Design and Analysis of Algorithms
CSE 2202

Department of Computer Science and Engineering
University of Dhaka

Recommended Textbooks

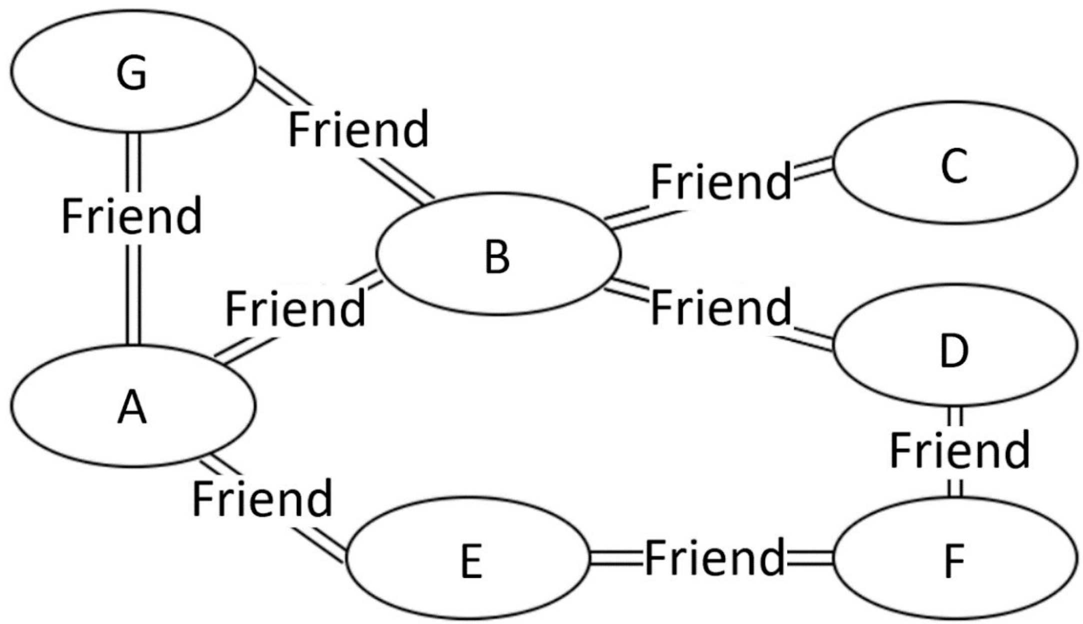
- Cormen, T.H., Leiserson, C.E., Rivest, R.L. and Stein, C., 2022. Introduction to algorithms. MIT press.
- Goodrich, M.T., Tamassia, R. and Goldwasser, M.H., 2013. Data structures and algorithms in Python. John Wiley & Sons Ltd.

Time complexity

- Amount of time needed for the algorithm to finish
- Best case
- Average case
- Worst case
- Not actual time: related to size of input.
- Big O notation

Graph

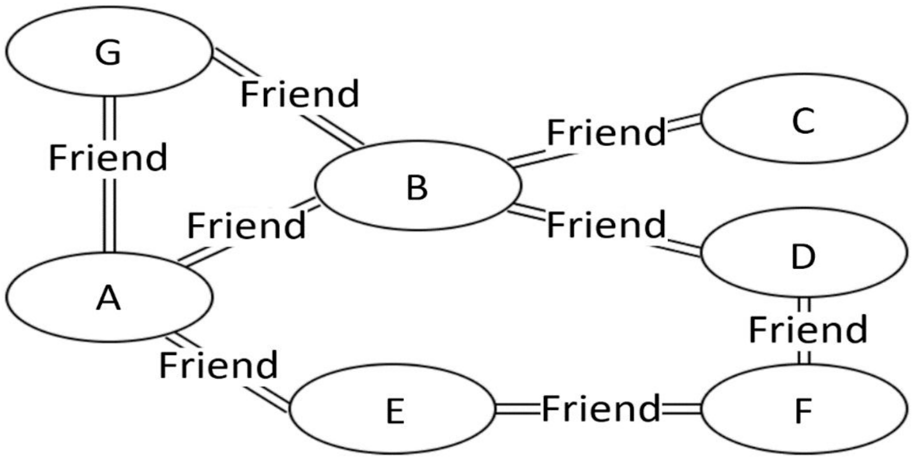
- Graph is probably the data structure that has the closest resemblance to our daily life.
- There are many types of graphs describing the relationships in real life.

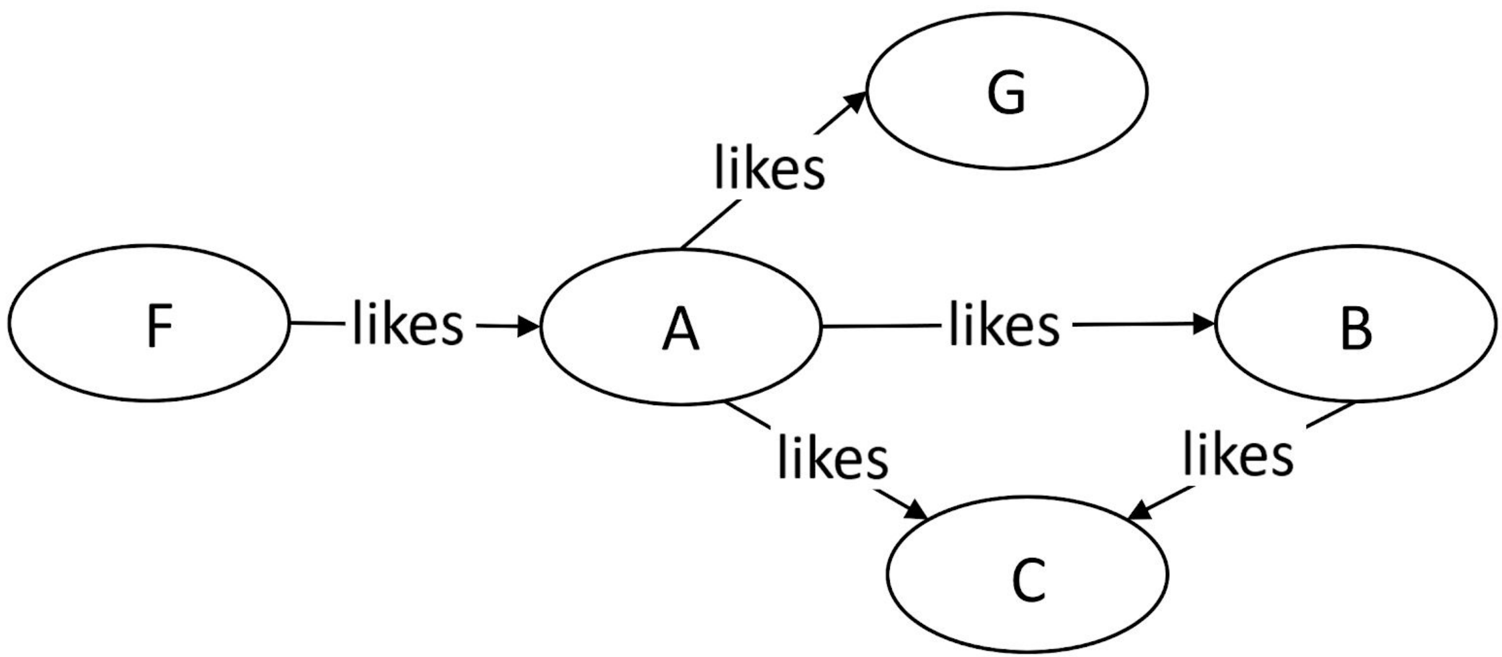


Graph Variations

- Variations:
  - A *connected graph* has a path from every vertex to every other
  - In an *undirected graph:*
    - Edge (u,v) = edge (v,u)
    - No self-loops
  - In a *directed* graph:
    - Edge (u,v) goes from vertex u to vertex v, notated $u \rightarrow v$

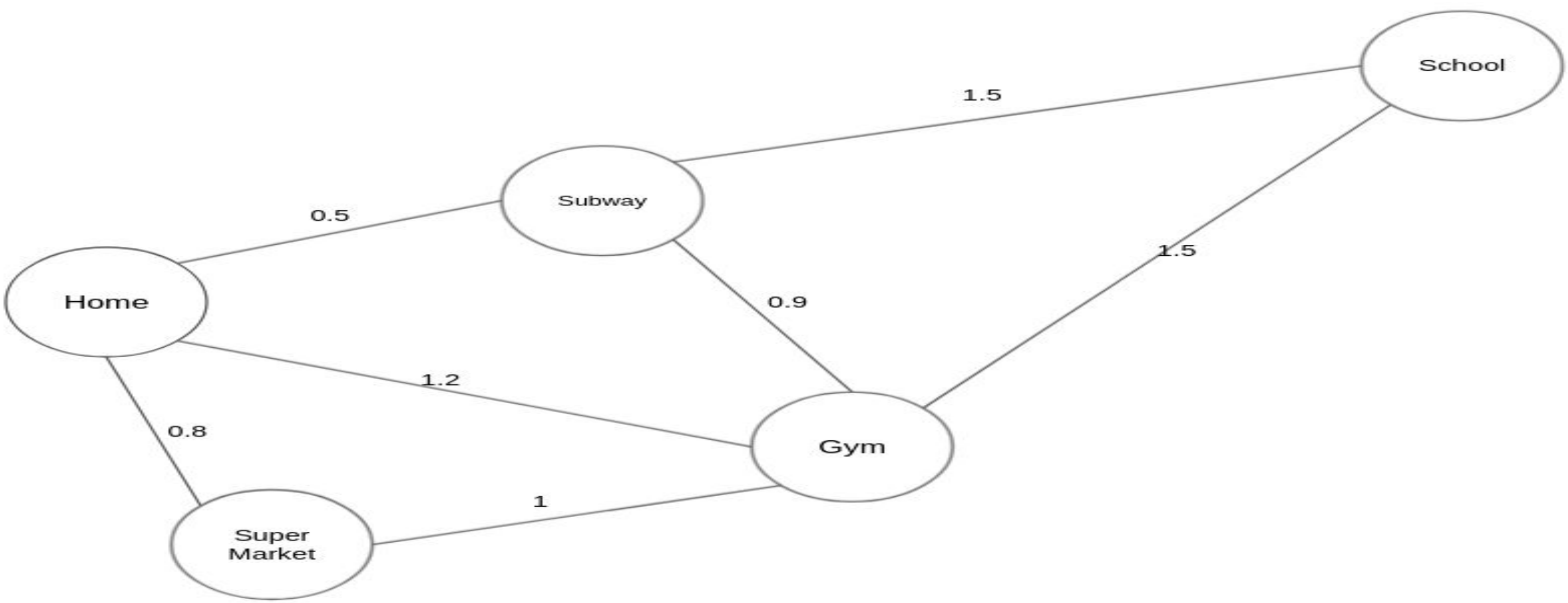
Graph Variations





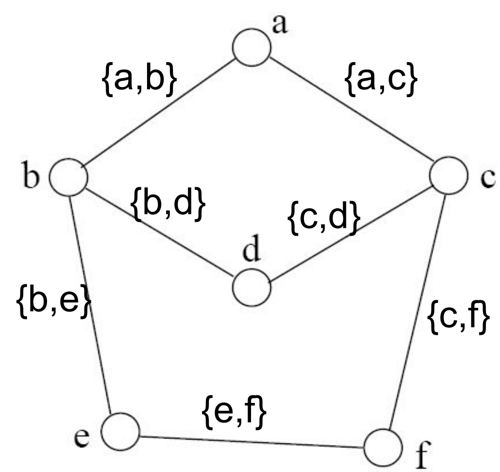
Graph Variations

- More variations:
  - A *weighted graph* associates weights with either the edges or the vertices
    - E.g., a road map: edges might be weighted w/ distance
  - A *multigraph* allows multiple edges between the same vertices
    - E.g., the call graph in a program (a function can get called from multiple points in another function)



Graph - Definition

- A graph G=(V, E) consists a set of vertices, V, and a set of edges, E.
- Each edge is a pair of *(v, w)*, where v, w belongs to V
- If the pair is unordered, the graph is undirected; otherwise it is directed



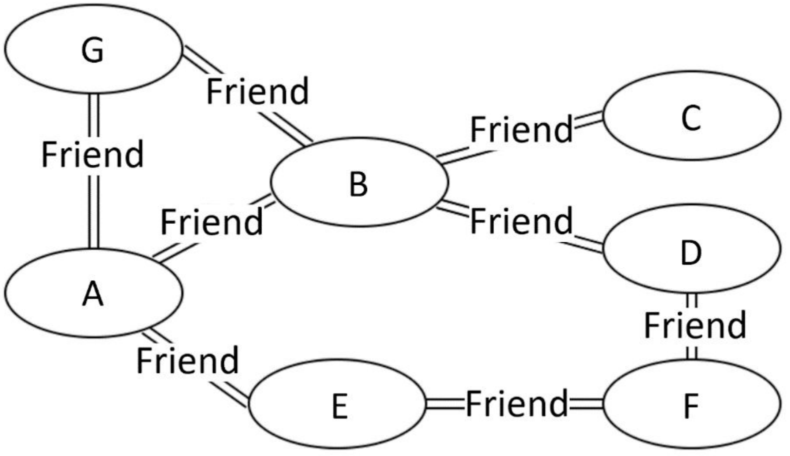
$$V = \{a, b, c, d, e, f\}$$

$$E = \{\{a, b\}, \{a, c\}, \{b, d\}, \{c, d\}, \{b, e\}, \{c, f\}, \{e, f\}\}$$

An undirected graph

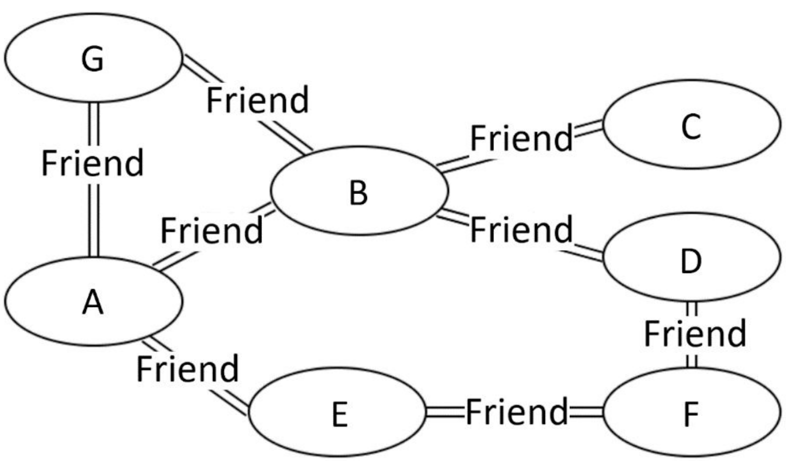
Definition

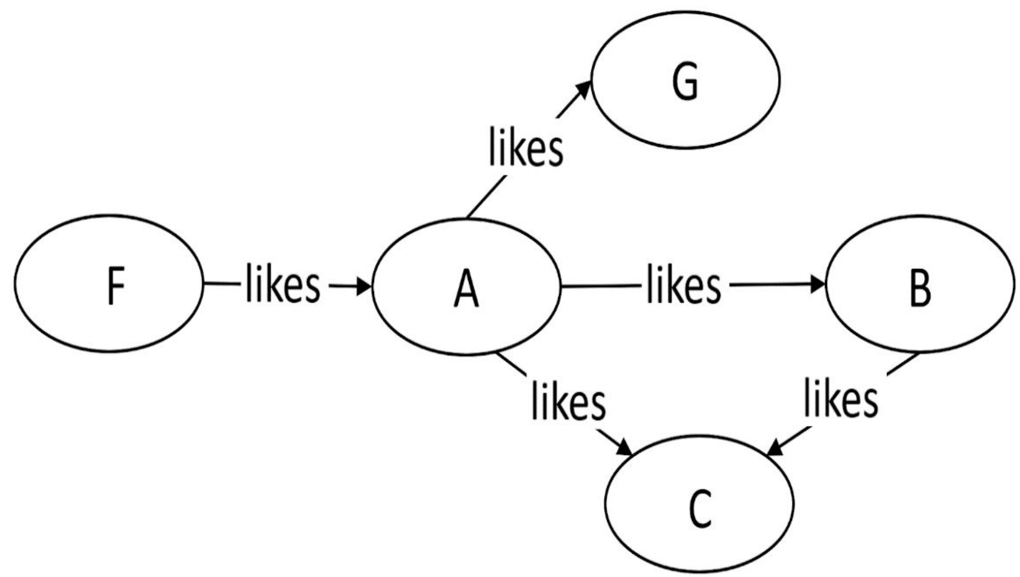
- **Path:** the sequence of vertices to go through from one vertex to another.
  - a path from A to C is [A, B, C], or [A, G, B, C], or [A, E, F, D, B, C].
- **Path Length:** the number of edges in a path.
- **Cycle:** a path where the starting point and endpoint are the same vertex.
  - [A, B, D, F, E] forms a cycle. Similarly, [A, G, B] forms another cycle.



Definition

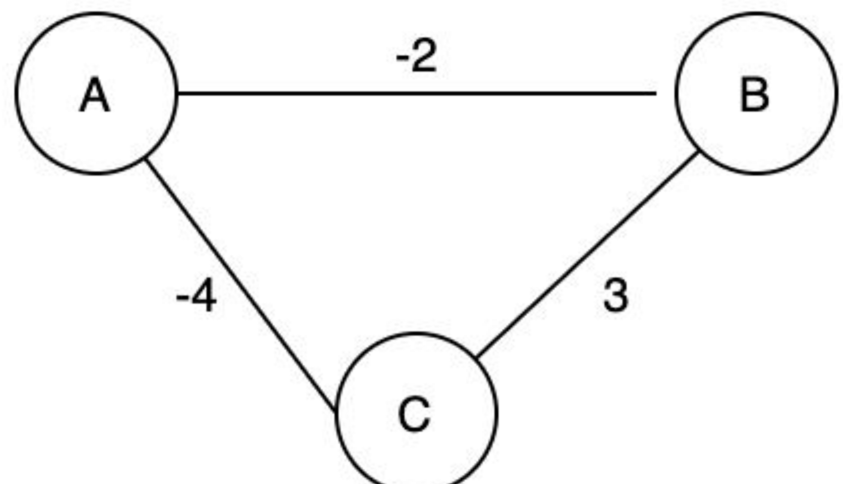
- **Degree of a Vertex:** the term “degree” applies to unweighted graphs. The degree of a vertex is the number of edges connecting the vertex.
  - the degree of vertex A is 3 because three edges are connecting it.
- **In-Degree:** “in-degree” is a concept in directed graphs. If the in-degree of a vertex is d, there are d directional edges incident to the vertex.
  - In Figure 2, A’s indegree is 1, i.e., the edge from F to A.
- **Out-Degree:** “out-degree” is a concept in directed graphs. If the out-degree of a vertex is d, there are d edges incident from the vertex.
  - A’s outdegree is 3, i,e, the edges A to B, A to C, and A to G.





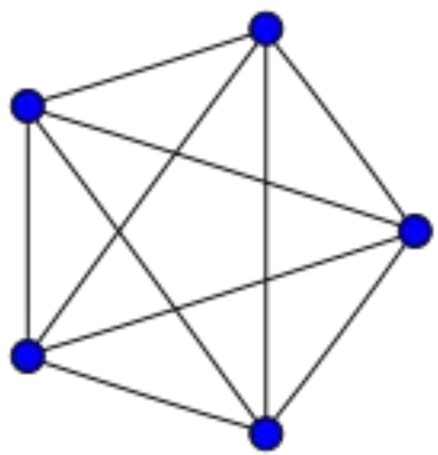
Definition

- **Connectivity**: if there exists at least one path between two vertices, these two vertices are connected.
  - A and C are connected because there is at least one path connecting them.
- **Negative Weight Cycle:** In a "weighted graph", if the sum of the weights of all edges of a cycle is a negative value, it is a negative weight cycle.
  - In the Figure the sum of weights is -3.



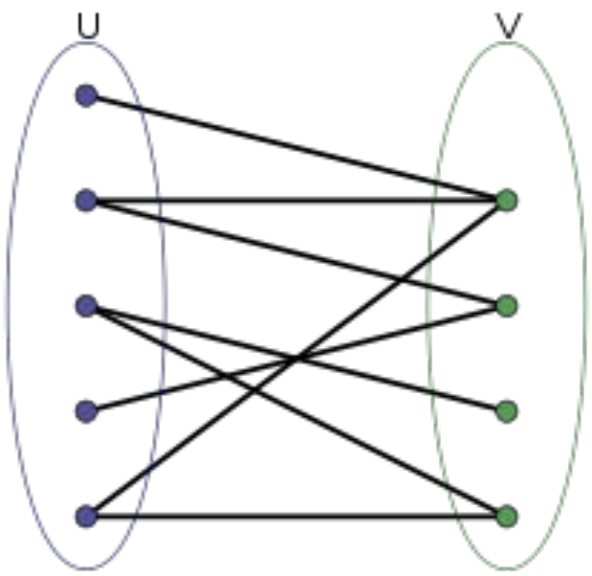
Definition

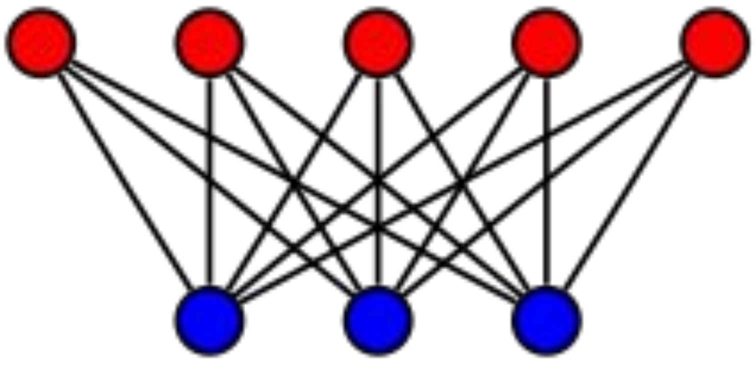
- Complete Graph
  - a complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge.
  - A complete digraph is a directed graph in which every pair of distinct vertices is connected by a pair of unique edges (one in each direction).
  - How many edges are there in an N-vertex complete graph?



Definition

- Bipartite Graph
- a bipartite graph (or bigraph) is a graph whose vertices can be divided into two disjoint and independent sets





Definition

- We will typically express running times in terms of |E| and |V| (often dropping the |’s)
  - If $|E| \approx |V|^2$ the graph is *dense*
  - If $|E| \approx |V|$ the graph is *sparse*
- If you know you are dealing with dense or sparse graphs, different data structures may make sense

Graph Representation

- Two popular computer representations of a graph. Both represent the vertex set and the edge set, but in different ways.

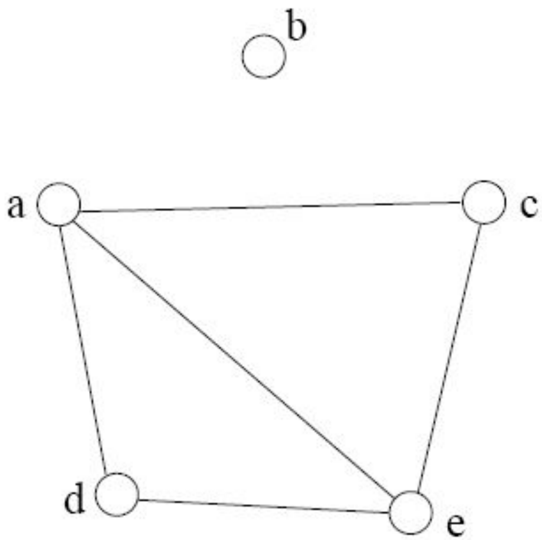
1. Adjacency Matrix

   Use a 2D matrix to represent the graph

1. Adjacency List

   Use a 1D array of linked lists

Adjacency Matrix



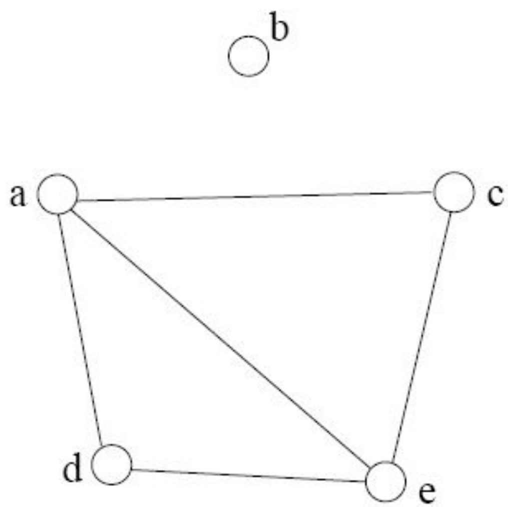
	a	b	c	d	e
a	0	0	1	1	1
b	0	0	0	0	0
c	1	0	0	0	1
d	1	0	0	0	1
e	1	0	1	1	0

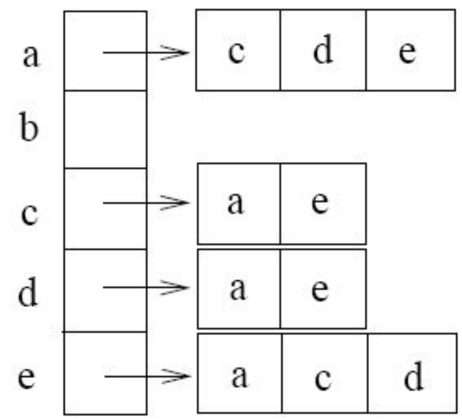
- I

Simple Questions on Adjacency Matrix

- Is there a direct link between A and B?
- What is the indegree and outdegree for a vertex A?
- How many nodes are directly connected to vertex A?
- Is it an undirected graph or directed graph?
- Suppose ADJ is an NxN matrix. What will be the result if we create another matrix ADJ2 where ADJ2=ADJxADJ?

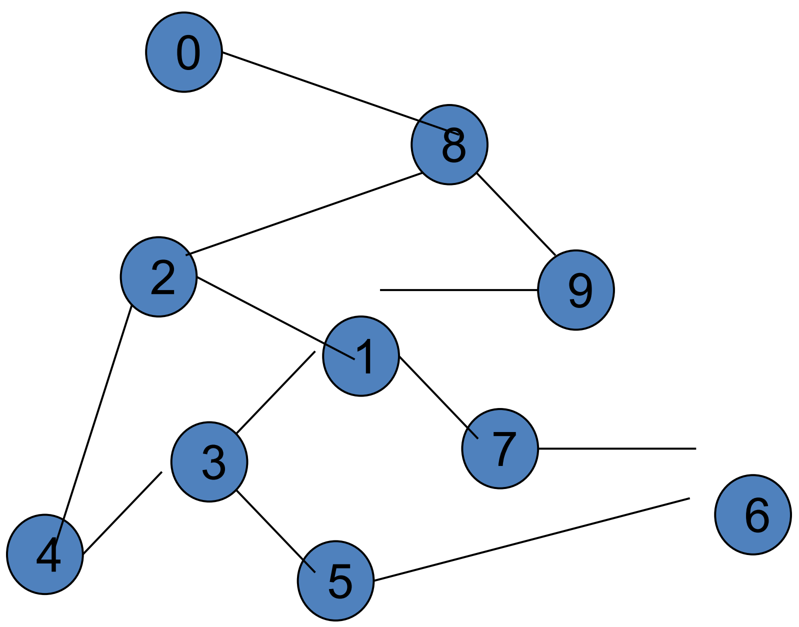
Adjacency List





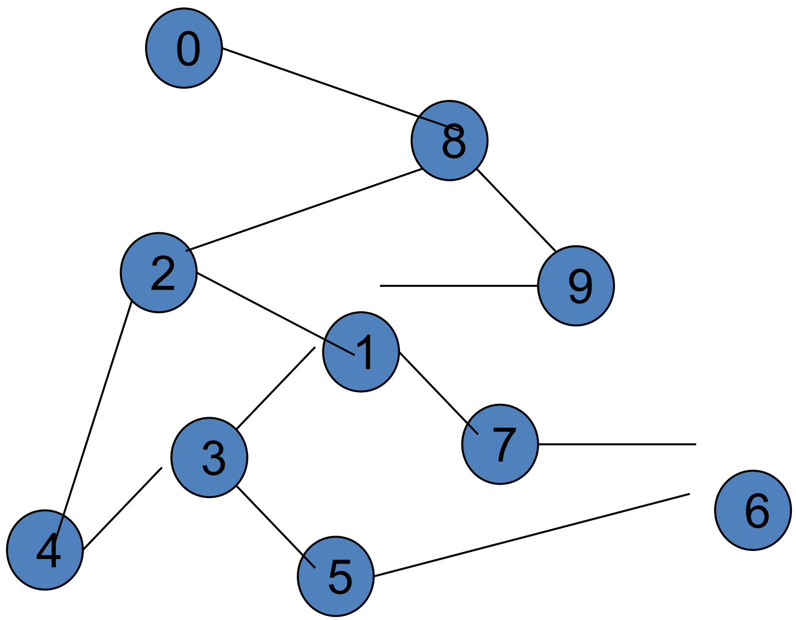
- If the graph is not dense, in other words, sparse, a better solution is an adjacency list

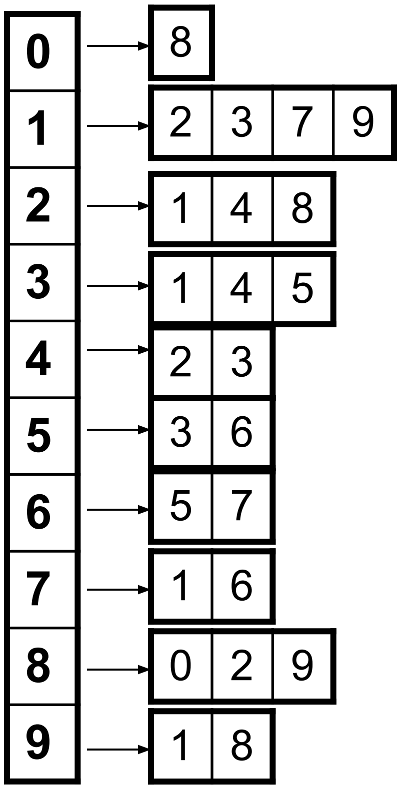
Adjacency Matrix Example



	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	0	0	0	0	0	0	0	1	0
1	0	0	1	1	0	0	0	1	0	1
2	0	1	0	0	1	0	0	0	1	0
3	0	1	0	0	1	1	0	0	0	0
4	0	0	1	1	0	0	0	0	0	0
5	0	0	0	1	0	0	1	0	0	0
6	0	0	0	0	0	1	0	1	0	0
7	0	1	0	0	0	0	1	0	0	0
8	1	0	1	0	0	0	0	0	0	1
9	0	1	0	0	0	0	0	0	1	0

Adjacency List Example





Storage of Adjacency List

- The array takes up $\Theta(n)$ space
- Define degree of *v*, deg(*v*), to be the number of edges incident to *v*. Then, the total space to store the graph is proportional to:

$$\sum_{\text{vertex } v} \deg(v)$$

- An edge *e={u,v}* of the graph contributes a count of 1 to deg(*u*) and contributes a count 1 to deg(*v*)
- Therefore, $\Sigma_{\text{vertex } v}\deg(v) = 2m$, where *m* is the total number of edges
- In all, the adjacency list takes up $\Theta(n+m)$ space
  - If $m = O(n^2)$ (i.e. dense graphs), both adjacent matrix and adjacent lists use $\Theta(n^2)$ space.
  - If $m = O(n)$, adjacent list outperform adjacent matrix

- However, one cannot tell in O(1) time whether two vertices are connected

Adjacency List vs. Matrix

- **Adjacency List**
  - More compact than adjacency matrices if graph has few edges
  - Requires more time to find if an edge exists
- **Adjacency Matrix**
  - Always require n^2 space
    - This can waste a lot of space if the number of edges are sparse
  - Can quickly find if an edge exists

Path between Vertices

- A path is a sequence of vertices (v_0, v_1, v_2,… v_k) such that:
  - For $0 \leq i < k$, $\{v_i, v_{i+1}\}$ is an edge

Note: a path is allowed to go through the same vertex or the same edge any number of times!

- The length of a path is the number of edges on the path

Types of paths

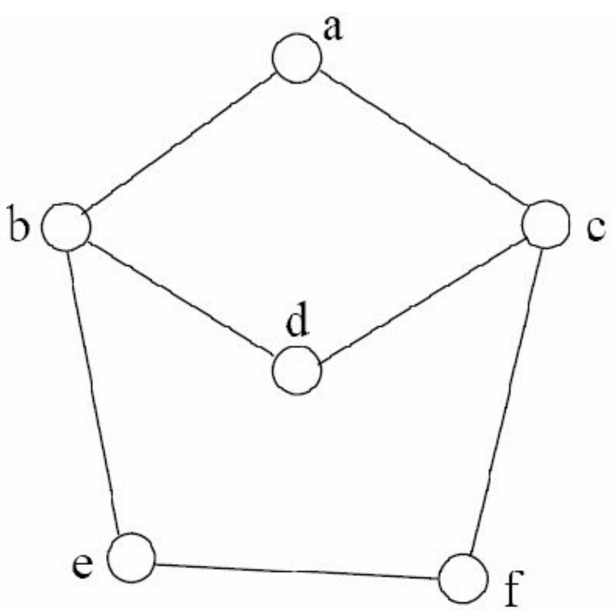
- A path is simple if and only if it does not contain a vertex more than once.
- A path is a cycle if and only if $v_0 = v_k$
  - The beginning and end are the same vertex!
- A path contains a cycle as its sub-path if some vertex appears twice or more

Path Examples

Are these paths?

Any cycles?

What is the path’s length?



1\. {a,c,f,e}

1\. {a,b,d,c,f,e}

1\. {a, c, d, b, d, c, f, e}

2\. {a,c,d,b,a}

1\. {a,c,f,e,b,d,c,a}

Exercises on Graph

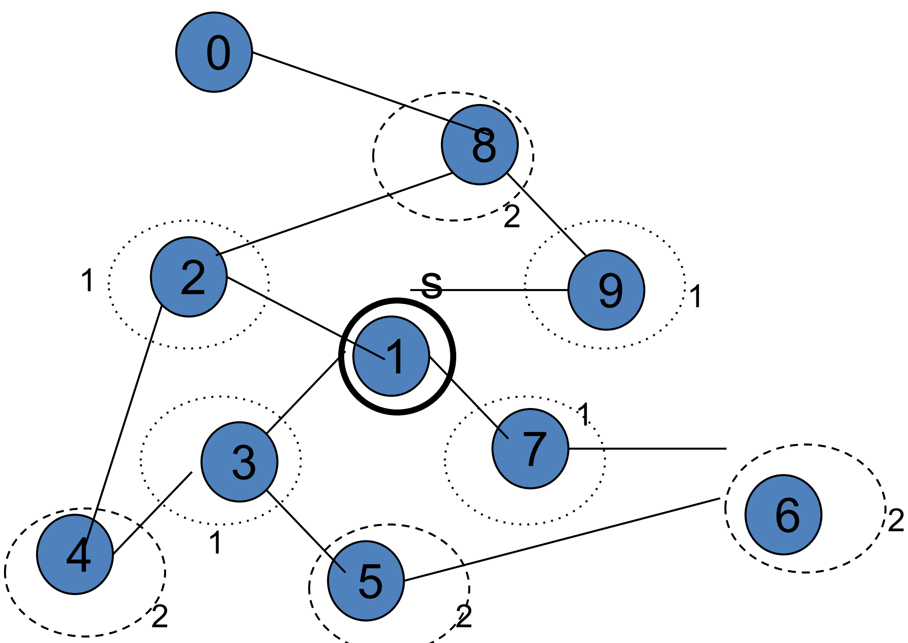
- CLRS – Chapter 22 – elementary Graph Algorithms
- Exercise you have to solve: (Page 593)
  - 22.1-5 (Square)
  - 22.1-6 (Universal Sink)

Graph Traversal

- Application example
  - Given a graph representation and a vertex **s** in the graph
  - Find paths from **s** to other vertices
- Two common graph traversal algorithms
  - Breadth-First Search (BFS)
    - Find the shortest paths in an unweighted graph
  - Depth-First Search (DFS)
    - Topological sort
    - Find strongly connected components

BFS and Shortest Path Problem

- Given any source vertex ***s***, BFS visits the other vertices at increasing distances away from s. In doing so, BFS discovers paths from s to other vertices
- What do we mean by "distance"? The number of edges on a path from s



Example

Consider s=vertex 1

Nodes at distance 1?
2, 3, 7, 9

Nodes at distance 2?
8, 6, 5, 4

Nodes at distance 3?
0

Graph Searching

- Given: a graph $G = (V, E)$, directed or undirected
- Goal: methodically explore every vertex and every edge
- Ultimately: build a tree on the graph
  - Pick a vertex as the root
  - Choose certain edges to produce a tree
  - Note: might also build a *forest* if graph is not connected

Breadth-First Search

- "Explore" a graph, turning it into a tree
  - One vertex at a time
  - Expand frontier of explored vertices across the *breadth* of the frontier
- Builds a tree over the graph
  - Pick a *source vertex* to be the root
  - Find ("discover") its children, then their children, etc.

Breadth-First Search

- Every vertex of a graph contains a color at every moment:
  - White vertices have not been discovered
    - All vertices start with white initially
  - Grey vertices are discovered but not fully explored
    - They may be adjacent to white vertices
  - Black vertices are discovered and fully explored
    - They are adjacent only to black and gray vertices
- Explore vertices by scanning adjacency list of grey vertices

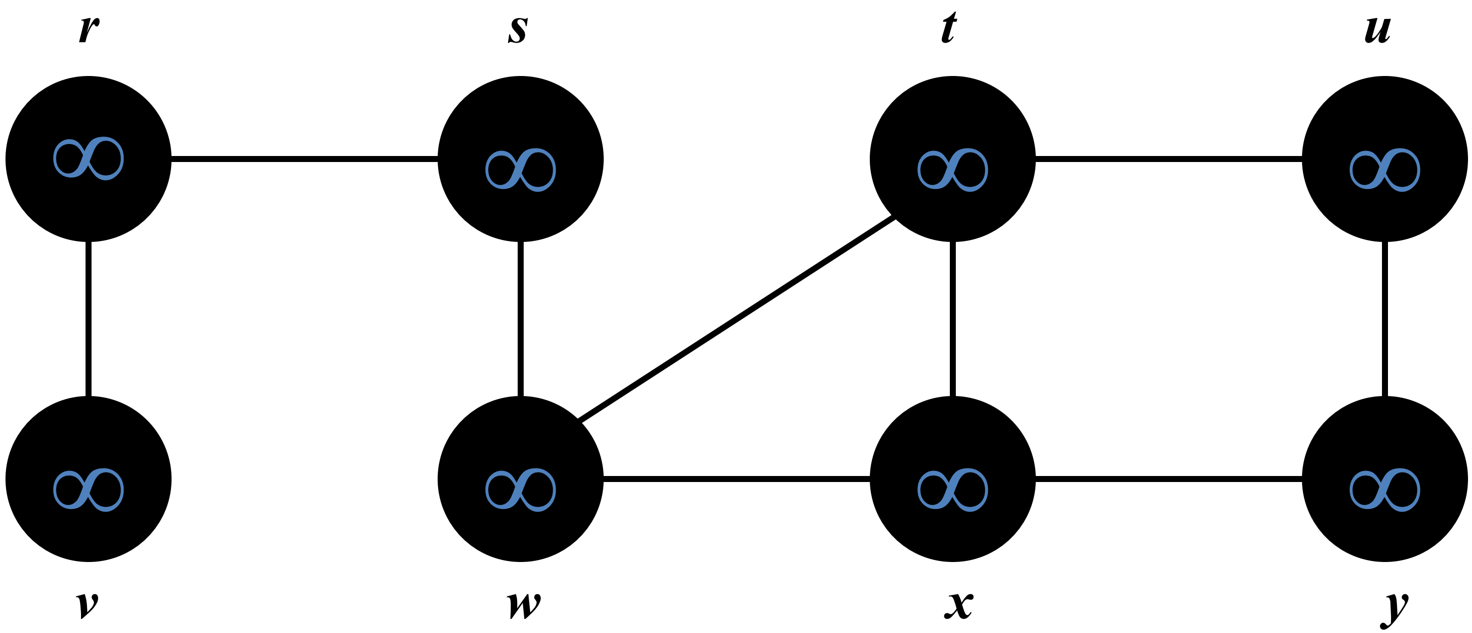
Breadth-First Search: The Code

```
Data: color[V], prev[V],d[V]

BFS(G) // starts from here
{
    for each vertex u ∈
   V-{s}
    {
       color[u]=WHITE;
    prev[u]=NIL;
    d[u]=inf;
    }
    color[s]=GRAY;
   d[s]=0; prev[s]=NIL;
   Q=empty;
   ENQUEUE(Q,s);
```

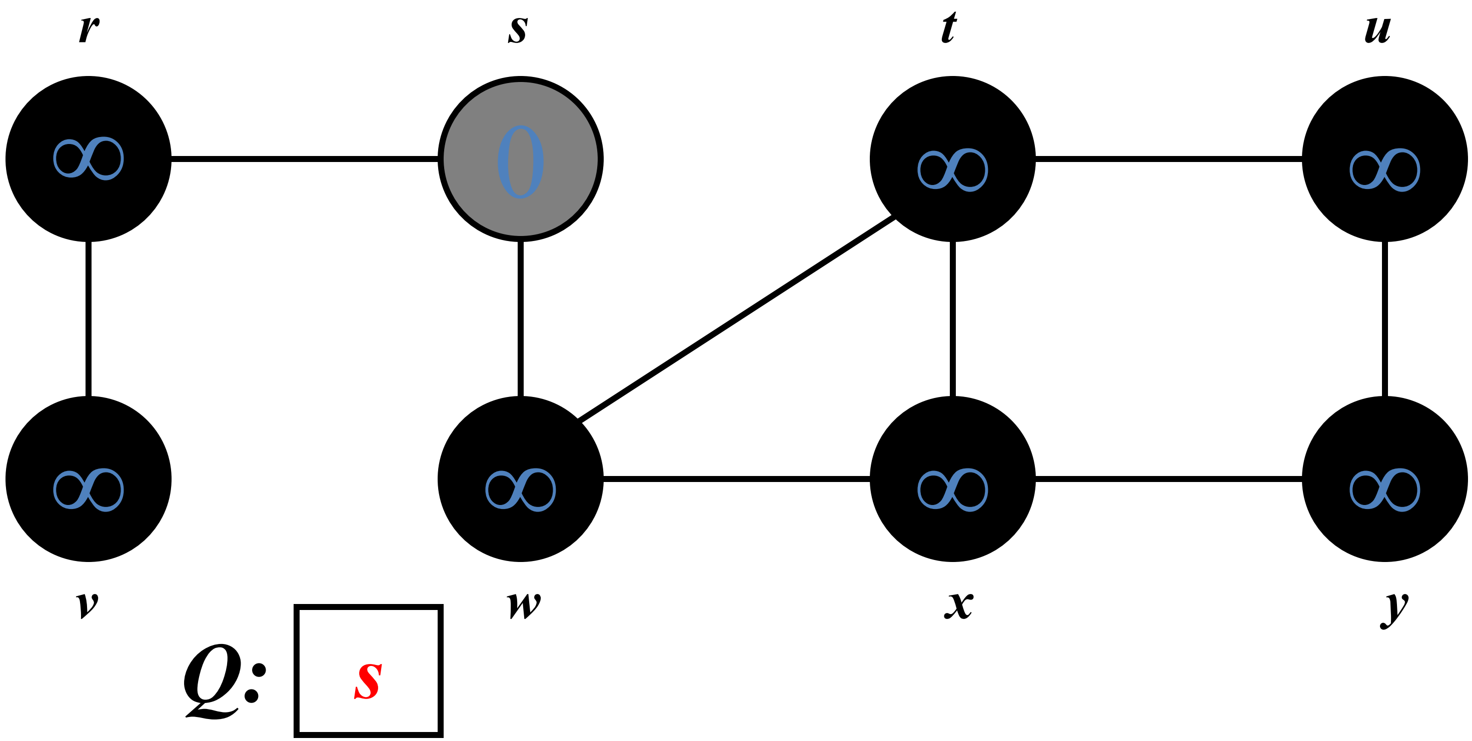
```
While(Q not empty)
{
   u = DEQUEUE(Q);
   for each v ∈ adj[u]{
     if (color[v] == WHITE){
         color[v] = GREY;
         d[v] = d[u] + 1;
         prev[v] = u;
         Enqueue(Q, v);
     }
   }
   color[u] = BLACK;
}
}
```

Breadth-First Search: Example



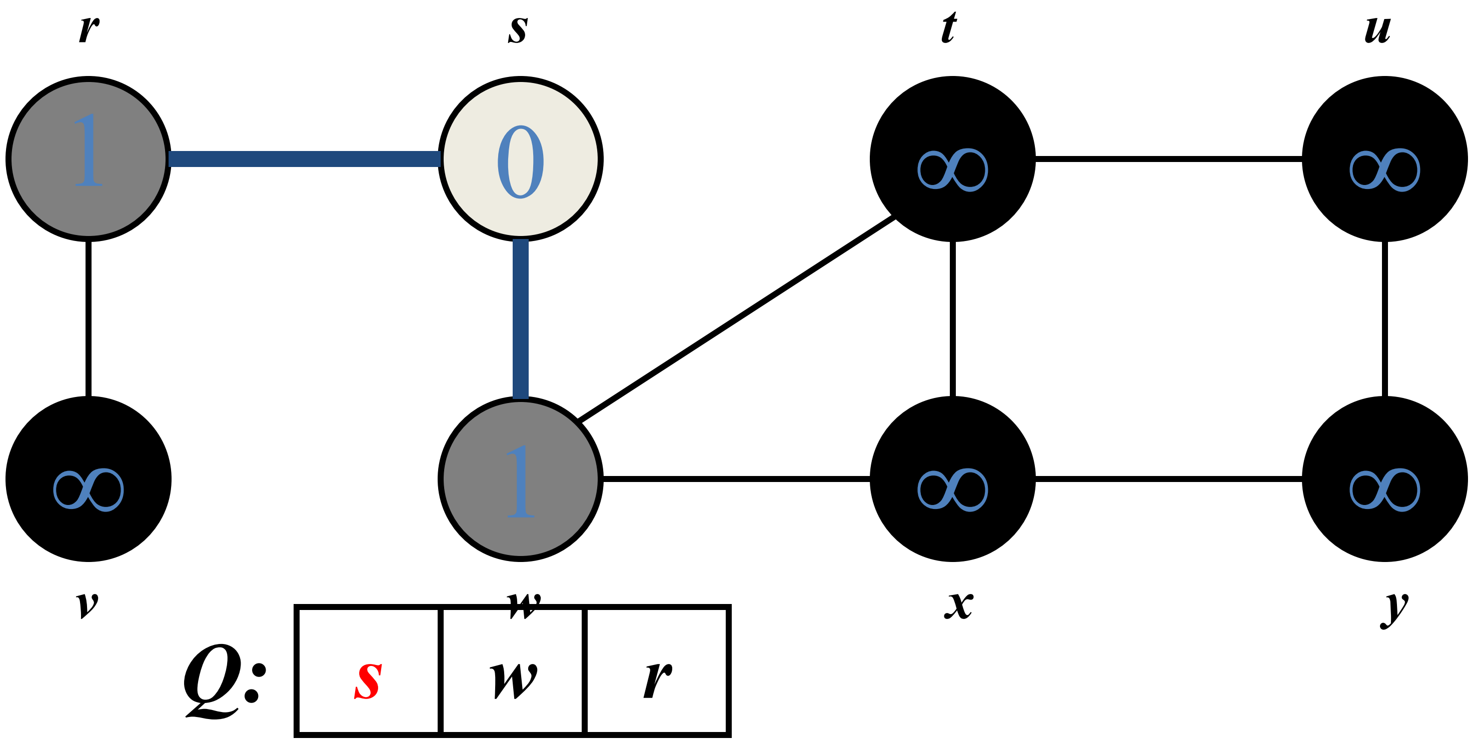
Vertex	r	s	t	u	v	w	x	y
color	W	W	W	W	W	W	W	W
d	∞	∞	∞	∞	∞	∞	∞	∞
prev	nil	nil	nil	nil	nil	nil	nil	nil

Breadth-First Search: Example



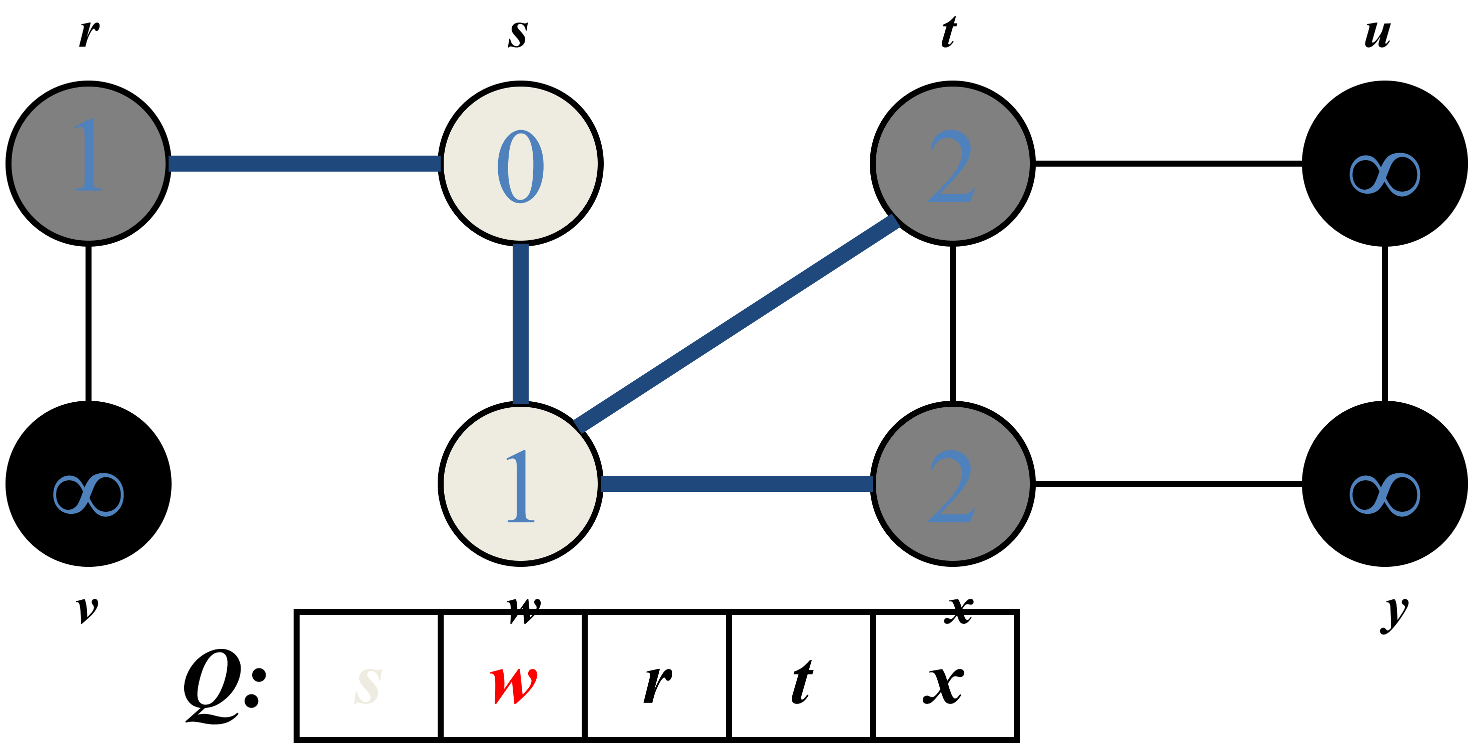
vertex	r	s	t	u	v	w	x	y
Color	W	**G**	W	W	W	W	W	W
d	∞	**0**	∞	∞	∞	∞	∞	∞
prev	nil	**nil**	nil	nil	nil	nil	nil	nil

Breadth-First Search: Example



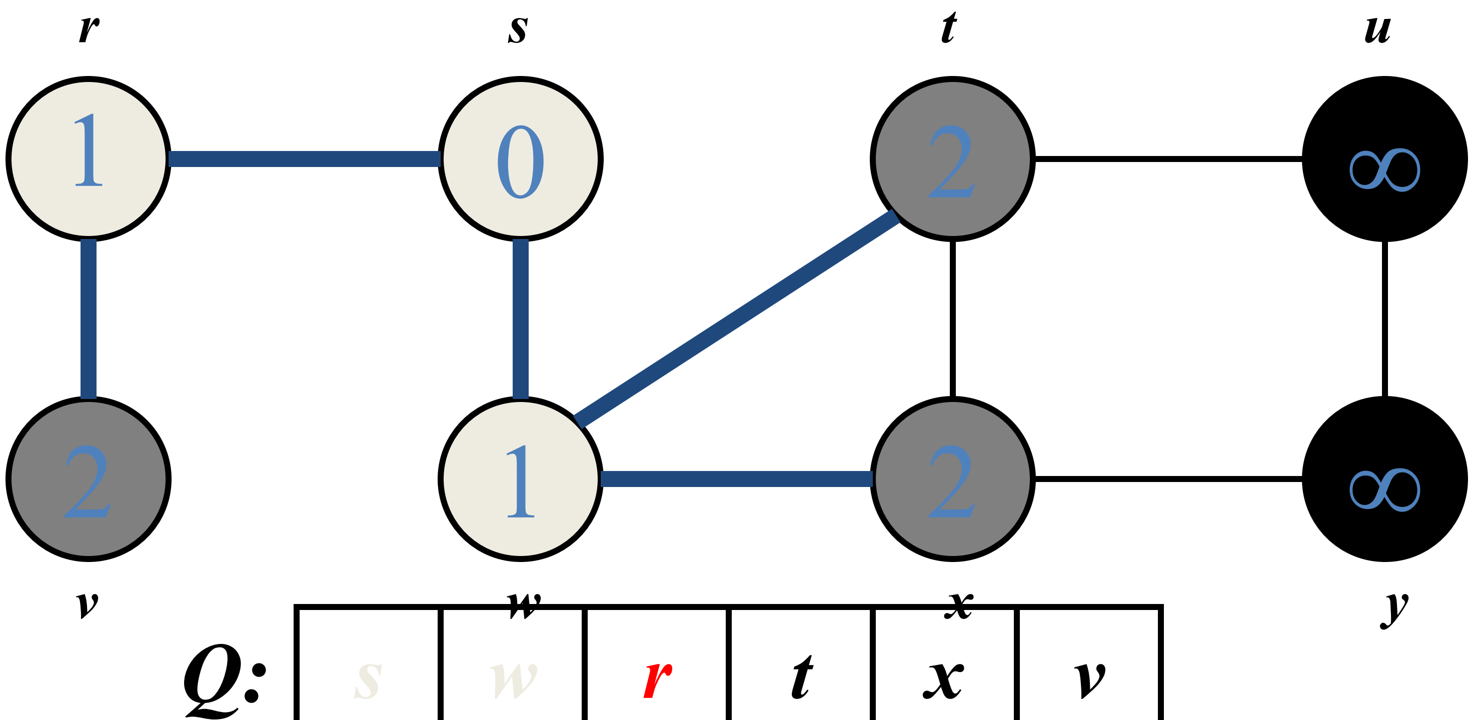
vertex	r	s	t	u	v	w	x	y
Color	G	B	W	W	W	G	W	W
d	1	0	∞	∞	∞	1	∞	∞
prev	s	nil	nil	nil	nil	s	nil	nil

Breadth-First Search: Example



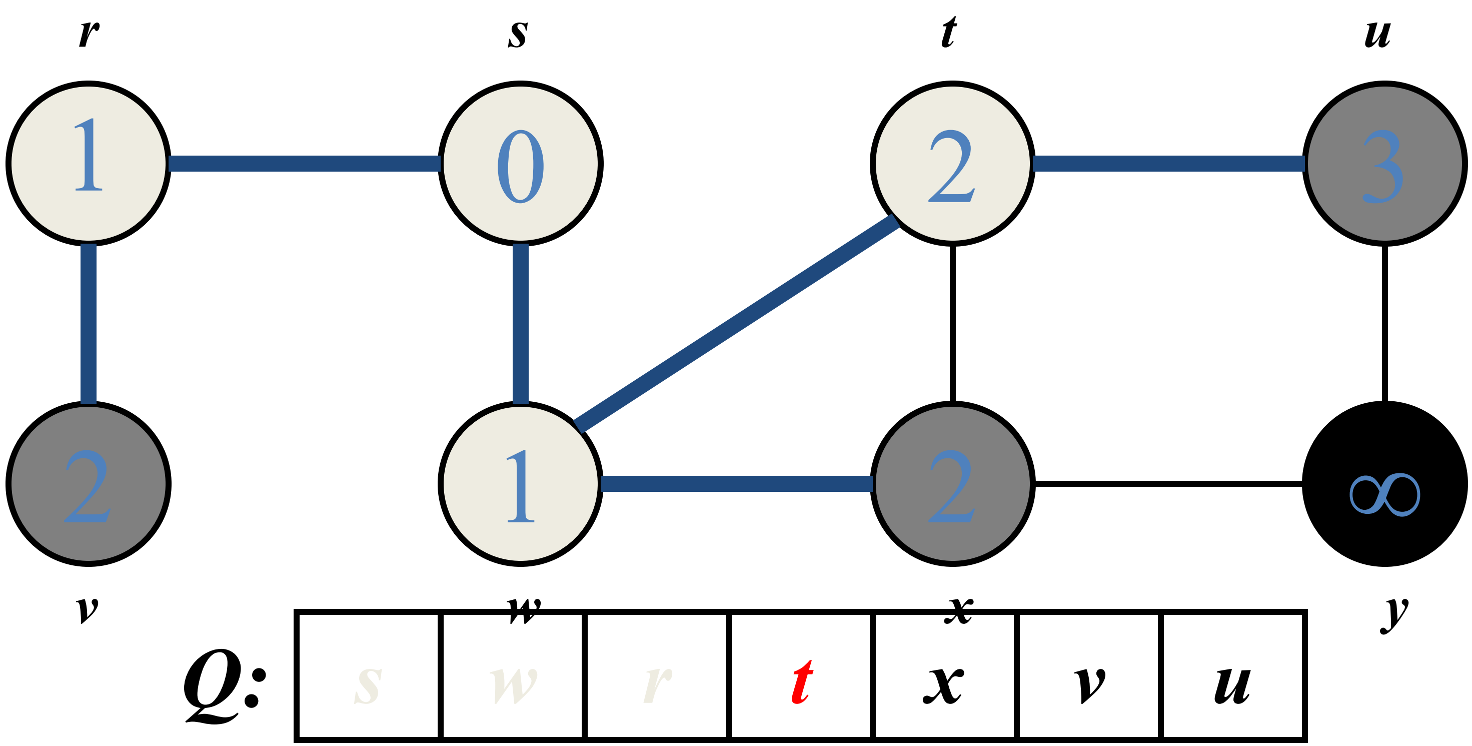
vertex	r	s	t	u	V	w	X	y
Color	G	B	**G**	W	W	**B**	**G**	W
d	1	0	**2**	∞	∞	**1**	**2**	∞
prev	s	nil	**w**	nil	nil	**s**	**w**	nil

Breadth-First Search: Example



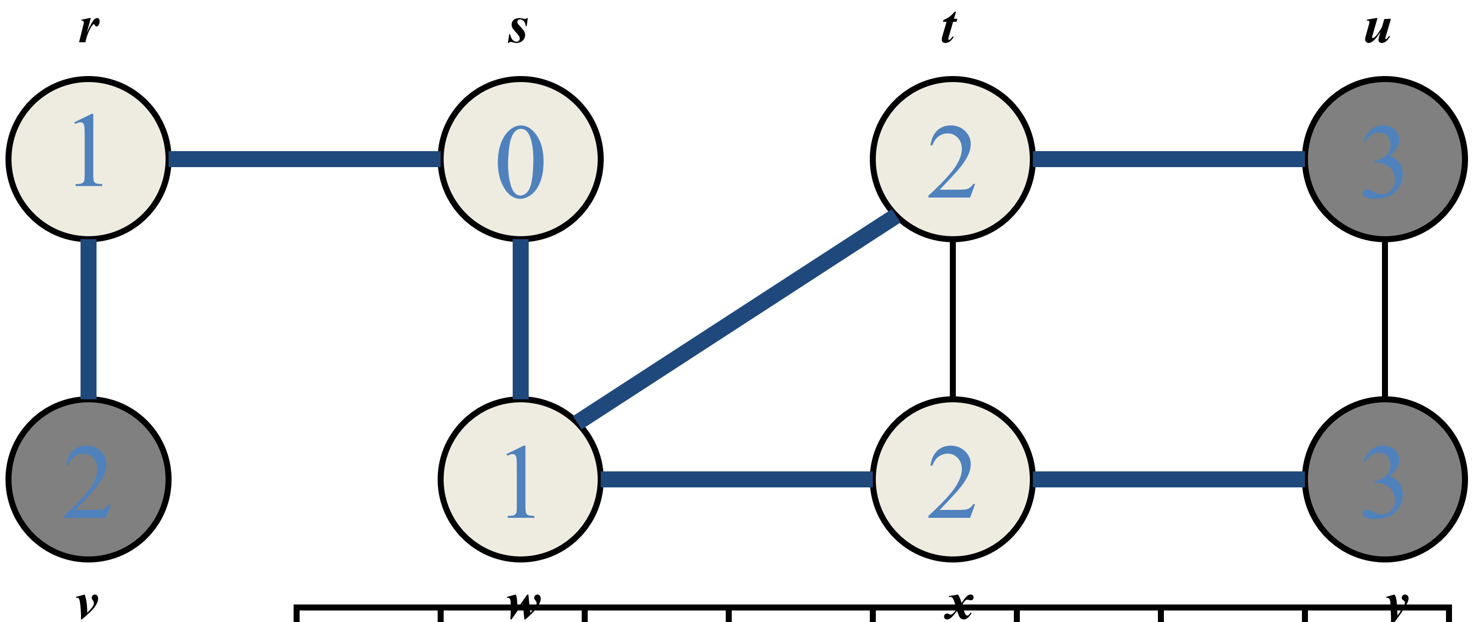
vertex	r	s	t	u	v	w	x	y
Color	**B**	B	G	W	**G**	B	G	W
d	**1**	0	2	∞	**2**	1	2	∞
prev	**s**	nil	w	nil	**r**	s	w	nil

Breadth-First Search: Example



vertex	r	s	t	u	v	w	x	y
Color	B	B	**B**	**G**	G	B	G	W
d	1	0	**2**	**3**	2	1	2	∞
prev	s	nil	**w**	**t**	r	s	w	nil

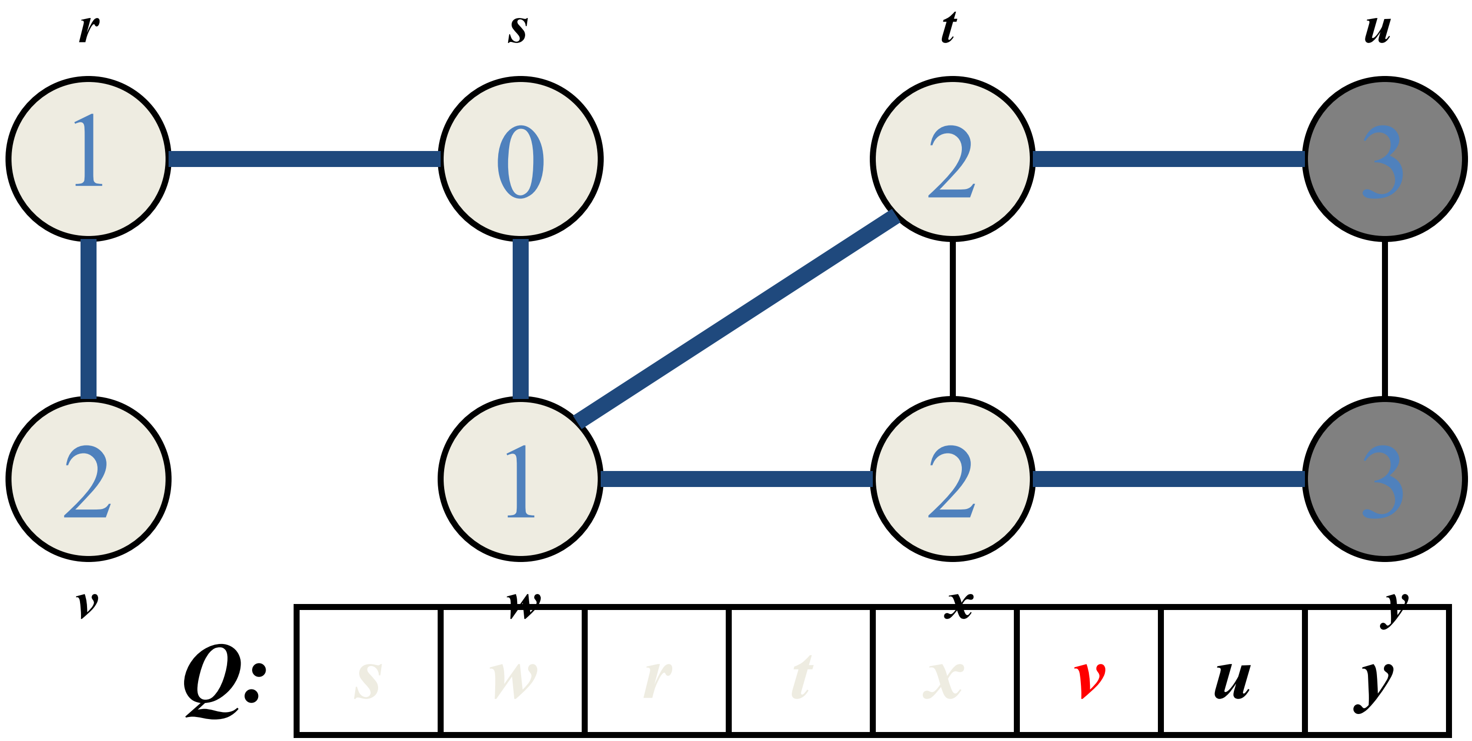
Breadth-First Search: Example



Q: | *s* | *w* | *r* | *t* | *x* | *v* | *u* | *y* |

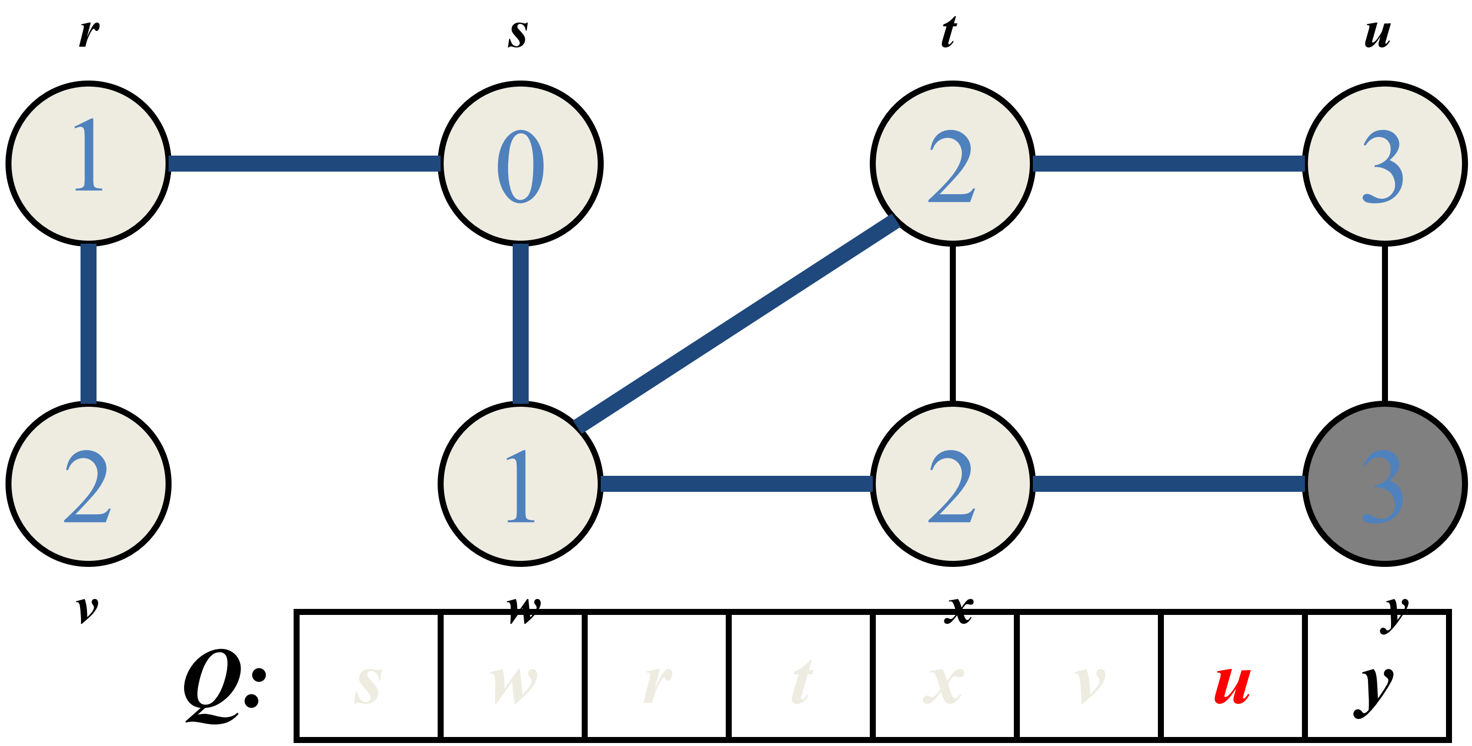
vertex	r	s	t	u	v	w	x	y
Color	B	B	B	G	G	B	**B**	**G**
d	1	0	2	3	2	1	**2**	**3**
prev	s	nil	w	t	r	s	**w**	**x**

Breadth-First Search: Example



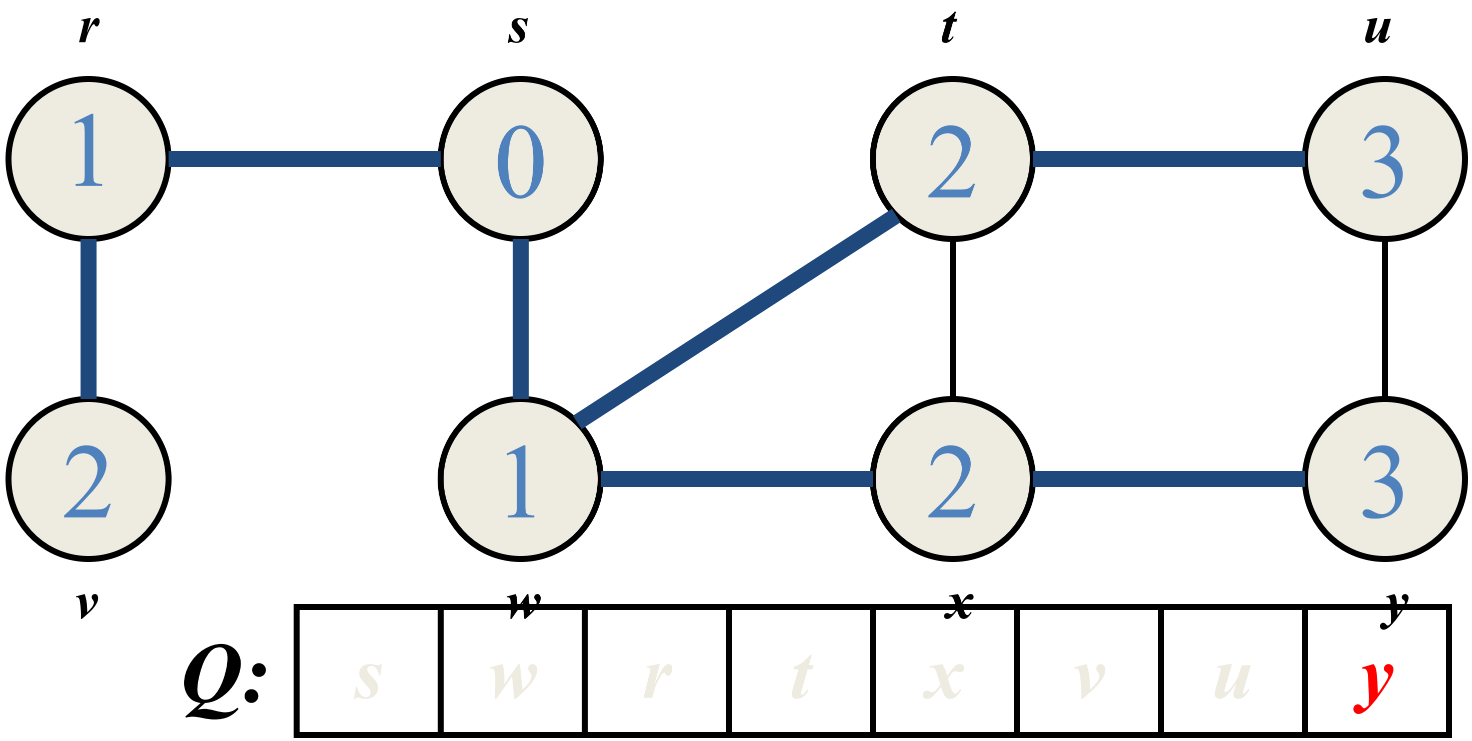
vertex	r	s	t	u	v	w	x	y
Color	B	B	B	G	**B**	B	B	G
d	1	0	2	3	**2**	1	2	3
prev	s	nil	w	t	**r**	s	w	x

Breadth-First Search: Example



vertex	r	s	t	u	v	w	x	y
Color	B	B	B	**B**	B	B	B	G
d	1	0	2	**3**	2	1	2	3
prev	s	nil	w	**t**	r	s	w	x

Breadth-First Search: Example



vertex	r	s	t	u	v	w	x	y
Color	B	B	B	G	B	B	B	**B**
d	1	0	2	3	2	1	2	**3**
prev	s	nil	w	t	r	s	w	**x**

BFS: The Code (again)

```
Data: color[V], prev[V],d[V]

BFS(G) // starts from here
{
    for each vertex u ∈
   V-{s}
    {
       color[u]=WHITE;
    prev[u]=NIL;
    d[u]=inf;
    }
    color[s]=GRAY;
   d[s]=0; prev[s]=NIL;
   Q=empty;
   ENQUEUE(Q,s);
```

```
While(Q not empty)
{
   u = DEQUEUE(Q);
   for each v ∈ adj[u]{
     if (color[v] == WHITE){
         color[v] = GREY;
         d[v] = d[u] + 1;
         prev[v] = u;
         Enqueue(Q, v);
     }
   }
   color[u] = BLACK;
 }
}
```

Breadth-First Search: Print Path

```
Data: color[V], prev[V],d[V]
```

```
Print-Path(G, s, v)
{
   if(v==s)
    print(s)
   else if(prev[v]==NIL)
    print(No path);
   else{
    Print-Path(G,s,prev[v]);
    print(v);
   }
}
```

BFS: Complexity

```
Data: color[V], prev[V],d[V]

BFS(G) // starts from here
{
    for each vertex u ∈
   V-{s}
    {
        color[u]=WHITE;
    prev[u]=NIL;
    d[u]=inf;
    }
    color[s]=GRAY;
   d[s]=0; prev[s]=NIL;
   Q=empty;
   ENQUEUE(Q,s);
```

O(V)

```
While(Q not empty)
{
   u = DEQUEUE(Q);
   for each v ∈ adj[u]{
    if(color[v] == WHITE){
          color[v] = GREY;
          d[v] = d[u] + 1;
          prev[v] = u;
          Enqueue(Q, v);
     }
   }
   color[u] = BLACK;
 }
}
```

u = every vertex, but only once

(Why?)

O(V)

What will be the running time?

Total running time: O(V+E)

Breadth-First Search: Properties

- BFS calculates the *shortest-path distance* to the source node
  - Shortest-path distance $\delta(s,v)$ = minimum number of edges from s to v, or ∞ if v not reachable from s
  - Proof given in the book (p. 472-5)
- BFS builds *breadth-first tree*, in which paths to root represent shortest paths in G
  - Thus can use BFS to calculate shortest path from one vertex to another in O(V+E) time

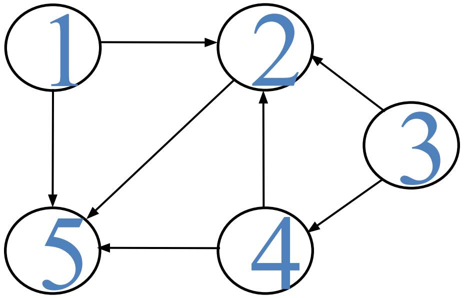
Application of BFS

- Find the shortest path in an undirected/directed unweighted graph.
- Find the bipartite-ness of a graph.
- Find cycle in a graph.
- Find the connectedness of a graph.
- And many more.

Exercises on BFS

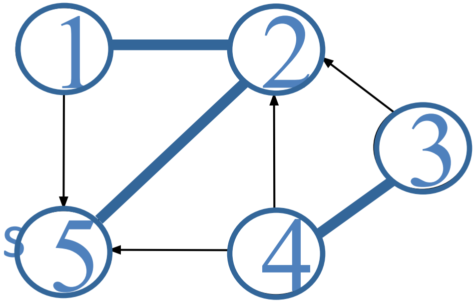
- CLRS – Chapter 22 – elementary Graph Algorithms
- Exercise you have to solve: (Page 602)
  - 22.2-7 (Wrestler)
  - 22.2-8 (Diameter)
  - 22.2-9 (Traverse)

Depth-First Search



- **Input:**
  - G = (V, E) (No source vertex given!)
- **Goal**:
  - Explore the edges of G to "discover" every vertex in V starting at the most current visited node
  - Search may be repeated from multiple sources
- **Output:**
  - 2 **timestamps** on each vertex:
    - d[v] = discovery time
    - f[v] = finishing time (done with examining v's adjacency list)
  - Depth-first forest

Depth-First Search

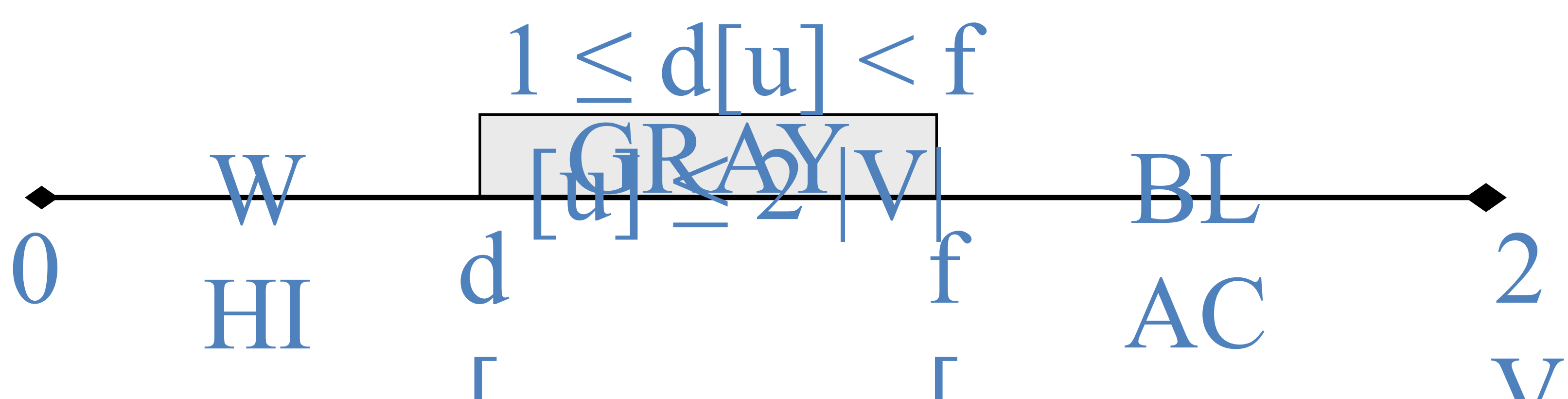


- Search "deeper" in the graph whenever possible
- Edges are explored out of the most recently discovered vertex v that still has unexplored edges

- After all edges of v have been explored, the search "backtracks" from the parent of v
- The process continues until all vertices reachable from the original source have been discovered
- If undiscovered vertices remain, choose one of them as a new source and repeat the search from that vertex
- DFS creates a "depth-first forest"

DFS Additional Data Structures

- Global variable: time-stamp
  - Incremented when nodes are discovered or finished
- color[u] – similar to BFS
  - White before discovery, gray while processing and black when finished processing
- prev[u] – predecessor of u
- d[u], f[u] – discovery and finish times



Depth-First Search: The Code

```
Data: color[V], time,
   prev[V],d[V], f[V]
DFS(G) // where prog starts
{
                                   Initialize
   for each vertex u ∈ V
   {
      color[u] = WHITE;
   prev[u]=NIL;
   f[u]=inf; d[u]=inf;
   }
   time = 0;
   for each vertex u ∈ V
     if (color[u] == WHITE)
         DFS_Visit(u);
}
```

```
DFS_Visit(u)
{
    color[u] = GREY;
    time = time+1;
    d[u] = time;
    for each v ∈ Adj[u]
    {
       if(color[v] == WHITE){
      prev[v]=u;
           DFS_Visit(v);}
    }
    color[u] = BLACK;
    time = time+1;
    f[u] = time;
}
```

Depth-First Search: The Code

```
    Data: color[V], time,
      prev[V],d[V], f[V]
DFS(G) // where prog starts
{
   for each vertex u ∈ V
   {
      color[u] = WHITE;
   prev[u]=NIL;
   f[u]=inf; d[u]=inf;
   }
   time = 0;
   for each vertex u ∈ V
     if (color[u] == WHITE)
         DFS_Visit(u);
}
```

```
DFS_Visit(u)
{
   color[u] = GREY;
   time = time+1;
   d[u] = time;
   for each v ∈ Adj[u]
   {
      if(color[v] == WHITE){
      prev[v]=u;
         DFS_Visit(v);}
   }
   color[u] = BLACK;
   time = time+1;
   f[u] = time;
}
```

What does `d[u]` represent?

Depth-First Search: The Code

```
Data: color[V], time,
   prev[V],d[V], f[V]
DFS(G) // where prog starts
{
   for each vertex u ∈ V
   {
      color[u] = WHITE;
   prev[u]=NIL;
   f[u]=inf; d[u]=inf;
   }
   time = 0;
   for each vertex u ∈ V
     if (color[u] == WHITE)
         DFS_Visit(u);
}
```

```
DFS_Visit(u)
{
   color[u] = GREY;
   time = time+1;
   d[u] = time;
   for each v ∈ Adj[u]
   {
      if(color[v] == WHITE){
     prev[v]=u;
         DFS_Visit(v);}
   }
   color[u] = BLACK;
   time = time+1;
   f[u] = time;
}
```

What does `f[u]` represent?

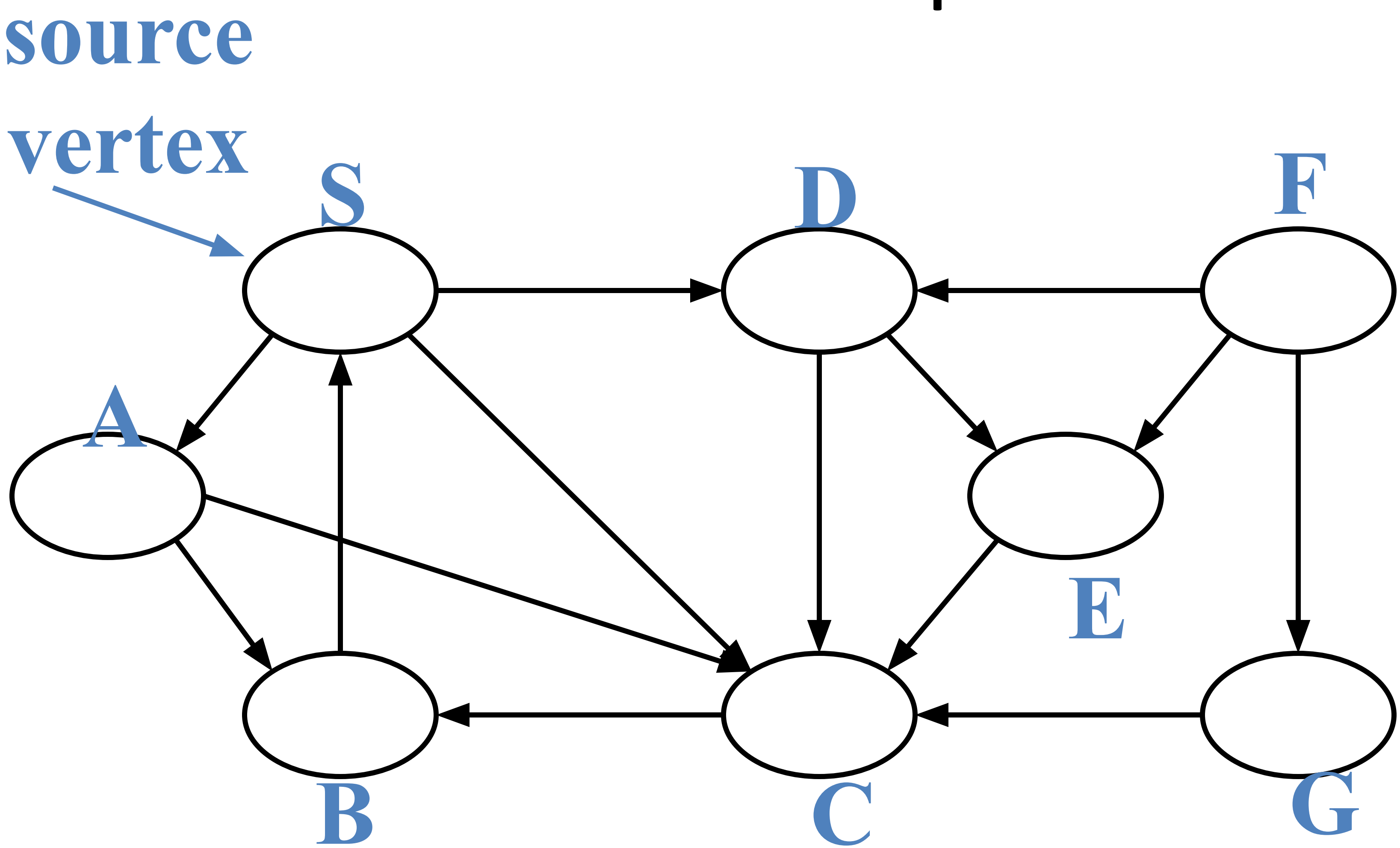
Depth-First Search: The Code

```
    Data: color[V], time,
       prev[V],d[V], f[V]
DFS(G)  // where prog starts
{
   for each vertex u ∈ V
   {
      color[u] = WHITE;
   prev[u]=NIL;
   f[u]=inf; d[u]=inf;
   }
   time = 0;
   for each vertex u ∈ V
     if (color[u] == WHITE)
         DFS_Visit(u);
}
```

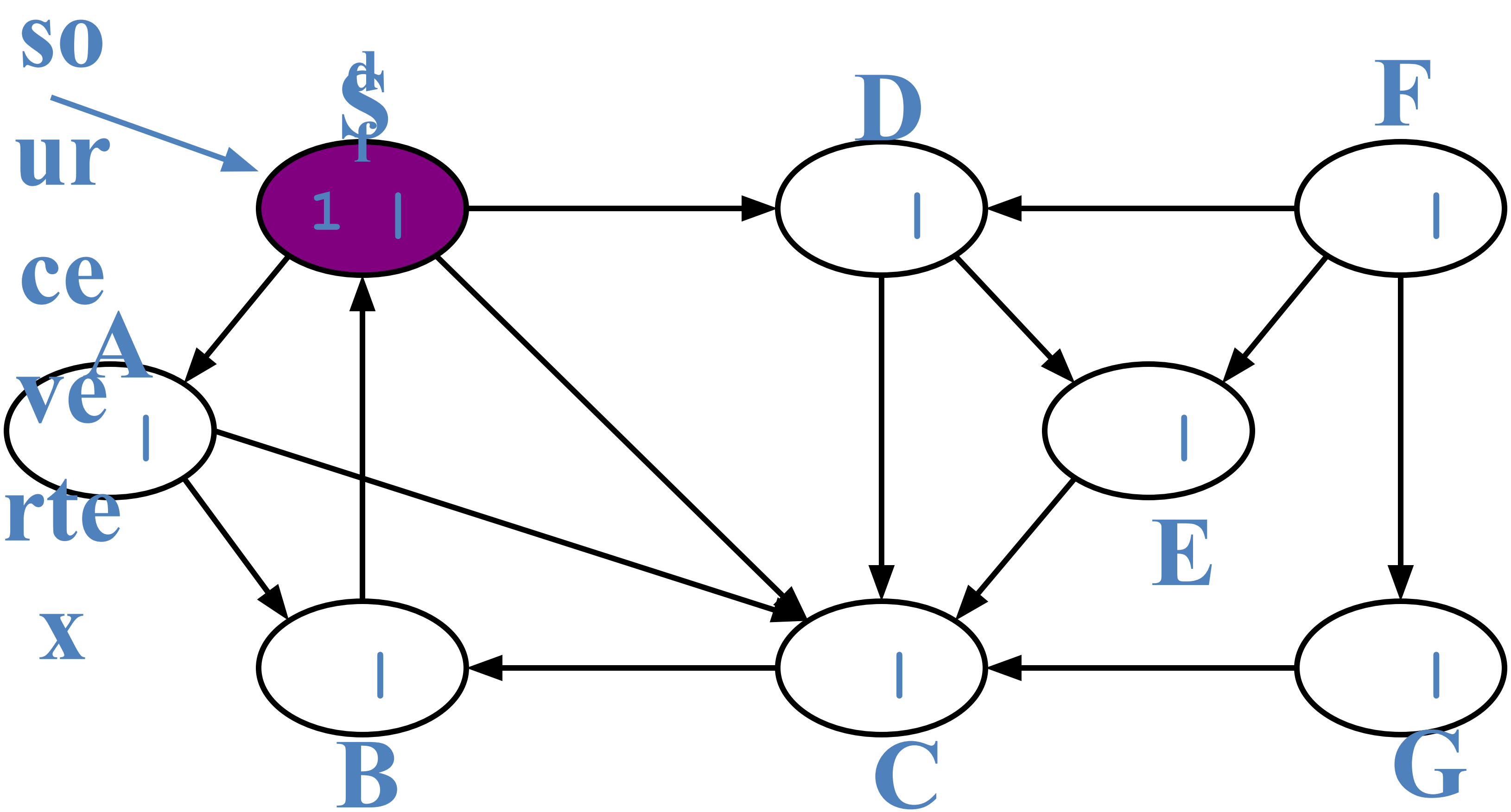
```
DFS_Visit(u)
{
   color[u] = GREY;
   time = time+1;
   d[u] = time;
   for each v ∈ Adj[u]
   {
      if(color[v] == WHITE){
     prev[v]=u;
         DFS_Visit(v);
   }}
   color[u] = BLACK;
   time = time+1;
   f[u] = time;
}
```

Will all vertices eventually be colored black?

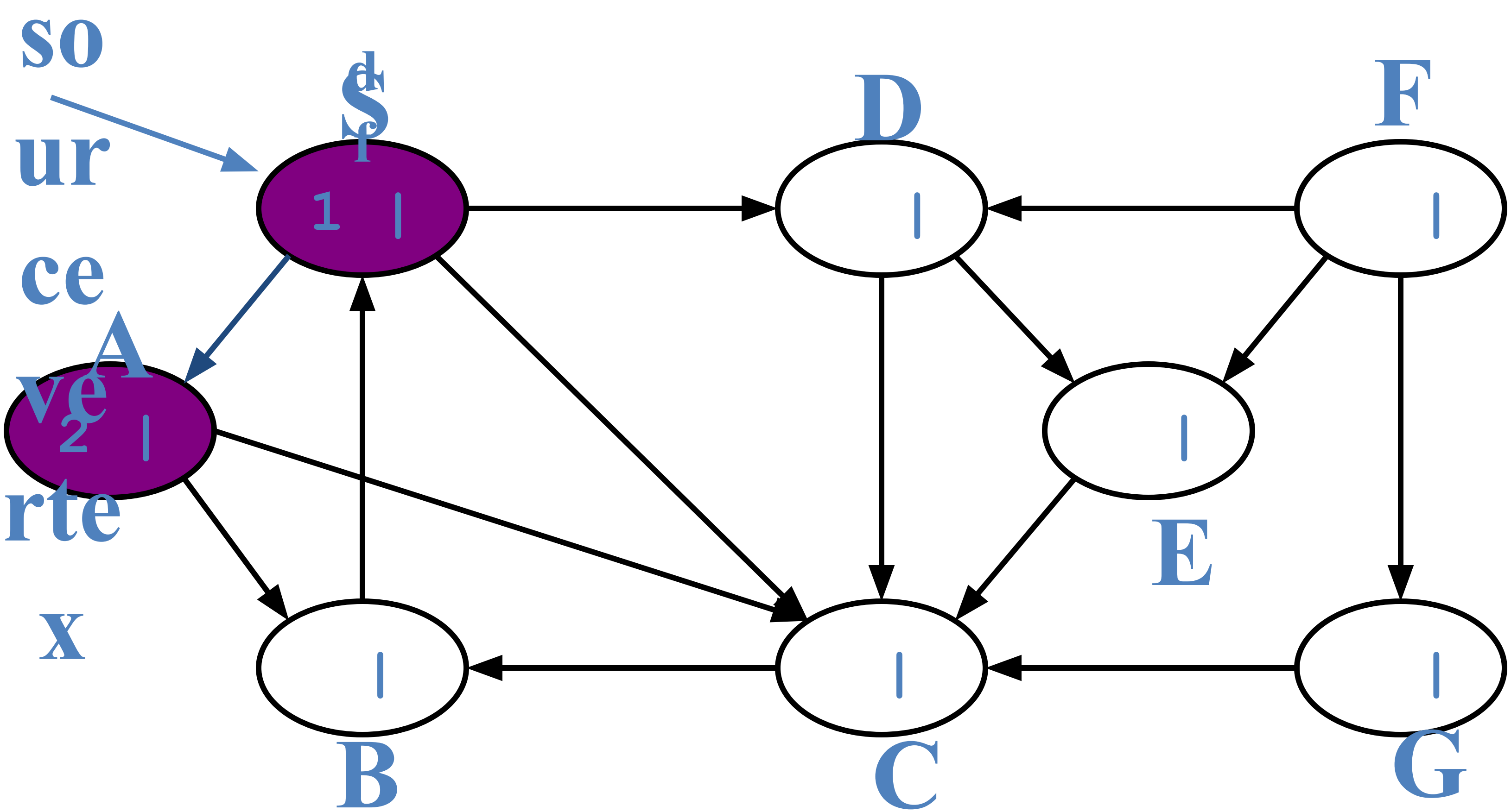
DFS Example



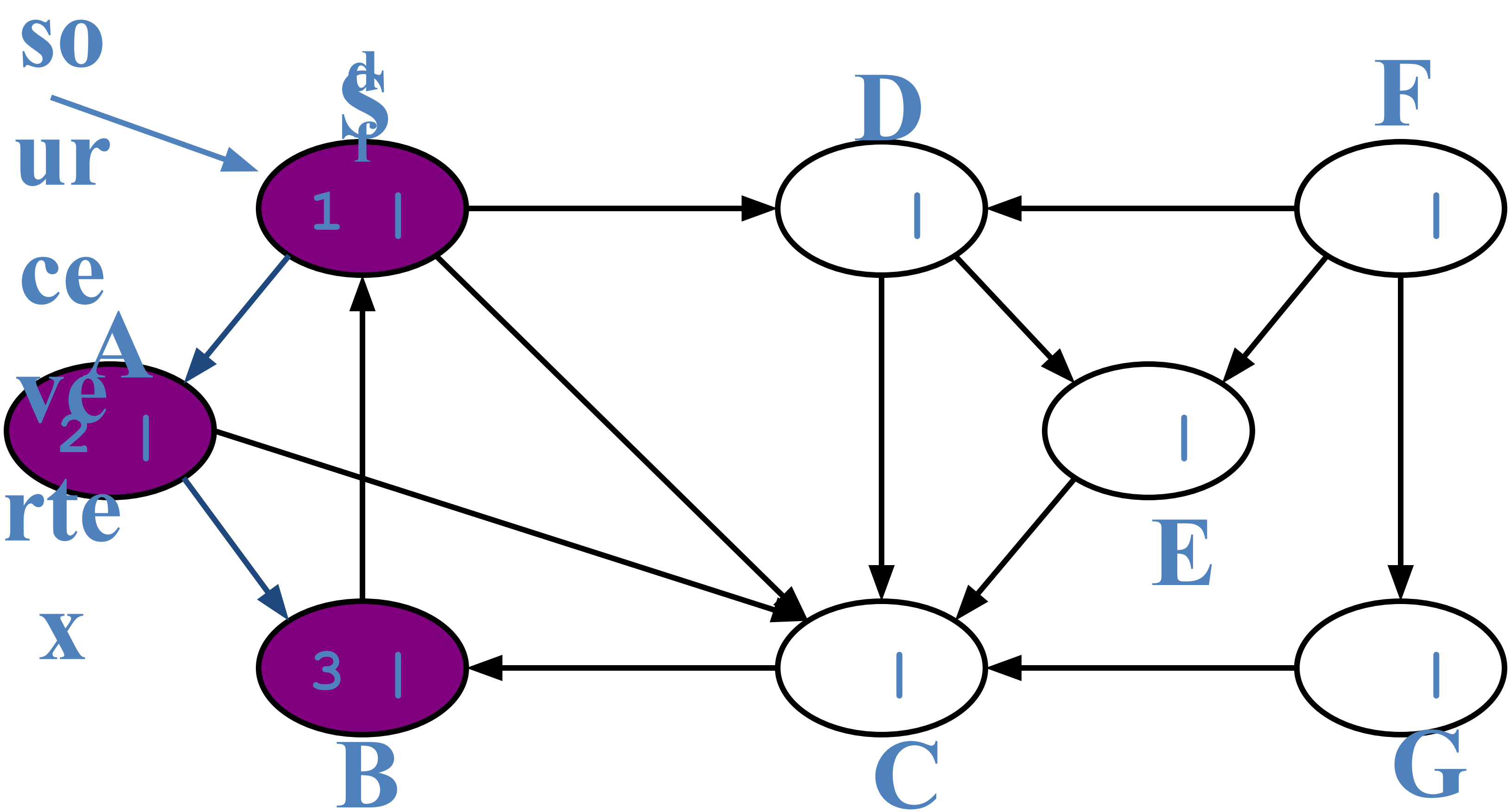
DFS Example



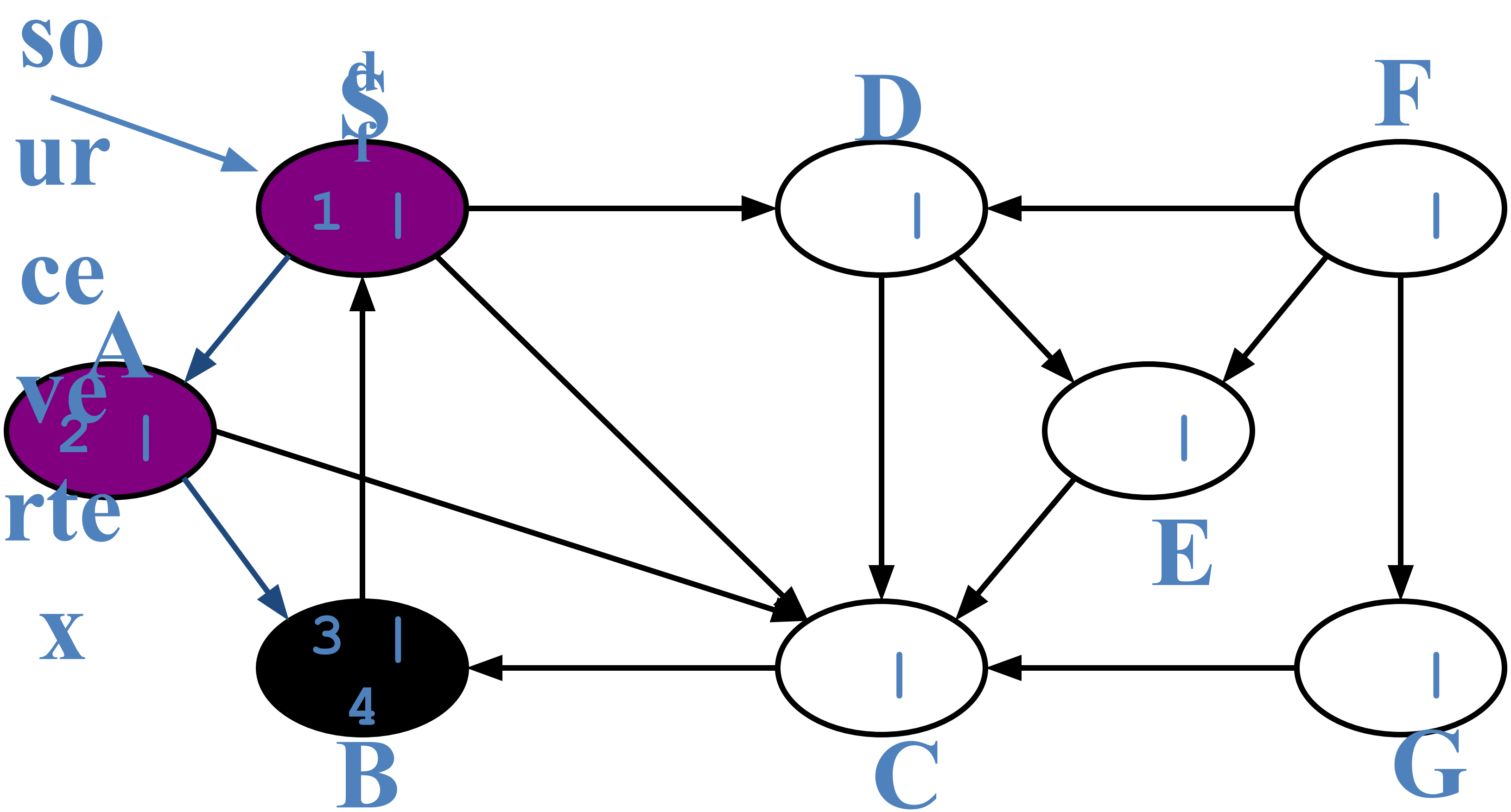
DFS Example



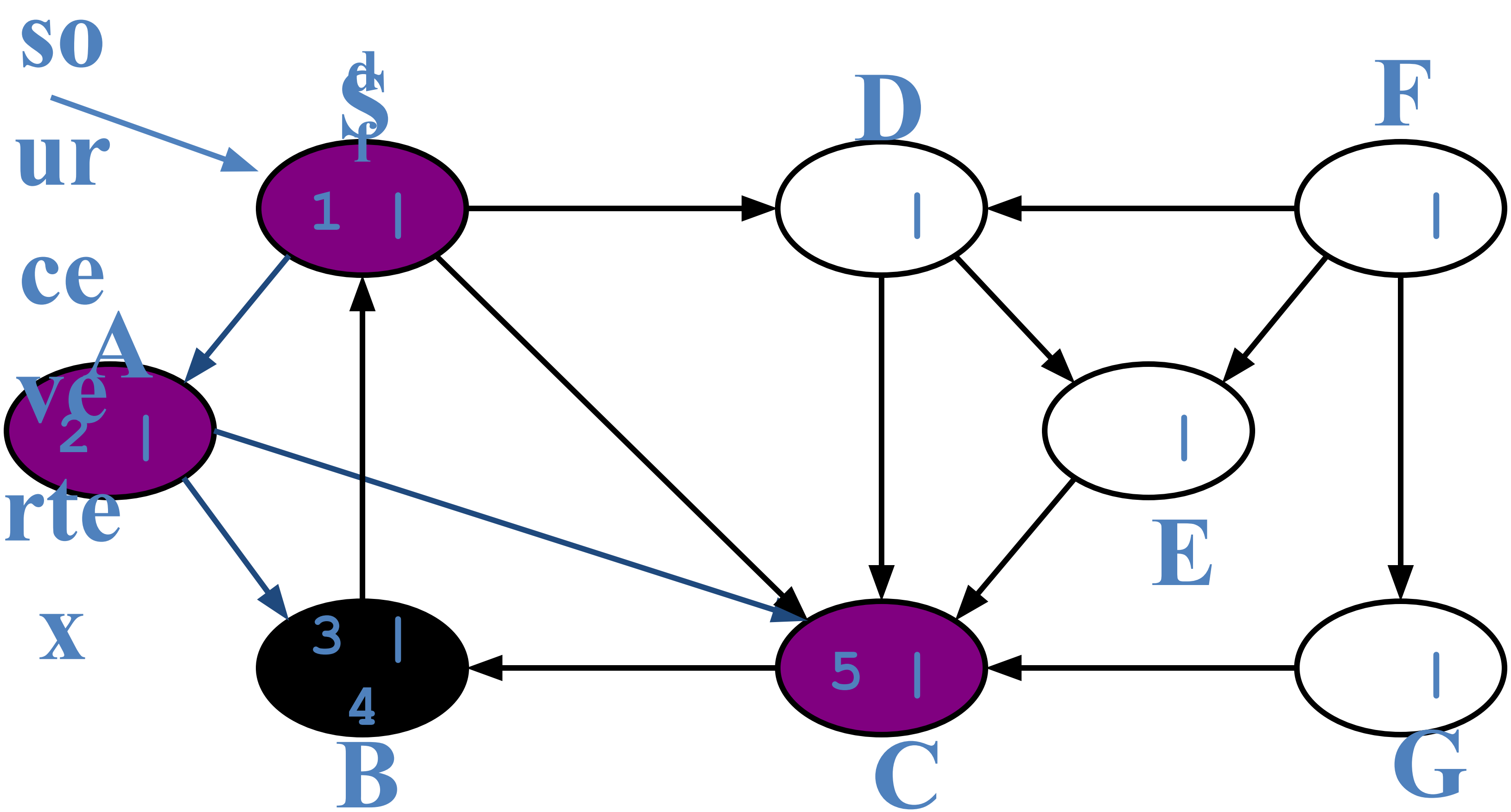
DFS Example



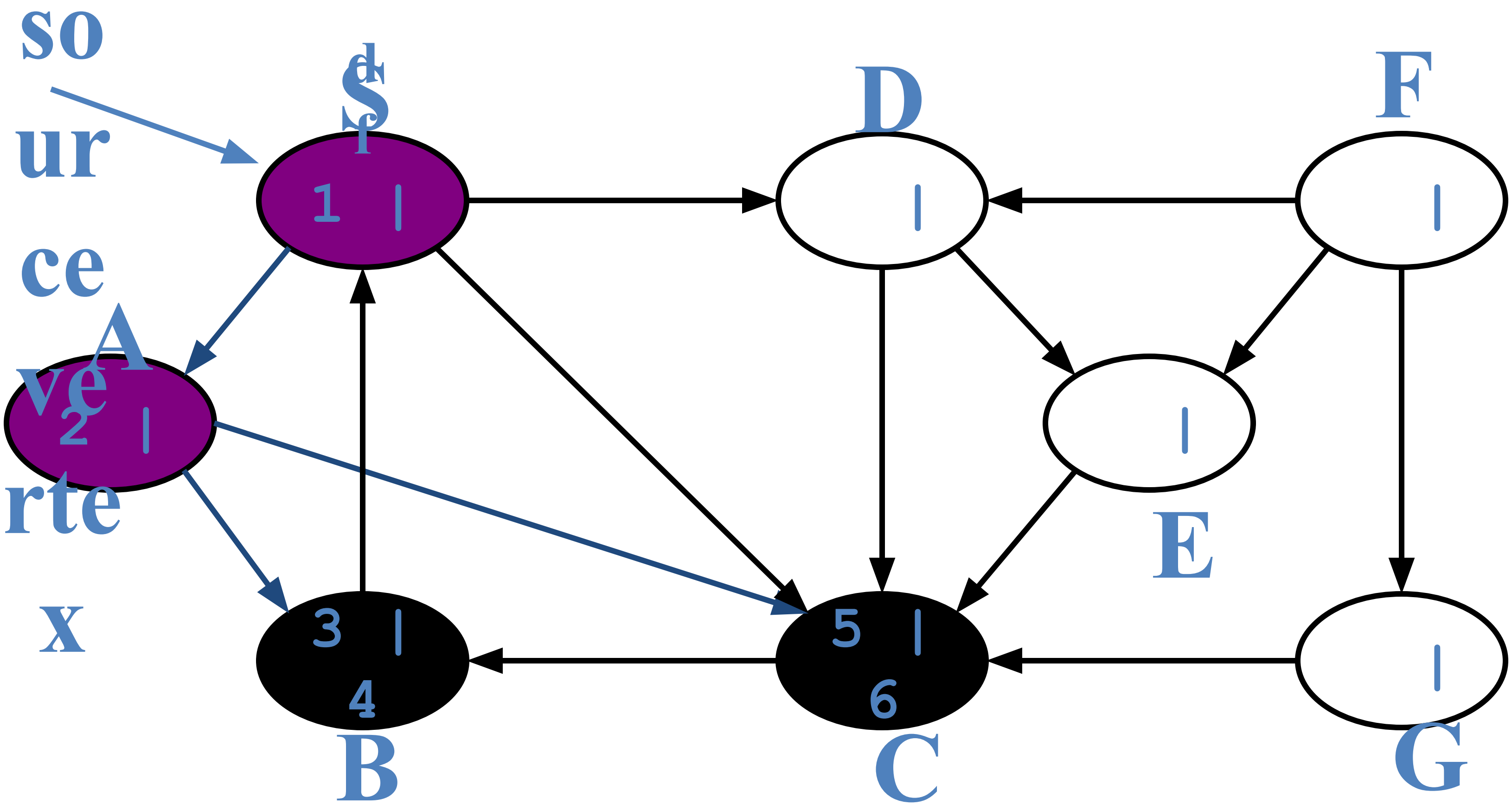
DFS Example



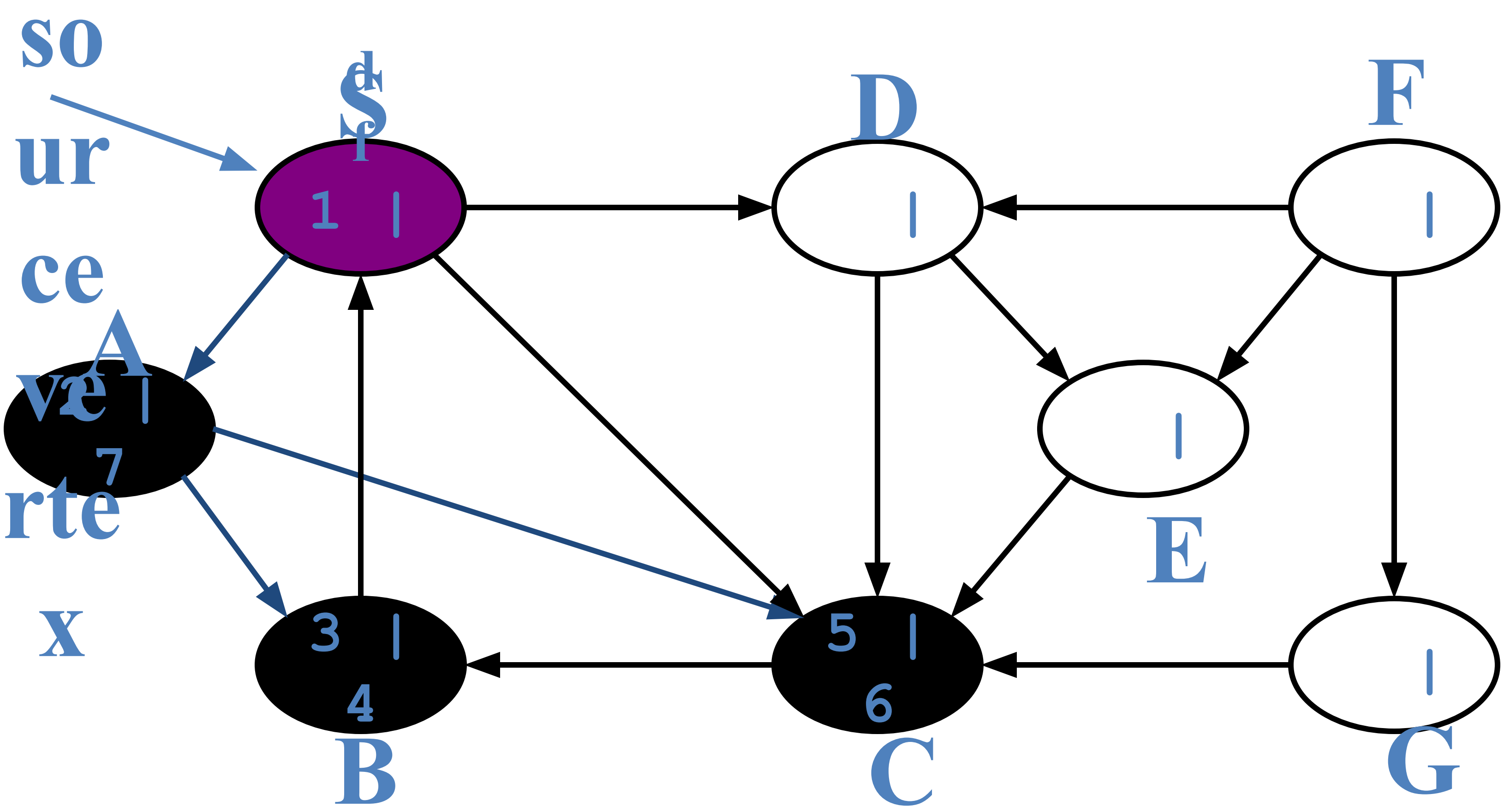
DFS Example



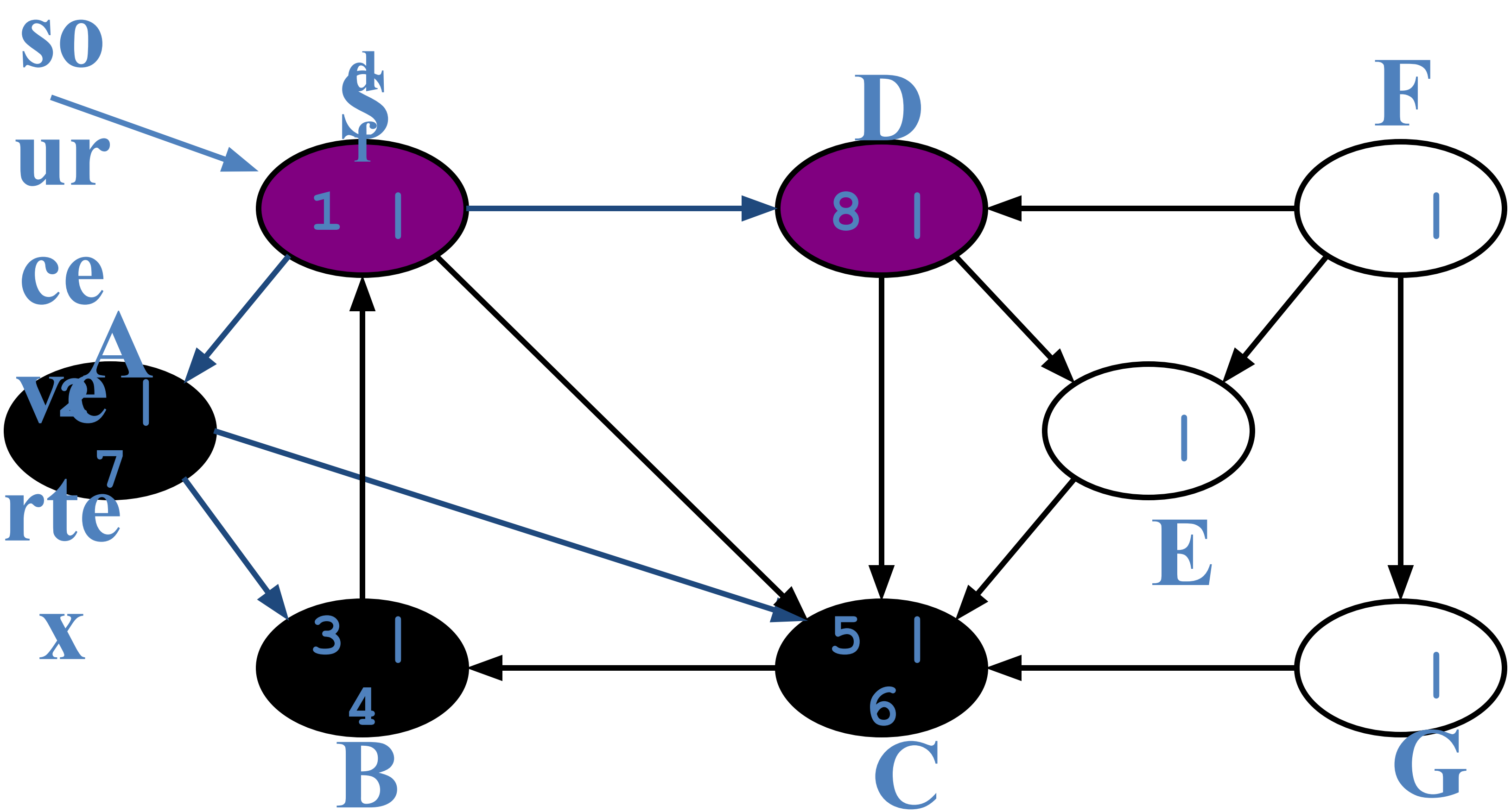
DFS Example



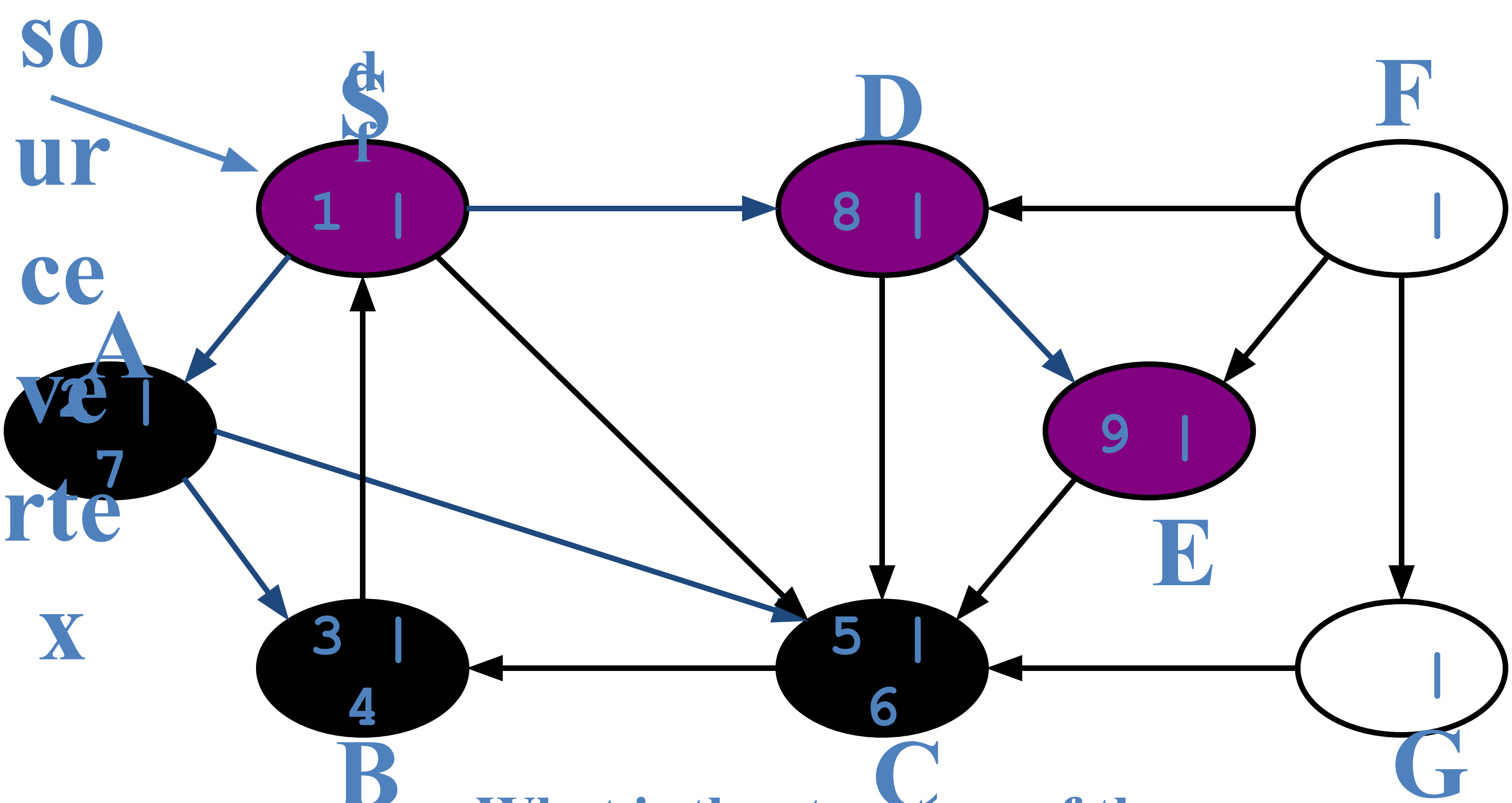
DFS Example



DFS Example



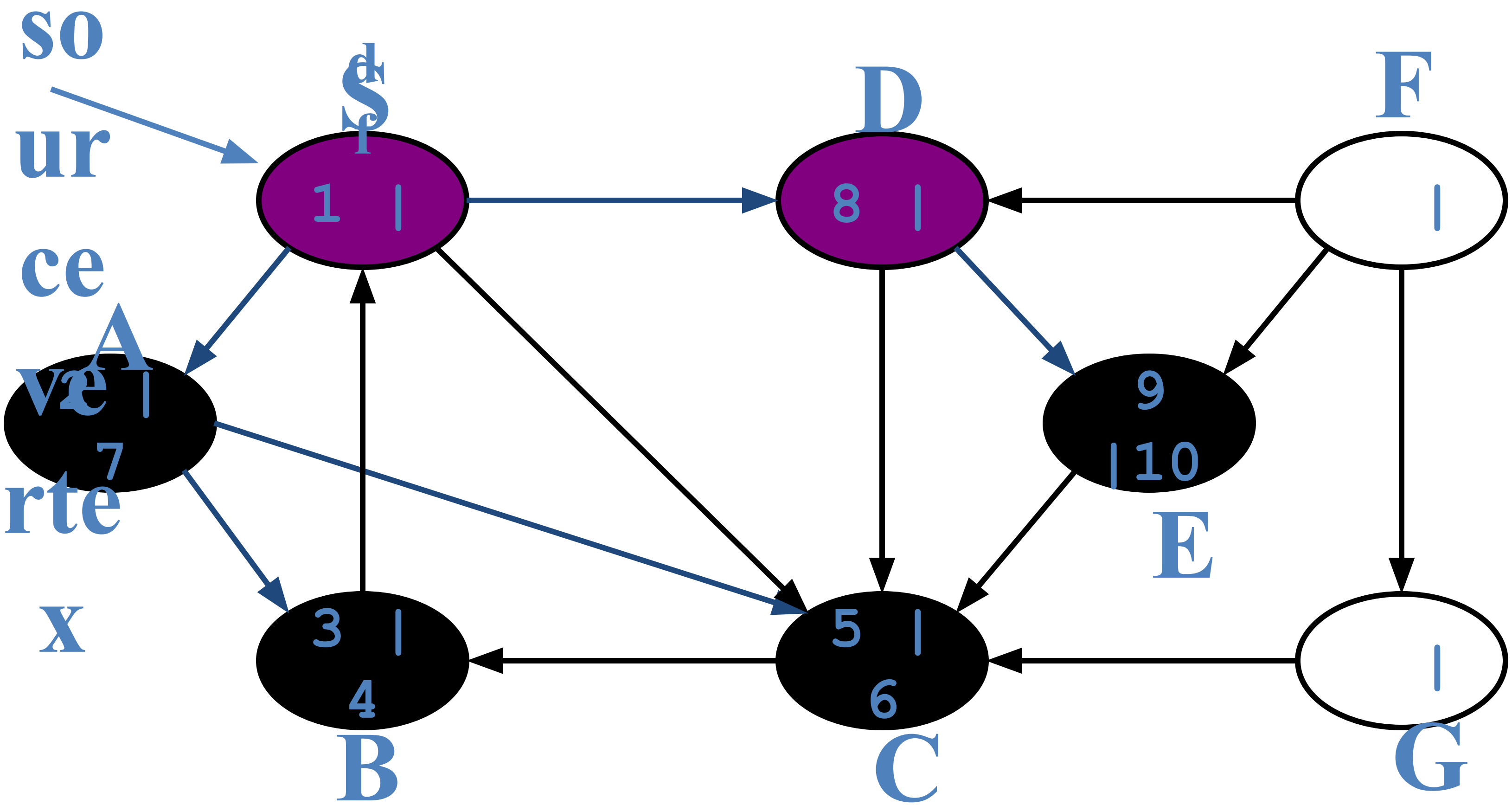
DFS Example



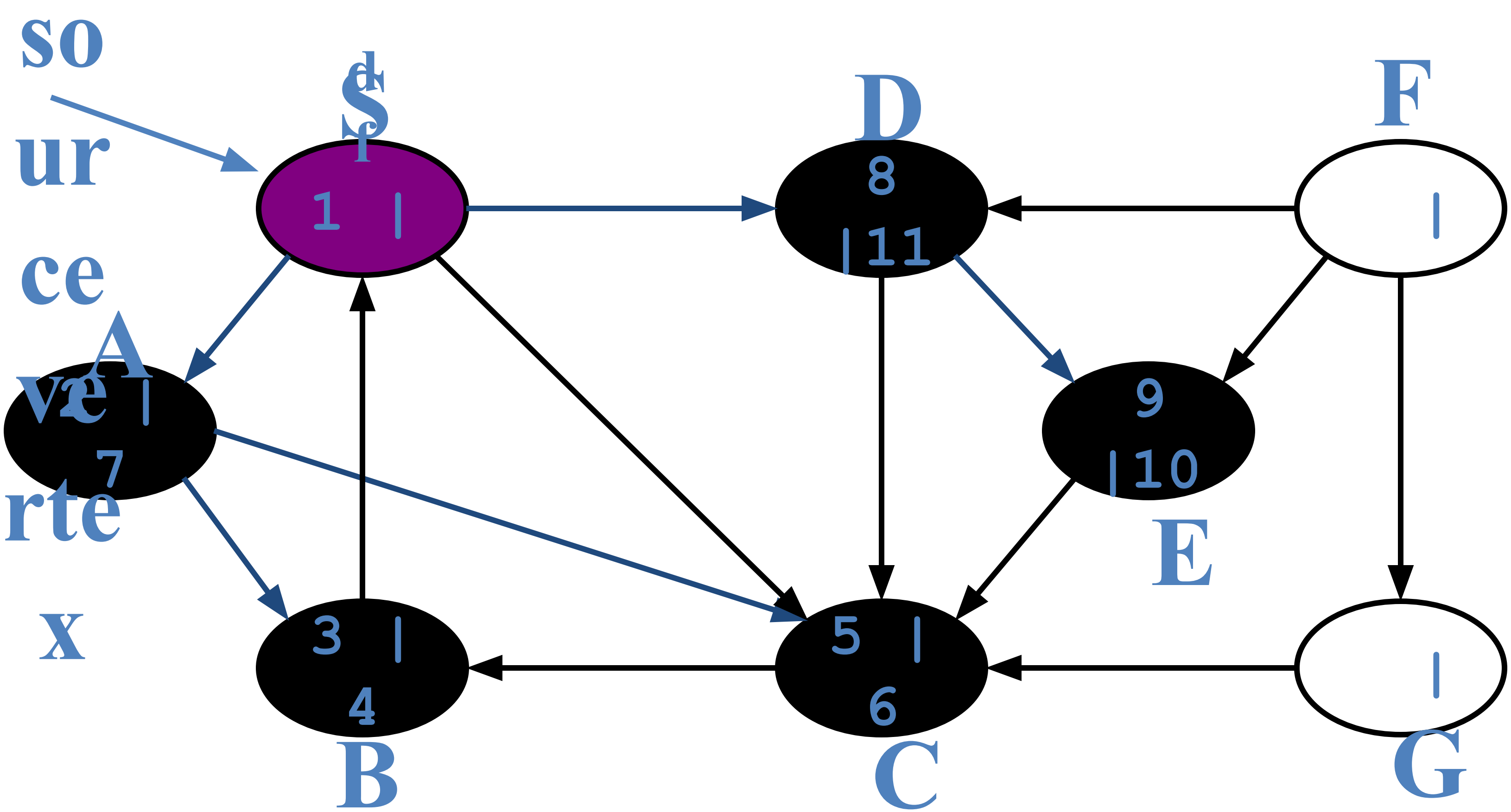
What is the structure of the grey vertices?

What do they represent?

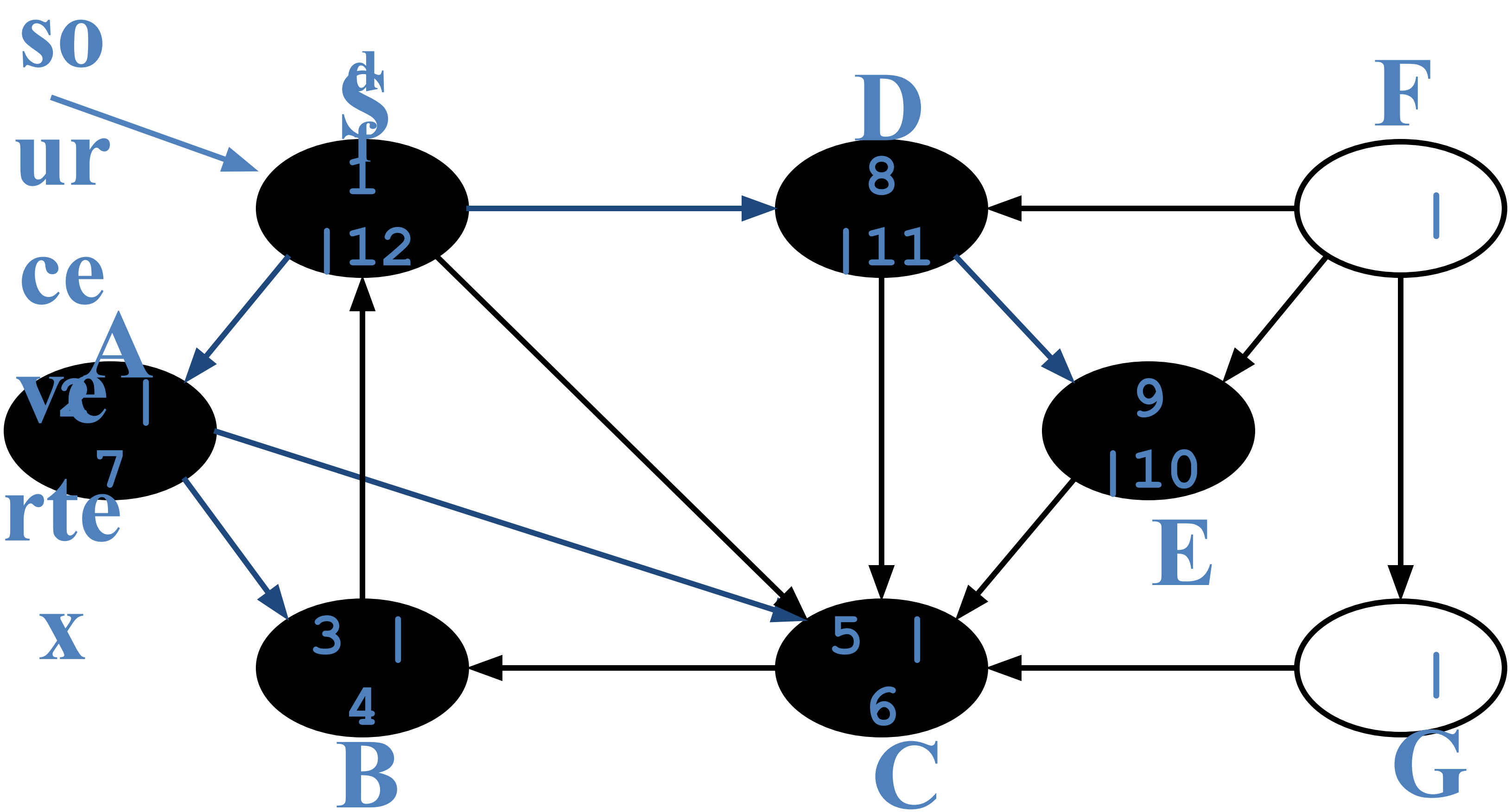
DFS Example



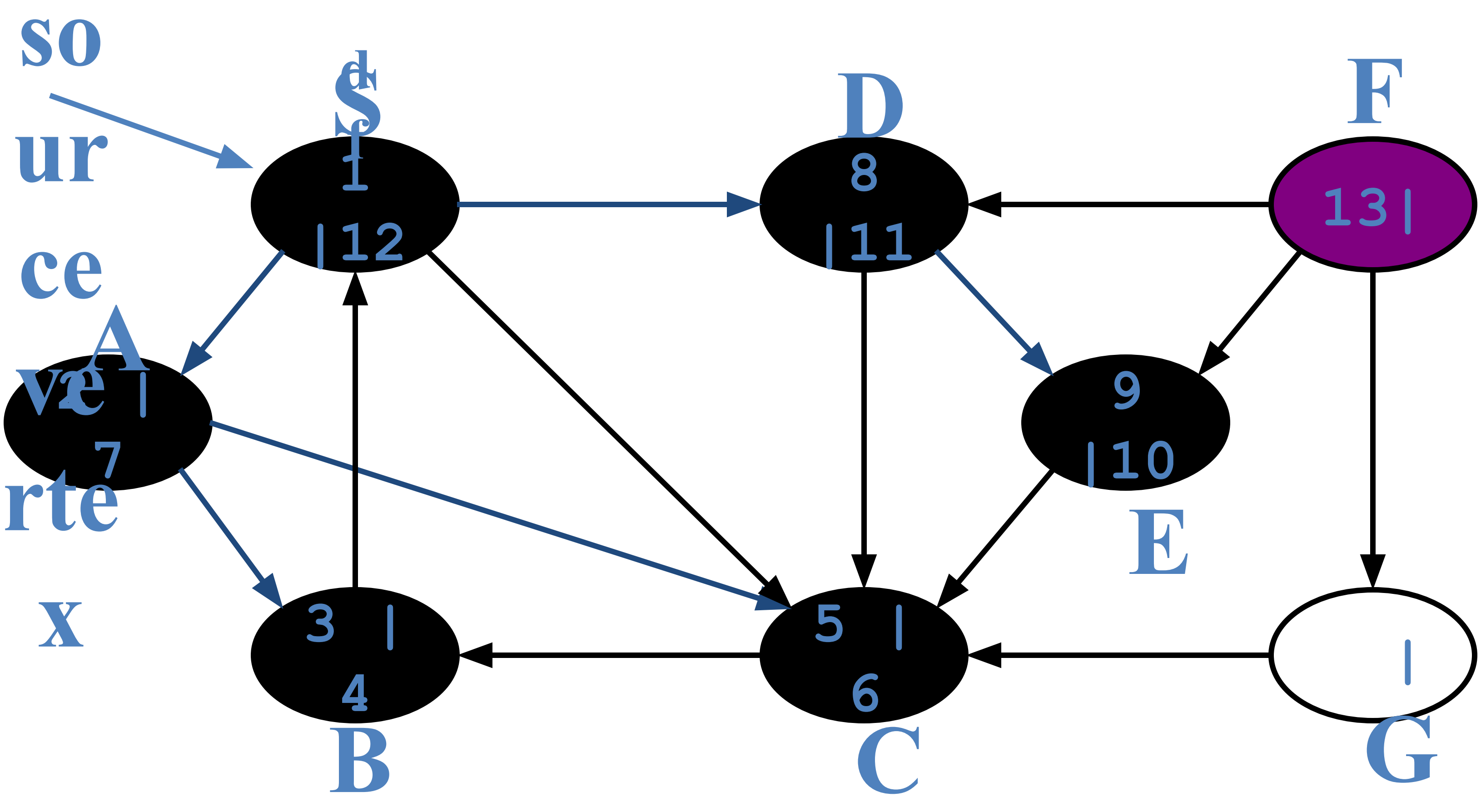
DFS Example



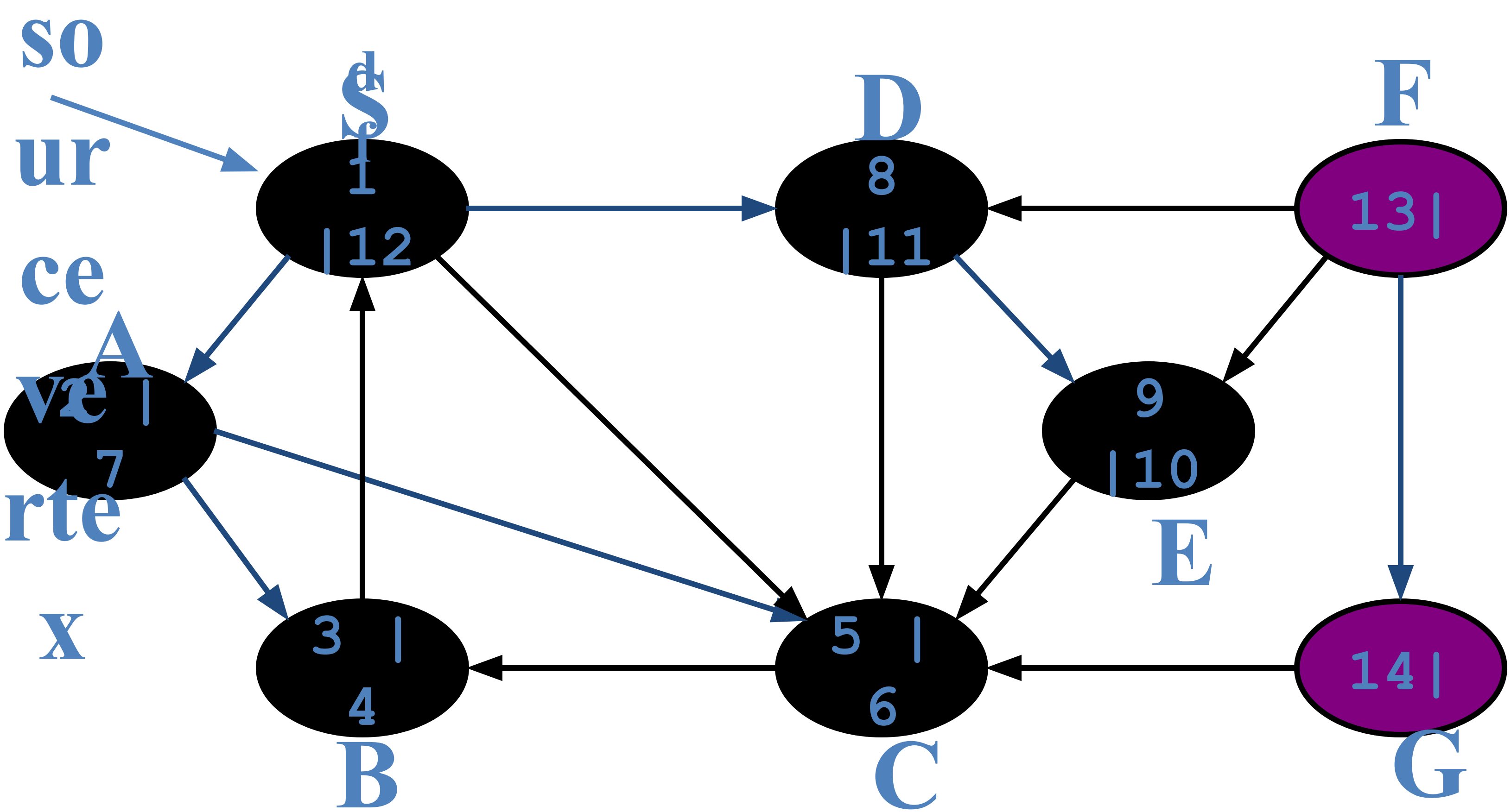
DFS Example



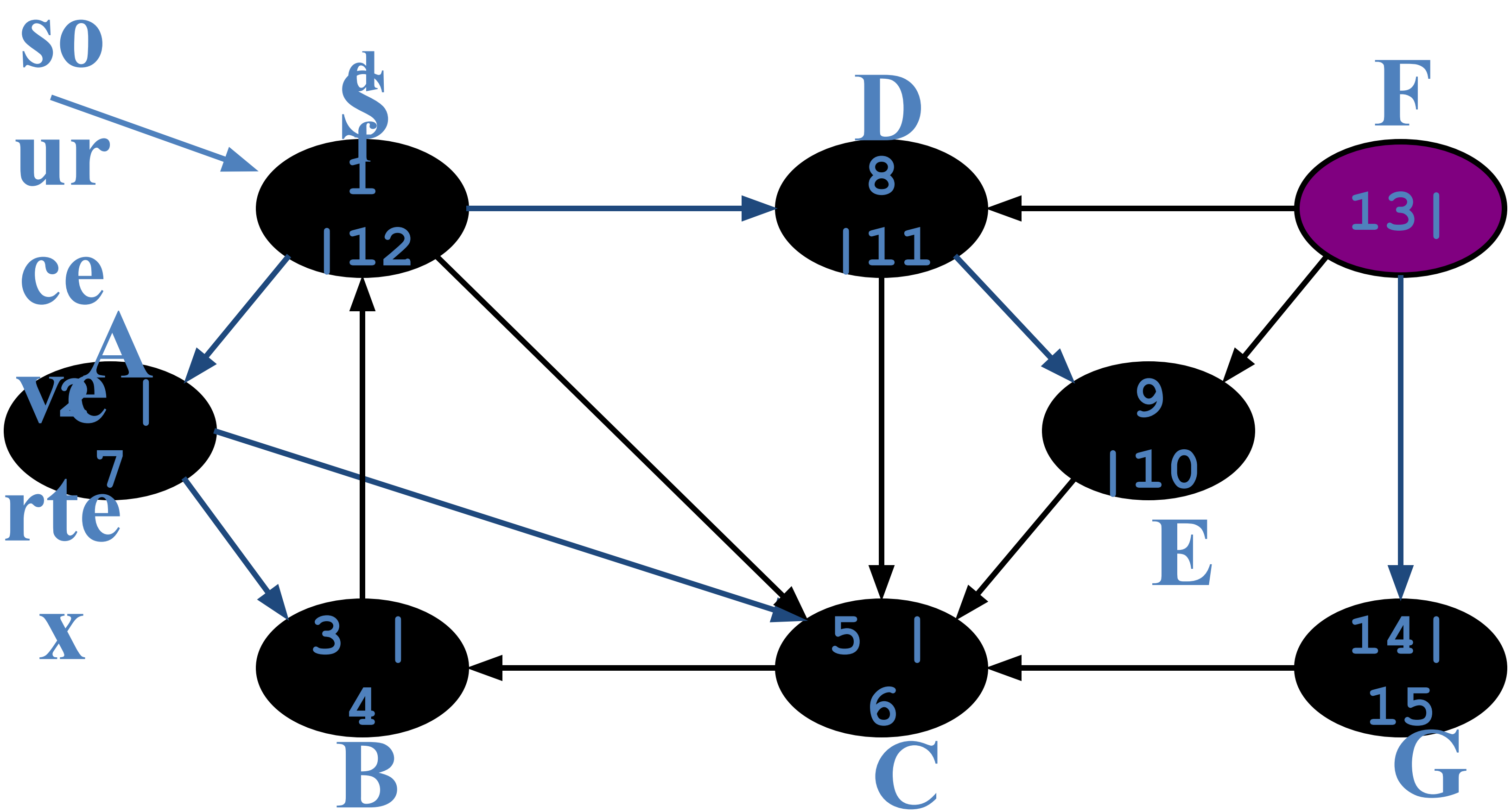
DFS Example



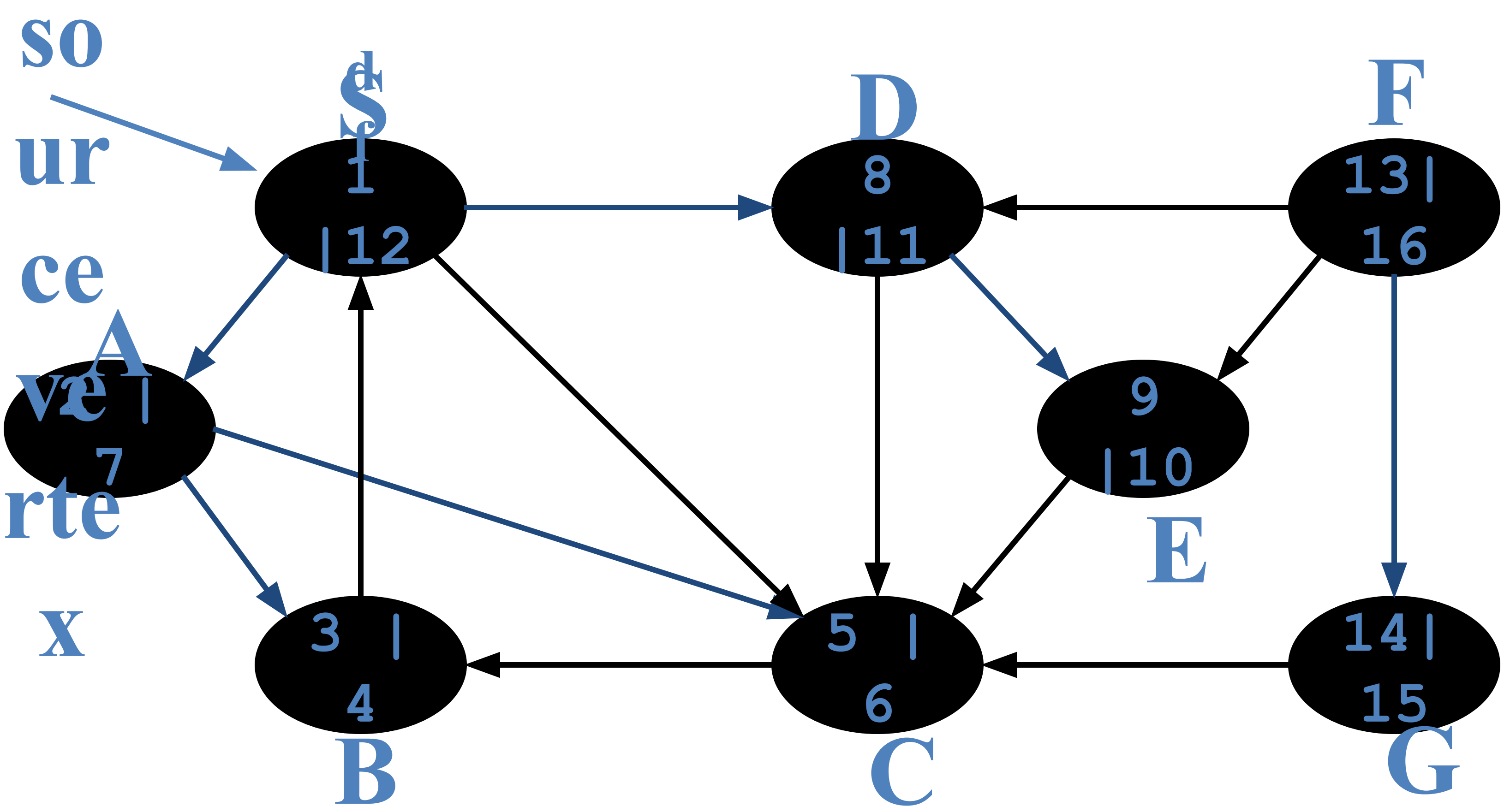
DFS Example



DFS Example



DFS Example



Depth-First Search: The Code

```
Data: color[V], time,
   prev[V],d[V], f[V]
DFS(G) // where prog starts
{
   for each vertex u ∈ V
   {
      color[u] = WHITE;
   prev[u]=NIL;
   f[u]=inf; d[u]=inf;
   }
   time = 0;
   for each vertex u ∈ V
     if (color[u] == WHITE)
         DFS_Visit(u);
}
```

```
DFS_Visit(u)
{
   color[u] = GREY;
   time = time+1;
   d[u] = time;
   for each v ∈ Adj[u]
   {
      if (color[v] == WHITE)
     prev[v]=u;
        DFS_Visit(v);
   }
   color[u] = BLACK;
   time = time+1;
   f[u] = time;
}
```

What will be the running time?

Depth-First Search: The Code

```
Data: color[V], time,
   prev[V],d[V], f[V]
DFS(G) // where prog starts
{
   for each vertex u ∈ V
   {
      color[u] = WHITE;
   prev[u]=NIL;
   f[u]=inf; d[u]=inf;
   }
   time = 0;
   for each vertex u ∈ V
     if (color[u] == WHITE)
        DFS_Visit(u);
}
```

O(V)

O(V)

```
DFS_Visit(u)
{
   color[u] = GREY;
   time = time+1;
   d[u] = time;
   for each v ∈ Adj[u]
   {
      if (color[v] == WHITE)
     prev[v]=u;
         DFS_Visit(v);
   }
   color[u] = BLACK;
   time = time+1;
   f[u] = time;
}
```

O(V)

Running time: $O(V^2)$ because call DFS_Visit on each vertex, and the loop over Adj[] can run as many as |V| times

Depth-First Search: The Code

```
Data: color[V], time,
  prev[V],d[V], f[V]
DFS(G) // where prog starts
{
   for each vertex u ∈ V
   {
      color[u] = WHITE;
   prev[u]=NIL;
   f[u]=inf; d[u]=inf;
   }
   time = 0;
   for each vertex u ∈ V
     if (color[u] == WHITE)
         DFS_Visit(u);
}
```

```
DFS_Visit(u)
{
   color[u] = GREY;
   time = time+1;
   d[u] = time;
   for each v ∈ Adj[u]
   {
      if (color[v] == WHITE)
     prev[v]=u;
         DFS_Visit(v);
   }
   color[u] = BLACK;
   time = time+1;
   f[u] = time;
}
```

BUT, there is actually a tighter bound.

How many times will DFS_Visit() actually be called?

Depth-First Search: The Code

```
   Data: color[V], time,
     prev[V],d[V], f[V]
DFS(G) // where prog starts
{
   for each vertex u ∈ V
   {
      color[u] = WHITE;
   prev[u]=NIL;
   f[u]=inf; d[u]=inf;
   }
   time = 0;
   for each vertex u ∈ V
     if (color[u] == WHITE)
         DFS_Visit(u);
}
```

```
DFS_Visit(u)
{
   color[u] = GREY;
   time = time+1;
   d[u] = time;
   for each v ∈ Adj[u]
   {
      if (color[v] == WHITE)
     prev[v]=u;
         DFS_Visit(v);
   }
   color[u] = BLACK;
   time = time+1;
   f[u] = time;
}
```

So, running time of DFS = O(V+E)

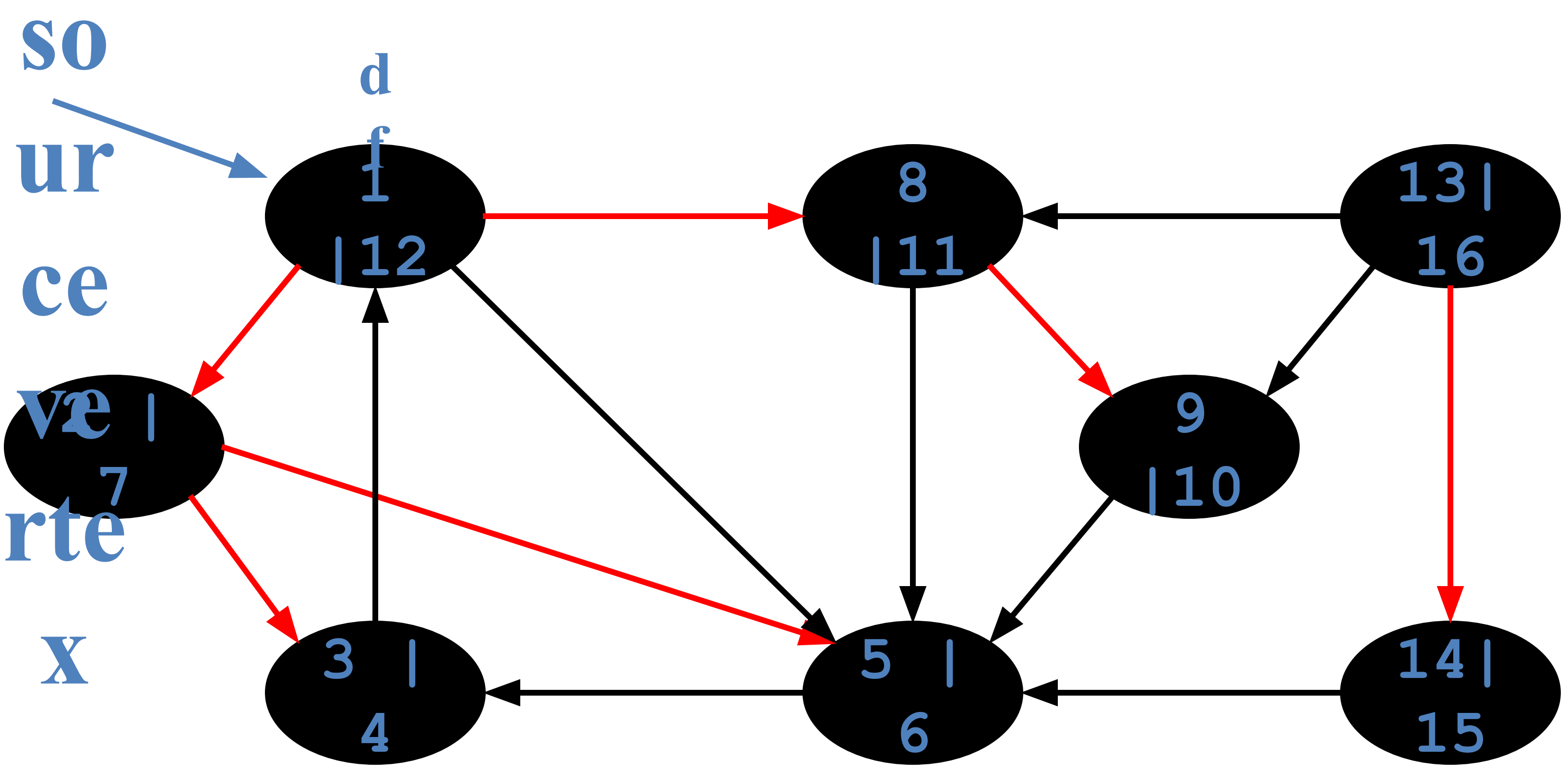
Depth-First Sort Analysis

- This running time argument is an informal example of *amortized analysis*
  - “Charge” the exploration of edge to the edge:
    - Each loop in DFS_Visit can be attributed to an edge in the graph
    - Runs once per edge if directed graph, twice if undirected
    - Thus loop will run in O(E) time, algorithm O(V+E)
      - Considered linear for graph, b/c adj list requires O(V+E) storage
  - Important to be comfortable with this kind of reasoning and analysis

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
    - The tree edges form a spanning forest
    - *Can tree edges form cycles? Why or why not?*
      - *No*

DFS Example

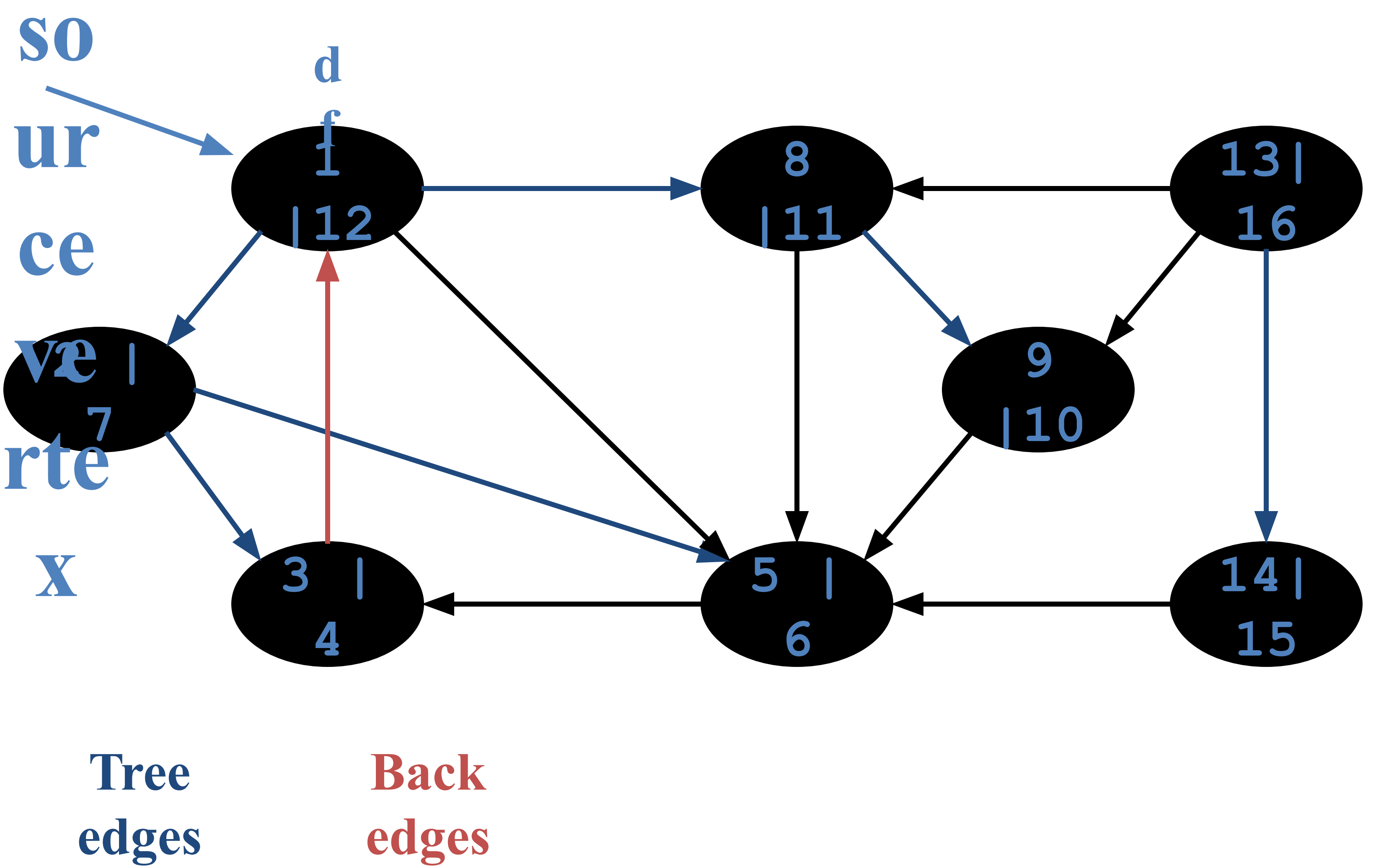


Tree edges

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
    - Encounter a grey vertex (grey to grey)
    - Self loops are considered as to be back edge.

DFS Example

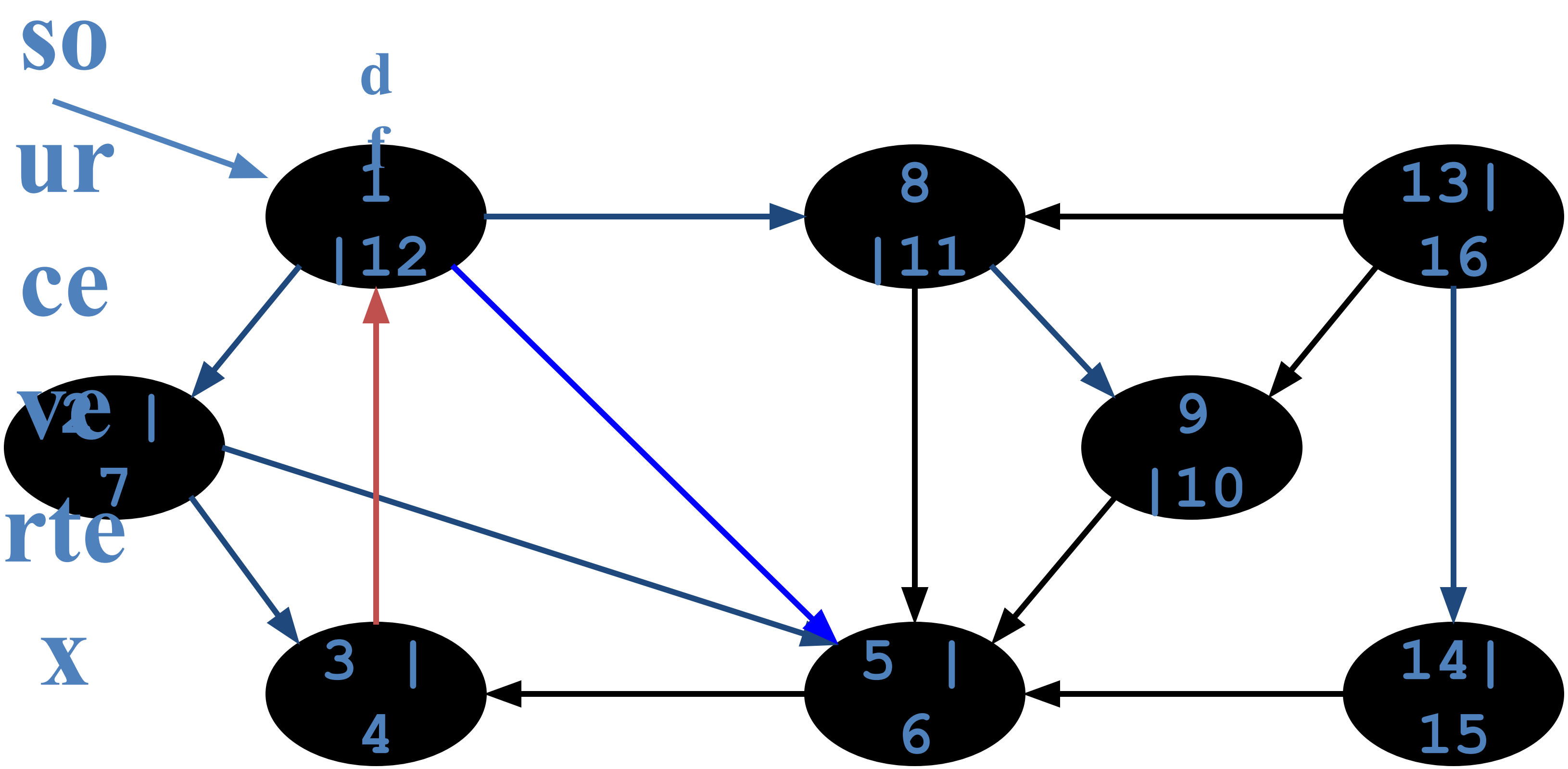


source vertex
d f
1 |12
8 |11
13| 16
2 | 7
9 |10
3 | 4
5 | 6
14| 15
Tree edges
Back edges

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
  - *Forward edge*: from ancestor to descendent
    - Not a tree edge, though
    - From grey node to black node

DFS Example



Tree edges

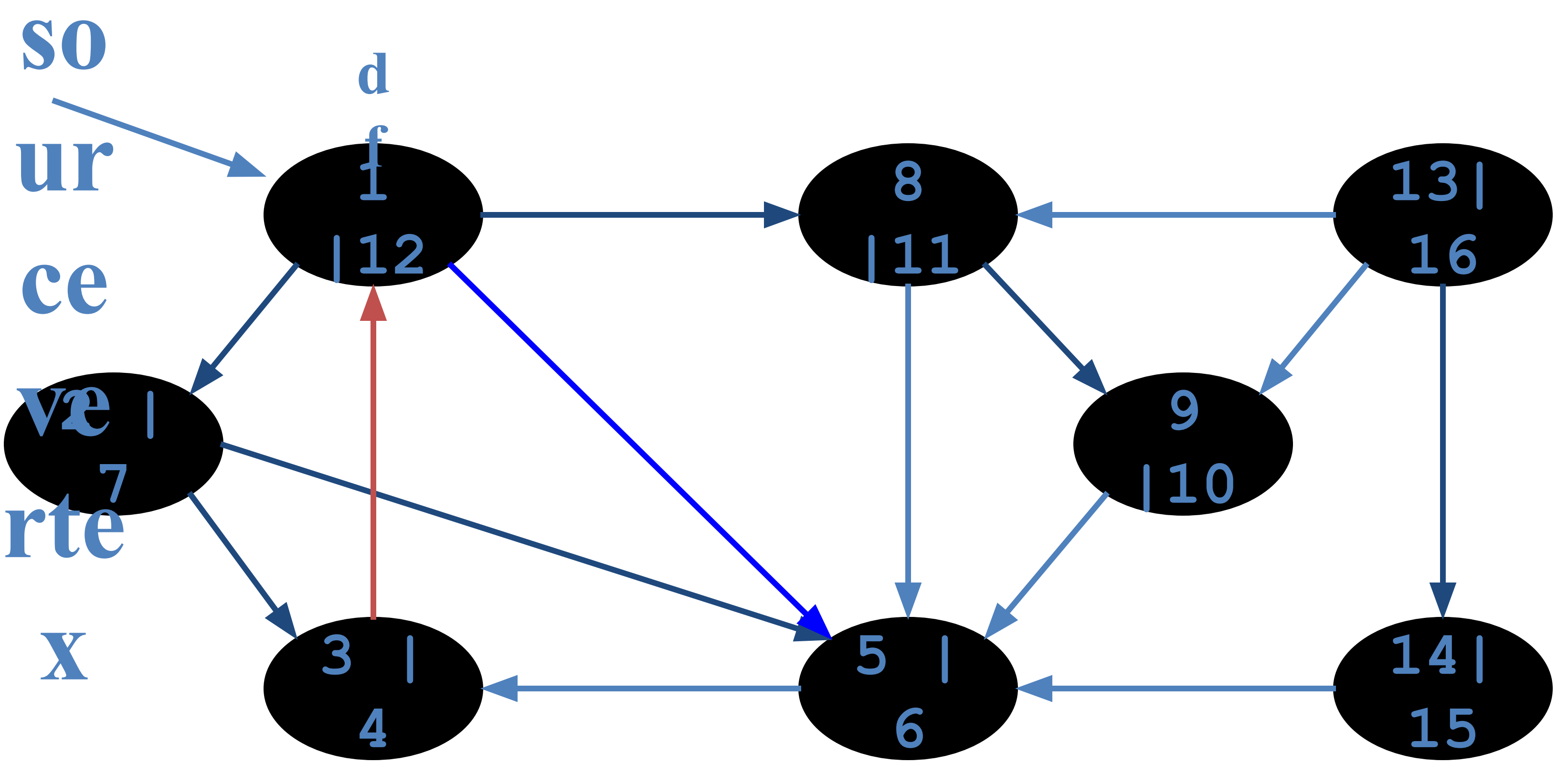
Back edges

Forward edges

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
  - *Forward edge*: from ancestor to descendent
  - *Cross edge*: between a tree or subtrees
    - From a grey node to a black node

DFS Example



Tree edges

Back edges

Forward edges

Cross edges

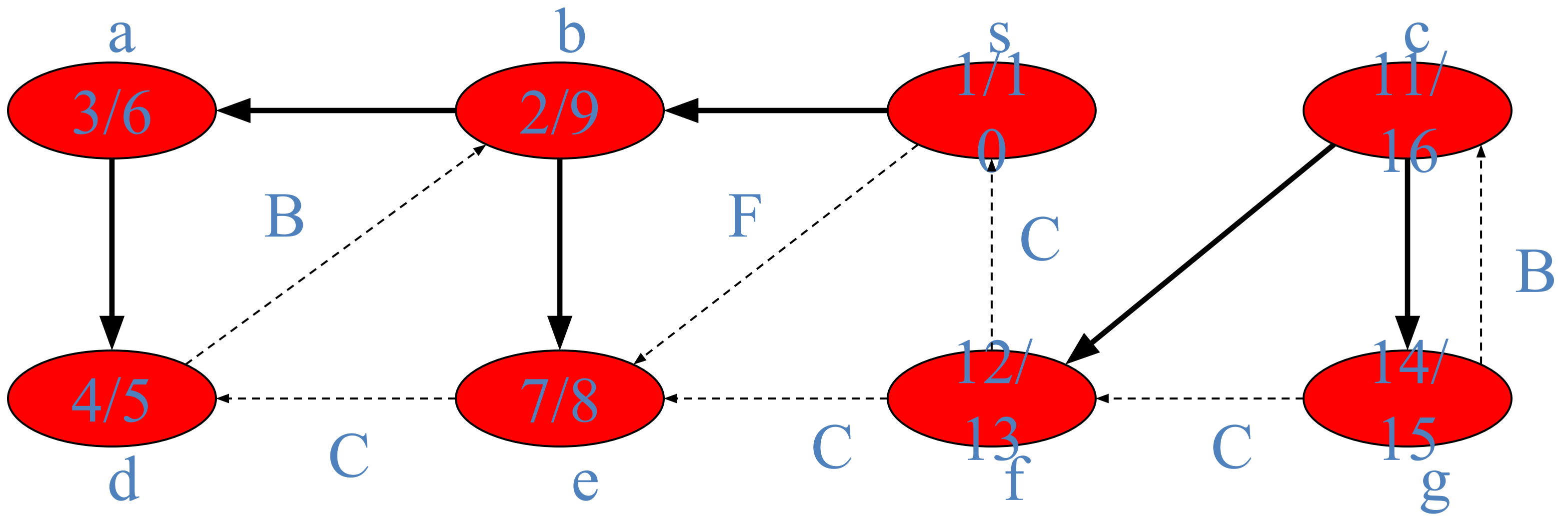
DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
  - *Forward edge*: from ancestor to descendent
  - *Cross edge*: between a tree or subtrees
- Note: tree & back edges are important; most algorithms don’t distinguish forward & cross

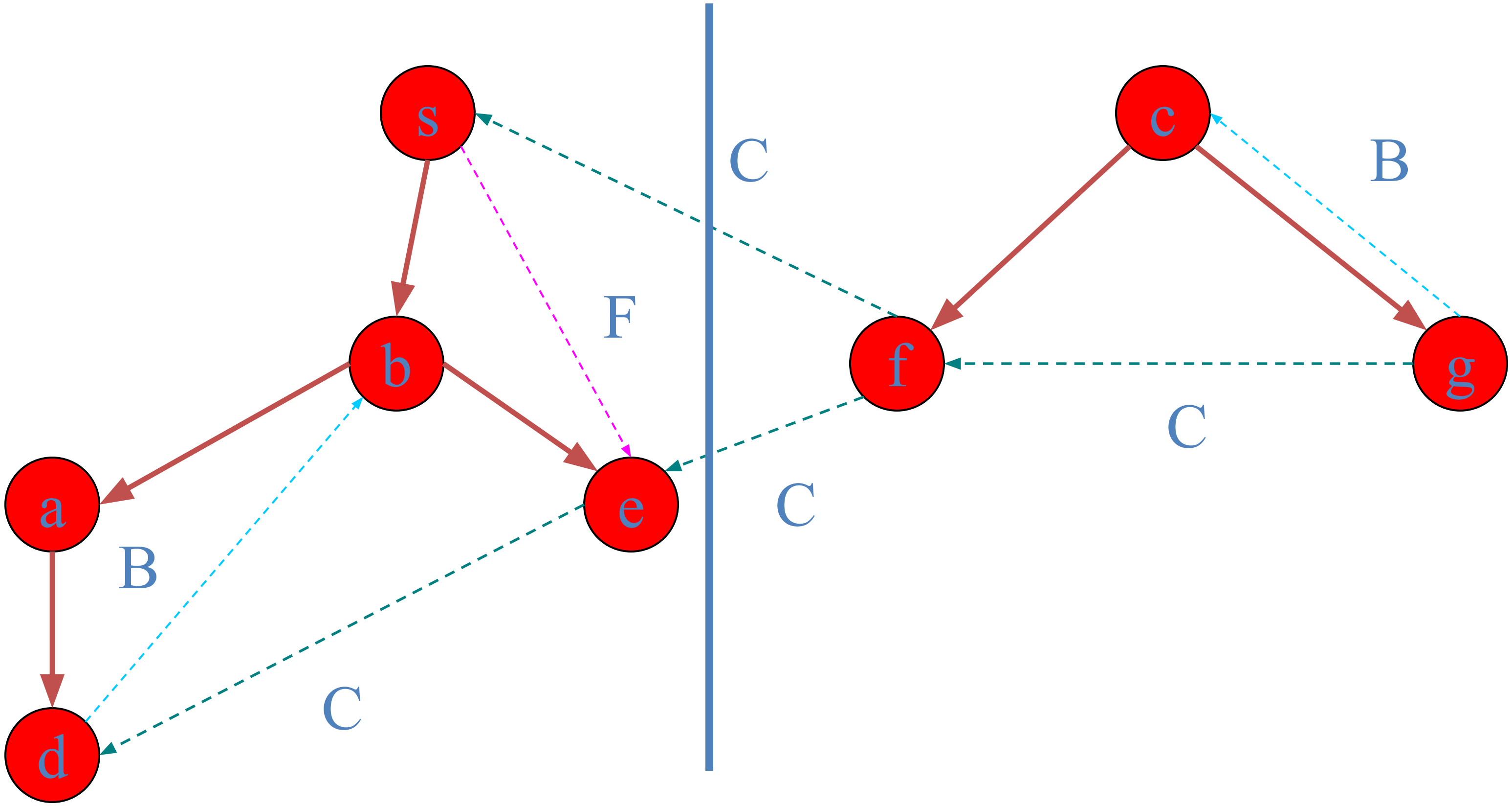
More about the edges

- Let (u,v) is an edge.
  - If (color[v] = WHITE) then (u,v) is a tree edge
  - If (color[v] = GRAY) then (u,v) is a back edge
  - If (color[v] = BLACK) then (u,v) is a forward/cross edge
    - Forward Edge: d[u]<d[v]
    - Cross Edge: d[u]>d[v]

Depth-First Search - Timestamps



Depth-First Search - Timestamps



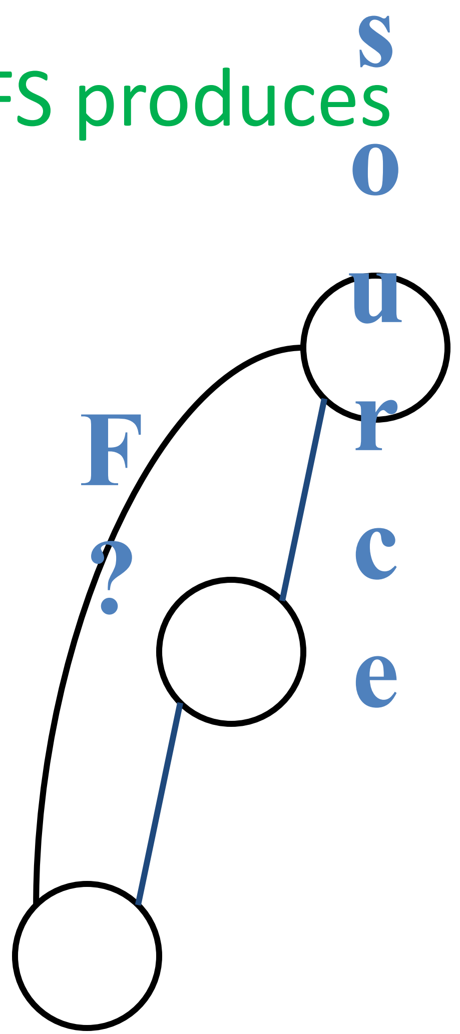
Depth-First Search: Detect Edge

```
Data: color[V], time,
  prev[V],d[V], f[V]
DFS(G) // where prog starts
{
   for each vertex u ∈ V
   {
      color[u] = WHITE;
   prev[u]=NIL;
   f[u]=inf; d[u]=inf;
   }
   time = 0;
   for each vertex u ∈ V
     if (color[u] == WHITE)
        DFS_Visit(u);
}
```

```
DFS_Visit(u)
{
   color[u] = GREY;
   time = time+1;
   d[u] = time;
   for each v ∈ Adj[u]
   {
   detect edge type using
   "color[v]"
      if(color[v] == WHITE){
     prev[v]=u;
        DFS_Visit(v);
   }}
   color[u] = BLACK;
   time = time+1;
   f[u] = time;
}
```

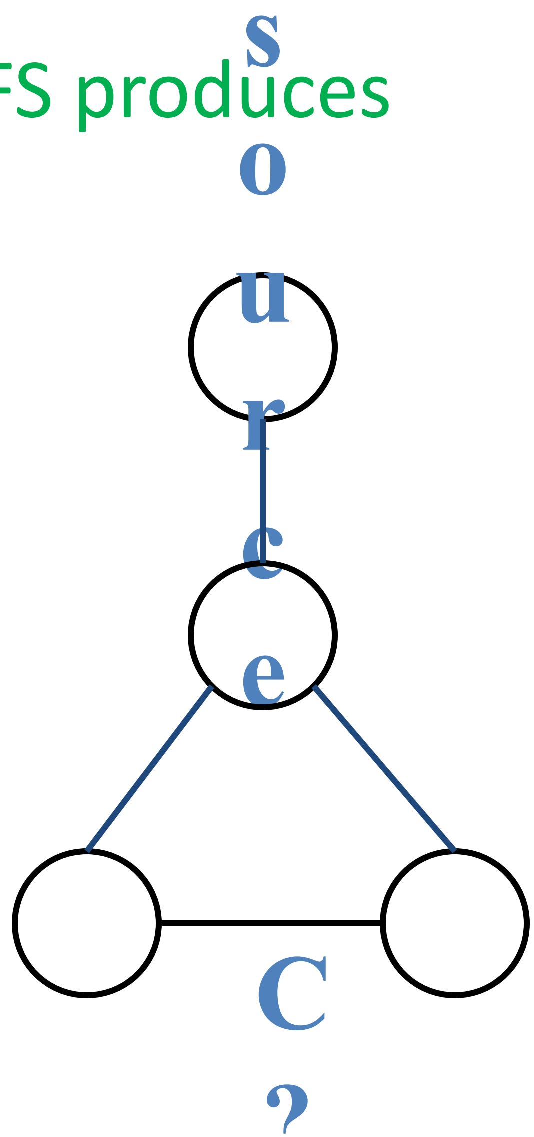
DFS: Kinds Of Edges

- Thm 22.10: If G is undirected, a DFS produces only tree and back edges
- Proof by contradiction:
  - Assume there's a forward edge
    - But F? edge must actually be a back edge (*why?*)



DFS: Kinds Of Edges

- Thm 22.10: If G is undirected, a DFS produces only tree and back edges
- Proof by contradiction:
  - Assume there's a cross edge
    - But C? edge cannot be cross:
    - must be explored from one of the vertices it connects, becoming a tree vertex, before other vertex is explored
    - So in fact the picture is wrong…both lower tree edges cannot in fact be tree edges



DFS And Graph Cycles

- Thm: An undirected graph is *acyclic* iff a DFS yields no back edges
  - If acyclic, no back edges (because a back edge implies a cycle
  - If no back edges, acyclic
    - No back edges implies only tree edges (*Why?*)
    - Only tree edges implies we have a tree or a forest
    - Which by definition is acyclic
- Thus, can run DFS to find whether a graph has a cycle

DFS And Cycles

How would you modify the code to detect cycles?

```
Data: color[V], time,
      prev[V],d[V], f[V]
DFS(G) // where prog starts
{
   for each vertex u ∈ V
   {
      color[u] = WHITE;
    prev[u]=NIL;
    f[u]=inf; d[u]=inf;
   }
   time = 0;
   for each vertex u ∈ V
     if (color[u] == WHITE)
         DFS_Visit(u);
}
```

```
DFS_Visit(u)
{
   color[u] = GREY;
   time = time+1;
   d[u] = time;
   for each v ∈ Adj[u]
   {
      if (color[v]==WHITE){
      prev[v]=u;
         DFS_Visit(v);
      }
}
   color[u] = BLACK;
   time = time+1;
   f[u] = time;
}
```

DFS And Cycles

What will be the running time?

```
Data: color[V], time,
      prev[V],d[V], f[V]
DFS(G) // where prog starts
{
    for each vertex u ∈ V
    {
        color[u] = WHITE;
     prev[u]=NIL;
     f[u]=inf; d[u]=inf;
    }
    time = 0;
    for each vertex u ∈ V
      if (color[u] == WHITE)
           DFS_Visit(u);
}
```

```
DFS_Visit(u)
{
    color[u] = GREY;
    time = time+1;
    d[u] = time;
    for each v ∈ Adj[u]
    {
        if (color[v]==WHITE){
        prev[v]=u;
            DFS_Visit(v);      }
        else {cycle exists;}
    }
    color[u] = BLACK;
    time = time+1;
    f[u] = time;
}
```

DFS And Cycles

- *What will be the running time for undirected graph to detect cycle?*
- A: O(V+E)
- We can actually determine if cycles exist in O(V) time
  - How??

DFS And Cycles

- *What will be the running time for undirected graph to detect cycle?*
- A: $O(V+E)$
- We can actually determine if cycles exist in $O(V)$ time:
  - In an undirected acyclic forest, $|E| \leq |V| - 1$
  - So count the edges: if ever see $|V|$ distinct edges, must have seen a back edge along the way

DFS And Cycles

- *What will be the running time for directed graph to detect cycle?*
- A: O(V+E)

Exercises on DFS

- CLRS Chapter 22 (Elementary Graph Algorithms)
- Exercise: (Page
  - 22.3-5 –Detect edge using d[u], d[v], f[u], f[v]
  - 22.3-12 – Connected Component
  - 22.3-13 – Singly connected

Some applications of BFS and DFS

- Topological Sort (Topic of Next Lecture)
- Euler Path (Topic of Next Lecture)
- Dictionary Search
- Mathematical Problem
- Grid Traversal

The idea of State/Node

- Parameters describing a scenario
- Useless Parameters
  - If value of the parameter change doesn’t affect the outcome
  - If value of the parameter can be derived from other parameters
- Useful Parameter
  - Not useless!!!

Example - States

- Grid Problem, direction change takes time
- Grid Problem, blocks alternating