CSE 2202 Design and Analysis of Algorithms – I

Greedy Algorithms

Greedy Algorithm

- Algorithms for optimization problems typically go through a sequence of steps, with a set of choices at each step.
- Greedy algorithms make the choice that looks best at the moment.
  - That is, it makes such a decision in the hope that this will lead to a globally optimal solution
- This locally optimal choice may lead to a globally optimal solution (i.e., an optimal solution to the entire problem).

When can we use Greedy algorithms?

We can use a greedy algorithm when the following are true:

1) **The greedy choice property:** A(greedy) choice.

2) **The optimal substructure property:** The optimal solution contains within its optimal solutions to subproblems.

An Activity Selection Problem (Conference Scheduling Problem)

- **Input: A set of activities S = {a_1,…, a_n}**
- *We have* ***n*** proposed activities that wish to use a resource, such as a lecture hall, which can serve only one activity at a time.
- Each activity has start time and a finish time
  - $a_i = (s_i, f_i)$
- Two activities are compatible if and only if their interval does not overlap
- **Output: a maximum-size subset of mutually compatible activities**

The Activity Selection Problem

- Here are a set of start and finish times

i	1	2	3	4	5	6	7	8	9	10	11
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	8	9	10	11	12	13	14

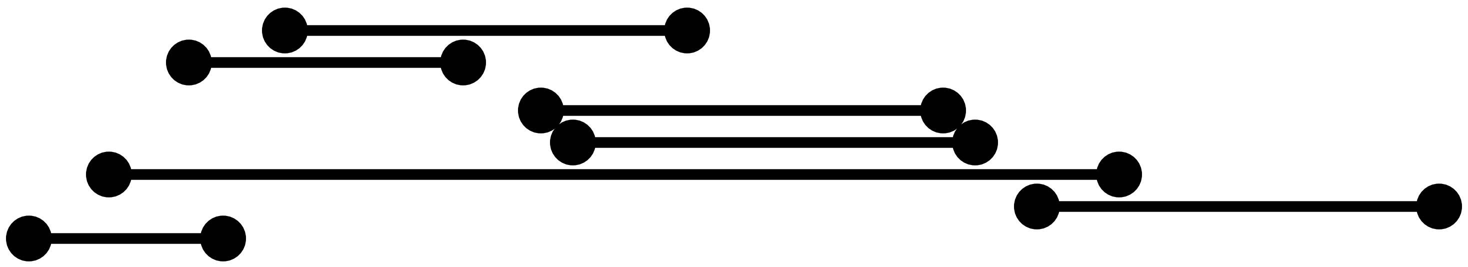
- What is the maximum number of activities that can be completed?
    - $\{a_3, a_9, a_{11}\}$ can be completed
    - But so can $\{a_1, a_4, a_8, a_{11}\}$ which is a larger set
    - But it is not unique, consider $\{a_2, a_4, a_9, a_{11}\}$

The Activity Selection Problem

Input: list of time-intervals L

Output: a non-overlapping subset S of the intervals

Objective: maximize |S|



3,7
2,4
5,8
6,9
1,11
10,12
0,3

The Activity Selection Problem

Input: list of time-intervals L

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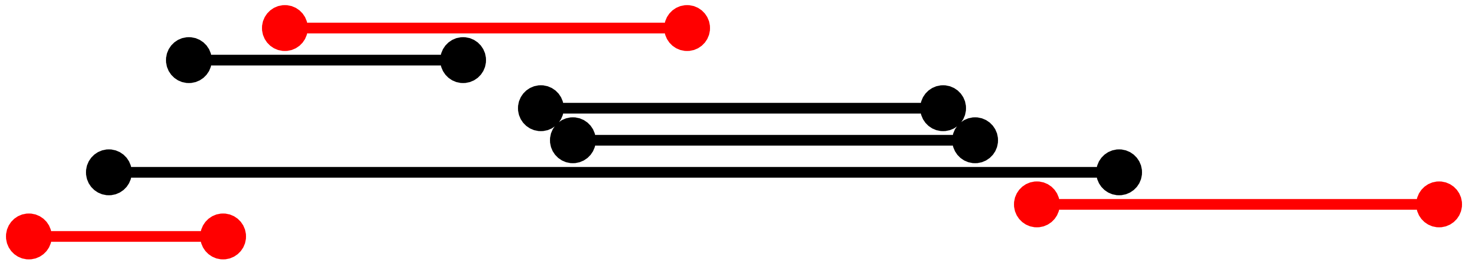
Objective: maximize |S|



Answer = 3

3,7
2,4
5,8
6,9
1,11
10,12
0,3

The Activity Selection Problem

Algorithm 1:

1. sort the activities by the starting time
2. pick the first activity ***"a"***
3. remove all activities conflicting with ***"a"***
4. repeat

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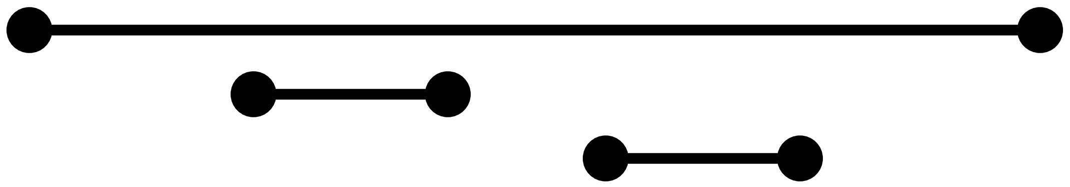

The Activity Selection Problem

Algorithm 1:

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2. ~~pick the first activity ***"a"***~~
3. ~~remove all activities conflicting with ***"a"***~~
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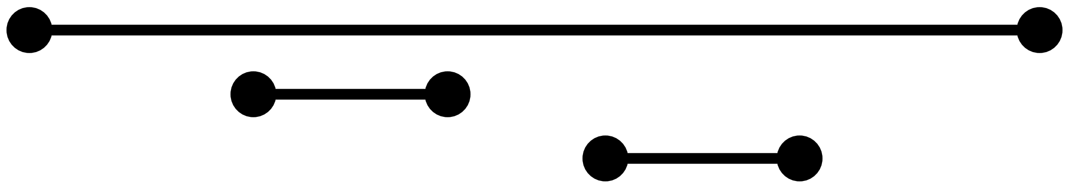

The Activity Selection Problem

Algorithm 2:

1. sort the activities by length
2. pick the shortest activity ***"a"***
3. remove all activities conflicting with ***"a"***
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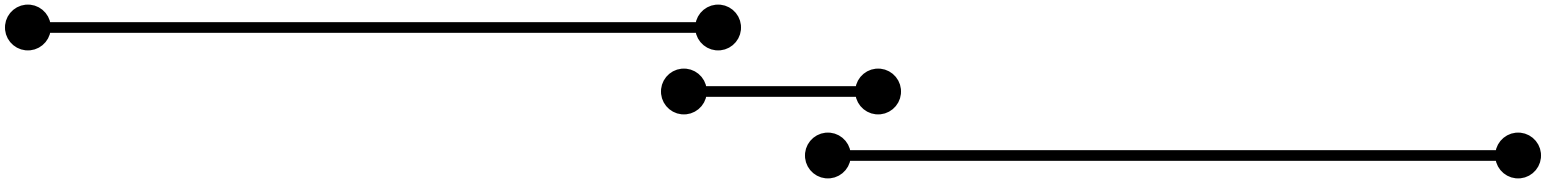

The Activity Selection Problem

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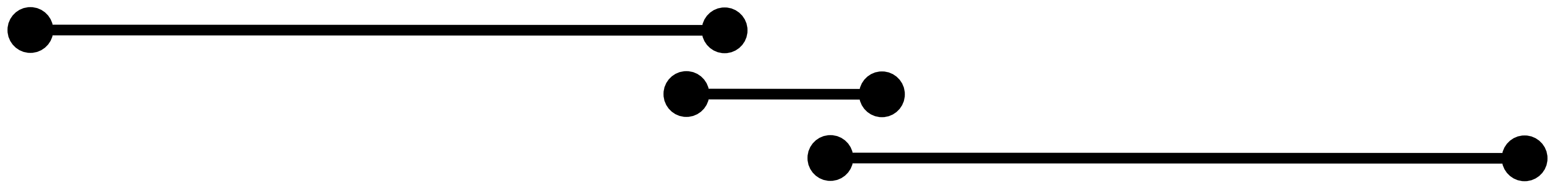
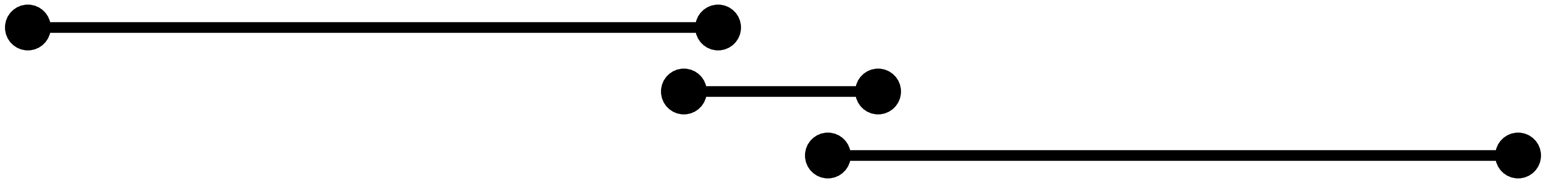


The Activity Selection Problem

Algorithm 3:

1. sort the activities by ending time
2. pick the activity which ends first
3. remove all activities conflicting with a
4. repeat



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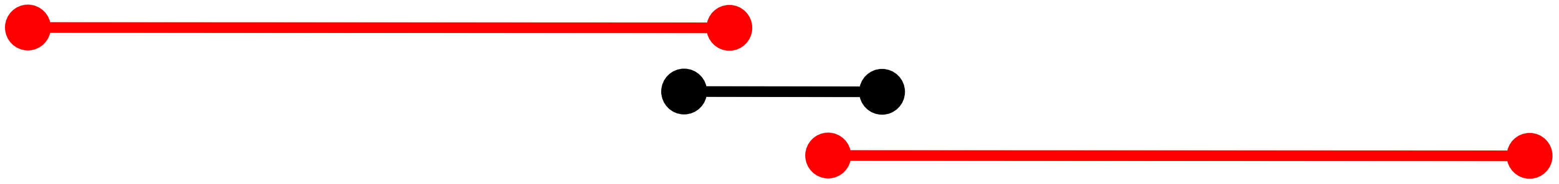

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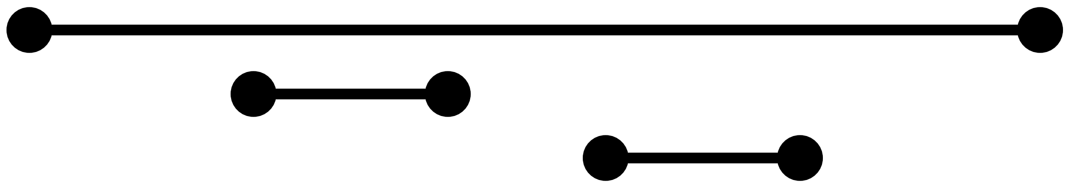

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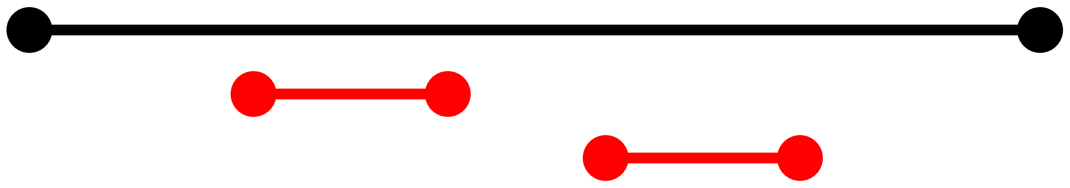

The Activity Selection Problem

Algorithm 3:

1. sort the activities by ending time
2. pick the activity “a” which ends first
3. remove all activities conflicting with “a”
4. repeat

Theorem:

Algorithm 3 gives an optimal solution to the activity selection problem.

Activity Selection Algorithm

Idea: At each step, select the activity with the smallest finish time that is compatible with the activities already chosen.

```
Greedy-Activity-Selector(s, f)
    n <- length[s]
    A <- {1}                {Automatically select first activity}
    j <- 1              {Last activity selected so far}
    for i <- 2 to n do
        if $s_i >= f_j$ then
            A <- A U {i}    {Add activity i to the set}
            j <- i          {record last activity added}
    return A
```

The idea is to always select the activity with the earliest finishing time, as it will free up the most time for other activities.

The Activity Selection Problem

- Here are a set of start and finish times

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- What is the maximum number of activities that can be completed?
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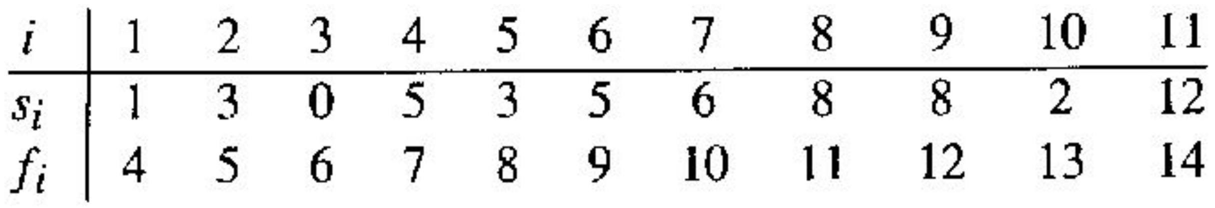
Interval Representation

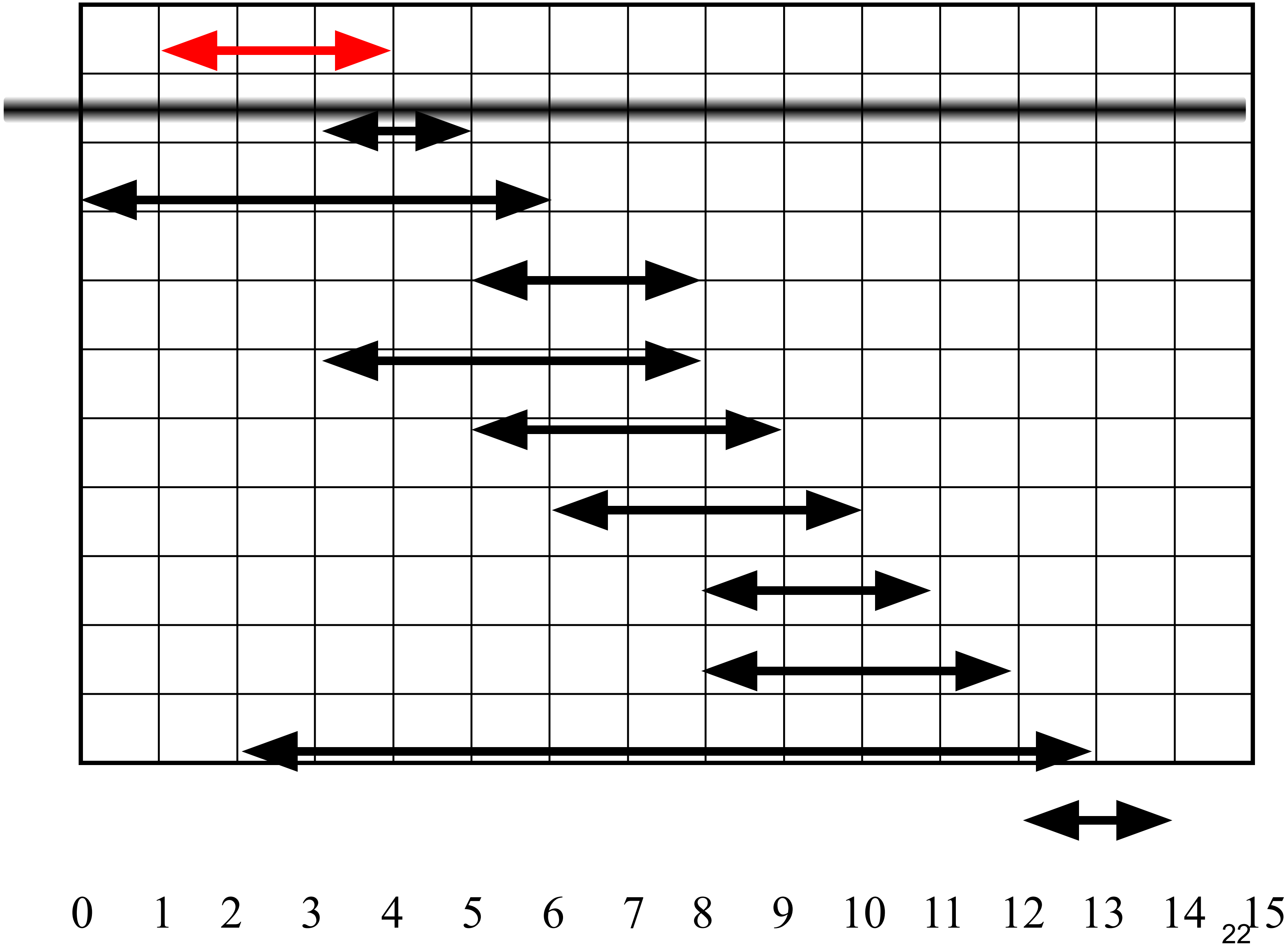


i	1	2	3	4	5	6	7	8	9	10	11
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	8	9	10	11	12	13	14

↔ Not Observed yet

↔ Added in optimal Solution

↔ Removed from the list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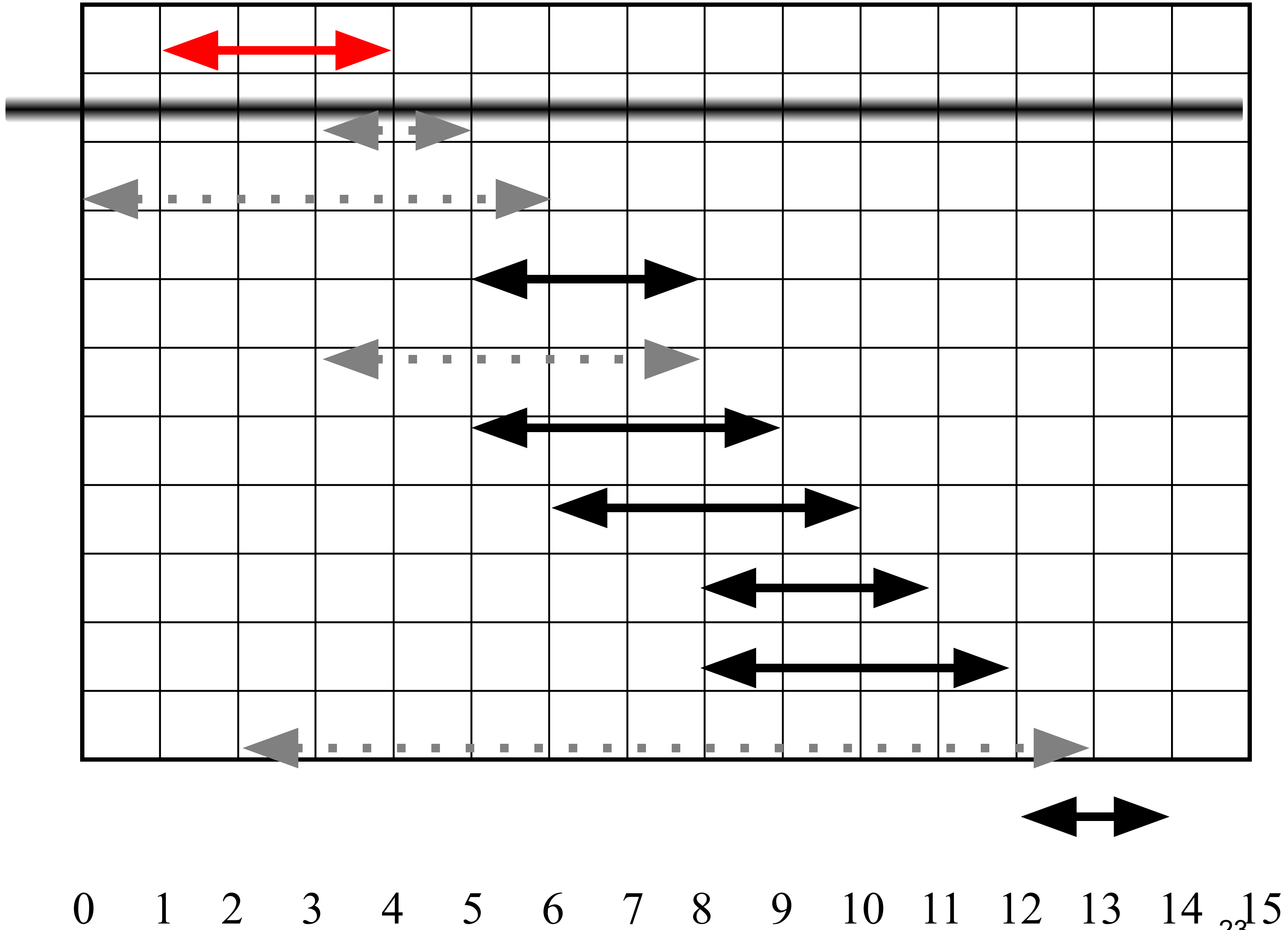
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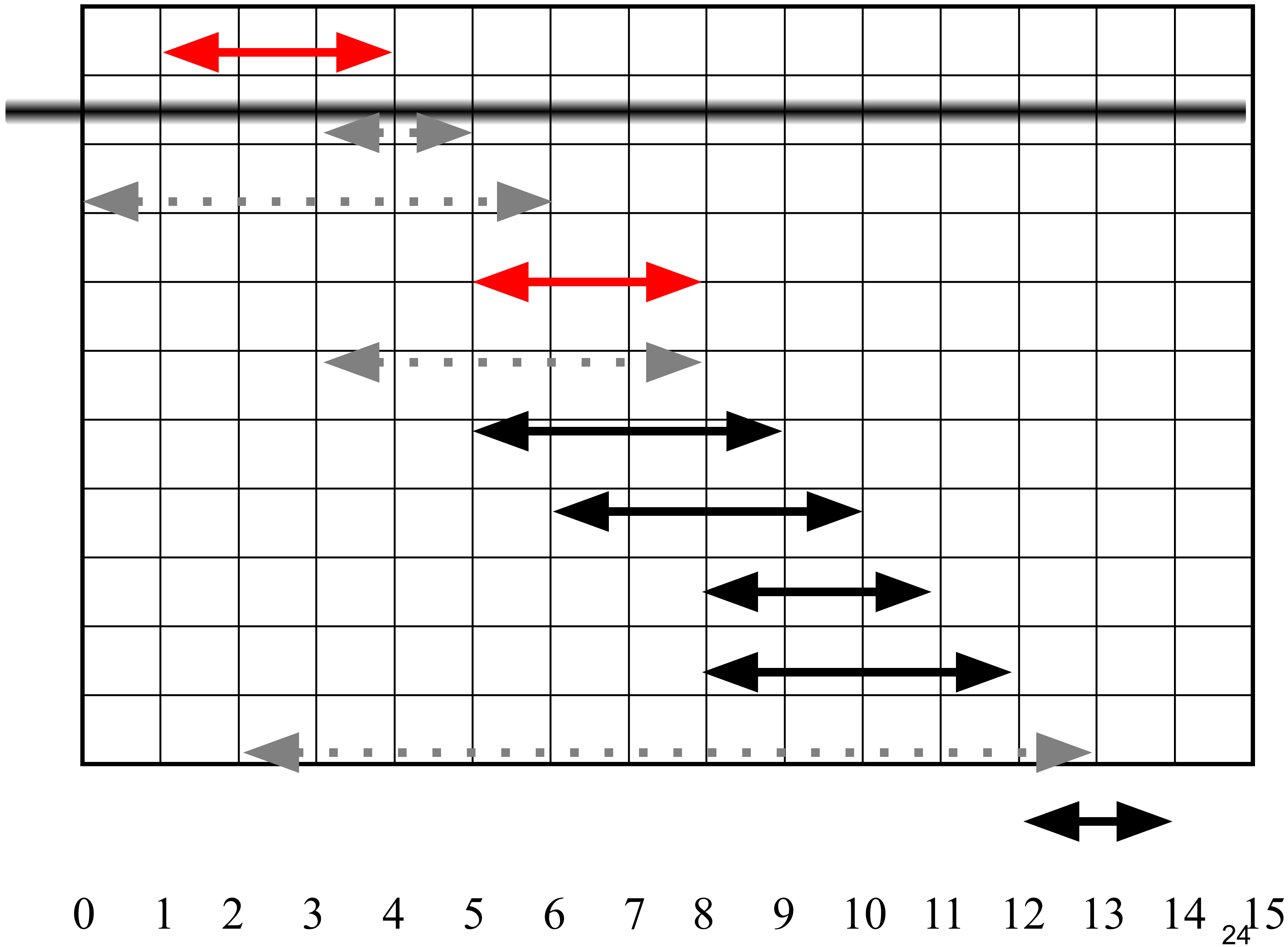
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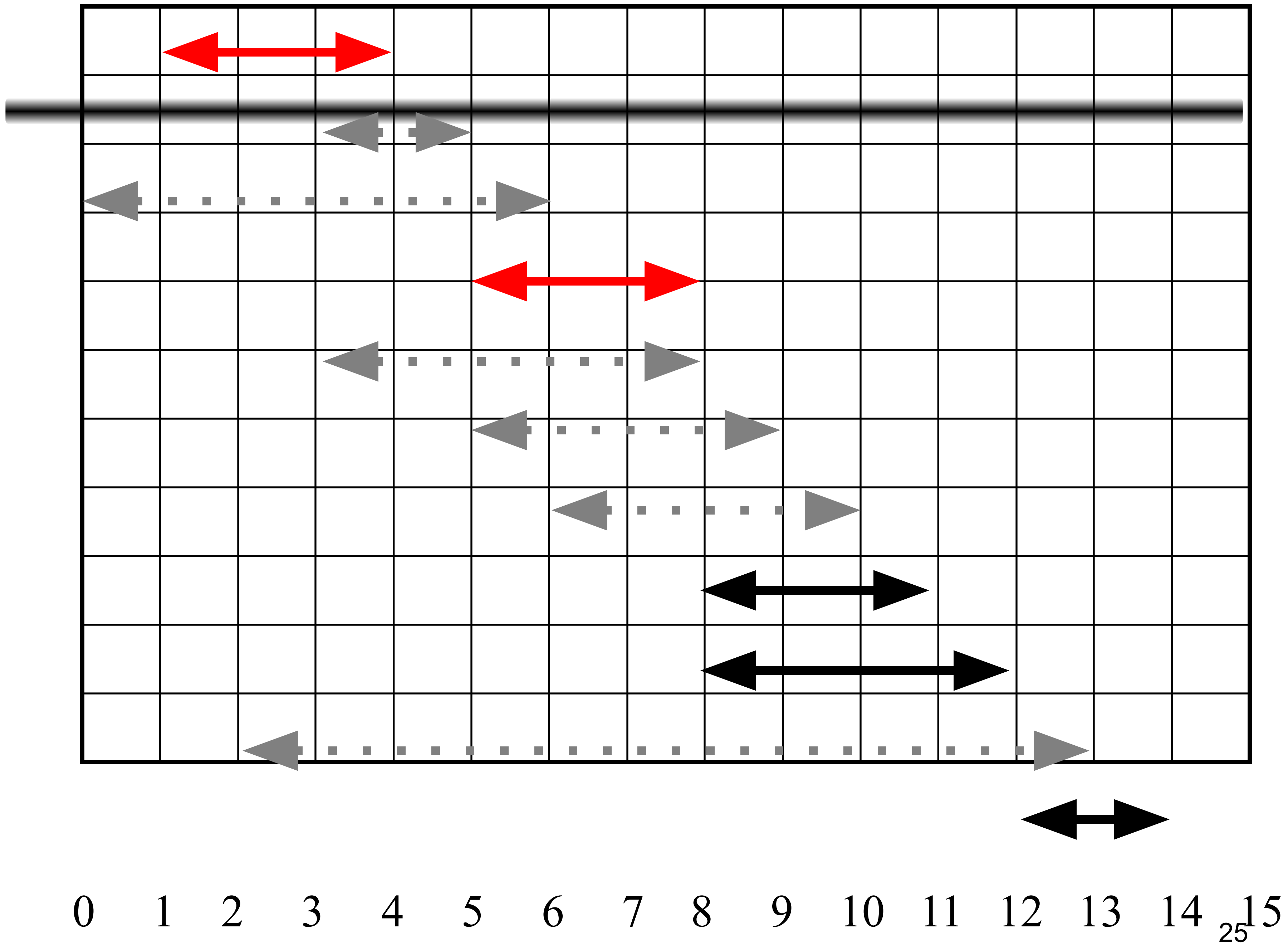
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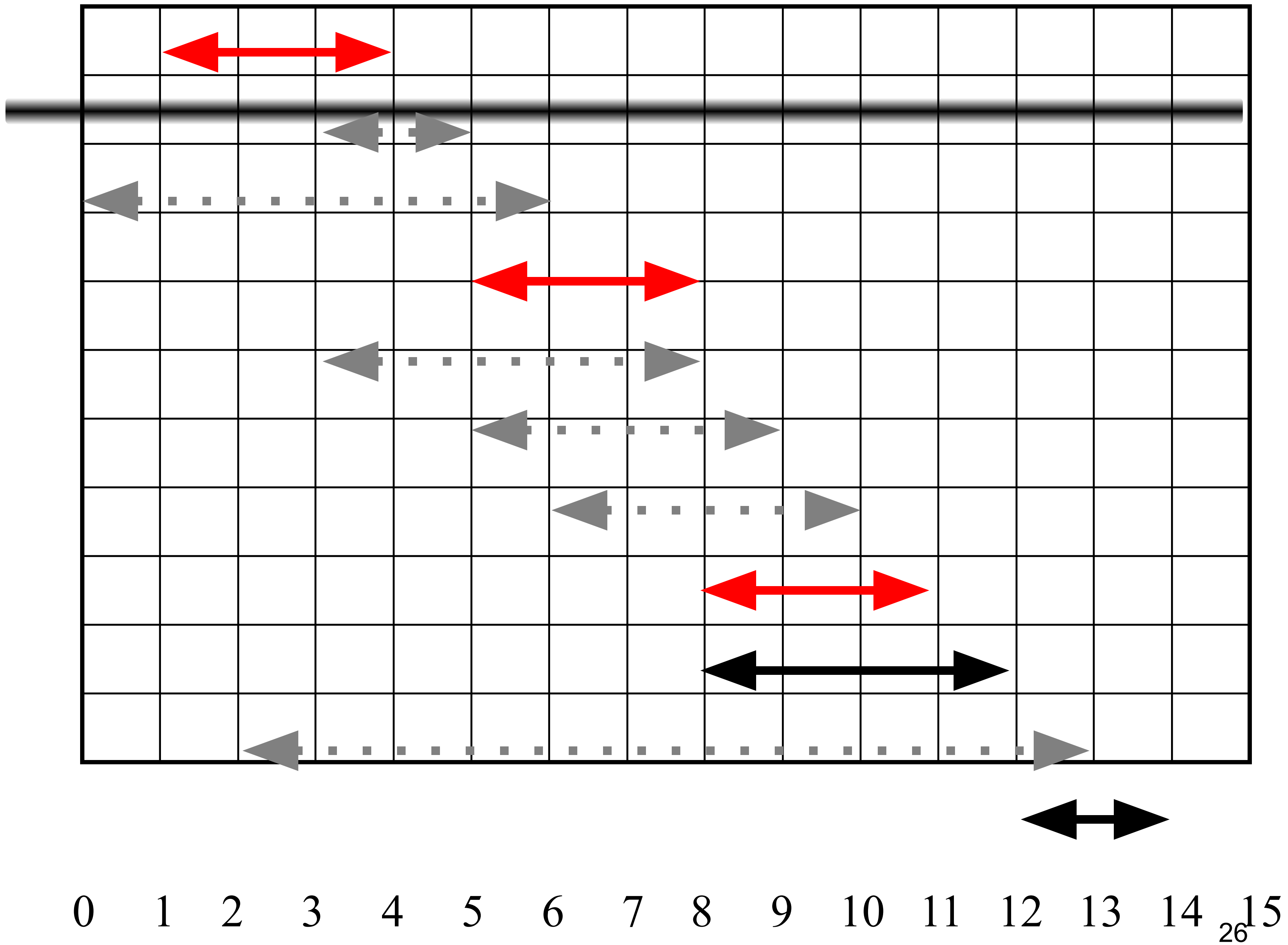
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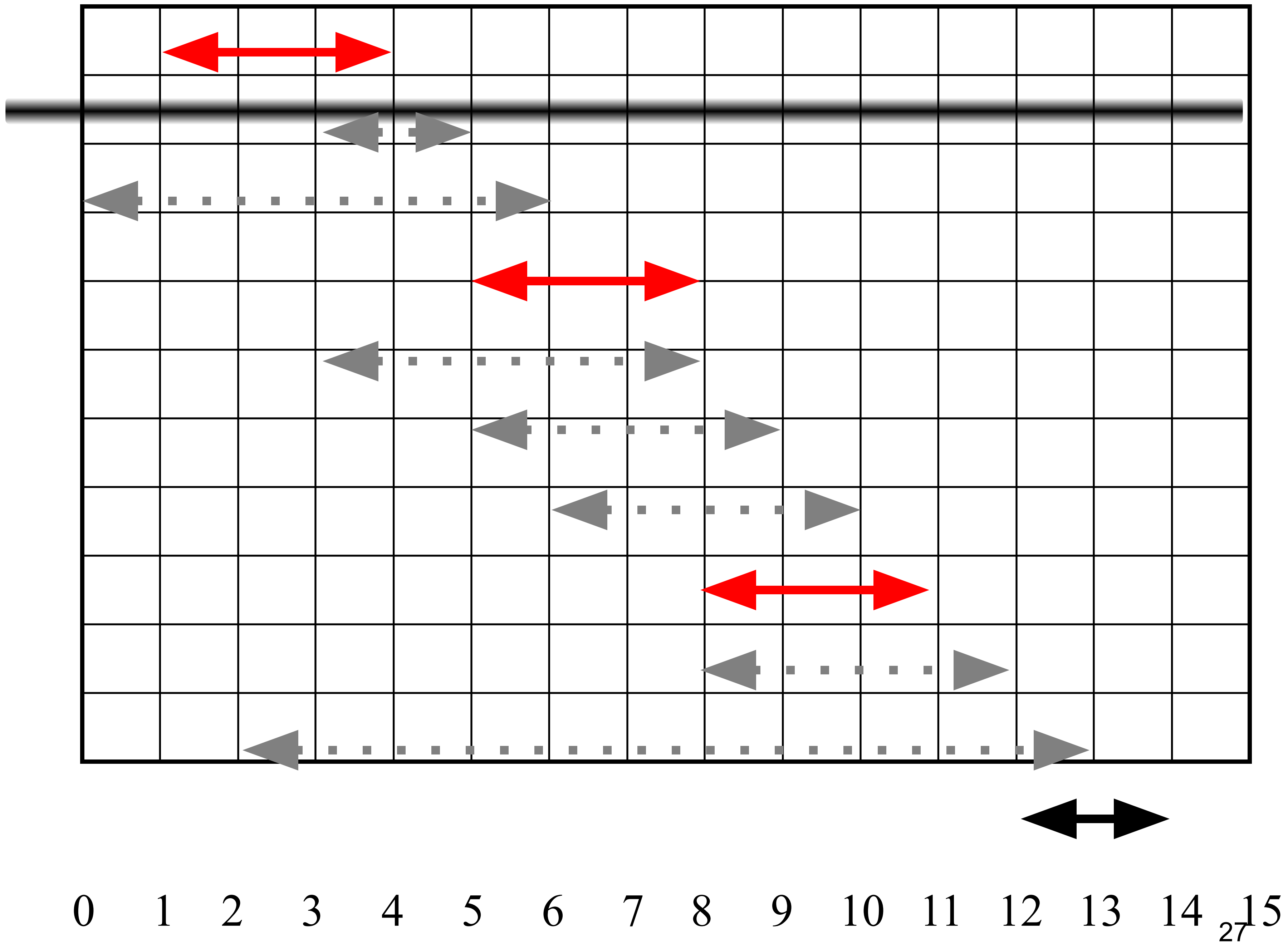
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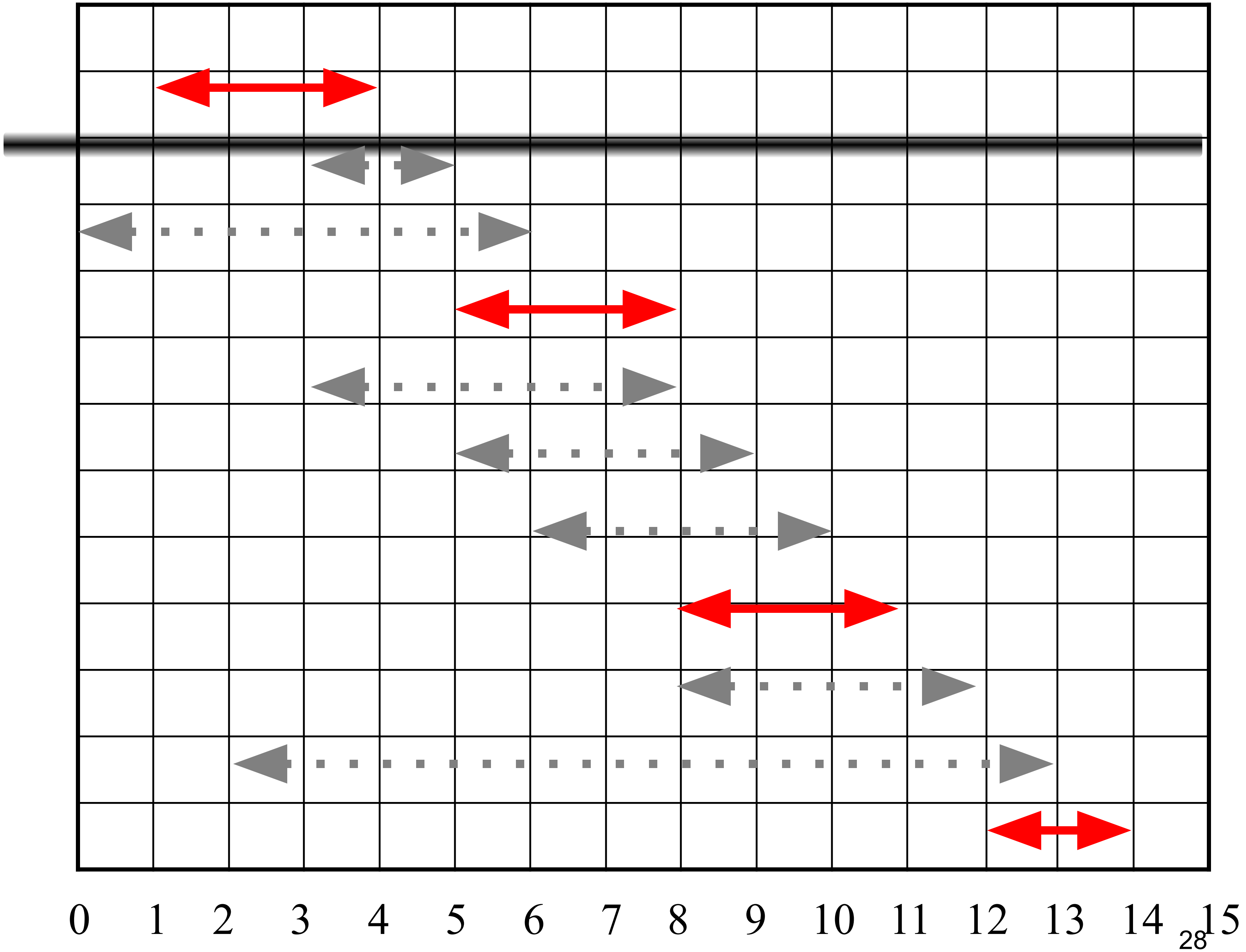
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Why this Algorithm is Optimal?

- We will show that this algorithm uses the following properties
  - The problem has the optimal substructure property
  - The algorithm satisfies the greedy-choice property
- Thus, it is Optimal

Optimal Substructure Property

- **Base Case:** For the smallest subproblem of size 1 (only one activity), the optimal solution is trivially the activity itself.

- **Inductive Hypothesis:** Assume that we have already proven that the optimal solution can be constructed for any subset of activities with size **k**, where **$1 \leq k \leq n - 1$**.

Optimal Substructure Property

Inductive Step: Now we want to prove that the optimal solution can be constructed for a subset of activities with size k + 1.

Let's consider the set of activities $\{A_1, A_2, ..., A_{k+1}\}$. Since the activities are sorted by finishing times,

the last activity in this set, A_{k+1}, will have the maximum finish time among all activities.

We have two cases:

First case: Activity A_{k+1} is included in the optimal solution.

- In this case, we need to find an optimal solution for the remaining activities $\{A_1, A_2, ..., A_k\}$ that are non-overlapping with A_{k+1}.
- By our inductive hypothesis, we know that an optimal solution can be constructed for these k activities.
- Combining A_{k+1} with this optimal solution gives us an optimal solution for the entire set $\{A_1, A_2, ..., A_{k+1}\}$.

Optimal Substructure Property

First case: Activity A_{k+1} is included in the optimal solution.

- In this case, we need to find an optimal solution for the remaining activities $\{A_1, A_2, ..., A_k\}$ that are non-overlapping with A_{k+1}.
- By our inductive hypothesis, we know that an optimal solution can be constructed for these k activities.
- Combining A_{k+1} with this optimal solution gives us an optimal solution for the entire set $\{A_1, A_2, ..., A_{k+1}\}$

Second Case: Activity A_{k+1} is not included in the optimal solution.

- In this case, we simply need to find an optimal solution for the activities $\{A_1, A_2, ..., A_k\}$, which we have already assumed possible by our inductive hypothesis.

Since we've covered both cases, we can conclude that the optimal solution for the set $\{A_1, A_2, ..., A_{k+1}\}$ can be constructed from the optimal solutions of the smaller subproblems $\{A_1, A_2, ..., A_k\}$,

Greedy-Choice Property

- Show there is an optimal solution that begins with a greedy choice (with activity 1, which as the earliest finish time)
- Suppose A ⊆ S in an optimal solution
  - Order the activities in A by finish time. The first activity in A is k
    - If k = 1, the schedule A begins with a greedy choice
    - If k ≠ 1, show that there is an optimal solution B to S that begins with the greedy choice, activity 1
  - Let B = A – {k} ∪ {1}
    - $f_1 \leq f_k$ □ activities in B are disjoint (compatible)
    - B has the same number of activities as A
    - Thus, B is optimal

Example of Greedy Algorithm

- Fractional Knapsack
- Huffman Coding
- Minimum Spanning Tree – Prims and Kruskal’s
- Activity Selection Problem
- Dijkstra’s Shortest Path Algorithm
- Network Routing
- Job sequencing with deadlines
- Coin change problems
- Graph Coloring: Greedy algorithms can be used to color a graph (though not necessarily optimally) by assigning the next available color to a vertex.

Designing Greedy Algorithms

1. Cast the optimization problem as one for which:
   - we make a choice and are left with only one subproblem to solve
2. Prove the GREEDY CHOICE
   - that there is always an optimal solution to the original problem that makes the greedy choice
3. Prove the OPTIMAL SUBSTRUCTURE:
   - the greedy choice + an optimal solution to the resulting subproblem leads to an optimal solution

Example: Making Change

- Instance: amount (in cents) to return to customer
- Problem: do this using fewest number of coins
- Example:
  - Assume that we have an unlimited number of coins of various denominations:
    - 1c (pennies), 5c (nickels), 10c (dimes), 25c (quarters), 1$ (loonies)
  - Objective: Pay out a given sum $5.64 with the smallest number of coins possible.

The Coin Changing Problem

- Assume that we have an unlimited number of coins of various *values*:
    - 1c (pennies), 5c (nickels), 10c (dimes), 25c (quarters), 1$ (loonies)
- Objective: Pay out a given sum S with the smallest number of coins possible.

- The greedy coin changing algorithm:
    - This is a Θ(m) algorithm where m = number of *values*.

```
while S > 0 do
   c := value of the largest coin no larger than S;
   num := S / c;
   pay out num coins of value c;
   S := S - num*c;
```

Example: Making Change

- E.g.:

  $5.64 = $2 +$2 + $1 +
  .25 + .25 + .10 +
  .01 + .01 + .01 +.01

Making Change – A big problem

- Example 2: Coins are valued \$.30, \$.20, \$.05, \$.01
  - Does not have greedy-choice property, since \$.40 is best made with two \$.20's, but the greedy solution will pick three coins (which ones?)

The Fractional Knapsack Problem

- **Given:** A set S of n items, with each item i having
  - b_i - a positive benefit
  - w_i - a positive weight
- **Goal:** Choose items with maximum total benefit but with weight at most W.
- If we are allowed to take fractional amounts, then this is the **fractional knapsack problem**.
  - In this case, we let x_i denote the amount we take of item i
  - Objective: maximize

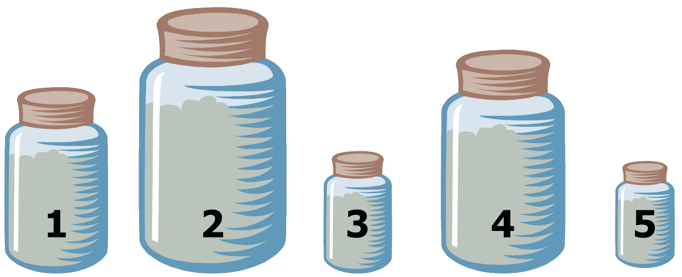
$$\sum_{i \in S} b_i (x_i / w_i)$$

  - Constraint:

$$\sum_{i \in S} x_i \le W, 0 \le x_i \le w_i$$

Example

- Given: A set S of n items, with each item i having
  - b_i - a positive benefit
  - w_i - a positive weight
- Goal: Choose items with maximum total benefit but with total weight at most W.



Items:	1	2	3	4	5
Weight:	4 ml	8 ml	2 ml	6 ml	1 ml
Benefit:	\$12	\$32	\$40	\$30	\$50
Value: (\$ per ml)	3	4	20	5	50

"knapsack"

10 ml

Solution: P

- 1 ml of 5 50\$
- 2 ml of 3 40\$
- 6 ml of 4 30\$
- 1 ml of 2 4\$

- Total Profit:124\$

The Fractional Knapsack Algorithm

- Greedy choice: Keep taking item with highest **value** (benefit to weight ratio)
  - Since $\sum_{i \in S} b_i (x_i / w_i) = \sum_{i \in S} (b_i / w_i) x_i$

Algorithm *fractionalKnapsack*(*S*, *W*)

Input: set *S* of items w/ benefit b_i and weight w_i; max. weight *W*

Output: amount x_i of each item *i* to maximize benefit w/ weight at most *W*

```
for each item i in S
    $x_i \leftarrow 0$
    $v_i \leftarrow b_i / w_i$          {value}
$w \leftarrow 0$                        {total weight}
while $w < W$
    remove item i with highest $v_i$
    $x_i \leftarrow \min\{w_i, W - w\}$
    $w \leftarrow w + \min\{w_i, W - w\}$
```

The Fractional Knapsack Algorithm

- Running time: Given a collection S of n items, such that each item i has a benefit b_i and weight w_i, we can construct a maximum-benefit subset of S, allowing for fractional amounts, that has a total weight W in O(nlogn) time.
  - Use heap-based priority queue to store S
  - Removing the item with the highest value takes O(logn) time
  - In the worst case, need to remove all items