Laplace transform of the function F(t) and is denoted by $\mathcal{L}\{F(t)\}$

Alternatively, let F(t) be a function of t defined for all $t > o$. Then the **Laplace transform** of F(t), denoted by $\mathcal{L}\{F(t)\}$ or f(s), is defined by $\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)dt$ (1)

provided that the integral (1) exists. 's' is a parameter which may be real or complex number. The **Laplace transform** of F(t) is said to exist if the integral $\int_0^\infty e^{-st} F(t)dt$ converges for some value of s; otherwise it does not exist.

1.4 Laplace transforms of some elementary functions

Laplace transforms of several elementary functions are determined just by the direct application of definition of the Laplace transform.

Example 1. Find the Laplace transforms of the following elementary functions :

(i) $F(t) = 1$, (ii) $F(t) = t$, (iii) $F(t) = t^n$, $n = 0, 1, 2, 3, \ldots\ldots$

(iv) $F(t) = e^{at}$, (v) $F(t) = \sin at$, (vi) $F(t) = \cos at$,

(vii) $F(t) = \sin h\, at$ and (viii) $F(t) = \cos h\, at$.

Solutions : By definition of the Laplace transform of a function F(t), we have $\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt$ (1)

(i) When $F(t) = 1$, equation (1) becomes IP

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1dt = \int_0^\infty e^{-st} dt = \lim_{p\to\infty} \int_0^p e^{-st} dt.$$

$$= \lim_{p\to\infty} \left[-\frac{1}{s} e^{-st}\right]_0^p = \lim_{p\to\infty} \left[-\frac{1}{s} e^{-sp} + \frac{1}{s}\right]$$

$$= \frac{1}{s} - 0 = \boxed{\frac{1}{s} \text{ if } s > o.}$$

M ~~(ii)~~ When $\boxed{F(t) = t,}$ equation (1) becomes IP

$$\mathcal{L}(t) = \int_0^\infty e^{-st}\,.\,t\,dt = \lim_{p\to\infty} \int_0^\infty t.\,e^{-st}\,dt$$

(Integrating by parts)

$$= \lim_{p\to\infty}\left[\frac{-t\,e^{-st}}{s}\right]_0^p + \lim_{p\to\infty}\frac{1}{s}\int_0^p e^{-st}\,dt$$

$$= \lim_{p\to\infty}\frac{-p}{se^{ps}} + 0 - \lim_{p\to\infty}\frac{1}{s^2}\left[e^{-st}\right]_0^p$$

$$= 0 - \lim_{p\to\infty}\frac{1}{s^2}\left[\frac{1}{e^{sp}} - 1\right] = \boxed{\frac{1}{s^2}} \text{ if } s>0.$$

M ~~(iii)~~ When $\boxed{F(t) = t^n}$ equation (1) becomes IP

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st}\,t^n dt \qquad (2)$$

Putting $st = y$ so that $sdt = dy \quad \therefore\ dt = \frac{1}{s}dy$.

Also $t^n = \frac{y^n}{s^n}$. **Limits** $\left.\begin{matrix} t=0 \\ y=0\end{matrix}\right\}$ $\left.\begin{matrix} t=\infty \\ y=\infty\end{matrix}\right\}$

Thus from (2) we have

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-y}\,\frac{y^n}{s^n}\cdot\frac{dy}{s}$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-y}\,y^n\,dy \text{ (using gamma function)}$$

$$= \boxed{\frac{1}{s^{n+1}}\,\Gamma(n+1)} \text{ if } n > -1 \text{ and } s > 0$$

Again we know that if n be a positive integer, then

$\Gamma(n+1) = \lfloor n$

Hence $\mathcal{L}\{t^n\} = \frac{\lfloor n}{s^{n+1}}$, when $n = 0, 1, 2, 3, \ldots\ldots\ldots$

M ~~(iv)~~ When $F(t) = e^{at}$ equation (1) becomes

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}\,.\,e^{at}dt = \int_0^\infty e^{-(s-a)t}\,dt$$

$$= -\frac{1}{(s-a)}\left[e^{-(s-a)t}\right]_0^\infty$$

$$= -\frac{1}{(s-a)}(0-1) = \boxed{\frac{1}{s-a}} \text{ if } s > a.$$

M (v) When $F(t) = \sin at$, equation (1) becomes.

IP

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at\, dt.$$

$$= \lim_{p\to\infty} \int_0^p e^{-st} \sin at\, dt.$$

$$= \lim_{p\to\infty} \left[\frac{e^{-st}(-s \sin at - a \cos at)}{s^2 + a^2} \right]_0^p$$

$$= \lim_{p\to\infty} \left[\frac{e^{-sp}(-s \sin ap - a \cos a p)}{s^2 + a^2} + \frac{a}{s^2 + a^2} \right]$$

$$= 0 + \frac{a}{s^2 + a^2} = \frac{a}{s^2 + a^2} \text{ if } s > 0.$$

M (vi) When $F(t) = \cos at$, equation (1) becomes

$$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at\, dt.$$

$$= \lim_{p\to\infty} \int_0^p e^{-st} \cos at\, dt.$$

$$= \lim_{p\to\infty} \left[\frac{e^{-st}(-s \cos at + a \sin at)}{s^2 + a^2} \right]_0^p$$

$$= \lim_{p\to\infty} \left[\frac{e^{-sp}(-s\cos ap + a \sin a p)}{s^2 + a^2} + \frac{s}{s^2 + a^2} \right]$$

$$= 0 + \frac{s}{s^2 + a^2} = \frac{s}{s^2 + a^2} \text{ if } s > 0.$$

When $F(t) = \sinh at$, equation (1) becomes

$$\mathcal{L}\{\sinh at\} = \int_0^\infty e^{-st} \sinh at\, dt.$$

$$= \int_0^\infty e^{-st} \cdot \frac{1}{2} (e^{at} - e^{-at})\, dt$$

$$= \frac{1}{2} \int_0^\infty e^{-st} \cdot e^{at}\, dt - \frac{1}{2} \int_0^\infty e^{-st} \cdot e^{-at}\, dt$$

$$= \frac{1}{2} \int_0^\infty e^{-(s-a)t}\, dt - \frac{1}{2} \int_0^\infty e^{-(s+a)t}\, dt$$

$\int (uv)\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx.$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty + \frac{1}{2}\left[\frac{e^{-(s+a)t}}{(s+a)}\right]_0^\infty$$

$$= \frac{1}{-2(s-a)}(0-1) + \frac{1}{2(s+a)}(0-1)$$

$$= \frac{1}{2(s-a)} - \frac{1}{2(s+a)} = \frac{a}{s^2-a^2} \text{ if } s > |a|$$

(viii) When F(t) = coshat, equation (1) becomes

$$\mathcal{L}\{\cosh at\} = \int_0^\infty e^{-st} \cosh at\, dt.$$

$$= \int_0^\infty e^{-st} \cdot \frac{1}{2}(e^{at} + e^{-at})\, dt.$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-a)t}\, dt + \frac{1}{2}\int_0^\infty e^{-(s+a)t}\, dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty - \frac{1}{2}\left[\frac{e^{-(s+a)t}}{(s+a)}\right]_0^\infty$$

$$= \frac{1}{-2(s-a)}(0-1) - \frac{1}{2(s+a)}(0-1)$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{s}{s^2-a^2} \text{ if } s > |a|.$$

Example 2. Find the Laplace transforms of the following functions:

(i) F(t) = t sin at (ii) F(t) = t cos at.

Solution : By definition of the Laplace transform of a function F (t), we have

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt \quad (1)$$

(i) When F(t) = t sin at, equation (1) becomes

$$\mathcal{L}\{t \sin at\} = \int_0^\infty e^{-st} \cdot t \sin at\, dt$$

$$= \int_0^\infty t \cdot e^{-st} \sin at\, dt$$

Integrating by parts, taking t as first function and e^{-st} sin at as the second function and applying the result

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx],$$

we get $\mathcal{L}\{t \sin at\} = \left[t \cdot \frac{e^{-st}}{s^2+a^2}(-s \sin at - a \cos at)\right]_0^\infty$

$$+ \int_0^\infty \frac{e^{-st}}{s^2+a^2}(s \sin at + a \cos at)\, dt$$

$$= 0 + \frac{s}{s^2+a^2}\int_0^\infty e^{-st} \sin at\, dt + \frac{a}{s^2+a^2}\int_0^\infty e^{-st} \cos at\, dt$$

$$= \frac{s}{s^2+a^2} \cdot \frac{a}{s^2+a^2} + \frac{a}{s^2+a^2} \cdot \frac{s}{s^2+a^2}$$ for $s > 0$ [using examples 1 (v) & 1 (vi)]

$$= \frac{2as}{(s^2+a^2)^2} \text{ for } s > o.$$

(ii) When F (t) = t cos at, equation (1) becomes

$$\mathcal{L}\{t \cos at\} = \int_0^\infty e^{-st} \cdot t \cos at\, dt$$

$$= \int_0^\infty t \cdot e^{-st} \cos at\, dt$$

Integrating by parts, taking t as first function and e^{-st} cos at as the second function and applying the result $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx],$

we get $\mathcal{L}\{t \cos at\} = \left[t \cdot \frac{e^{-st}}{s^2+a^2}(-s \cos at + a \sin at)\right]_0^\infty$

$$+ \frac{s}{s^2+a^2}\int_0^\infty e^{-st} \cos at\, dt - \frac{a}{s^2+a^2}\int_0^\infty e^{-st} \sin at\, dt$$

$$= 0 + \frac{s}{s^2+a^2} \cdot \frac{s}{s^2+a^2} - \frac{a}{s^2+a^2} \cdot \frac{a}{s^2+a^2} \text{ for } s > 0$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2} \text{ for } s>o.$$

a.S.E

Example 3. Find the Laplace transform of the function F(t) where $F(t) = \begin{cases} t, & o < t < 2 \\ 3, & t > 2. \end{cases}$

Solution : Here F (t) is not defined at $t = o$ and $t = 2$. By definition of the Laplace transform of a function F(t), we have

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

$$= \int_0^2 e^{-st} F(t)dt + \int_2^\infty e^{-st} F(t)\, dt$$

$$= \int_0^2 . e^{-st} .\ t\, dt + \int_2^\infty e^{-st} \cdot 3\, dt$$

$$= \int_0^2 \underset{u}{t}.\underset{v}{e^{-st}}\, dt + 3\int_2^\infty e^{-st}.\, dt$$

$$= \left[-\frac{t}{s} e^{-st}\right]_0^2 + \int_0^2 \frac{e^{-st}}{s}\, dt - \frac{3}{s}\left[e^{-st}\right]_2^\infty$$

$$= -\frac{2}{s} e^{-2s} + 0 - \frac{1}{s^2}\left[e^{-st}\right]_0^2 - \frac{3}{s}(0 - e^{-2s})$$

$$= -\frac{2}{s} e^{-2s} - \frac{1}{s^2} e^{-2s} + \frac{1}{s^2} + \frac{3e^{-2s}}{s}$$

$$= \frac{1}{s^2} + \frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2}.$$

Definition Sectional or Piecewise continuity

A function F(t) is called **sectionally continuous** or **Piecewise continuous** in a closed interval $a \le t \le b$ if the interval can be divided into a finite number of sub intervals in each of which the function remains continuous and possesses finite right and left hand limits.

Definition Functions of exponential order.

For a given positive integer N if there exist real

1. Some important Properties of Laplace transform

(A) Linearity property

Theorem 2. The Laplace transform is a linear transformation. i.e. if and λ_1 and λ_2 are any constants while $F_1(t)$ and $F_2(t)$ are functions with Laplace transform $f_1(s)$ and $f_2(s)$, respectively then

$$\mathcal{L}\{\lambda_1 F_1(t) + \lambda_2 F_2(t)\} = \lambda_1 \mathcal{L}\{F_1(t)\} + \lambda_2 \mathcal{L}\{F_2(t)\}.$$

Proof: Let $\mathcal{L}\{F_1(t)\} = f_1(s) = \int_0^\infty e^{-st} F_1(t)dt$

and $\mathcal{L}\{F_2(t)\} = f_2(s) = \int_0^\infty e^{-st} F_2(t)dt$

Now if λ_1 and λ_2 are any two constants then

$$\mathcal{L}\{\lambda_1 F_1(t) + \lambda_2 F_2(t)\} = \int_0^\infty e^{-st}\{\lambda_1 F_1(t) + \lambda_2 F_2(t)\}dt$$

$$= \lambda_1 \int_0^\infty e^{-st} F_1(t)dt + \lambda_2 \int_0^\infty e^{-st} F_2(t)dt$$

$$= \lambda_1 \mathcal{L}\{F_1(t)\} + \lambda_2 \mathcal{L}\{F_2(t)\}$$

$$= \lambda_1 f_1(s) + \lambda_2 f_2(s).$$

The result may be generalized for any number of functions and for the same number of arbitrary constants. i.e.

$$\mathcal{L}\left\{\sum_{r=1}^{n} \lambda_r F_r(t)\right\} = \sum_{r=1}^{n} \lambda_r \mathcal{L}\{F_r(t)\}.$$

(B) First translation (or shifting) property.

Theorem 3. If $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\{e^{at}F(t)\} = f(s-a).$$

Proof: By definition of the Laplace transform we have

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)dt = f(s)$$

$$\therefore \quad \mathcal{L}\{e^{at}F(t)\} = \int_0^\infty e^{-st}\{e^{at} F(t)\}dt$$

$$= \int_0^\infty e^{-(s-a)t} F(t)\}dt = f(s-a).$$

Second translation (or shifting) property

Theorem 4. If $\mathcal{L}\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$

then $\mathcal{L}\{G(t) = e^{-as} f(s)$.

Proof : By definition of the Laplace transform we have

$$\mathcal{L}\{F(t) = \int_0^\infty e^{-st} F(t)dt = f(s).$$

$$\therefore \quad \mathcal{L}\{G(t) = \int_0^\infty e^{-st} G(t)dt$$

$$= \int_0^a e^{-st} G(t)dt + \int_a^\infty e^{-st} G(t)dt$$

$$= \int_0^\infty e^{-st} (O)dt + \int_a^\infty e^{-st} F(t-a)dt.$$

$$= 0 + \int_a^\infty e^{-st} F(t-a)dt \quad (1)$$

Let $t - a = u \therefore dt = du$

Limits $\left.\begin{matrix} t = a \\ u = o \end{matrix}\right\}$ $\left.\begin{matrix} t = \infty \\ u = \infty \end{matrix}\right\}$

Thus from (1), we have

$$\mathcal{L}\{G(t)\} = \int_0^\infty e^{-s(u+a)} F(u) du$$

$$= e^{-sa} \int_0^\infty e^{-su} F(u)du = e^{-sa} f(s).$$

(D) The change of scale property

Theorem 5. If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$

Proof : By definition of the Laplace transform, we have

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)dt = f(s)$$

$$\therefore \quad \mathcal{L}\{F(at)\} = \int_0^\infty e^{-st} F(at)\, dt \quad (1)$$

Limits $\left.\begin{matrix} t = a \\ u = 0 \end{matrix}\right\}$ $\quad$ $\left.\begin{matrix} t = \infty \\ u = \infty \end{matrix}\right\}$

Thus from (1) we have

$$\therefore \quad \mathcal{L}\{F(at)\} = \int_0^\infty e^{\frac{-su}{a}} F(u).\frac{1}{a}du.$$

$$= \frac{1}{a}\int_0^\infty e^{\left(\frac{-s}{a}\right)u} F(u)du = \frac{1}{a} f\left(\frac{s}{a}\right).$$

C.S.E

Laplace transform of derivatives.

Theorem 6. If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{F'(t)\} = s f(s) - F(0)$.

Proof.: By definition of the Laplace transform

$$\text{we have } \mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = f(s)$$

$$\therefore \quad \mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t)dt \quad \text{(integrating by parts)}$$

$$= \Big[e^{-st} F(t)\Big]_0^\infty + s\int_0^\infty e^{-st} F(t)dt$$

$$= 0 - F(0) + sf(s) = sf(s) - F(0).$$

Theorem 7. If $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0).$$

Proof : By definition of the Laplace transform,

$$\text{we have } \mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)dt = f(s)$$

$$\therefore \mathcal{L}\{F''(t)\} = \int_0^\infty e^{-st} F''(t)dt \quad \text{(integrating by parts)}$$

$$= [e^{-st} F'(t)] + \int_0^\infty s\, e^{-st} F'(t)dt$$

$$= 0 - F'(0) + s\int_0^\infty e^{-st} F'(t)dt$$

$$= -F'(0) + s\{sf(s) - F(0)\}$$

$$= s^2 f(s) - sF(0) - F'(0).$$

Similarly, we can find the third order derivative of F(t).

Again the limit of the right hand side of (1) as

$s \to 0$ is $\lim_{s\to 0} sf(s) - F(o)$.

Thus we have $\lim_{t\to\infty} F(t) - F(0) = \lim_{s\to 0} sf(s) - F(0)$

$\therefore \quad \lim_{t\to\infty} F(t) = \lim_{s\to 0} sf(s).$

WORKED OUT EXAMPLES

Example 4. Find the Laplace transform of
$e^{4t} + 4t^3 - 2\sin 3t + 3\cos 5t.$

Solution : By applying linearity property, we have

$$\mathcal{L}\{e^{4t} + 4t^3 - 2\sin 3t + 3\cos 5t.\}$$

$$= \mathcal{L}\{e^{4t}\} + 4\,\mathcal{L}\{t^3\} - 2\mathcal{L}\{\sin 3t\} + 3\mathcal{L}\{\cos 5t\}$$

$$= \frac{1}{s-4} + 4.\frac{\lfloor 3}{s^4} - 2.\frac{3}{s^2+3^2} + 3.\frac{s}{s^2+5^2}$$

$$= \frac{1}{s-4} + \frac{24}{s^4} - \frac{6}{s^2+9} + \frac{3s}{s^2+25}.$$

Example 5. Find the Laplace transform of
$3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t.$

Solution : By applying linearity property, we have

$$\mathcal{L}\{3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\}$$

$$= 3\,\mathcal{L}\{t^4\} - 2\,\mathcal{L}\{t^3\} + 4\mathcal{L}\{e^{-3t}\} - 2\mathcal{L}\{\sin 5t\} + 3\mathcal{L}\{\cos 2t\}$$

$$= 3.\frac{\lfloor 4}{s^5} - 2.\frac{\lfloor 3}{s^4} + 4.\frac{1}{s+3} - 2.\frac{5}{s^2+5^2} + 3.\frac{s}{s^2+2^2}.$$

$$= \frac{72}{s^5} - \frac{12}{s^4} + \frac{4}{s+3} - \frac{10}{s^2+25} + \frac{3s}{s^2+4}.$$

Example 6. Find the Laplace transform of t^3e^{5t}

Solution : We have $\mathcal{L}\{(t^3) = \frac{\lfloor 3}{s^4} = \frac{6}{s^4} = f(s)$(say)

Then by using first shifting theorem, we get

$$\mathcal{L}\{t^3e^{5t}\} = f(s-5) = \frac{6}{(s-5)^4}.$$

Method–2

C. S. E

Example 7. Find the Laplace transform of

$e^{3t}(2\cos 5t - 3\sin 5t)$.

Solution : We have

$$\mathcal{L}\{2\cos 5t - 3\sin 5t\} = 2\,\mathcal{L}\{\cos 5t\} - 3\mathcal{L}\{\sin 5t\}$$

$$= 2.\frac{s}{s^2+5^2} - 3.\frac{5}{s^2+5^2}$$

$$= \frac{2s-15}{s^2+25} = f(s) \text{ (say)}$$

Then by applying the first shifting theorem we have

$$\mathcal{L}\{e^{3t}(2\cos 5t - 3\sin 5t)\}$$

$$= f(s-3) = \frac{2(s-3)-16}{(s-3)^2+25}$$

$$= \frac{2s-21}{s^2-6s+34}.$$

C. S. E

Example 8. By applying first shifting theorem prove that

(i) $\mathcal{L}\{t \sin at\} = \dfrac{2as}{(s^2+a^2)^2}$

(ii) $\mathcal{L}\{t \cos at\} = \dfrac{s^2-a^2}{(s^2+a^2)^2}$.

Proof : We have $\mathcal{L}\{t\} = \dfrac{1}{s^2} = f(s)$ (say)

Then by applying the first shifting theorem we get

$$\mathcal{L}\{t\,e^{iat}\} = f(s-ia) = \frac{1}{(s-ia)^2}$$

Or, $\mathcal{L}\{t\cos at + it\sin at\} = \dfrac{(s+ia)^2}{\{(s+ia)(s-ia)\}^2}$

Or, $\mathcal{L}\{t\cos at\} + i\,\mathcal{L}\{t\sin at\} = \dfrac{(s^2-a^2)+i(2as)}{(s^2+a^2)^2}$

Equating real and imaginary parts from both sides, we get

$\mathcal{L}\{t\cos at\} = \dfrac{s^2-a^2}{(s^2+a^2)^2}$ and

$\mathcal{L}\{t\sin at\} = \dfrac{2as}{[illegible]}$

Similar applications of the first shifting property leads us to the following results :

(1) $\mathcal{L}\{e^{at}t^n\} = \frac{\lfloor n}{(s-a)^{n+1}}$

(2) $\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$

(3) $\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$

(4) $\mathcal{L}\{e^{at}\sinh bt\} = \frac{b}{(s-a)^2-b^2}$

(5) $\mathcal{L}\{e^{at}\cosh bt\} = \frac{s-a}{(s-a)^2-b^2}$.

C.S.E

Example 9. Find the Laplace transform of F(t), where

$$F(t) = \begin{cases} \sin(t - \pi/3), & t > \pi/3 \\ 0 & , t < \pi/3 \end{cases}$$

Solution : Let $\varphi(t) = \sin t$

$$\therefore \varphi(t - \pi/3) = \sin(t - \pi/3)$$

$$F(t) = \begin{cases} \varphi(t - \frac{\pi}{3}), & t > \frac{\pi}{3} \\ 0 & , t < \pi/3 \end{cases}$$

Now $\mathcal{L}\{\varphi(t)\} = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1} = f(s)$ (say)

Then applying the second shifting theorem we get

$$\mathcal{L}\{F(t)\} = e^{-as} f(s)$$

$$= e^{-\frac{\pi s}{3}} \frac{1}{s^2+1} = \frac{e^{-\frac{\pi s}{3}}}{s^2+1}.$$

Example 10. Find the Laplace transforms of cosat and sinhat using the change of scale property.

Solution : (i) We know that

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1} = f(s) \text{ (say)}$$

$$\mathcal{L}\{\cos at\} = \frac{1}{a} f\left(\frac{s}{a}\right) = \frac{1}{a} \cdot \frac{\frac{s}{a}}{(s/a)^2 + 1}.$$

$$= \frac{s}{a^2} \cdot \frac{a^2}{s^2 + a^2} = \frac{s}{s^2 + a^2}.$$

(ii) We know that $\mathcal{L}\{\sinh t\} = \frac{1}{s^2 - 1} = f(s)$ (say)

$$\therefore \mathcal{L}\{\sinh at\} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

$$= \frac{1}{a} \cdot \frac{1}{\left(\frac{s}{a}\right)^2 - 1} = \frac{a}{s^2 - a^2}.$$

Example ~~11~~. Find the Laplace transform of cosat using theorem 7.

Solution : Let F (t) = cos at

Then $\left.\begin{array}{l} F'(t) = -a \sin at \\ F''(t) = -a^2 \cos at \end{array}\right\}$ $\left.\begin{array}{l} F(o) = 1 \\ F'(o) = o \end{array}\right\}$

Now by theorem 11, we have

$\mathcal{L}\{F''(t)\} = s^2 \mathcal{L}\{F(t)\} - s F(o) - F'(o).$

$\mathcal{L}\{-a^2 \cos at\} = s^2 \mathcal{L}\{\cos at\} - s.1 - o.$

Or, $-a^2 \mathcal{L}\{\cos at\} = s^2 \mathcal{L}\{\cos at\} - s.$

Or, $(s^2 + a^2) \mathcal{L}\{\cos at\} = s$

Or, $\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}.$

Example 12. Find the Laplace trasfrom of t^2 cos at using theorem 11.

Solution : Since $\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}, s > o$

$$\therefore \mathcal{L}\{t^2 \cos at\} = (-1)^2 \frac{d^2}{ds^2} \mathcal{L}\{\cos at\}$$

$$= \frac{d^2}{ds^2} \left\{\frac{s}{s^2 + a^2}\right\}$$

This can be proved by other methods also.

C.S.

Example 20. Prove that

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}.$$

Proof. Let $F(t) = \sin\sqrt{t}$.

Then $F'(t) = \dfrac{\cos\sqrt{t}}{2\sqrt{t}}$ and $F(o) = \sin 0 = 0$.

From the Laplace transform of the derivatives.

we have $\mathcal{L}\{F'(t)\} = s\mathcal{L}\{F(t)\} - F(0)$

$$\therefore \mathcal{L}\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = s\,\mathcal{L}\{\sin\sqrt{t}\} - 0$$

$$= s\,\frac{\sqrt{\pi}}{2s^{3/2}}\, e^{-\frac{1}{4s}}$$

Therefore, $\mathcal{L}\left\{\dfrac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \dfrac{\sqrt{\pi}}{s^{1/2}}\, e^{-\frac{1}{4s}} = \sqrt{\dfrac{\pi}{s}}\, e^{-\frac{1}{4s}}.$

This can be proved by other methods also.

Example 21. Prove that

$$\mathcal{L}\{e^{-at} J_0(bt)\} = \frac{1}{\sqrt{(s+a)^2 + b^2}}.$$

[D. U. M. Sc. (F) 1986]

Prof : By formula, we have

if $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F(at)\} = \dfrac{1}{a} f\left(\dfrac{s}{a}\right)$

$$\mathcal{L}\{J_0(t)\} = \frac{1}{s^2+1}$$

$$\therefore \mathcal{L}\{J_0(at)\} = \frac{1}{a}\cdot\frac{1}{\sqrt{\left(\frac{s}{a}\right)^2 + 1}} = \frac{1}{\sqrt{s^2+a^2}}$$

2.3 Some important properties of the inverse Laplace transform

1. Linearity property

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Theorem 1. If $\mathcal{L}\{F_1(t)\} = f_1(s)$ and $\mathcal{L}\{F_2(t)\} = f_2(s)$ and c_1 and c_2 are any two constants, then

$$\mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} = c_1 \mathcal{L}^{-1}\{f_1(s)\} + c_2 \mathcal{L}^{-1}\{f_2(s)\} = c_1 F_1(t) + c_2 F_2(t).$$

Proof : Given $\mathcal{L}\{F_1(t)\} = f_1(s)$ and $\mathcal{L}\{F_2(t)\} = f_2(s)$.

$\therefore$ by the definition of inverse Laplace transform, we have

$F_1(t) = \mathcal{L}^{-1}\{f_1(s)\}$ and $F_2(t) = \mathcal{L}^{-1}\{f_2(s)\}$.

Now $\mathcal{L}\{c_1F_1(t) + c_2F_2(t)\} = c_1\mathcal{L}\{F(t)\} + c_2\,\mathcal{L}\{F_2(t)\}$

$= c_1 f_1(s) + c_2 f_2(s).$

$$\therefore \quad \mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} = c_1 F_1(t) + c_2 F_2(t) = c_1 \mathcal{L}^{-1}\{f_1(s)\} + c_2 \mathcal{L}^{-1}\{f_2(s)\}$$

This result can be easily generalised.

The above theorem is illustrated by the following examples :

$$\mathcal{L}^{-1}\left\{\frac{2}{s-a} + \frac{3}{s^2} + \frac{4a}{s^2+a^2} + \frac{5s}{s^2-a^2}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} + 5\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\}$$

$$= 2e^{at} + 3t + 4\sin at + 5\cosh at.$$

$$\mathcal{L}^{-1}\left\{\frac{5}{(s-2)^2} + 2\tan^{-1}\frac{1}{s} + \frac{s+2}{s^2+2s+13}\right\}$$

$$= 5\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} + 2\mathcal{L}^{-1}\left\{\tan^{-1}\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\}$$

$$= 5t\,e^{2t} + \frac{2\sin t}{t} + e^{-2t}\cos 3t.$$

2. First translation (or shifting) property

Theorem 2. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{f(s-a)\} = e^{at}\,F(t).$$

Proof : By definition of Laplace transform we have

$$f(s) = \mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}\,F(t)dt$$

$$\therefore \quad f(s-a) = \int_0^\infty e^{-(s-a)t}\,F(t)dt$$

$$= \int_0^\infty e^{-st}.\,\{e^{at}\,F(t)dt$$

$$= \mathcal{L}\,\{e^{at}\,F(t)\}$$

$$\therefore \quad \mathcal{L}^{-1}\{f(s-a)\} = e^{at}\,F(t).$$

The above theorem is illustrated by the following examp

Since $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4^2}\right\} = \cos 4t.$

$$\mathcal{L}^{-1}\left\{\frac{s-2}{s^2-4s+20}\right\} = \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+4^2}\right\}$$

$$= e^{2t}\cos 4t\,.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+8s+16}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2}\right\}$$

$$= e^{-4t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^{-4t}.\,t = te^{-4t}.$$

3. Second translation (or shifting) property

Theorem 3. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\,\{e^{-as}\,F(s)\} = G(t) \text{ where } G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$$

Proof : By definition of Laplace transform, we have

$$f(s) = \int_0^\infty e^{-st}\,F(t)dt.$$

$$\therefore \quad e^{-as}f(s) = \int_0^{\infty} e^{-as} . e^{-st} F(t)dt$$

$$= \int_0^{\infty} e^{-s(t+a)} F(t)dt$$

Let $t + a = u$

$\therefore \quad dt = du$ and $t = u - a$

Limits $\left.\begin{matrix} t = 0 \\ u = a \end{matrix}\right\} \quad \left.\begin{matrix} t = \infty \\ u = \infty \end{matrix}\right\}$

$$\text{Thus } e^{-as}f(s) = \int_a^{\infty} e^{-su} F(u-a)du = \int_a^{\infty} e^{-st} F(t-a)dt$$

$$= \int_0^{a} e^{-st}(O)\, dt + \int_a^{\infty} e^{-st} F(t-a)\, dt$$

$$= \int_0^{\infty} e^{-st} G(t)dt = \mathcal{L}\{G(t)\}$$

$$\therefore \quad \mathcal{L}^{-1}\{e^{-as}f(s)\} = G(t).$$

The above theorem is illustrated by the following examples :

$$\text{Since } \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^4}\right\} = e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = e^{2t}.\frac{t^3}{\lfloor 3} = \frac{1}{6}t^3e^{2t}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{e^{-5s}}{(s-2)^4}\right\} = \begin{cases} \frac{1}{6}(t-5)^3 e^{2(t-5)}, & t>5 \\ 0 & , t<5. \end{cases}$$

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4. Change of scale property

Theorem 4. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{f(ks)\} = \frac{1}{k} F\left(\frac{t}{k}\right)$$

Proof : By the definition of Laplace transform

we have

$$f(s) = \mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st} F(t)\, dt.$$

$$\therefore \quad f(ks) = \int_0^\infty e^{-kst} F(t)dt$$

Let $u = kt \therefore du = kdt$ and $t = \frac{u}{k}$

Limits $\left.\begin{matrix} t=0 \\ u=0 \end{matrix}\right\} \left.\begin{matrix} t=\infty \\ u=\infty \end{matrix}\right\}$

$$\text{Thus } f(ks) = \int_0^\infty e^{-su} F\left(\frac{u}{k}\right) \cdot \frac{1}{k}du$$

$$= \frac{1}{k}\int_0^\infty e^{-su} F\left(\frac{u}{k}\right) du$$

$$= \frac{1}{k}\int_0^\infty e^{-st} F\left(\frac{t}{k}\right) dt$$

$$= \frac{1}{k}\mathcal{L}\left\{F\left(\frac{t}{k}\right)\right\}$$

$$= \mathcal{L}\left\{\frac{1}{k}F\left(\frac{t}{k}\right)\right\}$$

$$\therefore \quad \mathcal{L}^{-1}\{f(ks)\} = \frac{1}{k}F\left(\frac{t}{k}\right).$$

The above theorem is illustrated by the following examples

Since $\mathcal{L}^{-1}\left\{\frac{s}{s^2-3^2}\right\} = \cosh 3t$

we have $\mathcal{L}^{-1}\left\{\frac{5s}{25s^2-9}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{5s}{(5s)^2-3^2}\right\}$$

$$= \frac{1}{5}\cosh\frac{3t}{5}.$$

Also since $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} = \cos 2t,$

we have

$$\mathcal{L}^{-1}\left\{\frac{3s}{9s^2+4}\right\}$$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \sinh t$

$\therefore$ (i) $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2-1)}\right\} = \int_0^t \sinh u\,du = [\cosh u]_0^t = \cosh t - 1.$

(ii) $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\} = \int_0^t (\cosh u - 1)\,du = [\sinh u - u]_0^t$

$$= \sinh t - t - 0$$

$$= \sinh t - t.$$

(iii) $\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2-1)}\right\} = \int_0^t (\sinh u - u)\,du = \left[\cosh u - \frac{u^2}{2}\right]_0^t$

$$= \cosh t - 1 - \frac{t^2}{2} - 0$$

$$= \cosh t - 1 - \frac{t^2}{2}.$$

WORKED OUT EXAMPLES

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Example 3. Evaluate $\mathcal{L}^{-1}\left\{\frac{5s-6}{s^2+9} - \frac{s-15}{s^2-25}\right\}$

Solution : $\mathcal{L}^{-1}\left\{\frac{5s-6}{s^2+9} - \frac{s-15}{s^2-25}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{5s}{s^2+3^2} - \frac{6}{s^2+3^2} - \frac{s}{s^2-5^2} + \frac{15}{s^2-5^2}\right\}$$

$$= 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2-5^2}\right\}$$
$$+ 3\mathcal{L}^{-1}\left\{\frac{5}{s^2-5^2}\right\}$$

$$= 5\cos 3t - 2\sin 3t - \cosh 5t + 3\sinh 5t.$$

Example 4. Prove that

$$\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\} = 2e^{2t}(3\cos 4t + \sin 4t)$$

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Proof. $$\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\} = \mathcal{L}^{-1}\left\{\frac{6s-12+8}{s^2-4s+20}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{6(s-2)}{(s-2)^2+16}\right\} + \mathcal{L}^{-1}\left\{\frac{8}{(s-2)^2+16}\right\}$$

$$= 6\mathcal{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2+4^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2+4^2}\right\}$$

$$= 6e^{2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4^2}\right\} + 2e^{2t}\mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\}$$

$$= 6e^{2t}\cos 4t + 2e^{2t}\sin 4t$$

$$= 2e^{2t}(3\cos 4t + \sin 4t).$$

Example 5. Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+4)}\right\}$

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Solution : By theorem we have

$$\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{\sin 2t}{2}$$

$$\therefore \quad \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \int_0^t \frac{\sin 2u}{2}\,du$$

$$= \frac{1}{2}\left[-\frac{1}{2}\cos 2u\right]_0^t$$

$$= -\frac{1}{4}(\cos 2t - 1)$$

$$= \frac{1}{4}(1 - \cos 2t).$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \int_0^t \frac{1}{4}(1 - \cos 2u)\, du$$

$$= \frac{1}{4}\left[u - \frac{1}{2}\sin 2u\right]_0^t$$

$$= \frac{1}{4}(t - 0) - \frac{1}{8}(\sin 2t - 0)$$

$$= \frac{1}{4}t - \frac{1}{8}\sin 2t.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+4)}\right\} = \int_0^t \left(\frac{1}{4}u - \frac{1}{8}\sin 2u\right) du$$

$$= \left[\frac{1}{8}u^2 + \frac{1}{16}\cos 2u\right]_0^t$$

$$= \frac{1}{8}(t^2 - 0) + \frac{1}{16}(\cos 2t - 1)$$

$$= \frac{1}{8}t^2 + \frac{1}{16}\cos 2t - \frac{1}{16}.$$

Example 6. By expanding $e^{-k\sqrt{s}}$ where k is a positive constant, show that,

$$\mathcal{L}^{-1}\{e^{-k\sqrt{s}}\} = \frac{k}{2\sqrt{\pi}\, t^{\frac{3}{2}}}\, e^{-\frac{k^2}{4t}}.$$

D. U. M. Sc. (F) 1984

Proof :

$$e^{-k\sqrt{s}} = 1 - \frac{k\sqrt{s}}{|1} + \frac{k^2(\sqrt{s})^2}{|2} - \frac{k^3(\sqrt{s})^3}{|3} + \frac{k^4(\sqrt{s})^4}{|4} - \frac{k^5(\sqrt{s})^5}{|5} + \cdots$$

Example 12. Evaluate $\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

Solution : Let $\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3}$ (1)

$\therefore \quad 2s^2-4 = A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2)$ (2)

Putting s = – 1 in (2), we get

$$2-4 = 12A \therefore \boxed{A = -\frac{1}{6}}$$

Putting s = 2 in (2), we get

$$8-4 = -3B \therefore \boxed{B = -\frac{4}{3}}$$

Putting s = 3 in (2), we get

$$14 = 4C \therefore \boxed{C = \frac{7}{2}}$$

Thus from (1), we have

$$\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{-\frac{1}{6}}{s+1} + \frac{-\frac{4}{3}}{s-2} + \frac{\frac{7}{2}}{s-3}$$

$$\therefore \quad \mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$$

$$= -\frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{7}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}.$$

Example 13. Find the inverse Laplace transform of

$$\frac{2s^2-6s+5}{s^3-6s^2+11s-6}.$$

Solution : Let $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$

$$= \frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$$

$$= \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3} \qquad (1)$$

$$\therefore \quad 2s^2-6s+5 = A(s-2)(s-3) + B(s-3)(s-1) + C(s-1)(s-2) \qquad (2)$$

Putting $s = 1$ in (2), we get

$$2-6+5 = 2A, \; \therefore \; \boxed{A = \frac{1}{2}}$$

Putting $s = 2$ in (2), we get

$$8-12+5 = -B, \; \therefore \; \boxed{B = -1}$$

Putting $s = 3$ in (2), we get

$$18-18+5 = 2C \; \therefore \; \boxed{C = \frac{5}{2}}$$

Thus from (1), we have

$$\frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{1}{2}\cdot\frac{1}{s-1} - \frac{1}{s-2} + \frac{5}{2}\cdot\frac{1}{s-3}$$

Example . Evaluate $\mathcal{L}^{-1}\left\{\dfrac{5s+3}{(s-1)(s^2+2s+5)}\right\}$

Solution : Let $\dfrac{5s+3}{(s-1)(s^2+2s+5)} = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+2s+5}$ (1)

$\therefore \quad 5s+3 = A(s^2+2s+5) + (Bs+C)(s-1)$ (2)

Putting $s = 1$ in (2), we get

$8 = 8A \therefore \boxed{A = 1}$

Equating coefficients of s^2 from both sides of (2), we get

$0 = A + B \therefore 1 + B = 0$, Or, $\boxed{B = -1}$

Putting $s = 0$ in (2), we get

$3 = 5A - C \therefore C = 5 - 3 = 2$ or, $\boxed{C = 2}$

Thus from (1), we have

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1} + \frac{-s+2}{s^2+2s+5}$$

$$= \frac{1}{s-1} + \frac{-s+2}{(s+1)^2+2^2}$$

$$= \frac{1}{s-1} + \frac{-(s+1)+3}{(s+1)^2+2^2}$$

$$= \frac{1}{s-1} - \frac{(s+1)}{(s+1)^2+2^2} + \frac{3}{2}\cdot\frac{2}{(s+1)^2+2^2}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}$$

$$= e^t - e^{-t}\cdot\cos 2t + \frac{3}{2}e^{-t}\sin 2t.$$

Example . Prove that $\mathcal{L}^{-1}\left\{\dfrac{2s+1}{(s+2)^2(s-1)^2}\right\} = \dfrac{1}{3}t\,(e^t - e^{-2t})$

2.6 Definition of convolution

Let F(t) and G(t) be two functions of a class A, then the ***convolution*** of the two functions F(t) and G(t) denoted by $F * G$ is defined by the relation

$$F * G = \int_0^t F(u)\, G(t-u)\, du$$

This relation $F * G$ is also called the ***resultant*** or ***Faltung*** of F and G. Here we have to note the followings :

(1) $F * G$ is commutative i.e $F * G = G * F$

(ii) $F * G$ is associative i.e $(F * G) * H = F * (G * H)$

(iii) $F * G$ is distributive with respect to addition i.e $F * (G + H) = F * G + F * H$.

2.7 Convolution theorem (or convolution property)

Statement : If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $\mathcal{L}^{-1}\{g(s)\} = G(t)$, then

$$\mathcal{L}^{-1}\{f(s)\, g(s)\} = \int_0^t F(u)\, G(t-u)\, du = F * G.$$

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Proof : Here the required result follows if we can prove that

$$\mathcal{L}\left\{\int_0^t F(u)\, G(t-u)\, du\right\} = f(s)\, g(s) \qquad (1)$$

where $f(s) = \mathcal{L}\{F(t)\}$ and $g(s) = \mathcal{L}\{G(t)\}$.

Now $\mathcal{L}\left\{\int_0^t F(u)\, G(t-u)\, du\right\} = \int_{t=0}^{\infty} e^{-st} \left\{\int_{u=0}^{t} F(u)\, G(t-u)\, du\right\} dt$

$$= \int_{t=0}^{\infty} \int_{u=0}^{t} e^{-st} F(u)\, G(t-u)\, du dt$$

$$= \lim_{T \to \infty} s_T \text{ where } s_T = \int_{t=0}^{T} \int_{u=0}^{t} e^{-st} F(u)\, G(t-u)\, du dt \qquad (2)$$

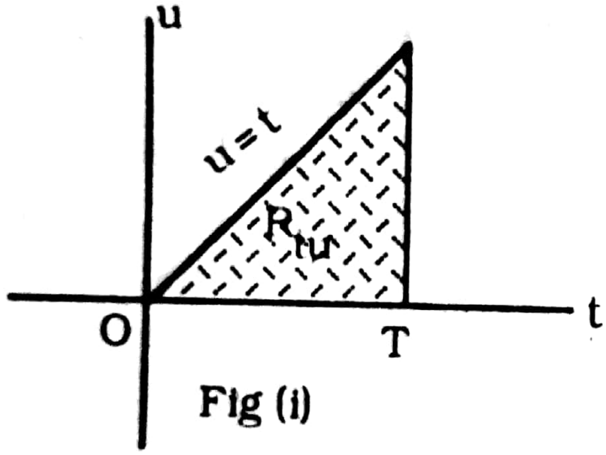

Fig (i)

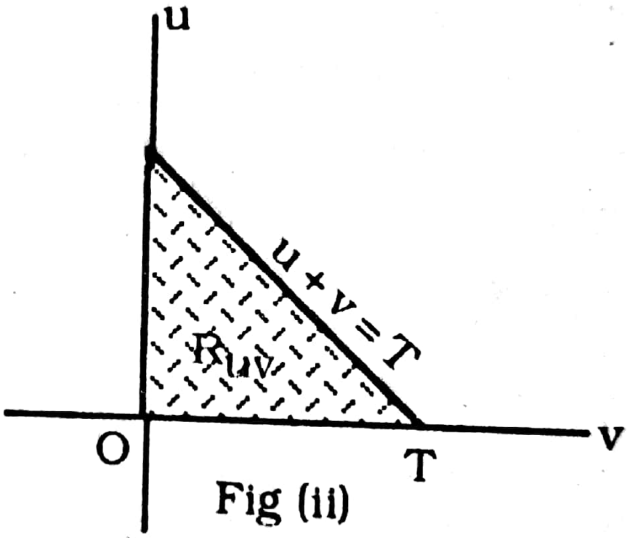

Fig (ii)

The region in the tu plane over which the integration (2) is defined is shown shaded in Fig (i). Letting $t - u = v$ or $t = u + v$, the shaded region R_{tu} of the tu plane is transformed into the shaded region R_{uv} of the uv plane shown in Fig (ii). Then by the theorem on transformation of multiple integrals, we have

$$s_T = \iint_{R_{tu}} e^{-st} F(u)\, G(t-u)\, du dt$$

$$= \iint_{R_{uv}} e^{-s(u+v)} .\, F(u)\, G(v) \frac{\partial(u,t)}{\partial(u,v)}\, du dv \qquad (3)$$

where the **Jacobian** of the transformation is

$$J = \frac{\partial(u,t)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\delta u} & \frac{\partial u}{\delta v} \\ \frac{\partial t}{\delta u} & \frac{\partial t}{\delta v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

Thus the right hand side of (3) is

$$s_T = \int_{v=0}^{T} \int_{u=0}^{T-v} e^{-s(u+v)} F(u)\, G(v)\, du dv \quad (4)$$

Let us define a new function

$$K(u, v) = \begin{cases} e^{-s(u+v)} F(u)\, G(v) & \text{if } u + v \le T \\ 0 & \text{if } u + v > T. \end{cases} \quad (5)$$

This function is defined over the square of Fig (iii). But, as indicated in (5) is zero over the unshaded portion of the square.

In terms of new function we can write (4) as

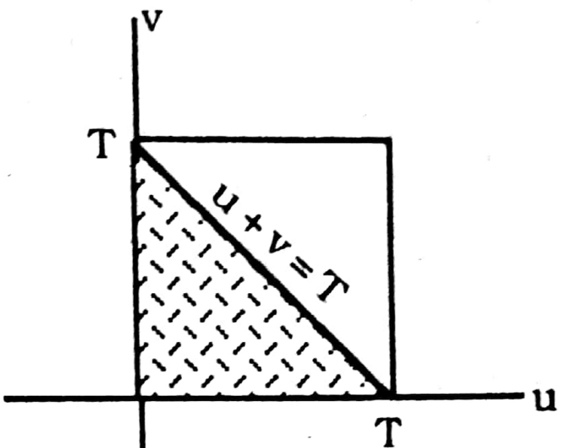

Fig (iii)

$$s_T = \int_{v=0}^{T} \int_{u=0}^{T} K(u, v)\, du dv$$

Then $\lim\limits_{T \to \infty} s_T = \int_0^{\infty} \int_0^{\infty} K(u, v)\, du dv$

$$= \int_0^{\infty} \int_0^{\infty} e^{-s(u+v)} F(u)\, G(v)\, du dv.$$

$$= \left\{ \int_0^{\infty} e^{-su} F(u) du \right\} \left\{ \int_0^{\infty} e^{-sv} G(v) dv \right\} = f(s)\, g(s)$$

which establishes the theorem.

Note : $F * G = G * F$.

$$F*G=\int_0^t F(u)\,G(t-u)\,du = \int_0^t F(t-v)\,G(v)\,dv$$

$$= \int_0^t G(v)\,F(t-v)\,dv = G * F.$$

$\therefore \; F * G = G * F.$

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Example 17. Evaluate $\mathcal{L}^{-1}\left\{\dfrac{3}{s^2(s+2)}\right\}$ by use of the convolution theorem.

Solution : We can write

$$\frac{3}{s^2(s+2)} = \frac{1}{s+2}\cdot\frac{3}{s^2}$$

Let $f(s) = \dfrac{1}{s+2}$ and $g(s) = \dfrac{3}{s^2}$

Since $F(t) = \mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s+2}\right\} = e^{-2t}$

and $G(t) = \mathcal{L}^{-1}\{g(s)\} = \mathcal{L}^{-1}\left\{\dfrac{3}{s^2}\right\} = 3\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\} = 3t$

Therefore, by using the convolution theorem,

we get $\mathcal{L}^{-1}\left\{\dfrac{3}{s^2(s+2)}\right\} = \displaystyle\int_0^t e^{-2u}\,3(t-u)\,du$

$$= 3\int_0^t te^{-2u}du - 3\int_0^t ue^{-2u}du.$$

$$= -\frac{3t}{2}\left[e^{-2u}\right]_0^t + \frac{3}{2}\left[ue^{-2u}\right]_0^t - \frac{3}{2}\int_0^t e^{-2u}du$$

$$= -\frac{3t}{2}\left(e^{-2t}-1\right) + \frac{3}{2}te^{-2t} - 0 + \frac{3}{4}\left[e^{-2u}\right]_0^t$$

$$= -\frac{3t}{2}e^{-2t} + \frac{3t}{2} + \frac{3t}{2}e^{-2t} + \frac{3}{4}e^{-2t} - \frac{3}{4}.$$

$$= \frac{3t}{2} + \frac{3}{4}e^{-2t} - \frac{3}{4}.$$

Hence $\mathcal{L}^{-1}\left\{\dfrac{3}{s^2(s+2)}\right\} = \dfrac{3t}{2} + \dfrac{3}{4}e^{-2t} - \dfrac{3}{4}.$

Example 18. Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\}$ by use of the convolution theorem.

Solution : We can write

$$\frac{1}{s^2(s^2+4)} = \frac{1}{s^2+4}\cdot\frac{1}{s^2}$$

Let $f(s) = \frac{1}{s^2+4}$ and $g(s) = \frac{1}{s^2}$

Then since $\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$

and $\mathcal{L}^{-1}\{g(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$

$\therefore$ By use of the convolution theorem

$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(u)\, G(t-u)\, du$, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \int_0^t \frac{1}{2}\sin 2u\,(t-u)\,du$$

$$= \frac{1}{2}t\int_0^t \sin 2u\,du - \frac{1}{2}\int_0^t u\sin 2u\,du$$

$$= \frac{1}{2}t\left[-\frac{1}{2}\cos 2u\right]_0^t + \frac{1}{4}\left[u\cos 2u\right]_0^t - \frac{1}{4}\int_0^t \cos 2u\,du$$

$$= -\frac{1}{4}t\,(\cos 2t - 1) + \frac{1}{4}t\cos 2t - 0 - \frac{1}{8}\left[\sin 2u\right]_0^t$$

$$= -\frac{1}{4}t\cos 2t + \frac{1}{4}t + \frac{1}{4}t\cos 2t - \frac{1}{8}\sin 2t + 0$$

$$= \frac{1}{4}t - \frac{1}{8}\sin 2t.$$

Hence $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \frac{1}{4}t - \frac{1}{8}\sin 2t.$

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Example 9. Evaluate $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$ by use of the convolution theorem. **[D. U. M. Sc(F) 1989]**

Solution : We can write $\frac{s}{(s^2+a^2)^2} = \frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}$

Then since $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$

Therefore, by using the convolution theorem

$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(u)\,G(t-u)\,du$, we have

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \cos au.\frac{\sin a(t-u)}{a}\,du$$

$$= \frac{1}{a}\int_0^t \cos au \sin(at-au)\,du$$

$$= \frac{1}{a}\int_0^t \cos au\,(\sin at\cos au - \cos at\sin au)\,du$$

$$= \frac{1}{a}\sin at\int_0^t \cos^2 au\,du - \frac{1}{a}\cos at\int_0^t \sin au\cos au\,du$$

$$= \frac{1}{2a}\sin at\int_0^t (1+\cos 2au)\,du - \frac{1}{2a}\cos at\int_0^t \sin 2au\,du$$

$$= \frac{\sin at}{2a}\left[u+\frac{\sin 2au}{2a}\right]_0^t + \frac{\cos at}{2a}\left[\frac{\cos 2au}{2a}\right]_0^t$$

$$= \frac{\sin at}{2a}\left[t+\frac{\sin 2at}{2a}-0\right] + \frac{\cos at}{4a^2}(\cos 2at-1)$$

$$= \frac{t\sin at}{2a} + \frac{\sin at}{2a}.\frac{2\sin at\cos at}{2a} - \frac{\cos at}{4a^2}.2\sin^2 at$$

$$= \frac{t\sin at}{2a} + \frac{\sin^2 at\cos at}{2a^2} - \frac{\sin^2 at\cos at}{2a^2} = \frac{t\sin at}{2a}.$$

Hence $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t\sin at}{2a}.$

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Example 20. Evaluate $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s+1)^2}\right\}$ by using the convolution theorem. **[D. U. M. Sc. (F) 1985]**

Solution : We can write $\dfrac{1}{s^2(s+1)^2} = \dfrac{1}{(s+1)^2} \cdot \dfrac{1}{s^2}$

Then since $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\} = t$ and $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^2}\right\} = te^{-t}$,

By using the convolution theorem

$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(u)\, G(t-u)du$, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t ue^{-u}(t-u)du$$

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$$= \int_0^t (ut-u^2)e^{-u}du$$

$$= [-(ut-u^2)e^{-u}]_0^t + \int_0^t (t-2u)\, e^{-u}\, du$$

$$= 0 + [-(t-2u)\, e^{-u}]_0^t + (0-2)\int_0^t e^{-u}du$$

$$= -t\,(e^{-t}-1) + 2\,(te^{-t}-0) + 2[e^{-u}]_0^t$$

$$= -te^{-t} + t + 2te^{-t} + 2e^{-t} - 2.$$

$$= te^{-t} + 2e^{-t} + t - 2$$

Therefore, $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s+1)^2}\right\} = te^{-t} + 2e^{-t} + t - 2.$

2.8 Heaviside's expansion formula

Statement : Let P(s) and Q(s) be polynomials in s where P(s)

27. Prove that

$$\mathcal{L}^{-1}\left\{\frac{11s^2-2s+5}{(s+1)(s-2)(2s-1)}\right\} = 5e^{2t} - \frac{3}{2}e^{\frac{t}{2}} + 2e^{-t}.$$

28. Prove that

$$\mathcal{L}^{-1}\left\{\frac{3s^3-3s^2-40s+36}{(s^2-4)^2}\right\} = (5t+3)e^{-2t} - 2te^{2t}.$$

29. Prove that

$$\mathcal{L}^{-1}\left\{\frac{2s^3-s^2-1}{(s+1)^2(s^2+1)^2}\right\} = \frac{1}{2}\sin t + \frac{1}{2}t\cos t - te^{-t}.$$

Evaluate the followings using the **convolution theorem :**

30. $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\}$

Answer : $\frac{1}{2}(\sin t - \cos t + e^{-t})$

31. $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$

Answer : $\frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t.$

32. $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+3)}\right\}$

Answer : $\frac{1}{3}t - \frac{1}{3\sqrt{3}}\sin\sqrt{3}t.$

33. $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-2)}\right\}$

Answer : $e^{2t} - e^{t}.$

34. $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$

Answer: $\frac{1}{2}\sin t - \frac{t}{2}\cos t$