CSE 211 (Theory of Computation)

Regular Languages

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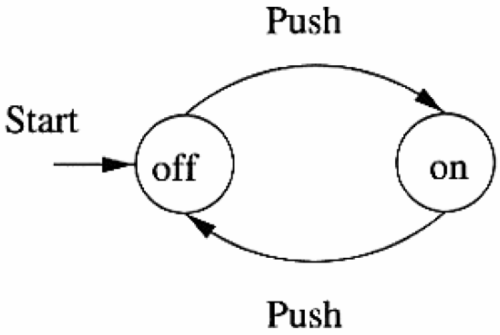
Example

Hopcroft, Motwani, and Ullman, Figure 1.1, p-3



A finite automaton modeling an on/off switch

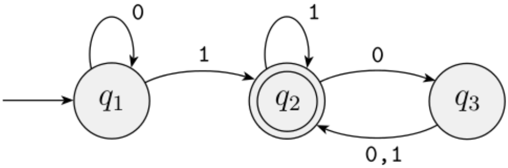


FIGURE 1.4
A finite automaton called M_1 that has three states

Finite Automata

Sipser, 1.1, p-34

- state diagram
- states
- start state
- accept state
- transitions

Finite Automata

Hopcroft, Motwani, and Ullman, 2.2, p-45

- deterministic finite automaton
- deterministic
- nondeterministic
- DFA

Formal Definition of a Finite Automaton

Sipser, Definition 1.5, p-35

DEFINITION 1.5

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
2. Σ is a finite set called the ***alphabet***,
3. $\delta \colon Q \times \Sigma \longrightarrow Q$ is the ***transition function***,[1]
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.[2]

- A is the set of all strings that machine M accepts.
- We say that A is the language of machine M.
- Write $L(M) = A$.
- We say that M recognizes A or that M accepts A.

Example

Sipser, Example 1.7, p-37

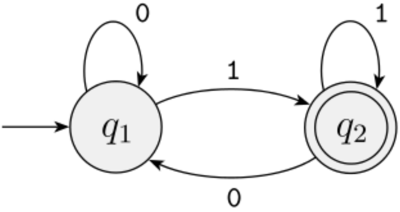


FIGURE 1.8
State diagram of the two-state finite automaton M_2

Example

Sipser, Example 1.9, p-38

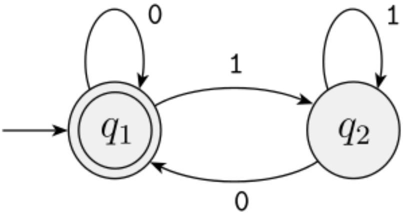


FIGURE 1.10

State diagram of the two-state finite automaton M_3

Example

Sipser, Example 1.11, p-38

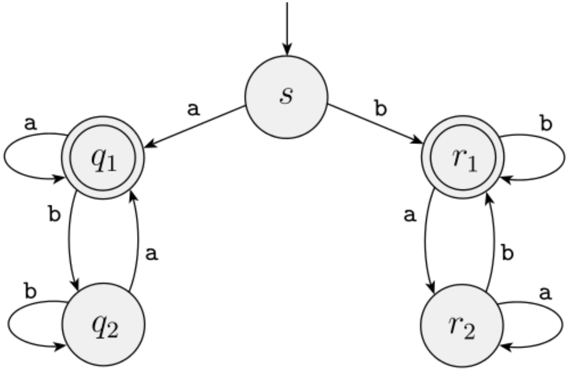


FIGURE 1.12
Finite automaton M_4

Example

Sipser, Example 1.13, p-39

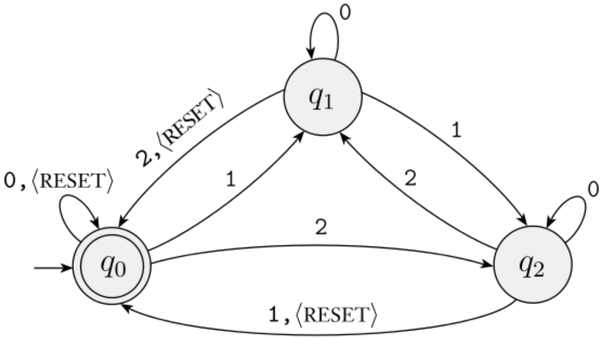


FIGURE 1.14
Finite automaton M_5

Example

Sipser, Example 1.15, p-40

- A generalization of Example 1.13.
- Same four-symbol alphabet Σ.
- For each $i \geq 1$ let A_i be the language of all strings where the sum of the numbers is a multiple of i.
- Except that the sum is reset to 0 whenever the symbol `<RESET>` appears.
- For each A_i we give a finite automaton B_i, recognizing A_i.

Example — *continued*

Sipser, Example 1.15, p-40

- We describe the machine B_i formally as follows.
- $B_i = (Q_i, \Sigma, \delta_i, q_0, \{q_0\})$, where Q_i is the set of i states $\{q_0, q_1, q_2, \ldots, q_{i-1}\}$.
- We design the transition function δ_i so that for each j, if B_i is in q_j.
- The running sum is j, modulo i.

- For each q_j let,

$$
\begin{aligned}
\delta_i(q_j, 0) &= q_j, \\
\delta_i(q_j, 1) &= q_k, \text{ where } k = j + 1 \text{ modulo } i, \\
\delta_i(q_j, 2) &= qk, \text{ where } k = j + 2 \text{ modulo } i, \text{ and} \\
\delta_i(q_j, \texttt{<RESET>}) &= q_0
\end{aligned}
$$

Formal Definition of Computation

Sipser, 1.1, p-40

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be a finite automaton.
- Let $w_1, w_2, \ldots, w_n$ be a string where each w_i is a member of the alphabet Σ.

Formal Definition of Computation — *continued*

Sipser, 1.1, p-40

- Then M accepts w if a sequence of states $r_0, r_1, r_2, \ldots, r_n$ in Q exists with three conditions:
  1. $r_0 = q_0$,
  2. $\delta(r_i, w_{i+1}) = r_{i+1}$, for $i = 0, \ldots, n-1$, and
  3. $r_n \in F$.

Formal Definition of Computation — *continued*

Sipser, 1.1, p-40

- Then M accepts w if a sequence of states $r_0, r_1, r_2, \ldots, r_n$ in Q exists with three conditions:
  1. $r_0 = q_0$,
  2. $\delta(r_i, w_{i+1}) = r_{i+1}$, for $i = 0, \ldots, n-1$, and
  3. $r_n \in F$.
- Condition 1 says that the machine starts in the start state.
- Condition 2 says that the machine goes from state to state according to the transition function.
- Condition 3 says that the machine accepts its input if it ends up in an accept state.
- We say that M recognizes language A if $A = \{w \mid M \text{ accepts } w\}$.

Formal Definition of Computation

Sipser, Definition 1.16, p-40

DEFINITION 1.16

A language is called a ***regular language*** if some finite automaton recognizes it.

Designing Finite Automata

Sipser, 1.1, p-41

- You have to figure out what you need to remember about the string as you are reading it.

Designing Finite Automata

Sipser, 1.1, p-41

- Suppose that the alphabet is $\{0, 1\}$ and that the language consists of all strings with an odd number of 1s.
- You want to construct a finite automaton E_1 to recognize this language.

Designing Finite Automata

Sipser, Figure 1.18, p-42

FIGURE 1.18
The two states q_{even} and q_{odd}

Designing Finite Automata

Sipser, Figure 1.19, p-42

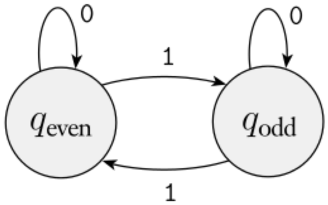


FIGURE 1.19
Transitions telling how the possibilities rearrange

Designing Finite Automata

Sipser, Figure 1.20, p-43

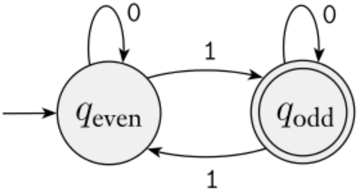


FIGURE 1.20
Adding the start and accept states

Example

Sipser, Example 1.21, p-43

- Design a finite automaton E_2 to recognize the regular language of all strings that contain the string 001 as a substring.
- For example, 0010, 1001, 001, and 11111110011111 are all in the language, but 11 and 0000 are not.

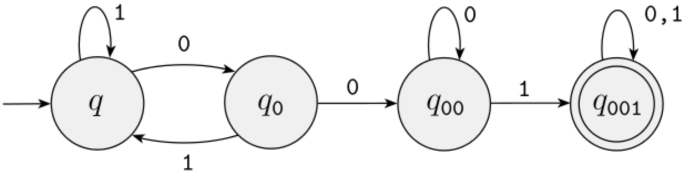


FIGURE 1.22
Accepts strings containing 001

Example

Hopcroft, Motwani, and Ullman, Example 2.1, p-46

- Let us formally specify a DFA that accepts all and only the strings of 0's and 1's that have the sequence 01 somewhere in the string.

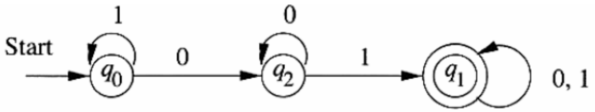
Example — *continued*

Hopcroft, Motwani, and Ullman, Example 2.1, p-46

- We can write this language L as:
  $\{w \mid w$ is of the form $x01y$ for some strings x and y consisting of 0's and 1's only.$\}$
- Another equivalent description, using parameters x and y to the left of the vertical bar, is:
  $\{x01y \mid x$ and y are any strings of 0's and 1's$\}$

Example — *continued*

Hopcroft, Motwani, and Ullman, Example 2.1, p-46



The transition diagram for the DFA accepting all strings with a substring 01

- Design a DFA to accept the language $L = \{w \mid w$ has both an even number of 0's and an even number of 1's$\}$

Example — *continued*

Hopcroft, Motwani, and Ullman, Example 2.4, p-51

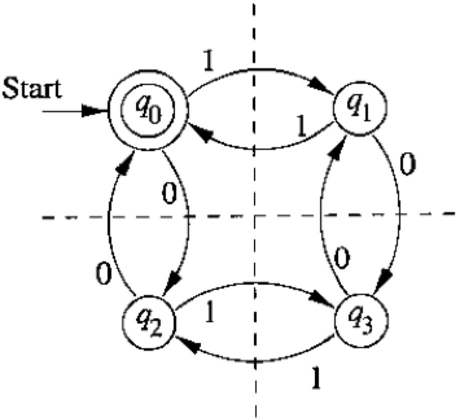
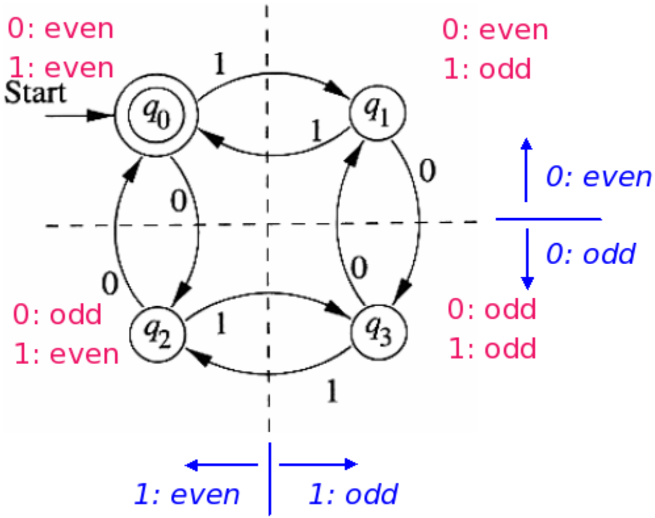


Figure 2.6: Transition diagram for the DFA of Example 2.4

Hopcroft, Motwani, and Ullman, Example 2.4, p-51



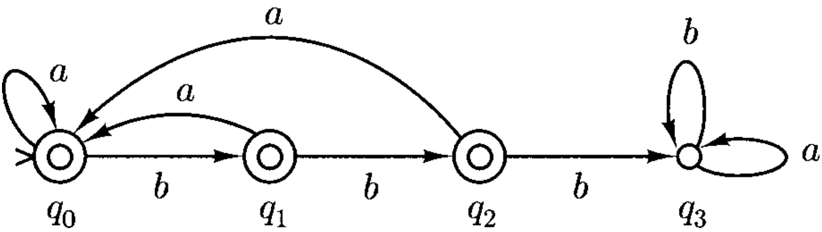
Example

Lewis and Papadimitriou, Example 2.1.2, p-59

- Design a deterministic finite automaton M that accepts the language
  $L(M) = \{w \in \{a, b\}^* :$
  w does not contain three consecutive b's$\}$.

Example — *continued*

Lewis and Papadimitriou, Example 2.1.2, p-59



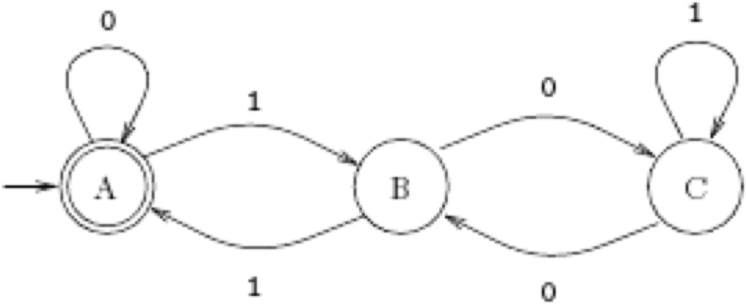
Example

http://math.stackexchange.com/questions/140283/why-does-this-fsm-accept-binary-numbers-divisible-by-three

- Design a DFA that accepts binary numbers that are divisible by three.

Example — *continued*

http://math.stackexchange.com/questions/140283/why-does-this-fsm-accept-binary-numbers-divisible-by-three



The Regular Operations

Sipser, 1.1, p-44

DEFINITION 1.23

Let A and B be languages. We define the regular operations ***union***, ***concatenation***, and ***star*** as follows:

- **Union**: $A \cup B = \{x |\ x \in A \text{ or } x \in B\}$.
- **Concatenation**: $A \circ B = \{xy |\ x \in A \text{ and } y \in B\}$.
- **Star**: $A^* = \{x_1 x_2 \ldots x_k |\ k \geq 0 \text{ and each } x_i \in A\}$.

Example

Sipser, Example 1.24, p-45

- Alphabet Σ be the standard 26 letters $\{a, b, ..., z\}$.
- $A = \{\text{good}, \text{bad}\}$ and $B = \{\text{boy}, \text{girl}\}$.

Example

Sipser, Example 1.24, p-45

- Alphabet Σ be the standard 26 letters $\{a, b, ..., z\}$.
- $A = \{\text{good}, \text{bad}\}$ and $B = \{\text{boy}, \text{girl}\}$.
- $A \cup B = \{\text{good}, \text{bad}, \text{boy}, \text{girl}\}$

Example

Sipser, Example 1.24, p-45

- Alphabet Σ be the standard 26 letters $\{a, b, ..., z\}$.
- $A = \{\text{good}, \text{bad}\}$ and $B = \{\text{boy}, \text{girl}\}$.
- $A \cup B = \{\text{good}, \text{bad}, \text{boy}, \text{girl}\}$
- $A \circ B = \{\text{goodboy}, \text{goodgirl}, \text{badboy}, \text{badgirl}\}$

Example

Sipser, Example 1.24, p-45

- Alphabet Σ be the standard 26 letters $\{a, b, ..., z\}$.
- $A = \{\text{good}, \text{bad}\}$ and $B = \{\text{boy}, \text{girl}\}$.
- $A \cup B = \{\text{good}, \text{bad}, \text{boy}, \text{girl}\}$
- $A \circ B = \{\text{goodboy}, \text{goodgirl}, \text{badboy}, \text{badgirl}\}$
-
$$
\begin{aligned}
A^* = \{&\epsilon, \text{good}, \text{bad}, \text{goodgood}, \text{goodbad}, \text{badgood}, \text{badbad}, \\
&\text{goodgoodgood}, \text{goodgoodbad}, \text{goodbadgood}, \\
&\text{goodbadbad}, \ldots\}
\end{aligned}
$$

- $\mathcal{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers.
- We say that $\mathcal{N}$ is closed under multiplication.
- We mean that for any x and y in $\mathcal{N}$, the product $x \times y$ also is in $\mathcal{N}$.
- In contrast, $\mathcal{N}$ is *not* closed under division.
- 1 and 2 are in $\mathcal{N}$ but $1/2$ is not.

- Generally speaking, a collection of objects is closed under some operation if applying that operation to members of the collection returns an object still in the collection.
- We show that the collection of regular languages is closed under all three of the regular operations.

THEOREM 1.25

The class of regular languages is closed under the union operation.

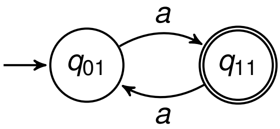
In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

- $\Sigma = \{a\}$
- $L_1 = \{\text{contains an odd number of } a\text{'s}\}$
  $L_2 = \{aa\}$
- Design automata M_1 and M_2 for L_1 and L_2 and then construct M which recognizes $L_1 \cup L_2$

- $L_1 = \{\text{contains an odd number of } a\text{'s}\}$
  $L_2 = \{aa\}$

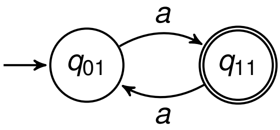
- $L_1 = \{$contains an odd number of a's$\}$
  $L_2 = \{aa\}$

M_1

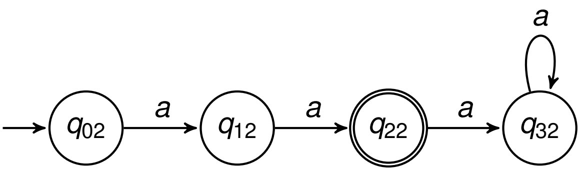


- $L_1 = \{\text{contains an odd number of } a\text{'s}\}$
  $L_2 = \{aa\}$

M_1



M_2



PROOF IDEA

- We have regular languages A_1 and A_2 and want to show that $A_1 \cup A_2$ also is regular.
- Because A_1 and A_2 are regular, we know that some finite automaton M_1 recognizes A_1 and some finite automaton M_2 recognizes A_2.
- To prove that $A_1 \cup A_2$ is regular, we demonstrate a finite automaton, call it M, that recognizes $A_1 \cup A_2$.

The Regular Operations — *continued*

Sipser, 1.1, p-45

- This is a proof by construction.
- We construct M from M_1 and M_2.
- Machine M must accept its input exactly when either M_1 or M_2 would accept it in order to recognize the union language.

The Regular Operations — *continued*

Sipser, 1.1, p-45

- It works by simulating both M_1 and M_2 and accepting if either of the simulations accept.
- How can we make machine M simulate M_1 and M_2?
- Perhaps it first simulates M_1 on the input and then simulates M_2 on the input.

- But we must be careful here!
- Once the symbols of the input have been read and used to simulate M_1 , we can't "rewind the input tape" to try the simulation on M_2.
- We need another approach.

- Pretend that you are M.
- As the input symbols arrive one by one, you simulate both M_1 and M_2 simultaneously.
- That way, only one pass through the input is necessary.

The Regular Operations — *continued*

Sipser, 1.1, p-45

- But can you keep track of both simulations with finite memory?
- All you need to remember is the state that each machine would be in if it had read up to this point in the input.
- Therefore, you need to remember a pair of states.

The Regular Operations — *continued*

Sipser, 1.1, p-45

- How many possible pairs are there?
- If M_1 has k_1 states and M_2 has k_2 states, the number of pairs of states, one from M_1 and the other from M_2, is the product $k_1 \times k_2$.
- This product will be the number of states in M, one for each pair.

- The transitions of M go from pair to pair, updating the current state for both M_1 and M_2.
- The accept states of M are those pairs wherein either M_1 or M_2 is in an accept state.

PROOF

- M_1 recognize A_1, where $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$.
- M_2 recognize A_2, where $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$.
- Construct M to recognize $A_1 \cup A_2$, where $M = (Q, \Sigma, \delta, q_0, F)$.

1. $Q = \{(r_1, r_2) \mid r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$.

- This set is the Cartesian product of sets Q_1 and Q_2 and is written $Q_1 \times Q_2$.
- It is the set of all pairs of states, the first from Q_1 and the second from Q_2.

The Regular Operations — *continued*

Sipser, 1.1, p-45

2. Σ, the alphabet, is the same as in M_1 and M_2.

- In this theorem and in all subsequent similar theorems, we assume for simplicity that both M_1 and M_2 have the same input alphabet Σ.
- The theorem remains true if they have different alphabets, Σ_1 and Σ_2.
- We would then modify the proof to let $\Sigma = \Sigma_1 \cup \Sigma_2$.

3. δ, the transition function, is defined as follows.

- For each $(r_1, r_2) \in Q$ and each $a \in \Sigma$, let

$$\delta\left((r_1, r_2), a\right) = \left(\delta_1(r_1, a), \delta_2(r_2, a)\right).$$

- Hence δ gets a state of M (which actually is a pair of states from M_1 and M_2), together with an input symbol, and returns M's next state.

4. q_0 is the pair (q_1, q_2).

The Regular Operations — *continued*

Sipser, 1.1, p-45

5. F is the set of pairs in which either member is an accept state of M_1 or M_2.

- We can write it as $F = \{(r_1, r_2) \mid r_1 \in F_1 \text{ or } r_2 \in F_2\}$.
- This expression is the same as $F = (F_1 \times Q_2) \cup (Q_1 \times F_2)$.

5. F is the set of pairs in which either member is an accept state of M_1 or M_2.

- We can write it as $F = \{(r_1, r_2) \mid r_1 \in F_1 \text{ or } r_2 \in F_2\}$.
- This expression is the same as $F = (F_1 \times Q_2) \cup (Q_1 \times F_2)$.
- Note that it is not the same as $F = F_1 \times F_2$.

THEOREM 1.26

The class of regular languages is closed under the concatenation operation.

In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

The Regular Operations — *continued*

Sipser, 1.1, p-47

- To prove this theorem, let's try something along the lines of the proof of the union case.
- As before, we can start with finite automata M_1 and M_2 recognizing the regular languages A_1 and A_2.

The Regular Operations — *continued*

- But now, instead of constructing automaton M to accept its input if either M_1 or M_2 accept, it must accept if its input can be broken into two pieces, where M_1 accepts the first piece and M_2 accepts the second piece.
- The problem is that M doesn't know where to break its input (i.e., where the first part ends and the second begins).

Nondeterministic Finite Automata

Lewis and Papadimitriou, 2.2, p-63

- Nondeterminism is an *inessential feature* of finite automata.
- Every nondeterministic finite automaton is equivalent to a deterministic finite automaton.
- Thus we shall profit from the powerful notation of nondeterministic finite automata.
- But we always know that, if we must, we can always go back and redo everything in terms of the lower-level language of ordinary, down-to-earth deterministic automata.

- $L = (ab \cup aba)^*$

- $L = (ab \cup aba)^*$
- As many as $(ab \cup aba)$'s you like.
- $(ab \cup aba)^* =$
  $(ab \cup aba)(ab \cup aba)(ab \cup aba)(ab \cup aba) \ldots (ab \cup aba)$

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- As many as $(ab \cup aba)$'s you like.
- $(ab \cup aba)^* =$
  $(ab \cup aba)(ab \cup aba)(ab \cup aba)(ab \cup aba) \ldots (ab \cup aba)$
- ab

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- As many as $(ab \cup aba)$'s you like.
- $(ab \cup aba)^* =$
  $(ab \cup aba)(ab \cup aba)(ab \cup aba)(ab \cup aba) \ldots (ab \cup aba)$
- ab *belongs*

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- As many as $(ab \cup aba)$'s you like.
- $(ab \cup aba)^* =$
  $(ab \cup aba)(ab \cup aba)(ab \cup aba)(ab \cup aba)\ldots(ab \cup aba)$
- ab belongs
- aba

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- aba *belongs*

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- aba *belongs*
- $ababa$

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- aba *belongs*
- $ababa$ *belongs*

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  $(ab \cup aba)(ab \cup aba)(ab \cup aba)(ab \cup aba)\ldots(ab \cup aba)$
- ab *belongs*
- aba *belongs*
- $ababa$ *belongs*
- $abaab$

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- $(ab \cup aba)^* =$
  $(ab \cup aba)(ab \cup aba)(ab \cup aba)(ab \cup aba) \ldots (ab \cup aba)$
- *ab* — *belongs*
- *aba* — *belongs*
- *ababa* — *belongs*
- *abaab* — *belongs*

- $L = (ab \cup aba)^*$
- As many as $(ab \cup aba)$'s you like.
- $(ab \cup aba)^* =$
  $(ab \cup aba)(ab \cup aba)(ab \cup aba)(ab \cup aba) \ldots (ab \cup aba)$
- ab belongs
- aba belongs
- $ababa$ belongs
- $abaab$ belongs
- $abab$

- $L = (ab \cup aba)^*$
- As many as $(ab \cup aba)$'s you like.
- $(ab \cup aba)^* =$
  $(ab \cup aba)(ab \cup aba)(ab \cup aba)(ab \cup aba) \ldots (ab \cup aba)$
- ab belongs
- aba belongs
- $ababa$ belongs
- $abaab$ belongs
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- $(ab \cup aba)^* =$
  $(ab \cup aba)(ab \cup aba)(ab \cup aba)(ab \cup aba) \ldots (ab \cup aba)$
- ab *belongs*
- aba *belongs*
- $ababa$ *belongs*
- $abaab$ *belongs*
- $abab$ *belongs*
- ϵ

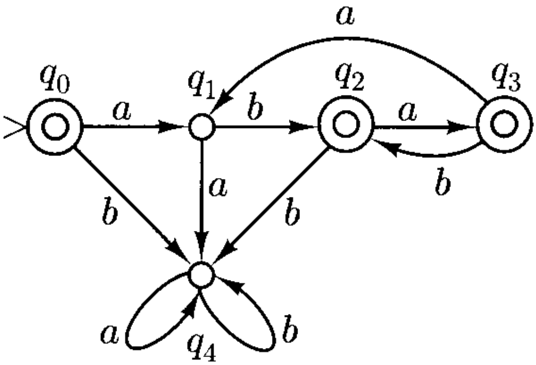
- $L = (ab \cup aba)^*$
- As many as $(ab \cup aba)$'s you like.
- $(ab \cup aba)^* =$
  $(ab \cup aba)(ab \cup aba)(ab \cup aba)(ab \cup aba) \ldots (ab \cup aba)$
- ab belongs
- aba belongs
- $ababa$ belongs
- $abaab$ belongs
- $abab$ belongs
- ϵ belongs

- $L = (ab \cup aba)^*$
- As many as $(ab \cup aba)$'s you like.
- $(ab \cup aba)^* =$
  $(ab \cup aba)(ab \cup aba)(ab \cup aba)(ab \cup aba) \ldots (ab \cup aba)$
- ab belongs
- aba belongs
- $ababa$ belongs
- $abaab$ belongs
- $abab$ belongs
- ϵ belongs
- $abababba$

- $L = (ab \cup aba)^*$
- As many as $(ab \cup aba)$'s you like.
- $(ab \cup aba)^* =$
  $(ab \cup aba)(ab \cup aba)(ab \cup aba)(ab \cup aba) \dots (ab \cup aba)$
- ab — belongs
- aba — belongs
- $ababa$ — belongs
- $abaab$ — belongs
- $abab$ — belongs
- ϵ — belongs
- $abababba$ — does not belong

Nondeterministic Finite Automata — *continued*

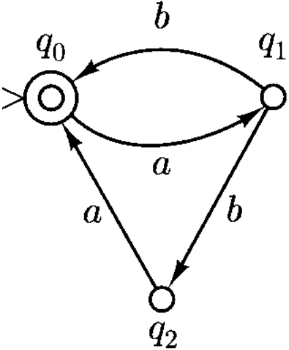
Lewis and Papadimitriou, Figure 2.4, p-64



Nondeterministic Finite Automata — *continued*

Lewis and Papadimitriou, Figure 2.5, p-65

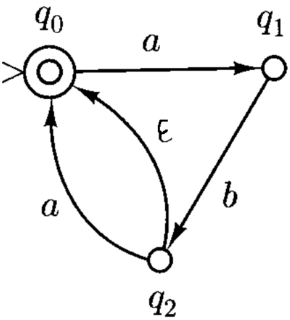
$L = (ab \cup aba)^*$



Nondeterministic Finite Automata — *continued*

Lewis and Papadimitriou, Figure 2.6, p-65

$L = (ab \cup aba)^*$



Nondeterministic Finite Automata — *continued*

Sipser, Figure 1.27, p-48

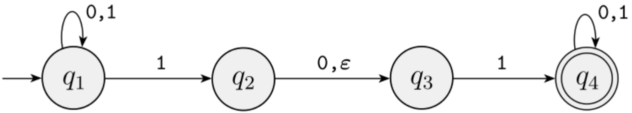


FIGURE 1.27
The nondeterministic finite automaton N_1

Nondeterministic Finite Automata — *continued*

Sipser, Figure 1.28, p-49

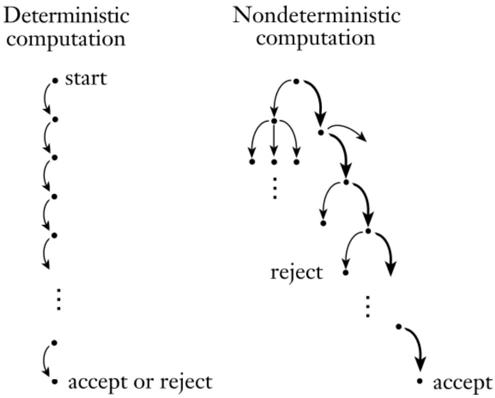


FIGURE 1.28
Deterministic and nondeterministic computations with an accepting branch

Nondeterministic Finite Automata — *continued*

Sipser, Figure 1.29, p-49

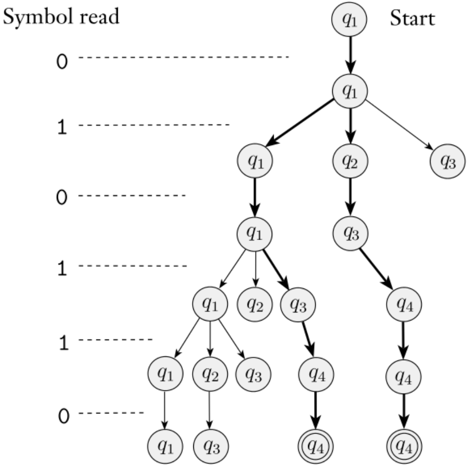
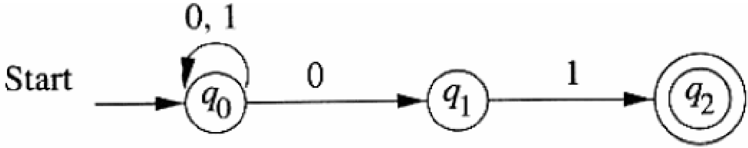


FIGURE 1.29
The computation of N_1 on input 010110

Example

Hopcroft, Motwani, and Ullman, Example 2.6, p-56

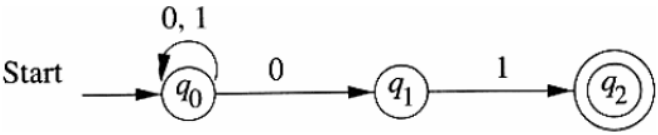
- Job of this automaton is to accept all and only the strings of 0's and 1's that end in 01.

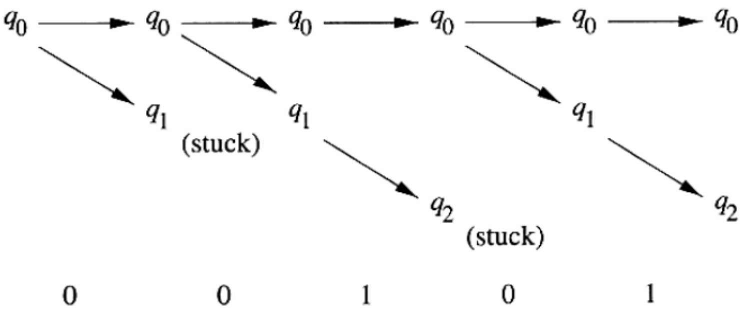


An NFA accepting all strings that end in 01

Example — *continued*

Hopcroft, Motwani, and Ullman, Example 2.6, p-56





Example

Sipser, Example 1.30, p-51

- Let A be the language consisting of all strings over $\{0, 1\}$ containing a 1 in the third position from the end.
- 000100 is in A but 0011 is not.

Example — *continued*

Sipser, Example 1.30, p-51

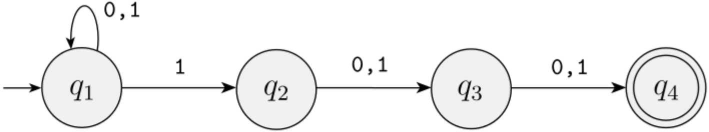


FIGURE 1.31
The NFA N_2 recognizing A

Nondeterministic Finite Automata — *continued*

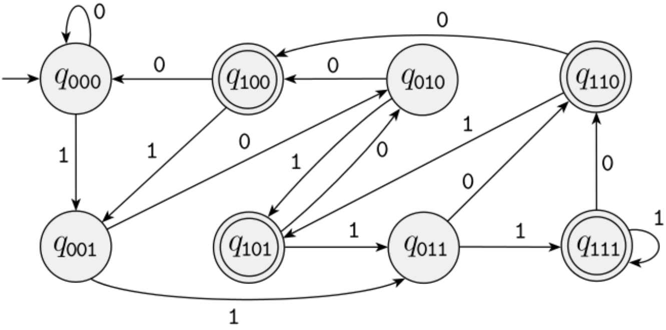


FIGURE 1.32
A DFA recognizing A

Nondeterministic Finite Automata — *continued*

Sipser, Example 1.33, p-52

- Accepts all strings of the form 0^k where k is a multiple of 2 or 3.
- N_3 accepts the strings ϵ, 00, 000, 0000, and 000000, but not 0 or 00000.

Nondeterministic Finite Automata — *continued*

Sipser, Example 1.33, p-52

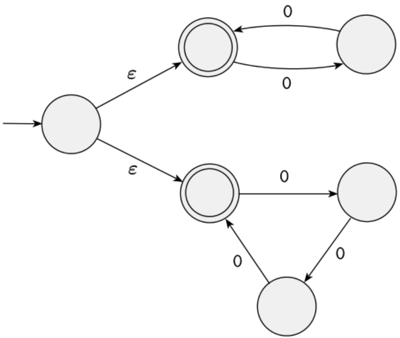


FIGURE 1.34
The NFA N_3

- Has an input alphabet $\{0\}$ consisting of a single symbol.
- An alphabet containing only one symbol is called a unary alphabet.

Nondeterministic Finite Automata — *continued*

Sipser, Example 1.35, p-52

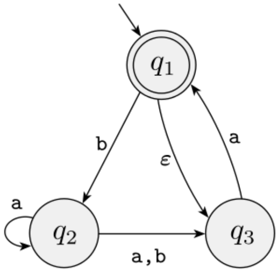


FIGURE 1.36
The NFA N_4

- It accepts the strings ϵ, a, baba, and baa.
- But that it doesn't accept the strings b, bb, and babba.

Formal Definition of a Nondeterministic Finite Automaton

Sipser, 1.2, p-53

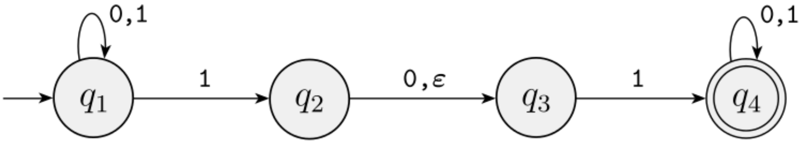
DEFINITION 1.37

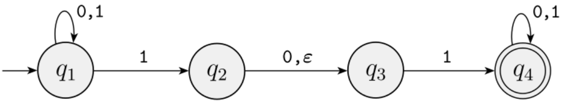
A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta\colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Formal Definition of. . . — *continued*

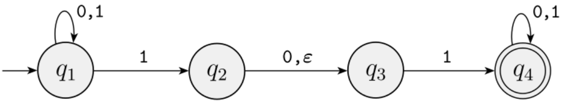
Sipser, Example 1.38, p-54





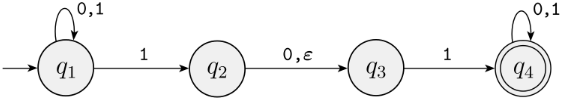
The formal description of N_1 is $(Q, \Sigma, \delta, q_1, F)$, where,

1. $Q = \{q_1, q_2, q_3, q_4\}$



The formal description of N_1 is $(Q, \Sigma, \delta, q_1, F)$, where,

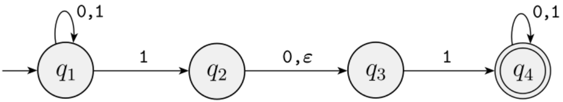
2. $\Sigma = \{0, 1\}$



The formal description of N_1 is $(Q, \Sigma, \delta, q_1, F)$, where,

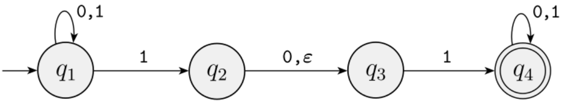
3. δ is given as

	0	1	ε
q_1	$\{q_1\}$	$\{q_1, q_2\}$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\{q_3\}$
q_3	$\emptyset$	$\{q_4\}$	$\emptyset$
q_4	$\{q_4\}$	$\{q_4\}$	$\emptyset$,



The formal description of N_1 is $(Q, \Sigma, \delta, q_1, F)$, where,

4. q_1 is the start state.



The formal description of N_1 is $(Q, \Sigma, \delta, q_1, F)$, where,

5. $F = \{q_4\}$.

Equivalence of NFAs AND DFAs

Sipser, 1.2, p-54

- Deterministic and nondeterministic finite automata recognize the same class of languages.
- Such equivalence is both surprising and useful.
- It is surprising because NFAs appear to have more power than DFAs, so we might expect that NFAs recognize more languages.
- It is useful because describing an NFA for a given language sometimes is much easier than describing a DFA for that language.
- Say that two machines are equivalent if they recognize the same language.

Equivalence of NFAs AND DFAs

Sipser, 1.2, p-55

THEOREM 1.39

Every nondeterministic finite automaton has an equivalent deterministic finite automaton.

Equivalence of NFAs AND DFAs — *continued*

Sipser, 1.2, p-55

PROOF IDEA

- If a language is recognized by an NFA, then we must show the existence of a DFA that also recognizes it.
- The idea is to convert the NFA into an equivalent DFA that simulates the NFA.
- Recall the “reader as automaton” strategy for designing finite automata.

Equivalence of NFAs AND DFAs — *continued*

Sipser, 1.2, p-55

- How would you simulate the NFA if you were pretending to be a DFA?
- What do you need to keep track of as the input string is processed?
- In the examples of NFA's, you kept track of the various branches of the computation by placing a finger on each state that could be active at given points in the input.
- You updated the simulation by moving, adding, and removing fingers according to the way the NFA operates.
- All you needed to keep track of was the set of states having fingers on them.

Equivalence of NFAs AND DFAs — *continued*

Sipser, 1.2, p-55

- If k is the number of states of the NFA, it has 2^k subsets of states.
- Each subset corresponds to one of the possibilities that the DFA must remember, so the DFA simulating the NFA will have 2^k states.
- Now we need to figure out which will be the start state and accept states of the DFA.
- What will be its transition function.
- We can discuss this more easily after setting up some formal notation.

PROOF

- Let $N = (Q, \Sigma, \delta, q_0, F)$ be the NFA recognizing some language A.
- We construct a DFA $M = (Q', \Sigma, \delta', {q_0}', F')$ recognizing A.

- Before doing the full construction, let's first consider the easier case wherein N has no ϵ arrows.
- Later we take the ϵ arrows into account.

Equivalence of NFAs AND DFAs — *continued*

Sipser, 1.2, p-55

1. $Q' = P(Q)$.

- Every state of M is a set of states of N.
- Recall that $P(Q)$ is the set of subsets of Q.

2. For $R \in Q'$ and $a \in \Sigma$, let
   $\delta'(R, a) = \{q \in Q \mid q \in \delta(r, a) \text{ for some} r \in R\}$.

- If R is a state of M, it is also a set of states of N.
- When M reads a symbol a in state R, it shows where a takes each state in R.
- Because each state may go to a set of states, we take the union of all these sets.
- Another way to write this expression is

$$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a).$$

Equivalence of NFAs AND DFAs — *continued*

Sipser, 1.2, p-55

3. ${q_0}' = \{q_0\}$.

- M starts in the state corresponding to the collection containing just the start state of N.

4. $F' = \{R \in Q' \mid R \text{ contains an accept state of } N\}$.

- The machine M accepts if one of the possible states that N could be in at this point is an accept state.

- Now we need to consider the ϵ arrows.
- To do so, we set up an extra bit of notation.
- For any state R of M, we define $E(R)$ to be the collection of states that can be reached from members of R by going only along ϵ arrows, including the members of R themselves.

Equivalence of NFAs AND DFAs — *continued*

Sipser, 1.2, p-55

- Formally, for $R \subseteq Q$ let

$$E(R) = \{q \mid q \text{ can be reached from } R \text{ by traveling along 0 or more } \epsilon \text{ arrows}\}.$$

- Then we modify the transition function of M to place additional fingers on all states that can be reached by going along ϵ arrows after every step.
- Replacing $\delta(r, a)$ by $E(\delta(r, a))$ achieves this effect.

Equivalence of NFAs AND DFAs — *continued*

Sipser, 1.2, p-55

- Thus $\delta'(R, a) = \{q \in Q \mid q \in E(\delta(r, a)) \text{ for some } r \in R\}$.
- Additionally, we need to modify the start state of M to move the fingers initially to all possible states that can be reached from the start state of N along the ϵ arrows.
- Changing $q_0{}'$ to be $E(\{q_0\})$ achieves this effect.

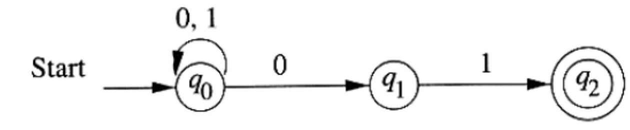
Equivalence of NFAs AND DFAs — *continued*

Sipser, 1.2, p-55

- We have now completed the construction of the DFA M that simulates the NFA N.
- The construction of M obviously works correctly.
- At every step in the computation of M on an input, it clearly enters a state that corresponds to the subset of states that N could be in at that point.
- Thus our proof is complete.

Example

Hopcroft, Motwani, and Ullman, Example 2.10, p-61



An NFA accepting all strings that end in 01

Example — *continued*

Hopcroft, Motwani, and Ullman, Example 2.10, p-61

	0	1
$\emptyset$	$\emptyset$	$\emptyset$
$\rightarrow \{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_1\}$	$\emptyset$	$\{q_2\}$
$*\{q_2\}$	$\emptyset$	$\emptyset$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
$*\{q_0, q_2\}$	$\{q_0, q_1\}$	$\{q_0\}$
$*\{q_1, q_2\}$	$\emptyset$	$\{q_2\}$
$*\{q_0, q_1, q_2\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$

Figure 2.12: The complete subset construction from Fig. 2.9

Example — *continued*

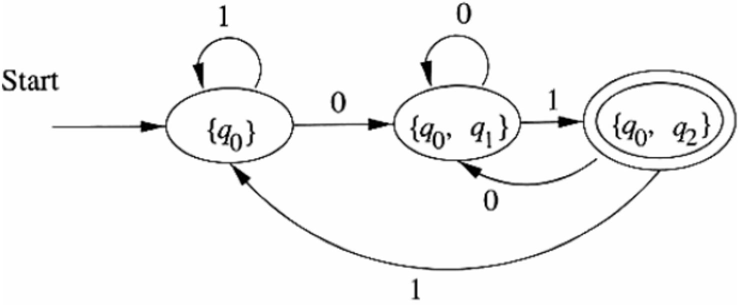
Hopcroft, Motwani, and Ullman, Example 2.10, p-61

	0	1
A	A	A
$\rightarrow B$	E	B
C	A	D
$*D$	A	A
E	E	F
$*F$	E	B
$*G$	A	D
$*H$	E	F

Renaming the states

Example — *continued*

Hopcroft, Motwani, and Ullman, Example 2.10, p-61



The DFA constructed from the NFA

Example

Sipser, Example 1.41, p-56

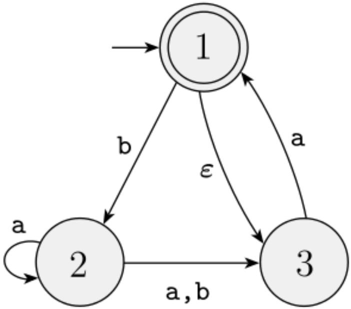


FIGURE 1.42
The NFA N_4

Example — *continued*

Sipser, Example 1.41, p-56

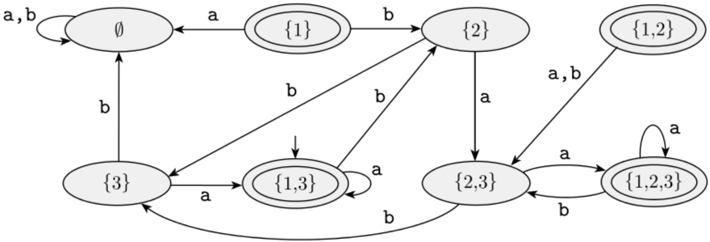


FIGURE 1.43
A DFA D that is equivalent to the NFA N_4

Example — *continued*

Sipser, Example 1.41, p-56

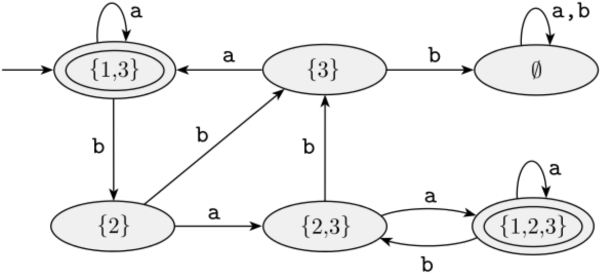
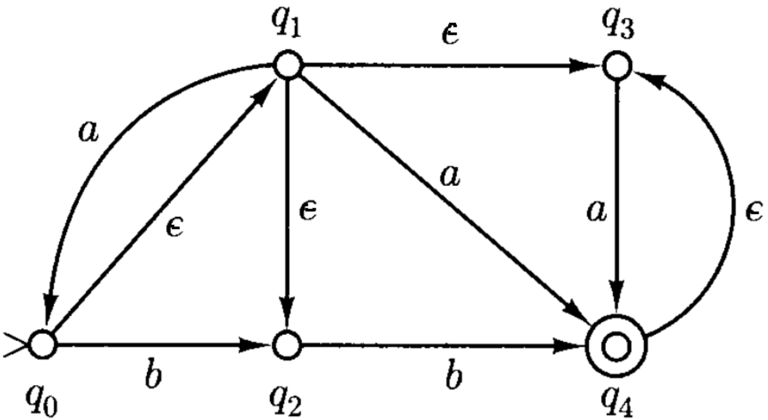


FIGURE 1.44
DFA D after removing unnecessary states

Example

Lewis and Papadimitriou, Example 2.2.3, p-70

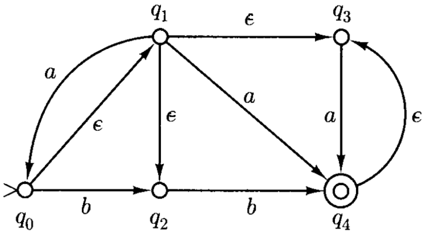
We find the DFA equivalent to the nondeterministic automaton.



Example — *continued*

$$N = (Q, \Sigma, \delta, q_0, F) \qquad D = (Q', \Sigma, \delta', q_0', F')$$

- Q' is the power set of Q.

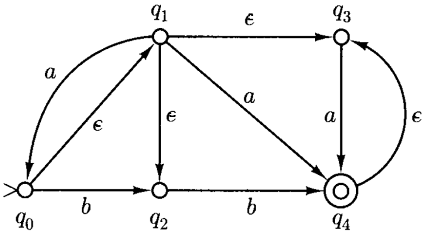


- Since N has 5 states, D will have $2^5 = 32$ states.
- However, only a few of these states will be relevant to the operation of D.

Example — *continued*

$$N = (Q, \Sigma, \delta, q_0, F) \qquad D = (Q', \Sigma, \delta', q_0', F')$$

- Q' is the power set of Q.

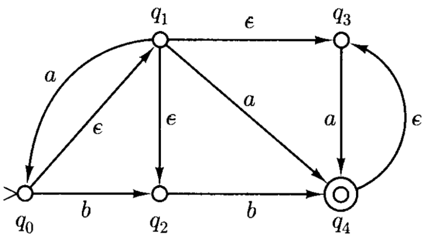


- Namely, those states that can be reached from state q_0' by reading some input string.
- Obviously, any state in D that is not reachable from q_0' is irrelevant to the operation of D.

Example — *continued*

$$N = (Q, \Sigma, \delta, q_0, F) \qquad D = (Q', \Sigma, \delta', {q_0}', F')$$

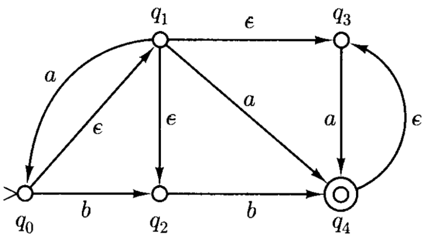
- Q' is the power set of Q.



- We shall build this by *lazy evaluation* on the subsets.

$$N = (Q, \Sigma, \delta, q_0, F) \qquad D = (Q', \Sigma, \delta', {q_0}', F')$$

- ${q_0}' = E(q_0)$.

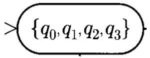


- ${q_0}' = E(q_0) = \{q_0, q_1, q_2, q_3\}$.

Example — *continued*

$$N = (Q, \Sigma, \delta, q_0, F) \qquad D = (Q', \Sigma, \delta', q_0', F')$$

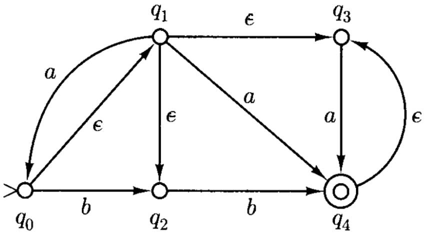
- ${q_0}' = E(q_0)$.



- ${q_0}' = E(q_0) = \{q_0, q_1, q_2, q_3\}$.

Example — *continued*

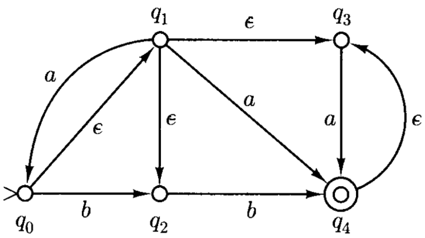
$N = (Q, \Sigma, \delta, q_0, F)$ $\qquad$ $D = (Q', \Sigma, \delta', {q_0}', F')$



- $\delta(q_0, a) \cup \delta(q_1, a) \cup \delta(q_2, a) \cup \delta(q_3, a) = \emptyset \cup \{q_0, q_4\} \cup \emptyset \cup \{q_4\} = \{q_0, q_4\}$

Example — *continued*

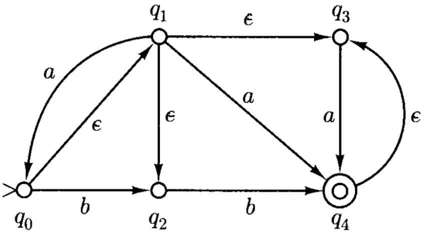
$N = (Q, \Sigma, \delta, q_0, F)$ $D = (Q', \Sigma, \delta', q_0{}', F')$



- $E(q_0) = \{q_0, q_1, q_2, q_3\}$, and $E(q_4) = \{q_3, q_4\}$.
- $\delta'(q_0{}', a) = \{q_0, q_1, q_2, q_3\} \cup \{q_3, q_4\} = \{q_0, q_1, q_2, q_3, q_4\}$.

Example — *continued*

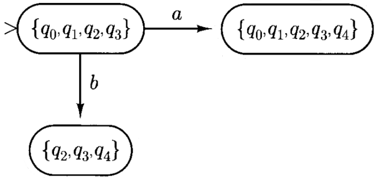
$N = (Q, \Sigma, \delta, q_0, F)$ $D = (Q', \Sigma, \delta', {q_0}', F')$



- Similarly, $\delta'({q_0}', b) = \{q_2, q_3, q_4\}$.

Example — *continued*

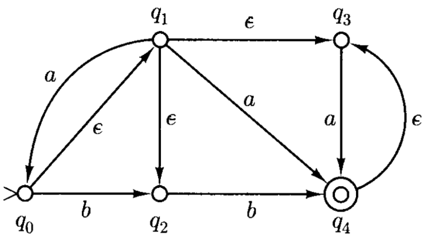
$$N = (Q, \Sigma, \delta, q_0, F) \qquad D = (Q', \Sigma, \delta', q_0{}', F')$$



- Similarly, $\delta'(q_0{}', b) = \{q_2, q_3, q_4\}$.

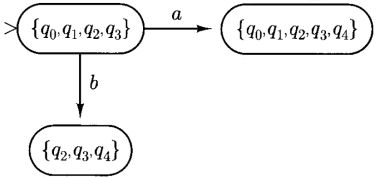
Example — *continued*

$$N = (Q, \Sigma, \delta, q_0, F) \qquad D = (Q', \Sigma, \delta', {q_0}', F')$$



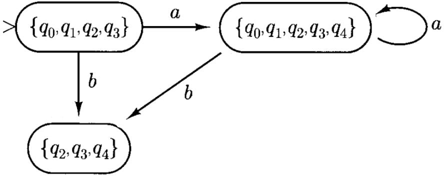
- We repeat the calculation for the newly introduced states.
- $\delta'(\{q_0, q_1, q_2, q_3, q_4\}, a) = \{q_0, q_1, q_2, q_3, q_4\}$, and
- $\delta'(\{q_0, q_1, q_2, q_3, q_4\}, b) = \{q_2, q_3, q_4\}$.

$$N = (Q, \Sigma, \delta, q_0, F) \qquad D = (Q', \Sigma, \delta', {q_0}', F')$$



- We repeat the calculation for the newly introduced states.
- $\delta'(\{q_0, q_1, q_2, q_3, q_4\}, a) = \{q_0, q_1, q_2, q_3, q_4\}$, and
- $\delta'(\{q_0, q_1, q_2, q_3, q_4\}, b) = \{q_2, q_3, q_4\}$.

$$N = (Q, \Sigma, \delta, q_0, F) \qquad D = (Q', \Sigma, \delta', {q_0}', F')$$



- We repeat the calculation for the newly introduced states.
- $\delta'(\{q_0, q_1, q_2, q_3, q_4\}, a) = \{q_0, q_1, q_2, q_3, q_4\}$, and
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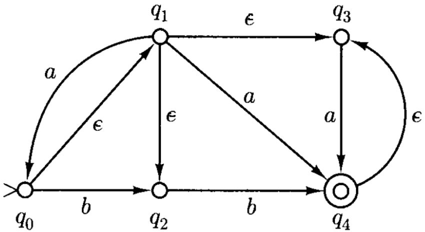
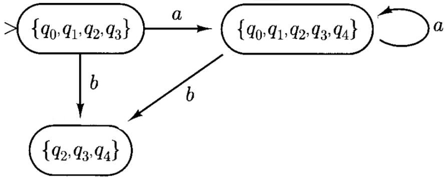
$N = (Q, \Sigma, \delta, q_0, F)$ $\qquad$ $D = (Q', \Sigma, \delta', q_0{}', F')$



- Also we get.
- $\delta'(\{q_2, q_3, q_4\}, a) = \{q_3, q_4\}$, and
- $\delta'(\{q_2, q_3, q_4\}, b) = \{q_3, q_4\}$.

$N = (Q, \Sigma, \delta, q_0, F)$ $D = (Q', \Sigma, \delta', q_0{}', F')$



- Also we get.
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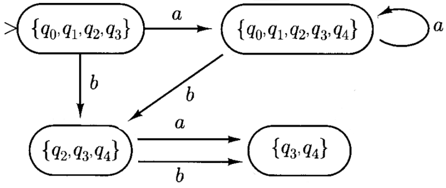
Example — *continued*

$$N = (Q, \Sigma, \delta, q_0, F) \qquad D = (Q', \Sigma, \delta', q_0{}', F')$$



- Also we get.
- $\delta'(\{q_2, q_3, q_4\}, a) = \{q_3, q_4\}$, and
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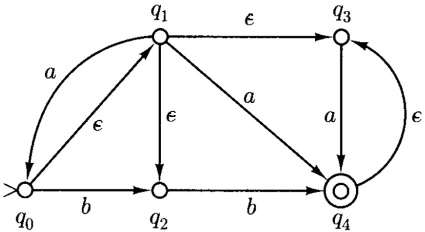
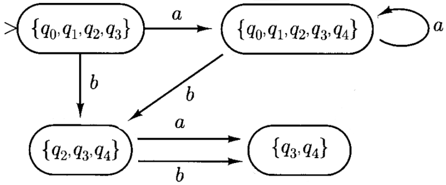
Example — *continued*

$$N = (Q, \Sigma, \delta, q_0, F) \qquad D = (Q', \Sigma, \delta', {q_0}', F')$$



- Next we get.
- $\delta'(\{q_3, q_4\}, a) = \{q_3, q_4\}$, and
- $\delta'(\{q_3, q_4\}, b) = \emptyset$.

$N = (Q, \Sigma, \delta, q_0, F)$ $D = (Q', \Sigma, \delta', q_0', F')$



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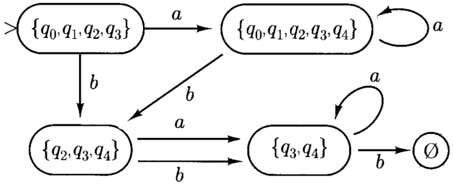

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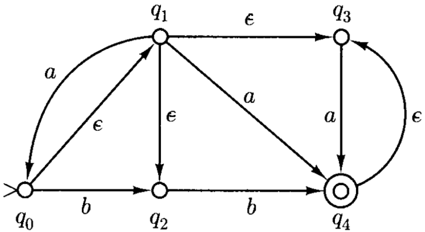
Example — *continued*

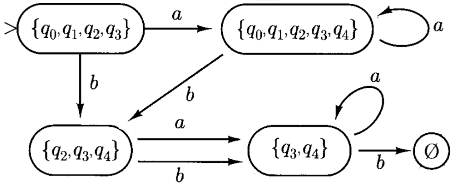
$$N = (Q, \Sigma, \delta, q_0, F) \qquad D = (Q', \Sigma, \delta', q_0', F')$$



- Finally, we get.
- $\delta'(\emptyset, a) = \delta'(\emptyset, b) = \emptyset$.

Example — *continued*

$$N = (Q, \Sigma, \delta, q_0, F) \qquad D = (Q', \Sigma, \delta', {q_0}', F')$$



- Finally, we get.
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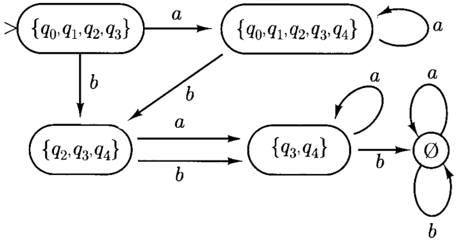
Example — *continued*

$$N = (Q, \Sigma, \delta, q_0, F) \qquad D = (Q', \Sigma, \delta', {q_0}', F')$$

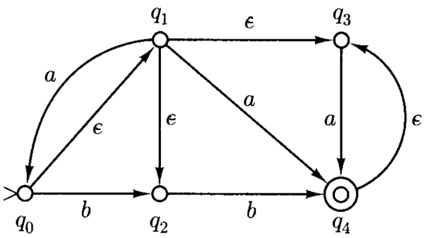


- Finally, we get.
- $\delta'(\emptyset, a) = \delta'(\emptyset, b) = \emptyset$.

Example — *continued*

$$N = (Q, \Sigma, \delta, q_0, F) \qquad D = (Q', \Sigma, \delta', {q_0}', F')$$

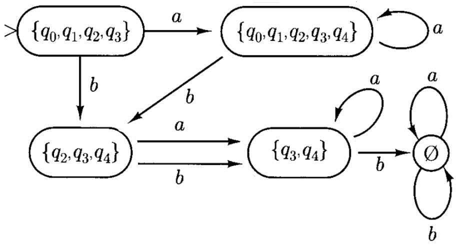
- F' is those sets of states that contain at least one accepting state of N.



- q_4 is the sole member of F.
- The set of final states, contains each set of states of which q_4 is a member.

$$N = (Q, \Sigma, \delta, q_0, F) \qquad D = (Q', \Sigma, \delta', {q_0}', F')$$

- F' is those sets of states that contain at least one accepting state of N.



- q_4 is the sole member of F.
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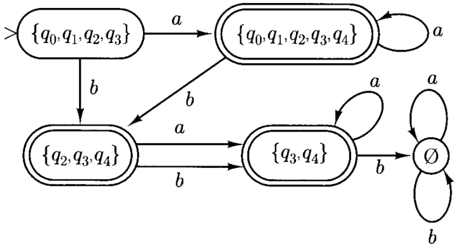
$$N = (Q, \Sigma, \delta, q_0, F) \qquad D = (Q', \Sigma, \delta', {q_0}', F')$$

- F' is those sets of states that contain at least one accepting state of N.



- The three states $\{q_0, q_1, q_2, q_3, q_4\}$, $\{q_2, q_3, q_4\}$, and $\{q_3, q_4\}$ are final.

COROLLARY 1.40

A language is regular if and only if some nondeterministic finite automaton recognizes it.

Closure under the Regular Operations

Sipser, 1.2, p-59

THEOREM 1.45

The class of regular languages is closed under the union operation.

Closure under the Regular Operations

Sipser, 1.2, p-59

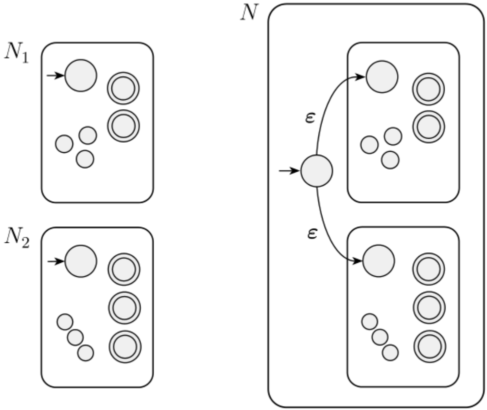
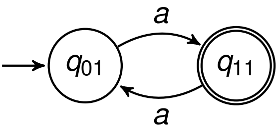


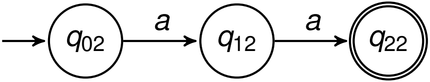
FIGURE 1.46
Construction of an NFA N to recognize $A_1 \cup A_2$

- $L_1 = \{\text{contains an odd number of } a\text{'s}\}$
  $L_2 = \{aa\}$

N_1



N_2



PROOF

- Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1.
- And $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.
- Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$.

1. $Q = \{q_0\} \cup Q_1 \cup Q_2$.

- The states of N are all the states of N_1 and N_2, with the addition of a new start state q_0.

2. The state q_0 is the start state of N.

3. The set of accept states $F = F_1 \cup F_2$.

- The accept states of N are all the accept states of N_1 and N_2.
- That way, N accepts if either N_1 accepts or N_2 accepts.

4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\epsilon$,

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & q \in Q_1 \\ \delta_2(q, a) & q \in Q_2 \\ \{q_1, q_2\} & q = q_0 \text{ and } a = \epsilon \\ \emptyset & q = q_0 \text{ and } a \neq \epsilon \end{cases}$$

Closure under the Regular Operations

Sipser, 1.2, p-60

THEOREM 1.47

The class of regular languages is closed under the concatenation operation.

Closure under the Regular Operations

Sipser, 1.2, p-60

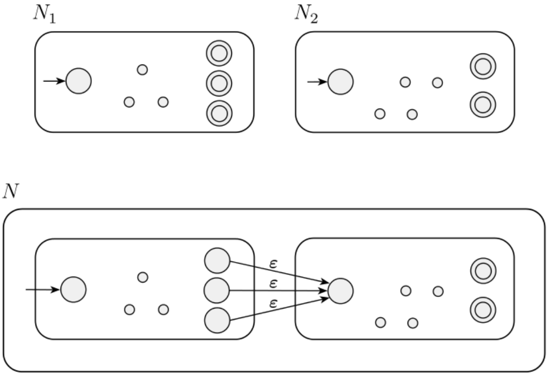
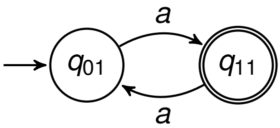


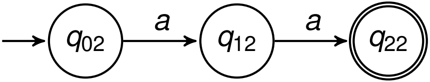
FIGURE 1.48
Construction of N to recognize $A_1 \circ A_2$

- $L_1 = \{\text{contains an odd number of } a\text{'s}\}$
  $L_2 = \{aa\}$

N_1



N_2



PROOF

- Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1.
- And $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.
- Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \circ A_2$.

1. $Q = Q_1 \cup Q_2$.

- The states of N are all the states of N_1 and N_2.

2. The state q_1 is the start state of N.

Equivalence of NFAs AND DFAs — *continued*

Sipser, 1.2, p-61

3. The set of accept states $F = F_2$.

- The accept states F are the same as the accept states of N_2.

Equivalence of NFAs AND DFAs — *continued*

Sipser, 1.2, p-61

4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\epsilon$,

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q, a) & q \in F_1 \text{ and } a \neq \epsilon \\ \delta_1(q, a) \cup \{q_2\} & q \in F_1 \text{ and } a = \epsilon \\ \delta_2(q, a) & q \in Q_2 \end{cases}$$

Closure under the Regular Operations

Sipser, 1.2, p-62

THEOREM 1.49

The class of regular languages is closed under the star operation.

Closure under the Regular Operations

Sipser, 1.2, p-62

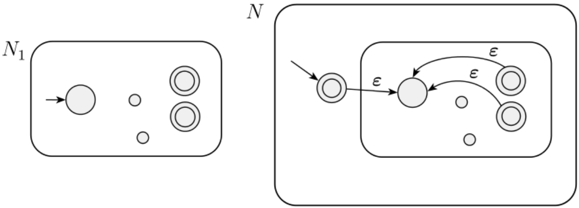
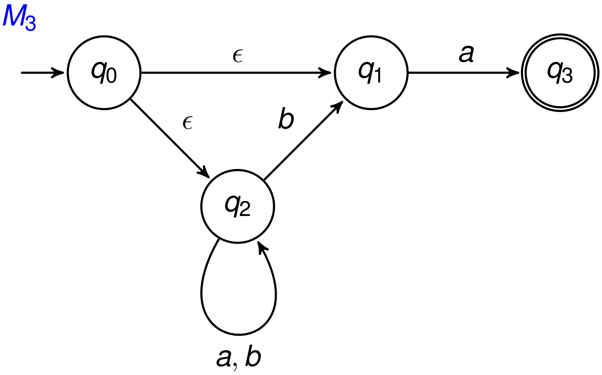


FIGURE 1.50
Construction of N to recognize A^*

- $\sum = \{a, b\}$, $L_3 = \{$ends in exactly one a at the end$\}$

- $\sum = \{a, b\}$, $L_3 = \{\text{ends in exactly one } a \text{ at the end}\}$



PROOF

- Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1.
- Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize A_1^*.

Equivalence of NFAs AND DFAs — *continued*

Sipser, 1.2, p-62

1. $Q = \{q_0\} \cup Q_1$.

- The states of N are the states of N_1 plus a new start state.

2. The state q_0 is the new start state.

3. $F = \{q_0\} \cup F_1$.

- The accept states are the old accept states plus the new start state.

4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\epsilon$,

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \text{ and } a \neq \epsilon \\ \delta_1(q,a) \cup \{q_1\} & q \in F_1 \text{ and } a = \epsilon \\ \{q_1\} & q \in q_0 \text{ and } a = \epsilon \\ \emptyset & q = q_0 \text{ and } a \neq \epsilon \end{cases}$$

Regular Expressions

Sipser, 1.3, p-63

- In arithmetic, we can use the operations $+$ and to build up expressions such as $(5 + 3) \times 4$.
- Similarly, we can use the regular operations to build up expressions describing languages.
- These are called regular expressions.
- An example is:

$$(0 \cup 1)0^*$$

Regular Expressions

Sipser, 1.3, p-64

DEFINITION 1.52

Say that R is a ***regular expression*** if R is

1. a for some a in the alphabet Σ,
2. ε,
3. $\emptyset$,
4. $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,
5. $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or
6. (R_1^*), where R_1 is a regular expression.

In items 1 and 2, the regular expressions a and ε represent the languages $\{a\}$ and $\{\varepsilon\}$, respectively. In item 3, the regular expression $\emptyset$ represents the empty language. In items 4, 5, and 6, the expressions represent the languages obtained by taking the union or concatenation of the languages R_1 and R_2, or the star of the language R_1, respectively.

Example

Sipser, Example 1.53, p-65

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

---

1. 0^*10^*

Example

Sipser, Example 1.53, p-65

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

1. 0^*10^*
   $\{w \mid w \text{ contains a single } 1\}$.

Example

Sipser, Example 1.53, p-65

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

---

1. 0^*10^*
   $\{w \mid w \text{ contains a single } 1\}$.
2. $\Sigma^*1\Sigma^*$

Example

Sipser, Example 1.53, p-65

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

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   $\{w \mid w \text{ contains a single } 1\}$.
2. $\Sigma^*1\Sigma^*$

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Example

Sipser, Example 1.53, p-65

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

1. 0^*10^*

   $\{w \mid w \text{ contains a single } 1\}$.
2. $\Sigma^*1\Sigma^*$

   $\{w \mid w \text{ has at least one } 1\}$.
3. $\Sigma^*001\Sigma^*$

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

1. 0^*10^*

   $\{w \mid w \text{ contains a single } 1\}$.
2. $\Sigma^*1\Sigma^*$

   $\{w \mid w \text{ has at least one } 1\}$.
3. $\Sigma^*001\Sigma^*$

   $\{w \mid w \text{ contains the string } 001 \text{ as a substring}\}$.

Example

Sipser, Example 1.53, p-65

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

---

1. 0^*10^*

   $\{w \mid w \text{ contains a single } 1\}$.
2. $\Sigma^*1\Sigma^*$

   $\{w \mid w \text{ has at least one } 1\}$.
3. $\Sigma^*001\Sigma^*$

   $\{w \mid w \text{ contains the string } 001 \text{ as a substring}\}$.
4. $1^*(01^+)^*$

Example

Sipser, Example 1.53, p-65

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

---

1. 0^*10^*

   $\{w \mid w$ contains a single 1$\}$.
2. $\Sigma^*1\Sigma^*$

   $\{w \mid w$ has at least one 1$\}$.
3. $\Sigma^*001\Sigma^*$

   $\{w \mid w$ contains the string 001 as a substring$\}$.
4. $1^*(01^+)^*$

- $\{w \mid$ every 0 in w is followed by at least one 1$\}$.

Example

Sipser, Example 1.53, p-65

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

---

1. 0^*10^*

   $\{w \mid w \text{ contains a single } 1\}$.
2. $\Sigma^*1\Sigma^*$

   $\{w \mid w \text{ has at least one } 1\}$.
3. $\Sigma^*001\Sigma^*$

   $\{w \mid w \text{ contains the string } 001 \text{ as a substring}\}$.
4. $1^*(01^+)^*$

- $\{w \mid \text{every } 0 \text{ in } w \text{ is followed by at least one } 1\}$.

5. $(\Sigma\Sigma)^*$

Example

Sipser, Example 1.53, p-65

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

---

1. 0^*10^*

   $\{w \mid w$ contains a single $1\}$.
2. $\Sigma^*1\Sigma^*$

   $\{w \mid w$ has at least one $1\}$.
3. $\Sigma^*001\Sigma^*$

   $\{w \mid w$ contains the string 001 as a substring$\}$.
4. $1^*(01^+)^*$

   - $\{w \mid$ every 0 in w is followed by at least one $1\}$.
5. $(\Sigma\Sigma)^*$

   $\{w \mid w$ is a string of even length$\}$.

Example — *continued*

Sipser, Example 1.53, p-65

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

---

6. $(\Sigma\Sigma\Sigma)^*$

Example — *continued*

Sipser, Example 1.53, p-65

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

6. $(\Sigma\Sigma\Sigma)^*$
   $\{w \mid \text{the length of } w \text{ is a multiple of } 3\}$.

Example — *continued*

Sipser, Example 1.53, p-65

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

---

6. $(\Sigma\Sigma\Sigma)^*$
   $\{w \mid \text{the length of } w \text{ is a multiple of } 3\}$.
7. $01 \cup 10$

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

---

6. $(\Sigma\Sigma\Sigma)^*$

   $\{w \mid \text{the length of } w \text{ is a multiple of } 3\}$.
7. $01 \cup 10$

   $\{01, 10\}$.

Example — *continued*

Sipser, Example 1.53, p-65

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

---

6. $(\Sigma\Sigma\Sigma)^*$

   $\{w \mid \text{the length of } w \text{ is a multiple of } 3\}$.
7. $01 \cup 10$

   $\{01, 10\}$.
8. $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1$

Example — *continued*

Sipser, Example 1.53, p-65

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

6. $(\Sigma\Sigma\Sigma)^*$

   $\{w \mid \text{the length of } w \text{ is a multiple of } 3\}$.
7. $01 \cup 10$

   $\{01, 10\}$.
8. $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1$

   $\{w \mid w \text{ starts and ends with the same symbol}\}$.

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

---

6. $(\Sigma\Sigma\Sigma)^*$

   $\{w \mid \text{the length of } w \text{ is a multiple of 3}\}$.
7. $01 \cup 10$

   $\{01, 10\}$.
8. $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1$

   $\{w \mid w \text{ starts and ends with the same symbol}\}$.
9. $(0 \cup \epsilon)1^* = 01^* \cup 1^*$

Example — *continued*

Sipser, Example 1.53, p-65

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

---

6. $(\Sigma\Sigma\Sigma)^*$

   $\{w \mid \text{the length of } w \text{ is a multiple of } 3\}$.

7. $01 \cup 10$

   $\{01, 10\}$.

8. $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1$

   $\{w \mid w \text{ starts and ends with the same symbol}\}$.

9. $(0 \cup \epsilon)1^* = 01^* \cup 1^*$

   The expression $0 \cup \epsilon$ describes the language $\{0, \epsilon\}$, so the concatenation operation adds either 0 or ϵ before every string in 1^*.

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

---

10. $(0 \cup \epsilon)(1 \cup \epsilon)$

Example — *continued*

Sipser, Example 1.53, p-65

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

10. $(0 \cup \epsilon)(1 \cup \epsilon)$
    $\{\epsilon, 0, 1, 01\}$.

Example — *continued*

Sipser, Example 1.53, p-65

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

10. $(0 \cup \epsilon)(1 \cup \epsilon)$

    $\{\epsilon, 0, 1, 01\}$.

11. $1^*\emptyset = \emptyset$

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    $\{\epsilon, 0, 1, 01\}$.

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    Concatenating the empty set to any set yields the empty set.

Example — *continued*

Sipser, Example 1.53, p-65

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

10. $(0 \cup \epsilon)(1 \cup \epsilon)$

    $\{\epsilon, 0, 1, 01\}$.
11. $1^*\emptyset = \emptyset$

    Concatenating the empty set to any set yields the empty set.
12. $\emptyset^* = \{\epsilon\}$

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11. $1^*\emptyset = \emptyset$

    Concatenating the empty set to any set yields the empty set.

12. $\emptyset^* = \{\epsilon\}$

    The star operation puts together any number of strings from the language to get a string in the result.

- In the following instances, we assume that the alphabet Σ is $\{0, 1\}$.

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10. $(0 \cup \epsilon)(1 \cup \epsilon)$

    $\{\epsilon, 0, 1, 01\}$.
11. $1^*\emptyset = \emptyset$

    Concatenating the empty set to any set yields the empty set.
12. $\emptyset^* = \{\epsilon\}$

    The star operation puts together any number of strings from the language to get a string in the result.

    If the language is empty, the star operation can put together 0 strings, giving only the empty string.

- If we let R be any regular expression, we have the following identities.

Regular Expressions — *continued*

Sipser, 1.3, p-63

- $R \cup \emptyset = R$.
- Adding the empty language to any other language will not change it.

Regular Expressions — *continued*

Sipser, 1.3, p-63

- $R \circ \epsilon = R$.
- Joining the empty string to any string will not change it.

Regular Expressions — *continued*

Sipser, 1.3, p-63

- However, exchanging $\emptyset$ and ϵ in the preceding identities may cause the equalities to fail.

- $R \cup \epsilon$ may not equal R.
- For example, if $R = 0$, then $L(R) = \{0\}$.
- But $L(R \cup \epsilon) = \{0, \epsilon\}$.

- $R \circ \emptyset$ may not equal R.
- For example, if $R = 0$, then $L(R) = \{0\}$.
- But $L(R \circ \emptyset) = \emptyset$.

$B = \{0^n 1^n \mid n \geq 0\}$

$C = \{w \mid w \text{ has an equal number of 0's and 1's}\}$

Nonregular Languages

Sipser, 1.4, p-77

$B = \{0^n 1^n \mid n \geq 0\}$

$C = \{w \mid w \text{ has an equal number of 0's and 1's}\}$

$D = \left\{ w \,\middle|\, \begin{array}{l} w \text{ has an equal number of occurrences of} \\ \text{01 and 10 as substrings} \end{array} \right\}$

Nonregular Languages

http://www.eecs.berkeley.edu/~sseshia/172/lectures/Slides3.pdf, Slide 31

$$D = \left\{ w \;\middle|\; \begin{array}{l} w \text{ has an equal number of occurrences of} \\ \text{01 and 10 as substrings} \end{array} \right\}$$

Nonregular Languages

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$$D = \left\{ w \;\middle|\; \begin{array}{l} w \text{ has an equal number of occurrences of} \\ 01 \text{ and } 10 \text{ as substrings} \end{array} \right\}$$

- 0110

Nonregular Languages

http://www.eecs.berkeley.edu/~sseshia/172/lectures/Slides3.pdf, Slide 31

$$D = \left\{ w \;\middle|\; \begin{array}{l} w \text{ has an equal number of occurrences of} \\ \text{01 and 10 as substrings} \end{array} \right\}$$

- 0110 *belongs*

Nonregular Languages

http://www.eecs.berkeley.edu/~sseshia/172/lectures/Slides3.pdf, Slide 31

$$D = \left\{ w \;\middle|\; \begin{array}{l} w \text{ has an equal number of occurrences of} \\ 01 \text{ and } 10 \text{ as substrings} \end{array} \right\}$$

- 0110 *belongs*
- 01100

Nonregular Languages

http://www.eecs.berkeley.edu/~sseshia/172/lectures/Slides3.pdf, Slide 31

$$D = \left\{ w \;\middle|\; \begin{array}{l} w \text{ has an equal number of occurrences of} \\ 01 \text{ and } 10 \text{ as substrings} \end{array} \right\}$$

- 0110 *belongs*
- 01100 *belongs*

Nonregular Languages

`http://www.eecs.berkeley.edu/~sseshia/172/lectures/Slides3.pdf`, Slide 31

$$D = \left\{ w \;\middle|\; \begin{array}{l} w \text{ has an equal number of occurrences of} \\ \text{01 and 10 as substrings} \end{array} \right\}$$

- 0110 *belongs*
- 01100 *belongs*
- 1101110011

Nonregular Languages

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- 0110 *belongs*
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- 0110 *belongs*
- 01100 *belongs*
- 1101110011 *belongs*
- ϵ

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- 0110 — *belongs*
- 01100 — *belongs*
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$$D = \left\{ w \;\middle|\; \begin{array}{l} w \text{ has an equal number of occurrences of} \\ \text{01 and 10 as substrings} \end{array} \right\}$$

- 0110 *belongs*
- 01100 *belongs*
- 1101110011 *belongs*
- ϵ *belongs*
- 10

Nonregular Languages

http://www.eecs.berkeley.edu/~sseshia/172/lectures/Slides3.pdf, Slide 31

$$D = \left\{ w \,\middle|\, \begin{array}{l} w \text{ has an equal number of occurrences of} \\ 01 \text{ and } 10 \text{ as substrings} \end{array} \right\}$$

- 0110 — *belongs*
- 01100 — *belongs*
- 1101110011 — *belongs*
- ϵ — *belongs*
- 10 — *does not belong*

Nonregular Languages

http://www.eecs.berkeley.edu/~sseshia/172/lectures/Slides3.pdf, Slide 31

$$D = \left\{ w \;\middle|\; \begin{array}{l} w \text{ has an equal number of occurrences of} \\ \text{01 and 10 as substrings} \end{array} \right\}$$

- 0110 *belongs*
- 01100 *belongs*
- 1101110011 *belongs*
- ϵ *belongs*
- 10 *does not belong*
- 110

Nonregular Languages

http://www.eecs.berkeley.edu/~sseshia/172/lectures/Slides3.pdf, Slide 31

$$D = \left\{ w \;\middle|\; \begin{array}{l} w \text{ has an equal number of occurrences of} \\ \text{01 and 10 as substrings} \end{array} \right\}$$

- 0110 *belongs*
- 01100 *belongs*
- 1101110011 *belongs*
- ϵ *belongs*
- 10 *does not belong*
- 110 *does not belong*

Nonregular Languages

http://www.eecs.berkeley.edu/~sseshia/172/lectures/Slides3.pdf, Slide 31

$$D = \left\{ w \;\middle|\; \begin{array}{l} w \text{ has an equal number of occurrences of} \\ 01 \text{ and } 10 \text{ as substrings} \end{array} \right\}$$

- 0110 — *belongs*
- 01100 — *belongs*
- 1101110011 — *belongs*
- ϵ — *belongs*
- 10 — *does not belong*
- 110 — *does not belong*
- 1101

Nonregular Languages

http://www.eecs.berkeley.edu/~sseshia/172/lectures/Slides3.pdf, Slide 31

$$D = \left\{ w \;\middle|\; \begin{array}{l} w \text{ has an equal number of occurrences of} \\ 01 \text{ and } 10 \text{ as substrings} \end{array} \right\}$$

- 0110 — *belongs*
- 01100 — *belongs*
- 1101110011 — *belongs*
- ϵ — *belongs*
- 10 — *does not belong*
- 110 — *does not belong*
- 1101 — *belongs*

Nonregular Languages

http://www.eecs.berkeley.edu/~sseshia/172/lectures/Slides3.pdf, Slide 31

$$D = \left\{ w \,\middle|\, \begin{array}{l} w \text{ has an equal number of occurrences of} \\ \text{01 and 10 as substrings} \end{array} \right\}$$

- 0110 *belongs*
- 01100 *belongs*
- 1101110011 *belongs*
- ϵ *belongs*
- 10 *does not belong*
- 110 *does not belong*
- 1101 *belongs*

- *w* should *toggle* between 0 and 1 an equal number of times.

Nonregular Languages

http://www.eecs.berkeley.edu/~sseshia/172/lectures/Slides3.pdf, Slide 31

$$D = \left\{ w \,\middle|\, \begin{array}{l} w \text{ has an equal number of occurrences of} \\ 01 \text{ and } 10 \text{ as substrings} \end{array} \right\}$$

$$= \left\{ w \,\middle|\, \begin{array}{l} w = 1,\ w = 0,\ w = \epsilon \text{ or } w \text{ starts with a } 0 \\ \text{and ends with a } 0 \text{ or } w \text{ starts with a } 1 \text{ and} \\ \text{ends with a } 1 \end{array} \right\}$$

Nonregular Languages

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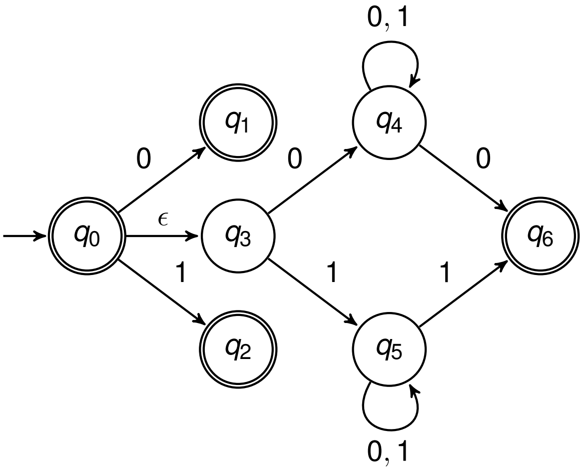
- $\epsilon \cup 0 \cup 1 \cup (0\Sigma^*0) \cup (1\Sigma^*1)$

Nonregular Languages

$\epsilon \cup 0 \cup 1 \cup (0\Sigma^*0) \cup (1\Sigma^*1)$

Nonregular Languages

$\epsilon \cup 0 \cup 1 \cup (0\Sigma^*0) \cup (1\Sigma^*1)$



The Pumping Lemma for Regular Languages

Sipser, 1.4, p-77

THEOREM 1.70

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

- $|s|$ represents the length of string s.
- y^i means that i copies of y are concatenated together.
- y^0 equals ϵ.

The Pumping Lemma for Regular Languages

Sipser, 1.4, p-77

THEOREM 1.70

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

- When s is divided into xyz, either x or z may be ϵ.
- But condition 2 says that $y \neq \epsilon$.
- Without condition 2 the theorem would be trivially true.

The Pumping Lemma for Regular Languages

Sipser, 1.4, p-77

THEOREM 1.70

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

- Condition 3 states that the pieces x and y together have length at most p.
- It is an extra technical condition that we occasionally find useful when proving certain languages to be nonregular.

The Pumping Lemma for Regular Languages — *continued*

Sipser, 1.4, p-77

PROOF IDEA

- Let $M = (Q, \Sigma, \delta, q_1, F)$ be a DFA that recognizes A.
- We assign the pumping length p to be the number of states of M.
- We show that any string s in A of length at least p may be broken into the three pieces xyz, satisfying our three conditions.
- What if no strings in A are of length at least p?
- Then our task is even easier because the theorem becomes vacuously true.
- Obviously the three conditions hold for all strings of length at least p if there aren't any such strings.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

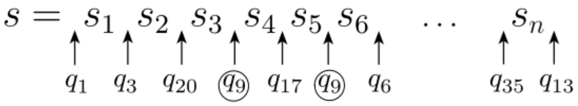


FIGURE 1.71

Example showing state q_9 repeating when M reads s

- If s in A has length at least p, consider the sequence of states that M goes through when computing with input s.
- It starts with q_1 the start state, then goes to, say, q_3, then, say, q_{20}, then q_9, and so on, until it reaches the end of s in state q_{13}.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

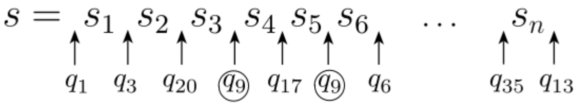


FIGURE 1.71
Example showing state q_9 repeating when M reads s

- With s in A, we know that M accepts s, so q_{13} is an accept state.
- If we let n be the length of s, the sequence of states $q_1, q_3, q_{20}, q_9, \ldots, q_{13}$ has length $n + 1$.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

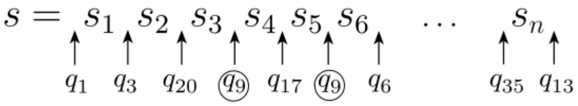


FIGURE 1.71
Example showing state q_9 repeating when M reads s

- Because *n* is at least *p*, we know that *n*+1 is greater than *p*, the number of states of *M*.
- Therefore, the sequence must contain a repeated state.
- This result is an example of the pigeonhole principle.

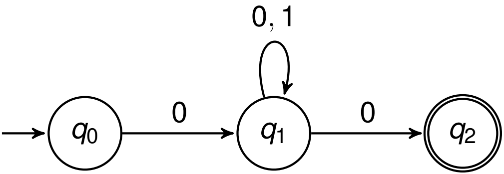
$$
\begin{array}{cccccccccccccccccc}
s = & & s_1 & & s_2 & & s_3 & & s_4 & & s_5 & & s_6 & & \ldots & & s_n & \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & q_1 & & q_3 & & q_{20} & & q_9 & & q_{17} & & q_9 & & q_6 & & q_{35} & & q_{13}
\end{array}
$$

FIGURE 1.71
Example showing state q_9 repeating when M reads s

- State q_9 is the one that repeats.

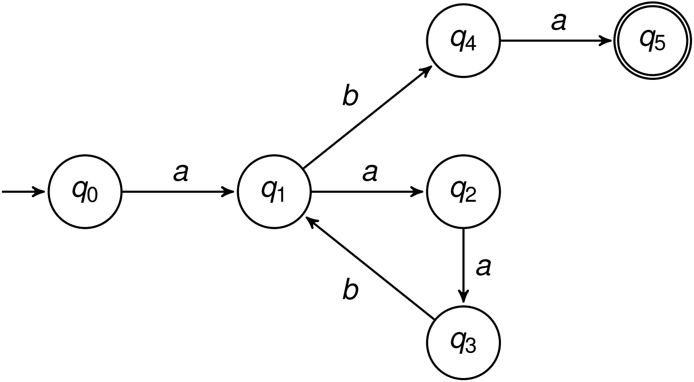
$L = \{w \mid w \text{ starts and ends with } 0, |w| \geq 2\}$

$L = 0\Sigma^*0$



The Pumping Lemma for Regular Languages

$L = a(aab)^*ba$



The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

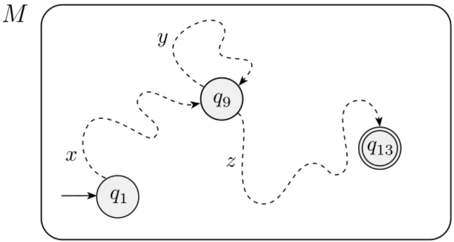


FIGURE 1.72
Example showing how the strings x, y, and z affect M

- Piece x is the part of s appearing before q_9.
- Piece y is the part between the two appearances of q_9.
- Piece z is the remaining part of s, coming after the second occurrence of q_9.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

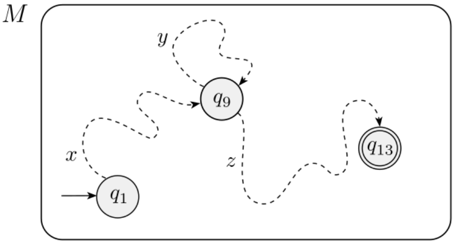


FIGURE 1.72
Example showing how the strings x, y, and z affect M

- x takes M from the state q_1 to q_9.
- y takes M from q_9 back to q_9.
- z takes M from q_9 to the accept state q_{13}.

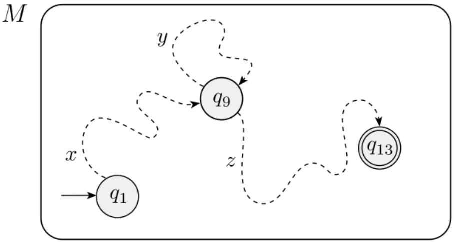


FIGURE 1.72
Example showing how the strings x, y, and z affect M

- Suppose that we run M on input $xyyz$.
- We know that x takes M from q_1 to q_9.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

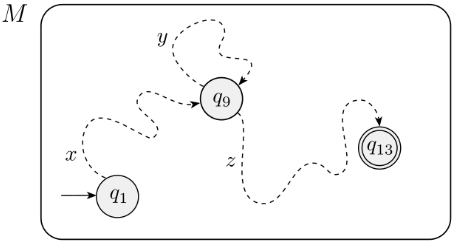


FIGURE 1.72
Example showing how the strings x, y, and z affect M

- Then the first y takes it from q_9 back to q_9 , as does the second y.
- Then z takes it to q_{13}.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

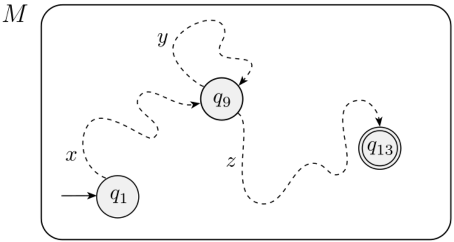


FIGURE 1.72
Example showing how the strings x, y, and z affect M

- With q_{13} being an accept state, M accepts input $xyyz$.
- Similarly, it will accept xy^iz for any $i > 0$.

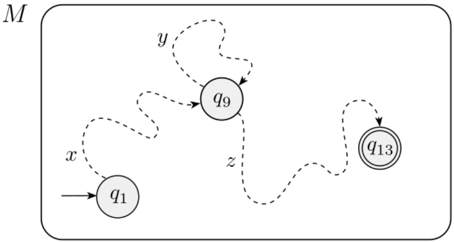


FIGURE 1.72
Example showing how the strings x, y, and z affect M

- For the case $i = 0$, $xy^iz = xz$, which is accepted for similar reasons.
- That establishes condition 1.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

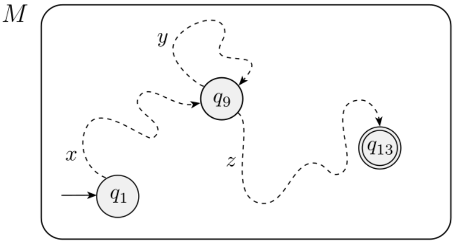


FIGURE 1.72
Example showing how the strings x, y, and z affect M

- Checking condition 2, we see that $|y| > 0$, as it was the part of s that occurred between two different occurrences of state q_9.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

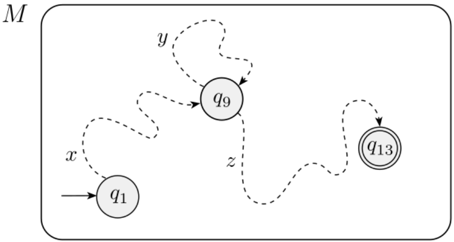


FIGURE 1.72
Example showing how the strings x, y, and z affect M

- In order to get condition 3, we make sure that q_9 is the first repetition in the sequence.
- By the pigeonhole principle, the first $p + 1$ states in the sequence must contain a repetition.
- Therefore, $|xy| \leq p$.

PROOF

- Let $M = (Q, \Sigma, \delta, q_1, F)$ be a DFA recognizing A and p be the number of states of M.
- Let $s = s_1 s_2 \ldots s_n$ be a string in A of length n, where $n \geq p$.
- Let $r_1, r_2, \ldots, r_{n+1}$ be the sequence of states that M enters while processing s.
- So $r_{i+1} = \delta(r_i, s_i)$ for $1 \leq i \leq n$.
- This sequence has length $n + 1$, which is at least $p + 1$.

- Among the first $p + 1$ elements in the sequence, two must be the same state.
- By the pigeonhole principle, we call the first of these r_j and the second r_ℓ.
- Because r_ℓ occurs among the first $p + 1$ places in a sequence starting at r_1, we have $\ell \leq p + 1$.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

- Let $x = s_1 \ldots s_{j-1}$.
- $y = s_j \ldots s_{\ell-1}$.
- $z = s_\ell \ldots s_n$.

- Let $x = s_1 \dots s_{j-1}$.
- $y = s_j \dots s_{\ell-1}$.
- $z = s_\ell \dots s_n$.
- x takes M from r_1 to r_j.
- y takes M from r_j to r_j.
- z takes M from r_j to r_{n+1}, which is an accept state, M must accept xy^iz for $i \geq 0$.

The Pumping Lemma... — *continued*

Sipser, 1.4, p-77

- We know that $j \neq \ell$, so $|y| > 0$.
- $\ell \leq p + 1$, so $|xy| \leq p$.
- Thus we have satisfied all conditions of the pumping lemma.

- To use the pumping lemma to prove that a language B is not regular, first assume that B is regular in order to obtain a contradiction.
- Then use the pumping lemma to guarantee the existence of a pumping length p such that all strings of length p or greater in B can be pumped.

- Next, find a string s in B that has length p or greater but that cannot be pumped.

- Finally, demonstrate that s cannot be pumped by considering all ways of dividing s into x, y, and z (taking condition 3 of the pumping lemma into account if convenient).
- For each such division, find a value i where $xy^iz \notin B$.

- This final step often involves grouping the various ways of dividing s into several cases and analyzing them individually.
- The existence of s contradicts the pumping lemma if B were regular.
- Hence B cannot be regular.

- Finding s sometimes takes a bit of creative thinking.
- You may need to hunt through several candidates for s before you discover one that works.
- Try members of B that seem to exhibit the “essence” of B’s nonregularity.

- If you are still uncomfortable on this topic, you may want to watch this presentation:

  Pumping Lemma For Regular by Didem Yalcin

Example

Sipser, Example 1.73, p-80

- Let B be the language $\{0^n1^n \mid n \geq 0\}$.
- We use the pumping lemma to prove that B is not regular.
- The proof is by contradiction.

- Assume to the contrary that B is regular.
- Let p be the pumping length given by the pumping lemma.

Example — *continued*

Sipser, Example 1.73, p-80

- Choose s to be the string $0^p 1^p$.
- Because s is a member of B and s has length more than p, the pumping lemma guarantees that s can be split into three pieces, $s = xyz$.
- Where for any $i \geq 0$ the string $xy^i z$ is in B.

Example — *continued*

Sipser, Example 1.73, p-80

- We consider three cases to show that this result is impossible.

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1. The string y consists only of 0s.

- In this case, the string $xyyz$ has more 0s than 1s and so is not a member of B, violating condition 1 of the pumping lemma.
- This case is a contradiction.

- We consider three cases to show that this result is impossible.

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2. The string y consists only of 1s.

- This case also gives a contradiction.

- We consider three cases to show that this result is impossible.

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3. The string y consists of both 0s and 1s.

- In this case, the string $xyyz$ may have the same number of 0s and 1s, but they will be out of order with some 1s before 0s.
- Hence it is not a member of B, which is a contradiction.

Example — *continued*

Sipser, Example 1.73, p-80

- Thus a contradiction is unavoidable if we make the assumption that B is regular.
- So B is not regular.

- Note that we can simplify this argument by applying condition 3 of the pumping lemma to eliminate cases 2 and 3.

Example

Sipser, Example 1.74, p-80

- $C = \{w \mid w \text{ has an equal number of 0s and 1s}\}$.
- We use the pumping lemma to prove that C is not regular.
- The proof is by contradiction.

Example — *continued*

Sipser, Example 1.74, p-80

- Assume to the contrary that C is regular.
- Let p be the pumping length given by the pumping lemma.
- Let s be the string 0^p1^p.
- With s being a member of C and having length more than p, the pumping lemma guarantees that s can be split into three pieces.
- $s = xyz$, where for any $i \geq 0$ the string xy^iz is in C.

Example — *continued*

Sipser, Example 1.74, p-80

- We would like to show that this outcome is impossible.

- But wait, it is possible!
- If we let x and z be the empty string and y be the string 0^p1^p, then xy^iz always has an equal number of 0s and 1s and hence is in C.
- So it seems that s can be pumped.

Example — *continued*

Sipser, Example 1.74, p-80

- Here condition 3 in the pumping lemma is useful.
- It stipulates that when pumping s, it must be divided so that $|xy| \leq p$.
- That restriction on the way that s may be divided makes it easier to show that the string $s = 0^p 1^p$ we selected cannot be pumped.
- If $|xy| \leq p$, then y must consist only of 0s, so $xyyz \notin C$.
- Therefore, s cannot be pumped.
- That gives us the desired contradiction.

Example

Sipser, Example 1.75, p-81

- $F = \{ww \mid w \in \{0,1\}^*\}$.
- We use the pumping lemma to prove that F is not regular.

- Assume to the contrary that F is regular.
- Let p be the pumping length given by the pumping lemma.
- Let s be the string 0^p10^p1.
- Because s is a member of F and s has length more than p, the pumping lemma guarantees that s can be split into three pieces, $s = xyz$, satisfying the three conditions of the lemma.

Example — *continued*

Sipser, Example 1.75, p-81

- We show that this outcome is impossible.
- Condition 3 is once again crucial because without it we could pump s if we let x and z be the empty string.
- With condition 3 the proof follows because y must consist only of 0s, so $xyyz \notin F$.

Example — *continued*

Sipser, Example 1.75, p-81

- Observe that we chose $s = 0^p10^p1$ to be a string that exhibits the "essence" of the nonregularity of F, as opposed to, say, the string 0^p0^p.
- Even though 0^p0^p is a member of F, it fails to demonstrate a contradiction because it can be pumped.

Example

Sipser, Example 1.76, p-82

- We demonstrate a nonregular unary language.
- $D = \left\{ 1^{n^2} \mid n \geq 0 \right\}$.
- We use the pumping lemma to prove that D is not regular.
- The proof is by contradiction.

Example — *continued*

Sipser, Example 1.76, p-82

- Assume to the contrary that D is regular.
- Let p be the pumping length given by the pumping lemma.

Example — *continued*

Sipser, Example 1.76, p-82

- Let s be the string 1^{p^2}.
- Because s is a member of D and s has length at least p, the pumping lemma guarantees that s can be split into three pieces, $s = xyz$.
- where for any $i \geq 0$ the string xy^iz is in D.

Example — *continued*

Sipser, Example 1.76, p-82

- We show that this outcome is impossible.
- The sequence of perfect squares:

$$0, 1, 4, 9, 16, 25, 36, 49, \ldots$$

- Note the growing gap between successive members of this sequence.
- Large members of this sequence cannot be near each other.

- Now consider the two strings xyz and xy^2z.
- These strings differ from each other by a single repetition of y.
- Consequently their lengths differ by the length of y.
- By condition 3 of the pumping lemma, $|xy| \leq p$ and thus $|y| \leq p$.

Example — *continued*

Sipser, Example 1.76, p-82

- We have $|xyz| = p^2$ and so $|xy^2z| \leq p^2 + p$.
- But $p^2 + p < p^2 + 2p + 1 = (p + 1)2$.
- Moreover, condition 2 implies that y is not the empty string and so $|xy^2z| > p^2$.
- Therefore, the length of xy^2z lies strictly between the consecutive perfect squares p^2 and $(p + 1)2$.
- Hence this length cannot be a perfect square itself.
- So we arrive at the contradiction $xy^2z \notin D$ and conclude that D is not regular.

Example

Sipser, Example 1.77, p-82

- Let E be the language $\{0^i 1^j \mid i > j\}$.
- We use the pumping lemma to prove that E is not regular.
- The proof is by contradiction.

Example — *continued*

Sipser, Example 1.77, p-82

- Assume that E is regular.
- Let p be the pumping length for E given by the pumping lemma.

Example — *continued*

Sipser, Example 1.77, p-82

- Let $s = 0^{p+1}1^p$.
- Then s can be split into xyz, satisfying the conditions of the pumping lemma.
- By condition 3, y consists only of 0s.

Example — *continued*

Sipser, Example 1.77, p-82

- Let's examine the string $xyyz$ to see whether it can be in E.
- Adding an extra copy of y increases the number of 0s.
- But, E contains all strings in 01 that have more 0s than 1s.
- So increasing the number of 0s will still give a string in E.
- No contradiction occurs.

Example — *continued*

Sipser, Example 1.77, p-82

- We need to try something else.
- The pumping lemma states that $xy^iz \in E$ even when $i = 0$.

Example — *continued*

Sipser, Example 1.77, p-82

- So let's consider the string $xy^0z = xz$.
- Removing string y decreases the number of 0s in s.
- Recall that s has just one more 0 than 1.
- Therefore, xz cannot have more 0s than 1s, so it cannot be a member of E.
- Thus we obtain a contradiction.

Example

Hopcroft, Motwani, and Ullman, Example 4.3, p-129

- Let us show that the language L_{pr} consisting of all strings of 1's whose length is a prime is not a regular language.
- Suppose it were.
- Then there would be a constant p satisfying the conditions of the pumping Lemma.

Example — *continued*

Hopcroft, Motwani, and Ullman, Example 4.3, p-129

- Consider some prime $n \geq p + 2$.
- There must be such an n, since there are an infinity of primes.

Example — *continued*

Hopcroft, Motwani, and Ullman, Example 4.3, p-129

- Let $w = 1^r$.
- By the pumping lemma, we can break $w = xyz$ such that $y \neq \epsilon$ and $|xy| \leq p$.
- Let $|y| = m$.
- Then $|xz| = n - m$.

Example — *continued*

Hopcroft, Motwani, and Ullman, Example 4.3, p-129

- Now consider the string $xy^{n-m}z$.
- This must be in L_{pr} by the pumping lemma, if L_{pr} really is regular.
- However,

$$\begin{aligned} |xy^{n-m}z| &= |xz| + (n-m)|y| \\ &= n - m + (n-m)m \\ &= (m+1)(n-m) \end{aligned}$$

- It looks like $|xy^{n-m}z|$ is not a prime, since it has two factors $(m+1)$ and $(n-m)$.

Example — *continued*

Hopcroft, Motwani, and Ullman, Example 4.3, p-129

- However, we must check that neither of these factors are 1.
- Since then $(m+1)(n-m)$ might be a prime after all.
- But $m+1>1$, since $y \neq \epsilon$ tells us $m \geq 1$.
- Also, $n-m \geq 1$, since $n \geq p+2$ was chosen, and $m \leq p$ since

$$m = |y| \leq |xy| \leq p$$

- Thus, $n-m \geq 2$.

- Again we have started by assuming the language in question was regular.
- We derived a contradiction by showing that some string not in the language was required by the pumping lemma to be in the language.
- Thus, we conclude that L_{pr} is not a regular language.

End of Slides