Standard Form

- Standard form

  - General form.

$$\min / \max \quad Z = c_1 x_1 + c_2 x_2 + \quad + c_n x_n$$

$$\begin{aligned} \text{St} \quad & a_{1,1} x_1 + a_{1,2} x_2 + \quad + a_{1,n} x_n \ \{\le, =, \ge\} \ b_1 \\ & a_{2,1} x_1 + a_{2,2} x_2 + \quad + a_{2,n} x_n \ \{\le, =, \ge\} \ b_2 \\ & \\ & a_{m,1} x_1 + a_{m,2} x_2 + \quad + a_{m,n} x_n \ \{\le, =, \ge\} \ b_m \\ & \{ x_i \ge 0, \ \ x_i \le 0, \ \ x_i \text{ is free} \} \end{aligned}$$

the sign may be either $\le$, $\ge$ or $=$

  - Standard form

$$\max \quad Z = c_1 x_1 + c_2 x_2 + \quad + c_n x_n$$

$$\begin{aligned} \text{St} \quad & a_{1,1} x_1 + a_{1,2} x_2 + \quad + a_{1,n} x_n = b_1 \\ & a_{2,1} x_1 + a_{2,2} x_2 + \quad + a_{2,n} x_n = b_2 \\ & \\ & a_{m,1} x_1 + a_{m,2} x_2 + \quad + a_{m,n} x_n = b_m \\ & x_1, x_2, \quad , x_n \ge 0 \end{aligned}$$

← all variables

matrix notation

$$\begin{aligned} \max \quad & c^T x \\ \text{St} \quad & Ax = b \\ & x \ge 0 \end{aligned}$$

  - motivation starting point of the simplex method

- Def An LP is in standard form if

1 The RHS is nonnegative, i.e. $b_i \geq 0 \quad i=1,\cdots,m$

2 Nonnegativity constraints for ALL variables

3 All remaining constraints are expressed as equality constraints

4 Maximization problem

Remark: Different textbooks and resources may use different definitions. We will stick to the definition above.

• Equivalence between problems

- Def Two optimization problems are (informally) equivalent if the optimal solution of one is readily obtained from the optimal solution of the other, and vice versa.

eg
$$\begin{array}{ll} \min & x^2+x \\ \text{s.t} & x \geq 0 \end{array} \iff \begin{array}{ll} \min & t \\ \text{s.t} & t = x^2+x \\ & x \geq 0 \end{array} \iff \begin{array}{ll} \min & t \\ \text{s.t} & t \geq x^2+x \\ & x \geq 0 \end{array}$$

eg
$$\begin{array}{ll} \max & 2x+y \\ \text{s.t} & x+y \leq 1 \\ & x \leq 0 \end{array} \iff \begin{array}{ll} \max & 2x+y \\ \text{s.t} & x+y+s=1 \\ & x \leq 0,\ s \geq 0 \end{array}$$

s: slack variable

$\Updownarrow$

$$\begin{array}{ll} -\min & -2x-y \\ \text{s.t.} & x+y \leq 1 \\ & x \leq 0 \end{array}$$

- Reformulation techniques

  - Inequalities and equalities:

$$a^T x \le b \quad \longleftrightarrow \quad -a^T x \ge -b$$

$$a^T x \le b \quad \longleftrightarrow \quad \begin{aligned} a^T x + s &= b \\ s &\ge 0 \end{aligned}$$

s: slack variable

$$a^T x \ge b \quad \longleftrightarrow \quad \begin{aligned} a^T x - s &= b \\ s &\ge 0 \end{aligned}$$

s: surplus variable

$$a^T x = b \quad \longleftrightarrow \quad \begin{cases} a^T x \ge b \\ a^T x \le b \end{cases}$$

  - Positivity / Negativity of variables.

$$\begin{cases} a^T x + b^T y = c \\ x \le 0 \end{cases} \quad \longleftrightarrow \quad \begin{cases} a^T(-w) + b^T y = c \\ w \ge 0 \end{cases}$$

$$a^T x + b^T y = c \quad \longleftrightarrow \quad \begin{cases} a^T(w - v) + b^T y = c \\ w \ge 0, \quad v \ge 0. \end{cases}$$

writing a free variable as the difference of two nonnegative variables.

  - Minimization and maximization

$$\begin{aligned} \max \quad & a^T x \\ \text{s.t.} \quad & x \in X \end{aligned} \quad \longleftrightarrow \quad \begin{aligned} \min \quad & -a^T x \\ \text{s.t.} \quad & x \in X \end{aligned}$$

eg.: $\begin{aligned} \min \quad & 3x \\ \text{s.t.} \quad & 1 \le x \le 2 \end{aligned}$ $\begin{cases} x^* = 1 \\ \text{opt val} = 3 \end{cases}$ $\qquad$ $\begin{aligned} \max \quad & -3x \\ \text{s.t.} \quad & 1 \le x \le 2 \end{aligned}$ $\begin{cases} x^* = 1 \\ \text{opt val} = -3 \end{cases}$

- Transform general form LP into standard form

eg. min $Z = 3x_1 + 2x_2 - x_3 + x_4$

St $x_1 + 2x_2 + x_3 - x_4 \le 5$ minimization

$-2x_1 - 4x_2 + x_3 + x_4 \le -1$ inequality

$x_1 \ge 0, \quad x_2 \le 0$ inequality, negative RHS

x_2 nonpositive, x_3, x_4 free

This is not a standard form LP.

1 RHS

$$-2x_1 - 4x_2 + x_3 + x_4 \le -1 \longrightarrow 2x_1 + 4x_2 - x_3 - x_4 \ge 1$$

$$\begin{aligned} \min \quad & Z = 3x_1 + 2x_2 - x_3 + x_4 \\ \text{St} \quad & x_1 + 2x_2 + x_3 - x_4 \le 5 \\ & 2x_1 + 4x_2 - x_3 - x_4 \ge 1 \\ & x_1 \ge 0, \quad x_2 \le 0 \end{aligned}$$

2 Equality constraints

"$\le$" — add a slack variable

$$x_1 + 2x_2 + x_3 - x_4 \le 5 \longrightarrow \begin{cases} x_1 + 2x_2 + x_3 - x_4 + s_1 = 5 \\ s_1 \ge 0 \end{cases}$$

$$\begin{aligned} \min \quad & Z = 3x_1 + 2x_2 - x_3 + x_4 \\ \text{St} \quad & x_1 + 2x_2 + x_3 - x_4 + s_1 = 5 \\ & 2x_1 + 4x_2 - x_3 - x_4 \ge 1 \\ & x_1 \ge 0, \quad x_2 \le 0, \quad s_1 \ge 0 \end{aligned}$$

" $\geq$ " — subtract a "surplus" variable

$$2x_1 + 4x_2 - x_3 - x_4 \geq 1 \longrightarrow \begin{cases} 2x_1 + 4x_2 - x_3 - x_4 - s_2 = 1 \\ s_2 \geq 0 \end{cases}$$

$$\min \quad z = 3x_1 + 2x_2 - x_3 + x_4$$

$$\begin{aligned} \text{st} \quad & x_1 + 2x_2 + x_3 - x_4 + s_1 = 5 \\ & 2x_1 + 4x_2 - x_3 - x_4 - s_2 = 1 \\ & x_1 \geq 0, \quad x_2 \leq 0, \quad s_1 \geq 0, \quad s_2 \geq 0 \end{aligned}$$

3 Variables

nonpositive variable — replace with its opposite

$$x_2 \leq 0 \rightarrow y_2 \geq 0$$

$$\min \quad z = 3x_1 - 2y_2 - x_3 + x_4$$

$$\begin{aligned} \text{st} \quad & x_1 - 2y_2 + x_3 - x_4 + s_1 = 5 \\ & 2x_1 - 4y_2 - x_3 - x_4 - s_2 = 1 \\ & x_1 \geq 0, \quad y_2 \geq 0, \quad s_1 \geq 0, \quad s_2 \geq 0 \end{aligned}$$

free variable replace with difference of two nonnegative vars

$$x_3 = y_3 - w_3, \quad \text{where } y_3 \geq 0 \text{ and } w_3 \geq 0$$

$$\min \quad z = 3x_1 - 2y_2 - y_3 + w_3 + x_4$$

$$\begin{aligned} \text{St} \quad & x_1 - 2y_2 + y_3 - w_3 - x_4 + s_1 = 5 \\ & 2x_1 - 4y_2 - y_3 + w_3 - x_4 - s_2 = 1 \\ & x_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ w_3 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0 \end{aligned}$$

$x_4 = y_4 - w_4$, where $y_4 \geq 0$ and $w_4 \geq 0$

$$\min \quad z = 3x_1 - 2y_2 - y_3 + w_3 + y_4 - w_4$$

$$\begin{aligned} \text{St} \quad & x_1 - 2y_2 + y_3 - w_3 - y_4 + w_4 + s_1 = 5 \\ & 2x_1 - 4y_2 - y_3 + w_3 - y_4 + w_4 - s_2 = 1 \\ & x_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ w_3 \geq 0,\ y_4 \geq 0,\ w_4 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0 \end{aligned}$$

4 Objective

$$\max \quad z = -3x_1 + 2y_2 + y_3 - w_3 - y_4 + w_4$$

$$\begin{aligned} \text{St} \quad & x_1 - 2y_2 + y_3 - w_3 - y_4 + w_4 + s_1 = 5 \\ & 2x_1 - 4y_2 - y_3 + w_3 - y_4 + w_4 - s_2 = 1 \\ & x_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ w_3 \geq 0,\ y_4 \geq 0,\ w_4 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0 \end{aligned}$$

e.g. Transform the following LP into standard form

$$\begin{aligned} \max\ & c^T x + d^T y \\ \text{s.t. } & Ax = b \\ & 0 \le y \le 1 \end{aligned}$$

where $1 = (1, \cdots, 1)^T$ is the vector of all ones. and $b \geq 0$.

1. RHS ✓
2. Equality: $y \le 1 \rightarrow \begin{cases} y + s = 1 \\ s \geq 0 \end{cases}$

$$\begin{aligned} \max\ & c^T x + d^T y \\ \text{s.t. } & Ax = b \\ & y + s = 1 \\ & y \geq 0,\ s \geq 0 \end{aligned}$$

3. Variables:

x free $\rightarrow \begin{cases} x = x^+ - x^- \\ x^+, x^- \geq 0 \end{cases}$

4. Objective ✓

$$\begin{aligned} \max\ & c^T x^+ - c^T x^- + d^T y \\ \text{s.t. } & Ax^+ - Ax^- = b \\ & y + s = 1 \\ & x^+, x^- \geq 0,\ y, s \geq 0. \end{aligned}$$

Or equivalently,

$$\max \begin{bmatrix} c \\ -c \\ d \\ 0 \end{bmatrix}^T \begin{bmatrix} x^+ \\ x^- \\ y \\ s \end{bmatrix}$$

$$\text{s.t. } \begin{bmatrix} A & -A & 0 & 0 \\ 0 & 0 & I & I \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ y \\ s \end{bmatrix} = \begin{bmatrix} b \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x^+ \\ x^- \\ y \\ s \end{bmatrix} \geq 0$$