Topological Strings and Twisted Supergravity

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MOTIVATION: want to formulate and prove statements like the following:

IIB string theory on $AdS_5 \times S^5$ $\simeq$ 4d $\mathcal{N}=4$ $U(N)$ SYM as $N \to \infty$

Strategy:

① TWIST each side

② Attach familiar mathematical objects to each side.

③ Make a statement using objects from ②

Today: What objects can we associate to a twist of type II?

Type II Strings $\xrightarrow{[\text{Costello-Li}]}$ TOPOLOGICAL STRINGS

3 field theories + how they interact

• worldsheet theory	① 2d (oriented) TQFT
• open string field theory	② twist of SUSY gauge theory
backreact ( ) closed open	⑤ ( ) ⑥
• closed string field theory	③ ...
∪|	∪|
SUGRA	Twists of SUGRA.

Q: Topological Strings ~~? ⇝ mathematical objects associated to ①, ②, ③!

I. THE WORLDSHEET

• physical theory: 2d σ-model coupled to 2d gravity.

• topological strings: topologically twisted σ-model where diffeos are gauged.

⇝ 2d ORIENTED EXTENDED TQFT. (TCFT).

The data of gauged diffeomorphisms are already present in modern formulation of TQFT

Defn - let $\mathrm{Bord}_{2,1}^{or}$ denote the dg cat w/

objects closed or'd 1-mflds

Def^{n} - let $\mathrm{Bord}^{or}_{2,1}$ denote the dg cat w/

objects closed or'd 1-mflds

morphisms: chains on the space of bordisms $C_\cdot(B(M,N))$

$$B((S^1)^k, (S^1)^{n-k}) = \coprod_g BDiff(\Sigma_{g,n})$$

$$= \coprod_g \mathcal{M}_{g,n} \sqcup \dots$$

$\Gamma_{g,n} = \pi_0 Diff(\Sigma_{g,n})$

$\mathcal{M}_{g,n} = B\Gamma_{g,n}$.

- A 2d TQFT is a symmetric monoidal functor

$$Z: \mathrm{Bord}^{or}_{2,1,0} \longrightarrow \mathrm{DGCat}.$$

[So homs depend on chains on $\mathcal{M}_{g,n}$.]

Thm (Costello, Lurie)

or'd 2d TQFTs $\simeq$ (smooth, proper) CALABI-YAU CATEGORIES

Def^{n} - let $\mathcal{C}$ be an A_∞-category. $\mathcal{C}$ is a CYd category if $\forall\ X, Y \in \mathcal{C}$,

$$\exists \langle -,- \rangle_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes \mathrm{Hom}_{\mathcal{C}}(Y,X) \longrightarrow \mathbb{C}[d]$$

nondegenerate, symmetric, cyclically inv.

Ex

① The B-model: let X be a (smooth) CY variety of dim^n d.

$\mathrm{Coh}(X)$ is a CY cat under Serre duality pairing.
CYd category

② The A-model: let M be a symplectic mfld of dim^n 2n.

$\mathrm{Fuk}_w(M)$ is a CY cat.

This talk: consider examples where pseudoholomorphic disks don't contribute.

Use $\mathrm{Fuk}^\circ(M)$ to denote the wide subcat w/

- objects: $L \subseteq M$ Lagrangian. CYn category.
- morphisms: $\mathrm{Hom}_{\mathrm{Fuk}^\circ(M)}(L_1, L_2) = \Omega^\bullet(L_1 \cap L_2)$.

CY str: wedge and integrate.

③ Mixed A-B model: $\mathrm{Fuk}^\circ(M) \otimes \mathrm{Coh}(X)$ is CY-(n+d).

! All known examples of top'l strings that

! All known examples of top'l strings that arise as twists of superstrings are of this flavor.

II. OPEN STRING FIELD THEORY

: in physical string
- input: a D-brane [cycle on target space w/ a vector bundle]
- output: gauge theory on support of brane.

$\mathcal{F} \in \mathcal{C} \rightsquigarrow \mathrm{End}_{\mathcal{C}}(\mathcal{F}, \mathcal{F})$ computes BRST cohomology of open string states. [Witten]

i.e.

1) $\mathcal{L} = \mathrm{End}_{\mathcal{C}}(\mathcal{F}, \mathcal{F})$ has an A_∞-str hence a L_∞-str.

2) inv't trace $\mathcal{L} \longrightarrow \mathbb{C}[d]$.

Suppose that further

1) $\mathcal{L}$ arises as sheaf of sections of some vb $E \to M$

2) L_∞-str maps given by polydifferential operators.

3) $\mathcal{L} \longrightarrow \mathbb{C}[d]$ factors through $\int_M$

Then $\mathcal{L}$ carries str of a $\mathbb{Z}/2$-graded BV theory.

Examples

SU(5)-inv't twist of IIB

① Let $\mathcal{C} = \mathrm{Coh}(\mathbb{C}^5)$. Fix hol coords z_i, $i = 1, \dots, 5$.

$$\begin{array}{c|ccccc} & z_1 & z_2 & z_3 & z_4 & z_5 \\ \hline N\ D3 & \times & \times & & & \end{array} \rightsquigarrow \mathcal{O}_{\mathbb{C}^2}^{\oplus N} \in \mathcal{C}.$$

Can compute using Koszul resolution that

sections of the normal bundle.

$$\mathrm{Ext}_{\mathcal{C}}(\mathcal{O}_{\mathbb{C}^2}^{\oplus N}) = \Omega^{0,\bullet}(\mathbb{C}^2)[\varepsilon_1, \varepsilon_2, \varepsilon_3] \otimes \mathfrak{gl}_N.$$

L_∞-str : $\Omega^{0,\bullet}(\mathbb{C}^2) \otimes \mathfrak{gl}_N$ has $\ell_1 = \bar\partial$, $\ell_2 = [-\wedge-] \otimes [-,-]_{\mathfrak{gl}_N}$.

$\mathbb{C}[\varepsilon_1, \varepsilon_2, \varepsilon_3]$ is a comm alg.

$$\begin{array}{ccc} \mathrm{Ext}_{\mathcal{C}}(\mathcal{O}_{\mathbb{C}^2}^{\oplus N}) & \longrightarrow & \mathbb{C}[5] \\ \rotatebox{90}{$\in$} & & \\ \alpha & \longmapsto & \int_{\mathbb{C}^{2|3}} dz_1 dz_2 \prod d\varepsilon_i \, \mathrm{Tr}(\alpha). \end{array}$$

We've found the holomorphic twist of 4d $\mathcal{N}=4$ SYM.

② $\mathcal{C}$ as above

$z_1 \quad z_2 \quad z_3 \quad z_4 \quad z_5$

② $\mathcal{C}$ as above

$$M\ D7 \quad \begin{array}{ccccc} z_1 & z_2 & z_3 & z_4 & z_5 \\ \hline \times & \times & \times & \times & \end{array} \quad \rightsquigarrow \quad \mathcal{O}_{\mathbb{C}^4}^{\oplus M}$$

$$\mathrm{Ext}_{\mathcal{C}}(\mathcal{O}_{\mathbb{C}^4}^{\oplus M}) \simeq \Omega^{0,*}(\mathbb{C}^4)[\varepsilon] \otimes \mathfrak{gl}_n$$

L_∞-str and Trace similar to before.

This recovers the holomorphic twist of 8d $\mathcal{N}=1$ SYM.

D5 $\rightsquigarrow$ hol twist of 6d $\mathcal{N}=(1,1)$ SYM.

Could also consider intersecting branes:

③ $\mathcal{C} = \mathrm{Fuk}^{\circ}(\mathbb{R}^4) \otimes \mathrm{Coh}(\mathbb{C}^3)$

	$\mathbb{R}^4$				$\mathbb{C}^3$		
	u	v	x	y	z_1	z_2	z_3
N D3			$\times$	$\times$		$\times$	
M D5		$\times$	$\times$		$\times$	$\times$	

$$\rightsquigarrow \quad \begin{array}{c}(\mathbb{R}^2, \mathcal{O}_{\mathbb{C}}^{\oplus N}) \\ \oplus \\ (\mathbb{R}^2, \mathcal{O}_{\mathbb{C}^2}^{\oplus M})\end{array} =: \mathcal{F}$$

$\mathrm{End}_{\mathcal{C}}(\mathcal{F})$ has 3 summands:

① D3-D3 strings : $\mathrm{End}_{\mathcal{C}}(\mathbb{R}^2, \mathcal{O}_{\mathbb{C}}^{\oplus N}) = \Omega^{\bullet}_{\mathbb{R}^2} \otimes \Omega^{0,*}_{\mathbb{C}}[\varepsilon_1, \varepsilon_3] \otimes \mathfrak{gl}_N$.

② D5-D5 strings : $\mathrm{End}_{\mathcal{C}}(\mathbb{R}^2, \mathcal{O}_{\mathbb{C}^2}^{\oplus M}) = \Omega^{\bullet}_{\mathbb{R}^2} \otimes \Omega^{0,*}_{\mathbb{C}^2}[\varepsilon_1] \otimes \mathfrak{gl}_M$

③ bifundamentals : $\mathrm{Hom}_{\mathcal{C}}((\mathbb{R}^2, \mathcal{O}_{\mathbb{C}}^{\oplus N}), (\mathbb{R}^2, \mathcal{O}_{\mathbb{C}^2}^{\oplus M})) \oplus \mathrm{Hom}_{\mathcal{C}}((\mathbb{R}^2, \mathcal{O}_{\mathbb{C}^2}^{\oplus M}), (\mathbb{R}^2, \mathcal{O}_{\mathbb{C}^2}^{\oplus N}))$ $= \boxed{\Omega^{\bullet}_{\mathbb{R}} \otimes \Omega^{0,*}_{\mathbb{C}}[\varepsilon_1] \otimes T^*\mathrm{Hom}(\mathbb{C}^M, \mathbb{C}^N)}$

3 kinds of brackets:

- ① × ① ⟶ ① ; ② × ② ⟶ ② $\rightsquigarrow$ ① is Kapustin twist of 4d $\mathcal{N}=4$; ② is rank (1,1) twist of 6d $\mathcal{N}=(1,1)$.

- (① ⊕ ②) × ③ ⟶ ③ $\rightsquigarrow$ encodes coupling of free hypermultiplet in hol twist of 3d $\mathcal{N}=4$ to bulk gauge fields.

- ③ × ③ ⟶ ① ⊕ ② $\rightsquigarrow$ enhanced gauge symmetry

  [hypers behave like off-diagonal components of a supergroup.]

More global picture:

To a CY category, can attach a shifted symplectic stack.

Recall – $\mathrm{Perf}: \mathrm{PrSt} \longrightarrow \mathrm{DGCat}^{op}$ has a right adjoint.

Def$^{\underline{n}}$ – For $\mathcal{C}$ a (smooth) dg category, its **Moduli of objects** $\mathcal{M}_{\mathcal{C}}$ is the image under the above right adjoint.

Explicitly: $\mathcal{M}_{\mathcal{C}}(\mathrm{Spec}\, A) = \mathrm{Hom}_{\mathrm{DGCat}}(\mathcal{C}, \mathrm{Perf}\, A)$.

Th$^{\underline{m}}$ (Brav–Dyckerhoff)

1) $\mathcal{F} \in \mathcal{C}$, $T[-1]_{\mathcal{F}} \mathcal{M}_{\mathcal{C}} = \mathrm{End}_{\mathcal{C}}(\mathcal{F}, \mathcal{F})$ as dglas.

2) if $\mathcal{C}$ is CYd, then $\mathcal{M}_{\mathcal{C}}$ is **(2−d)-shifted symplectic**

III CLOSED STRING FIELD THEORY

- input: closed string states of worldsheet theory
- output: field theory on target that recovers SUGRA at low energies.

Let $\mathcal{C}$ be a CY cat and Z the corresponding TFT.

$Z(S^1) = HH^{\bullet}(\mathcal{C})$ is the space of states on S^1.

$\mathrm{Diff}_0(S^1) \curvearrowright Z(S^1)$ b/c of coupling to gravity.

$\mathrm{Vect}_0(S^1) \curvearrowright Z(S^1)$ homotopically trivially.

Want to consider $Z(S^1)^{\mathrm{Diff}_0(S^1)} = Z(S^1)^{U(1)}$.

cyclic cochains.

Thus, space of closed string states is $HC^{\bullet}(\mathcal{C})$.

This will be the fields of our closed string field theory.

Examples

(1) Let $\mathcal{C} = \mathrm{Coh}(\mathbb{C}^d)$. We derive the closed string sector of the B-model **Kodaira–Spencer gravity / BCOV theory**.

Thm - [Wilwacher - Calaque; Kontsevich]

There is a L_∞-equivalence $HC^\cdot(\mathcal{E}) \simeq \left(PV^{\cdot,\cdot}(\mathbb{C}^d)[[t]],\ \bar\partial + t\partial,\ [-,-]_{SN} \right)$

degree 2   divergence operator.

Fields will be $PV^{\cdot,\cdot}(\mathbb{C}^d)[[t]][1]$ (shifted so $PV^{\cdot,\cdot}$ has ghost # 1 in Lie grading)

L_∞ str: $\ell_1 = \bar\partial + t\partial$, $\ell_2 = [-,-]_{SN}$

This theory is **Degenerate** [Butson-Yoo] it has an odd Poisson structure.

- Kernel: $(\partial \otimes 1)\, \delta_{\Delta \subset \mathbb{C}^d \times \mathbb{C}^d} \in PV^{\cdot,\cdot}(X)^{\boxtimes 2}$

$\quad = \Pi$

Powers of t do not appear! Most fields will pair to zero.

(2) let $\mathcal{C} = \mathrm{Fuk}^\circ(\mathbb{R}^{2n})$. In this talk we will take it as an ansatz that the closed string A-model is $\Omega^\cdot(\mathbb{R}^{2n})$ as an abelian theory.

Rel^n to Twisted SUGRA:

In example (1) above, most fields pair to 0 under Π.

let $\Sigma \in PV^{\cdot,\cdot}(X)[[t]]$, write $\Sigma = \sum_k t^k \mu_k$.

Only the duals of $\mu_0^{i,j}$, $\mu_0^{d-1-i,\,d-j}$ will pair to something nonzero.

These fields propagate.

Def^n - **Minimal BCOV** is the smallest subcomplex of the fields that

contains propagating fields

Explicitly: $\bigoplus_{i,j \leq d-1} t^{?} PV^{i,j}(X)[1]$.

- is a Lie subalgebra

- has a Poisson tensor that makes the inclusion a Poisson map.

Thm [Saberi - Williams]

The free limit of the SU(5) twist of IIB SUGRA

> The free limit of the $SU(5)$ twist of IIB SUGRA maps to minimal BCOV on $\mathbb{C}^3$,

$$
\begin{array}{ccccc}
\text{odd} & \text{even} & \text{odd} & \cdots & \text{even} \quad \text{odd} \\
 & & PV^0 & & PV^0 \\
 & & PV^1 \to tPV^0 & & PV^1 \to tPV^0 \\
t^{-2}PV^5 \to & t^{-1}PV^4 \to & PV^3 & \xrightarrow{\partial} & PV^2 \to tPV^1 \to t^2 PV^0 \\
 & t^{-1}PV^5 \to & PV^4 & \xrightarrow{\partial} & PV^3 \to tPV^2 \to t^2PV^1 \to t^3 PV^0 \\
 & & PV^5 & \xrightarrow{\partial} & PV^4 \to tPV^3 \to t^2PV^2 \to t^3PV^0 \to t^4 PV^0
\end{array}
$$

Global Picture: For $\mathcal{C}$ a smooth CYd category, let $\hat{\mathcal{M}}_{CY,\mathcal{C}}$ denote formal moduli problem describing defs of $\mathcal{C}$ as a CY cat.

Thm [Brav-Rozenblyum] $\mathbb{T}[-1]_{\mathcal{C}}\hat{\mathcal{M}}_{CY,\mathcal{C}} = HN_\cdot(\mathcal{C})[1-d]$.

Using CY str, $HN_\cdot(\mathcal{C})[1-d] \simeq \underline{HC^\cdot(\mathcal{C})[1]}$.

Thm [Folklore; Barannikov-Kontsevich (Costello-Li); Grantcharov]

> $\hat{\mathcal{M}}_{CY,\mathcal{C}}$ is $(5-2d)$-shifted Poisson.

Idea: - $HP_\cdot(\mathcal{C})[2-d]$ is $(6-2d)$-symplectic.

- Non commutative period map $\hat{\mathcal{M}}_{CY,\mathcal{C}} \longrightarrow HP_\cdot(\mathcal{C})[2-d]$. Lagrangian.

[jt work w/ Pavel Safronov]

IV CLOSED-OPEN MAP:

input : closed string field and a brane
output : deformation of worldvolume theory on brane.

① let $\mathcal{C}$ be a A_∞-cat, $\mathcal{F} \in \mathcal{C}$, $\mathcal{C}$ CY

inclusion of subcat gen'd by $\mathcal{F}$ induces $HH_\cdot(End_{\mathcal{C}}(\mathcal{F})) \longrightarrow HH_\cdot(\mathcal{C})$.

Taking linear duals and S^1-invts: $HC^\cdot(\mathcal{C}) \longrightarrow HC^\cdot(End_{\mathcal{C}}(\mathcal{F}))$.

<u>example</u> B-model w/ D3 branes:

$$HC^\cdot(End_{\mathcal{C}}(\mathcal{F})) = HC^\cdot(\Omega^{0,\cdot}_{\mathbb{C}^2}[\varepsilon_1,\varepsilon_2,\varepsilon_3]\otimes \mathfrak{gl}_N)$$

$$\underset{HKR}{\simeq} \underline{PV^{\cdot,\cdot}(\mathbb{C}^2)[\varepsilon_i,\partial_{\varepsilon_i}]}[[t]].$$

Map $\quad PV^{\bullet,\bullet}(\mathbb{C}^5)[[t]] \longrightarrow PV^{\bullet,\bullet}(\mathbb{C}^2_{z_1,z_2})[\varepsilon_i, \partial_{\varepsilon_i}][[t]]$

given by a Fourier-transform (to first order)

$$z_1, z_2, \partial_{z_1}, \partial_{z_2} \longmapsto z_1, z_2, \partial_{z_1}, \partial_{z_2}.$$

$$z_i, \ i = 3,4,5 \longmapsto \partial_{\varepsilon_i}$$

$$\partial_{z_i} \longmapsto \varepsilon_i .$$

(2) There is a map of complexes $HC^\bullet(\mathrm{End}_{\mathcal{C}}(\mathcal{F})) \longrightarrow CE^\bullet(\mathrm{End}_{\mathcal{C}}(\mathcal{F}))$. Taking such a polyvector field to an action functional by which it couples to $\mathrm{End}_{\mathcal{C}}(\mathcal{F})$.

examples

(1) $\quad PV(\mathbb{C}^5) \longrightarrow CE^\bullet(\Omega^{0,\bullet}_{\mathbb{C}^2}[\varepsilon_1, \varepsilon_2, \varepsilon_3] \otimes \mathfrak{gl}_N)$

$$\partial_{z_2}\partial_{z_3} \longmapsto \int_{\mathbb{C}^{2|3}} \mathrm{Tr}(\alpha\, \varepsilon_3 \partial_{\varepsilon_2} \alpha)\, dz_1 dz_2$$

$H^\bullet\left(\begin{matrix}10d\ N=(2,0)\\ \text{susy alg}\end{matrix}, Q\right) \to$ closed string field theory.

Turns z_2 into a top'l plane; theory is deformed to Kapustin twist.

$$z_2 z_3 \longmapsto \int \mathrm{Tr}(\alpha\, z_2 \partial_{\varepsilon_3} \alpha)$$

Localizes theory onto $z_2 = 0$.

Theory is a 2d gauged BF-system.

(2) $\quad PV(\mathbb{C}^5) \longrightarrow CE^\bullet(\Omega^{0,\bullet}_{\mathbb{C}^4}[\varepsilon] \otimes \mathfrak{gl}N)$

$$z_4 z_5 \longmapsto \int \mathrm{Tr}(\alpha\, z_4 \partial_{\varepsilon_5} \alpha)$$

Localizes theory onto $z_4 = 0$

Result is hol CS on $\mathbb{C}^3$.

(3) $\quad \Omega^\bullet(\mathbb{R}^4) \otimes PV^{\bullet,\bullet}(\mathbb{C}^3) \longrightarrow CE^\bullet(D3-D5 \text{ system}).$

$$z_1 z_3 \longmapsto$$

D3-D3 $\rightsquigarrow$ 2d BF

D5-D5 $\rightsquigarrow$ 4d CS

D3-D5 $\rightsquigarrow$ TQM.

Global Picture

$\mathcal{C}$ a smooth CYd cat.

[expand around $\mathcal{F}$]

$$\begin{tikzcd} HN_\cdot(\mathcal{C})[1-d] \arrow[r, "\text{[Brav-Rozenblyum]}"'] & SVect(M_{\mathcal{C}})[1-d] \arrow[r] & CE^\cdot(\mathrm{End}\,\mathcal{F}, \mathrm{End}\,\mathcal{F}[1]) \arrow[d] \\ & & Sym^{\leq 2}(\mathrm{End}\,\mathcal{F}, \mathrm{End}\,\mathcal{F}[1]) \end{tikzcd}$$

:

$$\begin{tikzcd} HC_\cdot(\mathcal{C})[2-d] \arrow[r, "B"] \arrow[d] & HN_\cdot(\mathcal{C})[1-d] \arrow[d] \\ Ham(M_{\mathcal{C}})[2-d] \arrow[r, "d"] & SVect(M_{\mathcal{C}})[1-d] \end{tikzcd}$$