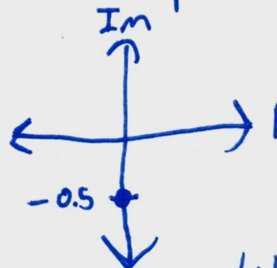
Ex 5.1.1 (p. 302)

For an LTI system w/ impulse response $h(n) = \left(\frac{1}{2}\right)^n u(n)$ and input $x(n) = Ae^{-j\frac{\pi}{2}n}$ find the transfer function $H(w)$ and the output $y(n)$.

$$H(w) = \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^n e^{-jwn} u(n) = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n e^{-jwn} = \sum_{n=0}^{\infty} \left(\tfrac{1}{2} e^{-jw}\right)^n = \boxed{\frac{1}{1 - \frac{1}{2}e^{-jw}}}$$

System is LTI ∴ no frequency change. $x(n)$ has one freq component $w = \frac{\pi}{2}$

$$H(\pi/2) = \frac{1}{1 - \underbrace{\frac{1}{2}e^{-jw}}} \rightarrow$$ indicates change in magnitude + phase to an input at $w = \pi/2$

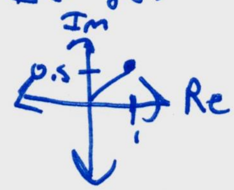


$\therefore \frac{1}{2}e^{-jw} = -0.5j$

$$H(\pi/2) = \frac{1}{1 + \frac{1}{2}j}$$

$$|H(\pi/2)| = \frac{|1|}{|1 + \frac{1}{2}j|} = \frac{1}{\sqrt{1^2 + \frac{1}{2}^2}} = \frac{1}{\sqrt{5/4}} = \sqrt{\frac{1}{5/4}} = \frac{2}{\sqrt{5}}$$

$$\angle H(\pi/2) = \angle(1) - \angle(1 + \tfrac{1}{2}j) = 0 - \tan^{-1}(1/2 \div 1) = -26.6^\circ$$



$$y(n) = A\frac{2}{\sqrt{5}} e^{-j\frac{\pi}{2}n} e^{+j26.6^\circ} = \boxed{\frac{2A}{\sqrt{5}} e^{-j\left(\frac{\pi}{2}n - 26.6^\circ\right)}}$$

Ex 5.4.3 (p. 335)

Given the following difference equation for a lowpass filter

$$y(n) = 0.9\,y(n-1) + 0.1\,x(n)$$

a. Determine the order of the y terms + x terms.

y: Order 1 b/c largest k $y(n-k) = 1 = N$

x: order $0 = M$

b. Convert to a highpass system.

Recall that a lowpass $y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k)$

can be transformed to a highpass $y(n) = -\sum_{k=1}^{N} (-1)^k a_k\, y(n-k) + \sum_{k=0}^{M} (-1)^k b_k\, x(n-k)$

$$\therefore \boxed{y(n) = -0.9\,y(n-1) + 0.1\,x(n)}$$

c. Determine the frequency response of the new highpass filter and verify that it is a highpass filter using the frequency response.

freq response $\rightarrow H(w) = \dfrac{Y(w)}{X(w)}$

$$y(n) = -0.9\,y(n-1) + 0.1\,x(n)$$

$$\Big\updownarrow \mathcal{F}$$

$$Y(w) = -0.9e^{-jw}\,Y(w) + 0.1\,X(w)$$

$$Y(w) + 0.9e^{-jw}\,Y(w) = 0.1\,X(w)$$

$$H(w) = \frac{Y(w)}{X(w)} = \boxed{\frac{0.1}{1 + 0.9e^{-jw}}}$$

Highpass $|H(\infty)| > |H(0)|$

$|H(0)| = \dfrac{0.1}{1.9} \approx 0.05$

$|H(\infty)| = \dfrac{0.1}{1} = 0.1 \leftarrow$ larger

$\therefore$ highpass

Problem 6.9 (p. 441)

Given an analog signal $x_a(t) = e^{-j2\pi F_0 t} u(t)$

a. Determine the signal's spectrum $X_a(F)$.

$$X_a(F) = \int_{-\infty}^{\infty} x_a(t)\, e^{-j2\pi F t}\, dt = \int_0^{\infty} e^{-j2\pi F_0 t} e^{-j2\pi F t}\, dt$$

$$= \int_0^{\infty} e^{-j2\pi (F+F_0)t}\, dt = \frac{1}{-j2\pi(F+F_0)} \underbrace{\left[e^{-j2\pi(F+F_0)t} \right]_{t=0}^{\infty}}_{[0-1]=-1}$$

$$\boxed{X_a(F) = \frac{1}{j2\pi(F+F_0)}}$$

b. Compute the spectrum of the sampled signal $x(n) = x_a(nT)$ $T = 1/F_s$.

$$x(n) = e^{-j2\pi \frac{F_0}{F_s} n} u(n)$$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi f n} = \sum_{n=0}^{\infty} e^{-j2\pi \frac{F_0}{F_s} n} e^{-j2\pi f n}$$

$$= \sum_{n=0}^{\infty} e^{-j2\pi\left(f + \frac{F_0}{F_s}\right) n} = \sum_{n=0}^{\infty} \Big[\underbrace{e^{-j2\pi\left(f+\frac{F_0}{F_s}\right)}}_{r}\Big]^n = \frac{1}{1-r}$$

$$= \boxed{\frac{1}{1 - e^{-j2\pi\left(f + \frac{F_0}{F_s}\right)}}}$$

Ex 5.5.5 (p. 354)

Given $y(n) = \{1, 7/10\}$ $\quad x(n) = \{1, -7/10, 1/10\}$

a. Determine the system function $H(z)$.

$Y(z) = 1 + \frac{7}{10}z^{-1}$ $\quad X(z) = 1 - \frac{7}{10}z^{-1} + \frac{1}{10}z^{-2}$ $\quad H(z) = \frac{Y(z)}{X(z)} = \boxed{\frac{1+\frac{7}{10}z^{-1}}{1-\frac{7}{10}z^{-1}+\frac{1}{10}z^{-2}}}$

b. Determine the system's poles, zeros, and ROC. Is the system stable?

$$\frac{1+\frac{7}{10}z^{-1}}{1-\frac{7}{10}z^{-1}+\frac{1}{10}z^{-2}} \cdot \frac{z^2}{z^2} = \frac{z^2+\frac{7}{10}z}{z^2-\frac{7}{10}z+\frac{1}{10}} = \frac{z(z+\frac{7}{10})}{(z-\frac{1}{2})(z-\frac{1}{5})}$$

$\boxed{\text{Poles: } z = 1/5,\ 1/2 \quad \text{Zeros: } z = -7/10,\ 0}$

ROC: $(|z| > \frac{1}{5}) \cup (|z| > \frac{1}{2}) = \boxed{|z| > 1/2}$ $\boxed{\text{Stable}}$ b/c ROC contains unit circle. Alternatively, all poles w/in unit circle.

c. Determine the inverse system function $H^{-1}(z)$. Is it stable?

$H(z) = \frac{B(z)}{A(z)} \therefore H^{-1}(z) = \frac{A(z)}{B(z)} = \boxed{\frac{1-\frac{7}{10}z^{-1}+\frac{1}{10}z^{-2}}{1+\frac{7}{10}z^{-1}}}$

Poles: $z = 0, -7/10$

ROC: $|z| > 7/10$ $\therefore \boxed{\text{Stable}}$

d. Determine the in/out difference eqn for the original system.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+\frac{7}{10}z^{-1}}{1-\frac{7}{10}z^{-1}+\frac{1}{10}} \rightarrow [1 + \frac{-7}{10}z^{-1} + \frac{1}{10}z^{-2}]Y(z) = [1+\frac{7}{10}z^{-1}]X(z)$$

$$\xrightarrow{Z^{-1}} y(n) - \frac{7}{10}y(n-1) + \frac{1}{10}y(n-2) = x(n) + \frac{7}{10}x(n-1) \rightarrow \boxed{y(n) = \frac{7}{10}y(n-1) - \frac{1}{10}y(n-2) + x(n) + \frac{7}{10}x(n-1)}$$

e. Determine the original system's impulse response.

$h(n) = Z^{-1}\{H(z)\}$. $\frac{1+\frac{7}{10}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{5}z^{-1})} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-\frac{1}{5}z^{-1}}$

$A(1-\frac{1}{5}z^{-1}) + B(1-\frac{1}{2}z^{-1}) = 1+\frac{7}{10}z^{-1}$

$A + B = 1 \rightarrow B = 1 - A$

$\frac{-A}{5} + \frac{-(1-A)}{2} = \frac{7}{10} \rightarrow 3A = 12$

$\boxed{A = 4 \quad B = -3}$

$$H(z) = \frac{4}{1-\frac{1}{2}z^{-1}} + \frac{-3}{1-\frac{1}{5}z^{-1}} \xrightarrow{Z^{-1}} \boxed{h(n) = [4(\tfrac{1}{2})^n - 3(\tfrac{1}{5})^n]u(n)}$$