# 11

## 11.3

4

▶ **Example 4** Find the angle between a diagonal of a cube and one of its edges.

***Solution.*** Assume that the cube has side $a$, and introduce a coordinate system as shown in Figure 11.3.6. In this coordinate system the vector

$$\mathbf{d} = a\mathbf{i} + a\mathbf{j} + a\mathbf{k}$$

is a diagonal of the cube and the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ run along the edges. By symmetry, the diagonal makes the same angle with each edge, so it is sufficient to find the angle between $\mathbf{d}$ and $\mathbf{i}$ (the direction angle $\alpha$). Thus,

$$\cos\alpha = \frac{\mathbf{d}\cdot\mathbf{i}}{\|\mathbf{d}\|\,\|\mathbf{i}\|} = \frac{a}{\|\mathbf{d}\|} = \frac{a}{\sqrt{3a^2}} = \frac{1}{\sqrt{3}}$$

and hence

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 0.955 \text{ radian} \approx 54.7^\circ \quad ◀$$

5

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{e}_1)\mathbf{e}_1 + (\mathbf{v}\cdot\mathbf{e}_2)\mathbf{e}_2 \qquad (7)$$

In this formula we call $(\mathbf{v}\cdot\mathbf{e}_1)\mathbf{e}_1$ and $(\mathbf{v}\cdot\mathbf{e}_2)\mathbf{e}_2$ the ***vector components*** of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively; and we call $\mathbf{v}\cdot\mathbf{e}_1$ and $\mathbf{v}\cdot\mathbf{e}_2$ the ***scalar components*** of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively. If $\theta$ denotes the angle between $\mathbf{v}$ and $\mathbf{e}_1$, and the angle between $\mathbf{v}$ and $\mathbf{e}_2$ is $\pi/2$ or less, then the scalar components of $\mathbf{v}$ can be written in trigonometric form as

$$\mathbf{v}\cdot\mathbf{e}_1 = \|\mathbf{v}\|\cos\theta \quad \text{and} \quad \mathbf{v}\cdot\mathbf{e}_2 = \|\mathbf{v}\|\sin\theta \qquad (8)$$

(Figure 11.3.8$b$). Moreover, the vector components of $\mathbf{v}$ can be expressed as

$$(\mathbf{v}\cdot\mathbf{e}_1)\mathbf{e}_1 = (\|\mathbf{v}\|\cos\theta)\mathbf{e}_1 \quad \text{and} \quad (\mathbf{v}\cdot\mathbf{e}_2)\mathbf{e}_2 = (\|\mathbf{v}\|\sin\theta)\mathbf{e}_2 \qquad (9)$$

▶ **Example 5** Let

$$\mathbf{v} = \langle 2, 3 \rangle, \quad \mathbf{e}_1 = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle, \quad \text{and} \quad \mathbf{e}_2 = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

Find the scalar components of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$ and the vector components of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$.

***Solution.*** The scalar components of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$ are

$$\mathbf{v} \cdot \mathbf{e}_1 = 2\left(\frac{1}{\sqrt{2}}\right) + 3\left(\frac{1}{\sqrt{2}}\right) = \frac{5}{\sqrt{2}}$$

$$\mathbf{v} \cdot \mathbf{e}_2 = 2\left(-\frac{1}{\sqrt{2}}\right) + 3\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

so the vector components are

$$(\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 = \frac{5}{\sqrt{2}}\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle \frac{5}{2}, \frac{5}{2} \right\rangle$$

$$(\mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2 = \frac{1}{\sqrt{2}}\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle -\frac{1}{2}, \frac{1}{2} \right\rangle \quad ◀$$

7

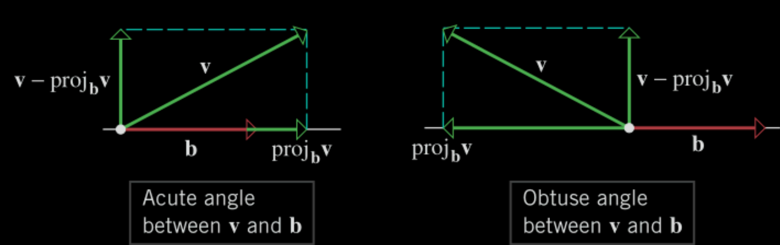
The orthogonal projection of $\mathbf{v}$ on an arbitrary nonzero vector $\mathbf{b}$ can be obtained by normalizing $\mathbf{b}$ and then applying Formula (11); that is,

$$\operatorname{proj}_{\mathbf{b}}\mathbf{v} = \left(\mathbf{v} \cdot \frac{\mathbf{b}}{\|\mathbf{b}\|}\right)\left(\frac{\mathbf{b}}{\|\mathbf{b}\|}\right)$$

which can be rewritten as

$$\operatorname{proj}_{\mathbf{b}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} \tag{12}$$

Geometrically, if $\mathbf{b}$ and $\mathbf{v}$ have a common initial point, then $\operatorname{proj}_{\mathbf{b}}\mathbf{v}$ is the vector that is determined when a perpendicular is dropped from the terminal point of $\mathbf{v}$ to the line through $\mathbf{b}$ (illustrated in Figure 11.3.10 in two cases).




▶ **Figure 11.3.10**

Moreover, it is evident from Figure 11.3.10 that if we subtract $\operatorname{proj}_{\mathbf{b}}\mathbf{v}$ from $\mathbf{v}$, then the resulting vector

$$\mathbf{v} - \operatorname{proj}_{\mathbf{b}}\mathbf{v}$$

will be orthogonal to $\mathbf{b}$; we call this the ***vector component of* $\mathbf{v}$ *orthogonal to* $\mathbf{b}$**.

**Example 7** Find the orthogonal projection of $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ on $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j}$, find the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$.

*Solution.* We have

$$\mathbf{v} \cdot \mathbf{b} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} + 2\mathbf{j}) = 2 + 2 + 0 = 4$$
$$\|\mathbf{b}\|^2 = 2^2 + 2^2 = 8$$

Thus, the orthogonal projection of $\mathbf{v}$ on $\mathbf{b}$ is

$$\operatorname{proj}_{\mathbf{b}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} = \frac{4}{8}(2\mathbf{i} + 2\mathbf{j}) = \mathbf{i} + \mathbf{j}$$

and the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$ is

$$\mathbf{v} - \operatorname{proj}_{\mathbf{b}}\mathbf{v} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) - (\mathbf{i} + \mathbf{j}) = \mathbf{k}$$

# 11.4

## 3

**GEOMETRIC PROPERTIES OF THE CROSS PRODUCT**

The following theorem shows that the cross product of two vectors is orthogonal to both factors. This property of the cross product will be used many times in the following sections.

**11.4.4 THEOREM** *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in 3-space, then:*

(*a*) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$  ($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{u}$)

(*b*) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$  ($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{v}$)

**Example 3** Find a vector that is orthogonal to both of the vectors $\mathbf{u} = \langle 2, -1, 3 \rangle$ and $\mathbf{v} = \langle -7, 2, -1 \rangle$.

*Solution.* By Theorem 11.4.4, the vector $\mathbf{u} \times \mathbf{v}$ will be orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. We compute that

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -7 & 2 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ -7 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ -7 & 2 \end{vmatrix}\mathbf{k} = -5\mathbf{i} - 19\mathbf{j} - 3\mathbf{k}$$

# 11.5

2

**11.5.1 THEOREM**

(*a*) *The line in 2-space that passes through the point $P_0(x_0, y_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b\rangle = a\mathbf{i} + b\mathbf{j}$ has parametric equations*

$$x = x_0 + at, \quad y = y_0 + bt \tag{1}$$

(*b*) *The line in 3-space that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b, c\rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ has parametric equations*

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \tag{2}$$

**▶ Example 2**

(a) Find parametric equations of the line $L$ passing through the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

(b) Where does the line intersect the $xy$-plane?

***Solution (a).*** The vector $\overrightarrow{P_1P_2} = \langle 3, -4, 8\rangle$ is parallel to $L$ and the point $P_1(2, 4, -1)$ lies on $L$, so it follows from (2) that $L$ has parametric equations

$$x = 2 + 3t, \quad y = 4 - 4t, \quad z = -1 + 8t \tag{3}$$

Had we used $P_2$ as the point on $L$ rather than $P_1$, we would have obtained the equations

$$x = 5 + 3t, \quad y = -4t, \quad z = 7 + 8t$$

Although these equations look different from those obtained using $P_1$, the two sets of equations are actually equivalent in that both generate $L$ as $t$ varies from $-\infty$ to $+\infty$. To

***Solution (b).*** It follows from (3) in part (a) that the line intersects the $xy$-plane at the point where $z = -1 + 8t = 0$, that is, when $t = \frac{1}{8}$. Substituting this value of $t$ in (3) yields the point of intersection $(x, y, z) = \left(\frac{19}{8}, \frac{7}{2}, 0\right)$. ◀

▶ **Example 3** Let $L_1$ and $L_2$ be the lines

$$L_1\colon x = 1 + 4t, \quad y = 5 - 4t, \quad z = -1 + 5t$$
$$L_2\colon x = 2 + 8t, \quad y = 4 - 3t, \quad z = 5 + t$$

(a) Are the lines parallel?

(b) Do the lines intersect?

***Solution (a).*** The line $L_1$ is parallel to the vector $4\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, and the line $L_2$ is parallel to the vector $8\mathbf{i} - 3\mathbf{j} + \mathbf{k}$. These vectors are not parallel since neither is a scalar multiple of the other. Thus, the lines are not parallel.

***Solution (b).*** For $L_1$ and $L_2$ to intersect at some point $(x_0, y_0, z_0)$ these coordinates would have to satisfy the equations of both lines. In other words, there would have to exist values $t_1$ and $t_2$ for the parameters such that

$$x_0 = 1 + 4t_1, \quad y_0 = 5 - 4t_1, \quad z_0 = -1 + 5t_1$$

and

$$x_0 = 2 + 8t_2, \quad y_0 = 4 - 3t_2, \quad z_0 = 5 + t_2$$

This leads to three conditions on $t_1$ and $t_2$,

$$\begin{aligned} 1 + 4t_1 &= 2 + 8t_2 \\ 5 - 4t_1 &= 4 - 3t_2 \\ -1 + 5t_1 &= 5 + t_2 \end{aligned} \tag{4}$$

Thus, the lines intersect if there are values of $t_1$ and $t_2$ that satisfy all three equations, and the lines do not intersect if there are no such values. You should be familiar with methods for solving systems of two linear equations in two unknowns; however, this is a system of three linear equations in two unknowns. To determine whether this system has a solution we will solve the first two equations for $t_1$ and $t_2$ and then check whether these values satisfy the third equation.

We will solve the first two equations by the method of elimination. We can eliminate the unknown $t_1$ by adding the equations. This yields the equation

$$6 = 6 + 5t_2$$

from which we obtain $t_2 = 0$. We can now find $t_1$ by substituting this value of $t_2$ in either the first or second equation. This yields $t_1 = \frac{1}{4}$. However, the values $t_1 = \frac{1}{4}$ and $t_2 = 0$ do not satisfy the third equation in (4), so the lines do not intersect. ◀

and for the equation in 3-space we define them as

$$\mathbf{r} = \langle x, y, z \rangle, \quad \mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle, \quad \mathbf{v} = \langle a, b, c \rangle \tag{8}$$

Substituting (7) and (8) in (5) and (6), respectively, yields the equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \tag{9}$$

in both cases. We call this the ***vector equation of a line*** in 2-space or 3-space. In this equation, $\mathbf{v}$ is a nonzero vector parallel to the line, and $\mathbf{r}_0$ is a vector whose components are the coordinates of a point on the line.

▶ **Example 6** Find an equation of the line in 3-space that passes through the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

***Solution.*** The vector

$$\overrightarrow{P_1P_2} = \langle 3, -4, 8 \rangle$$

is parallel to the line, so it can be used as $\mathbf{v}$ in (9). For $\mathbf{r}_0$ we can use either the vector from the origin to $P_1$ or the vector from the origin to $P_2$. Using the former yields

$$\mathbf{r}_0 = \langle 2, 4, -1 \rangle$$

Thus, a vector equation of the line through $P_1$ and $P_2$ is

$$\langle x, y, z \rangle = \langle 2, 4, -1 \rangle + t\langle 3, -4, 8 \rangle$$

If needed, we can express the line parametrically by equating corresponding components on the two sides of this vector equation, in which case we obtain the parametric equations in Example 2 (verify). ◀

# 11.6

1

**PLANES DETERMINED BY A POINT AND A NORMAL VECTOR**

A plane in 3-space can be determined uniquely by specifying a point in the plane and a vector perpendicular to the plane (Figure 11.6.2). A vector perpendicular to a plane is called a ***normal*** to the plane.

Suppose that we want to find an equation of the plane passing through $P_0(x_0, y_0, z_0)$ and perpendicular to the vector $\mathbf{n} = \langle a, b, c\rangle$. Define the vectors $\mathbf{r}_0$ and $\mathbf{r}$ as

$$\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle \quad \text{and} \quad \mathbf{r} = \langle x, y, z\rangle$$

It should be evident from Figure 11.6.3 that the plane consists precisely of those points $P(x, y, z)$ for which the vector $\mathbf{r} - \mathbf{r}_0$ is orthogonal to $\mathbf{n}$; or, expressed as an equation,

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \tag{1}$$

If preferred, we can express this vector equation in terms of components as

$$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0 \tag{2}$$

from which we obtain

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{3}$$

This is called the ***point-normal form*** of the equation of a plane. Formulas (1) and (2) are

▶ **Example 1** Find an equation of the plane passing through the point $(3, -1, 7)$ and perpendicular to the vector $\mathbf{n} = \langle 4, 2, -5\rangle$.

***Solution.*** From (3), a point-normal form of the equation is

$$4(x - 3) + 2(y + 1) - 5(z - 7) = 0 \tag{4}$$

If preferred, this equation can be written in vector form as

$$\langle 4, 2, -5\rangle \cdot \langle x - 3, y + 1, z - 7\rangle = 0 \ ◀$$

2

▶ **Example 2** Determine whether the planes

$$3x - 4y + 5z = 0 \quad \text{and} \quad -6x + 8y - 10z - 4 = 0$$

are parallel.

***Solution.*** It is clear geometrically that two planes are parallel if and only if their norma are parallel vectors. A normal to the first plane is

$$\mathbf{n}_1 = \langle 3, -4, 5 \rangle$$

and a normal to the second plane is

$$\mathbf{n}_2 = \langle -6, 8, -10 \rangle$$

Since $\mathbf{n}_2$ is a scalar multiple of $\mathbf{n}_1$, the normals are parallel, and hence so are the planes.

3

**Example 3** Find an equation of the plane through the points $P_1(1, 2, -1)$, $P_2(2, 3, 1)$, nd $P_3(3, -1, 2)$.

***olution.*** Since the points $P_1$, $P_2$, and $P_3$ lie in the plane, the vectors $\overrightarrow{P_1P_2} = \langle 1, 1, 2 \rangle$ nd $\overrightarrow{P_1P_3} = \langle 2, -3, 3 \rangle$ are parallel to the plane. Therefore,

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 2 & -3 & 3 \end{vmatrix} = 9\mathbf{i} + \mathbf{j} - 5\mathbf{k}$$

s normal to the plane, since it is orthogonal to both $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$. By using this normal nd the point $P_1(1, 2, -1)$ in the plane, we obtain the point-normal form

$$9(x - 1) + (y - 2) - 5(z + 1) = 0$$

vhich can be rewritten as

$$9x + y - 5z - 16 = 0 \quad ◀$$

4

▶ **Example 4** Determine whether the line

$$x = 3 + 8t, \quad y = 4 + 5t, \quad z = -3 - t$$

is parallel to the plane $x - 3y + 5z = 12$.

***Solution.*** The vector $\mathbf{v} = \langle 8, 5, -1 \rangle$ is parallel to the line and the vector $\mathbf{n} = \langle 1, -3, 5 \rangle$ is normal to the plane. For the line and plane to be parallel, the vectors $\mathbf{v}$ and $\mathbf{n}$ must be orthogonal. But this is not so, since the dot product

$$\mathbf{v} \cdot \mathbf{n} = (8)(1) + (5)(-3) + (-1)(5) = -12$$

is nonzero. Thus, the line and plane are not parallel. ◀

5

▸ **Example 5** Find the intersection of the line and plane in Example 4.

***Solution.*** If we let $(x_0, y_0, z_0)$ be the point of intersection, then the coordinates of this point satisfy both the equation of the plane and the parametric equations of the line. Thus,

$$x_0 - 3y_0 + 5z_0 = 12 \tag{7}$$

and for some value of $t$, say $t = t_0$,

$$x_0 = 3 + 8t_0, \quad y_0 = 4 + 5t_0, \quad z_0 = -3 - t_0 \tag{8}$$

Substituting (8) in (7) yields

$$(3 + 8t_0) - 3(4 + 5t_0) + 5(-3 - t_0) = 12$$

Solving for $t_0$ yields $t_0 = -3$ and on substituting this value in (8), we obtain

$$(x_0, y_0, z_0) = (-21, -11, 0)$$ ◂

6

## INTERSECTING PLANES

Two distinct intersecting planes determine two positive angles of intersection—an (acute) angle $\theta$ that satisfies the condition $0 \le \theta \le \pi/2$ and the supplement of that angle (Figure 11.6.7*a*). If $\mathbf{n}_1$ and $\mathbf{n}_2$ are normals to the planes, then depending on the directions of $\mathbf{n}_1$ and $\mathbf{n}_2$, the angle $\theta$ is either the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ or the angle between $\mathbf{n}_1$ and $-\mathbf{n}_2$ (Figure 11.6.7*b*). In both cases, Theorem 11.3.3 yields the following formula for the *acute angle $\theta$ between the planes*:

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} \tag{9}$$

▸ **Example 6** Find the acute angle of intersection between the two planes

$$2x - 4y + 4z = 6 \quad \text{and} \quad 6x + 2y - 3z = 4$$

***Solution.*** The given equations yield the normals $\mathbf{n}_1 = \langle 2, -4, 4\rangle$ and $\mathbf{n}_2 = \langle 6, 2, -3\rangle$. Thus, Formula (9) yields

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{|-8|}{\sqrt{36}\sqrt{49}} = \frac{4}{21}$$

from which we obtain

$$\theta = \cos^{-1}\left(\frac{4}{21}\right) \approx 79^\circ$$ ◂

7

▶ **Example 7** Find an equation for the line $L$ of intersection of the planes in Example 6.

***Solution.*** First compute $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 2, -4, 4\rangle \times \langle 6, 2, -3\rangle = \langle 4, 30, 28\rangle$. Since $\mathbf{v}$ is orthogonal to $\mathbf{n}_1$, it is parallel to the first plane, and since $\mathbf{v}$ is orthogonal to $\mathbf{n}_2$, it is parallel to the second plane. That is, $\mathbf{v}$ is parallel to $L$, the intersection of the two planes. To find a point on $L$ we observe that $L$ must intersect the $xy$-plane, $z = 0$, since $\mathbf{v} \cdot \langle 0, 0, 1\rangle = 28 \neq 0$. Substituting $z = 0$ in the equations of both planes yields

$$\begin{aligned} 2x - 4y &= 6 \\ 6x + 2y &= 4 \end{aligned}$$

with solution $x = 1$, $y = -1$. Thus, $P(1, -1, 0)$ is a point on $L$. A vector equation for $L$ is

$$\langle x, y, z\rangle = \langle 1, -1, 0\rangle + t\langle 4, 30, 28\rangle \quad ◀$$

9

**11.6.2 THEOREM** *The distance $D$ between a point $P_0(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is*

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \qquad (10)$$

▶ **Example 9** The planes

$$x + 2y - 2z = 3 \quad \text{and} \quad 2x + 4y - 4z = 7$$

are parallel since their normals, $\langle 1, 2, -2\rangle$ and $\langle 2, 4, -4\rangle$, are parallel vectors. Find t distance between these planes.

***Solution.*** To find the distance $D$ between the planes, we can select an *arbitrary* point in one of the planes and compute its distance to the other plane. By setting $y = z = 0$ in the equation $x + 2y - 2z = 3$, we obtain the point $P_0(3, 0, 0)$ in this plane. From (10), the distance from $P_0$ to the plane $2x + 4y - 4z = 7$ is

$$D = \frac{|(2)(3) + 4(0) + (-4)(0) - 7|}{\sqrt{2^2 + 4^2 + (-4)^2}} = \frac{1}{6} \quad ◀$$

▶ **Example 10** It was shown in Example 3 of Section 11.5 that the lines

$$L_1: x = 1 + 4t, \quad y = 5 - 4t, \quad z = -1 + 5t$$
$$L_2: x = 2 + 8t, \quad y = 4 - 3t, \quad z = 5 + t$$

are skew. Find the distance between them.

***Solution.*** Let $P_1$ and $P_2$ denote parallel planes containing $L_1$ and $L_2$, respectively (Figure 11.6.10). To find the distance $D$ between $L_1$ and $L_2$, we will calculate the distance from a point in $P_1$ to the plane $P_2$. Since $L_1$ lies in plane $P_1$, we can find a point in $P_1$ by finding a point on the line $L_1$; we can do this by substituting any convenient value of $t$ in the parametric equations of $L_1$. The simplest choice is $t = 0$, which yields the point $Q_1(1, 5, -1)$.

The next step is to find an equation for the plane $P_2$. For this purpose, observe that the vector $\mathbf{u}_1 = \langle 4, -4, 5\rangle$ is parallel to line $L_1$, and therefore also parallel to planes $P_1$ and $P_2$. Similarly, $\mathbf{u}_2 = \langle 8, -3, 1\rangle$ is parallel to $L_2$ and hence parallel to $P_1$ and $P_2$. Therefore, the cross product

$$\mathbf{n} = \mathbf{u}_1 \times \mathbf{u}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -4 & 5 \\ 8 & -3 & 1 \end{vmatrix} = 11\mathbf{i} + 36\mathbf{j} + 20\mathbf{k}$$

is normal to both $P_1$ and $P_2$. Using this normal and the point $Q_2(2, 4, 5)$ found by setting $t = 0$ in the equations of $L_2$, we obtain an equation for $P_2$:

$$11(x - 2) + 36(y - 4) + 20(z - 5) = 0$$

or

$$11x + 36y + 20z - 266 = 0$$

The distance between $Q_1(1, 5, -1)$ and this plane is

$$D = \frac{|(11)(1) + (36)(5) + (20)(-1) - 266|}{\sqrt{11^2 + 36^2 + 20^2}} = \frac{95}{\sqrt{1817}}$$

# 12

## 12.1

### 4

▶ **Example 4** Find the natural domain of

$$\mathbf{r}(t) = \langle \ln|t-1|, e^t, \sqrt{t} \rangle = (\ln|t-1|)\mathbf{i} + e^t\mathbf{j} + \sqrt{t}\mathbf{k}$$

***Solution.*** The natural domains of the component functions

$$x(t) = \ln|t-1|, \quad y(t) = e^t, \quad z(t) = \sqrt{t}$$

are

$$(-\infty, 1) \cup (1, +\infty), \quad (-\infty, +\infty), \quad [0, +\infty)$$

respectively. The intersection of these sets is

$$[0, 1) \cup (1, +\infty)$$

(verify), so the natural domain of $\mathbf{r}(t)$ consists of all values of $t$ such that

$$0 \le t < 1 \quad \text{or} \quad t > 1 \;◀$$

6

**Example 6** Sketch the graph and a radius vector of

(a) $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}, \quad 0 \le t \le 2\pi$

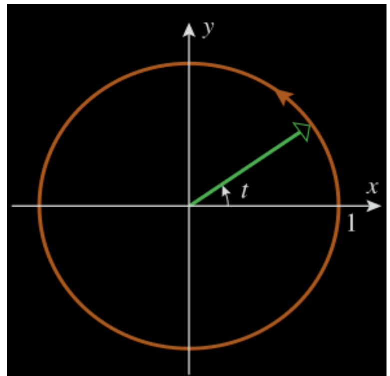
(b) $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2\mathbf{k}, \quad 0 \le t \le 2\pi$



*Solution (a).* The corresponding parametric equations are

$$x = \cos t, \quad y = \sin t \qquad (0 \le t \le 2\pi)$$

so the graph is a circle of radius 1, centered at the origin, and oriented counterclockwise. The graph and a radius vector are shown in Figure 12.1.7.



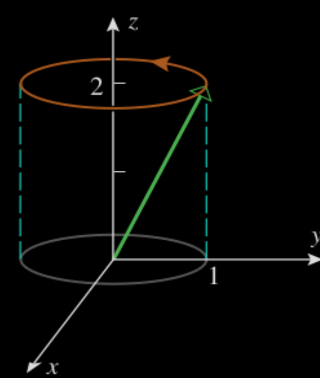

$\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$

igure 12.1.7

*Solution (b).* The corresponding parametric equations are

$$x = \cos t, \quad y = \sin t, \quad z = 2 \qquad (0 \le t \le 2\pi)$$

From the third equation, the tip of the radius vector traces a curve in the plane $z = 2$, and from the first two equations, the curve is a circle of radius 1 centered at the point $(0, 0, 2)$ and traced counterclockwise looking down the $z$-axis. The graph and a radius vector are




$\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2\mathbf{k}$

# 12.2

1

**Example 1** Let $\mathbf{r}(t) = t^2\mathbf{i} + e^t\mathbf{j} - (2\cos \pi t)\mathbf{k}$. Then

$$\lim_{t\to 0} \mathbf{r}(t) = \left(\lim_{t\to 0} t^2\right)\mathbf{i} + \left(\lim_{t\to 0} e^t\right)\mathbf{j} - \left(\lim_{t\to 0} 2\cos \pi t\right)\mathbf{k} = \mathbf{j} - 2\mathbf{k}$$

Alternatively, using the angle bracket notation for vectors,

$$\lim_{t\to 0} \mathbf{r}(t) = \lim_{t\to 0} \langle t^2, e^t, -2\cos \pi t\rangle = \left\langle \lim_{t\to 0} t^2, \lim_{t\to 0} e^t, \lim_{t\to 0}(-2\cos \pi t)\right\rangle = \langle 0, 1, -2\rangle$$

2

**12.2.3 DEFINITION** If $\mathbf{r}(t)$ is a vector-valued function, we define the ***derivative of* r *with respect to t*** to be the vector-valued function $\mathbf{r}'$ given by

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \tag{4}$$

The domain of $\mathbf{r}'$ consists of all values of $t$ in the domain of $\mathbf{r}(t)$ for which the limit exists.

**▶ Example 2** Let $\mathbf{r}(t) = t^2\mathbf{i} + e^t\mathbf{j} - (2\cos \pi t)\mathbf{k}$. Then

$$\begin{aligned}\mathbf{r}'(t) &= \frac{d}{dt}(t^2)\mathbf{i} + \frac{d}{dt}(e^t)\mathbf{j} - \frac{d}{dt}(2\cos \pi t)\mathbf{k} \\ &= 2t\mathbf{i} + e^t\mathbf{j} + (2\pi \sin \pi t)\mathbf{k} \quad ◀\end{aligned}$$

3

**▶ Example 3** Find parametric equations of the tangent line to the circular helix

$$x = \cos t, \quad y = \sin t, \quad z = t$$

where $t = t_0$, and use that result to find parametric equations for the tangent line at the point where $t = \pi$.

***Solution.*** The vector equation of the helix is

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$$

so we have

$$\mathbf{r}_0 = \mathbf{r}(t_0) = \cos t_0 \mathbf{i} + \sin t_0 \mathbf{j} + t_0 \mathbf{k}$$

$$\mathbf{v}_0 = \mathbf{r}'(t_0) = (-\sin t_0)\mathbf{i} + \cos t_0 \mathbf{j} + \mathbf{k}$$

It follows from (5) that the vector equation of the tangent line at $t = t_0$ is

$$\begin{aligned} \mathbf{r} &= \cos t_0 \mathbf{i} + \sin t_0 \mathbf{j} + t_0 \mathbf{k} + t[(-\sin t_0)\mathbf{i} + \cos t_0 \mathbf{j} + \mathbf{k}] \\ &= (\cos t_0 - t \sin t_0)\mathbf{i} + (\sin t_0 + t \cos t_0)\mathbf{j} + (t_0 + t)\mathbf{k} \end{aligned}$$

Thus, the parametric equations of the tangent line at $t = t_0$ are

$$x = \cos t_0 - t \sin t_0, \quad y = \sin t_0 + t \cos t_0, \quad z = t_0 + t$$

In particular, the tangent line at $t = \pi$ has parametric equations

$$x = -1, \quad y = -t, \quad z = \pi + t$$

The graph of the helix and this tangent line are shown in Figure 12.2.5. ◀

4

▶ **Example 4** Let

$$\mathbf{r}_1(t) = (\tan^{-1} t)\mathbf{i} + (\sin t)\mathbf{j} + t^2\mathbf{k}$$

and

$$\mathbf{r}_2(t) = (t^2 - t)\mathbf{i} + (2t - 2)\mathbf{j} + (\ln t)\mathbf{k}$$

The graphs of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ intersect at the origin. Find the degree measure of the acute angle between the tangent lines to the graphs of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ at the origin.

***Solution.*** The graph of $\mathbf{r}_1(t)$ passes through the origin at $t = 0$, where its tangent vector is

$$\mathbf{r}_1'(0) = \left\langle \frac{1}{1+t^2}, \cos t, 2t \right\rangle \bigg|_{t=0} = \langle 1, 1, 0 \rangle$$

The graph of $\mathbf{r}_2(t)$ passes through the origin at $t = 1$ (verify), where its tangent vector is

$$\mathbf{r}_2'(1) = \left\langle 2t - 1, 2, \frac{1}{t} \right\rangle \bigg|_{t=1} = \langle 1, 2, 1 \rangle$$

By Theorem 11.3.3, the angle $\theta$ between these two tangent vectors satisfies

$$\cos\theta = \frac{\langle 1, 1, 0 \rangle \cdot \langle 1, 2, 1 \rangle}{\|\langle 1, 1, 0 \rangle\| \, \|\langle 1, 2, 1 \rangle\|} = \frac{1 + 2 + 0}{\sqrt{2}\sqrt{6}} = \frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2}$$

It follows that $\theta = \pi/6$ radians, or $30°$. ◀

5

$$\frac{d}{dt}[\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)] = \mathbf{r}_1(t)\cdot\frac{d\mathbf{r}_2}{dt} + \frac{d\mathbf{r}_1}{dt}\cdot\mathbf{r}_2(t)$$

$$\frac{d}{dt}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)] = \mathbf{r}_1(t)\times\frac{d\mathbf{r}_2}{dt} + \frac{d\mathbf{r}_1}{dt}\times\mathbf{r}_2(t)$$

**12.2.8 THEOREM** *If $\mathbf{r}(t)$ is a differentiable vector-valued function in 2-space or 3-space and $\|\mathbf{r}(t)\|$ is constant for all $t$, then*

$$\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0 \tag{8}$$

*that is, $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal vectors for all $t$.*

8,9

**▶ Example 8** Evaluate the definite integral $\int_0^2 (2t\mathbf{i} + 3t^2\mathbf{j})\,dt$.

***Solution.*** Integrating the components yields

$$\int_0^2 (2t\mathbf{i} + 3t^2\mathbf{j})\,dt = t^2\Big]_0^2\,\mathbf{i} + t^3\Big]_0^2\,\mathbf{j} = 4\mathbf{i} + 8\mathbf{j}$$

***Alternative Solution.*** The function $\mathbf{R}(t) = t^2\mathbf{i} + t^3\mathbf{j}$ is an antiderivative of the integrand since $\mathbf{R}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$. Thus, it follows from (17) that

$$\int_0^2 (2t\mathbf{i} + 3t^2\mathbf{j})\,dt = \mathbf{R}(t)\Big]_0^2 = t^2\mathbf{i} + t^3\mathbf{j}\Big]_0^2 = (4\mathbf{i} + 8\mathbf{j}) - (0\mathbf{i} + 0\mathbf{j}) = 4\mathbf{i} + 8\mathbf{j} \quad ◀$$

**▶ Example 9** Find $\mathbf{r}(t)$ given that $\mathbf{r}'(t) = \langle 3, 2t\rangle$ and $\mathbf{r}(1) = \langle 2, 5\rangle$.

***Solution.*** Integrating $\mathbf{r}'(t)$ to obtain $\mathbf{r}(t)$ yields

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \int \langle 3, 2t\rangle\,dt = \langle 3t, t^2\rangle + \mathbf{C}$$

where $\mathbf{C}$ is a vector constant of integration. To find the value of $\mathbf{C}$ we substitute $t = 1$ and use the given value of $\mathbf{r}(1)$ to obtain

$$\mathbf{r}(1) = \langle 3, 1\rangle + \mathbf{C} = \langle 2, 5\rangle$$

so that $\mathbf{C} = \langle -1, 4\rangle$. Thus,

$$\mathbf{r}(t) = \langle 3t, t^2\rangle + \langle -1, 4\rangle = \langle 3t - 1, t^2 + 4\rangle \quad ◀$$

12.3

$$\left\|\frac{d\mathbf{r}}{dt}\right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad \text{or} \quad \left\|\frac{d\mathbf{r}}{dt}\right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

2-space 3-space

Substituting these expressions in (2) and (4) leads us to the following theorem.

**12.3.1 THEOREM** *If $C$ is the graph in 2-space or 3-space of a smooth vector-valued function $\mathbf{r}(t)$, then its arc length $L$ from $t = a$ to $t = b$ is*

$$L = \int_a^b \left\|\frac{d\mathbf{r}}{dt}\right\| dt \qquad (5)$$

2

**Example 2** Find the arc length of that portion of the circular helix

$$x = \cos t, \quad y = \sin t, \quad z = t$$

m $t = 0$ to $t = \pi$.

*lution.* Set $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k} = \langle \cos t, \sin t, t\rangle$. Then

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$$

**ns**

heorem 12.3.1 the arc length of the helix is

$$L = \int_0^\pi \left\|\frac{d\mathbf{r}}{dt}\right\| dt = \int_0^\pi \sqrt{2}\, dt = \sqrt{2}\pi \blacktriangleleft$$

**Example 4** Find a change of parameter $t = g(\tau)$ for the circle

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} \qquad (0 \le t \le 2\pi)$$

such that

(a) the circle is traced counterclockwise as $\tau$ increases over the interval [0, 1];

(b) the circle is traced clockwise as $\tau$ increases over the interval [0, 1].

***Solution (a).*** The given circle is traced counterclockwise as $t$ increases. Thus, if we choose $g$ to be an increasing function, then it will follow from the relationship $t = g(\tau)$ that $t$ increases when $\tau$ increases, thereby ensuring that the circle will be traced counterclockwise as $\tau$ increases. We also want to choose $g$ so that $t$ increases from 0 to $2\pi$ as $\tau$ increases from 0 to 1. A simple choice of $g$ that satisfies all of the required criteria is the linear function graphed in Figure 12.3.5*a*. The equation of this line is

$$t = g(\tau) = 2\pi\tau \tag{7}$$

which is the desired change of parameter. The resulting representation of the circle in terms of the parameter $\tau$ is

$$\mathbf{r}(g(\tau)) = \cos 2\pi\tau\,\mathbf{i} + \sin 2\pi\tau\,\mathbf{j} \qquad (0 \le \tau \le 1)$$

***Solution (b).*** To ensure that the circle is traced clockwise, we will choose $g$ to be a decreasing function such that $t$ decreases from $2\pi$ to 0 as $\tau$ increases from 0 to 1. A simple choice of $g$ that achieves this is the linear function

$$t = g(\tau) = 2\pi(1 - \tau) \tag{8}$$

graphed in Figure 12.3.5*b*. The resulting representation of the circle in terms of the parameter $\tau$ is

$$\mathbf{r}(g(\tau)) = \cos(2\pi(1-\tau))\mathbf{i} + \sin(2\pi(1-\tau))\mathbf{j} \qquad (0 \le \tau \le 1)$$

which simplifies to (verify)

$$\mathbf{r}(g(\tau)) = \cos 2\pi\tau\,\mathbf{i} - \sin 2\pi\tau\,\mathbf{j} \qquad (0 \le \tau \le 1)$$

6

**▶ Example 6** Find the arc length parametrization of the circular helix

$$\mathbf{r} = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k} \tag{13}$$

that has reference point $\mathbf{r}(0) = (1, 0, 0)$ and the same orientation as the given helix.

***Solution.*** Replacing $t$ by $u$ in $\mathbf{r}$ for integration purposes and taking $t_0 = 0$ in Formula (10), we obtain

$$\mathbf{r} = \cos u\mathbf{i} + \sin u\mathbf{j} + u\mathbf{k}$$

$$\frac{d\mathbf{r}}{du} = (-\sin u)\mathbf{i} + \cos u\mathbf{j} + \mathbf{k}$$

$$\left\|\frac{d\mathbf{r}}{du}\right\| = \sqrt{(-\sin u)^2 + \cos^2 u + 1} = \sqrt{2}$$

$$s = \int_0^t \left\|\frac{d\mathbf{r}}{du}\right\| du = \int_0^t \sqrt{2}\, du = \sqrt{2}u\Big]_0^t = \sqrt{2}t$$

Thus, $t = s/\sqrt{2}$, so (13) can be reparametrized in terms of $s$ as

$$\mathbf{r} = \cos\left(\frac{s}{\sqrt{2}}\right)\mathbf{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\mathbf{j} + \frac{s}{\sqrt{2}}\mathbf{k}$$

8

**▶ Example 8** Recall from Formula (9) of Section 11.5 that the equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \tag{14}$$

is the vector form of the line that passes through the terminal point of $\mathbf{r}_0$ and is parallel to the vector $\mathbf{v}$. Find the arc length parametrization of the line that has reference point $\mathbf{r}_0$ and the same orientation as the given line.

***Solution.*** Replacing $t$ by $u$ in (14) for integration purposes and taking $t_0 = 0$ in Formula (10), we obtain

$$\mathbf{r} = \mathbf{r}_0 + u\mathbf{v} \quad \text{and} \quad \frac{d\mathbf{r}}{du} = \mathbf{v} \qquad \boxed{\text{Since } \mathbf{r}_0 \text{ is constant}}$$

It follows from this that

$$s = \int_0^t \left\|\frac{d\mathbf{r}}{du}\right\| du = \int_0^t \|\mathbf{v}\|\, du = \|\mathbf{v}\|u\Big]_0^t = t\|\mathbf{v}\|$$

This implies that $t = s/\|\mathbf{v}\|$, so (14) can be reparametrized in terms of $s$ as

$$\mathbf{r} = \mathbf{r}_0 + s\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right) \quad ◀ \tag{15}$$

# 12.4

1

**UNIT TANGENT VECTORS**

Recall that if $C$ is the graph of a *smooth* vector-valued function $\mathbf{r}(t)$ in 2-space or 3-space, then the vector $\mathbf{r}'(t)$ is nonzero, tangent to $C$, and points in the direction of increasing parameter. Thus, by normalizing $\mathbf{r}'(t)$ we obtain a unit vector

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \tag{1}$$

that is tangent to $C$ and points in the direction of increasing parameter. We call $\mathbf{T}(t)$ the ***unit tangent vector*** to $C$ at $t$.

▶ **Example 1** Find the unit tangent vector to the graph of $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$ at the point where $t = 2$.

***Solution.*** Since

$$\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$$

we obtain

$$\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{\|\mathbf{r}'(2)\|} = \frac{4\mathbf{i} + 12\mathbf{j}}{\sqrt{160}} = \frac{4\mathbf{i} + 12\mathbf{j}}{4\sqrt{10}} = \frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$$

The graph of $\mathbf{r}(t)$ and the vector $\mathbf{T}(2)$ are shown in Figure 12.4.2. ◀

2

**UNIT NORMAL VECTORS**

Recall from Theorem 12.2.8 that if a vector-valued function $\mathbf{r}(t)$ has constant norm, then $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal vectors. In particular, $\mathbf{T}(t)$ has constant norm 1, so $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are orthogonal vectors. This implies that $\mathbf{T}'(t)$ is perpendicular to the tangent line to $C$ at $t$, so we say that $\mathbf{T}'(t)$ is ***normal*** to $C$ at $t$. It follows that if $\mathbf{T}'(t) \neq \mathbf{0}$, and if we normalize $\mathbf{T}'(t)$, then we obtain a unit vector

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \tag{2}$$

**▶ Example 2** Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$ for the circular helix

$$x = a\cos t, \quad y = a\sin t, \quad z = ct$$

where $a > 0$.

***Solution.*** The radius vector for the helix is

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\mathbf{k}$$

(Figure 12.4.4). Thus,

$$\mathbf{r}'(t) = (-a\sin t)\mathbf{i} + a\cos t\,\mathbf{j} + c\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + c^2} = \sqrt{a^2 + c^2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\frac{a\sin t}{\sqrt{a^2+c^2}}\mathbf{i} + \frac{a\cos t}{\sqrt{a^2+c^2}}\mathbf{j} + \frac{c}{\sqrt{a^2+c^2}}\mathbf{k}$$

$$\mathbf{T}'(t) = -\frac{a\cos t}{\sqrt{a^2+c^2}}\mathbf{i} - \frac{a\sin t}{\sqrt{a^2+c^2}}\mathbf{j}$$

$$\|\mathbf{T}'(t)\| = \sqrt{\left(-\frac{a\cos t}{\sqrt{a^2+c^2}}\right)^2 + \left(-\frac{a\sin t}{\sqrt{a^2+c^2}}\right)^2} = \sqrt{\frac{a^2}{a^2+c^2}} = \frac{a}{\sqrt{a^2+c^2}}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} = -(\cos t\,\mathbf{i} + \sin t\,\mathbf{j})$$

Note that the $\mathbf{k}$ component of the principal unit normal $\mathbf{N}(t)$ is zero for every value of $t$, so this vector always lies in a horizontal plane, as illustrated in Figure 12.4.5. We leave it as an exercise to show that this vector actually always points toward the $z$-axis. ◀

3

▶ **Example 3** The circle of radius $a$ with counterclockwise orientation and centered at the origin can be represented by the vector-valued function

$$\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} \qquad (0 \le t \le 2\pi) \tag{8}$$

Parametrize this circle by arc length and find $\mathbf{T}(s)$ and $\mathbf{N}(s)$.

***Solution.*** In (8) we can interpret $t$ as the angle in radian measure from the positive $x$-axis to the radius vector (Figure 12.4.8). This angle subtends an arc of length $s = at$ on the circle, so we can reparametrize the circle in terms of $s$ by substituting $s/a$ for $t$ in (8). This yields

$$\mathbf{r}(s) = a\cos(s/a)\mathbf{i} + a\sin(s/a)\mathbf{j} \qquad (0 \le s \le 2\pi a)$$

To find $\mathbf{T}(s)$ and $\mathbf{N}(s)$ from Formulas (6) and (7), we must compute $\mathbf{r}'(s)$, $\mathbf{r}''(s)$, and $\|\mathbf{r}''(s)\|$. Doing so, we obtain

$$\mathbf{r}'(s) = -\sin(s/a)\mathbf{i} + \cos(s/a)\mathbf{j}$$

$$\mathbf{r}''(s) = -(1/a)\cos(s/a)\mathbf{i} - (1/a)\sin(s/a)\mathbf{j}$$

$$\|\mathbf{r}''(s)\| = \sqrt{(-1/a)^2\cos^2(s/a) + (-1/a)^2\sin^2(s/a)} = 1/a$$

Thus,

$$\mathbf{T}(s) = \mathbf{r}'(s) = -\sin(s/a)\mathbf{i} + \cos(s/a)\mathbf{j}$$

$$\mathbf{N}(s) = \mathbf{r}''(s)/\|\mathbf{r}''(s)\| = -\cos(s/a)\mathbf{i} - \sin(s/a)\mathbf{j}$$

so $\mathbf{N}(s)$ points toward the center of the circle for all $s$ (Figure 12.4.9). This makes sense geometrically and is also consistent with our earlier observation that in 2-space the unit normal vector is the inward normal. ◀

## BINORMAL VECTORS IN 3-SPACE

If $C$ is the graph of a vector-valued function $\mathbf{r}(t)$ in 3-space, then we define the ***binormal vector*** to $C$ at $t$ to be

$$\boxed{\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)} \tag{9}$$

It follows from properties of the cross product that $\mathbf{B}(t)$ is orthogonal to both $\mathbf{T}(t)$ and $\mathbf{N}(t)$ and is oriented relative to $\mathbf{T}(t)$ and $\mathbf{N}(t)$ by the right-hand rule. Moreover, $\mathbf{T}(t) \times \mathbf{N}(t)$ is a unit vector since

$$\|\mathbf{T}(t) \times \mathbf{N}(t)\| = \|\mathbf{T}(t)\|\|\mathbf{N}(t)\|\sin(\pi/2) = 1$$

Thus, $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$ is a set of three mutually orthogonal unit vectors.

Formula (9) expresses $\mathbf{B}(t)$ in terms of $\mathbf{T}(t)$ and $\mathbf{N}(t)$. Alternatively, the binormal $\mathbf{B}(t)$ can be expressed directly in terms of $\mathbf{r}(t)$ as

$$\mathbf{B}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|} \tag{11}$$

and in the case where the parameter is arc length it can be expressed in terms of $\mathbf{r}(s)$ as

$$\mathbf{B}(s) = \frac{\mathbf{r}'(s) \times \mathbf{r}''(s)}{\|\mathbf{r}''(s)\|} \tag{12}$$

# 12.5

**12.5.1 DEFINITION** If $C$ is a smooth curve in 2-space or 3-space that is parametrized by arc length, then the ***curvature*** of $C$, denoted by $\kappa = \kappa(s)$ ($\kappa$ = Greek "kappa"), is defined by

$$\kappa(s) = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{r}''(s)\| \tag{1}$$

1

**▶ Example 1** In Example 3 of Section 12.4 we showed that the circle of radius $a$, centered at the origin, can be parametrized in terms of arc length as

$$\mathbf{r}(s) = a\cos\left(\frac{s}{a}\right)\mathbf{i} + a\sin\left(\frac{s}{a}\right)\mathbf{j} \qquad (0 \le s \le 2\pi a)$$

Thus,

$$\mathbf{r}''(s) = -\frac{1}{a}\cos\left(\frac{s}{a}\right)\mathbf{i} - \frac{1}{a}\sin\left(\frac{s}{a}\right)\mathbf{j}$$

and hence from (1)

$$\kappa(s) = \|\mathbf{r}''(s)\| = \sqrt{\left[-\frac{1}{a}\cos\left(\frac{s}{a}\right)\right]^2 + \left[-\frac{1}{a}\sin\left(\frac{s}{a}\right)\right]^2} = \frac{1}{a}$$

so the circle has constant curvature $1/a$. ◀

2

**FORMULAS FOR CURVATURE**

Formula (1) is only applicable if the curve is parametrized in terms of arc length. The following theorem provides two formulas for curvature in terms of a general parameter $t$.

**12.5.2 THEOREM** *If $\mathbf{r}(t)$ is a smooth vector-valued function in 2-space or 3-space, then for each value of $t$ at which $\mathbf{T}'(t)$ and $\mathbf{r}''(t)$ exist, the curvature $\kappa$ can be expressed as*

(*a*) $$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} \tag{2}$$

(*b*) $$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \tag{3}$$

3

► **Example 3** Find $\kappa(t)$ for the circular helix

$$x = a\cos t, \quad y = a\sin t, \quad z = ct$$

where $a > 0$.

***Solution.*** The radius vector for the helix is

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\,\mathbf{k}$$

Thus,

$$\mathbf{r}'(t) = (-a\sin t)\mathbf{i} + a\cos t\,\mathbf{j} + c\mathbf{k}$$
$$\mathbf{r}''(t) = (-a\cos t)\mathbf{i} + (-a\sin t)\mathbf{j}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & c \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = (ac\sin t)\mathbf{i} - (ac\cos t)\mathbf{j} + a^2\mathbf{k}$$

Therefore,

$$\|\mathbf{r}'(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + c^2} = \sqrt{a^2 + c^2}$$

and

$$\begin{aligned} \|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= \sqrt{(ac\sin t)^2 + (-ac\cos t)^2 + a^4} \\ &= \sqrt{a^2c^2 + a^4} = a\sqrt{a^2 + c^2} \end{aligned}$$

so

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{a\sqrt{a^2 + c^2}}{\left(\sqrt{a^2 + c^2}\right)^3} = \frac{a}{a^2 + c^2}$$

Note that $\kappa$ does not depend on $t$, which tells us that the helix has constant curvature.

**Example 4** The graph of the vector equation

$$\mathbf{r} = 2\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} \quad (0 \le t \le 2\pi)$$

is the ellipse in Figure 12.5.2. Find the curvature of the ellipse at the endpoints of the major and minor axes, and use a graphing utility to generate the graph of $\kappa(t)$.

***Solution.*** To apply Formula (3), we must treat the ellipse as a curve in the $xy$-plane of an $xyz$-coordinate system by adding a zero $\mathbf{k}$ component and writing its equation as

$$\mathbf{r} = 2\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + 0\mathbf{k}$$

It is not essential to write the zero $\mathbf{k}$ component explicitly as long as you assume it to be there when you calculate a cross product. Thus,

$$\mathbf{r}'(t) = (-2\sin t)\mathbf{i} + 3\cos t\,\mathbf{j}$$

$$\mathbf{r}''(t) = (-2\cos t)\mathbf{i} + (-3\sin t)\mathbf{j}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin t & 3\cos t & 0 \\ -2\cos t & -3\sin t & 0 \end{vmatrix} = [(6\sin^2 t) + (6\cos^2 t)]\mathbf{k} = 6\mathbf{k}$$

Therefore,

$$\|\mathbf{r}'(t)\| = \sqrt{(-2\sin t)^2 + (3\cos t)^2} = \sqrt{4\sin^2 t + 9\cos^2 t}$$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = 6$$

so

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{6}{[4\sin^2 t + 9\cos^2 t]^{3/2}} \tag{7}$$

The endpoints of the minor axis are $(2, 0)$ and $(-2, 0)$, which correspond to $t = 0$ and $t = \pi$, respectively. Substituting these values in (7) yields the same curvature at both points, namely,

$$\kappa = \kappa(0) = \kappa(\pi) = \frac{6}{9^{3/2}} = \frac{6}{27} = \frac{2}{9}$$

The endpoints of the major axis are $(0, 3)$ and $(0, -3)$, which correspond to $t = \pi/2$ and $t = 3\pi/2$, respectively; from (7) the curvature at these points is

$$\kappa = \kappa\left(\frac{\pi}{2}\right) = \kappa\left(\frac{3\pi}{2}\right) = \frac{6}{4^{3/2}} = \frac{3}{4}$$

12.6

2

▸ **Example 2** A particle moves through 3-space in such a way that its velocity is

$$\mathbf{v}(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$$

Find the coordinates of the particle at time $t = 1$ given that the particle is at the point $(-1, 2, 4)$ at time $t = 0$.

***Solution.*** Integrating the velocity function to obtain the position function yields

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int (\mathbf{i} + t\mathbf{j} + t^2\mathbf{k})\,dt = t\mathbf{i} + \frac{t^2}{2}\mathbf{j} + \frac{t^3}{3}\mathbf{k} + \mathbf{C} \tag{5}$$

where $\mathbf{C}$ is a vector constant of integration. Since the coordinates of the particle at time $t = 0$ are $(-1, 2, 4)$, the position vector at time $t = 0$ is

$$\mathbf{r}(0) = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} \tag{6}$$

It follows on substituting $t = 0$ in (5) and equating the result with (6) that

$$\mathbf{C} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

Substituting this value of $\mathbf{C}$ in (5) and simplifying yields

$$\mathbf{r}(t) = (t-1)\mathbf{i} + \left(\frac{t^2}{2} + 2\right)\mathbf{j} + \left(\frac{t^3}{3} + 4\right)\mathbf{k}$$

Thus, at time $t = 1$ the position vector of the particle is

$$\mathbf{r}(1) = 0\mathbf{i} + \frac{5}{2}\mathbf{j} + \frac{13}{3}\mathbf{k}$$

so its coordinates at that instant are $\left(0, \frac{5}{2}, \frac{13}{3}\right)$. ◂

(Figure 12.6.3). The displacement vector, which describes the change in position of t particle during the time interval, can be obtained by integrating the velocity function fro $t_1$ to $t_2$:

$$\Delta \mathbf{r} = \int_{t_1}^{t_2} \mathbf{v}(t)\, dt = \int_{t_1}^{t_2} \frac{d\mathbf{r}}{dt}\, dt = \mathbf{r}(t)\Big]_{t_1}^{t_2} = \mathbf{r}(t_2) - \mathbf{r}(t_1)$$ Displacement (

It follows from Theorem 12.3.1 that we can find the distance $s$ traveled by a particle ov a time interval $t_1 \le t \le t_2$ by integrating the speed over that interval, since

$$s = \int_{t_1}^{t_2} \left\| \frac{d\mathbf{r}}{dt} \right\| dt = \int_{t_1}^{t_2} \|\mathbf{v}(t)\|\, dt$$ Distance traveled (

▶ **Example 3** Suppose that a particle moves along a circular helix in 3-space so that its position vector at time $t$ is

$$\mathbf{r}(t) = (4\cos \pi t)\mathbf{i} + (4\sin \pi t)\mathbf{j} + t\mathbf{k}$$

Find the distance traveled and the displacement of the particle during the time interval $1 \le t \le 5$.

***Solution.*** We have

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = (-4\pi \sin \pi t)\mathbf{i} + (4\pi \cos \pi t)\mathbf{j} + \mathbf{k}$$

$$\|\mathbf{v}(t)\| = \sqrt{(-4\pi \sin \pi t)^2 + (4\pi \cos \pi t)^2 + 1} = \sqrt{16\pi^2 + 1}$$

Thus, it follows from (9) that the distance traveled by the particle from time $t = 1$ to $t = 5$ is

$$s = \int_1^5 \sqrt{16\pi^2 + 1}\, dt = 4\sqrt{16\pi^2 + 1}$$

Moreover, it follows from (8) that the displacement over the time interval is

$$\begin{aligned} \Delta \mathbf{r} &= \mathbf{r}(5) - \mathbf{r}(1) \\ &= (4\cos 5\pi \mathbf{i} + 4\sin 5\pi \mathbf{j} + 5\mathbf{k}) - (4\cos \pi \mathbf{i} + 4\sin \pi \mathbf{j} + \mathbf{k}) \\ &= (-4\mathbf{i} + 5\mathbf{k}) - (-4\mathbf{i} + \mathbf{k}) = 4\mathbf{k} \end{aligned}$$

which tells us that the change in the position of the particle over the time interval was 4 units straight up. ◀

**12.6.2 THEOREM** *If a particle moves along a smooth curve C in 2-space or 3-space, then at each point on the curve velocity and acceleration vectors can be written as*

$$\mathbf{v} = \frac{ds}{dt}\mathbf{T} \qquad \mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N} \tag{10–11}$$

*where s is an arc length parameter for the curve, and* **T**, **N**, *and* $\kappa$ *denote the unit tangent vector, unit normal vector, and curvature at the point (Figure 12.6.4).*

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} \tag{14}$$

In this formula the scalars $a_T$ and $a_N$ are called the ***tangential scalar component of acceleration*** and the ***normal scalar component of acceleration***, and the vectors $a_T\mathbf{T}$ and $a_N\mathbf{N}$ are called the ***tangential vector component of acceleration*** and the ***normal vector component of acceleration***.

**12.6.3 THEOREM** *If a particle moves along a smooth curve C in 2-space or 3-space, then at each point on the curve the velocity* **v** *and the acceleration* **a** *are related to* $a_T$, $a_N$, *and* $\kappa$ *by the formulas*

$$a_T = \frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|} \qquad a_N = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|} \qquad \kappa = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|^3} \tag{15–17}$$

**▶ Example 4** Suppose that a particle moves through 3-space so that its position vector at time $t$ is

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$

(The path is the twisted cubic shown in Figure 12.1.5.)

(a) Find the scalar tangential and normal components of acceleration at time $t$.

(b) Find the scalar tangential and normal components of acceleration at time $t = 1$.

(c) Find the vector tangential and normal components of acceleration at time $t = 1$.

(d) Find the curvature of the path at the point where the particle is located at time $t = 1$.

***Solution (a).*** We have

$$\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = 2\mathbf{j} + 6t\mathbf{k}$$

$$\|\mathbf{v}(t)\| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\mathbf{v}(t) \cdot \mathbf{a}(t) = 4t + 18t^3$$

$$\mathbf{v}(t) \times \mathbf{a}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2\mathbf{i} - 6t\mathbf{j} + 2\mathbf{k}$$

Thus, from (15) and (16)

$$a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|} = \frac{4t + 18t^3}{\sqrt{1 + 4t^2 + 9t^4}}$$

$$a_N = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \frac{\sqrt{36t^4 + 36t^2 + 4}}{\sqrt{1 + 4t^2 + 9t^4}} = 2\sqrt{\frac{9t^4 + 9t^2 + 1}{9t^4 + 4t^2 + 1}}$$

***Solution (b).*** At time $t = 1$, the components $a_T$ and $a_N$ in part (a) are

$$a_T = \frac{22}{\sqrt{14}} \approx 5.88 \quad \text{and} \quad a_N = 2\sqrt{\frac{19}{14}} \approx 2.33$$

***Solution (c).*** Since $\mathbf{T}$ and $\mathbf{v}$ have the same direction, $\mathbf{T}$ can be obtained by normalizing $\mathbf{v}$, that is,

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}$$

At time $t = 1$ we have

$$\mathbf{T}(1) = \frac{\mathbf{v}(1)}{\|\mathbf{v}(1)\|} = \frac{\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}}{\|\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\|} = \frac{1}{\sqrt{14}}(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$

From this and part (b) we obtain the vector tangential component of acceleration:

$$a_T(1)\mathbf{T}(1) = \frac{22}{\sqrt{14}}\mathbf{T}(1) = \frac{11}{7}(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = \frac{11}{7}\mathbf{i} + \frac{22}{7}\mathbf{j} + \frac{33}{7}\mathbf{k}$$

To find the normal vector component of acceleration, we rewrite $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ as

$$a_N\mathbf{N} = \mathbf{a} - a_T\mathbf{T}$$

Thus, at time $t = 1$ the normal vector component of acceleration is

$$\begin{aligned} a_N(1)\mathbf{N}(1) &= \mathbf{a}(1) - a_T(1)\mathbf{T}(1) \\ &= (2\mathbf{j} + 6\mathbf{k}) - \left(\frac{11}{7}\mathbf{i} + \frac{22}{7}\mathbf{j} + \frac{33}{7}\mathbf{k}\right) \\ &= -\frac{11}{7}\mathbf{i} - \frac{8}{7}\mathbf{j} + \frac{9}{7}\mathbf{k} \end{aligned}$$

***Solution (d).*** We will apply Formula (17) with $t = 1$. From part (a)

$$\|\mathbf{v}(1)\| = \sqrt{14} \quad \text{and} \quad \mathbf{v}(1) \times \mathbf{a}(1) = 6\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$$

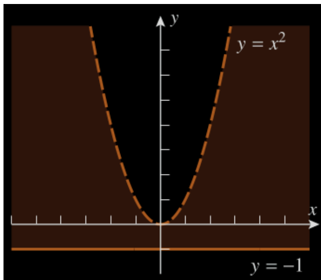
Thus, at time $t = 1$

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3} = \frac{\sqrt{76}}{(\sqrt{14})^3} = \frac{1}{14}\sqrt{\frac{38}{7}} \approx 0.17 \quad ◀$$

# 13

## 13.1

1




The solid boundary line is included in the domain, while the dashed boundary is not

**▶ Example 1** Let $f(x, y) = \sqrt{y+1} + \ln(x^2 - y)$. Find $f(e, 0)$ and sketch the natural domain of $f$.

***Solution.*** By substitution,

$$f(e, 0) = \sqrt{0+1} + \ln(e^2 - 0) = \sqrt{1} + \ln(e^2) = 1 + 2 = 3$$

To find the natural domain of $f$, we note that $\sqrt{y+1}$ is defined only when $y \geq -1$, while $\ln(x^2 - y)$ is defined only when $0 < x^2 - y$ or $y < x^2$. Thus, the natural domain of $f$ consists of all points in the $xy$-plane for which $-1 \leq y < x^2$. To sketch the natural domain, we first sketch the parabola $y = x^2$ as a "dashed" curve and the line $y = -1$ as a solid curve. The natural domain of $f$ is then the region lying above or on the line $y = -1$ and below the parabola $y = x^2$ (Figure 13.1.1). ◀

7

**▶ Example 7** Describe the level surfaces of

(a) $f(x, y, z) = x^2 + y^2 + z^2$ (b) $f(x, y, z) = z^2 - x^2 - y^2$

***Solution (a).*** The level surfaces have equations of the form

$$x^2 + y^2 + z^2 = k$$

For $k > 0$ the graph of this equation is a sphere of radius $\sqrt{k}$, centered at the origin; for $k = 0$ the graph is the single point $(0, 0, 0)$; and for $k < 0$ there is no level surface (Figure 13.1.9).

***Solution (b).*** The level surfaces have equations of the form

$$z^2 - x^2 - y^2 = k$$

As discussed in Section 11.7, this equation represents a cone if $k = 0$, a hyperboloid of two sheets if $k > 0$, and a hyperboloid of one sheet if $k < 0$ (Figure 13.1.10). ◀

# 13.2

1

▶ **Example 1** Figure 13.2.3*a* shows a computer-generated graph of the function

$$f(x, y) = -\frac{xy}{x^2 + y^2}$$

The graph reveals that the surface has a ridge above the line $y = -x$, which is to be expected since $f(x, y)$ has a constant value of $\frac{1}{2}$ for $y = -x$, except at $(0, 0)$ where $f$ is undefined (verify). Moreover, the graph suggests that the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ along a line through the origin varies with the direction of the line. Find this limit along

(a) the $x$-axis  (b) the $y$-axis  (c) the line $y = x$

(d) the line $y = -x$  (e) the parabola $y = x^2$

***Solution (a).*** The $x$-axis has parametric equations $x = t$, $y = 0$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x, y) \to (0, 0) \\ (\text{along } y = 0)}} f(x, y) = \lim_{t \to 0} f(t, 0) = \lim_{t \to 0} \left(-\frac{0}{t^2}\right) = \lim_{t \to 0} 0 = 0$$

which is consistent with Figure 13.2.3*b*.

***Solution (b).*** The $y$-axis has parametric equations $x = 0$, $y = t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x, y) \to (0, 0) \\ (\text{along } x = 0)}} f(x, y) = \lim_{t \to 0} f(0, t) = \lim_{t \to 0} \left(-\frac{0}{t^2}\right) = \lim_{t \to 0} 0 = 0$$

which is consistent with Figure 13.2.3*b*.

***Solution (c).*** The line $y = x$ has parametric equations $x = t$, $y = t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,\, y)\to(0,\, 0)\\ (\text{along } y = x)}} f(x, y) = \lim_{t\to 0} f(t, t) = \lim_{t\to 0}\left(-\frac{t^2}{2t^2}\right) = \lim_{t\to 0}\left(-\frac{1}{2}\right) = -\frac{1}{2}$$

which is consistent with Figure 13.2.3*b*.

***Solution (d).*** The line $y = -x$ has parametric equations $x = t$, $y = -t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,\, y)\to(0,\, 0)\\ (\text{along } y = -x)}} f(x, y) = \lim_{t\to 0} f(t, -t) = \lim_{t\to 0}\frac{t^2}{2t^2} = \lim_{t\to 0}\frac{1}{2} = \frac{1}{2}$$

which is consistent with Figure 13.2.3*b*.

***Solution (e).*** The parabola $y = x^2$ has parametric equations $x = t$, $y = t^2$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,\, y)\to(0,\, 0)\\ (\text{along } y = x^2)}} f(x, y) = \lim_{t\to 0} f(t, t^2) = \lim_{t\to 0}\left(-\frac{t^3}{t^2 + t^4}\right) = \lim_{t\to 0}\left(-\frac{t}{1 + t^2}\right) = 0$$

This is consistent with Figure 13.2.3*c*, which shows the parametric curve

$$x = t, \quad y = t^2, \quad z = -\frac{t}{1 + t^2}$$

superimposed on the surface. ◂

7

**LIMITS AT DISCONTINUITIES**

Sometimes it is easy to recognize when a limit does not exist. For example, it is evident that

$$\lim_{(x,y)\to(0,0)} \frac{1}{x^2 + y^2} = +\infty$$

which implies that the values of the function approach $+\infty$ as $(x, y) \to (0, 0)$ along any smooth curve (Figure 13.2.9). However, it is not evident whether the limit

$$\lim_{(x,y)\to(0,0)} (x^2 + y^2)\ln(x^2 + y^2)$$

exists because it is an indeterminate form of type $0 \cdot \infty$. Although L'Hôpital's rule cannot be applied directly, the following example illustrates a method for finding this limit by converting to polar coordinates.

▸ **Example 7** Find $\lim_{(x,y)\to(0,0)} (x^2 + y^2)\ln(x^2 + y^2)$.

***Solution.*** Let $(r, \theta)$ be polar coordinates of the point $(x, y)$ with $r \geq 0$. Then we have

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2$$

# 13.3

## 5

▶ **Example 5** Let $f(x, y) = x^2y + 5y^3$.

(a) Find the slope of the surface $z = f(x, y)$ in the $x$-direction at the point $(1, -2)$.
(b) Find the slope of the surface $z = f(x, y)$ in the $y$-direction at the point $(1, -2)$.

***Solution (a).*** Differentiating $f$ with respect to $x$ with $y$ held fixed yields

$$f_x(x, y) = 2xy$$

Thus, the slope in the $x$-direction is $f_x(1, -2) = -4$; that is, $z$ is decreasing at the rate of 4 units per unit increase in $x$.

***Solution (b).*** Differentiating $f$ with respect to $y$ with $x$ held fixed yields

$$f_y(x, y) = x^2 + 15y^2$$

Thus, the slope in the $y$-direction is $f_y(1, -2) = 61$; that is, $z$ is increasing at the rate of 61 units per unit increase in $y$. ◀

## 8

▶ **Example 8** Suppose that $D = \sqrt{x^2 + y^2}$ is the length of the diagonal of a rectangle whose sides have lengths $x$ and $y$ that are allowed to vary. Find a formula for the rate of change of $D$ with respect to $x$ if $x$ varies with $y$ held constant, and use this formula to find the rate of change of $D$ with respect to $x$ at the point where $x = 3$ and $y = 4$.

***Solution.*** Differentiating both sides of the equation $D^2 = x^2 + y^2$ with respect to $x$ yields

$$2D\frac{\partial D}{\partial x} = 2x \quad \text{and thus} \quad D\frac{\partial D}{\partial x} = x$$

Since $D = 5$ when $x = 3$ and $y = 4$, it follows that

$$5\left.\frac{\partial D}{\partial x}\right|_{x=3,\,y=4} = 3 \quad \text{or} \quad \left.\frac{\partial D}{\partial x}\right|_{x=3,\,y=4} = \frac{3}{5}$$

Thus, $D$ is increasing at a rate of $\frac{3}{5}$ unit per unit increase in $x$ at the point $(3, 4)$. ◀

## HIGHER-ORDER PARTIAL DERIVATIVES

Suppose that $f$ is a function of two variables $x$ and $y$. Since the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are also functions of $x$ and $y$, these functions may themselves have partial derivatives. This gives rise to four possible ***second-order*** partial derivatives of $f$, which are defined by

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

Differentiate twice with respect to $x$.

Differentiate twice with respect to $y$.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

Differentiate first with respect to $x$ and then with respect to $y$.

Differentiate first with respect to $y$ and then with respect to $x$.

**Example 12** Find the second-order partial derivatives of $f(x, y) = x^2y^3 + x^4y$.

***olution.*** We have

$$\frac{\partial f}{\partial x} = 2xy^3 + 4x^3y \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x^2y^2 + x^4$$

o that

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2xy^3 + 4x^3y) = 2y^3 + 12x^2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x^2y^2 + x^4) = 6x^2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x^2y^2 + x^4) = 6xy^2 + 4x^3$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2xy^3 + 4x^3y) = 6xy^2 + 4x^3 \blacktriangleleft$$

# 13.5

1

► **Example 1** Suppose that

$$z = x^2 y, \quad x = t^2, \quad y = t^3$$

Use the chain rule to find $dz/dt$, and check the result by expressing $z$ as a function of $t$ and differentiating directly.

***Solution.*** By the chain rule [Formula (5)],

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2xy)(2t) + (x^2)(3t^2)$$
$$= (2t^5)(2t) + (t^4)(3t^2) = 7t^6$$

Alternatively, we can express $z$ directly as a function of $t$,

$$z = x^2 y = (t^2)^2(t^3) = t^7$$

and then differentiate to obtain $dz/dt = 7t^6$. However, this procedure may not always be convenient. ◄

2

► **Example 2** Suppose that

$$w = \sqrt{x^2 + y^2 + z^2}, \quad x = \cos\theta, \quad y = \sin\theta, \quad z = \tan\theta$$

Use the chain rule to find $dw/d\theta$ when $\theta = \pi/4$.

***Solution.*** From Formula (6) with $\theta$ in the place of $t$, we obtain

$$\frac{dw}{d\theta} = \frac{\partial w}{\partial x}\frac{dx}{d\theta} + \frac{\partial w}{\partial y}\frac{dy}{d\theta} + \frac{\partial w}{\partial z}\frac{dz}{d\theta}$$
$$= \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x)(-\sin\theta) + \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2y)(\cos\theta)$$
$$+ \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2z)(\sec^2\theta)$$

When $\theta = \pi/4$, we have

$$x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad z = \tan\frac{\pi}{4} = 1$$

Substituting $x = 1/\sqrt{2}$, $y = 1/\sqrt{2}$, $z = 1$, $\theta = \pi/4$ in the formula for $dw/d\theta$ yields

$$\left.\frac{dw}{d\theta}\right|_{\theta=\pi/4} = \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2})\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2})\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)(2)(2)$$
$$= \sqrt{2} \text{ ◄}$$

3

**Example 3** Given that

$$z = e^{xy}, \quad x = 2u + v, \quad y = u/v$$

find $\partial z/\partial u$ and $\partial z/\partial v$ using the chain rule.

***Solution.***

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (ye^{xy})(2) + (xe^{xy})\left(\frac{1}{v}\right) = \left[2y + \frac{x}{v}\right]e^{xy} \\
&= \left[\frac{2u}{v} + \frac{2u+v}{v}\right]e^{(2u+v)(u/v)} = \left[\frac{4u}{v} + 1\right]e^{(2u+v)(u/v)} \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (ye^{xy})(1) + (xe^{xy})\left(-\frac{u}{v^2}\right) \\
&= \left[y - x\left(\frac{u}{v^2}\right)\right]e^{xy} = \left[\frac{u}{v} - (2u+v)\left(\frac{u}{v^2}\right)\right]e^{(2u+v)(u/v)} \\
&= -\frac{2u^2}{v^2}e^{(2u+v)(u/v)}
\end{aligned}$$

4

▶ **Example 4** Suppose that

$$w = e^{xyz}, \quad x = 3u + v, \quad y = 3u - v, \quad z = u^2 v$$

Use appropriate forms of the chain rule to find $\partial w/\partial u$ and $\partial w/\partial v$.

***Solution.*** From the tree diagram and corresponding formulas in Figure 13.5.3 we obtain

$$\frac{\partial w}{\partial u} = yze^{xyz}(3) + xze^{xyz}(3) + xye^{xyz}(2uv) = e^{xyz}(3yz + 3xz + 2xyuv)$$

and

$$\frac{\partial w}{\partial v} = yze^{xyz}(1) + xze^{xyz}(-1) + xye^{xyz}(u^2) = e^{xyz}(yz - xz + xyu^2)$$

If desired, we can express $\partial w/\partial u$ and $\partial w/\partial v$ in terms of $u$ and $v$ alone by replacing $x$, $y$, and $z$ by their expressions in terms of $u$ and $v$. ◀

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▶ **Example 5** Suppose that $w = x^2 + y^2 - z^2$ and

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

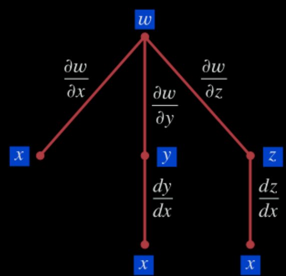
Use appropriate forms of the chain rule to find $\partial w/\partial \rho$ and $\partial w/\partial \theta$.

***Solution.*** From the tree diagram and corresponding formulas in Figure 13.5.4 we obtain

$$\begin{aligned}
\frac{\partial w}{\partial \rho} &= 2x \sin\phi \cos\theta + 2y \sin\phi \sin\theta - 2z \cos\phi \\
&= 2\rho \sin^2\phi \cos^2\theta + 2\rho \sin^2\phi \sin^2\theta - 2\rho \cos^2\phi \\
&= 2\rho \sin^2\phi (\cos^2\theta + \sin^2\theta) - 2\rho \cos^2\phi \\
&= 2\rho (\sin^2\phi - \cos^2\phi) \\
&= -2\rho \cos 2\phi \\
\frac{\partial w}{\partial \theta} &= (2x)(-\rho \sin\phi \sin\theta) + (2y)\rho \sin\phi \cos\theta \\
&= -2\rho^2 \sin^2\phi \sin\theta \cos\theta + 2\rho^2 \sin^2\phi \sin\theta \cos\theta \\
&= 0
\end{aligned}$$

This result is explained by the fact that $w$ does not vary with $\theta$. You can see this directly by expressing the variables $x$, $y$, and $z$ in terms of $\rho$, $\phi$, and $\theta$ in the formula for $w$. (Verify that $w = -\rho^2 \cos 2\phi$.) ◀

6




$$\frac{dw}{dx} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{dy}{dx} + \frac{\partial w}{\partial z}\frac{dz}{dx}$$

**Figure 13.5.5**

▶ **Example 6** Suppose that

$$w = xy + yz, \quad y = \sin x, \quad z = e^x$$

Use an appropriate form of the chain rule to find $dw/dx$.

***Solution.*** From the tree diagram and corresponding formulas in Figure 13.5.5 we obtain

$$\begin{aligned}\frac{dw}{dx} &= y + (x+z)\cos x + ye^x \\ &= \sin x + (x + e^x)\cos x + e^x \sin x\end{aligned}$$

This result can also be obtained by first expressing $w$ explicitly in terms of $x$ as

$$w = x \sin x + e^x \sin x$$

and then differentiating with respect to $x$; however, such direct substitution is not always possible. ◀

7

**13.5.3 THEOREM** *If the equation $f(x, y) = c$ defines $y$ implicitly as a differentiab function of $x$, and if $\partial f/\partial y \neq 0$, then*

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} \qquad (1$$

▶ **Example 7** Given that

$$x^3 + y^2x - 3 = 0$$

find $dy/dx$ using (14), and check the result using implicit differentiation.

***Solution.*** By (14) with $f(x, y) = x^3 + y^2x - 3$,

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{3x^2 + y^2}{2yx}$$

Alternatively, differentiating implicitly yields

$$3x^2 + y^2 + x\left(2y\frac{dy}{dx}\right) - 0 = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{3x^2 + y^2}{2yx}$$

which agrees with the result obtained by (14). ◀

8

**13.5.4 THEOREM** *If the equation $f(x, y, z) = c$ defines $z$ implicitly as a differentiable function of $x$ and $y$, and if $\partial f/\partial z \neq 0$, then*

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}$$

▶ **Example 8** Consider the sphere $x^2 + y^2 + z^2 = 1$. Find $\partial z/\partial x$ and $\partial z/\partial y$ at the point $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$.

***Solution.*** By Theorem 13.5.4 with $f(x, y, z) = x^2 + y^2 + z^2$,

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} = -\frac{2x}{2z} = -\frac{x}{z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z} = -\frac{2y}{2z} = -\frac{y}{z}$$

At the point $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$, evaluating these derivatives gives $\partial z/\partial x = -1$ and $\partial z/\partial y = -\frac{1}{2}$.

# Definition

## Real valued & vector valued function

Sure ✅ Here are the **definitions** clearly:

---

🔹 Function (Scalar-valued Function):

A **function** is a rule that assigns **each element** ( x ) in a set ( D ) (called the **domain**) to **exactly one element** ( f(x) ) in a set ( R ) (called the **range**).

[ f: D \to R]

If both ( D ) and ( R ) are subsets of real numbers, then ( f ) is called a **real-valued function**.

🔹 Vector-Valued Function:

A **vector-valued function** is a function whose **output is a vector** rather than a single real number.
It assigns each input ( t ) (usually a real number) to a vector

## what does the derivative r'(t) of a vector valued function represent geometrically ?

## how does its magnitude relate to the curve?

2. Geometric Meaning

- The derivative **r′(t)** gives a **tangent vector** to the curve traced by r(t) at the point corresponding to that t.
- It shows **the direction** in which the curve is heading and **how fast** the position vector is changing.

So geometrically:

> 🧭 **r′(t)** is a **velocity vector** tangent to the curve at the point r(t)

The **magnitude** of the derivative,represents the **speed** of a particle moving along the curve — that is, **how fast the position is changing**.

## what is contour plot and what is level curve

A **level curve** (also called an **isocontour**) is a curve along which a **function of two variables** has a **constant value**. x2+y2=k

A **contour plot** is a **graphical representation** that shows **many level curves** of a function f(x,y) on the same coordinate plane.

So — A **contour plot** is a **collection of level curves** for different values of kkk.

## explain the geometric meaning of partial derivatives and directional derivatives

| Partial Derivative | Meaning | Direction | Geometric Interpretation |
| --- | --- | --- | --- |

| | | | |
|---|---|---|---|
| ( f_x ) | Rate of change in (x) direction | Along x-axis | Slope in the x-direction |
| ( f_y ) | Rate of change in (y) direction | Along y-axis | Slope in the y-direction |

Directional derivatives

➢ Geometric Meaning:

- It gives the **rate of change of the surface** z=f(x,y) as you move **in the direction of** u=<a,b>
- Imagine walking on a hilly surface — Duf tells you **how steep the slope** is **in the direction you're walking**.

## describe the relation between gradient and directional derivative

### 🔹 3. Geometric Relationship

Because it's a dot product:

$$D_{\mathbf{u}} f = |\nabla f|\,|\mathbf{u}|\,\cos\theta = |\nabla f|\cos\theta$$

(where $\theta$ is the angle between $\nabla f$ and $\mathbf{u}$).

So geometrically:

- $D_{\mathbf{u}} f$ is the **component** of $\nabla f$ in the direction of $\mathbf{u}$.
- It tells **how fast** $f$ is changing when you move along $\mathbf{u}$.