Matrices

Def: • An $m\times n$ <u>matrix</u> is a rectangular array of mn numbers arranged in m rows and n columns.

$$\begin{array}{cc} A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} & \text{row} \\ \quad \text{column} & \end{array} = [a_{ij}]$$

e.g.:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 8 \end{bmatrix}$$ is a 2-by-3 matrix.

• If $m=n$, then A is called a <u>square matrix</u>. In this case $a_{11}, \cdots, a_{nn}$ form the <u>(main) diagonal</u> of A.

e.g.:

$$\begin{array}{c} B = \begin{bmatrix} -1 & 0 \\ 4 & \sqrt{2} \end{bmatrix} \\ \text{diagonal} \end{array}$$ is a square matrix of order 2.

• For a square matrix A, if all off-diagonal entries are zero, i.e., $a_{ij} = 0$ for all $i \neq j$, then A is called a <u>diagonal matrix</u>.

e.g.:

$$C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

• The diagonal matrix with all ones in the diagonal is called <u>the identity matrix</u>, usually denoted by I_n.

The matrix with all zero entries is called <u>the zero matrix</u>, usually denoted by $O_{m\times n}$.

e.g.:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad O_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- When $n=1$, the matrix is called a column vector. When $m=1$, the matrix is called a row vector.

  e.g.: $v=\begin{bmatrix}1\\2\\3\end{bmatrix}$, $w=\begin{bmatrix}2 & 4\end{bmatrix}$, $s=10$

  scalar $(m=n=1)$.

- Two $m\times n$ matrices A and B are equal if $a_{ij}=b_{ij}$ for all $i=1,\cdots,m$ and $j=1,\cdots,n$.

- Addition / Subtraction

Def: For two $m\times n$ matrices $A=[a_{ij}]$ and $B=[b_{ij}]$, the sum/difference of A and B is defined as

$$A\pm B=[a_{ij}\pm b_{ij}]$$

e.g.: $\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}+\begin{bmatrix}2 & -2 & 0\\1 & 0 & -1\end{bmatrix}=\begin{bmatrix}3 & 0 & 3\\5 & 5 & 5\end{bmatrix}$. $\begin{bmatrix}2\\1\end{bmatrix}-\begin{bmatrix}3\\5\end{bmatrix}=\begin{bmatrix}-1\\-4\end{bmatrix}$.

Matrices of different sizes cannot be added together.

Prop. 1. $A+B=B+A$

2. $(A+B)+C=A+(B+C)$

- Scalar multiplication

Def: For a scalar α and a matrix $A=[a_{ij}]$, the scalar multiplication is defined as

$$\alpha A=[\alpha a_{ij}].$$

e.g.: $3\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}=\begin{bmatrix}3 & 6\\9 & 12\end{bmatrix}$. $-\begin{bmatrix}1 & -2\\0 & 3\end{bmatrix}=\begin{bmatrix}-1 & 2\\0 & -3\end{bmatrix}$.

Prop. 3. $\alpha(\beta A)=(\alpha\beta)A$

4. $(\alpha+\beta)A=\alpha A+\beta A$

5. $\alpha(A+B)=\alpha A+\alpha B$

- Transpose

Def: For an $m\times n$ matrix $A=[a_{ij}]$, the <u>transpose</u> of A is an $n\times m$ matrix defined as

$$A^T=[a_{ji}].$$

e.g.: $A=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$, $A^T=\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}$

Prop: 6. $(A^T)^T=A$

7. $(A+B)^T=A^T+B^T$

8. $(\alpha A)^T=\alpha A^T$.

- Multiplication

Def: For matrices $A_{m\times p}=[a_{ij}]$ and $B_{p\times n}=[b_{ij}]$, the <u>product</u> of A and B is defined as

$$A_{m\times p}B_{p\times n}=C_{mn}=[c_{ij}],\quad \text{where}\quad c_{ij}=\sum_{k=1}^{p}a_{ip}b_{pj}=a_{i1}b_{1j}+\cdots+a_{ip}b_{pj}.$$

e.g.:

$$A=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix},\quad B=\begin{bmatrix}1&-1\\2&0\\0&1\end{bmatrix}\quad AB=\begin{bmatrix}1\times1+2\times2+3\times0 & 1\times(-1)+2\times0+3\times1\\4\times1+5\times2+6\times0 & 4\times(-1)+5\times0+6\times1\end{bmatrix}=\begin{bmatrix}5&2\\14&2\end{bmatrix}.$$

e.g.: $A=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$ $u=\begin{bmatrix}1\\2\\-1\end{bmatrix}$

$$Au=\begin{bmatrix}1\times1+2\times2+3\times(-1)\\4\times1+5\times2+6\times(-1)\end{bmatrix}=\begin{bmatrix}2\\8\end{bmatrix}$$

$$=1\begin{bmatrix}1\\4\end{bmatrix}+2\begin{bmatrix}2\\5\end{bmatrix}-\begin{bmatrix}3\\6\end{bmatrix}$$

e.g. $A=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$, $I_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $AI_3=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$

$I_2=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $I_2A=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$.

$$I_mA=A=AI_n.$$

$$O_{k\times m}A_{m\times n}=O_{k\times n},\quad A_{m\times n}O_{n\times k}=O_{m\times k}.$$

Prop. 9. $A(BC) = (AB)C$

10. $A(\alpha B) = \alpha(AB)$

11. $A(B+C) = AB + AC$

12. $(A+B)C = AC + BC$

13. $(AB)^T = B^T A^T$

Application: Linear System

The linear system of m equations in n variables

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

can be compactly represented as

$$Ax = b,$$

where $\underset{\substack{\uparrow \\ \text{coefficient matrix}}}{A_{m\times n}} = [a_{ij}]$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$.

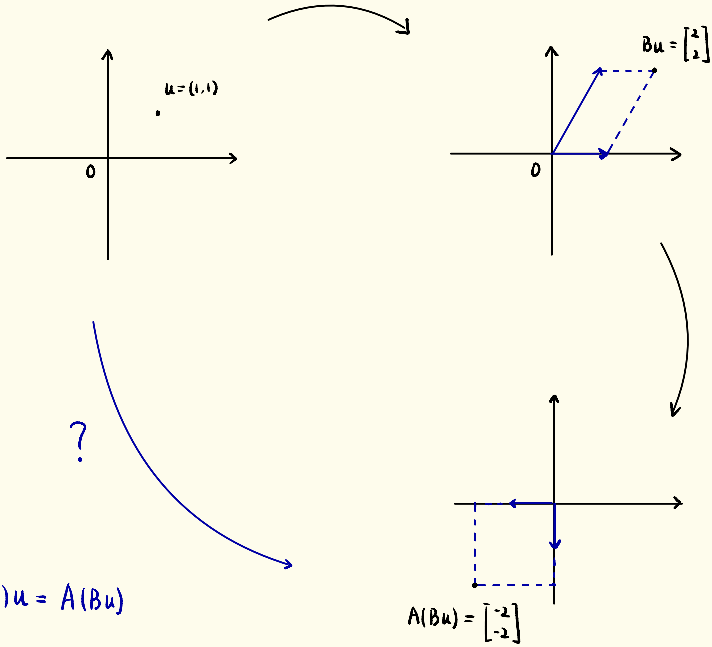
The matrix $[A \mid b]$ is called the <u>augmented matrix</u>.

e.g.: $\begin{cases} x_1 + 2x_2 + x_3 = -1 \\ 4x_1 + 5x_2 \qquad = 6 \end{cases} \iff \underset{\substack{\uparrow \\ A}}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \end{bmatrix}} \underset{\substack{\uparrow \\ x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{\substack{\uparrow \\ b}}{\begin{bmatrix} -1 \\ 6 \end{bmatrix}}$

The augmented matrix is $\left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 4 & 5 & 0 & 6 \end{array}\right]$.

Intepretation:

$B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$



$$Bu = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$$A(Bu) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}.$$

$(AB)u = A(Bu)$

- Block matrices

$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ $\quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ where A_{ij}, B_{ij} are submatrices.

If the sizes match, then

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

That is, the product is calculated as if A_{ij} and B_{ij} are scalars.

The number of blocks can be generalized.

e.g.:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 2 & 3 \end{array}\right] \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \\ \hline 5 & 6 \\ 7 & 8 \end{array}\right]$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} + \begin{bmatrix} 31 & 36 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \\ 31 & 36 \end{bmatrix}$$

Let

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \underbrace{\begin{bmatrix} | & | & & | \\ A_1 & A_2 & \cdots & A_n \\ | & | & & | \end{bmatrix}}_{\text{columns of } A} = \left.\begin{bmatrix} - a_1^T - \\ - a_2^T - \\ \vdots \\ - a_m^T - \end{bmatrix}\right\} \text{rows of } A.$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$

Then,

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots \ a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 A_1 + x_2 A_2 + \cdots + x_n A_n$$

$$= \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_n^T x \end{bmatrix}$$