Sarah Mahl
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EXAM #2

1.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{pmatrix}_{3\times 4}$$

(a) $\mathcal{U}$ is a 3x3 matrix. Σ is a 3x4 matrix. V is a 4x4 matrix. This is because $\mathcal{U}$ is formed from the unit-length eigenvectors of $A \cdot A^*$ ($3\times4 \cdot 4\times3 = 3\times3$). Σ is formed from the diagonal matrix of the square roots of the eigenvalues of A^*A and has the same dimensions as A. V is formed from the unit-length eigenvectors of A^*A ($4\times3 \cdot 3\times4 = 4\times4$).

(b) Σ is not a square matrix, so it will have a column of zeros. This is because A^*A has 4 eigenvalues (since it is a 4x4 matrix), but one of them is most likely zero. This gives us three singular values for A.

$\mathcal{U}$ is 3x1
Σ is 1x1
V is 1x4
→ rows of A are not linearly independent, & A has rank 1 (thus one singular value)

(c) For the compact SVD, Σ has dimensions 3x3. $\mathcal{U}$ has dimensions 2x3 and V has dimensions 3x2. This is because removing a column from Σ also means that we have to remove a column from $\mathcal{U}$ and a row from V. They also all have to be able to multiply ($A = \underset{M\times M}{\mathcal{U}}\underset{M\times N}{\Sigma}\underset{N\times N}{V^T}$)

→ can't just take off a row

(d) The pseudoinverse of the system $Ax=b$, given b, would tell us the approximate solution to the system such that the vector x minimizes $\|Ax-b\|_2^2$. A has 2 linearly dependent equations and one linearly independent equation. This means that the system has no solution, so the pseudoinverse finds an x that is as close to a "solution" as possible.

2.

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}, \quad V = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$\rightarrow$ A is a 2x2 matrix

$$\lambda_{A^*A} = (2\sqrt{2})^2, (\sqrt{2})^2 = 8, 2$$

$$A = U\Sigma V^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}^*$$

$$= \begin{pmatrix} 1(2\sqrt{2})+0 & 0(1)+0(\sqrt{2}) \\ 0(2\sqrt{2})+1(0) & 0(0)+1(\sqrt{2}) \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

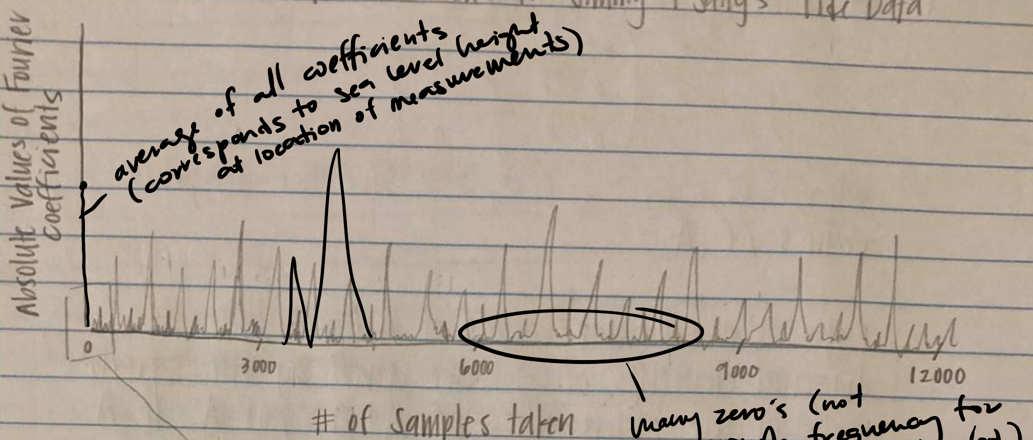
$$= \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 2\sqrt{2}(1/\sqrt{2})+0(\frac{-1}{\sqrt{2}}) & 2\sqrt{2}(\frac{+1}{\sqrt{2}})+0(\frac{1}{\sqrt{2}}) \\ 0(\frac{1}{\sqrt{2}})+\sqrt{2}(\frac{-1}{\sqrt{2}}) & 0(\frac{+1}{\sqrt{2}})+\sqrt{2}(\frac{1}{\sqrt{2}}) \end{pmatrix}$$

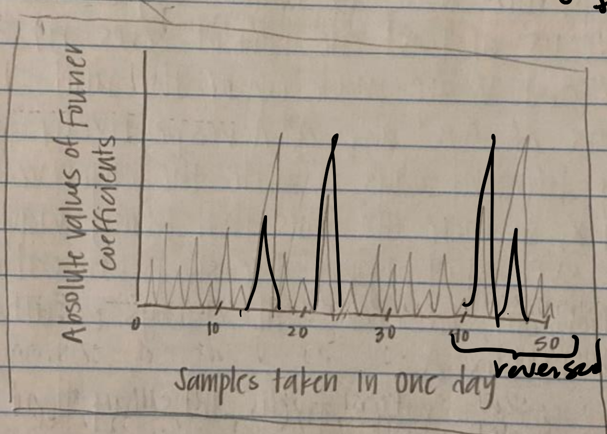
$$\boxed{A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}}$$

3. $50 \frac{\text{measurements}}{\text{day}} \cdot 240 \text{ days} = 12{,}000$ measurements

Because the two moons orbit at different intervals, we will assume the tides peak 4 times (a high and low tide for the first moon, and a high and low tide for the second). These will most likely end up being a lowest tide, low tide, high tide, and highest tide since the second moon affects the tide more than the first moon (which will increase the effects on the tide of the first moon superpositionally). The x-axis of the plot will give us the absolute value of the Fourier coefficients, where the Fourier coefficients can be found in the vector fhat $= \langle \hat{f}(0), \hat{f}(1), \hat{f}(2), \dots, \hat{f}(\frac{N}{2}), \hat{f}(-\frac{N}{2}+1), \hat{f}(-\frac{N}{2}+2), \dots, \hat{f}(-1) \rangle$ with $N = 12{,}000$ measurements. The value for $\hat{f}(0)$ gives us the average of the tide data on the interval $[0, N]$

Plot of Fourier Coefficients for Jimmy & Sally's Tide Data





Fourier coefficients of first moon are smaller than coefficients of second moon

reversed are the complex conjugates

$$\frac{0.48 \text{ hr}}{} \cdot \frac{60 \text{ min}}{1 \text{ hr}} = x \text{ minutes}$$

$\frac{50 \text{ measurements}}{24 \text{ hours/day}}$ = measurements were taken every 0.48 hours

The peaks are where the tide data was recorded to either be at a high or a low for the time that measurement was taken. This can tell us when the high and low tides occur, and the Fourier coefficients can tell us when the water will be the deepest or the most shallow. I "zoomed" into one day's worth of data to better show the peaks since it would be very difficult to see those on a plot of all the Fourier coefficients for the entire data set.

4. (a) ~~True, By the definition of~~ $\cancel{\Sigma}$,

$$\cancel{\Sigma = \begin{pmatrix} \theta_1 & 0 & 0 & 0 \\ 0 & \theta_2 & \ldots & 0 \\ 0 & & \theta_r & 0 \\ 0 & 0 & \ldots & 0 \end{pmatrix}}$$

~~where~~ $\cancel{\theta_1 \geq \theta_2 \geq \ldots \geq \theta_r \geq 0}$ ~~are the singular values of A~~

True. If $\lambda = 0$ is an eigenvalue of A, then it's also an eval of A^*A & thus a singular value of A.
↳ we would have a row of zeros

False. The singular values of A (and thus the diagonal elements of Σ) are given by the square roots of the eigenvalues of A^*A, not A. If A has an eigenvalue $\lambda = 0$, we cannot say whether this will give us at least one row of zeros in Σ.

(b) False. $\mathcal{U}$ and V are made up of the unit-length eigenvectors of AA^* and A^*A respectively. There are many different ways to write the eigenvectors of a matrix, so there are thus that many ways to write the SVD. Multiplying the singular value decomposition out, however, will always result in the same matrix A.

(c) With two singular values given, let's say that A is a 2x2 matrix.

$$\det\begin{bmatrix} x_1-\lambda & x_2 \\ x_3 & x_4-\lambda \end{bmatrix} = \text{characteristic equation} - \sqrt{9}, \sqrt{4} - 3, 2$$

$$(\lambda-3)(\lambda-2) = 0$$

$$(x_1-\lambda)(x_4-\lambda) - x_3x_2 = (\lambda-3)(\lambda-2)$$

$$x_1x_4 - x_1\lambda - x_4\lambda + \lambda^2 - x_3x_2 = \lambda^2 - 2\lambda - 3\lambda + 6$$

$$\lambda^2 - x_1\lambda - x_4\lambda - x_3x_2 - x_1x_4 = \lambda^2 - 2\lambda - 3\lambda + 6$$

$$\lambda^2 - (x_1+x_4)\lambda - x_3x_2 - x_1x_4 = \lambda^2 - 5\lambda + 6$$

$$x_1 + x_4 = 5 \qquad x_3x_2 + x_1x_4 = -6$$

$$\Rightarrow x_1 = 2,\ x_4 = 3,\ x_2 = 4,\ x_3 = -3$$

Answer on next page

A possible matrix for A is

$$\boxed{A = \begin{pmatrix} 2 & 4 \\ -3 & 3 \end{pmatrix}} \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(d) Given a set of 1024 points, one would find the curve of best fit using the linear least squares problem for the functions $y_1(x) = \cos(x)$ and $y_2(x) = \cos(2x)$. This would create the system $Ax = b$, where A is a matrix formed by $y_1(x)$ and $y_2(x)$ and b is the column vector of the 1024 data points. Solving for x would give us $x = A^{-1}b$, and the pseudoinverse would solve the linear least squares problem to find the vector x that would minimize $\|Ax - b\|_2^2$. This would give us the curve of best fit.

Let data be in the form (x_j, y_j), $j \in [1, 1024]$

for $Ax = b$,

$$A = \begin{pmatrix} \cos(x_1) & \cos(2x_1) \\ \cos(x_2) & \cos(2x_2) \\ \vdots & \vdots \\ \cos(x_n) & \cos(x_n) \end{pmatrix}$$

$$b = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$x = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

The curve of best fit is then $y(x) = c_1 \cos(x) + c_2 \cos(2x)$

I, Sarah Mahl, completed this exam by myself with my notes, codes, and course materials without using any other internet or human resources, signed

Sarah Mahl