Gauss-Jordan Reduction

- RREF

Def: An $m \times n$ matrix A is in reduced row echelon form (RREF) if

a) All zero rows, if any, are at the bottom of the matrix.

b) The first nonzero entry, called the leading entry, in each nonzero row is a 1.

c) If rows i and $i+1$ are two successive nonzero rows, then the leading entry of row $i+1$ is to the right of the leading entry of row i.

d) If a column contains a leading entry of some row, then all other entries in that column are zero.

e.g.:

$$\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad \times$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \checkmark$$

- why RREF ?

Consider a linear system $Ax=b$. If the augmented matrix $[A \mid b]$ is in RREF, then the solution set can be easily expressed.

e.g. 1.

$$[A \mid b] = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right] \quad \text{corresponds to} \quad \begin{cases} x_1 = 2 \\ x_2 = 3 \end{cases}.$$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is the unique solution to the linear system $Ax=b$.

e.g. 2:

$$[A \mid b] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{corresponds to} \quad \begin{cases} x_1 \qquad -x_4 = 2 \\ \qquad x_3 + 2x_4 = 3 \\ \qquad\qquad 0 = 0 \end{cases}$$

x_1, x_3 : basic variable

x_2, x_4 : free variable.

All solutions to the linear system can be expressed as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2+x_4 \\ x_2 \\ 3-2x_4 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{\text{parametric vector form}}$$

- How to get RREF?

Def: An <u>elementary row operation</u> on a matrix A is any of the following operations.

Type I: Interchange two rows.

Type II: Multiply a row by a number $c \neq 0$.

Type III: Add a multiple of a row to another row.

e.g.:

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 3 & -2 & 1 & 5 \\ 4 & 2 & 3 & 4 \end{bmatrix}$$

$$A \xrightarrow{R_1 \leftrightarrow R_3} B, \qquad A \xrightarrow{-2R_3 \to R_3} C, \qquad A \xrightarrow{-3R_1 + R_2 \to R_2} D$$

$$B = \begin{bmatrix} 4 & 2 & 3 & -4 \\ 3 & -2 & 1 & 5 \\ 1 & 2 & 0 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 3 & -2 & 1 & 5 \\ -8 & -4 & -6 & 8 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -8 & 1 & -4 \\ 4 & 2 & 3 & -4 \end{bmatrix}$$

Prop: The inverse operation of an elementary row operation is an elementary row operation of the same type.

Def: An $m \times n$ matrix A is <u>row equivalent</u> to an $m \times n$ matrix B if B can be obtained from A by applying a finite sequence of elementary row operations to A.

e.g.: In the above example, A, B, C, D are row equivalent.

Thm: Every $m \times n$ matrix can be transformed to RREF by a finite sequence of elementary row operations. That is, every matrix is row equivalent to a matrix in RREF.

1. Find the first nonzero column — pivotal column.
2. Find the first nonzero entry in the pivotal column — pivot.
3. If the row in which the pivot is in is not the first row, interchange this row with the first row.
4. Multiply the first row by the reciprocal of the pivot so that its leading entry becomes 1.
5. Add suitable multiples of the first row to all its other rows to zero out the pivot column except for the pivot.
6. Ignore the first row and repeat Steps 1-5, with an additional Step 5'.

   5'. Add suitable multiples of the first row to all the ignored rows to zero out the pivot column of the ignored rows.

   until the RREF is obtained.

e.g.:

$$\begin{array}{rl} & \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \end{bmatrix} \\ & \uparrow \text{ pivot column} \end{array} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1 \to R_1} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \end{bmatrix}$$

pivot →

$$\xrightarrow{-3R_1+R_3 \to R_3} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 2 & -4 & 4 & 2 & -6 \end{bmatrix} \xrightarrow{\frac{1}{3}R_2 \to R_2} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 4/3 & -5/3 \\ 0 & 2 & -4 & 4 & 2 & -6 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3 \to R_3} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 4/3 & -5/3 \\ 0 & 0 & 0 & 0 & -2/3 & -8/3 \end{bmatrix} \xrightarrow{3R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & -2 & 3 & 6 & 0 \\ 0 & 1 & -2 & 2 & 4/3 & -5/3 \\ 0 & 0 & 0 & 0 & -2/3 & -8/3 \end{bmatrix}$$

$$\xrightarrow{-\frac{3}{2}R_3 \to R_3} \begin{bmatrix} 1 & 0 & -2 & 3 & 6 & 0 \\ 0 & 1 & -2 & 2 & 4/3 & -5/3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow[-\frac{4}{3}R_3+R_2 \to R_2]{-6R_3+R_1 \to R_1} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \text{ RREF}$$

Thm: Let $Ax=b$ and $Cx=d$ be two linear systems, each consisting of m equations in n unknowns. If the augmented matrices $[A\,|\,b]$ and $[C\,|\,d]$ are row equivalent, then both linear systems have no solutions or they have exactly the same solutions.

Application: solve linear system of equations.

e.g. 1.
$$\begin{cases} 3x_1 + 5x_2 - 4x_3 = 7 \\ -3x_1 - 2x_2 + 4x_3 = -1 \\ 6x_1 + x_2 - 8x_3 = -4. \end{cases}$$

pivot

$$[A\,|\,b] = \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \leftarrow \text{augmented matrix}$$

elementary row operations

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{5}{3} & -\frac{4}{3} & \frac{7}{3} \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{array}\right]$$

$$\sim \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} \underbrace{x_1 \;\; x_2}_{\text{basic variable}} & x_3 & \text{RHS} \end{array} \end{array} \leftarrow \text{reduced row echelon form (RREF)}$$

x_3: free variable

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 - 1 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} \leftarrow \text{parametric vector form}$$

Solution set: $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} : x_3 \in \mathbb{R} \right\}$

e.g.2: Solve
$$\begin{cases} x + 2y - 3z = 2 \\ x + 3y + z = 7 \\ x + y - 7z = 3 \end{cases}$$

$$[A \mid b] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 1 & 3 & 1 & 7 \\ 1 & 1 & -7 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -11 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Note that the last row of the augmented matrix indicates that $0 = 1$. This linear system has no solution.

e.g.3: A linear system of the form $Ax = 0$ is called a <u>homogeneous system</u>. A homogeneous system always has the solution $x = 0$. We call $x = 0$ the <u>trivial solution</u>. Any nonzero solution to the system is called a <u>nontrivial solution</u>.

$$\begin{cases} 3u + 5v - 4w = 0 \\ -3u - 2v + 4w = 0 \\ 6u + v - 8w = 0 \end{cases}$$

$$[A \mid b] = \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underbrace{u \quad v}_{\text{basic variable}} \qquad \underset{\text{free variable}}{w}$$

The solutions are:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \frac{4}{3}w \\ 0 \\ w \end{bmatrix} = w \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}.$$

Each $w \neq 0$ corresponds to a nontrivial solution.