then the value function is:

$$V_j(a(i)) = \max_{c,a(i+1)} \{u(c(i)) + s(i)V_{j+1}(a(i+1)) + (1-s(i))\phi(a(i+1)^{-\theta})\}$$

(If s(i)¿0 for every i, it occurs that there is self-imposed borrowing constraint).

5 Problem 5: Extension III: Consumption, Savings and Portfolio Choice with Deterministic Labor Income

5.1 Interpret the code and write a pseudo code.

Provided code may be rewritten as pseudo code below:

1. Set exogenous values
    (a) ρ - utility discount rate, θ - relative risk aversion, r^f - return on risk free asset, μ - expected return on risky asset, σ - standard deviation of return on risky asset, x_0 - initial cash-on-hand, n - number of simulations, T - maximum age, T_r - retirement age
    (b) lifetime non-stochastic human capital income:
    $[y_t] = \begin{cases} income & \text{if} \quad t < T_r \\ replacement\_rate * income & \text{otherwise} \end{cases}$
    (c) human capital:
    $h_T = 0, \quad$ for t in (T-1):1 $[hk_t] = \frac{hk_{t+1}+y_{t+1}}{1+r^f}$
2. Calculate policy function
    (a) $\hat{\alpha} = \frac{ln\mu - ln(1+r^f)+0.5\sigma^2}{\theta\sigma^2}$ - fraction of risky asset in total portfolio (constant over lifetime)
    (b) $mup = \mathbf{E}(ln(1+r^P) = \hat{\alpha} \cdot ln(1+\mu)+(1-\hat{\alpha})ln(1+r^P)+0.5\hat{\alpha}(1-\hat{\alpha})\sigma^2$ - expected logarithm return on portfolio
    (c) $\sigma^{P2} = \hat{\alpha}^2\sigma^2$ - variance of return on portfolio
    (d) Marginal Propensity to Consume out of cash-on-hand:
    $m_T = 1$
    for t in (T-1):1 $[b_t] = \frac{1}{m_{t+1}}(\beta exp((1-\theta)(mup + 0.5\sigma^2(1-\theta))))^{1/\theta}$
    $[mpc_t] = [\frac{1}{b_t}]$
3. Compute simulation (10 000 times)
    (a) Randomize possible states of the world ε_t~N(0,1)
    (b) Calculate rate of return on risky asset given states
    $[r_t] = exp(\varepsilon_t\sigma + ln(1+\mu)) - 1$
    (c) Solve optimization problem
    $x_1 = x_0$ - initial cash-on-hand
    $\mathcal{W}_1 = x_1 + hk_1$ - initial total wealth
    (d) for t in 1:T
        i. $c_t = mpc_t \cdot \mathcal{W}_t$ - consumption
        ii. $x_t = \mathcal{W}_t - hk_t$
        iii. $r_1^P = r^f + \hat{\alpha}(r_t - r^f)$ - return on portfolio
        iv. $a_t = \frac{x_t}{1+r_t^P}$ - assets
        v. $S_t = \mathbf{W}_t - c_t$ - total sacings (including human capital)
        vi. $s_t = x_t - c_t$ - financial savings
        vii. if $t < T$ then:
        $\alpha_t = \frac{\hat{\alpha} \cdot S_t}{s_t}$ - fraction of risky asset in financial savings
        $r_{t+1}^P = r^f + \hat{\alpha}(r_{t+1} - r_f)$
        $\mathbf{W}_{t+1} = S_t(1 + r_{t+1}^P)$

(e) Present first iteration

(f) Update average from simulations

4. Present average from simulations

5.2 Results

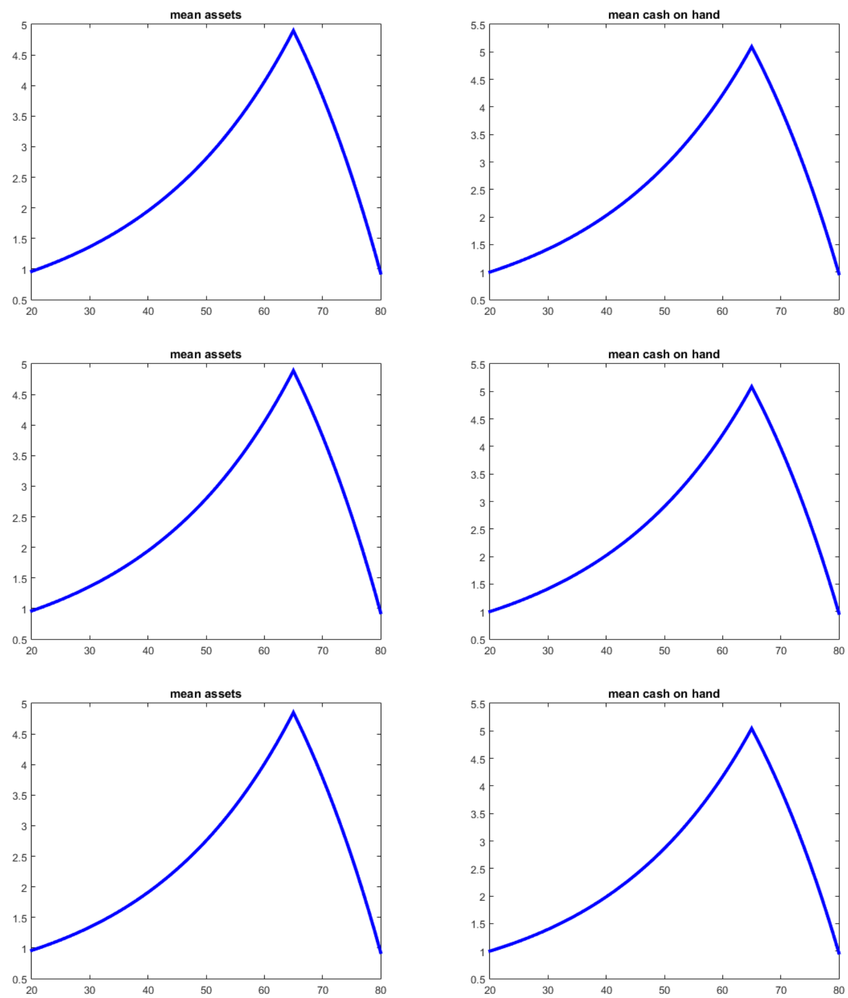


Figure 12: Mean assets (on the left) and cash on hand (on the right) for patameteres (from top to bottom): a) $\rho = 0.02, \theta = 150$, b)$\rho = 0.02, \theta = 150$, c) $\rho = 0.04, \theta = 150$.

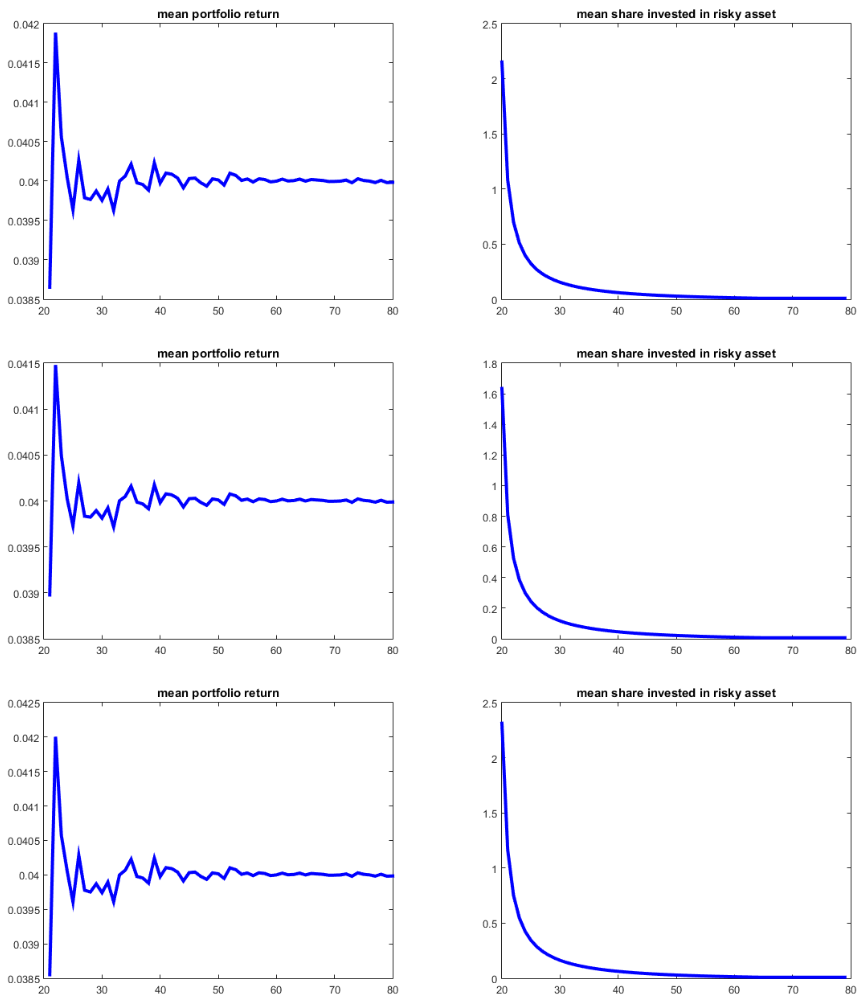


Figure 13: Mean portfolio return (on the left) and shares of financial assets invested in risky asset (on the right) for parameteres (from top to bottom): a) $\rho = 0.02, \theta = 150$, b)$\rho = 0.02, \theta = 150$, c) $\rho = 0.04, \theta = 150$.

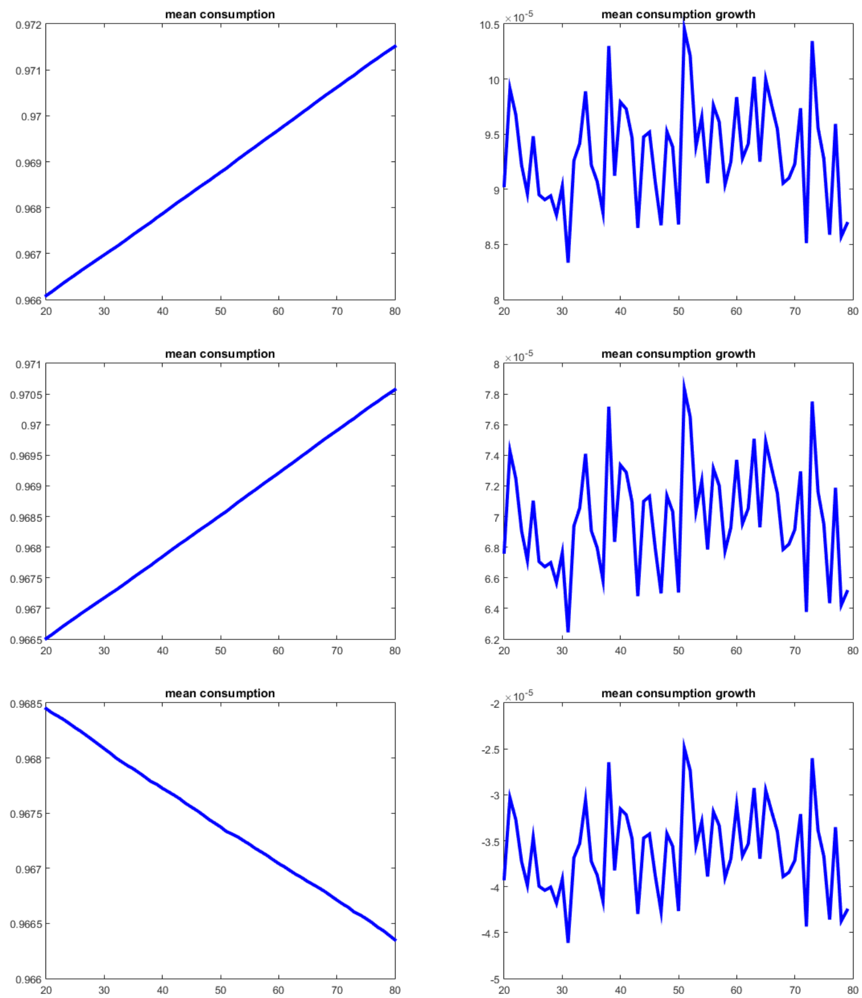


Figure 14: Mean consumption (on the left) and consumption growth (on the right) for parameters (from top to bottom): a) $\rho = 0.02, \theta = 150$, b)$\rho = 0.02, \theta = 150$, c) $\rho = 0.04, \theta = 150$.

The shape of plots of average asset as well as cash on hand over lifetime is in line with intuition. A household puts aside and invests until retirement to consume out any treasury before death. The graphs do not strongly depend on utility discount rate ρ nor the relative risk aversion θ. This is because of the fact that their values depend mostly on deterministic labor and later retirement income. Of course they vary with investment rate, however these fluctuations have marginal effect, especially at that high relative risk aversion.
At the same time average share of risk asset in the portfolio varies strongly over lifetime with very high values, even over 1, at the beginning. Though at

first sight it may be surprising, the share of risk asset in financial savings might exceed one, as the individual in fact manage his total savings, which including human capital - future deterministic labor income. Household, as a matter of fact, does not own money at that particular time. However these investment might be understood as financed by a kind of debt, which will be paid back during the lifetime. Moreover this is portfolio share over total wealth $al\hat{p}ha$, which is constant over lifetime. With age the fraction of human capital in total savings decreases in favor of cash-on-hand, that's why the portfolio share over financial savings decreases.
Higher risk aversion, according to intuition, results in lower shares invested in risky asset. Value of θ isn't that large by mistake. Thanks to that, we avoid computations in which household runs out a debt excessively huge with large amount of shares invested in risky assets, what in real life isn't observable due regulatory and institutional constraints.
For discounted utility of consumption $\rho = 0.02$ smaller than rate of return on risk-free asset $\mathring{=}0.04$, households in two first cases prefer to consume less at the beginning and more later. On the other hand, when discount factor of utility exceeds possible market returns households prefer to consume more immediately. Then the shape of plot of mean consumption growth is similar to previous cases, but has negative values.

5.3 Rule of thumb

Mentioned rule of thumb is in line with general trend of exact optimization results. With time passing by households should invest less in risky assets (stock) and more in the risk-free ones (bonds). Their diminishing time horizon doesn't allow to smooth highly deviating portfolio returns between periods in case of any drop.
Carried optimization showed that this trend should be rather hyperbolic than linear, thus this rule does not fit perfectly to the results. However in real life it might be successfully utilised because of its simplicity.

5.4 Epstein-Zin-Weil preferences

5.4.1 a)

In the baseline formulation of the problem, agents have CRRA preferences, given by:

$$u(c_t) = \frac{1}{1-\theta} c_t^{1-\theta} \tag{21}$$

which obvious remark that $\lim_{\theta \to 1} u(c_t) = \ln c_t$. Now we want to consider EZW preferences, i.e.:

$$u(c_t) = \Big((1-\tilde{\beta}) c_t^{\rho} + \tilde{\beta} \big(\mathbb{E}_t (u_{t+1}^{\alpha}) \big)^{\frac{\rho}{\alpha}} \Big)^{\frac{1}{\rho}} \tag{22}$$

We should note, however, that EZW preferences (eq. 22) may be considered generalisation of CRRA (eq. 21). In other words, CRRA is just a special case of EZW. Indeed, taking $\rho = \alpha$ yields preference equivalent to CRRA.

In quantitative analyses, it is always reasonable to generalise. **Substituting EZW (with α and ρ parameters) for CRRA means, in detail, taking an agent with $RRA = 1 - \alpha$ and aggregating his utility from current consumption with utility for certainty equivalent (CE) of future consumption by a CES aggregator with $ES = \frac{1}{1-\rho}$.**

Such a formulation allows for distinguishing between agent's risk aversion and intertemporal elasticity of substitution without loss of any particular characteristics of the previously assumed CRRA function.

5.4.2 b)

Contrary to the instruction, we will conduct a step-by-step Bellman equation solution, instead of using guess-and-verify method.

Under EZW preferences agents face a linear programming problem given by a following Bellman equation:

$$V_t(w_t) = \max_{c_t, \alpha_t} \Big((1 - \tilde{\beta}) c_t^{\rho} + \tilde{\beta} \big(\mathbb{E}_t (V_{t+1}(w_{t+1})^{\alpha}) \big)^{\frac{\rho}{\alpha}} \Big)^{\frac{1}{\rho}} \tag{23}$$

Given budget constraint and the fact that EZW value function is homogenous in wealth (i.e., $V_t(w_t) = \phi_t w_t$), we obtain:

$$V_t(w_t) = \max_{c_t, \alpha_t} \Big((1 - \tilde{\beta}) c_t^{\rho} + \tilde{\beta} \underbrace{\mathbb{E}\big(\phi_{t+1}^{\alpha} (\alpha_t R_{t+1}^{p})^{\alpha} \big)^{\frac{\rho}{\alpha}}}_{=\mu_t^{\rho}} (w_t - c_t)^{\rho} \Big)^{\frac{1}{\rho}} \tag{24}$$

Differentiating with respect to consumption gives the first FOC:

$$c_t^{\rho - 1} = \frac{\delta}{1 - \delta} (w_t - c_t)^{\rho - 1} \mu_t^{\rho} \tag{25}$$

From homogeneity we obtain that consumption is a linear function of wealth ($c_t = \psi_t w_t$). Substituting to the equation above and transforming yields:

$$\mu_t^{\rho} = \frac{1 - \tilde{\beta}}{\tilde{\beta}} \Big(\frac{\psi_t}{1 - \psi_t} \Big)^{\rho - 1} \tag{26}$$

Now substituting to the value function we get:

$$\phi_t = (1 - \tilde{\beta})^{\frac{1}{\rho}} \Big(\frac{c_t}{w_t} \Big)^{\frac{1-\rho}{\rho}} \tag{27}$$

Which, after substituting to eq. 26 and transforming, yields Euler equation in a following form:

$$\mathbb{E} \Bigg(\Big(\tilde{\beta} (\frac{c_{t+1}}{c_t})^{\rho - 1} R_{t+1}^{p} \Big)^{\frac{\alpha}{\rho}} \Bigg)^{\frac{\rho}{\alpha}} \tag{28}$$

Now optimising with respect to the portfolio choice α_t may be simplified by transforming our Bellman equation into:

$$\max_{\alpha_t} \mathbb{E}\Big((\phi_{t+1} R^p_{t+1})^{\alpha}\Big)^{\frac{1}{\alpha}} \tag{29}$$

Hence FOC for a given asset $i \in \{1, 2\}$ goes:

$$\mathbb{E}\big(\phi^{\alpha}_{t+1}(R^p_{t+1})^{\alpha-1}(R^i_{t+1} - R^1_{t+1})\big) = 0 \tag{30}$$

Now, we can transform the formula for ϕ_{t+1}:

$$\phi_{t+1} = \frac{(1-\tilde{\beta})^{\frac{1}{\rho}}}{w_t(1-\psi_t)^{\frac{\rho-1}{\rho}}}\Big(\frac{c_{t+1}}{R^p_{t+1}}\Big)^{\frac{\rho-1}{\rho}} \tag{31}$$

Finally, substituting (31) into (30) yields:

$$\mathbb{E}\Big(\tilde{\beta}^{\frac{\alpha}{\rho}}\big(\frac{c_{t+1}}{c_t}\big)^{\frac{\alpha}{\rho}(\rho-1)}(R^p_{t+1})^{\frac{\alpha}{\rho}-1} R^i_{t+1}\Big) = 1 \tag{32}$$

Which is second formulation of Euler equation.

5.4.3 c)

5.4.4 d)

Actually, we do not need computer to check whether EZW can be reduced to CRRA. Note that when we have $\alpha = \rho$, then after substituting to equation (32) this gives indeed:

$$\mathbb{E}\Big(\big(\frac{c_{t+1}}{c_t}\big)^{\rho-1} R^i_{t+1}\Big) = 1 \tag{33}$$

6 ∅

No Problem 6 in given Project. So this is a dummy section just to keep numeration correct.

7 Problem 7: Summary of Papers

7.1 Cocco, Gomes & Mænhunt, 2001

Note: From now on, the paper will be referred to as CGM.

7.1.1 Aim of the paper

The primary aim of the CGM paper is to establish, calibrate and analyse an OLG model that would be able to rationalise popular rules promoted by financial advisers that with age investors should decrease amount of stock and move their resources towards bonds, T-bills, or generally, no-risk assets. The authors try to design a reliable, data-adherent quantitative framework for comparative welfare analysis of different portfolio choice strategies. Their approach includes, i.a., risky labour income and complex formulation of agent heterogeneity which includes age and education.

7.1.2 Model formulation

Agents optimise a following utility function:

$$\mathbb{E}\sum_{t=1}^{T}\delta^{t-1}\Big(\prod_{j=0}^{t-1}p_j\Big)\frac{C_{it}^{1-\gamma}}{1-\gamma} \tag{34}$$

where p_j is a probability of survival from j to $j+1$, δ is a discount factor, and γ is RRA coefficient. We should note that the function given in 34 does **not** include bequest.

Labour income is modelled with a following equation:

$$\ln(Y_{it}) = f(t, Z_{it}) + v_{it} + \epsilon_{it} \tag{35}$$

$f(t, Z_{it})$ is a deterministic function responsible for modelling agent heterogeneity (its arguments are age and a vector of individual-specific characteristics, such as education). ϵ_{it} is an iid, zero-mean normal, idiosyncratic income shock. v_{it}, on the other hand, is a random walk component that captures, roughly speaking, state of the economy - in the paper it is called *persistent shock*. Formally:

$$v_{it} = v_{i,t-1} + \xi_t + \omega_{it} \tag{36}$$

Both ξ_t and ω_{it} are normal with zero mean. Finally, old age pension is deterministic and given by a fixed fraction λ of the last working period wage:

$$\ln(Y_{i,t>K}) = \ln(\lambda) + f(K, Z_{iK}) + v_{iK} \tag{37}$$

There are two classes of assets in the model: risky equity and riskfree bonds (in CGM: *T-bills*). Excess return is given by:

$$R_{t+1} - \overline{R_f} = \mu + \eta_{t+1} \tag{38}$$

Importantly, $Corr(\xi_t, \eta_t) = \rho \neq 0$. Agents face zero borrowing constraints in each type of asset.

All in all, choice variables (*jumpers*) are consumption and fraction of wealth invested in stocks (C_{it}, α_{it}), state variable is wealth X_{it}[3], and agent optimisation problem is given by a following Bellman equation:

$$V_{it}(X_{it}) = \max_{C_{it},\alpha_{it}} u(C_{it}) + \delta p_t \mathbb{E} V_{i,t+1}(X_{i,t+1}) \tag{39}$$

$$\text{such that} \tag{40}$$

$$X_{i,t+1} = Y_{i,t+1} + (X_{it} - C_{it})\big(\alpha_{it} R_{t+1} + (1-\alpha_{it})\overline{R_f}\big) \tag{41}$$

The model is calibrated as to assure highest possible fit to the data. Parameters in equations 35 and 36 are estimated on the basis of PSID data. Asset mean returns and volatilities are also adjusted to the real life data.

7.1.3 Findings

Having once established a quantitative framework, CGM aim to compare different portfolio strategies. CGM compute the optimal, endogenous portfolio choice strategy and then compare it with three exogenous ones proposed, respectively, by Samuelson (1969), Merton (1971), and Malkiel (1996):

$$\alpha = \frac{\mu}{\gamma \sigma_\eta^2} = \text{const} \tag{42}$$

$$\frac{\alpha_t W_t}{W_t + PDV_t \mathbb{Y}} = \frac{\mu}{\gamma \sigma_\eta^2} \tag{43}$$

$$\alpha_t = \frac{100 - t}{100} \tag{44}$$

where $\mathbb{Y}$ in 43 represents stream of all future labour incomes.
CGM find what follows:

- **Merton heuristic**: agents lose 0.152 p. points of annual consumption
- **Malkiel heuristic**: agents lose 0.637 p. points of annual consumption
- **Samuelson heuristic**: agents lose 1.531 p. points of annual consumption

Summing up, one might note that accounting for the (at least deterministic) part of labour income stream impacts optimal portfolio choice in a significant way, while volatility of such a stream is not a crucial factor. This is why static Samuelson's heuristic yields most severe losses, while dynamic heuristics of Merton and Malkiel do not differ much.

The key contribution of CGM is undoubtely an insight into forces shaping the asset curve over time. CGM argue that labour income is treated by the agents as a substitute of the risk-free asset. Building on that, CGM optimise performing

[3]Formally, there are two more states: the first is time (trivial), and the second is random walk component v_{it} which can be normalised to 1

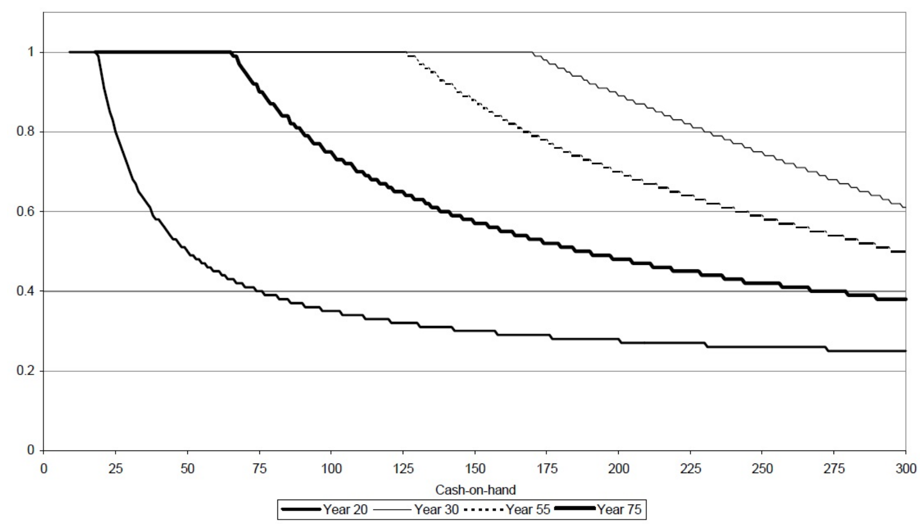


Figure 15: Optimal portfolio choice *vs* wealth for agents of different ages (α_t *vs* W_t for given ts)

a backward induction and show that **for agents with little wealth, labour income constitutes a substantial part of assets, being at the same time substitute for the risk-free asset**, which makes them more inclined to invest in equity. On the other hand, **wealthy agents receive only small fraction of their assets in the form of labour income**, so optimality condition requires them to invest more in bonds. Hence, CGM argue, we establish that $\frac{\partial \alpha_t}{\partial W_t} < 0$.

Result on portfolio-wealth dependence leads to important remarks on the relation between α_t and agents age. As agents get older, present value of their expected stream of labour income shrinks, which leads to the diminishing of implicit risk-free asset holdings[4] relative to the wealth to be invested. This makes elderly agents more inclined to invest in bonds so to make up for diminishing implicit risk-free asset holdings. Formally, $\frac{\partial \alpha_t}{\partial t} < 0$.

Arguments presented above are straightforward for the retirement stage, where agent faces no labour income risk. However, as we can see in Figure 15, they remain valid also for the working agents.

7.2 Lee M. Lockwood (2018)

7.2.1 Aim of the paper

In the paper *'Incidental Bequests and the Choice to Self-Insure Late-Life Risks'* the author is proposing and testing if the bequest motive can be a new explanation for the fact that many retirees self-insure. To achieve that he estimates few versions of a life-cycle model with retirement and he checks if they can

[4] Remember that labour income is a substitute for risk-free asset!

match retirees' choices in terms of savings and long-term care insurances. The author also wants to answer the question if bequest motives are a significant improvement to the standard life-cycle models.

7.2.2 Model

The model is similar to the one we discussed during classes. Preferences are such that the individual maximizes expected discounted utility from consumption and bequests, that is:

$$EU_t = u(c_t) + E_t\left[\sum_{a=t+1}^{T+1} \beta^{a-t}(\prod_{s=t}^{a-1}(1-\delta_s))[(1-\delta_a)u(c_a) + \delta_a v(b_a)]\right] \tag{45}$$

With standard CRRA utility from consumption and utility from bequest equal to:

$$v(b) = (\frac{\phi}{1-\phi})^{\sigma} \frac{(\frac{\phi}{1-\phi}c_b + b)^{1-\sigma}}{1-\sigma} \tag{46}$$

For$\phi\epsilon(0,1)$ which is the marginal propensity to bequest and where δ_s is the stochastic probability that an (s-1) year-old will die before age s.

In the model individuals can be in one of five health states:

- healthy
- requiring home health care
- requiring assisted living facility care
- requiring nursing home care
- dead

And this health state together with age, sex and income quintile influence the future medical care costs and future health state.

The model analyze people decision after they retire, so individuals who are eligible to purchase long-term care insurance (based on parameters listed before) make a once-and-for-all decision about whether to buy long-term care insurance.

There are also government transfers in proposed model with social insurance programs that ensure a minimum standard of living after paying for any medical care by putting a floor under net wealth.

7.2.3 Data

The main data set that was used by the author is the Health and Retirement Study (HRS), a longitudinal survey performed in US on a representative sample population over 50 years old. The HRS surveys more than 22,000 Americans every two years.

Six waves were used, which occur in even-numbered years from 1998–2008. The analysis was restricted to single retirees who are at least 65 years old in

1998 and who do not miss any of the 1998–2008 interviews while they are alive. Hence the resulting sample contained only 3,386 individuals.

To enrich the data the author takes supplement information from the National Long-Term Care Survey (NLTCS). Mainly prices of medical care. Also some other sources were used to set a values to some specific parameters.

7.2.4 Results

In this paper author compared the results of the model with and without bequest motives for different settings of parameters. For a model with bequest motives even relatively small bequest motives have a large effect on purchases of insurance against late-life risks. This model fits the data significantly better than the model without bequest motives, which does very poorly at that field.

The other interesting result is visible when the author introduced the one hundred percent estate and gift tax in his model. In this case long-term care insurance ownership rate decreases from 24.0 percent to 4.3 percent, when compared to standard simulation of the model.

What is more bequest motives significantly reduce Purchases of annuities. The other results are the same as during classes, introduction of bequest motives decreases the consumption spending and slow down the wealth decline for older generations.

The main conclusion is that the introduction of bequest motives in models changes a lot in terms of obtained results, so they should be used more often in research and literature. The other conclusion is that policies that affect the behavior of retirees, are likely to have significant effects on the economy, so bequest motives should not be trivialized.