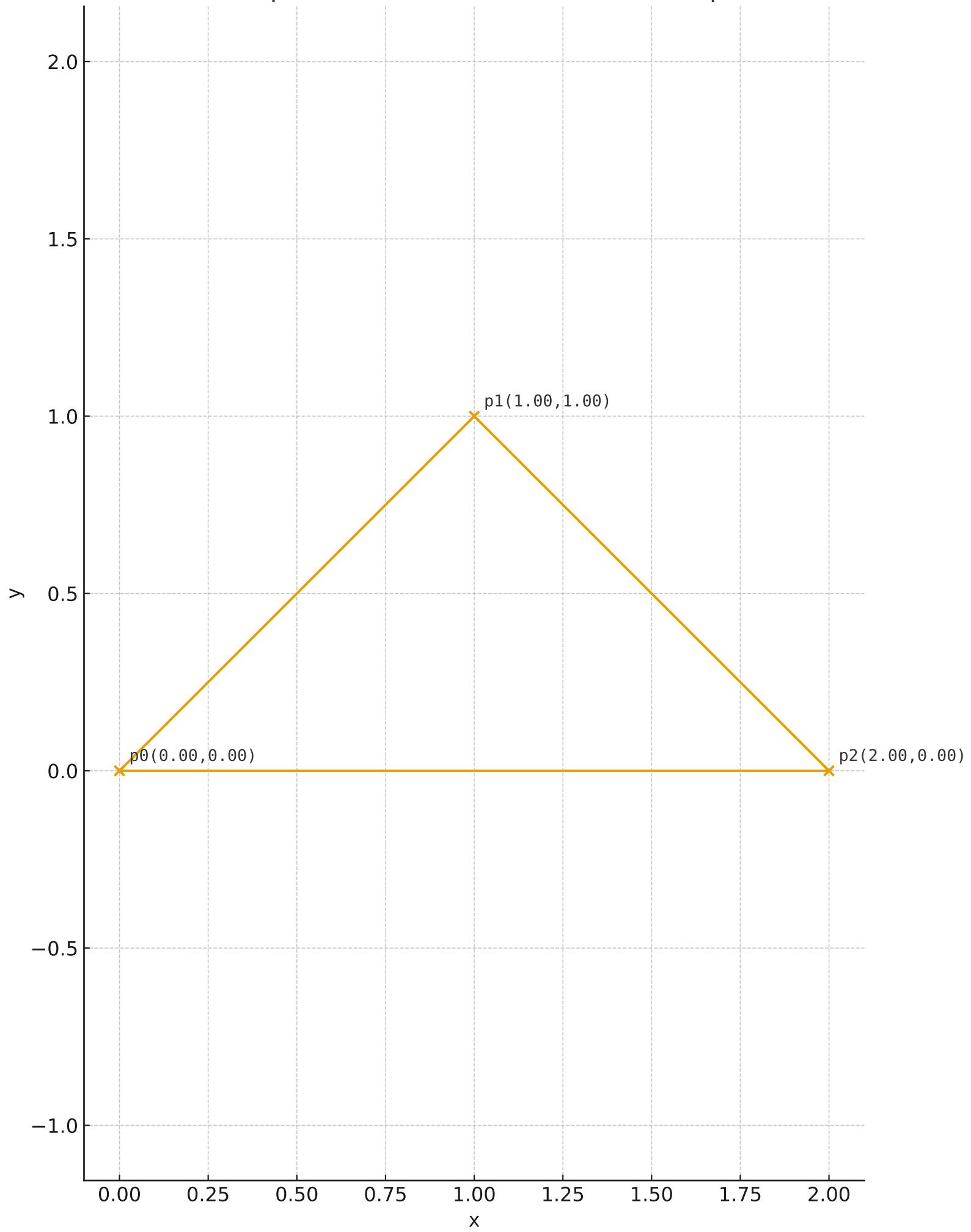


Point-Line Duality in R^2: Quick Reference

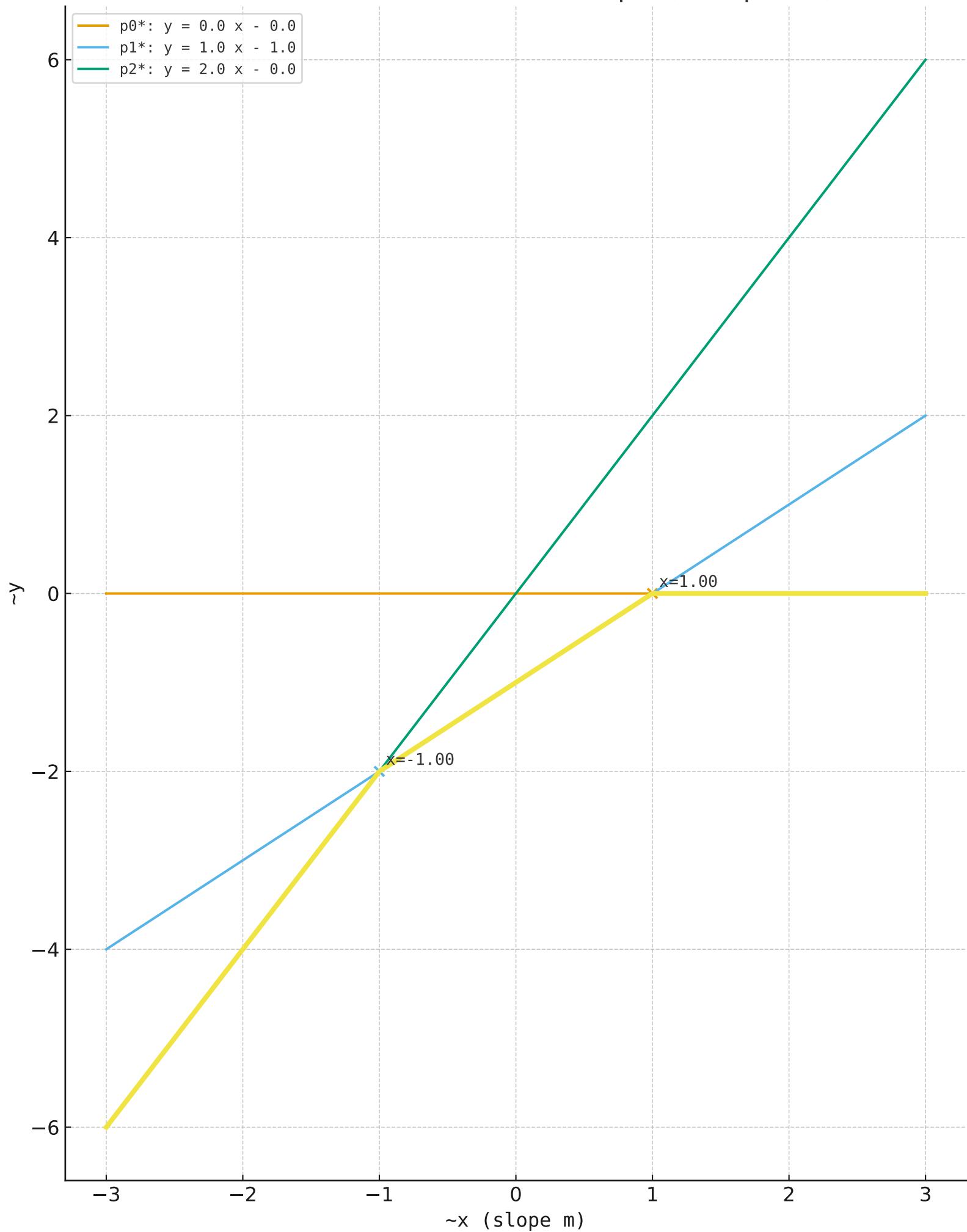
- Dual convention used here: $p=(a,b) \leftrightarrow p^*: \sim y = a^* \sim x - b$.
- Lines: $L: y = m*x + c \leftrightarrow L^* = (m, -c)$.
- Invariants: (1) Incidence preserved. (2) Above/below flips. (3) Involution.
- Upper convex hull of points \leftrightarrow lower envelope of dual lines.
- Envelope at slope m : $E(m) = \min_i (a_i*m - b_i)$.
- Supporting line of slope m has intercept $c^* = -E(m)$.
- Breakpoints where two lines tie: $a_i*m - b_i = a_j*m - b_j \rightarrow$ hull vertices.

Note: Pure vertical lines $x = \text{const}$ are not representable as $y = m*x + c$. Rotate slightly or switch to projective duality if needed.

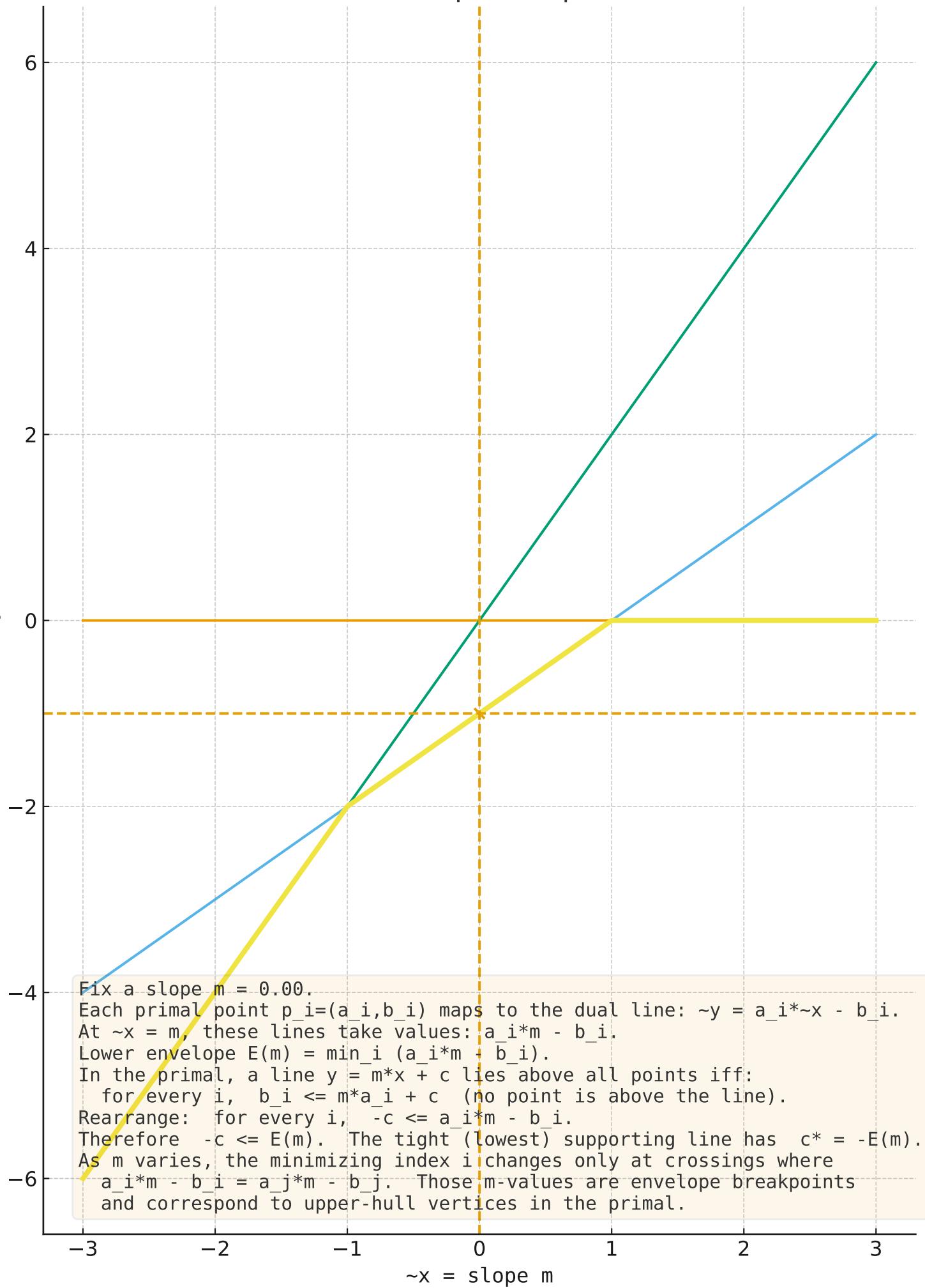
Primal: points and convex hull (example A)



Dual lines and lower envelope (example A)



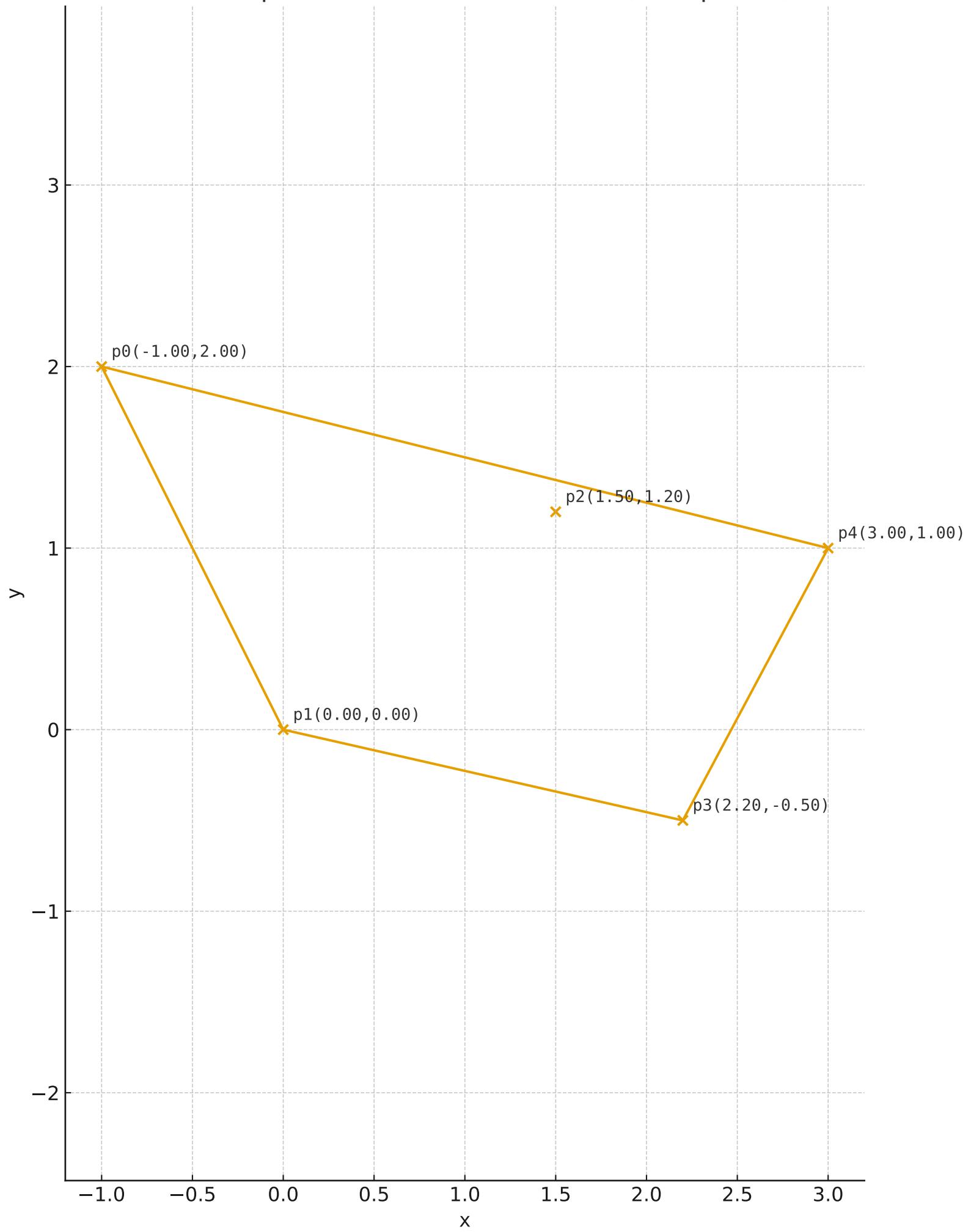
Where the lower-envelope inequalities come from



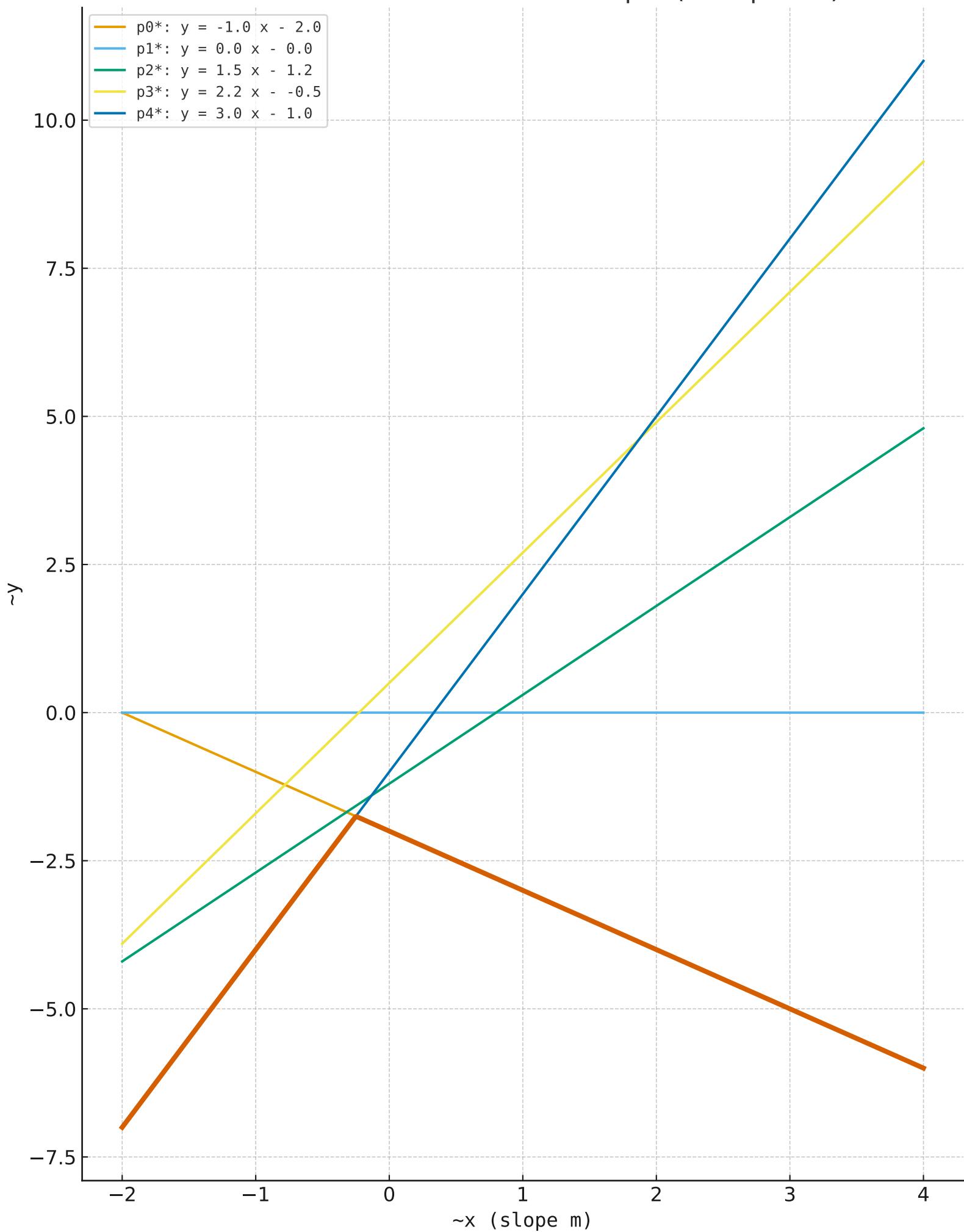
Incidence and Above/Below Flip

- Incidence: $p=(a,b)$ on $L: y = m*x + c$.
- Evaluate p^* : $\sim y = a*\sim x - b$ at $\sim x = m \rightarrow \sim y = a*m - b = -c$.
- Hence $(m, -c) = L^*$ lies on p^* .
- Above/below flip: $b > m*a + c \leftrightarrow -c < a*m - b$.
- Collinearity vs concurrency: collinear points \leftrightarrow concurrent dual lines.

Primal: points and convex hull (example B)



Dual lines and lower envelope (example B)



Hull <-> Envelope: Inequalities, step by step

- 1) Map $p_i = (a_i, b_i) \rightarrow \sim y = a_i * \sim x - b_i$.
- 2) Fix slope m ; primal lines are $y = m * x + c$.
- 3) Above all points iff $b_i \leq m * a_i + c$ for every i .
- 4) Rearrange: $-c \leq a_i * m - b_i$ for every i .
- 5) Tightest support: $c^* = -\min_i (a_i * m - b_i)$.
- 6) Lower envelope value at $\sim x = m$ equals $\min_i (a_i * m - b_i)$.
- 7) Breakpoints where two lines tie map to upper-hull vertices.

Beyond R^2 and Handling Degeneracies

- Higher-d: (a, b) in $R^{d-1} \times R$ $\leftrightarrow \sim x_d = a \cdot \sim x - b$.
- Hyperplane $\sim x_d = m \cdot \sim x + c$ $\leftrightarrow (m, -c)$.
- Applications: halfspace intersection \leftrightarrow upper hull; k-sets \leftrightarrow k-levels.
- Degeneracies: rotate to avoid verticals, or use projective duality.