Graphical solution

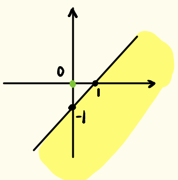
- Linear functions in 2-D.
  - General form of a line.
    - $l: a_1x_1 + a_2x_2 = b \quad (a_1 \neq 0 \text{ or } a_2 \neq 0)$

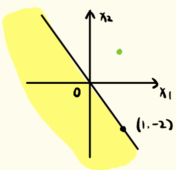
x_1-intercept: $\frac{b}{a_1}$ (if $a_1 \neq 0$) l is a horizontal line if $a_1 = 0$.	x_2-intercept: $\frac{b}{a_2}$ (if $a_2 \neq 0$) l is a vertical line if $a_2 = 0$.

  - It takes two points to draw a line.
  - normal vector: $\vec{n} = (a_1, a_2)$

    $\vec{n}$ is perpendicular to l.

  - Linear inequality in $\mathbb{R}^2$.
    - A linear inequality in $\mathbb{R}^2$ corresponds to a half plane.
    - Two steps to find the half plane $H^- := \{(x_1, x_2) \mid a_1x_1 + a_2x_2 \leq b\}$.
      - Draw $a_1x_1 + a_2x_2 = b$, which divides $\mathbb{R}^2$ into two half planes.
      - Find a point $P(\bar{x}_1, \bar{x}_2)$ not on $a_1x_1 + a_2x_2 = b$. If $a_1\bar{x}_1 + a_2\bar{x}_2 < b$, then H^- is the half plane containing P. If $a_1\bar{x}_1 + a_2\bar{x}_2 > b$, then H^- is the half plane not containing P.
    - e.g.: $-x_1 + x_2 \leq -1$ $\qquad\qquad$ $2x_1 + x_2 \leq 0$

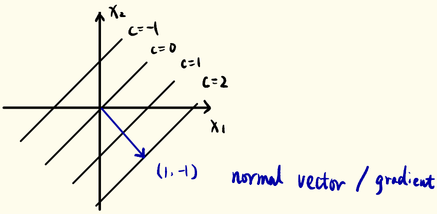




- Level curves (a.k.a. contours)

  - For each scalar c in the range of f, $L_c := \{x \in \mathbb{R}^n \mid f(x) = c\}$ is a level curve.
  - When f is a linear function, the level curves of f are parallel lines.

    e.g.: $f(x_1, x_2) = x_1 - x_2$, $L_c = \{x \in \mathbb{R}^2 \mid x_1 - x_2 = c\}$.

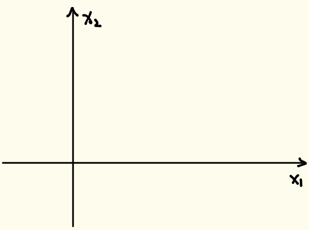


  - The gradient of a linear function $f(x) = a^T x$ is the vector a.
  - The gradient is a normal vector, and it is perpendicular to the level curves.
  - The function value increases along the direction of the gradient.

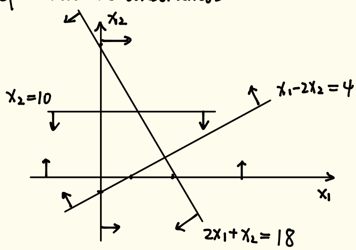
- Graphical Solution of LP in 2-D

eg
$$
\begin{aligned}
\max \quad & z = 2x_1 + 3x_2 \\
\text{s.t.} \quad & x_1 - 2x_2 \le 4 \\
& 2x_1 + x_2 \le 18 \\
& x_2 \le 10 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

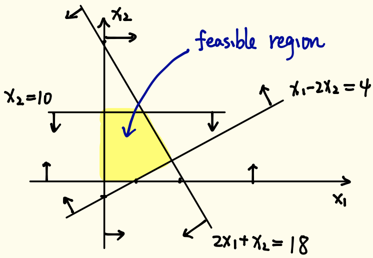
- Step 1 Define the coordinate system



- Step 2 Plot the constraints

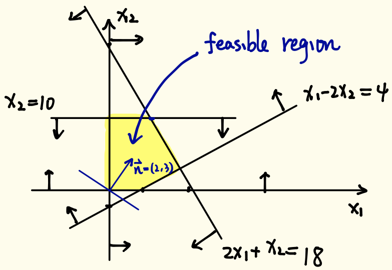


- Step 3 Identify the feasible region



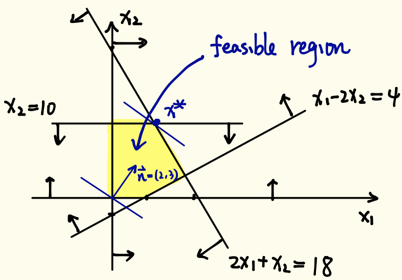
If the intersection is empty, the problem is infeasible.

- Step 4 Identify the gradient of the objective function

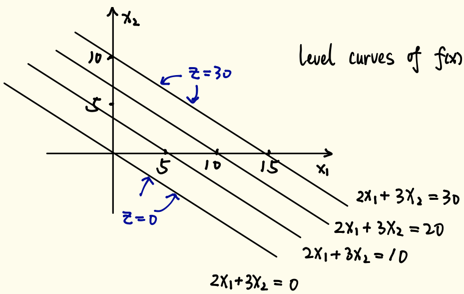


$z = 2x_1 + 3x_2$

- Step 5 Identify the optimal solution(s)



$z = 2x_1 + 3x_2$



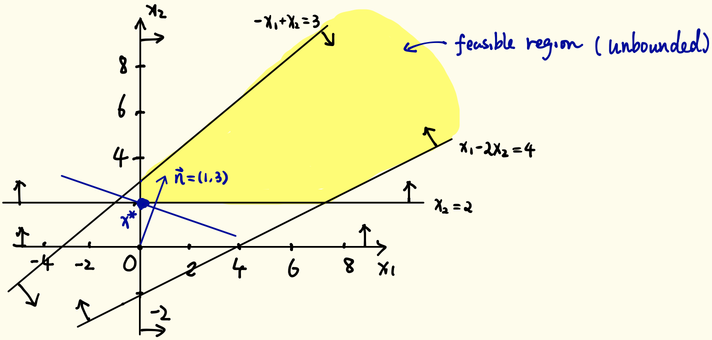
level curves of $f(x) = 2x_1 + 3x_2$

$$\begin{cases} 2x_1^* + x_2^* = 18 \\ x_2^* = 10 \end{cases} \Rightarrow x^* = (4, 10) \qquad z^* = 2x_1^* + 3x_2^* = 38$$

- Unbounded case

e.g.: $\min \quad z = x_1 + 3x_2$

$$\begin{aligned} \text{s.t.} \quad x_1 - 2x_2 &\le 4 \\ -x_1 + x_2 &\le 3 \\ x_2 &\ge 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$



As the problem is a minimization problem, the point should be pushed towards the direction of $-\vec{n}$.

optimal solution $x^* = (0, 2)$, optimal value. $z^* = 6$

Remark: 1. Even though the feasible region of the above problem is unbounded, the problem still has a finite optimal value. We say that the problem is bounded if it has a finite optimal value.

2. Suppose that the objective is changed to maximize $z = x_1 + 3x_2$, then obviously the objective value can be larger than any given number M. In this case, we say that the optimal value is $+\infty$ and the problem is unbounded.

3. In summary, unbounded feasible region $\not\Rightarrow$ unbounded problem

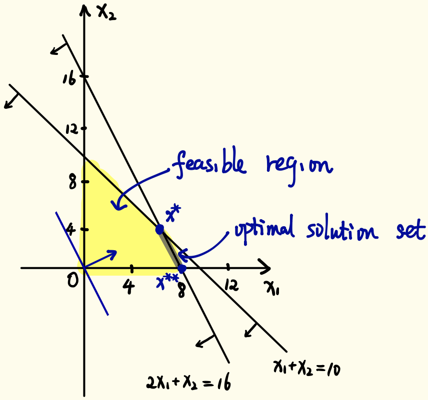
- Alternative optimal solution

e.g.: max $z = 6x_1 + 3x_2$

s.t. $2x_1 + x_2 \le 16$

$x_1 + x_2 \le 10$

$x_1, x_2 \ge 0$



$x^* = (6, 4)$ is an optimal solution

$x^{**} = (8, 0)$ is also an optimal solution

All points between x^* and x^{**} are optimal solutions.

And they all result in the same optimal value $z^* = 48$

Optimal solution set $\left\{ \underbrace{\lambda \begin{bmatrix} 6 \\ 4 \end{bmatrix} + (1-\lambda) \begin{bmatrix} 8 \\ 0 \end{bmatrix}}_{\text{convex combination of } x^* \text{ and } x^{**}} \quad 0 \le \lambda \le 1 \right\}$

Remark: 1. Similar to linear system, we may have no optimal solution, one optimal solution, or infinitely many optimal solutions. No other cases.

2. The optimal value is always unique if it exists.