Chapter 7
Small World Network Models

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Collective dynamics of 'small-world' networks

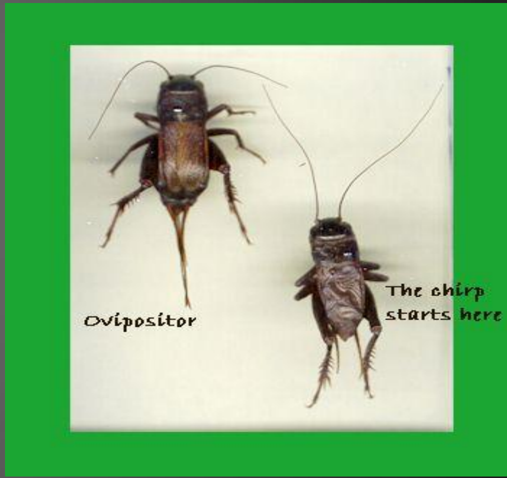
---------*Nature*, 393, 440-442, 1998

D. J. Watts ***S. H. Strogatz***

Cornell Univ.

Synchronization is a good topic to nurture new ideas

- Watt's work on small worlds was inspired by investigation on a synchronization problem:
- *How crickets synchronize their chirping?*



Watts's memory

- Back when Steve and I started our work together, we didn't know any of this. Neither of us had the foggiest idea about anything about social nets.
- We both knew some physics. Graph theory was also a mystery. Effectively a branch of pure mathematics, it can be divided roughly into two components – the almost obvious and the utterly impenetrable.
- I learned the obvious stuff out of a textbook, and after some futile struggling with the rest, convinced myself that it wasn't very interesting anyway.

- All this profundity of ignorance left us in sth of an awkward place. We were reasonably certain that someone must have thought about this problem before, and we worried that we might waste a lot of time reinventing the wheel.
  - But we also thought that if we went out looking for it, we might get discouraged by how much had already been done, or else trapped into thinking about the problem from the same perspective and so get stuck on the very same things that other people had.

(Not) Re-inventing the wheel

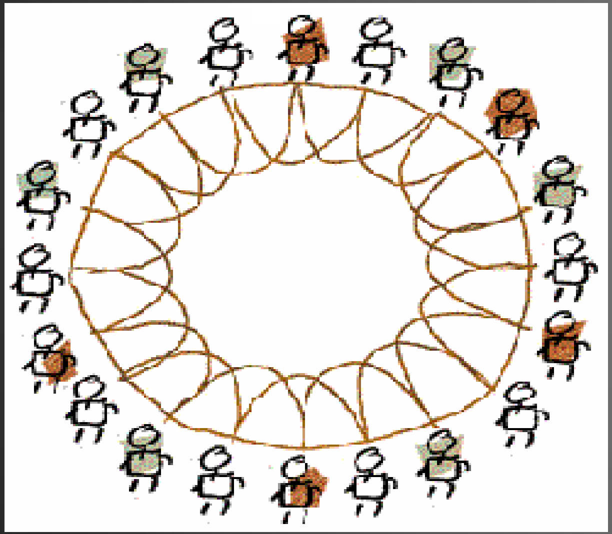
Made up mind

- In Jan. 1996, we made up our minds: we would go it alone. Telling almost nobody and reading virtually nothing, we would drop the crickets project and have a go at building some very simple models of social networks to look for features like the small-world.
- No doubt feeling that he needed to protect me from myself, Steve insisted that we give it only 4 months-a single semester-after which, if we hadn't made some significant progress, we would concede defeat and return to the crickets.
- At worst, my graduation would be delayed by a semester, and **if it would make me happy, why not?**

WS 'Small-World' Model

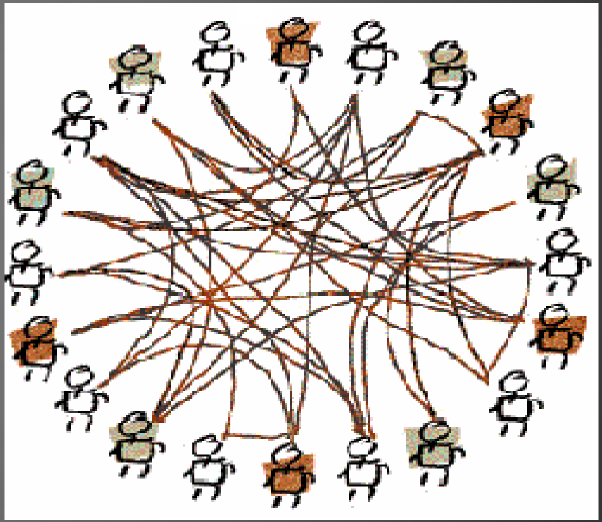
------*Nature*, 393, 440-442, 1998

Regular Graph



Large L & C

Random Graph

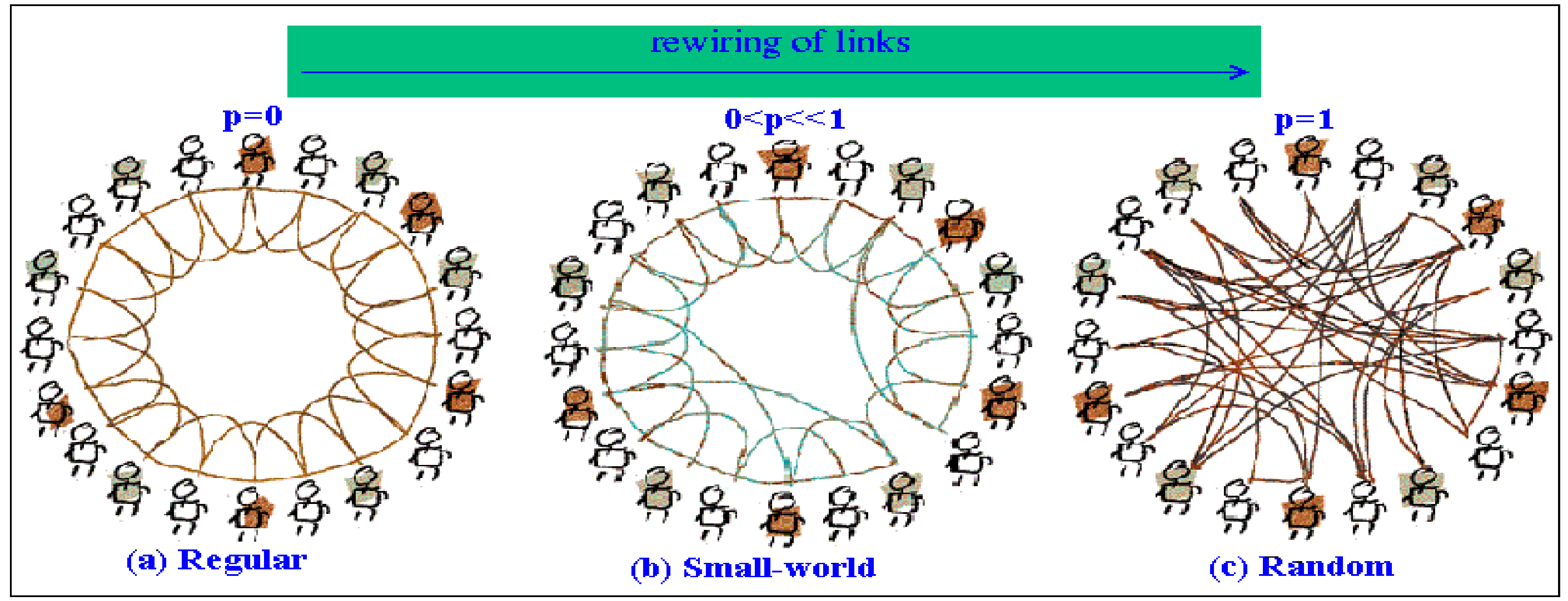


Small L & C

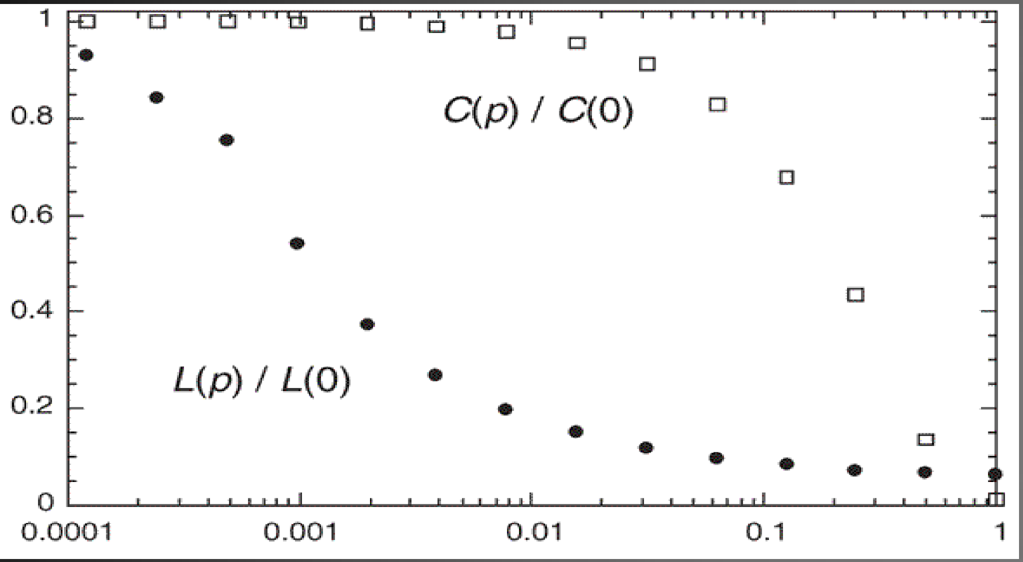
Bridging the gap (Small L but Large C)

Model Description

- Start from an regular graph
- Randomly rewiring with probability p

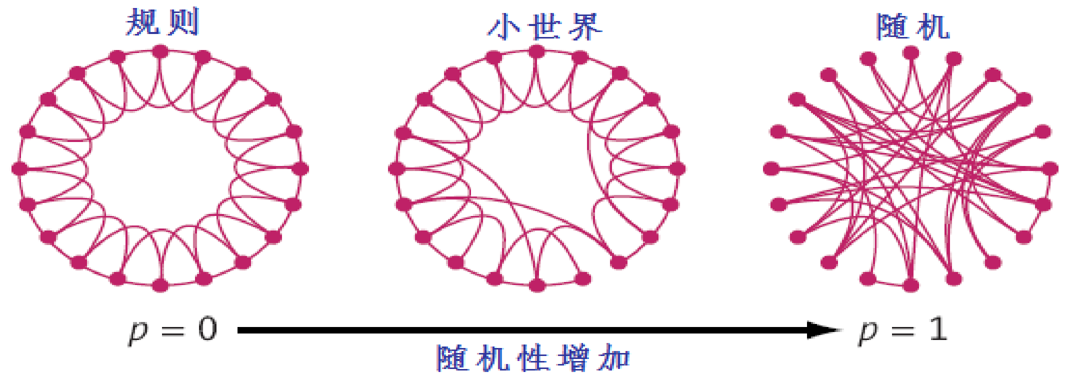


Model Analysis



- p=0, Large C(0), L(0)
- p=1, Small C(1), L(1)
- 0<p<<1, Large C, Small L:

  C(p)~C(0), L(p)<<L(0)

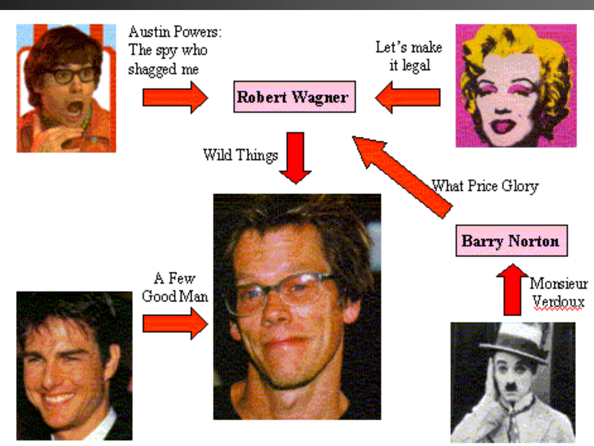


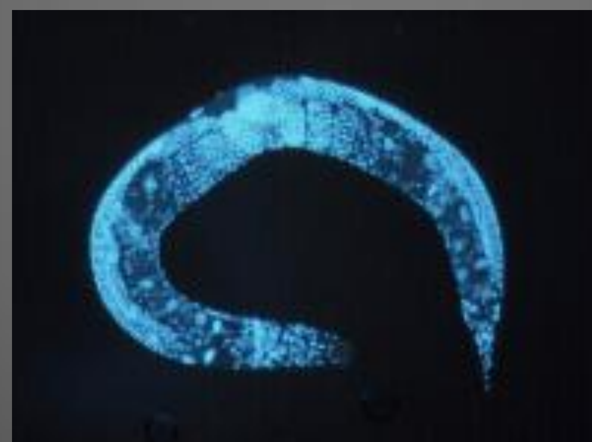
- Normalization
- logarithmic plot
- Average

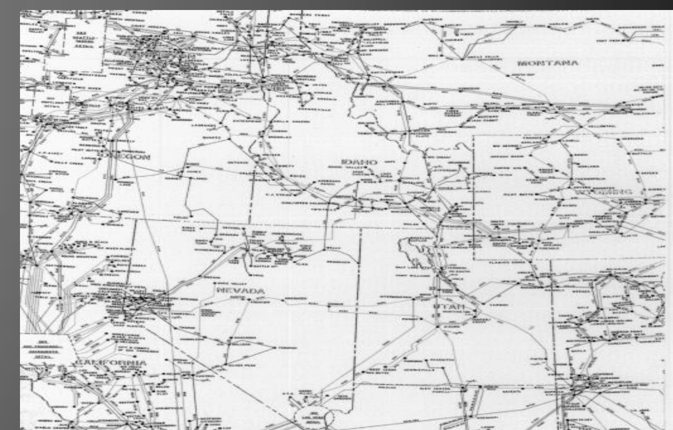
Model Verification

Table 1 Empirical examples of small-world networks

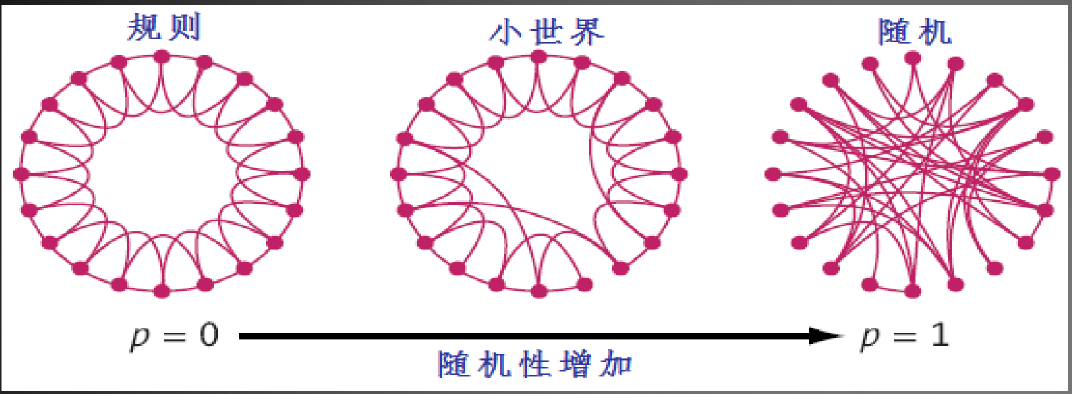
	L_{actual}	L_{random}	C_{actual}	C_{random}
Film actors	3.65	2.99	0.79	0.00027
Power grid	18.7	12.4	0.080	0.005
C. elegans	2.65	2.25	0.28	0.05



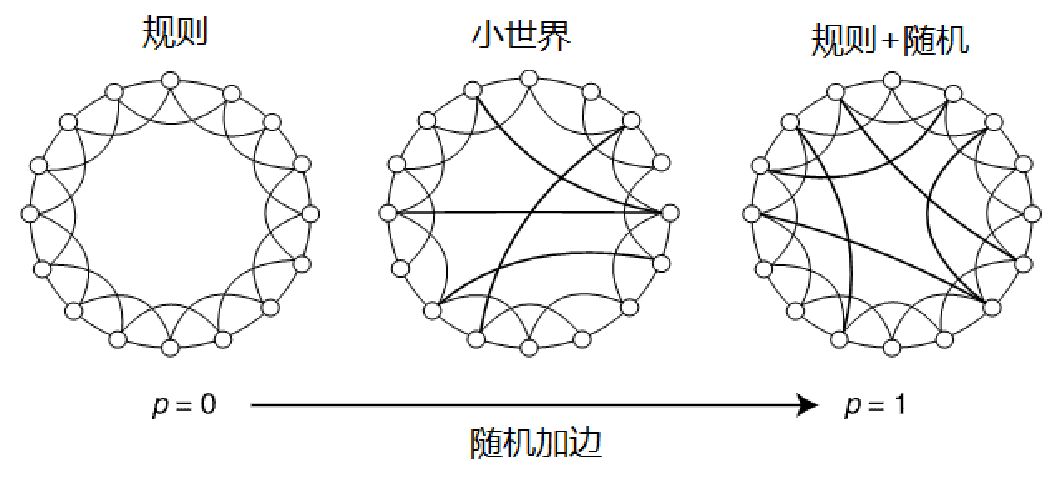




NW Small-World Model



- WS model:
- Expected number of shortcuts 0.5NKp



- NW model:
- Randomly add 0.5NKp shortcuts

Theoretical Analysis

- Clustering coefficient

$$\tilde{C}_{WS}(p) \approx \frac{3(K-2)}{4(K-1)}(1-p)^3 = C_{nc}(1-p)^3$$

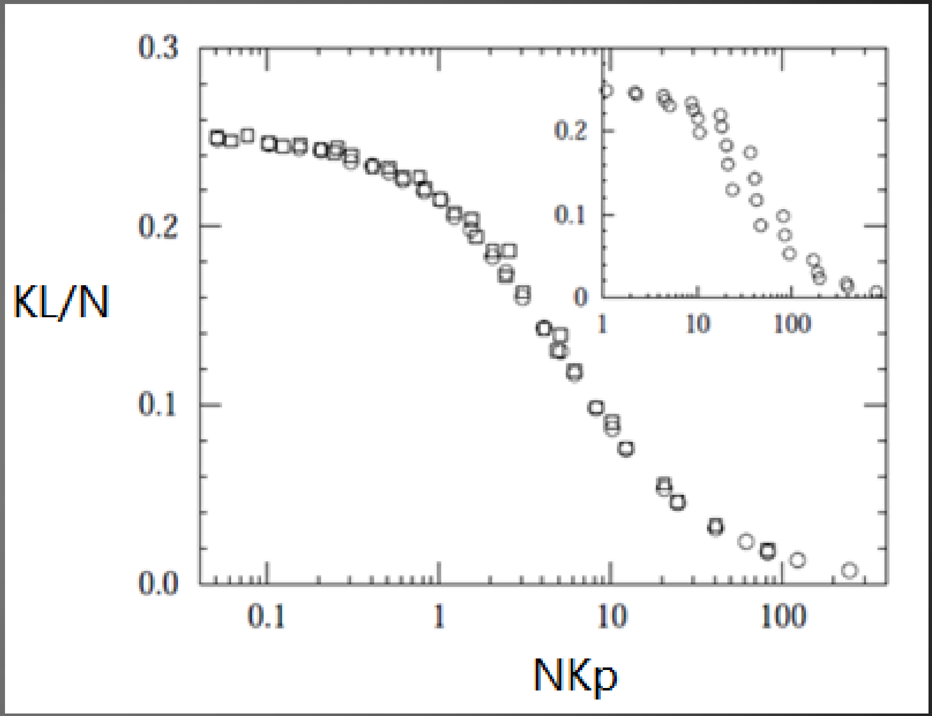
$$\tilde{C}_{NW}(p) = \frac{3(K-2)}{4(K-1)+4Kp(p+2)}$$

Theoretical Analysis

- Average Distance

$$L = \frac{N}{K} f(NKp)$$

$$\frac{KL}{N} = f(NKp)$$



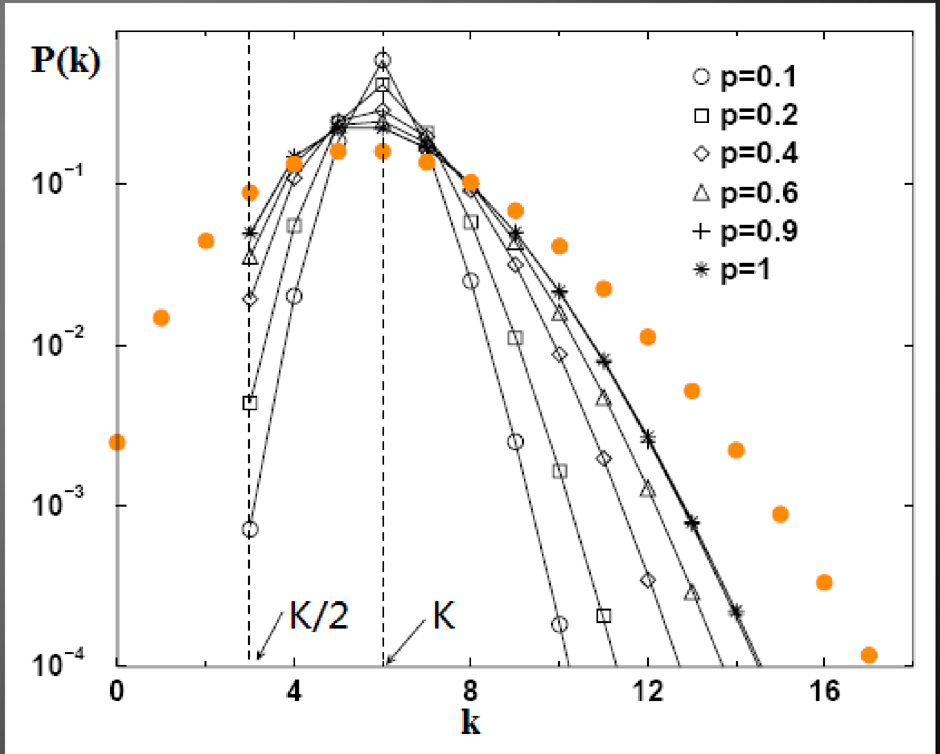
Universal scaling function of NW model

Theoretical Analysis

$$P(k) = \sum_{n=0}^{\min(k-K/2, K/2)} \binom{K/2}{n} (1-p)^n p^{K/2-n} \frac{(pK/2)^{k-(K/2)-n}}{(k-(K/2)-n)!} e^{-pK/2} \qquad k \geq K/2$$

$$P(k) = 0$$

- Degree distribution of WS model



It takes a lot of randomness to ruin the clustering, but a very small amount to overcome locality

Could a network which is so strongly locally structured be at the same time a small world?
Yes. You don't need more than a few random links.

Could a network which is so strongly locally structured be at the same time a small world?
Yes. You don't need more than a few random links.

Going Beyond Facebook

Network	Size	$\langle k \rangle$	ℓ	ℓ_{rand}	C	C_{rand}
WWW, site level, undir.	153 127	35.21	3.1	3.35	0.1078	0.00023
Internet, domain level	3015–6209	3.52–4.11	3.7–3.76	6.36–6.18	0.18–0.3	0.001
Movie actors	225 226	61	3.65	2.99	0.79	0.00027
LANL co-authorship	52 909	9.7	5.9	4.79	0.43	1.8×10^{-4}
MEDLINE co-authorship	1 520 251	18.1	4.6	4.91	0.066	1.1×10^{-5}
SPIRES co-authorship	56 627	173	4.0	2.12	0.726	0.003
NCSTRL co-authorship	11 994	3.59	9.7	7.34	0.496	3×10^{-4}
Math. co-authorship	70 975	3.9	9.5	8.2	0.59	5.4×10^{-5}
Neurosci. co-authorship	209 293	11.5	6	5.01	0.76	5.5×10^{-5}
E. coli, substrate graph	282	7.35	2.9	3.04	0.32	0.026
E. coli, reaction graph	315	28.3	2.62	1.98	0.59	0.09
Ythan estuary food web	134	8.7	2.43	2.26	0.22	0.06
Silwood Park food web	154	4.75	3.40	3.23	0.15	0.03
Words, co-occurrence	460.902	70.13	2.67	3.03	0.437	0.0001
Words, synonyms	22 311	13.48	4.5	3.84	0.7	0.0006
Power grid	4941	2.67	18.7	12.4	0.08	0.005
C. Elegans	282	14	2.65	2.25	0.28	0.05

Albert and Barabási, Reviews of Modern Physics **74**,47 (2002)

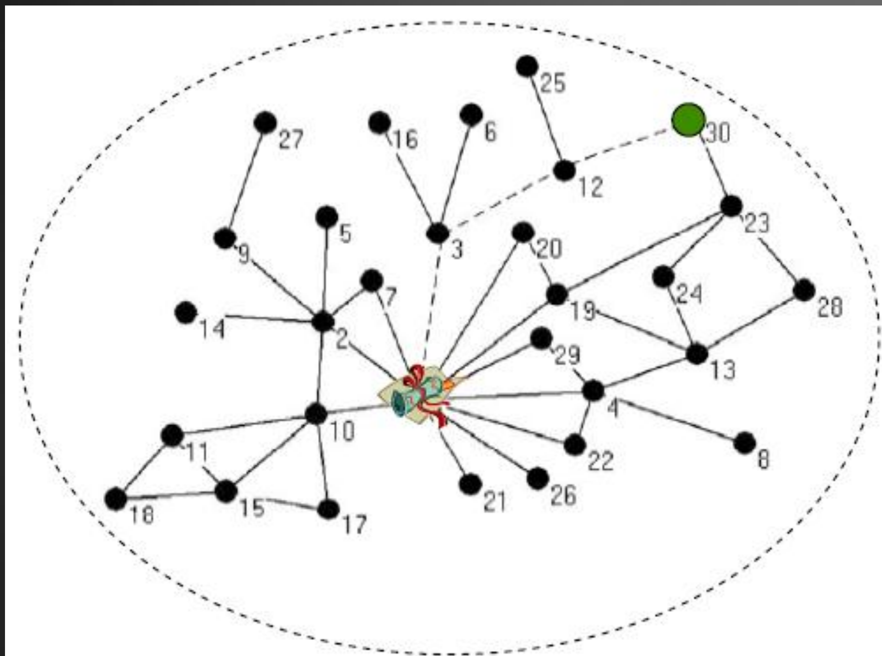
Six Degrees (Stanley Milgram)

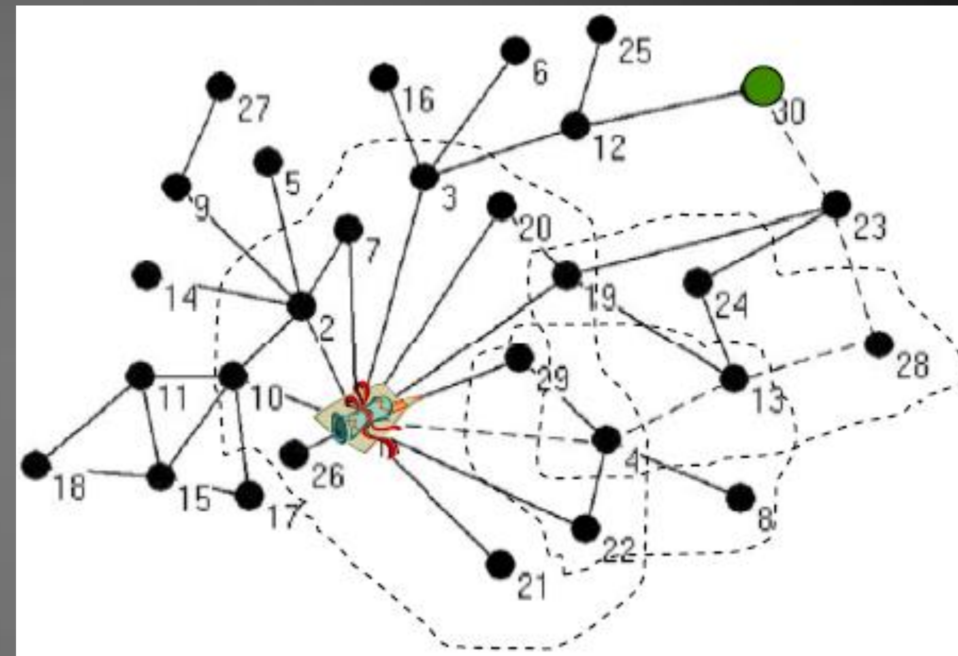
Revisiting Milgram's Experiment

Milgram's experiment revisited

- What did Milgram's experiment show?
  - (a) There are short paths in large networks that connect individuals
  - (b) People are able to find these short paths using a simple, greedy, decentralized algorithm
- WS small world model take care of (a)
- Kleinberg: what about (b)?

Global vs. Local Search





1 Start node, 30 Destination node

Path obtained is shown in dashed line

John Kleinberg, Nature 406(2000), 845

Navigation in a small world

It is easier to find short chains between points in some networks than others.

The small-world phenomenon — the principle that most of us are linked by short chains of acquaintances — was first investigated as a question in sociology[1,2] and is a feature of a range of networks arising in nature and technology[3–5]. Experimental study of the phenomenon[1] revealed that it has two fundamental components: first, such short chains are ubiquitous, and second, individuals operating with purely

tions follow an inverse-square distribution, there is a decentralized algorithm that achieves a very rapid delivery time; T is bounded by a function proportional to $(\log N)^2$. The algorithm achieving this bound is a 'greedy' heuristic: each message holder forwards the message across a con-

nection that brings it as close as possible to the target in lattice distance. Moreover, $\alpha = 2$ is the only exponent at which any decentralized algorithm can achieve a delivery time bounded by any polynomial in $\log N$: for every other exponent, an asymptotically much larger delivery time is required, regardless of the algorithm employed (Fig. 1b).

These results indicate that efficient navi-

John Kleinberg Cornell Univ.

- A. B., Cornell Univ., 1993
- Ph. D., MIT, 1996
- 美国科学院院士、工程院院士

‘Jon Kleinberg is a computer scientist with a reputation for tackling important, practical problems and, in the process, deriving deep mathematical insights’

Watts' View

- Jon is the proverbial rocket scientist –the kind who will hear about a problem in a lecture for the first time and by the end will understand it better than the lecturer.

The decentralized search algorithm

- Given a source s and a destination t, the search algorithm
  1. knows the positions of the nodes on the grid (**geography information**)
  2. knows the neighbors and shortcuts of the current node (**local information**)
  3. operates greedily, each time moving as close to t as possible (**greedy operation**)
  4. knows the neighbors and shortcuts of all nodes seen so far (**history information**)

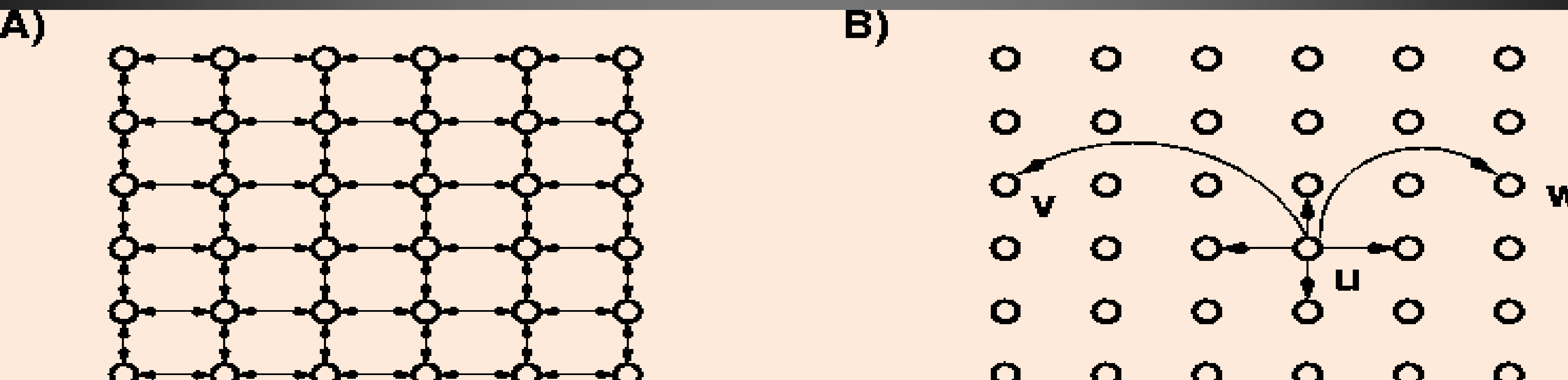
Kleinberg's Model

- n*n square
- p: lattice distance for local neighbors
- q: number of long-range contacts

$$d(u,v) = d((i,j),(k,l)) = |k - i| + |l - j|$$

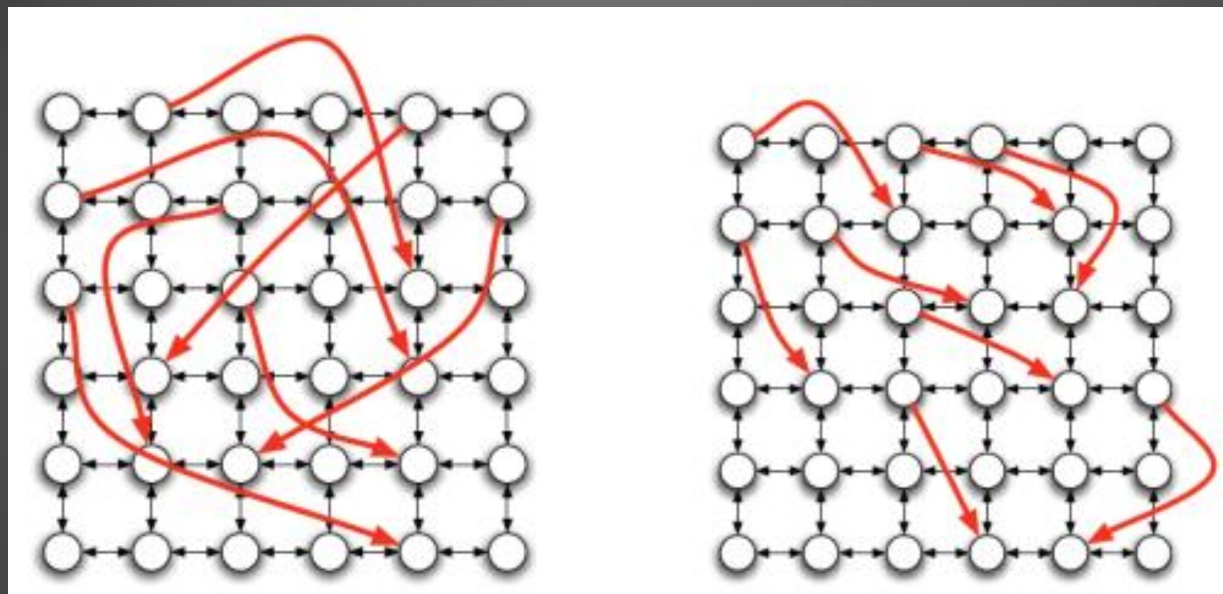
$$P \sim [d(u,v)]^{\alpha} / \sum_{v} [d(u,v)]^{\alpha}$$

n=6; p=1; q=2



Kleinberg's Model

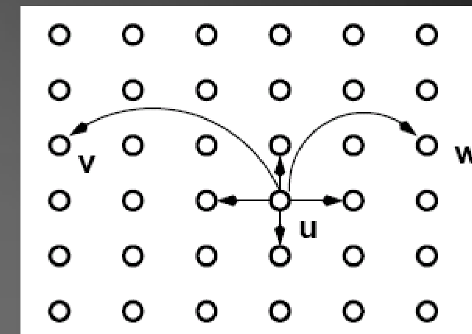
$$P \sim [d(u,v)]^{a} / \sum_{v} [d(u,v)]^{a}$$



- α is close to 0: plenty of short paths exists, but can't be found
- α is very large: short paths don't exist, but navigated easily
- Can we have the optimal balance?

Kleinberg’s Elegant Finding

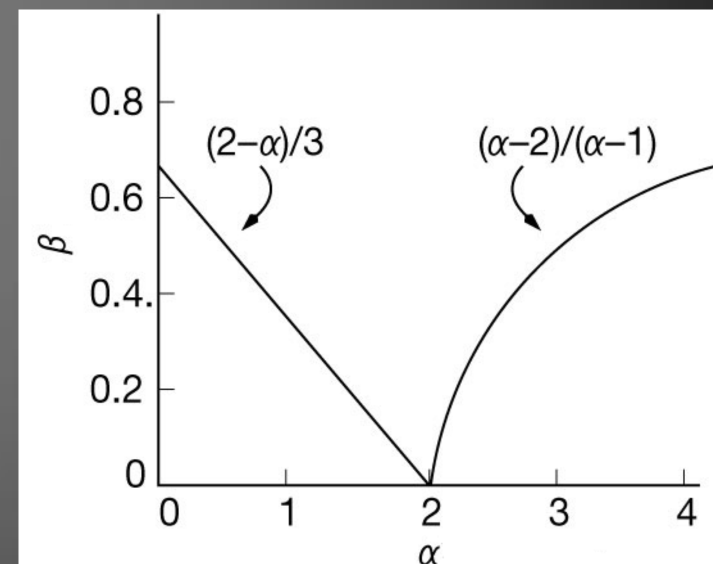
$$P \sim [d(u,v)]^{a} / \mathring{a}_{v}[d(u,v)]^{a}$$



$$a = 2 \qquad T \leq c_2 \left(\log N\right)^2$$

$$0\,?\; a \quad 2 \qquad T \geq c_\alpha N^{(2-\alpha)/3}$$

$$a > 2 \qquad T \geq c_\alpha N^{(\alpha-2)/(\alpha-1)}$$



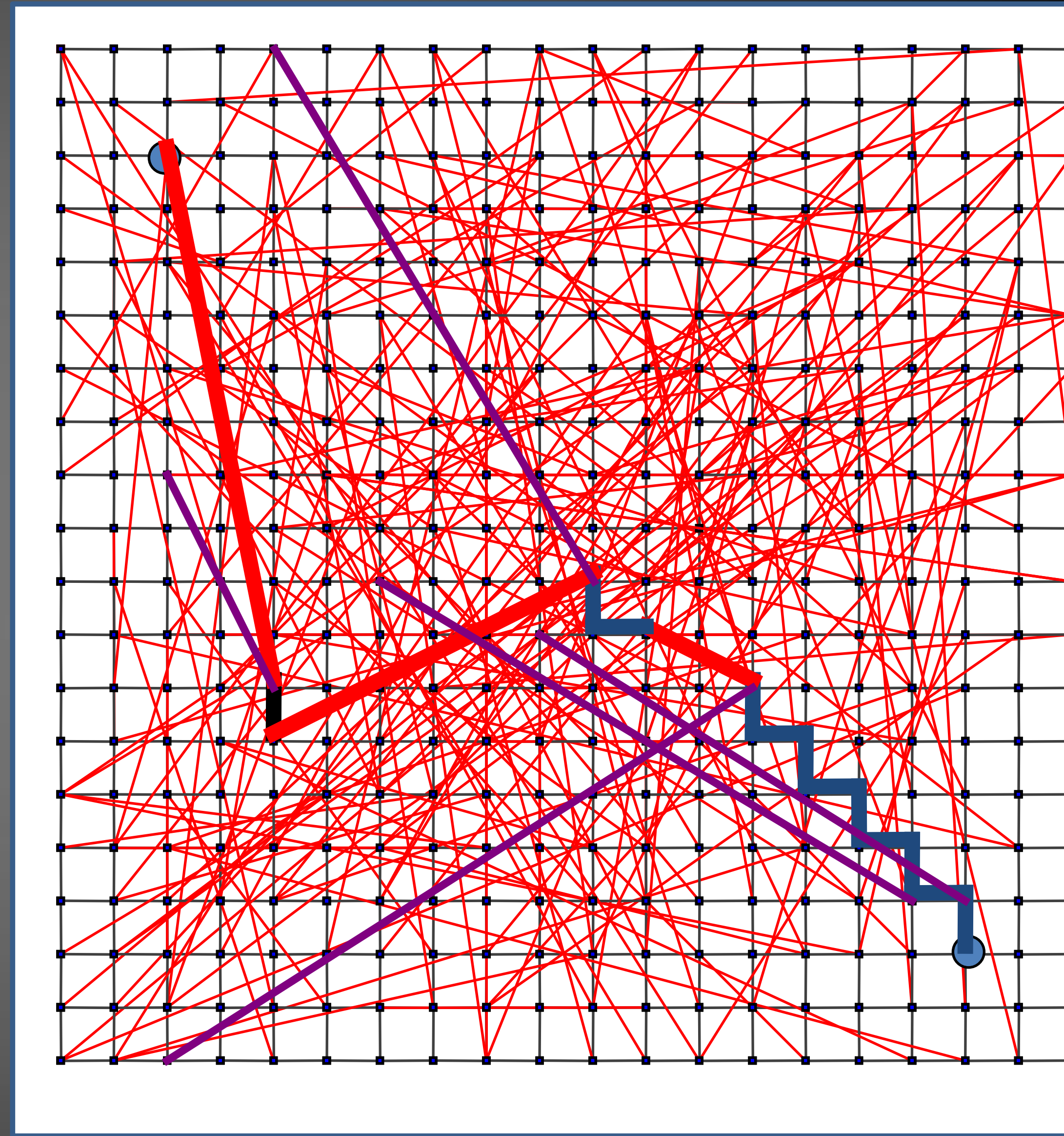
$$P \sim [d(u,v)]^{a} / å_{v} [d(u,v)]^{a}$$

$$0 \; ? \; a \quad 2$$

$$P \sim \frac{1}{d^2}$$

plenty of short paths exists ,
but can’t be found

$$T \geq c_{\alpha} N^{(2-\alpha)/3}$$



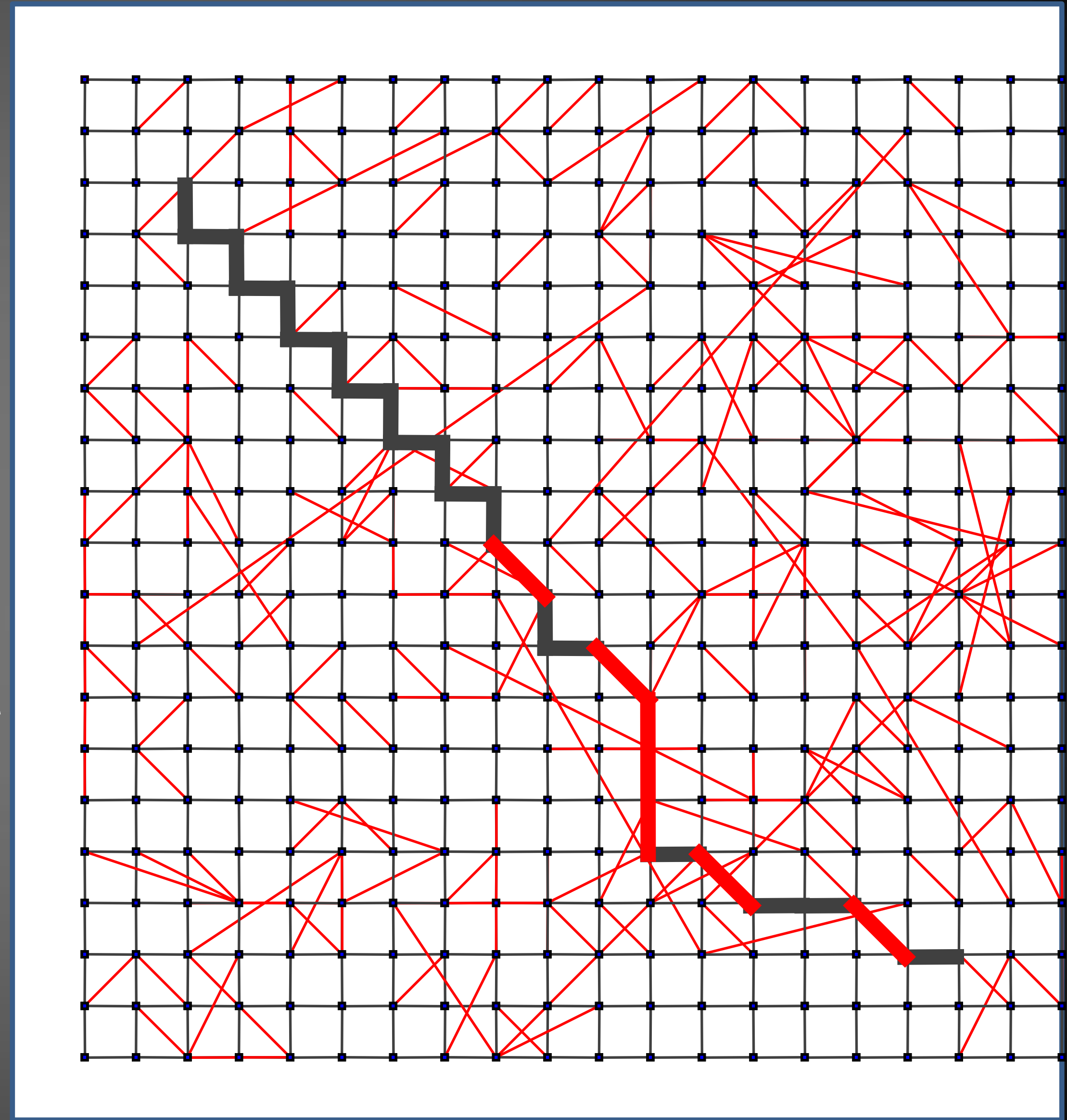
$$P \sim [d(u,v)]^{a} / \text{å}_{v} [d(u,v)]^{a}$$

$$a > 2$$

$$P \sim \frac{1}{d^4}$$

- short paths don’t exist, but navigated easily

$$T \geq c_{\alpha} N^{(\alpha-2)/(\alpha-1)}$$



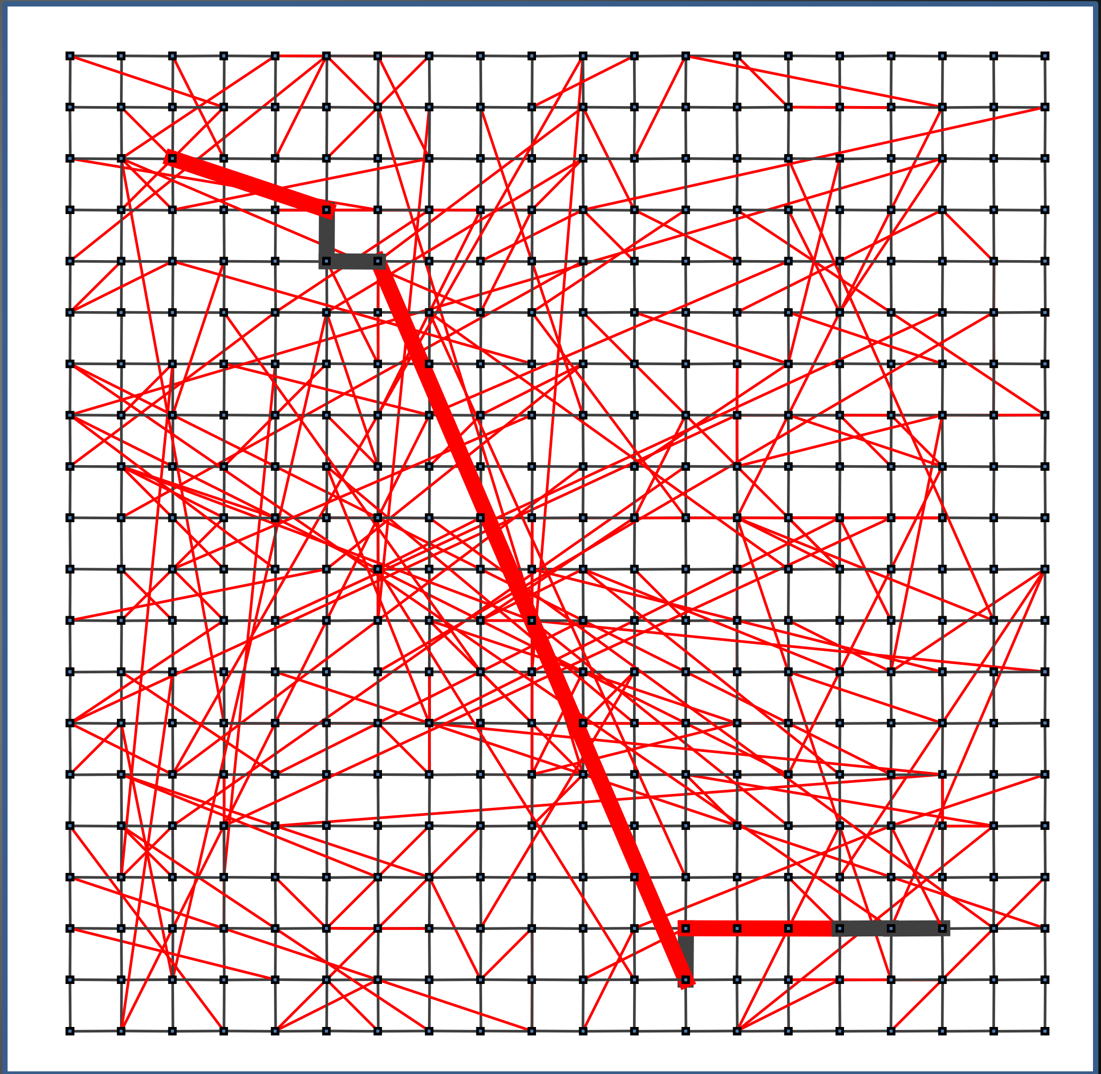
Links balanced between long and short range

$$P \sim [d(u,v)]^{a} / \sum_{v} [d(u,v)]^{a}$$

$$a = 2$$

$$P \sim \frac{1}{d^2}$$

$$T \leq c_2 (\log N)^2$$



Why Two of All Numbers

Optimality of a=2

Divide the network into logarithmically growing shells:

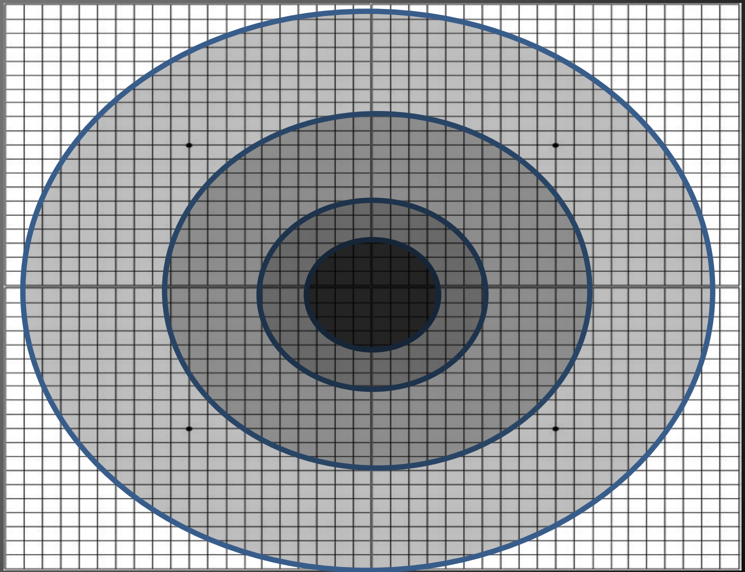
$$S_j = S(d = 2^j) \propto d^2$$

The probability of a rewired edge into the *j*-th shell

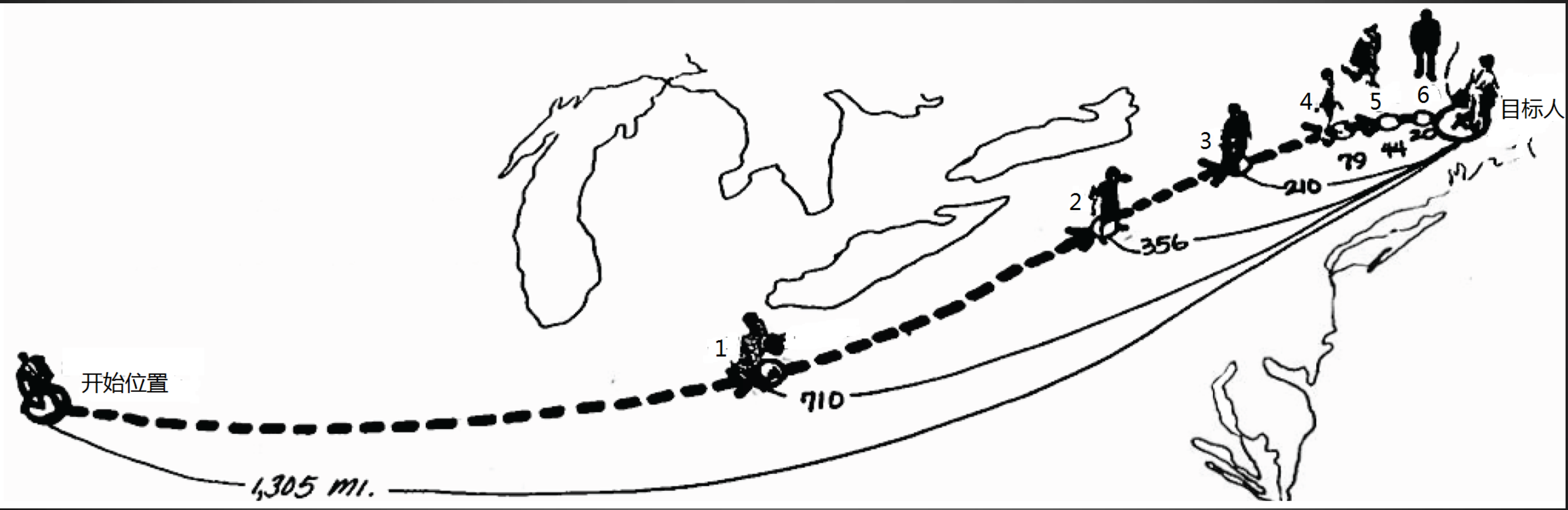
$$P_j \propto S_j d^{-\alpha}$$

At $\alpha = 2$ long-range contacts are evenly distributed over distance scales

You are likely to have a contact half way through $\alpha = 2$



Revisiting Milgram's Experiment

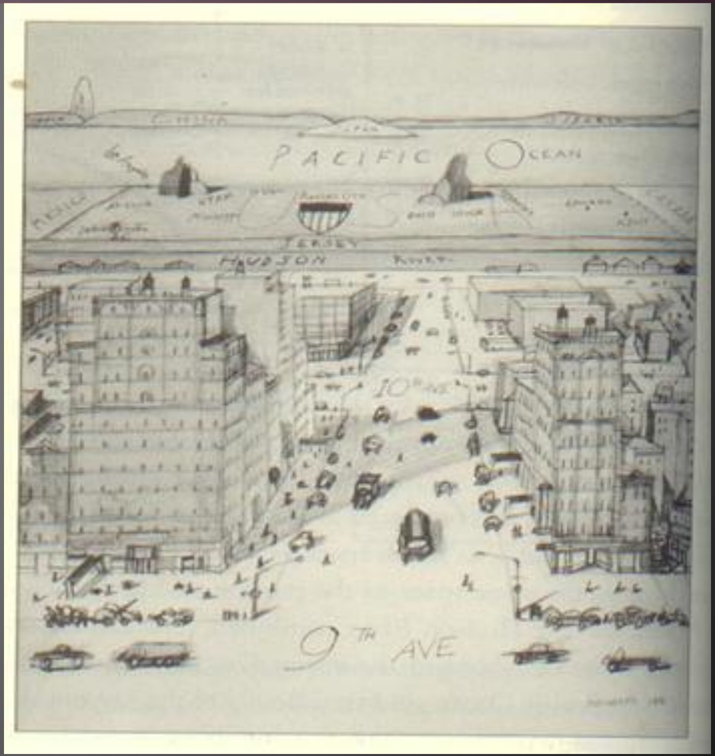


View of the World from 9th Avenue

- 9th Avenue
- An entire city block
- the portion of Manhattan west of 10th Avenue and the Hudson River
- the entire United States west of the Hudson
- the rest of the world
- An individual on 9th Avenue is likely to have the same number of friends in each region (scale)

How Many People From Over the Ocean Do You Know

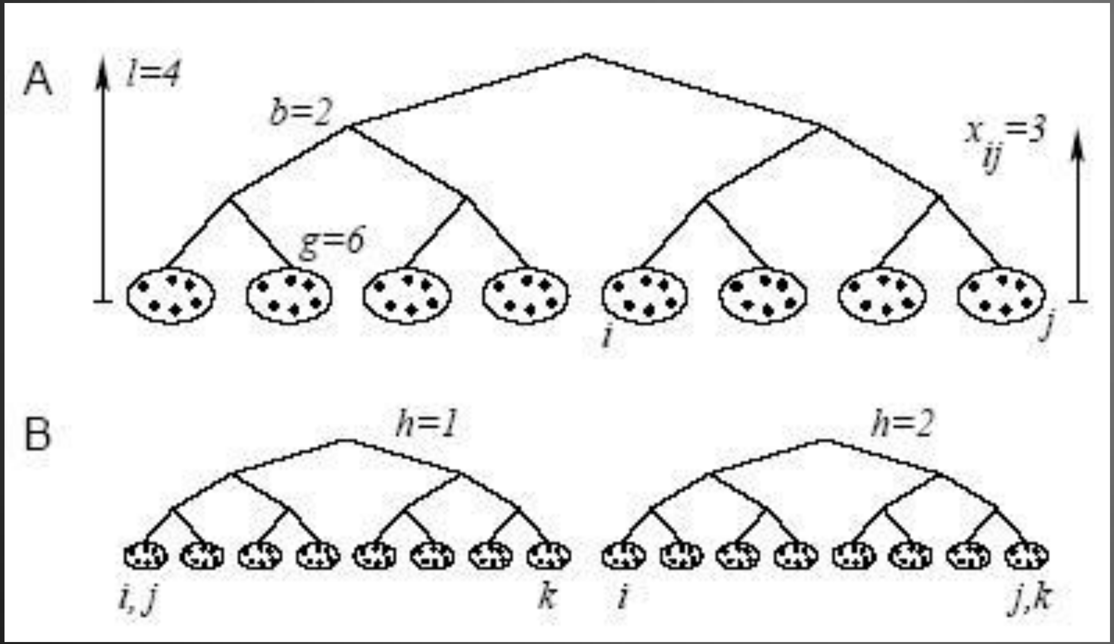
Just as many as you know from down the street



Saul Steinberg, "View of the World from 9th Avenue"

A Hierarchical Model

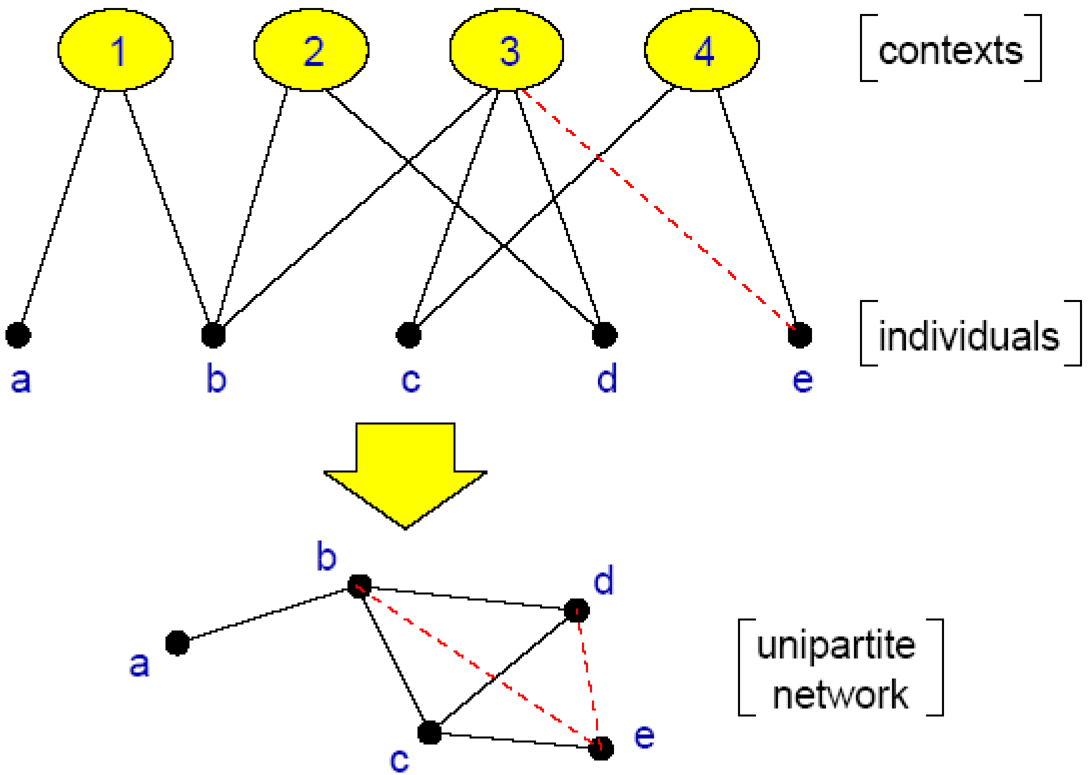
- Lattice captures geographic distance. How do we capture social distance (e.g. occupation)?
- Hierarchical organization of groups
  - distance h(i,j) = height of Least Common Ancestor



$$e^{-a\,x_{ij}}$$

Watts, D. J., Dodds, P. S., and Newman, M. E. J., Identity and search in social networks, *Science* 296, 1302–1305 (2002)

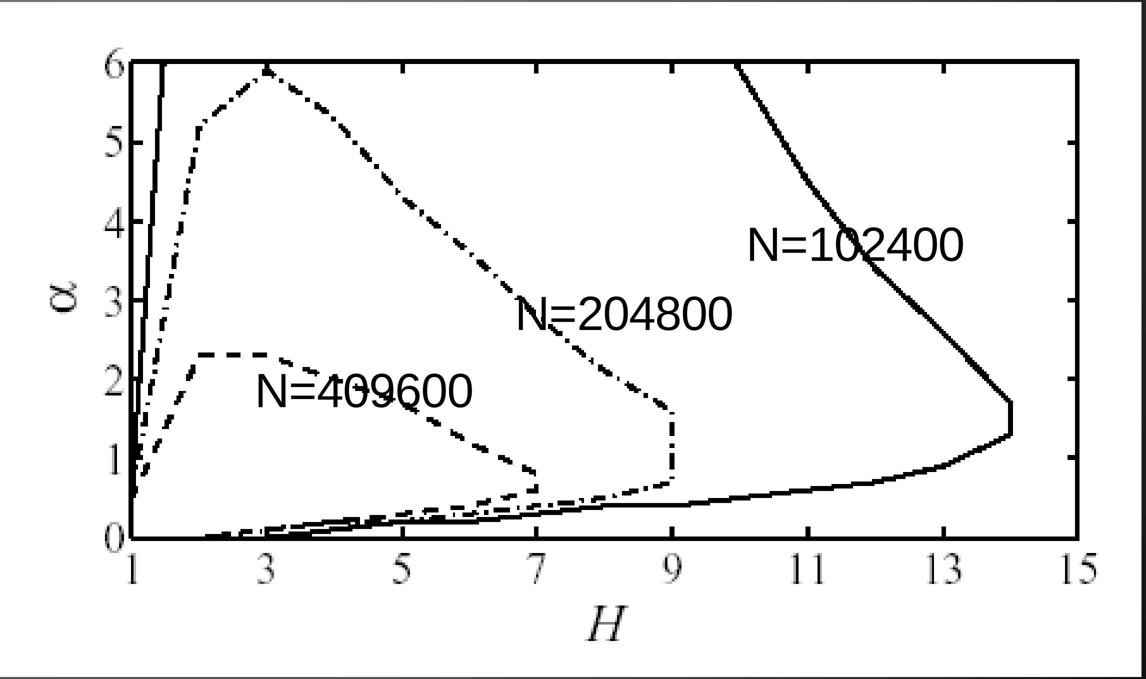
Social distance—Bipartite networks:



A Hierarchical Model

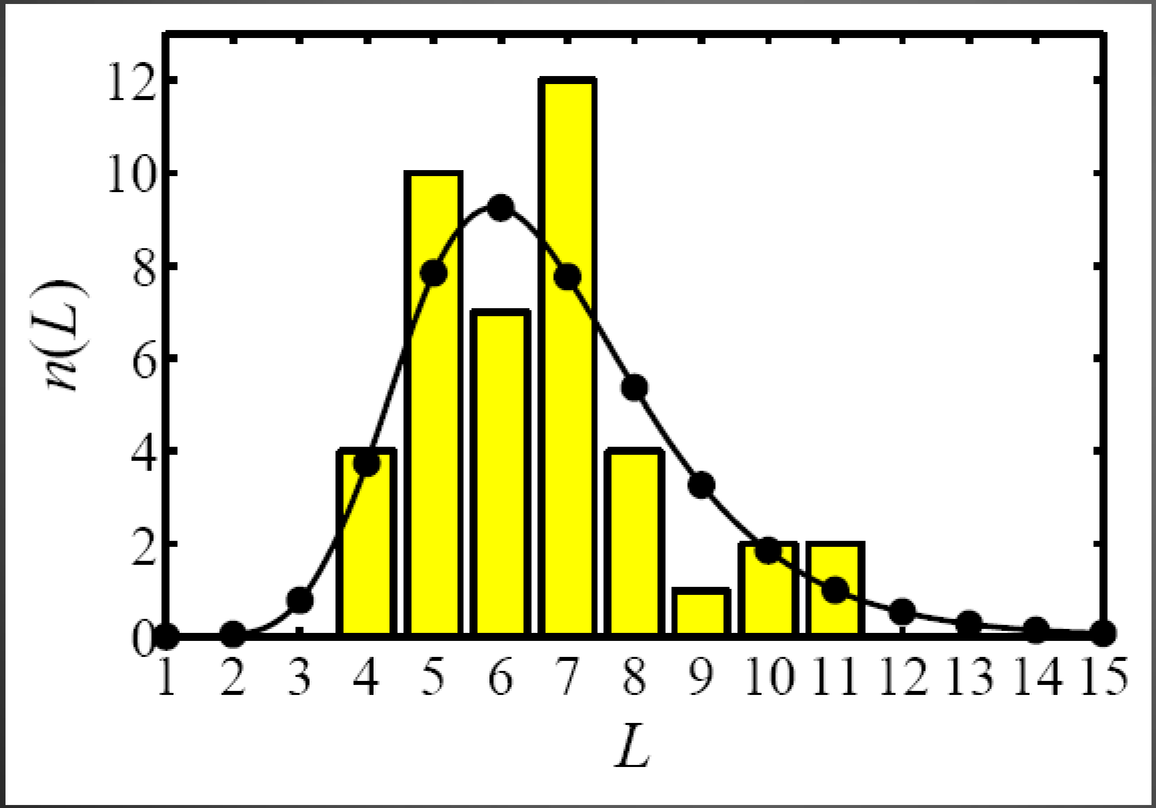
- Message chains fail at each node with probability p
- Network is ‘searchable’ if a fraction r of messages reach the target

$$q = \left\langle (1-p)^L \right\rangle_L \geq r$$



A Hierarchical Model

Fits Milgram's data well



Model parameters:
N = 10^8
z = 300
g = 100
b = 10
α= 1, H = 2

L_{model}= 6.7
L_{data} = 6.5

Thank You !

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