Math → 16.09.18

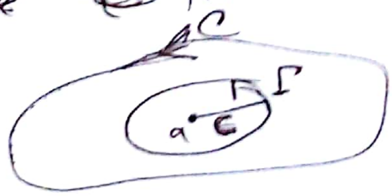
1. state and prove cauchy theorem

2. Evaluate $\oint_C \frac{dz}{z-a}$ where c is any simple closed curve C and $z = a$ is (a) outside c (b) inside c

Solution: If a is outside c, then $f(z) = \frac{1}{z-a}$ is analytic every-where inside and on c.

Hence by cauchy-theorem $\oint \frac{dz}{z-a} = 0$

(b) suppose a is inside c and let Γ be a circle of radius ϵ with centre at $z = a$ so that Γ is inside C then



$$\oint_C \frac{dz}{z-a} = \int_\Gamma \frac{dz}{z-a} \longrightarrow (1)$$

Now On Γ, $|z-a| = \epsilon$ or $z-a = \epsilon e^{i\theta}$ i.e. $z = a+\epsilon e^{i\theta}$

$0 \leq \theta \leq 2\pi$

Thus since $dz = i\epsilon e^{i\theta} d\theta$

The right side of (1) becomes

$$\int_{\theta=0}^{2\pi} \frac{i\epsilon e^{i\theta} d\theta}{\epsilon e^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i$$

∴ required value.

H.W Varify cachy's theorem for the function

(a) $3z^2+iz-4$ (b) $3\cosh(z+2)$ if c is the square with vertices at ~~$[I], -[I]$~~ $i\pm i, -i\pm i$

inside

If c is circle $|z-2|=5$ (a) determine whether $\oint \frac{dz}{z-3}=0$ (b) Does your ~~yo~~ answer to contradict cauchy's theorem

state and prove cauchy's theorem

statement: Let $f(z)$ be analytic in a Region R and on it's boundary c. then

$$\oint_c f(z)\,dz = 0$$

This fundamental theorem, often called cachy's integral theorem or breifly Cauchy's theorem. ~~is valid for both simply and multiply connected regions.~~

prove: Since $f(z)=u+iv$ is analytic and has a contineous derivate

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}$$

it follows that the partial derivates

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$$
$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

are continuom inside and on c. Thus Green's theorem can be applied and we have

$$\oint_c f(z)\,dz = \oint_c (u+iv)(dx+idy)$$

$$= \oint_c u\,dx - v\,dy + i\oint_c v\,dx + u\,dy$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= 0$$

using the Cauchy-Riemann equation (1) and (2). [proved]

Problem-3:

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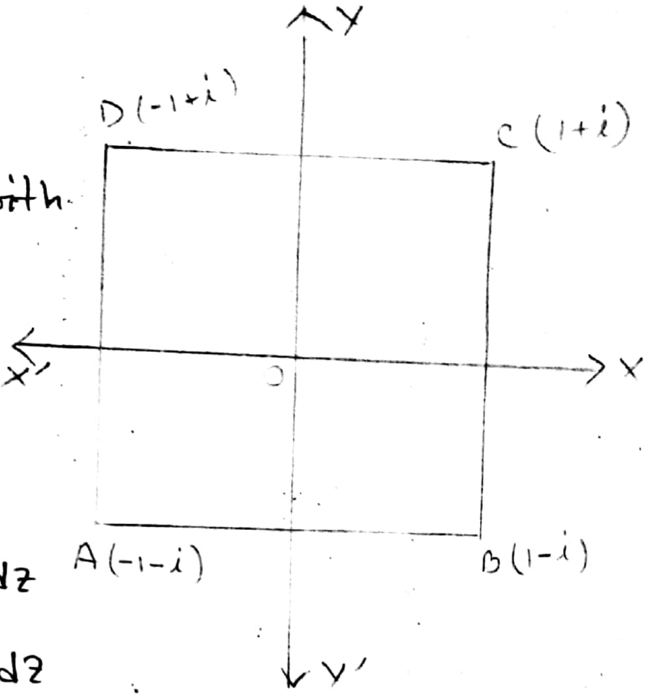
Verify cauchy's theorem for the function (a) $3z^2+iz-4$ (b) $5\sin 2z$. (c) $3\cosh(z+2)$ if c is the square with vertices at $1\pm i, -1\pm i$.

Soln-a:

let, $f(z) = 3z^2+iz-4$

where, c is the square with vertices at $1\pm i$, $-1\pm i$.



By cauchy's theorem we get,

$$\oint_c f(z)\,dz = 0$$

$$\therefore \oint_c f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CD} f(z)\,dz + \int_{DA} f(z)\,dz$$

$$= \int_{-1-i}^{1-i}(3z^2+iz-4)\,dz + \int_{1-i}^{1+i}(3z^2+iz-4)\,dz + \int_{1+i}^{-1+i}(3z^2+iz-4)\,dz + \int_{-1+i}^{-1-i}(3z^2+iz-4)\,dz$$

$$= \left[z^3+\frac{iz^2}{2}-4z\right]_{-1-i}^{1-i} + \left[z^3+\frac{iz^2}{2}-4z\right]_{1-i}^{1+i} + \left[z^3+\frac{iz^2}{2}-4z\right]_{1+i}^{-1+i} + \left[z^3+\frac{iz^2}{2}-4z\right]_{-1+i}^{-1-i}$$

$$= (1-i)^3 + \frac{i(1-i)^2}{2} - 4(1-i) - (-1-i)^3 - \frac{i(-1-i)^2}{2} + 4(-1-i) + (1+i)^3 + \frac{i(1+i)^2}{2} - 4(1+i) - (1-i)^3 - \frac{i(1-i)^2}{2} + 4(1-i) + (-1+i)^3 + \frac{i(-1+i)^2}{2} - 4(-1+i) - (1+i)^3 - \frac{i(1+i)^2}{2} + 4(1+i) + (-1-i)^3 + \frac{i(-1-i)^2}{2} - 4(-1-i) - (-1+i)^3 - \frac{i(-1+i)^2}{2} + 4(-1+i)$$

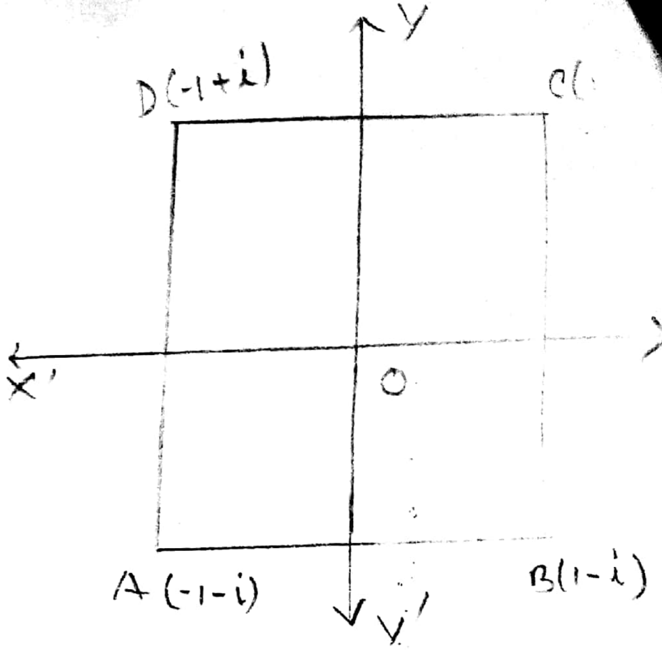
$\therefore \oint_c f(z)\,dz = 0$ verified the cauchy's theorem.

Solⁿ - 6:

Let, $f(z) = 5\sin 2z$.

By cauchy's theorem we get,

$\oint_c f(z)\,dz = 0$; where c is the square with vertices at $1 \pm i$ and $-1 \pm i$



$$\therefore \oint_c f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CD} f(z)\,dz + \int_{DA} f(z)\,dz$$

$$= \int_{-1-i}^{1-i} 5\sin 2z\,dz + \int_{1-i}^{1+i} 5\sin 2z\,dz + \int_{1+i}^{-1+i} 5\sin 2z\,dz + \int_{-1+i}^{-1-i} 5\sin 2z\,dz$$

$$= \frac{5}{2}\left[-\cos 2z\right]_{-1-i}^{1-i} + \frac{5}{2}\left[-\cos 2z\right]_{1-i}^{1+i} + \frac{5}{2}\left[-\cos 2z\right]_{1+i}^{-1+i} + \frac{5}{2}\left[-\cos 2z\right]_{-1+i}^{-1-i}$$

$$= \frac{5}{2}\left[-\cos 2(1-i) + \cos 2(-1-i) - \cos 2(1+i) + \cos 2(1-i) - \cos 2(-1+i) + \cos 2(1+i) - \cos 2(-1-i) + \cos 2(-1+\right.$$

$$\Rightarrow \oint_c f(z)\,dz = \frac{5}{2} \times 0$$

$$\Rightarrow \oint_c f(z)\,dz = 0$$

$\therefore$ Verified cauchy's theorem.

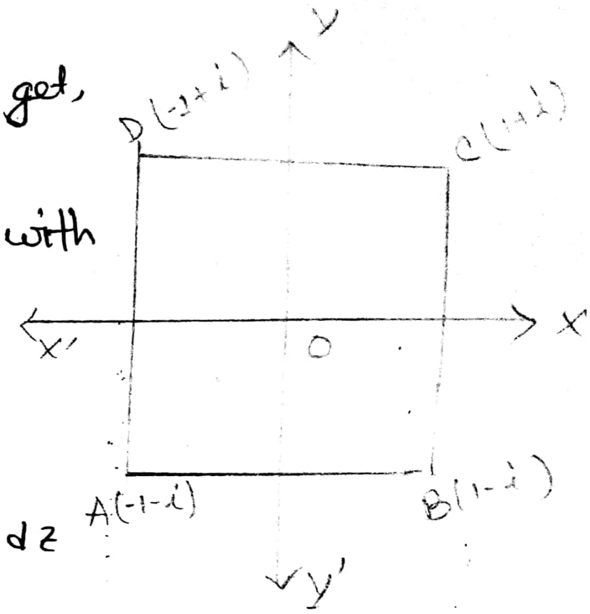
Sol^n - C:

Let, $f(z) = 3\cosh(z+2)$.

By cauchy's theorem we get,

$\oint_c f(z)\, dz = 0$.

where c is the squarre with verticees at $1 \pm i, -1 \pm i$



$$\therefore \oint_c f(z)\, dz = \int_{AB} f(z)\, dz + \int_{BC} f(z)\, dz + \int_{CD} f(z)\, dz + \int_{DA} f(z)\, dz$$

$$= \int_{-1-i}^{1-i} 3\cosh(z+2)\, dz + \int_{1-i}^{1+i} 3\cosh(z+2)\, dz + \int_{1+i}^{-1+i} 3\cosh(z+2)\, dz + \int_{-1+i}^{-1-i} 3\cosh(z+2)\, dz$$

$$= \Big[3\sinh(z+2)\Big]_{-1-i}^{1-i} + \Big[3\sinh(z+2)\Big]_{1-i}^{1+i} + \Big[3\sinh(z+2)\Big]_{1+i}^{-1+i} + \Big[3\sinh(z+2)\Big]_{-1+i}^{-1-i}$$

$$= 3\big[\sinh(1-i+2) - \sin(-1-i+2) + \sin(1+i+2) - \sinh(1-i+2) + \sinh(-1+i+2) - \sinh(1+i+2) + \sinh(-1-i+2) - \sin(-1+i+2)\big]$$

$$= 3\big[\cancel{\sinh(3-i)} - \cancel{\sinh(1-i)} + \cancel{\sinh(3+i)} - \cancel{\sin(3-i)} + \cancel{\sinh(1+i)} - \cancel{\sinh(3+i)} + \cancel{\sinh(1-i)} - \cancel{\sinh(1+i)}\big]$$

$= 3 \times 0$

$= 0$

$\therefore \oint_c f(z)\, dz = 0$

Verified cauchy's theorem.

Proplem - 4 :

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Verify cauchy's theorem for the function $z^3 - iz^2 - 5z$ if c is ⓐ the circle $|z| = 1$ ⓑ the circle $|z-1| = 2$ ⓒ the ellipse $|z-3i| + |z+3i| = 20$.

Solⁿ-a:

Given that, $f(z) = z^3 - iz^2 - 5z + 2i$

By cauchy's theorem, $\oint_c f(z)\,dz = 0$

where c is the circle $|z| = 1$

Let, $z = e^{i\theta}$ where, $0 \leq \theta \leq 2\pi$.

$\Rightarrow dz = ie^{i\theta}\,d\theta$.

Now,

$$\oint_c f(z)\,dz = \oint_c (z^3 - iz^2 - 5z + 2i)\,dz$$

$$= \int_0^{2\pi} \left(e^{3i\theta} - ie^{2i\theta} - 5e^{i\theta} + 2i\right) ie^{i\theta}\,d\theta$$

$$= i\left[\frac{e^{4i\theta}}{4i} - \frac{ie^{3i\theta}}{3i} - \frac{5e^{2i\theta}}{2i} + \frac{2ie^{i\theta}}{i}\right]_0^{2\pi}$$

$$= i\left\{\frac{1}{4i}\left(e^{8i\pi} - e^0\right) - \frac{1}{3}\left(e^{6i\pi} - e^0\right) - \frac{5}{2i}\left(e^{4i\pi} - e^0\right) + 2\left(e^{2i\pi} - e^0\right)\right\}$$

$$= i\left\{\frac{1}{4i}\times 0 - \frac{1}{3}\times 0 - \frac{5}{2i}\times 0 + 2\times 0\right\}$$

$$= i\times 0$$

$$= 0$$

$\therefore \oint_c f(z)\,dz = 0$

Thus, verified cauchy's theorem.

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Solⁿ ÷ b:

Given that,

$f(z) = z^3 - iz^2 - 5z + 2i$

let, $z - 1 = 2e^{i\theta}$

$\Rightarrow z = 2e^{i\theta} + 1$

$\Rightarrow dz = 2ie^{i\theta}\, d\theta$; where, $0 \le \theta \le 2\pi$.

By cauchy's theorem,

$\oint_c f(z)\cdot dz = 0$, where c is the circle

$|z-1| = 2$.

Now, $\oint_c f(z)\, dz = \oint_c (z^3 - iz^2 - 5z + 2i)\, dz$

$\Rightarrow \oint_c f(z)\, dz = \int_0^{2\pi} \left[\left\{(2e^{i\theta}+1)^3 - i(2e^{i\theta}+1)^2 - 5(2e^{i\theta}+1) + 2i\right\} \cdot 2ie^{i\theta} d\theta\right]$

$= \int_0^{2\pi} \left\{(8e^{3i\theta} + 12e^{2i\theta} + 6e^{i\theta} + 1) - i(4e^{2i\theta} + 4e^{i\theta} + 1) - 10e^{i\theta} - 5 + 2i\right\} 2ie^{i\theta} d\theta$

$= \int_0^{2\pi} \left(8e^{3i\theta} + 12e^{2i\theta} - 4 + i - 8e^{i\theta} - i4e^{2i\theta}\right) 2ie^{i\theta} d\theta$

$= \int_0^{2\pi} \left(16ie^{4i\theta} + 24ie^{3i\theta} - 8ie^{i\theta} - 2e^{i\theta} - 16ie^{2i\theta} + 8e^{3i\theta}\right) d\theta$

$= \left[\frac{16ie^{5i\theta}}{5i} + \frac{24ie^{4i\theta}}{4i} - \frac{8ie^{2i\theta}}{2i} - \frac{2e^{2i\theta}}{2i} - \frac{16ie^{3i\theta}}{3i} + \frac{8e^{4i\theta}}{4i}\right]_0^{2\pi}$

$= \frac{16}{5}(e^{10\pi i} - e^0) + 6(e^{8\pi i} - e^0) - 4(e^{4\pi i} - e^0) - \frac{1}{i}(e^{4\pi i} - e^0) - \frac{16}{3}(e^{6\pi i} - e^0) + \frac{2}{i}(e^{8\pi i} - e^0)$

$= \frac{16}{5}(1-1) + 6(1-1) - 4(1-1) - \frac{1}{i}(1-1) - \frac{16}{3}(1-1) + \frac{2}{i}(1-1)$

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$\frac{16}{5} \times 0 + 6 \times 0 - 4 \times 0 - \frac{1}{i} \times 0 - \frac{16}{3} \times 0 + \frac{2}{i} \times 0$

$= 0$

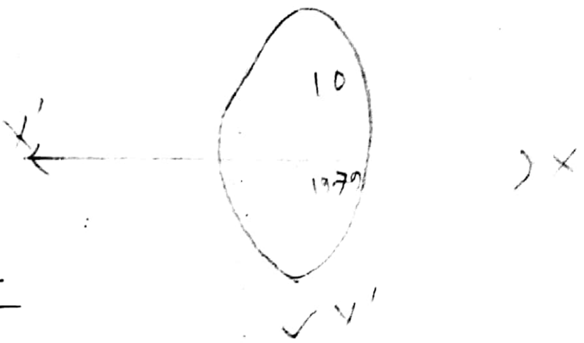
$\therefore \oint_C f(z)\, dz = 0$

Thus, verified cauchy's theorem.

Solⁿ - c:

The ellipse $|z-3i| + |z+3i| = 20$, let, $f(z) = z^3 - iz^2 - 5z + 2i$

By cauchy's theorem, $\oint_C f(z)\, dz = 0$, where c is the ellipse $|z-3i| + |z+3i| = 20$.



Now, $|z-3i| + |z+3i| = 20$

$\Rightarrow \sqrt{x^2 + (y-3)^2} + \sqrt{x^2 + (y+3)^2} = 20$

$\Rightarrow \sqrt{x^2 + (y-3)^2} = 20 - \sqrt{x^2 + (y+3)^2}$

$\Rightarrow x^2 + y^2 - 6y + 9 = 400 - 40\sqrt{x^2 + (y+3)^2} + x^2 + y^2 + 6y + 9$

$\Rightarrow 40\sqrt{x^2 + (y+3)^2} = 12y + 400$

$\Rightarrow 10\sqrt{x^2 + (y+3)^2} = 3y + 100$

$\Rightarrow 100(x^2 + y^2 + 6y + 9) = 9y^2 + 600y + 10000$

$\Rightarrow 100x^2 + 91y^2 = 9100$

$\Rightarrow \frac{x^2}{(\sqrt{91})^2} + \frac{y^2}{(10)^2} = 1$

Here, the major axis $= 19.08$

the minor axis $= 20$.

et, $z = e^{i\theta}$, $dz = ie^{i\theta}\,d\theta$, where $0 \le \theta \le 2\pi$

From cauchy's theorem we get,

$\oint_c f(z)\,dz = 0$

i.e $\oint_c f(z)\,dz = \oint_c (z^3 - iz^2 - 5z + 2i)\,dz$

$= \int_0^{2\pi} (e^{3i\theta} - ie^{2i\theta} - 5e^{i\theta} + 2i)\, ie^{i\theta}\,d\theta$

$= i\int_0^{2\pi} (e^{4i\theta} - ie^{3i\theta} - 5e^{2i\theta} + 2ie^{i\theta})\,d\theta$

$= i\left[\frac{e^{4i\theta}}{4i} - \frac{1}{3}e^{3i\theta} - \frac{5}{2i}e^{2i\theta} + \frac{2ie^{i\theta}}{i}\right]_0^{2\pi}$

$= i\left[\frac{1}{4i}(e^{8i\pi} - e^0) - \frac{1}{3}(e^{6i\pi} - e^0) - \frac{5}{2i}(e^{4i\pi} - e^0) + 2(e^{2\pi i} - e^0)\right]$

$= i\left[\frac{1}{4i}\times 0 - \frac{1}{3}\times 0 - \frac{5}{2i}\times 0 + 2\times 0\right]$

$= i \times 0$

$= 0$

$\therefore \oint_c f(z)\,dz = 0$

Hence, the function satisfies cauchy theorem.

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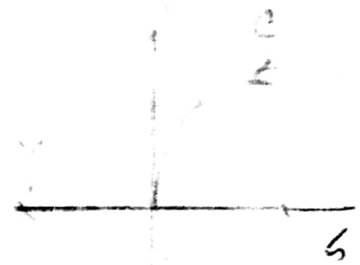
Problem-5:

If c is the circle $|z-2|=5$ (a) determine whether $\oint_c \frac{dz}{z-3}=0$ (b) Does your Answer to contradict cauchy's theorem.



Soln-a:

Given, $\oint_c \frac{dz}{z-3} = 0$

where, c is the circle $|z-2|=5$

let, $z-2 = 5e^{i\theta}$ where, $0 \le \theta \le 2\pi$

$\Rightarrow z = 5e^{i\theta}+2$

$\Rightarrow dz = 5ie^{i\theta}d\theta$

$$\therefore \oint_c \frac{dz}{z-3} = \int_0^{2\pi} \frac{5ie^{i\theta}d\theta}{5e^{i\theta}+2-3}$$

$$= \int_0^{2\pi} \frac{5ie^{i\theta}.d\theta}{5e^{i\theta}-1}$$

$$= \left[\ln(5e^{i\theta}-1)\right]_0^{2\pi}$$

$$= \ln\left[5e^{2i\pi}-1\right] - \ln\left[5e^{0}-1\right]$$

$$= \ln(5-1) - \ln(5-1)$$

$$= \ln|4| - \ln|4|$$

$$= 0$$

Soln-b:

No, its satisfies the cauchy's theorem