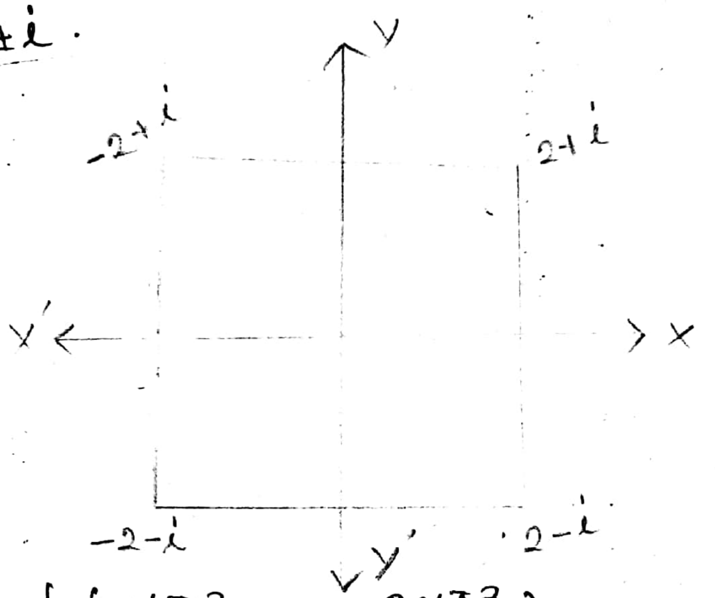
Problem - 7

Evalute $\frac{1}{2\pi i}\oint_c \frac{\cos\pi z}{z^2-1}\,dz$ arround a rectangle verties at (a) $2\pm i, -2\pm i$ (b) $-i, 2-i, 2+i, i$.

Sol^n - a:

Let, $f(z) = \cos\pi z$ which is analytic inside on C.
The points $z^2=1 \Rightarrow z=\pm 1$. inside the rectangle with verties $2\pm i, -2\pm i$.



$$\text{Here, } \frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)}$$

$$= \frac{1}{2(z-1)} - \frac{1}{2(z+1)}$$

$$\therefore \frac{1}{2\pi i}\oint_c \frac{\cos\pi z}{z^2-1}\,dz = \frac{1}{4\pi i}\oint_c \left(\frac{\cos\pi z}{z-1} - \frac{\cos\pi z}{z+1}\right)dz$$

By cauchy's integral formula with $a=1$ and $b=-1$ we have,

$$f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)\,dz}{z-a}$$

Here, $f(z) = \cos\pi z$
and. $a=1$.
$\Rightarrow f(1) = \cos\pi.1$
$\Rightarrow f(1) = -1$

$$\therefore \frac{1}{2\pi i}\oint_c \frac{f(z)\,dz}{z-a} = f(a)$$

$$\Rightarrow \frac{1}{4\pi i}\oint_c \frac{\cos\pi z}{z-1}\,dz = \frac{1}{2}(-1)$$

$$= -\frac{1}{2}.$$

$$\text{and } f(b) = \frac{1}{2\pi i}\oint_c \frac{f(z)\,dz}{z-b}$$

$$\frac{1}{2\pi i}\oint_c \frac{f(z)}{z-b}\,dz = f(b)$$

Here, $f(z) = \cos \pi z$
and $b = -1$

$$f(-1) = \cos(-\pi) = -1$$

$$\Rightarrow \frac{1}{2\pi i}\oint_c \frac{\cos \pi z}{z+1}\,dz = f(-1)$$

$$\Rightarrow \frac{1}{4\pi i}\oint_c \frac{\cos \pi z}{z+1}\,dz = \frac{1}{2}(-1) = -\frac{1}{2}.$$

$$\therefore \frac{1}{2\pi i}\oint_c \frac{\cos \pi z}{z^2-1}\,dz = \frac{1}{4\pi i}\oint_c \frac{\cos \pi z}{z-1}\,dz - \frac{1}{4\pi i}\oint_c \frac{\cos \pi z}{z+1}\,d$$

$$= -\frac{1}{2} - \left(-\frac{1}{2}\right)$$

$$= -\frac{1}{2} + \frac{1}{2}$$

$$= 0 \quad \text{Ans.}$$

Problem - 8 ... if $t > 0$ and c is

Problem - 10

Find the value of (a) $\oint_C \frac{\sin^6 z}{z-\frac{\pi}{6}} dz$, (b) $\oint_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz$ if c is the circle $|z|=1$.

Solⁿ- a:

Let, $f(z) = \sin^6 z$ which is analytic inside on c. The point $z = \frac{\pi}{6}$ inside the circle $|z|=1$.
Hence by cauchy's integral formula we get,

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a} dz$$

$$\Rightarrow \oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\Rightarrow \oint_C \frac{\sin^6 z}{z-\pi/6} dz = 2\pi i f\left(\frac{\pi}{6}\right)$$

$$= 2\pi i \cdot \frac{1}{64}$$

$$= \frac{\pi i}{32} \text{ Ans.}$$

Here, $f(z) = \sin^6 z$

$$\Rightarrow f\left(\frac{\pi}{6}\right) = \left(\sin\frac{\pi}{6}\right)^6$$

$$= \left(\frac{1}{2}\right)^6$$

$$= \frac{1}{64}$$

Solⁿ - b:

Let, $f(z) = \sin^6 z$ which is analytic and inside on c. The point $z = \frac{\pi}{6}$ inside the circle $|z|=1$.
Hence by cauchy's integral formula we get,

$$f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^{n+1}}$$

$$\Rightarrow \oint_C \frac{f(z)\,dz}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^n(a)$$

Here, $f(z) = \sin^6 z$, $a = \frac{\pi}{6}$, and $n = 2$

$$\Rightarrow f'(z) = \underset{u}{6\sin^5 z} \cdot \underset{v}{\cos z}$$

$$\Rightarrow f''(z) = 30\sin^4 z \cdot \cos^2 z - 6\sin^6 z.$$

$$\Rightarrow f''\left(\frac{\pi}{6}\right) = 30 \cdot \frac{1}{16} \cdot \frac{3}{4} - 6 \cdot \frac{1}{64}$$

$$= \frac{84}{64} = \frac{21}{16}.$$

$$\therefore \oint_C \frac{\sin^6 z}{\left(z - \frac{\pi}{6}\right)^3}\, dz = \frac{2\pi i}{2!} f''\left(\frac{\pi}{6}\right)$$

$$= \frac{2\pi i}{2} \cdot \frac{21}{16}$$

$$= \frac{21\pi i}{16} \quad \underline{\text{Ans.}}$$

Problem - 11

If each of the following function expanded into a taylor series about the indicated points, what would be the region of convergence. Do not perform the expansion.

(a) $\sin z / z^2+4$; $z = 0$ (b) $z/(e^z+1)$; $z=0$

(c) $(z+3)/(z-1)(z-4)$; $z=2$ (d) $e^{-z}\sinh(z+2)$; $z=0$

(e) $e^z/z(z-1)$; $z = 4i$. (f) $\coth 2z$; $z=0$

(g) $\sec\pi z$; $z = 1$

Soln - a: Let, $f(z) = \dfrac{\sin z}{z^2+4}$; $z = 0$

and let, $z = u$ then

$$f(z) = \frac{\sin u}{u^2+4} = \frac{1}{u^2+4}\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \cdots\right)$$

$$= \frac{1}{z^2+4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)$$

The series convergences for the values of

$z^2+4=0$

$\Rightarrow z^2 = -4$

$\Rightarrow z^2 = 4i^2$

$\Rightarrow z = \pm 2i$

$\Rightarrow |z| = |2i|$

$\therefore |z| < 2$

$\therefore$ The series convergence for all value of $|z| < 2$ (Ans).

Problem-12

Expand $f(z) = 1/(z-3)$ in a laurent series valid for (a) $|z|<3$ (b) $|z|>3$.

Solⁿ-b:

Here, $f(z) = \frac{1}{z-3}$ where, $|z|>3$ $\Rightarrow \frac{3}{|z|} < 1$

$$= \frac{1}{z\left(1-\frac{3}{z}\right)}$$

$$= \frac{1}{z}\left(1-\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1+\frac{3}{z}+\frac{9}{z^2}+\frac{27}{z^3}+\cdots\right)$$

$$= \frac{1}{z}+\frac{3}{z^2}+\frac{9}{z^3}+\frac{27}{z^4}+\cdots$$ Ans.

Solⁿ-a:

Here, $f(z) = \frac{1}{z-3}$ where, $|z|<3$ $\Rightarrow \frac{|z|}{3} < 1$

$$= \frac{1}{3\left(\frac{z}{3}-1\right)} = -\frac{1}{3\left(1-\frac{z}{3}\right)}$$

$$= -\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1}$$

$$= -\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\cdots\right)$$

$$= -\frac{1}{3}-\frac{z}{9}-\frac{z^2}{27}-\frac{z^3}{81}-\cdots$$

Ans.

Problem - 13

Expand $f(z) = \frac{z}{(z-1)(2-z)}$ in a Laurent series v...

(a) $|z|<1$, (b) $1<|z|<2$, (c) $|z|>2$ (d) $|z-1|>1$, (e) $0<|z-$...

Solⁿ - a:

$$f(z) = \frac{z}{(z-1)(2-z)}$$

If, $|z|<1$

$\Rightarrow |z|<2$

$\Rightarrow \frac{|z|}{2}<1$.

$$= \frac{1}{(z-1)(2-1)} + \frac{2}{(2-1)(2-z)}$$

$$= \frac{1}{(z-1)} + \frac{2}{(2-z)}$$

$$= -\frac{1}{1-z} + \frac{2}{2(1-z/2)}$$

$$= -(1-z)^{-1} + (1-z/2)^{-1}$$

$$= \left(-1-z-z^2-z^3-\cdots\right) + \left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)$$

$$= -1-z-z^2-z^2-\cdots+1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots$$

$$= -\frac{1}{2}z - \frac{3}{4}z^2 - \frac{7}{8}z^3 - \frac{15}{16}z^4 - \cdots$$ Ans.

Solⁿ - b:

Here, $f(z) = \frac{z}{(z-1)(2-z)} = \frac{1}{z-1} + \frac{2}{2-z}$.

If. $1<|z|<2$

$\Rightarrow 1<|z|$

$\Rightarrow \frac{1}{|z|}<1$

$$\frac{1}{z-1} = \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$

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$$= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$

and, $\dfrac{2}{2-z} = \dfrac{2}{2\left(1-\frac{z}{2}\right)}$ If, $|z|<2$

$\Rightarrow \dfrac{|z|}{2} < 1$

$$= \left(1-\frac{z}{2}\right)^{-1}$$

$$= 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots$$

The required laurent expansion valid for $1<|z|<2$ is,

$$f(z) = \frac{z}{(z-1)(2-z)} = \cdots + \frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots$$

Ans.

Solⁿ - c:

$$f(z) = \frac{z}{(z-1)(2-z)} = \frac{1}{z-1} + \frac{2}{2-z}$$

If, $|z|>2$

$\Rightarrow |z|>1$

$\Rightarrow \dfrac{1}{|z|} < 1$

$$\frac{1}{z-1} = \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right)$$

$$= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$

and, $\dfrac{2}{2-z} = \dfrac{2}{-z\left(1-\frac{2}{z}\right)}$

$$= -\frac{2}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= -\frac{2}{z}\left(1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \cdots\right)$$

$$= -\frac{2}{z} - \frac{4}{z^2} - \frac{8}{z^3} - \frac{16}{z^3} - \cdots$$

The required laurent exponsion valid for

$$\frac{z}{(z-1)(2-z)} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots - \frac{2}{z} - \frac{4}{z^2} - \frac{8}{z^3} - \cdots$$

$$= -\frac{1}{z} - \frac{3}{z^2} - \frac{7}{z^3} - \frac{15}{z^4} - \cdots$$

Ans.

Sol^n – d:

Here,

$$f(z) = \frac{z}{(z-1)(2-z)}$$

where, $|z-1| > 1$

$\Rightarrow |u| > 1$

$\Rightarrow \frac{1}{|u|} < 1$

let, $z - 1 = u$

$\Rightarrow z = 1 + u$

$$\therefore \frac{z}{(z-1)(2-z)} = \frac{u+1}{u(1-u)} = \frac{1}{u} + \frac{2}{1-u}$$

$$= \frac{1}{u} + \frac{2}{-u(1-\frac{1}{u})}$$

$$= \frac{1}{u} - \frac{2}{u}\left(1 - \frac{1}{u}\right)^{-1}$$

$$= \frac{1}{u} - \frac{2}{u}\left(1 + \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \cdots\right)$$

$$= \frac{1}{u} - \frac{2}{u} - \frac{2}{u^2} - \frac{2}{u^3} - \frac{2}{u^4} - \cdots$$

$$= -\frac{1}{u} - \frac{2}{u^2} - \frac{2}{u^3} - \frac{2}{u^4} - \cdots$$

$$= -\frac{1}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} - \frac{2}{(z-1)^4} - \cdots$$

Ans.

Solⁿ – e:

$$f(z) = \frac{z}{(z-1)(2-z)}.$$

let, $z-2 = u$

$$\therefore \frac{z}{(z-1)(2-z)} = \frac{u+2}{(u+1)(-u)} = -\frac{2}{u} + \frac{1}{u+1}$$

$$= -\frac{2}{u} + (1+u)^{-1}$$

If, $0 < |z-2| < 1$

$\Rightarrow |u| < 1$.

$$= -\frac{2}{u} + (1 - u + u^2 - u^3 + u^4 - \cdots)$$

$$= -\frac{2}{u} + 1 - u + u^2 - u^3 + u^4 - \cdots$$

$$= 1 - \frac{2}{z-2} - (z-2) + (z-2)^2 - (z-2)^3 + \cdots$$

Ans.

Problem:– 14

Expand $f(z) = \frac{1}{z(z-2)}$ in a laurent series valid for

ⓐ $0 < |z| < 2$ ⓑ $|z| > 2$.

Solⁿ – a:

$$f(z) = \frac{1}{z(z-2)} = \frac{1}{-2z(1-\frac{z}{2})}$$

where, $0 < |z| < 2$

$\Rightarrow |z| < 2$

$\Rightarrow \frac{|z|}{2} < 1$

$$= -\frac{1}{2z}(1 - z/2)^{-1}$$

$$= -\frac{1}{2z}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right)$$

$$= -\frac{1}{2z} - \frac{1}{4} - \frac{z}{8} - \frac{z^2}{16} - \cdots$$

Ans.

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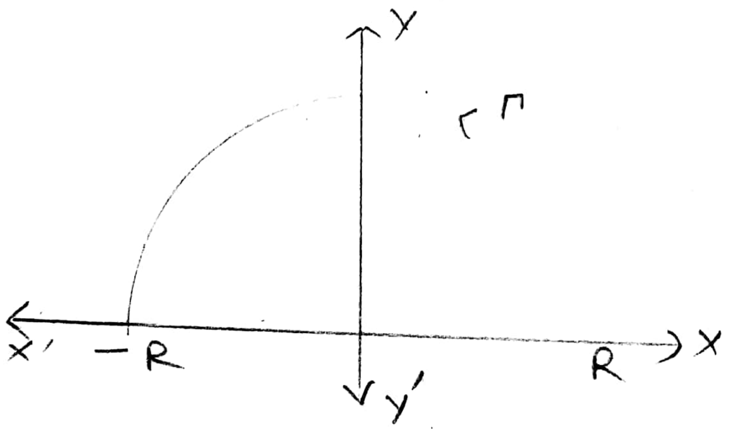
Problem 17

Prove that $\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$.

Solution:

consider, $\oint_c \frac{dz}{z^4+1}$; where c is the closed contour (semi circle) as shown in the fig.



$z^4+1=0$

$\Rightarrow z^4 = -1$

$= \cos\pi + i\sin\pi$

$= \cos(2n+1)\pi + i\sin(2n+1)\pi$

$= e^{i(2n+1)\pi}$

$\Rightarrow z = e^{i(2n+1)\pi/4}$, where, $n=0,1,2,3$

$\therefore z = e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4}$

All poles are simple pole. Only the poles $z = e^{i\pi/4}$, $e^{3i\pi/4}$, and $e^{3i\pi/4}$ inside the contour.

Residue at $z = e^{i\pi/4}$ is $a_1 = \lim_{z\to e^{i\pi/4}} \frac{(z-e^{i\pi/4})}{z^4+1}$ form $\frac{0}{0}$

$= \lim_{z\to e^{i\pi/4}} \frac{1}{4z^3}$

$$= \frac{1}{4}\frac{1}{(e^{i\pi/4})^3} = \frac{1}{4}e^{-\frac{3i\pi}{4}}.$$

Residue at $z = e^{3i\pi/4}$ is $b_1 = \lim_{z\to e^{3i\pi/4}} \left\{\frac{(z-e^{i3\pi/4})}{z^4+1}\right.$

$$= \lim_{z\to e^{3i\pi/4}} \frac{1}{4z^3} = \frac{1}{4}\cdot\frac{1}{(e^{3i\pi/4})^3}$$

$$= \frac{1}{4}\cdot e^{-9i\pi/4}.$$

Residue at $z = e^{5i\pi/4}$ is $e_1 = \lim_{z\to e^{5i\pi/4}} \left\{\frac{(z-e^{5i\pi/4})}{(z^4+1)}\right\}$

$$= \lim_{z\to e^{5i\pi/4}} \frac{1}{4z^3} = \frac{1}{4}\frac{1}{(e^{5i\pi/4})^3}$$

$$= \frac{1}{4}e^{-15i\pi/4}$$

By residue th^m,

$\oint_c f(z)\,dz = 2\pi i$ [sum of the residue]

$\oint_c f(z)\,dz = 2\pi i\left[\frac{1}{4}e^{-3i\pi/4} + \frac{1}{4}e^{-9i\pi/4} + \frac{1}{4}e^{-15i\pi/4}\right]$

$$\Rightarrow \oint_c f(z)\,dz = \frac{\pi i}{2}\left[e^{-3i\pi/4} + e^{-9i\pi/4} + e^{-15i\pi/4}\right]$$

$$= \frac{\pi i}{2}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} + \cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4} + \cos\frac{15\pi}{4} - i\sin\frac{15\pi}{4}\right]$$

$$= \frac{\pi i}{2}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]$$

$$= \frac{\pi i}{2}\left(-\frac{2i}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}.$$

$$\therefore \int_{\Gamma} \frac{dz}{z^4+1} + \int_{-R}^{R} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow \lim_{R\to\infty} \int_{\Gamma} \frac{dz}{z^4+1} + \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow 0 + \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow \int_{0}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$$

[Proved]