MATH 3450 Exam #2 Solutions Winter 2021

1a. A is 3×4, so

$\mathcal{U}$ is 3×3

Σ is 3×4

V is 4×4

1b. A is 3×4, but the rows are not lin indep, so

$\mathcal{U}$ is 3×1

Σ is 1×1

V is 1×4

1c. The matrix A has rank 1, so that means A has one singular value. Thus

$$\Sigma = \begin{pmatrix} \square & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where the $\square$ is the singular value (calculations show $\square = 2\sqrt{105}$)

1d. It depends.

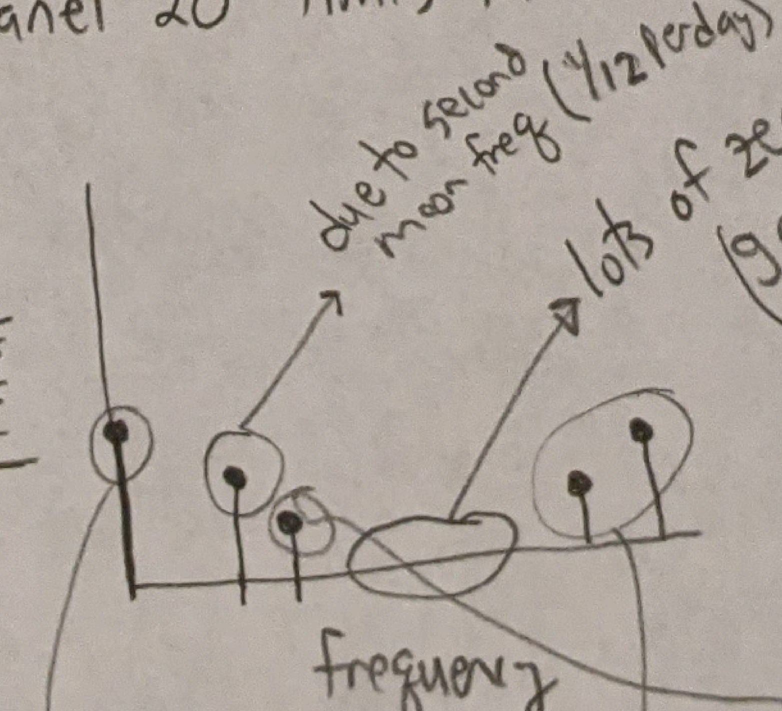
- If b makes the system consistent, then the pseudo inverse would find the vector x with smallest ℓ_2 norm that satisfies $Ax=b$. (In this case the system is undetermined, so there are an infinite number of solns.)
- If b makes the system inconsistent, then the pseudoinverse would find the vector x that minimizes $\|Ax-b\|_2^2$. (In this case the system is overdetermined, so there aren't any solns, but this tells us the least squares soln.)

2. $A = \mathcal{U} * \Sigma * V^T$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$$

3. Jimmy + Sally collect 12,000 measurements. They are going to measure the mean water height and the change in tides due to the two moons. The F. coeffs due to the first moon are smaller in magnitude than the F. coeffs of the second moon. The first moon orbits the planet 24 times in 240 days. The second moon orbits the planet 20 times in 240 days.



graph not to scale because there are no high frequency waves

due to first moon frequency ($1/10$ per day)

frequency zero. This corresponds to the mean sea level at the measurement location

complex conjugates of the $1/10$ and $1/12$ frequencies

4. $A = \mathcal{U}\Sigma V^T$

- Σ is the diagonal matrix with the square roots of the evals of A^TA on its diagonal
- $\mathcal{U}$ is formed by the unit-length evecs of AA^T
- V is formed by the unit-length evecs of A^TA
  - use $\mathcal{U} = \Sigma^{+} VA \rightarrow$ to ensure evec signs work properly.

5a. True. If $\lambda = 0$ is an eval of A, then it will also be an eval of A^TA and therefore zero will be a "singular value" of A.

5b. False. There is a sign choice for each of the eigenvectors in $\mathcal{U}$ and V. This means there will be at least 2 equivalent SVD for any matrix A.

5c. $A = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

5d. Let the data be of the form (x_j, y_j) for $j = 1, 1024$. Consider

$$Ax = b$$

where

$$x = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$A = \begin{pmatrix} \cos(x_1) & \cos(2x_1) \\ \cos(x_2) & \cos(2x_2) \\ \vdots & \\ \cos(x_n) & \cos(2x_n) \end{pmatrix}$$

$$b = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

Use the pseudoinverse of A to find c_1 and c_2.

The curve of best fit is then

$$y(x) = c_1\cos(x) + c_2\cos(2x)$$