state and prove cauchy Theorem.

Evaluate $\oint_C \frac{dz}{z-a}$ where C is

any simple closed curve C, and

$z=a$ is (a) outside C (b) inside C

ution:

(a) If a is outside C, then $f(z)=\frac{1}{(z-}$

is analytic everywhere inside and

C. Hence, by cauchy's theorem

$$\oint_C \frac{dz}{z-a} = 0$$

suppose a is inside C and let

Γ be a circle of

radius ϵ with

centre at $z=a$ so that Γ is

$|z-a| = \varepsilon$, or $z - a = \varepsilon e^{i\theta}$

i.e $z = a + \varepsilon e^{i\theta}$, $0 \le \theta \le 2\pi$

Thus since $dz = i\varepsilon e^{i\theta} d\theta$.

the right side of (1) becomes,

$$\int_{\theta=0}^{2\pi} \frac{i\varepsilon e^{i\theta} d\theta}{\varepsilon e^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i.$$

(Ans)

H.W verify cauchy's Theorem for the function (a) $3z^2 + iz - 4$ (b) $3\cos(z+2)$ if c is the square with vertices at $1 \pm i$, $-1 \pm i$.

If C is the circle $|z-2| = 5$ (a)

Determine whether $\oint_C \frac{dz}{z-3} = 0$

Does your answer to contradict Cauchy's theorem.

$$\boxed{\text{Accounting}}$$

Account

Debit → left   credit → Right

	Inc	Dec
Asset →	(+) Dr. ✓	(−) Cr.
Liability →	(−) Dr.	(+) Cr. ✓
Capital →	(−) Dr.	(+) Cr ✓
Revenue →	(−) Dr.	(+) Cr. ✓
Expense →	(+) Dr. ✓	(−) Cr.
Drawing →	(+) Dr. ✓	(−) Cr.

20.09.18

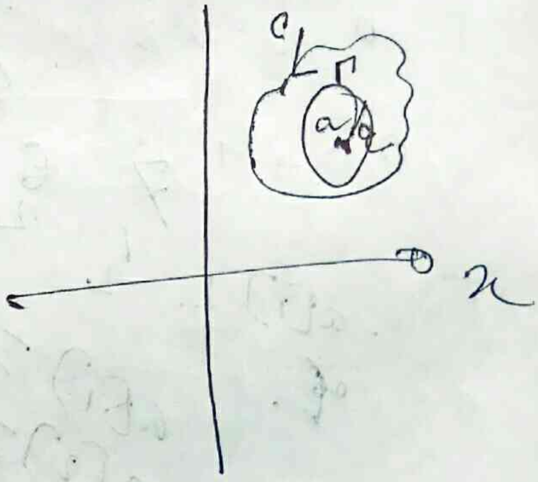
MATH

CH-09

Cauchy Integral formula

statement: If $f(z)$ is analytic inside and on the boundary c of a simply connected region R, then the cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-a} dz$$



proof: The function $\frac{f(z)}{z-a}$ is analytic inside on c except at the point $z = a$. Then we have,

$$\oint_c \frac{f(z)}{z-a} . dz = \oint_{c} \frac{f(z)}{z-a} . dz \quad \text{--- (1)}$$

circle of radius ϵ with centre
at a. Then an equation for
is $|z-a| = \epsilon$, where $0 \le \theta < 2\pi$.

Substituting $z = a + \epsilon e^{i\theta}$, $dz = i\epsilon e^{i\theta}$, the integral on the
right of (1) becomes

$$\oint \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+\epsilon e^{i\theta})}{\epsilon e^{i\theta}}\,d\theta$$

$$= i\int_0^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta$$

Thus we have from (1).

$$\oint_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta$$

Taking the limit of both sides of (2) and making use of the continuity of f(z), we have

$$\oint_C \frac{f(z)}{z-a}\, dz = \lim_{\varepsilon \to 0} i \int_0^{2\pi} f(a+\varepsilon e^{i\theta})\, d\theta$$

$$= i \int_0^{2\pi} \lim_{\varepsilon \to 0} f(a+\varepsilon e^{i\theta})\, d\theta$$

$$= i \int_0^{2\pi} f(a)\, d\theta = 2\pi i f(a)$$

o, that, we have,

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz$$

auchy's Integral formula for th derivative.

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$$f^n(a) = \frac{n!}{2\pi j} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz, \quad n = 1, 2, 3 \ldots$$

Laurent Series
Taylor's Series } H.W

Formula

Algorithm

Back tracking

23.09.18

Sum of subsets → Picture

Hamiltone →

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MATH

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Evalute $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz$ of C is

(a) the circle $|z|=3$ (b) the circle $|z|=1$.

solution: Let, $f(z)=e^z$, $f(z)$ is analytic inside on c. The point $z=2$ lies in the circle $|z|=3$. Hence by cauchy's integral formula we get,

$$f(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

where, $a=2$, $f(z)=e^z$

$$\therefore f(2)=e^2$$

$$\Rightarrow f(2)=\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz$$

$$\Rightarrow \frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz=e^2$$

Let, $f(z) = e^z$, $f(z)$ is analytic
inside on C.

The point lies outside the circle
$|z| = 1$. Hence by cauchy's theorem
we get,

$$\oint_C \frac{f(z)}{z-a}\, dz = 0$$

$$\Rightarrow \frac{1}{2\pi i} \oint_C \frac{e^z}{z-a} \cdot dz = 0$$

Find the value of $\oint_C \frac{\sin^6 z}{z - \pi/6}\, dz$

(ii) $\oint_C \frac{\sin^6 z}{(z - \pi/6)^n}\, dz$

$f(a) = \oint_C \frac{f(z)}{z-a}\, dz$

$a = \pi/6$, $f(z) = \sin^6 z$

$\therefore f(\pi/6) = \sin^6 \pi/6$

$= \left(\frac{1}{2}\right)^6 = 1/64$

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$$f(\pi/6) = \frac{1}{2\pi i}\oint_c \frac{\sin^6 z}{z-\pi/6}\,dz$$

$$\Rightarrow \oint_c \frac{\sin^6 z}{z-\pi/6} = 2\pi i \cdot 1/64$$

$$= \frac{\pi i}{32}$$

4 $$f^n(a) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-a)^n}\,dz$$

$f(z) = \sin^6 z$

$f'(z) = 6\sin^5 z \cdot \cos z$

$f''(z) = 30\cos^4 z = 30 \cdot \cos^2 z \cdot \cos^2 z$

$f''(z) = \cancel{30}$ $= 30(1-\sin^2 z) \cdot \cos^2 z$

$= 30\{(1-\sin^2 z) \cdot 2\cos z(-\sin z) +$

?? $\cos^2 z(-2\sin z \cdot \cos z)$

$\Rightarrow \cancel{30\ 30\sin^2}$ 30 $\{-2\cos z \sin z + 2\sin^3 z \cos z + 2\sin z \cos^3 z\} -$

$= 30\sin^4 z \cos^2 z - 6\sin^6 z$

$\therefore f''(\pi/6) = 21/16$

$$f''(\pi/6) = \frac{2!}{2\pi i}\oint_c \frac{\sin^6 z}{(z-\pi/6)^3}\, dz$$

State and prove Residue theorem.

Statement: If $f(z)$ is analytic inside and on a simple closed curve c except at a finite number of points $a, b, c \ldots$ inside c at which the residues are $a_{-1}, b_{-1}, c_{-1}, \ldots$ respectively, then $\oint_c f(z)\, dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \ldots)$ ... i.e $2\pi i$ times the sum of the residues at all ~~sig~~ singularities enclosed by c.

Residue: ~~$\oint_c f(z)\, dz$~~

$$a_{-1} = \lim_{z \to a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}}$$

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Expand $f(z) = \sin z$ in a Taylor series about $z = \pi/4$ 6.20

Expand Laurent series @ $\frac{e^{2z}}{(z-1)^3}$ 6.22

Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent series, valid for (a) $1 < |z| < 3$ (b) $|z| > 3$ (c) $0 < |z+1| < 2$ (d) $|z| < 1$ 6.29

Prove that, $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)}$ 7.12

Evaluate $\int_0^{\infty} \frac{dx}{x^6+1}$ 7.11

Residue Theorem. 7.2

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Laplace transformation

Defination: If the kernel $k(s, t)$ is define as,

$$k(s, t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{st} & \text{for } t \geq 0 \end{cases}$$

then $f(s) = \int_0^{\alpha} e^{-st} F(t)\, dt$ ...... (1)

The function $f(s)$ defined by the integral (1) is called the Laplace transform of the function $F(t)$ and is denoted by $\mathcal{L}\{F(t)\}$

Find the Laplace transform of the following ~~e~~elementary function.

(i) $F(t) = 1$

(ii) $F(t) = t$

Mathematical methods

Volume two

Abden Rahman

(iii) $f(t) = t^n$

(iv) $f(t) = e^{at}$

(v) $f(t) = \sin at$

(vi) $f(t) = \cos at$

(vii) $f(t) = \sinh at$

(viii) $f(t) = \cosh at$

Solution: (i) By the definition of the Laplace transform of a function $f(t)$ we have.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt. \quad (1)$$

when $f(t) = 1$, then,

$$\mathcal{L}\{1\} = \int_0^{\infty} e^{-st}.1\,dt = \int_0^{\infty} e^{-st}.\,dt$$

$$= \lim_{p \to \infty} \int_0^{p} e^{-st}\,dt$$

$$= \lim_{p \to \infty} \left[\frac{e^{-st}}{-s} \right]_0^p$$

$$= \lim_{p \to \infty} \left[-\frac{e^{-sp}}{s} + \frac{1}{s} \right]$$

$$= \frac{1}{s} - 0$$

$$= \frac{1}{s}$$

(ii) By the defination of the Laplace transform of a function $f(t)$ we have.

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^{\alpha} e^{-st} F(t)\, dt \quad \cdots (1)$$

When $F(t) = t$, then we have,

$$\mathcal{L}\{t\} = \int_0^{\alpha} e^{-st} \cdot t \, dt$$

$$= \lim_{p \to \alpha} \int_0^{p} t \cdot e^{-st} \, dt$$

$$= \lim_{p \to \alpha} \left[-\frac{te^{-st}}{s} \right]_0^{p} + \lim_{p \to \alpha} \frac{1}{s} \int_0^{p} e^{-st} \, dt$$

[Integrating by parts]

$$= \lim_{p \to \alpha} \frac{-p}{se^{ps}} + 0 - \lim_{p \to \alpha} \frac{1}{s^2} \left[e^{-st} \right]_0^{p}$$

$$= 0 - \lim_{p \to \alpha} \frac{1}{s^2} \left[\frac{1}{e^{sp}} - 1 \right]$$

$$= \frac{1}{s^2} \text{ if } s > 0.$$

(iii) $f(t) = t^n$

$\mathcal{L}\{f(t)\} = \mathcal{L}(t^n) = \frac{1}{s^{n+1}} \Gamma(n+1)$

(iv) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

(v) $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

(vi) $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$

(vii) $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$

(viii) $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$

Analogs to Digital converter

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Math | 08.10.18

find the laplace theorem of the function $f(t)$ where $f(t) = \begin{cases} t & 0<t<2 \\ 3 & t>2 \end{cases}$

Solution: Here, $f(t)$ is not defined at $t=0$ and $t=2$. By the definition of the laplace transform of a function $f(t)$, we have,

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) \cdot dt$$

$$= \int_0^{2} e^{-st} f(t) dt + \int_2^{\infty} e^{-st} f(t) dt$$

$$= \int_0^2 e^{-st}\cdot t\,dt + \int_2^{\infty} e^{-st}\cdot 3\,dt$$

$$= \int_0^2 t e^{-st}\,dt + 3\int_2^{\infty} e^{-st}\,dt$$

$$= \cancel{\left[t\cdot\frac{e^{-st}}{-s} - e^{-st}\right]_0^2 + 3\left[\frac{e^{-st}}{-s}\right]_2^{\infty}}$$

$u = t$
$v = e^{-st}$

$$= \left[-\frac{t}{s}e^{-st}\right]_0^2 + \int_0^2 \frac{e^{-st}}{s}\,dt - \frac{3}{s}\left[e^{-st}\right]_2^{\infty}$$

$$= -\frac{2}{s}e^{-2s} + 0 - \frac{1}{s^2}\left[e^{-st}\right]_0^2 - \frac{3}{s}\left(0 - e^{-2s}\right)$$

$$= -\frac{2}{s}e^{-2s} - \frac{1}{s^2}e^{-2s} + \frac{1}{s^2} + \frac{3}{s}e^{-2s}$$

$$= \frac{1}{s^2} + \frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2}$$

Some Important properties:

① Linearity property

If c_1 and c_2 are any constants which $f_1(t)$ and $f_2(t)$ are functions with Laplace transforms $f_1(s)$ and $f_2(s)$ respectively, then.

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\}$$

$$= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}$$

$$= c_1 f_1(s) + c_2 f_2(s)$$

Example:

$$\mathcal{L}\{4t - 3\cos 2t + 5e^{-t}\}$$

$$= 4\mathcal{L}\{t^2\} - 3\mathcal{L}\{\cos 2t\} + 5\mathcal{L}\{e^{-t}\}$$

$$= 4 \cdot \frac{2!}{s^3} - 3 \cdot \frac{s}{s^2+4} + 5 \cdot \frac{1}{s+1}$$

$$= \frac{8}{s^3} - \frac{3s}{s^2+9} + \frac{5}{s+1}$$

2. First translation or shifting property:

If $\mathcal{L}\{F(t)\} = f(s)$ then,

$$\mathcal{L}\{e^{at} F(t)\} = f(s-a)$$

Example: $\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+9}$ $\boxed{a=-1}$

$$\mathcal{L}\{e^{-t} \cos 2t\} = \frac{s+1}{(s+1)^2+9}$$

$$= \frac{s+1}{s^2+2s+5}$$

3. Second translation or shifting property: If $\mathcal{L}\{F(t)\} = f(s)$

and $G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$

$$\mathcal{L}\{G(t)\} = e^{-as} f(s)$$

Example: $\mathcal{L}\{t^3\} = \frac{3!}{s^4}$, the laplace transform of the function

$$g(t) = \begin{cases} (t-2)^3, & t > 2 \\ 0 & t < 2 \end{cases}$$

$$\mathcal{L}\{g(t)\} = e^{-2s} \cdot \frac{3!}{s^4}$$

Laplace transform of derivatives:

If $\mathcal{L}\{F(t)\} = f(s)$ then,

$$\mathcal{L}\{F'(t)\} = s f(s) - F(0)$$

Example: $F(t) = \cos 3t$, $\mathcal{L}\{F(t)\} = \frac{s}{s^2+9}$

$$\mathcal{L}\{F'(t)\} = \mathcal{L}\{-3\sin 3t\} = s\left(\frac{s}{s^2+9}\right) - 1$$

$$= \frac{-9}{s^2+9}$$

find the laplace transform of,

(i) $e^{4t} + 4t^3 - 2\sin 3t + 3\cos 5t$

(ii) $t^3 e^{5t}$

(iii) $e^{3t}(2\cos 5t - 3\sin 5t)$

(iv) $t\sin at$.

(v) $e^{at}\cosh bt$

(vi) $f(t) = \begin{cases} \sin(t - \pi/3), & t > \pi/3 \\ 0, & t < \frac{\pi}{3} \end{cases}$

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(B) $\mathcal{L}\{f(t)\} = f(s)$ then $\mathcal{L}\{f^{n}(t)\}$

$= s^2 f(s) - s f(0) - f'(0)$

$\mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0)$

$= s g(s) - g(0).$

let, $g(t) = f'(t)$ then,

$\mathcal{L}\{f^{n}(t)\} = s\mathcal{L}\{f'(t)\} - f'(0)$

$= s[s\,\mathcal{L}\{f(t)\} - f(0)] - f'(0)$

$= s^3\mathcal{L}\{f(t)\} - s f(0) - f'(0)$

$= s^2 f(s) - s f(0) - f'(0)$

Find $\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$

Solution: let, $f(t) = \sin\sqrt{t}$, $f(0) = 0$

Then, $f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$

$$\mathcal{L}\{f'(t)\} = \frac{1}{2}\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$$

$$= \frac{1}{2}\, s f(s) - f(0)$$

$$= \frac{\sqrt{\pi}}{2 s^{1/2}} e^{-1/4s}$$

$$\therefore \mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{s^{1/2}} e^{-\frac{1}{4s}}$$

Inverse Laplace transform:

$f(s)$	$\mathcal{L}^{-1}\{f(s)\} = f(t)$
$\frac{1}{s}$ $\longrightarrow$	1
$\frac{1}{s^2}$ $\longrightarrow$	t

$$\frac{1}{s-a} \rightarrow e^{at}$$

$$\frac{1}{s^2+a^2} \Rightarrow \frac{\sin at}{a}$$

Linearity property: If c_1 and c_2 are any constants while $f_1(s)$ and $f_2(s)$ are the laplace transforms of $F_1(t)$ and $F_2(t)$ respectively. Then

$$\mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} =$$

$$c_1 \mathcal{L}^{-1}\{f_1(s)\} + c_2 \mathcal{L}^{-1}\{f_2(s)\}$$

$$= c_1 F_1(t) + c_2 F_2(t)$$

Example: ~~$\mathcal{L}^{-1}\left\{\frac{9}{s-2}\right\} = \mathcal{L}^{-1}$~~

$$\mathcal{L}^{-1}\left\{\frac{9}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+9}\right\}$$

$$\boxed{\begin{aligned} \mathcal{L}\{1\} &= 1/s \\ \mathcal{L}\{t\} &= \frac{1}{s^2} \\ \mathcal{L}\{t^2\} &= \frac{2}{s^3} \end{aligned}}$$

$$= 4\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 3\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\}$$

$$+ 5\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= 4e^{2t} - 3\cos 4t + \frac{5}{2}\sin 2t$$

2. First translation or shifting property:

If $\mathcal{L}\{f(s)\} = F(t)$ then $\mathcal{L}^{-1}\{f(s-a)\} = e^{at}F(t)$

Example:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\}$$

$$= \frac{1}{2}e^{t}\sin 2t$$

3. Second translation property:

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ then $\mathcal{L}^{-1}$

$$\left\{e^{-as}f(s)\right\} = \begin{cases} F(t-a), & t>a \\ 0, & t<a \end{cases}$$

Example: $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

$$\mathcal{L}^{-1}\left\{\frac{e^{-\frac{\pi s}{3}}}{s^2+1}\right\} = \begin{cases} \sin(t-\pi/3), & t>\frac{\pi}{3} \\ 0, & t<\frac{\pi}{3} \end{cases}$$

4. <u>Change of scale property:</u>

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ then

$$\mathcal{L}^{-1}\{f(ks)\} = \frac{1}{k} F\left(\frac{t}{k}\right)$$

Example: $\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} = \cos 4t$

$$\mathcal{L}^{-1}\left\{\frac{2s}{(2s)^2+16}\right\} = \frac{1}{2}\cos\left(\frac{4t}{2}\right)$$

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Complex → A
Laplace → B

property:

Evaluate $\mathcal{L}^{-1}\left\{\frac{5s-6}{s^2+9} - \frac{s-15}{s^2-25}\right\}$

Solⁿ:

$$\mathcal{L}^{-1}\left\{\frac{5s-6}{s^2+9} - \frac{s-15}{s^2-25}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{5s}{s^2+3^2} - \frac{6}{s^2+3^2} - \frac{s}{s^2-5^2} + \frac{15}{s^2-5^2}\right\}$$

$$= 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2-5^2}\right\} + 3\mathcal{L}^{-1}\left\{\frac{5}{s^2-5^2}\right\}$$

$$= 5\cos 3t - 2\sin 3t - \cosh 5t + 3\sinh 5t$$

Ans

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Division by s: $\mathcal{L}^{-1}\{f(s)\} = F(t)$

$$\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du$$

Evaluate: $\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+9)}\right\}$

solution: By theorum we get,

$$\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{\sin 2t}{2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\} = \int_0^t \frac{\sin 2u}{2}\,du$$

$$= \frac{1}{2}\left[-\frac{1}{2}\cos 2u\right]_0^t$$

$$= -\frac{1}{4}\left[\cos 2u\right]_0^t$$

$$= -\frac{1}{4}\left[\cos 2t - \cos 0\right).$$

$$= \frac{1}{9}(1-\cos 2t)$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+9)}\right\} = \int_0^t \frac{1}{9}(1-\cos 2u)\,du$$

$$= \frac{1}{9}\left[u - \frac{\sin 2u}{2}\right]_0^t$$

$$= \frac{1}{9}(t-0) - \frac{1}{8}(\sin 2t - 0)$$

$$= \frac{1}{9}t - \frac{1}{8}\sin 2t$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+9)}\right\} = \int_0^t \left(\frac{1}{9}u - \frac{1}{8}\sin 2u\right)du$$

$$= \left[\frac{1}{4}\cdot\frac{u^2}{2} + \frac{1}{8}\cdot\frac{\cos 2u}{2}\right]_0^t$$

$$= \frac{1}{8}(t^2-0) + \frac{1}{16}(\cos 2t - 1)$$

$$= \frac{1}{8}t^2 + \frac{1}{16}\cos 2t - \frac{1}{16}$$ Ans.

Partial Fraction Decompositions:

Evaluate $\mathcal{L}^{-1}\left\{\frac{2s^2-9}{(s+1)(s-2)(s-3)}\right\}$

Soln:

Let, $\frac{2s^2-9}{(s+1)(s-2)(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3}$ ——— (1)

$\therefore 2s^2-9 = A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2) \cdots$ (2)

putting $s=-1$, ~~in (2)~~ in (2), we get

$2-9 = 12A$

$\Rightarrow A = -\frac{7}{12}$

putting $s=2$ in (2) we get,

$8-9 = -3B$, $B = -\frac{1}{3}$

putting $s=3$ in ② we get,

$19=4c$ $\therefore c=7/2$

Thus from ① we get,

$$\frac{2s^2-9}{(s+1)(s-2)(s-3)}=\frac{-1/6}{s+1}+\frac{-9/3}{s-2}+\frac{7/2}{s-3}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{2s^2-9}{(s+1)(s-2)(s-3)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{-1/6}{s+1}+\frac{-9/3}{s-2}+\frac{7/2}{s-3}\right\}$$

$$=-1/6\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}-9/3\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}+\frac{7}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$=-1/6\,e^{-t}-\frac{9}{3}e^{2t}+7/2\,e^{3t}$$ Ans.

Evaluate $\mathcal{L}^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}$

__Sol^n:__ let, $\frac{5s+3}{(s-1)(s^2+2s+5)}$

$$= \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5} \quad \text{(1)}$$

$\therefore 5s+3 = A(s^2+2s+5) + (Bs+c)(s-1) \quad \text{(2)}$

putting $s=1$ in (2) we get,

$8 = 8A \therefore A = 1$

Equating coefficient of s^2 from both sides of (2) we get,

$0 = A+B$

$\Rightarrow 0 = 1+B$

$\Rightarrow B = -1$

putting, $s=0$ in (2) we get

$3 = 5A - C \therefore C = 5 - 3 = 2$

Thus from ① we get,

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1} + \frac{-s+2}{s^2+2s+5}$$

$$= \frac{1}{s-1} + \frac{-s+2}{(s+1)^2+2^2}$$

$$= \frac{1}{s-1} + \frac{-(s+1)+3}{(s+1)^2+2^2}$$

$$= \frac{1}{s-1} - \frac{s+1}{(s+1)^2+2^2} + \frac{3}{2}\frac{2}{(s+1)^2+2^2}$$

$$\mathcal{L}^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}$$

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$= t - e^{-t}\cos 2t + \frac{3}{2} e^{-t} \sin 2t$

11/ $f(s) = \frac{2}{s^2+2}$

$f(s+1) = \frac{2}{(s+1)^2+2} = e^{-t} \sin 2t$

$\mathcal{L}^{-1}\{f(s)\} = F(t)$

$\mathcal{L}^{-1}\{f(s-a) = e^{at} F(t)$

Integral Transforms & Boundary value problem

Ms. ADDIB Fatima

Ans. take

4) state and prove convolution theorem

Statement: If $L^{-1}\{f(s)\} = F(t)$ and $L^{-1}\{g(s)\} = G(t)$

then $L^{-1}\{f(s)\,g(s)\} = \int_0^t F(u)\,G(t-u)\,du = F * G$

Example: Evaluate $L^{-1}\left\{\frac{s}{s^2(s+2)}\right\}$ by using convolution theorem.

6)

Soln: let, $f(s) = \frac{1}{s+2}$ and

$$g(s) = 3/s^2$$

$$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= e^{-2t} = F(t)$$

$$\mathcal{L}^{-1}\{g(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s^2}\right\} = 3t$$

$$= G(t)$$

~~$G(t-u)$~~

$$\mathcal{L}^{-1}\{f(s) \cdot g(s)\}$$

$$= \int_0^t e^{-2u} \cdot 3(t-u)$$

Therefore, by using the convolution theorem, we get,

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2(s+2)}\right\} = \int_0^t e^{-2u} \cdot 3(t-u)\, du$$

$$= 3\int_0^t t e^{-2u} du - 3\int_0^t u e^{-2u} du$$

$$= -\frac{3t}{2}\left[e^{-2u}\right]_0^t + \frac{3}{2}\left[ue^{-2u}\right]_0^t - \frac{3}{2}\int_0^t e^{-2u} \cdot du$$

$$= -\frac{3t}{2}\left(e^{-2t}-1\right) + \frac{3}{2}te^{-2t} - 0 + \frac{3}{4}\left[e^{-2u}\right]_0^t$$

$$= -\frac{3t}{2}e^{-2t} + \frac{3t}{2} + \frac{3t}{2}e^{-2t} + 3/4\, e^{-2t} - 3/4$$

$$= \frac{3t}{2} + \frac{3}{9}e^{-2t} - \frac{3}{9}.$$

Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+9)}\right\}$ by using convolution theorem.

Evaluate $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$ "

" $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$ "

MATH

Applications of Laplace transform To differential Equations

Solve the following differential equation by using Laplace transform.

$$\frac{dy}{dt} - 3y = 0, \ y(0) = 1$$

Soln: Given that, $\frac{dy}{dt} - 3y = 0, \ y(0) = 1$

The given differential equation can be written as

$$y' - 3y = 0 \ \cdots\cdots \ (1)$$

Taking the Laplace transform of both sides of (1) we get,

$$\mathcal{L}\{y'\} - \cancel{3\{}\, 3\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$\Rightarrow sY(s) - y(0) - 3Y(s) = 0$$

$$\Rightarrow sY(s) - 1 - 3Y(s) = 0, \text{ since, } y(0) = 1$$

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$\Rightarrow (s-3) Y(s) = 1$

$\Rightarrow Y(s) = \frac{1}{s-3}$ ...... (2)

Now, taking the inverse Laplace transform of both sides of (2) we get.

$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$

$\Rightarrow y(t) = e^{3t}$ Ans.

solve the

$y''(t) + y(t) = ;\ y(0) = 1,$

$y'(0) = -2$

solⁿ: Given differential equation is,

$y''(t) + y(t) = t$ ...... (1)

$y(0)=1,\ y'(0)$

Taking the Laplace

both sides of (1)

given conditions

$\mathcal{L}\{y''(t)\}+\mathcal{L}$

$\Rightarrow s^2 y - s\,y(0)$

$\Rightarrow s^2 y - s + 2$

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$$\Rightarrow (s^2+1)y = s - 2 + \frac{1}{s^2} = \frac{s^3 - 2s^2 + 1}{s^2}$$

$$\Rightarrow y = \frac{s^3 - 2s^2 + 1}{s^2(s^2+1)}$$

Now, $\frac{s^3 - 2s^2 + 1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$

$$\Rightarrow s^3 - 2s^2 + 1 = As(s^2+1) + B(s^2+1) + Cs^3 + Ds^2 \text{ ——— } (2)$$

Equating the coefficients of s^3 from both sides of (2), we get, $C = 1$

putting $s=0$ in (2) we get,

$$B = 1$$

Equating the coefficients of s^2 from both sides of (2) we get,

$-2 = B + D$

or $-2 = 1 + D$

$\therefore D = -3$

Equating the coefficient of s from both sides we get,

$0 = A + 0$

$\Rightarrow A = 0$

$\therefore y = \frac{1}{s^2} + \frac{s-3}{s^2+1}$

$= \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}$

Taking inverse Laplace transform of both sides,

$\mathcal{L}^{-1}\{y\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$\Rightarrow Y(t) = t + \cos t - 3\sin t$ which is the required solution.

solve : $y''(t) + a^2 y(t) = t$;

$y(0) = 1,\ y'(0) = -2$