3.2 Application of Laplace transform in solving ordinary linear differential equations with constant coefficients.

WORKED OUT EXAMPLES

Example 1. Solve the following differential equation by using Laplace transform :

Given that,

$$\frac{dy}{dt} - 3y = o; \; y(o) = 1.$$

Solution : The given differential equation can be written as $y' - 3y = o \quad (1)$

Taking the Laplace transform of both sides of (1) we get

$\mathcal{L}\{y'\} - 3\,\mathcal{L}\{y\} = \mathcal{L}\,\{o\}$

or, $sY(s) - y(o) - 3Y(s) = 0$

Since $\mathcal{L}\{F'(t)\} = sf(s) - F(o)$

or, $sY(s) - 1 - 3Y(s) = o$ since $y(o) = 1$

or, $(s - 3)\,Y(s) = 1$

or, $Y(s) = \dfrac{1}{s-3} \qquad (2)$

Now taking the inverse Laplace transform of both sides of (2), we get

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

or, $y(t) = e^{3t}$. since $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}, s > a.$

Example 2. Solve the differential equation

$y' + 2y = e^t;\ y(o) = 1$

by using Laplace transform.

Solution : The given differential equation is

$$y' + 2y = e^t \qquad (1)$$

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^t\}$$

$$sY(s) - y(o) + 2Y(s) = \frac{1}{s-1}$$

$$\text{or, } sY(s) - 1 + 2Y(s) = \frac{1}{s-1}$$

$$\text{or, } (s+2)Y(s) = 1 + \frac{1}{s-1} = \frac{s}{s-1}$$

$$\text{or, } Y(s) = \frac{s}{(s-1)(s+2)} = \frac{1}{3}\cdot\frac{1}{s-1} + \frac{2}{3}\cdot\frac{1}{s+2} \qquad (2)$$

Now taking inverse Laplace transform of both sides of (2); we get

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

Taking the inverse Laplace ...

we get

$$\mathcal{L}^{-1}\{X(s)\} = c\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^3}\right\}$$

or, $x(t) = ce^{-t} + \frac{1}{2}t^2e^{-t}$ since $\mathcal{L}^{-1}\left\{\frac{\lfloor n-1}{(s+a)^n}\right\} = = t^{n-1}e^{-at}$.

The constant c can be determined (n = 1, 2 , ...) only if an initial condition is given.

C. S. E. **Example 6.** Solve following differential equation by using Laplace transform : $Y''(t) + Y(t) = t;\ Y(o) = 1,\ Y'(o) = -2$

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Solution : The given differential equation is

$Y''(t) + Y(t) = t$ (1)

Taking the Laplace transfrom of both sides of (1) and using the given conditions we get $\mathcal{L}\{Y''(t)\} + \mathcal{L}\{Y(t)\} = \mathcal{L}(t)$

or, $s^2y - sY(o) - Y'(o) + y = \frac{1}{s^2}$

or, $s^2y - s + 2 + y = \frac{1}{s^2}$

or, $(s^2 + 1)y = s - 2 + \frac{1}{s^2} = \frac{s^3 - 2s^2 + 1}{s^2}$

or, $y = \frac{s^3 - 2s^2 + 1}{s^2(s^2 + 1)}$

Now $\frac{s^3 - 2s^2 + 1}{s^2(s^2 + 1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 1}$

or, $s^3 - 2s^2 + 1 = As(s^2 + 1) + B(s^2 + 1) + Cs^3 + Ds^2$ (2)

Equating the coefficients of s^3 from both sides of (2), we get $C = 1$ Putting $s = o$, in (2), we get $B = 1$.

Equating the coefficients of s^2 from both sides of (2), we get $-2 = B + D$, or, $-2 = 1 + D \therefore D = -3$.

Equating the coefficients of s from both sides of (2), we get $0 = A + 0 \therefore A = 0$

$$\therefore y = 0 + \frac{1}{s^2} + \frac{s - 3}{s^2 + 1} = \frac{1}{s^2} + \frac{s}{s^2 + 1} - \frac{3}{s^2 + 1} \quad (3)$$

Taking the inverse Laplace transform of both sides of (3), we get

$$\mathcal{L}^{-1}\{y\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left(\frac{s}{s^2 + 1}\right) - 3\,\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\}$$

or, $Y(t) = t + \cos t - 3\sin t$ which is the required solution.

Example 7. Solve the following differential equation by