Quantitative Macroeconomics - Homework 2

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29 September 2019

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1 Exercise 1: Function Approximation: Univariate

1.1 Taylor approximation of the exponential function

Question: Approximate f(x) = $x^{0.321}$ with a Taylor series around x = 1. Compare your approximation over the domain (0,4). Compare when you use up to 1; 2; 5 and 20 order approximations.https://www.overleaf.com/project/5d939541553ab000018bf816 Discuss your results.

Plot:

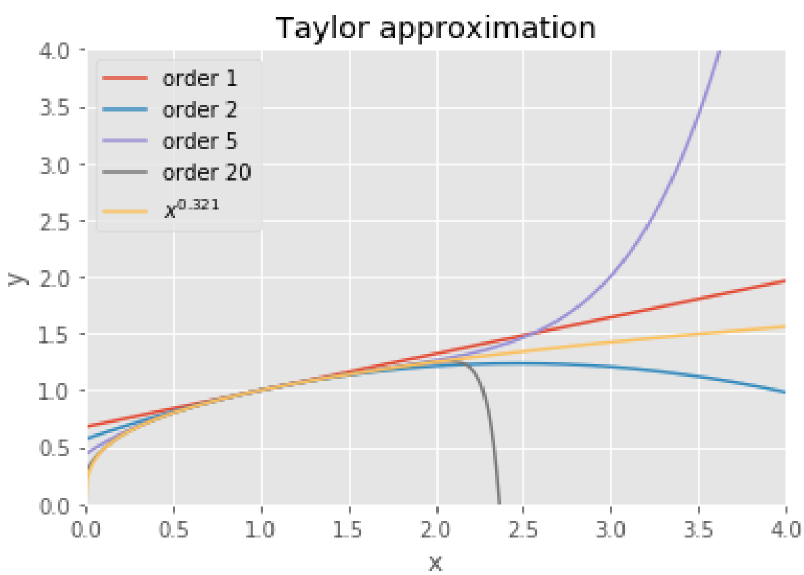


Figure 1: Taylor approximation of the exponential function

Comment: As we can see our approximation is quite good around x = 1. However if we go further away from x our approximations are becoming less accurate. We can also notice that higher order Taylor functions are giving worse results than lower order Taylor functions, which could be surprising.

1.2 Taylor approximation of the ramp function

Question: Approximate the ramp function f(x) = $\frac{x+|x|}{2}$ with a Taylor series around x = 2. Compare your approximation over the domain (-2,6). Compare when you use up to 1; 2; 5 and 20 order approximations. Discuss your results.

$$f(x) = \frac{x + |x|}{2} \tag{1}$$

Plot:

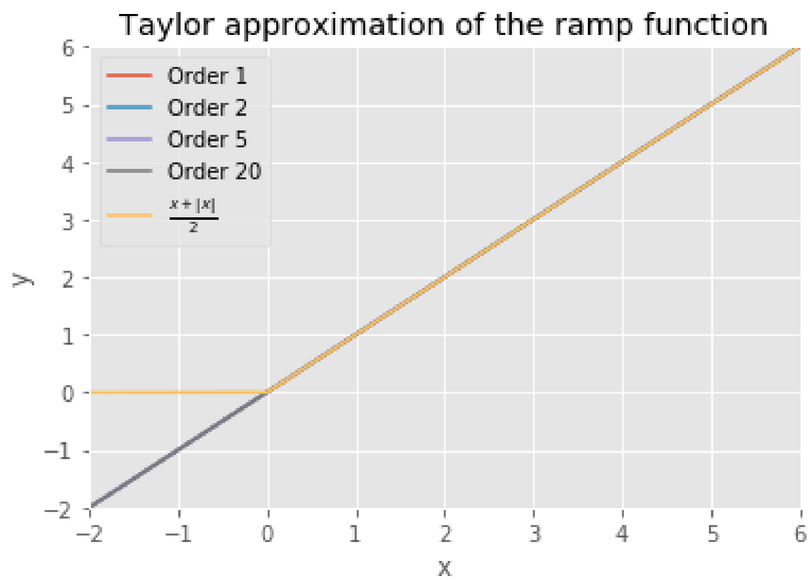


Figure 2: Taylor approximation of the ramp function

Comment: Again our approximation is accurate around x = 2, but it this case it is also accurate for all 'x' that are greater than 0. However, in point (0,0) we have a kink because of an absolute value, that is why for x $\leq$ 0 our approximation fails completely. Our Taylor series approximation just can not deal with ramp functions.

1.3 Miscellaneous approximations

Approximate these three functions: $e^{\frac{1}{x}}$, the runge function $\frac{1}{1+25x^2}$, and the ramp function $\frac{x+|x|}{2}$ for the domain x[-1; 1] with:

- Evenly spaced interpolation nodes and a cubic polynomial. Redo with monomials of order 5 and 10. Plot the exact function and the three approximations in the same graph. Provide an additional plot that reports the errors as the distance between the exact function and the approximand.

- Chebyshev interpolation nodes and a cubic polynomial. Redo with monomials of order 5 and 10. Plot the exact function and the three approximations in the same graph. Provide an additional plot that reports the errors as the distance between the exact function and the approximand.

- Chebyshev interpolation nodes and Chebyshev polynomial of order 3, 5 and 10. How does it compare to the previous results? Report your approximation and errors.

  Comment:

  In all the subpoints of Exercise 1.3 we need to use various types of interpolation nodes and polynomials to check how proper would be the approximation of the three mentioned functions in the selected nodes and outside them.

  Chebyshev approach to interpolation nodes differ from evenly spaced ones. They are more closely spaced near the endpoints of the interpolation interval [-1, 1] and less in its centre, thus allowing the approximations to have less extreme errors near the ends of the approximation. $x_j = cos(\frac{2j-1}{2n}\pi), j = 1, 2, ..., n$

  Chebyshev polynomials (of the first kind) are trigonometric polynomials $\psi_j(x) : [1, 1][1, 1]$ defined by $\psi_j(x) = cos(narccosx)$ and are orthogonal with respect to the weight function $(1 - x^2)^{-0.5}$

1.3.1 a) Evenly spaced interpolation nodes and a cubic polynomial

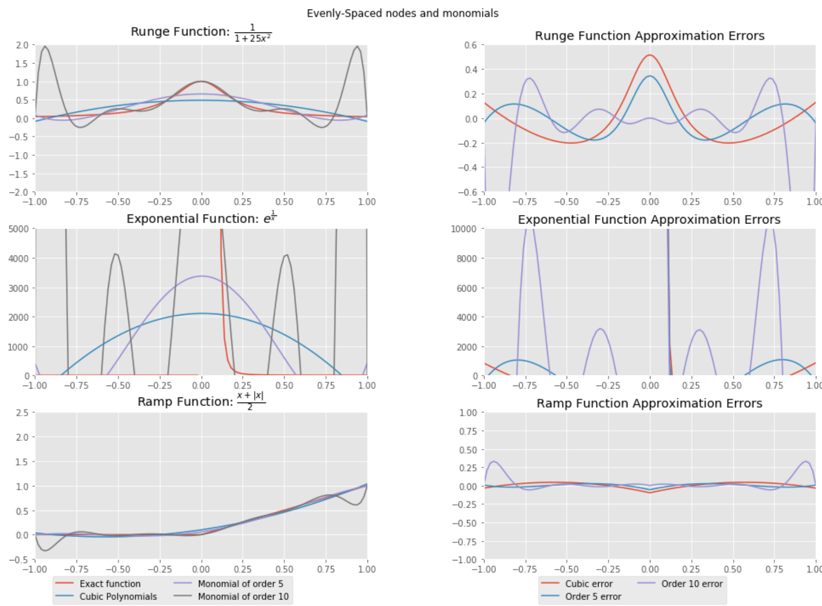


Figure 3: Function approximation

As we can see, the approximations using cubic polynomials of exponential function, ramp function and runge function on evenly-spaced nodes was more or less accurate in the interpolation nodes depending on the order of the polynomials. These approximations are shown on the plots in the left part of Figure 3. However, there were certain errors in function approximation, which are shown in the right part of Figure 3. The polynomials of higher orders were more accurate in the nodes, but outside them the errors made were usually more extreme.

The polynomials of exponential function yielded high approximation errors as a result of the function not being continuous. This is a special case.

1.3.2 b) Czebyszow interpolation nodes and a cubic polynomial

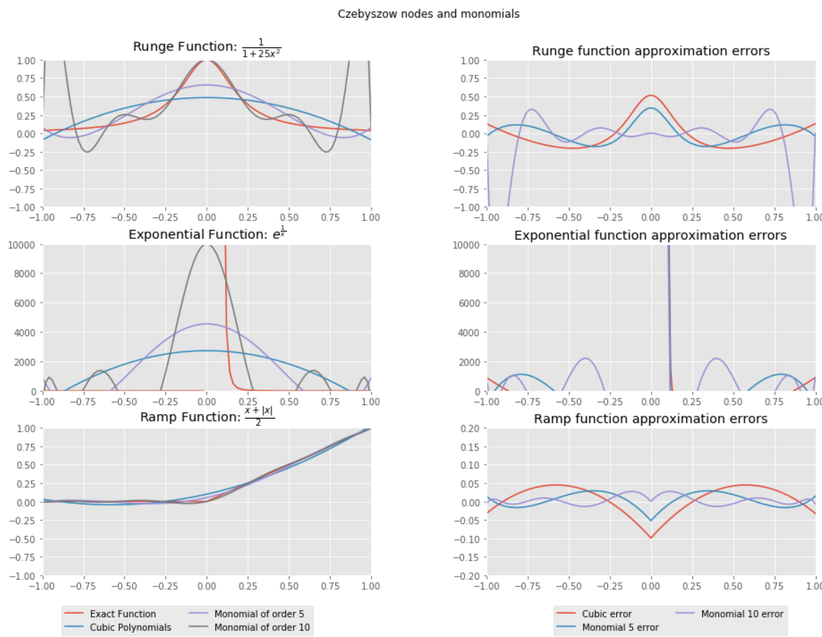


Figure 4: Function approximation

The approximations using cubic polynomials of exponential function, rump function and runge function on Chebyshev nodes gave results that slightly differ from a).

1.3.3 c) Chebyshev interpolation nodes and Chebychev polynomials

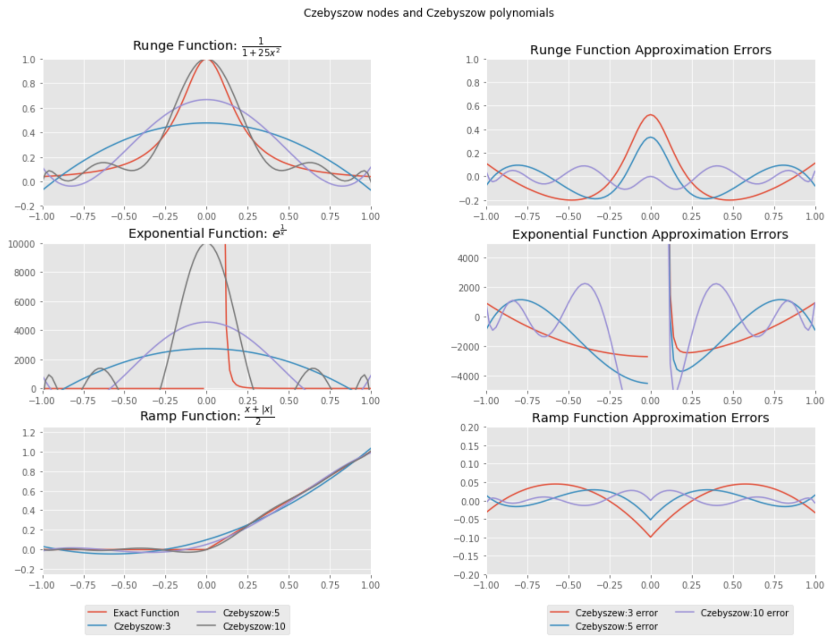


Figure 5: Function approximation

The approximations using Chebyshev polynomials of exponential function, rump function and runge function on Chebyshev nodes gave results that differ from a) and b).

2 Question 2: Function Approximation: Multivariate

2.1 Elasticity of substitution

Show that σ is the ES (hint: show this analytically).

$$f(k,h) = ((1-\alpha)k^{\frac{\sigma-1}{\sigma}} + \alpha h^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} \tag{2}$$

Now we calculate marginal productivity of labour:

$$MPL = \frac{\partial f(k,h)}{\partial h} = \alpha h^{\frac{-1}{\sigma}}((1-\alpha)k^{\frac{\sigma-1}{\sigma}} + \alpha h^{\frac{\sigma-1}{\sigma}})^{\frac{1}{\sigma-1}} \tag{3}$$

And marginal productivity of capital:

$$MPK = \frac{\partial f(k,h)}{\partial k} = (1-\alpha)k^{\frac{-1}{\sigma}}((1-\alpha)k^{\frac{\sigma-1}{\sigma}} + \alpha h^{\frac{\sigma-1}{\sigma}})^{\frac{1}{\sigma-1}} \tag{4}$$

We divide both marginal productivities:

$$\frac{MPL}{MPK} = \frac{\alpha h^{\frac{-1}{\sigma}}}{(1-\alpha)k^{\frac{-1}{\sigma}}} \tag{5}$$

Now we take a log:

$$log(\frac{MPL}{MPK}) = log(\frac{\alpha}{1-\alpha}) + \frac{1}{\sigma}log(\frac{k}{h}) \tag{6}$$

To receive elasticity of substitution we need to take derivative of the above equation with respect to $log(\frac{h}{k})$. Hence we get:

$$ES = \sigma \tag{7}$$

2.2 Labour share

We know that labour share is defined by:

$$LS = \frac{hw}{f(k,h)} \tag{8}$$

On competitive markets wage in this case is our MPH from previous subpoint:

$$w = MPH = (1-\alpha)k^{\frac{-1}{\sigma}}((1-\alpha)k^{\frac{\sigma-1}{\sigma}} + \alpha h^{\frac{\sigma-1}{\sigma}})^{\frac{1}{\sigma-1}} \tag{9}$$

Finally we get:

$$LS = \frac{\alpha h^{\frac{\sigma-1}{\sigma}}}{((1-\alpha)k^{\frac{\sigma-1}{\sigma}} + \alpha h^{\frac{\sigma-1}{\sigma}})^{\sigma}} \tag{10}$$

2.3 Multidimensional Chebyshev approximation (includes points 2.5 and 2.6)

We start with approximating CES with a 20-nodes Chebyshev polynomials technique. of deg. 3. First of all, the true CES function is plotted in Figure 6.

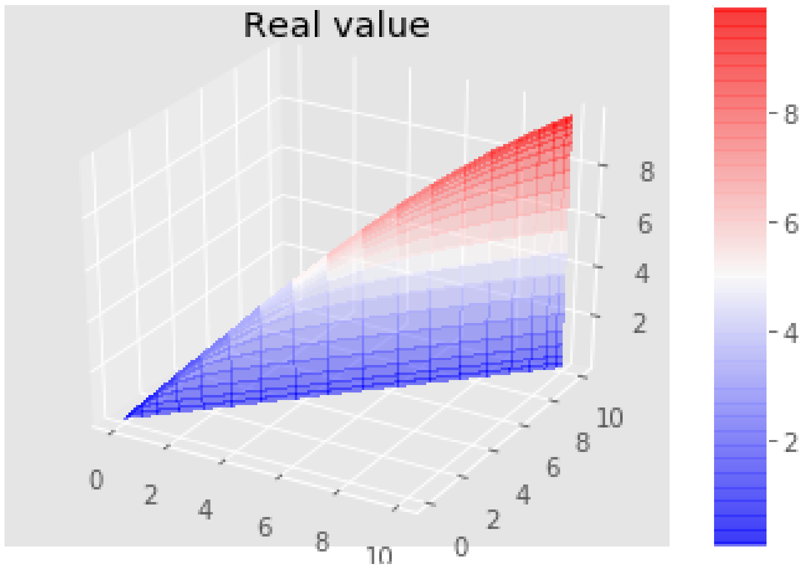
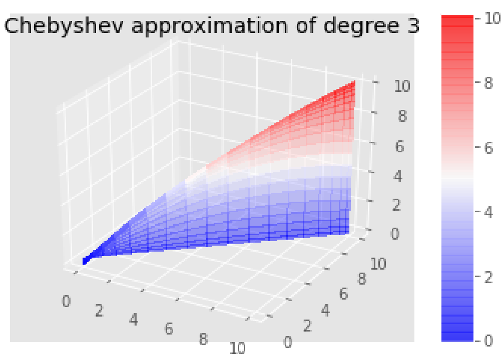
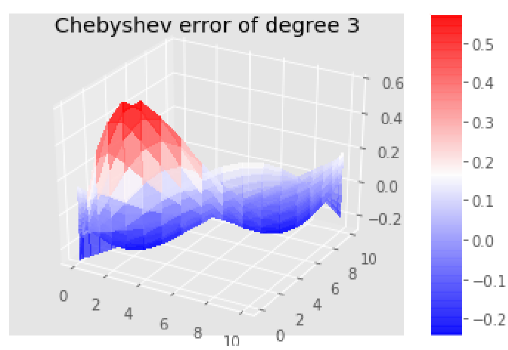


Figure 6: True CES function value

Figure 7 shows Chebyshev approximation of degree 3 alongside approximation error.



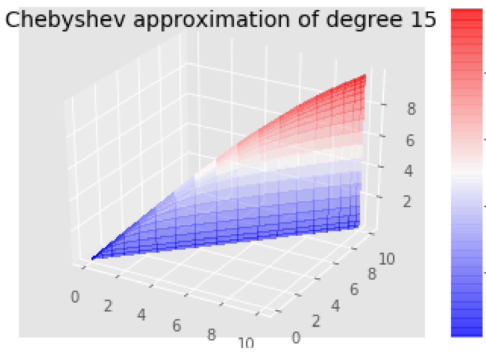
(a) Chebyshev approximation of deg. 3, $\sigma = 0.25$



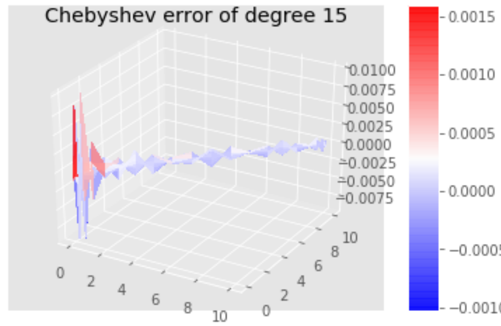
(b) Chebyshev approximation error

Figure 7: Multivariate Chebyshev approximation: 20 nodes, degree 3, $\sigma = 0.25$

Figure 8 shows Chebyshev approximation of degree 15 alongside approximation error.



(a) Chebyshev approximation of deg. 15, $\sigma = 0.25$



(b) Chebyshev approximation error

Figure 8: Multivariate Chebyshev approximation: 20 nodes, degree 15, $\sigma = 0.25$

Figures 9, 10, and 11 replicate previous approximations, but incorporating $\sigma = 5$

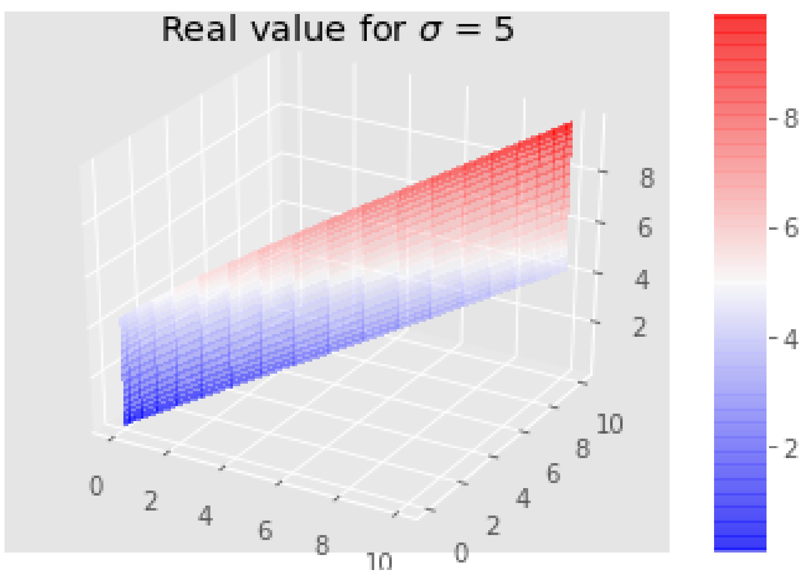
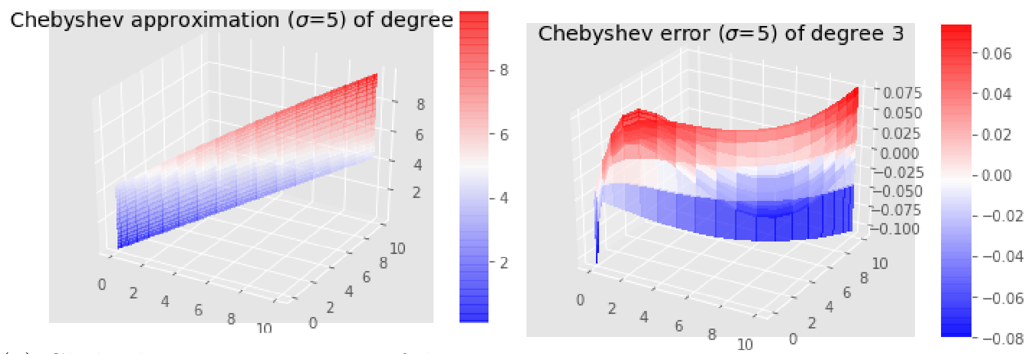


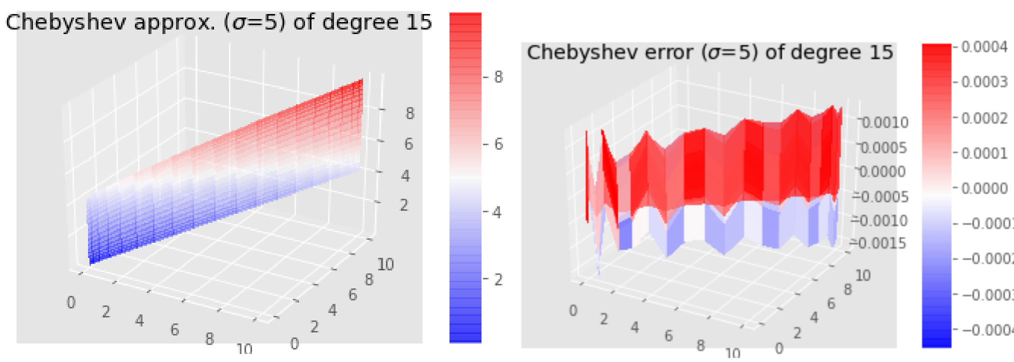
Figure 9: True CES function value with $\sigma = 5$



(a) Chebyshev approximation of deg. 15, $\sigma = 5$

(b) Chebyshev approximation error

Figure 10: Multivariate Chebyshev approximation: 20 nodes, degree 15, $\sigma = 5$



(a) Chebyshev approximation of deg. 15, $\sigma = 5$

(b) Chebyshev approximation error

Figure 11: Multivariate Chebyshev approximation: 20 nodes, degree 15, $\sigma = 5$

Finally, we are ask to replicate computations for $\sigma = 1$, but then CES exponent would diverge to infinity. Therefore, we assume that we should consider $\sigma = 10$. Figures 12, 13, and 14 show respective plots.

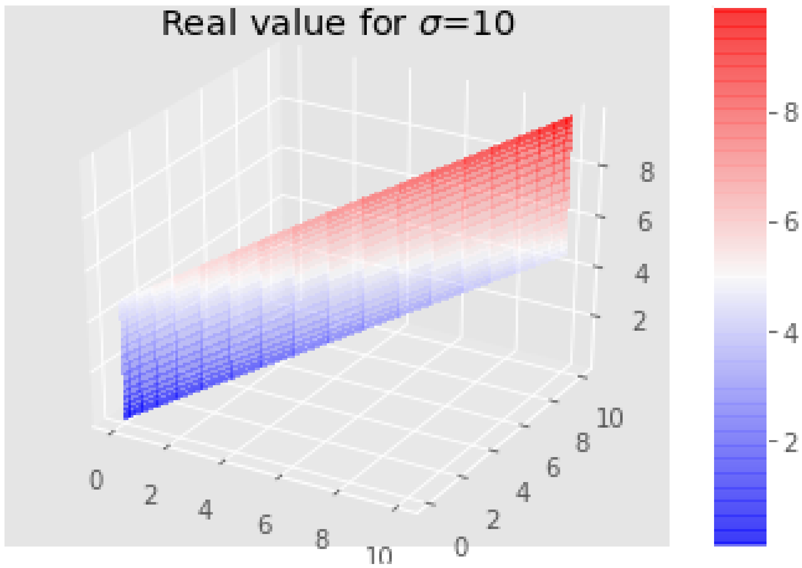
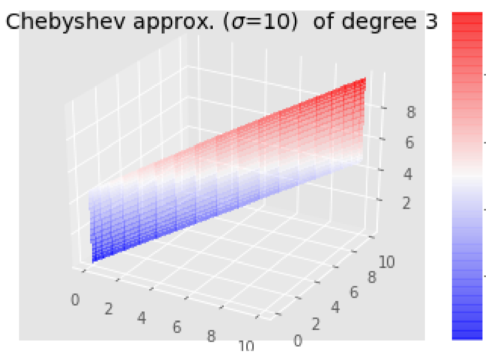
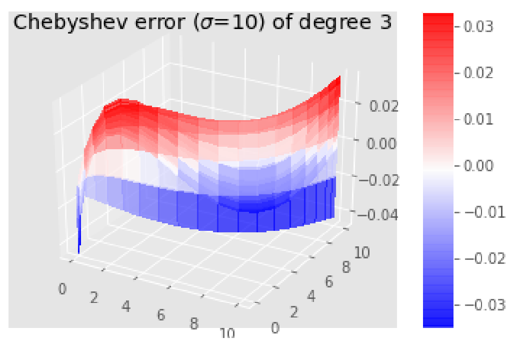


Figure 12: True CES function value with $\sigma = 10$

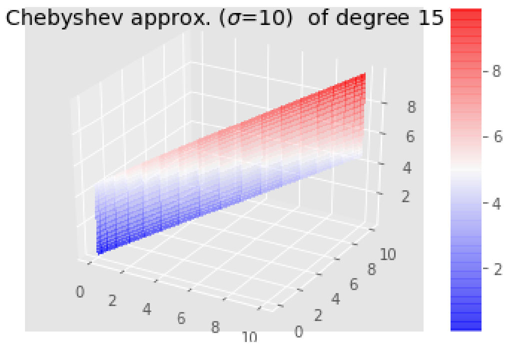


(a) Chebyshev approximation of deg. 15, $\sigma = 10$

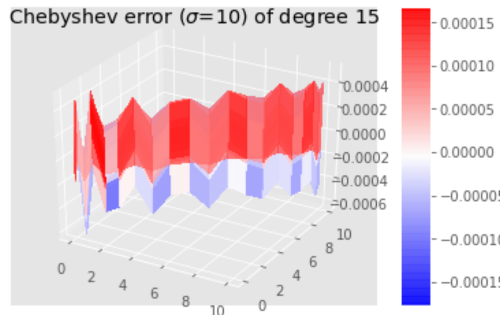


(b) Chebyshev approximation error

Figure 13: Multivariate Chebyshev approximation: 20 nodes, degree 15, $\sigma = 10$



(a) Chebyshev approximation of deg. 15, $\sigma = 10$



(b) Chebyshev approximation error

Figure 14: Multivariate Chebyshev approximation: 20 nodes, degree 15, $\sigma = 10$