1.10. Table of Laplace transform theorems

No	Operation	F (t)	$\mathcal{L}\{F(t)\} = f(s)$
1.	Linearity property	$a_1 F_1(t) + a_2 F_2(t)$	$a_1 \mathcal{L}\{F_1(t)\} + a_2 \mathcal{L}\{F_2(t)\}$
2.	First translation or Shifting property	$e^{at} F(t)$	$f(s - a)$
3.	Second translation or Shifting property	$G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$	$e^{-as} f(s)$
4.	Change of scale property	$F(at)$	$\frac{1}{a} f\left(\frac{s}{a}\right)$
5.	Differentiation theorems	$F'(t)$ $F^n(t)$	$s f(s) - F(0)$ $s^n f(s) - \sum_{r=0}^{n-1} s^{n-r-1} F^r(0)$
6.	Multiplication theorems	$t F(t)$ $t^n F(t)$	$-\frac{d}{ds} f(s)$ $(-1)^n \frac{d^n}{ds^n} f(s)$
7.	Division theorem	$\frac{1}{t} F(t)$	$\int_s^\infty f(u)\,du$
8.	Integral theorem	$\int_0^t F(u)\,du$	$\frac{1}{s} f(s)$
9.	Initial-value theorem	$\lim_{t \to 0} F(t)$	$= \lim_{s \to \infty} s\,\mathcal{L}\{F(t)\}$ $= \lim_{s \to \infty} s\,f(s)$
10.	Final-value theorem	$\lim_{t \to \infty} F(t)$	$= \lim_{s \to 0} s\,\mathcal{L}\{F(t)\}$ $= \lim_{s \to 0} s\,f(s)$
11.	Fundamental theorem for periodic function	$\mathcal{L}\{F(t)\} = \frac{\int_0^T e^{-st} F(t)\,dt}{1 - e^{-sT}}$ F (t) is a periodic function of period T	

functions.

No	F (t)	$\mathcal{L}\{F(t)\} = f(s)$
1.	1	$\frac{1}{s}$ $s>0$
2.	t	$\frac{1}{s^2}$ $s>o$
3.	t^n $n=0, 1. 2, 3.$ ------	$\frac{\underline{\lvert n}}{s^{n+1}}$ $s>o$ where $\underline{\lvert n} = 1. 2. 3.---, n$ and $\underline{\lvert 0} = 1.$
4.	e^{at}	$\frac{1}{s-a}$ $s>a$
5.	sin at	$\frac{a}{s^2+a^2}$ $s>o$
6.	cos at	$\frac{s}{s^2+a^2}$ $s>o$
7.	sin h at	$\frac{a}{s^2-a^2}$ $s>\lvert a\rvert$
8.	cos h at	$\frac{s}{s^2-a^2}$ $s>\lvert a\rvert$
9.	t sin at	$\frac{2as}{(s^2+a^2)^2}$ $s>0$
10.	t cos at	$\frac{s^2-a^2}{(s^2+a^2)^2}$ $s>o$
11.	$t^n e^{at}$ $(n = 1, 2, 3, ----)$	$\frac{\underline{\lvert n}}{(s-a)^{n+1}}$ $s>a$
12.	e^{at} sin bt	$\frac{b}{(s-a)^2+b^2}$
13.	e^{at} cos bt	$\frac{s-a}{(s-a)^2+b^2}$
14.	e^{at} sinh bt	$\frac{b}{(s-a)^2-b^2}$
15.	e^{at} cos h bt	$\frac{s-a}{(s-a)^2-b^2}$

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	and $\lfloor 0 = 1$.
e^{at}	$\frac{1}{s-a}$ $s>a$
$\sin at$	$\frac{a}{s^2+a^2}$ $s>o$
$\cos at$	$\frac{s}{s^2+a^2}$ $s>o$
$\sin h\ at$	$\frac{a}{s^2-a^2}$ $s>\lvert a\rvert$
$\cos h\ at$	$\frac{s}{s^2-a^2}$ $s>\lvert a\rvert$
$t \sin at$	$\frac{2as}{(s^2+a^2)^2}$ $s>0$
$t \cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$ $s>o$
$t^n e^{at}$ $(n = 1, 2, 3, ----)$	$\frac{\lfloor n}{(s-a)^{n+1}}$ $s>a$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at} \sinh bt$	$\frac{b}{(s-a)^2-b^2}$
$e^{at} \cos h\ bt$	$\frac{s-a}{(s-a)^2-b^2}$

No	F (t)	$\mathcal{L}\{F(t)\} = f(s)$
1.	$J_0(at)$	$\dfrac{1}{\sqrt{s^2 + a^2}}$
2.	$J_n(at)$	$\dfrac{(\sqrt{s^2 + a^2} - s)^n}{a^n \sqrt{s^2 + a^2}}$
3.	$\sin\sqrt{t}$	$\dfrac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$
4.	$\dfrac{\cos\sqrt{t}}{\sqrt{t}}$	$\sqrt{\dfrac{\pi}{s}}\, e^{-\frac{1}{4s}}$
5.	$\mathrm{erf}(\sqrt{t})$	$\dfrac{1}{s\sqrt{s + 1}}$
6.	$\mathrm{erf}(t)$	$\dfrac{e^{s^2/4}}{s}\, \mathrm{erfc}\,(s/2)$
7.	$\mathrm{Si}\,(t)$	$\dfrac{1}{s} \tan^{-1}\dfrac{1}{s}$
8.	$\mathrm{Ci}\,(t)$	$\dfrac{\log(s^2 + 1)}{2s}$
9.	$\mathrm{Ei}\,(t)$	$\dfrac{\log(s + 1)}{s}$
10.	$U(t - a)$	$\dfrac{e^{-as}}{s}$

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2.3 Some important properties of the inverse Laplace transform

1. Linearity property

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Theorem 1. If $\mathcal{L}\{F_1(t)\} = f_1(s)$ and $\mathcal{L}\{F_2(t)\} = f_2(s)$ and c_1 and c_2 are any two constants, then

$$\begin{aligned}\mathcal{L}^{-1}\{c_1f_1(s) + c_2f_2(s)\} &= c_1\mathcal{L}^{-1}\{f_1(s)\} + c_2\mathcal{L}^{-1}\{f_2(s)\} \\ &= c_1F_1(t) + c_2F_2(t).\end{aligned}$$

Proof : Given $\mathcal{L}\{F_1(t)\} = f_1(s)$ and $\mathcal{L}\{F_2(t)\} = f_2(s)$.

$\therefore$ by the definition of inverse Laplace transform, we have

$F_1(t) = \mathcal{L}^{-1}\{f_1(s)\}$ and $F_2(t) = \mathcal{L}^{-1}\{f_2(s)\}$.

$$\begin{aligned}\text{Now } \mathcal{L}\{c_1F_1(t) + c_2F_2(t)\} &= c_1\mathcal{L}\{F(t)\} + c_2\mathcal{L}\{F_2(t)\} \\ &= c_1f_1(s) + c_2f_2(s).\end{aligned}$$

$$\begin{aligned}\therefore\quad \mathcal{L}^{-1}\{c_1f_1(s) + c_2f_2(s)\} &= c_1F_1(t) + c_2F_2(t) \\ &= c_1\mathcal{L}^{-1}\{f_1(s)\} + c_2\mathcal{L}^{-1}\{f_2(s)\}\end{aligned}$$

This result can be easily generalised.

The above theorem is illustrated by the following examples :

$$\mathcal{L}^{-1}\left\{\frac{2}{s-a} + \frac{3}{s^2} + \frac{4a}{s^2+a^2} + \frac{5s}{s^2-a^2}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} + 5\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\}$$

$$= 2e^{at} + 3t + 4\sin at + 5\cosh at.$$

$$\mathcal{L}^{-1}\left\{\frac{5}{(s-2)^2} + 2\tan^{-1}\frac{1}{s} + \frac{s+2}{s^2+2s+13}\right\}$$

$$= 5\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} + 2\mathcal{L}^{-1}\left\{\tan^{-1}\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\}$$

$$= 5t\,e^{2t} + \frac{2\sin t}{t} + e^{-2t}\cos 3t.$$

$$= \mathcal{L}^{-1}\left\{\frac{3s}{(3s)^2 + 2^2}\right\}$$

$$= \frac{1}{3}\cos\frac{2t}{3}.$$

5. Inverse Laplace transform of derivatives

Theorem 5. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ then

$$\mathcal{L}^{-1}\{f^n(s)\} = \mathcal{L}^{-1}\left\{\frac{d^n}{ds^n} f(s)\right\} = (-1)^n t^n F(t).$$

where $n = 1, 2, 3, \ldots\ldots\ldots\ldots\ldots\ldots$

Proof : From Laplace transform we have

if $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{t^n F(t)\} = (-1)^n f^n(s)$

where $f^n(s) = \dfrac{d^n}{ds^n} f(s)$.

$\therefore \quad \mathcal{L}^{-1}\{(-1)^n f^n(s)\} = t^n F(t)$

Or, $(-1)^n \mathcal{L}^{-1}\{f^n(s)\} = t^n F(t)$.

Or, $\{(-1)^n\}^2 \mathcal{L}^{-1}\{f^n(s)\} = (-1)\, t^n F(t)$

Or, $(-1)^{2n} \mathcal{L}^{-1}\{f^n(s)\} = (-1)^n t^n F(t)$

Or, $\mathcal{L}^{-1}\{f^n(s)\} = (-1)^n t^n F(t)$ since $(-1)^{2n} = 1$.

The above theorem is illustrated by the following examples :

Since $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2^2}\right\} = \cos 2t.$

and $\dfrac{d}{ds}\left(\dfrac{s}{s^2+4}\right) = \dfrac{(s^2+4) - 2s^2}{(s^2+4)^2} = \dfrac{4-s^2}{(s^2+4)^2}$

$\therefore \quad \mathcal{L}^{-1}\left\{\dfrac{4-s^2}{(s^2+4)^2}\right\} = (-1)t\cos 2t = -t\cos 2t$

6. Inverse Laplace transform of integrals

Theorem 6. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, Then

$$\mathcal{L}^{-1}\left\{\int_s^\infty f(u)\,du\right\} = \frac{F(t)}{t}$$

Proof : From Laplace transform, we have

$$\text{if } \mathcal{L}\{F(t)\} = f(s), \text{ then } \mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)du$$

$$\therefore \quad \mathcal{L}^{-1}\left\{\int_s^\infty f(u)du\right\} = \frac{F(t)}{t}.$$

The above theorem is illustrated by the following example :

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\} = t - \sin t$$

$$\mathcal{L}^{-1}\left\{\int_s^\infty \left(\frac{1}{u^2} - \frac{1}{u^2+1}\right) du\right\} = \mathcal{L}^{-1}\left\{\left[-\frac{1}{u} - \tan^{-1} u\right]_s^\infty\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{\pi}{2} + \tan^{-1} s\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s} - \cot^{-1} s\right\}$$