# Lecture 1

## Review of Simple Linear Regression

Reading: ISLR 2021 Chapter 3.1

*DSA 8020 Statistical Methods II*



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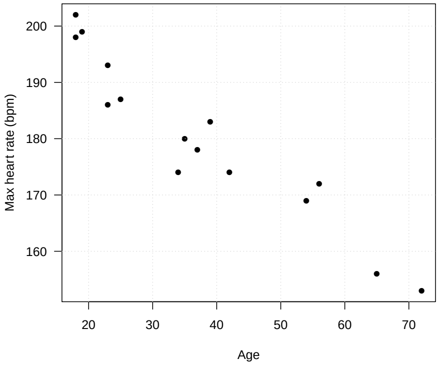
# Agenda

1. Simple Linear Regression
2. Parameter Estimation
3. Residual Analysis
4. Confidence/Prediction Intervals
5. Hypothesis Testing

# What is Regression Analysis?

**Regression analysis**: A set of statistical procedures for estimating the relationship between response variable and predictor variable(s)




Simple linear regression: The relationship between the response variable and the predictor variable is approximately linear

## Simple Linear Regression (SLR)

$y$: response variable; $x$: predictor variable

- In SLR we **assume** there is a **linear relationship** between $x$ and $y$:

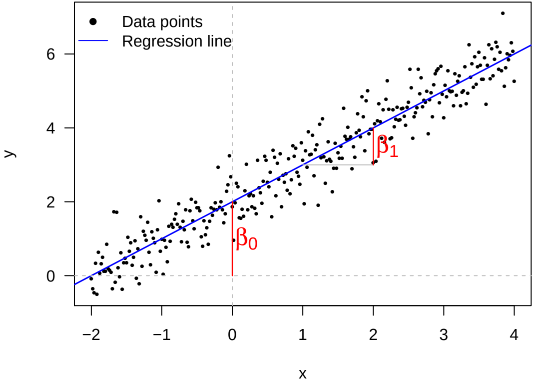
$$y = \beta_0 + \beta_1 x + \varepsilon$$

- We need to estimate $\beta_0$ (intercept) and $\beta_1$ (slope) based on observed data $\{x_i, y_i\}_{i=1}^n$

- We can use the estimated regression equation to
  - make predictions
  - study the relationship between response and predictor
  - control the response

- Yet we need to quantify our estimation uncertainty regarding the linear relationship

# Regression equation: $y = \beta_0 + \beta_1 x$




- $\beta_0$: $\mathrm{E}[y]$ when $x = 0$
- $\beta_1$: $\mathrm{E}[\Delta y]$ when $x$ increases by 1

# Assumptions about the Random Error $\varepsilon$

In order to estimate $\beta_0$ and $\beta_1$, we make the following assumptions about $\varepsilon$

- $\mathrm{E}[\varepsilon_i] = 0$
- $\mathrm{Var}[\varepsilon_i] = \sigma^2$
- $\mathrm{Cov}[\varepsilon_i, \varepsilon_j] = 0, \quad i \neq j$

Therefore, we have

$$\mathrm{E}[y_i] = \beta_0 + \beta_1 x_i, \text{ and}$$
$$\mathrm{Var}[y_i] = \sigma^2$$

The regression line $\beta_0 + \beta_1 x$ represents the **conditional mean curve** whereas $\sigma^2$ measures the magnitude of the **variation** around the regression curve

## Estimation: Method of Least Squares

For given observations $\{x_i, y_i\}_{i=1}^n$, choose $\beta_0$ and $\beta_1$ to minimize the *sum of squared errors*:

$$\ell(\beta_0, \beta_1) = \sum_{i=1}^{n} \left(y_i - (\beta_0 + \beta_1 x_i)\right)^2$$

Solving the above minimization problem requires some knowledge from Calculus (see notes `LS_SLR.pdf`)

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \quad (1)$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} \quad (2)$$

We also need to **estimate** $\sigma^2$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^n (y_i - \hat{y}_i)^2}{n-2}, \quad (3)$$

where

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i \quad (4)$$

# Example: Maximum Heart Rate vs. Age

The maximum heart rate `MaxHeartRate` of a person is often said to be related to age `Age` by the equation:

$$\text{MaxHeartRate} = 220 - \text{Age}.$$

Suppose we have 15 people of varying ages are tested for their maximum heart rate (bpm)

1. Compute the estimates for the regression coefficients, $\hat{\beta}_0$ and $\hat{\beta}_1$, using Equations (1) and (2)
2. Compute the fitted values $\{\hat{y}_i\}_{i=1}^{n}$ using Equation (4)
3. Compute the estimate for $\sigma$ by applying the square root of Equation (3)

# Maximum Heart Rate vs. Age

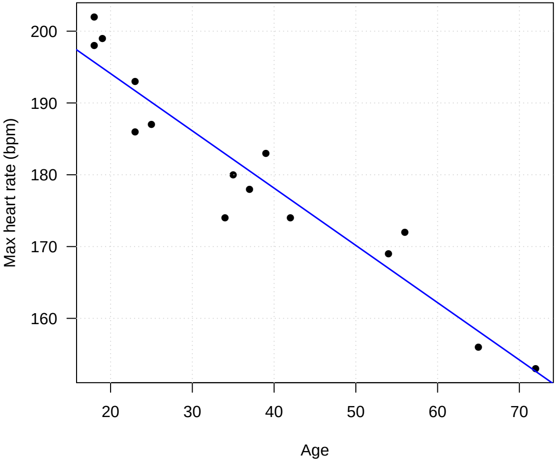




Output from R (RStudio)

```
> fit <- lm(MaxHeartRate ~ Age)
> summary(fit)

Call:
lm(formula = MaxHeartRate ~ Age)

Residuals:
    Min      1Q  Median      3Q     Max
-8.9258 -2.5383  0.3879  3.1867  6.6242

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 210.04846    2.86694   73.27  < 2e-16 ***
Age          -0.79773    0.06996  -11.40 3.85e-08 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.578 on 13 degrees of freedom
Multiple R-squared:  0.9091,    Adjusted R-squared:  0.9021
F-statistic:   130 on 1 and 13 DF,  p-value: 3.848e-08
```

# Assessing Linear Regression Fit




**Question:** Is linear relationship between max heart rate and age reasonable? ⇒ Residual Analysis

# Residuals

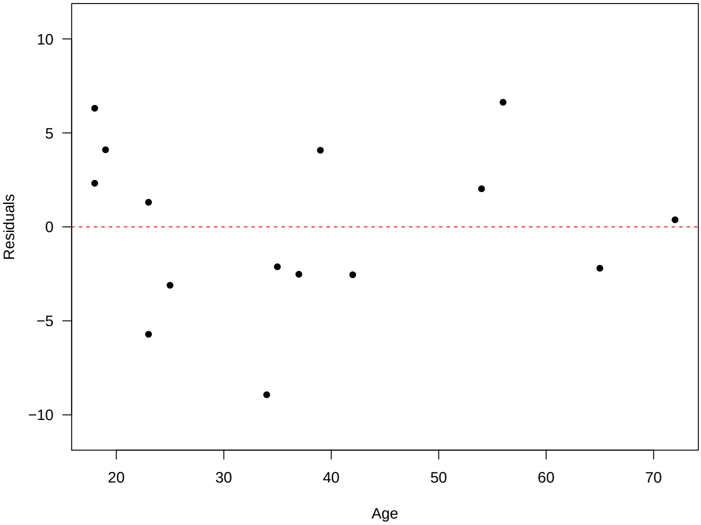
- The residuals are the differences between the observed and fitted values:

$$e_i = y_i - \hat{y}_i,$$

  where $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$

- Residuals are very useful in assessing the appropriateness of the assumptions on $\varepsilon_i$. Recall
  - $\mathrm{E}[\varepsilon_i] = 0$
  - $\mathrm{Var}[\varepsilon_i] = \sigma^2$
  - $\mathrm{Cov}[\varepsilon_i, \varepsilon_j] = 0, \quad i \neq j$

# Residuals Against Predictor Plot



# Interpreting Residual Plots

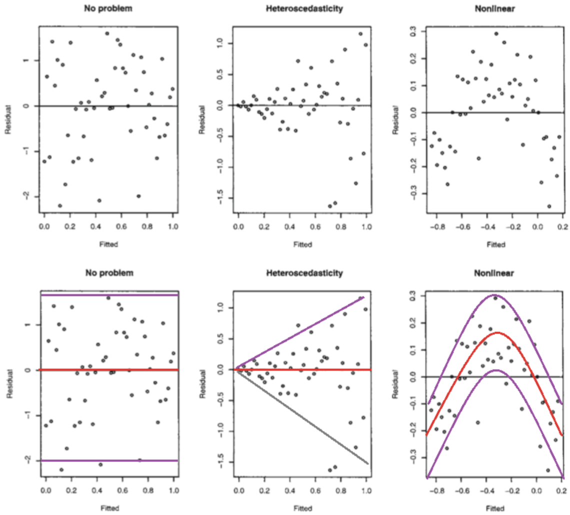
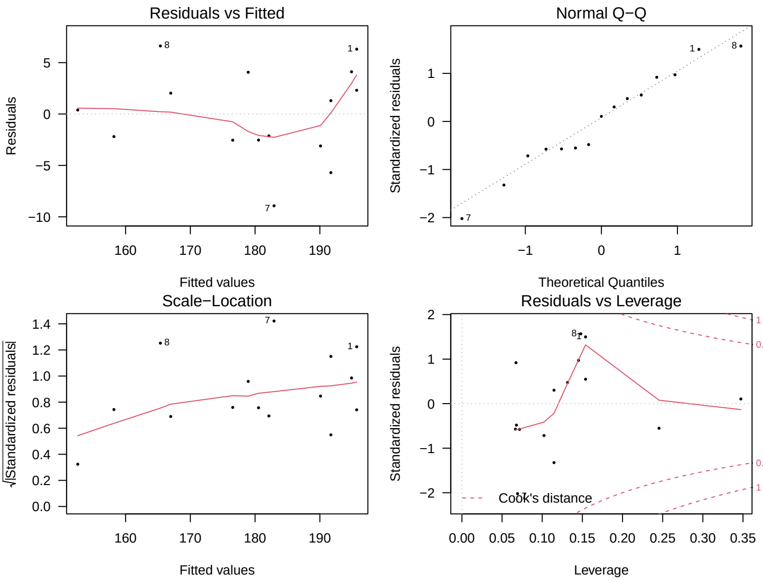


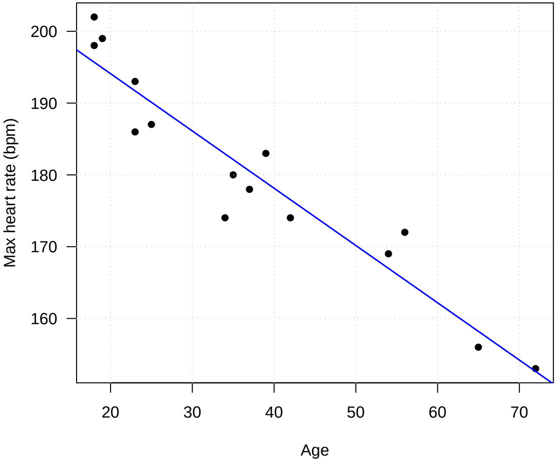

Figure courtesy of Faraway's Linear Models with R (2014, p. 74).

# Diagnostic Plots in R



# How (Un)certain We Are?






**Can we formally quantify our estimation uncertainty?** $\Rightarrow$

We need additional (distributional) assumption on $\varepsilon$

# Normal Error Regression Model



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Recall

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

- Further assume $\varepsilon_i \sim \mathrm{N}(0, \sigma^2) \Rightarrow y_i|x_i \sim \mathrm{N}(\beta_0 + \beta_1 x_i, \sigma^2)$
- With normality assumption, we can derive the **sampling distribution** of $\hat{\beta}_1$ and $\hat{\beta}_0 \Rightarrow$

$$\frac{\hat{\beta}_1 - \beta_1}{\hat{SE}(\hat{\beta}_1)} \sim t_{n-2}, \quad \hat{SE}(\hat{\beta}_1) = \frac{\hat{\sigma}}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}}$$

$$\frac{\hat{\beta}_0 - \beta_0}{\hat{SE}(\hat{\beta}_0)} \sim t_{n-2}, \quad \hat{SE}(\hat{\beta}_0) = \hat{\sigma}\sqrt{\left(\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2}\right)}$$

where $t_{n-2}$ denotes the Student's t distribution with $n-2$ degrees of freedom

# Confidence Intervals for $\beta_0$ and $\beta_1$



- Recall $\frac{\hat{\beta}_1 - \beta_1}{\hat{SE}(\hat{\beta}_1)} \sim t_{n-2}$, we use this fact to construct a **confidence interval (CI)** for $\beta_1$:

$$\left[\hat{\beta}_1 - t_{\alpha/2,n-2}\hat{SE}(\hat{\beta}_1), \hat{\beta}_1 + t_{\alpha/2,n-2}\hat{SE}(\hat{\beta}_1)\right],$$

  where $\alpha$ is the **confidence level** and $t_{\alpha/2,n-2}$ denotes the $1 - \alpha/2$ percentile of a student's t distribution with $n - 2$ degrees of freedom

- Similarly, we can construct a CI for $\beta_0$:

$$\left[\hat{\beta}_0 - t_{\alpha/2,n-2}\hat{SE}(\hat{\beta}_0), \hat{\beta}_0 + t_{\alpha/2,n-2}\hat{SE}(\hat{\beta}_0)\right]$$

# Confidence Interval of $\mathrm{E}(y_{new})$



- We often interested in estimating the **mean** response for an unobserved predictor value, say, $x_{new}$. Therefore we would like to construct CI for $\mathrm{E}[y_{new}]$, the corresponding **mean response**

- We need sampling distribution of $\widehat{\mathrm{E}(y_{new})}$ to form CI:
  - $\frac{\widehat{\mathrm{E}(y_{new})}-\mathrm{E}(y_{new})}{\hat{SE}(\widehat{\mathrm{E}(y_{new})})} \sim t_{n-2}, \quad \hat{SE}(\widehat{\mathrm{E}(y_{new})}) = \hat{\sigma}\sqrt{\left(\frac{1}{n} + \frac{(x_{new}-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)}$
  - CI:

$$\left[\hat{y}_{new} - t_{\alpha/2,n-2}\hat{SE}(\widehat{\mathrm{E}(y_{new})}), \hat{y}_{new} + t_{\alpha/2,n-2}\hat{SE}(\widehat{\mathrm{E}(y_{new})})\right]$$

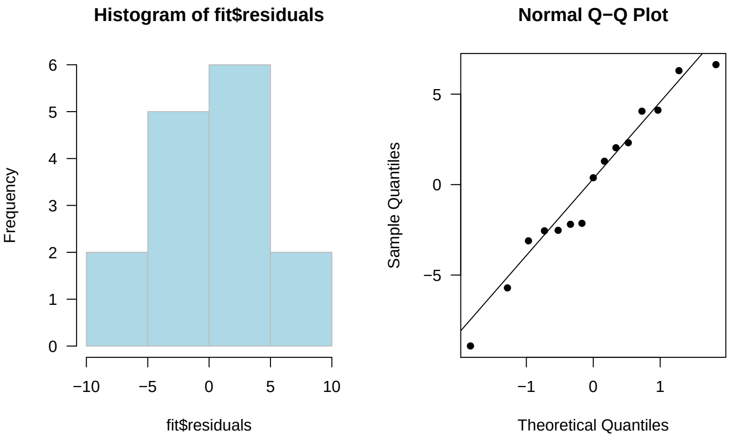
- **Quiz:** Use this formula to construct CI for $\beta_0$

# Prediction Interval of $y_{new}$



- Suppose we want to predict the response of a future observation $y_{new}$ given $x = x_{new}$
- We need to account for added variability as a new observation does not fall directly on the regression line (i.e., $y_{new} = \mathrm{E}[y_{new}] + \varepsilon_{new}$)
- Replace $\hat{SE}(\widehat{\mathrm{E}(y_{new})})$ by $\hat{SE}(\hat{y}_{new}) = \hat{\sigma}\sqrt{\left(1 + \frac{1}{n} + \frac{(x_{new} - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right)}$ to construct CIs for $Y_{new}$

# Assessing Normality Assumption on $\varepsilon$




The Q-Q plot is more effective in detecting subtle departures from normality, especially in the tails.

# Maximum Heart Rate vs. Age Revisited



The maximum heart rate `MaxHeartRate` ($HR_{max}$) of a person is often said to be related to age `Age` by the equation:

$$HR_{max} = 220 - \text{Age}.$$

Suppose we have 15 people of varying ages are tested for their maximum heart rate (bpm)

| Age | 18 | 23 | 25 | 35 | 65 | 54 | 34 | 56 | 72 | 19 | 23 | 42 | 18 | 39 | 37 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $HR_{max}$ | 202 | 186 | 187 | 180 | 156 | 169 | 174 | 172 | 153 | 199 | 193 | 174 | 198 | 183 | 178 |

- Construct the 95% CI for $\beta_1$
- Compute the estimate for mean `MaxHeartRate` given `Age` $= 40$ and construct the associated 90% CI
- Construct the prediction interval for a new observation given `Age` $= 40$

## Maximum Heart Rate vs. Age: Hypothesis Test for Slope

1. $H_0 : \beta_1 = 0$ vs. $H_a : \beta_1 \neq 0$
2. Compute the **test statistic**: $t^* = \frac{\hat{\beta}_1 - 0}{\hat{SE}(\hat{\beta}_1)} = \frac{-0.7977}{0.06996} = -11.40$
3. Compute $p$-**value**: $\mathrm{P}(|t^*| \geq |t_{obs}|) = 3.85 \times 10^{-8}$
4. Compare to $\alpha$ and draw conclusion:

> Reject $H_0$ at $\alpha = .05$ level, evidence suggests a negative linear relationship between `MaxHeartRate` and `Age`

# Maximum Heart Rate vs. Age: Hypothesis Test for Intercept





1. $H_0 : \beta_0 = 0$ vs. $H_a : \beta_0 \neq 0$
2. Compute the **test statistic**: $t^* = \frac{\hat{\beta}_0 - 0}{\hat{SE}(\hat{\beta}_0)} = \frac{210.0485}{2.86694} = 73.27$
3. Compute $p$**-value**: $\mathrm{P}(|t^*| \geq |t_{obs}|) \simeq 0$
4. Compare to $\alpha$ and draw conclusion:

Reject $H_0$ at $\alpha = .05$ level, evidence suggests evidence suggests the intercept (the expected `MaxHeartRate` at age 0) is different from $0$

## Summary



In this lecture, we reviewed

- Simple Linear Regression: $y = \beta_0 + \beta_1 x + \varepsilon,\ \varepsilon \sim \mathrm{N}(0, \sigma^2)$
- Method of Least Squares for parameter estimation

$$\hat{\boldsymbol{\beta}} = \underset{\boldsymbol{\beta}=(\beta_0,\beta_1)}{\operatorname{argmin}} \sum_{i=1}^{n} (y_i - (\beta_0 + \beta_1 x_i))^2$$

- Residual analysis to check model assumptions
- Confidence/Prediction Intervals and Hypothesis Testing

# R Funcations



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- Fitting linear models

  `object <- lm(formula, data)` where the formula is specified via `y ~ x` $\Rightarrow y$ is modeled as a linear function of $x$

- Summary of Fits and Diagnostic Plots

  ```
  summary(object);plot(object)
  ```

- Making Predictions and Their Intervals

  ```
  predict(object, newdata, interval)
  ```

- Confidence Intervals for Model Parameters

  ```
  confint(object)
  ```