MATH 3450 Exam #1 Winter 2021 Solns

1a. $f(x) = x^4 - x - 2$

$f(0) = -2$ ✓

$f(2) = 12$

$c = \frac{1}{2}(0+2) = 1$

$f(1) = -2$

$c = \frac{1}{2}(1+2) = \frac{3}{2}$

$f\left(\frac{3}{2}\right) = \frac{81}{16} - \frac{3}{2} - \frac{4}{2} > 0$

$c = \frac{1}{2}\left(1 + \frac{3}{2}\right) = \frac{5}{4}$

1b. $x_0 = 0$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_1 = 0 - \frac{0^4 - 0 - 2}{4 \cdot 0^3 - 1} = \boxed{-2}$

1c. The approx obtained by the bisection method is more accurate because it is closer to the exact value.

1d. Newton's method will converge more rapidly because it will converge to $x = -1$, a root of multiplicity one quadratically while bisection will only converge linearly.

2a. $I = \int_1^5 \frac{24}{3+x}\,dx$

$I \approx \frac{5-1}{2}\left(\frac{1}{2}f(1) + f(3) + \frac{1}{2}f(5)\right)$

$2\left(\frac{1}{2}(6) + 4 + \frac{1}{2}(3)\right) = 17$

2b. $E_{Trap} \le \frac{(b-a)^3}{12N^2}\left|f''(x^*)\right|$

$f''(x) = 48(3+x)^{-3}$

$\max_{x \in [1,5]} f''(x) = 48 \cdot 4^{-3} = 3/4$

$E_{Trap} \le \frac{4^3}{12N^2} \cdot \frac{3}{4} = \frac{4}{N^2} < 1 \times 10^{-8}$

$\frac{4}{1 \times 10^{-8}} < N^2$

$20{,}000 < N$

One would need to use more than approximately 20,000 trapezoids to ensure the error $< 1 \times 10^{-8}$

2d. The trapezoid rule will converge faster because it converges quadratically while LER converges linearly.

3. Let N_x = number samples in x

N_y = number samples in y

$f(x,y) = \cos^2(x^2 + y^3)$

$\Delta x = \frac{7-2}{N_x}$ $\quad \Delta y = \frac{3-1}{N_y}$

```
sum = 0.0
for j = 1:Nx
    for K = 1:Ny
        x = random # in [2,7]
        y = random # in [1,3]
        sum += f(x,y)
    end
end
return Δx·Δy·sum
```

4a. $\hat{g}(0) = 6$

$\hat{g}(2) = -7i$

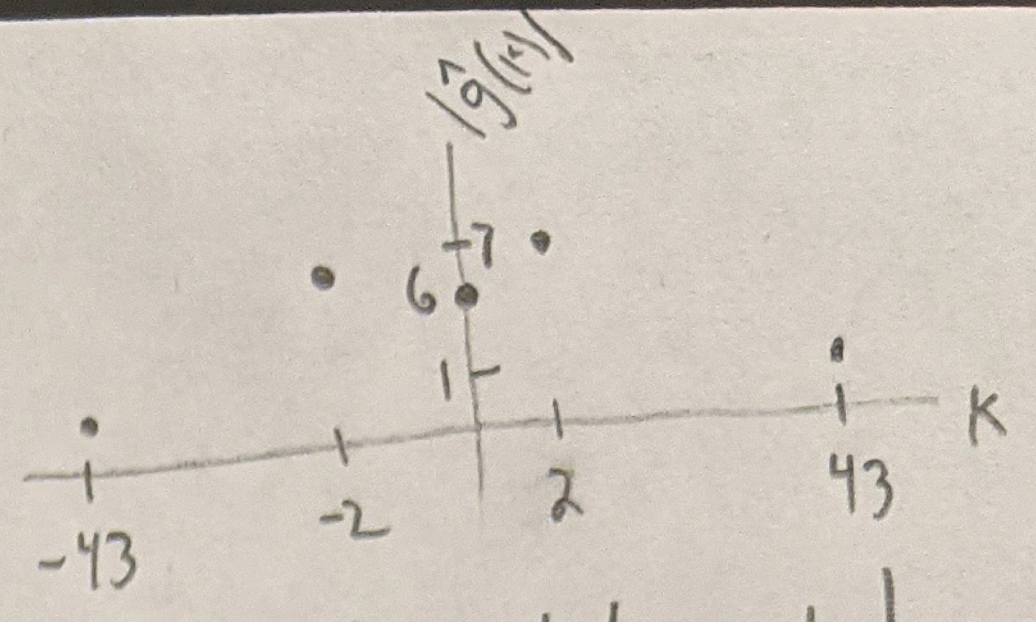
$\hat{g}(-2) = 7i$

$\hat{g}(43) = 1$

$\hat{g}(-43) = 1$

$\hat{g}(k) = 0$ for all other integers k

4b.



plot not to scale!

4c. $xvec = \langle 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \dots \frac{(N-1)}{N}2\pi \rangle$

gvec = g.(xvec)

ghat = fft(gvec)

4d. 87 or more F. coeffs are required to accurately recover the discrete version of $g(x)$.

4e. The ghats obtained via the fft will only differ from the exact values by round off error because $g(x)$ is periodic, assuming more than 86 gridpts are used.

5a. False. The size of the system does not determine the rate of convergence. However, it is likely that Sally's code will take longer to evaluate

5b. $Ax = b$

$$\begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

G.S. will not converge because this matrix is symmetric with positive diagonal elements, but is not positive definite because the evals are $\lambda = 6, -4$

Jacobi: $N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad P = \begin{pmatrix} 0 & -5 \\ -5 & 0 \end{pmatrix}$

$N^{-1}P = \begin{pmatrix} 0 & -5 \\ -5 & 0 \end{pmatrix} \longrightarrow$ has evals $\lambda = \pm 5$ which are greater than one.