# The Inverse of a Matrix

Addition ⟺ Subtraction    Multiplication ⟺ Division

The inverse of a matrix is the counterpart of the reciprocal of a scalar

Def: An $n \times n$ matrix $A$ is called <u>nonsingular</u> or <u>invertible</u> if there exists an $n \times n$ matrix $B$ such that

$$AB = BA = I_n.$$

The matrix $B$ is called the <u>inverse</u> of $A$, denoted by $A^{-1}$. If no such matrix $B$ exists, then $A$ is called <u>singular</u> or <u>noninvertible</u>.

eg $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A, B$ are both invertible, $A^{-1} = B$ and $B^{-1} = A$

e.g.: A matrix with a zero row is noninvertible.

Note When $A$ is invertible, $A^{-1}$ is uniquely defined

Prop:

a) If $A$ is nonsingular, then $A^{-1}$ is nonsingular and $(A^{-1})^{-1} = A$.

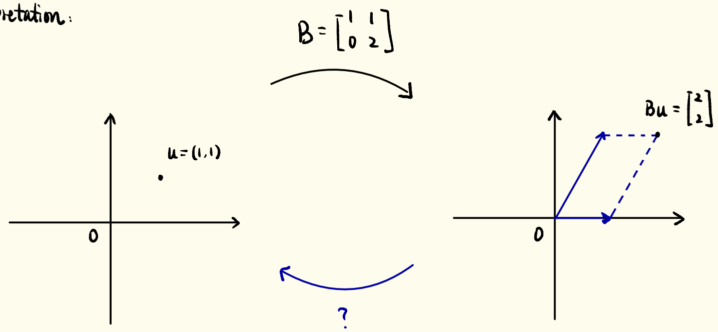
b) If $A$ and $B$ are nonsingular and of the same size, then $AB$ is nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.

c) If $A$ is nonsingular, then $A^T$ is nonsingular and $(A^T)^{-1} = (A^{-1})^T$.

Extension of b) If $A_1, A_2, \cdots, A_p$ are $n \times n$ invertible matrices, then $A_1 A_2 \cdots A_p$ is invertible,

and $(A_1 A_2 \cdots A_p)^{-1} = A_p^{-1} \cdots A_2^{-1} A_1^{-1}$.

Intepretation:




$$Bu = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$u = B^{-1}(Bu)$

- Elementary matrix

Def: An $n \times n$ elementary matrix is a matrix obtained from the identy matrix $I_n$ by performing a single elementary row operation.

Recall that there are three types of elementary row operations.

Type I: Interchange two rows.

Type II: Multiply a row by a number $c \neq 0$.

Type III: Add a multiple of a row to another row.

e.g.:

$$\begin{tikzcd}[column sep=small]
 & I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \arrow[dl, "R_1 \leftrightarrow R_3"'] \arrow[d, "-2R_3 \to R_3"] \arrow[dr, "-3R_1 + R_2 \to R_2"] & \\
E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} & E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} & E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 3 & -2 & 1 & 5 \\ 4 & 2 & 3 & 4 \end{bmatrix} \arrow[dl, "R_1 \leftrightarrow R_3"'] \arrow[d, "-2R_3 \to R_3"] \arrow[dr, "-3R_1 + R_2 \to R_2"] & \\
B = \begin{bmatrix} 4 & 2 & 3 & 4 \\ 3 & -2 & 1 & 5 \\ 1 & 2 & 0 & 3 \end{bmatrix} = E_1 A & C = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 3 & -2 & 1 & 5 \\ -8 & -4 & -6 & 8 \end{bmatrix} = E_2 A & D = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -8 & 1 & -4 \\ 4 & 2 & 3 & -4 \end{bmatrix} = E_3 A
\end{tikzcd}$$

Thm:

1. Performing an elementary row operation on a matrix $A$ is the same as left multiplying $A$ by the corresponding elementary matrix.

2. Matrices $A$ and $B$ are row equivalent if and only if there exist elementary matrices $E_1, E_2, \cdots, E_{k-1}, E_k$ such that $B = E_k E_{k-1} \cdots E_2 E_1 A$.

3. An elementary matrix is nonsingular, and its inverse is an elementary matrix of the same type.

4. An $n \times n$ matrix $A$ is nonsingular if and only if $A$ is row equivalent to $I_n$.

e.g.: $$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} \\ 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ is invertible.

- How to compute $A^{-1}$?

When $A_{n\times n}$ is invertible, it is row equivalent to $I_n$. Then, there exist elementary matrices $E_1, E_2, \cdots, E_{k-1}, E_k$ such that $I_n = E_k E_{k-1} \cdots E_2 E_1 A$. Therefore, $(E_k E_{k-1} \cdots E_2 E_1)^{-1} = A$ and $A^{-1} = E_k E_{k-1} \cdots E_2 E_1$.

Algorithm

$$[A \mid I_n] \xrightarrow[\text{elementary row operation}]{} E_1 [A \mid I_n] \to E_2 E_1 [A \mid I_n] \to \cdots \to E_k E_{k-1} \cdots E_2 E_1 [A \mid I_n]$$

$$= [E_k E_{k-1} \cdots E_2 E_1 A \mid E_k E_{k-1} \cdots E_2 E_1]$$

$$= [I_n \mid A^{-1}]$$

eg $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$.