Digital Image Processing (CSE/ECE 478)

Lecture # 08: Discrete Fourier Transform

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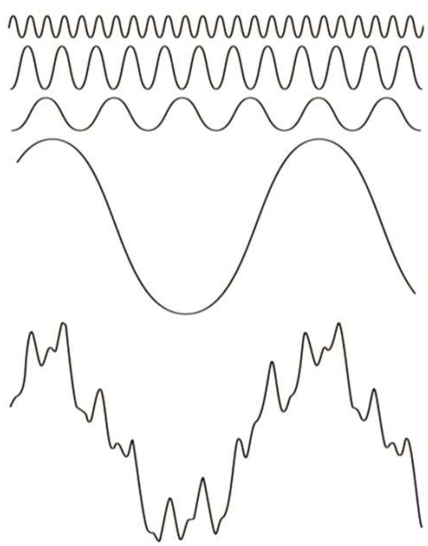
Basic Idea

- *Any periodically repeated function can be expressed of the sum of sines/cosines of different frequencies, each multiplied by a different coefficient*

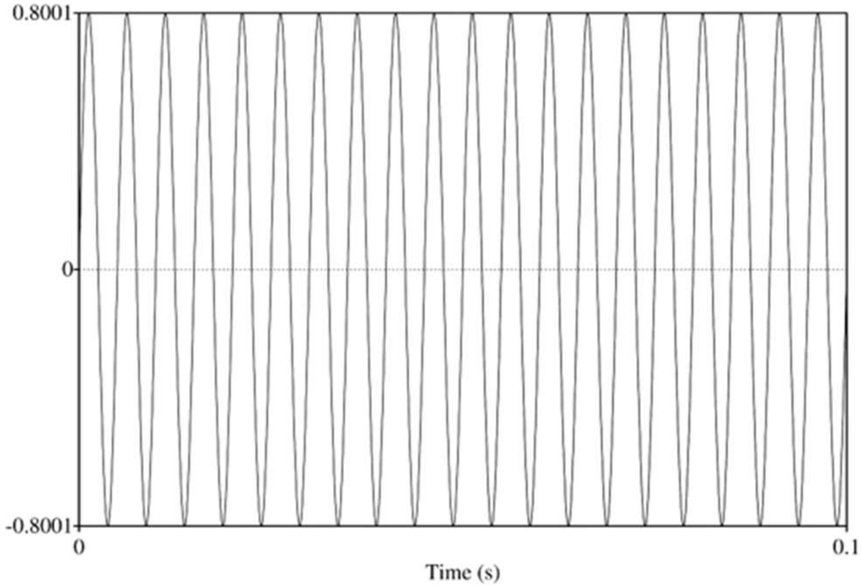


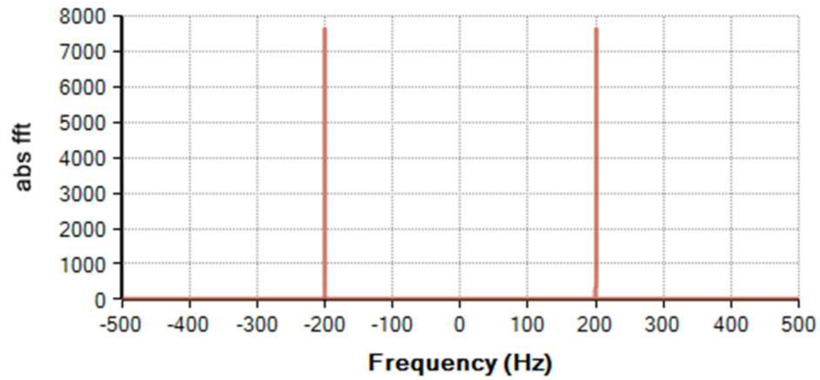
Jean Baptiste Joseph Fourier(1768~1830)

Transform based approach

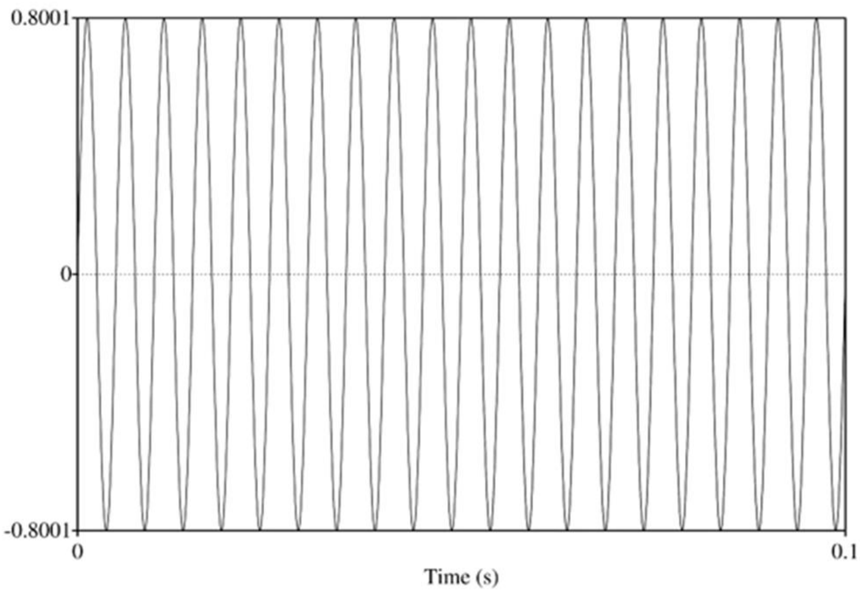
- A problem is defined in one setting
  - E.g. denoising a given rectangular image
- Transform the problem to a new domain (to a different basis)
  - Where it is more easily solvable
- Solve the problem in transformed setting
- Transform the solution back to original domain

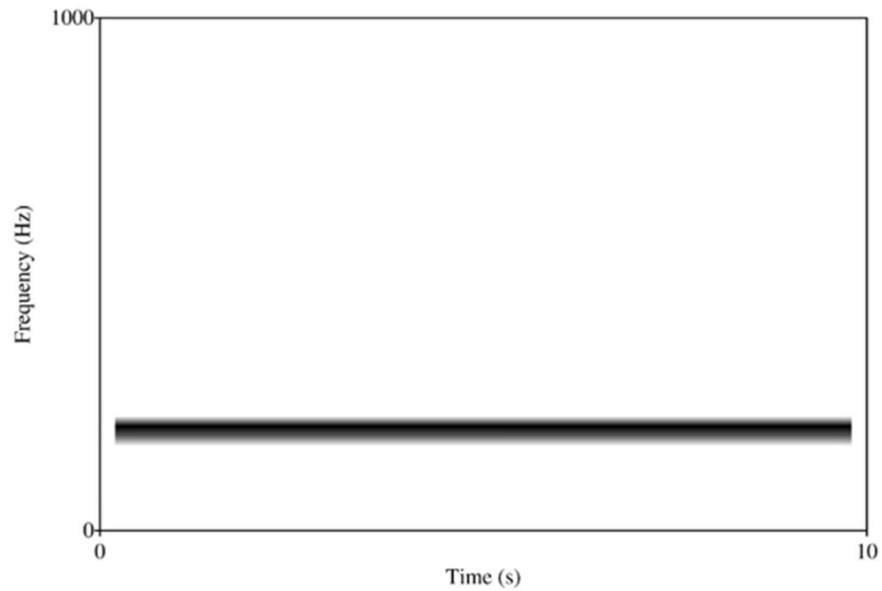
Motivation


0.8001
0
-0.8001
0
0.1
Time (s)


8000
7000
6000
5000
4000
3000
2000
1000
0
abs fft
-500
-400
-300
-200
-100
0
100
200
300
400
500
Frequency (Hz)

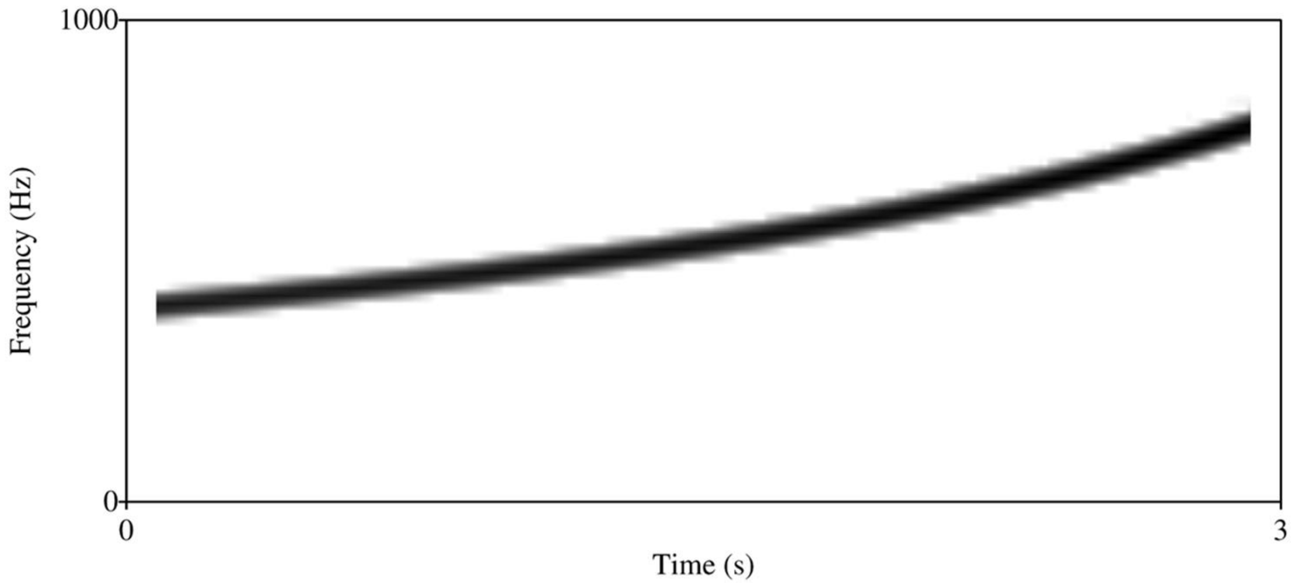
Motivation





Courtesy: floridalinguistics.com

Motivation



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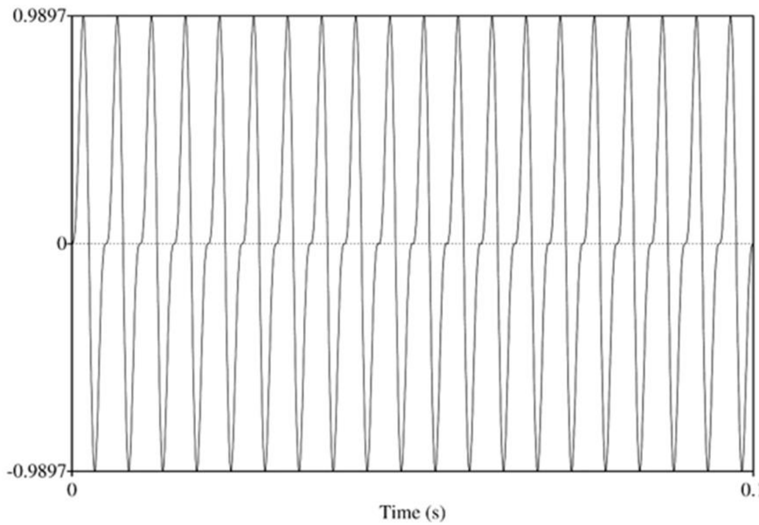
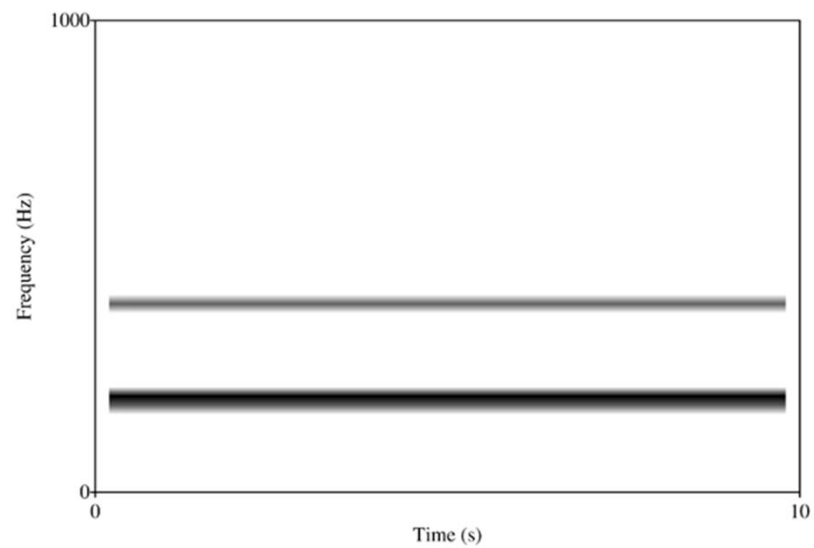
Motivation





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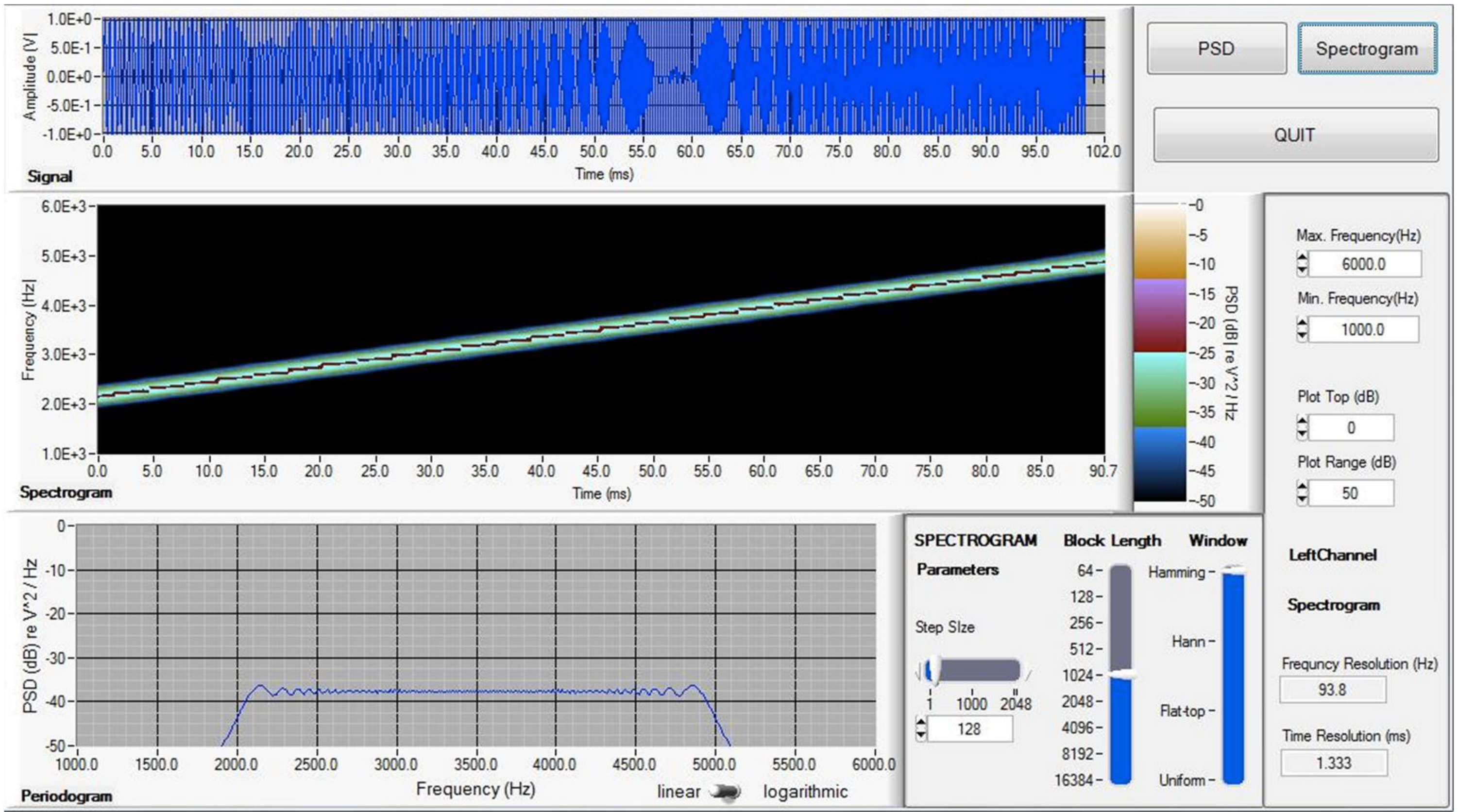
Motivation



Courtesy: benthowave.com

Motivation

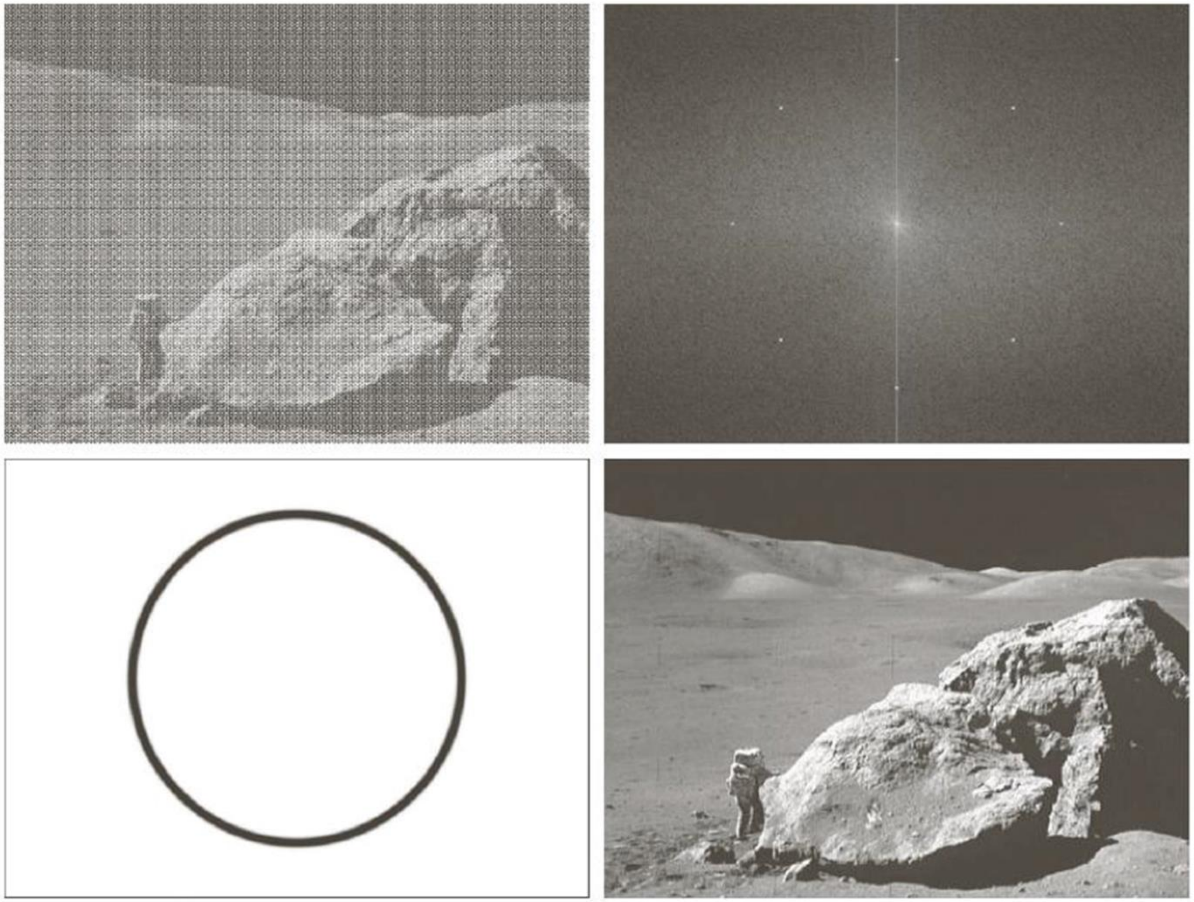


Image courtesy: Gonzalez and Woods

Todays lecture

- Understand the intuition behind DFT
- Discuss with some illustrative examples

Fourier Transform (FT) – 1D

FT

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

Inverse FT

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

Pair

Discrete Fourier Transform (DFT) – 1D

DFT

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M}$$

$For\ x = 0, 1, \ldots, (M-1)$

Inverse DFT

$$f(x) = \sum_{u=0}^{M-1} F(u)e^{j2\pi ux/M}$$

$For\ u = 0, 1, \ldots, (M-1)$

‘

Important Terms

- Magnitude spectrum

$$|F(u)| = (R^2(u) + I^2(u))^{1/2}$$

- Phase Spectrum

$$\phi(u) = tan^{-1}\left(\frac{I(u)}{R(u)}\right)$$

- Power Spectrum

$$P(u) = |F(u)|^2$$

Relationship between u and x

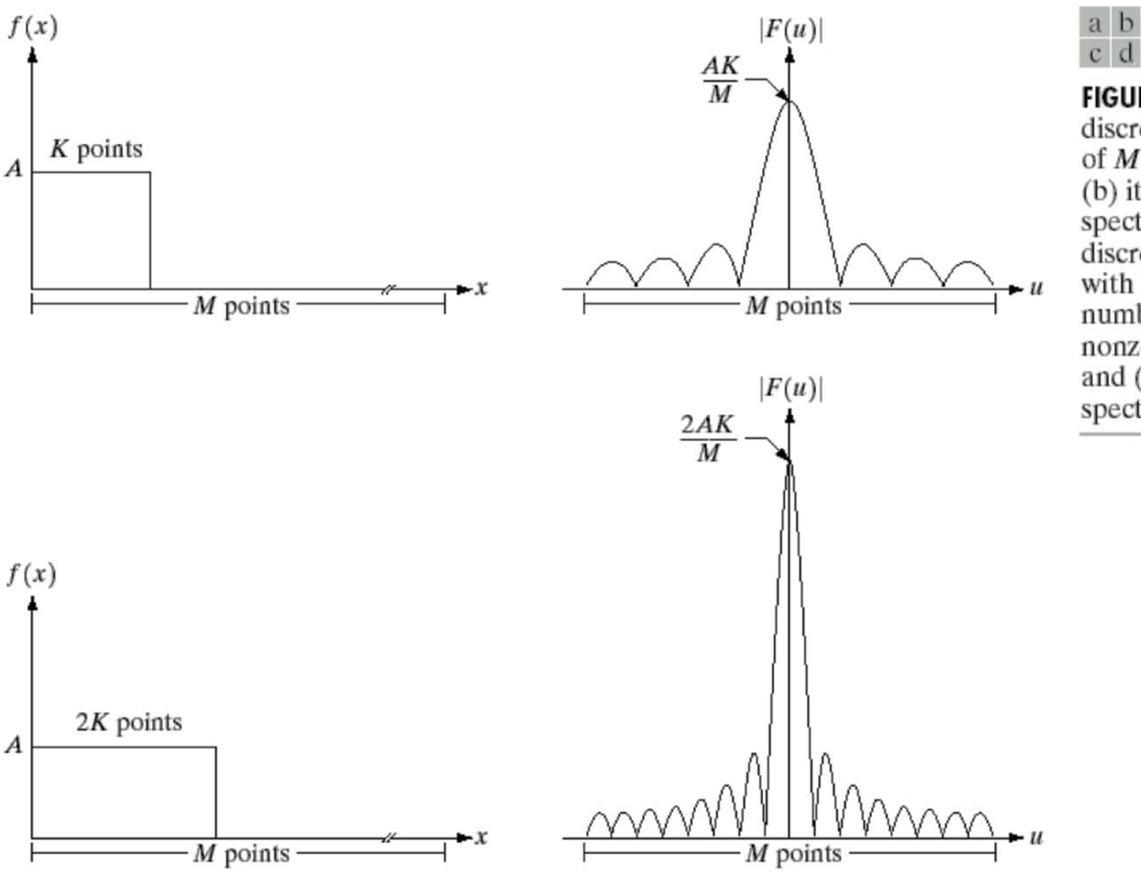


FIGURE 4.2 (a) A discrete function of M points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

Fourier Transform – 2D

DFT

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

Inverse DFT

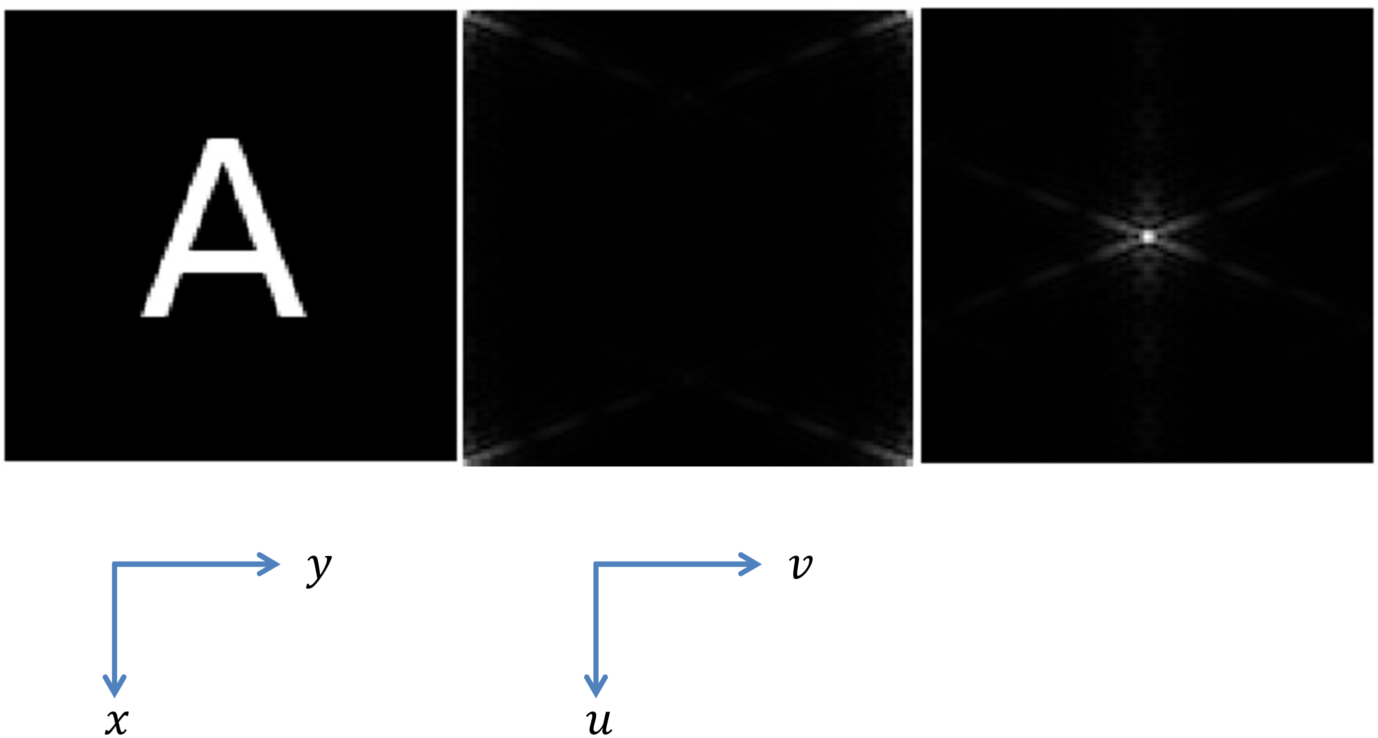
$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$

Discrete Fourier Transform (DFT) – 2D

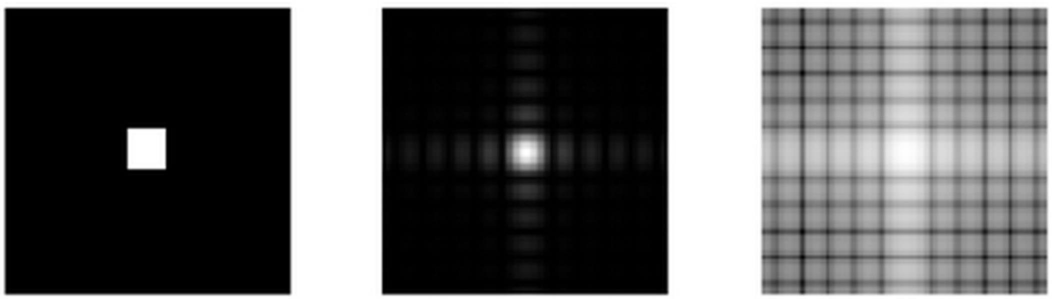
DFT $$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

Inverse DFT $$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$$

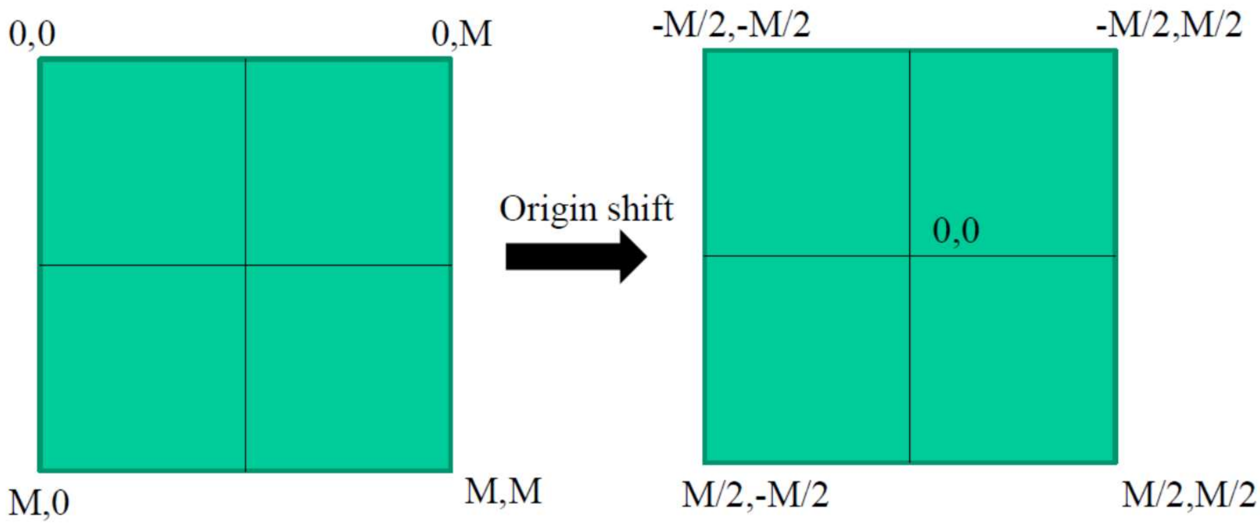
Discrete Fourier Transform (DFT) – 2D



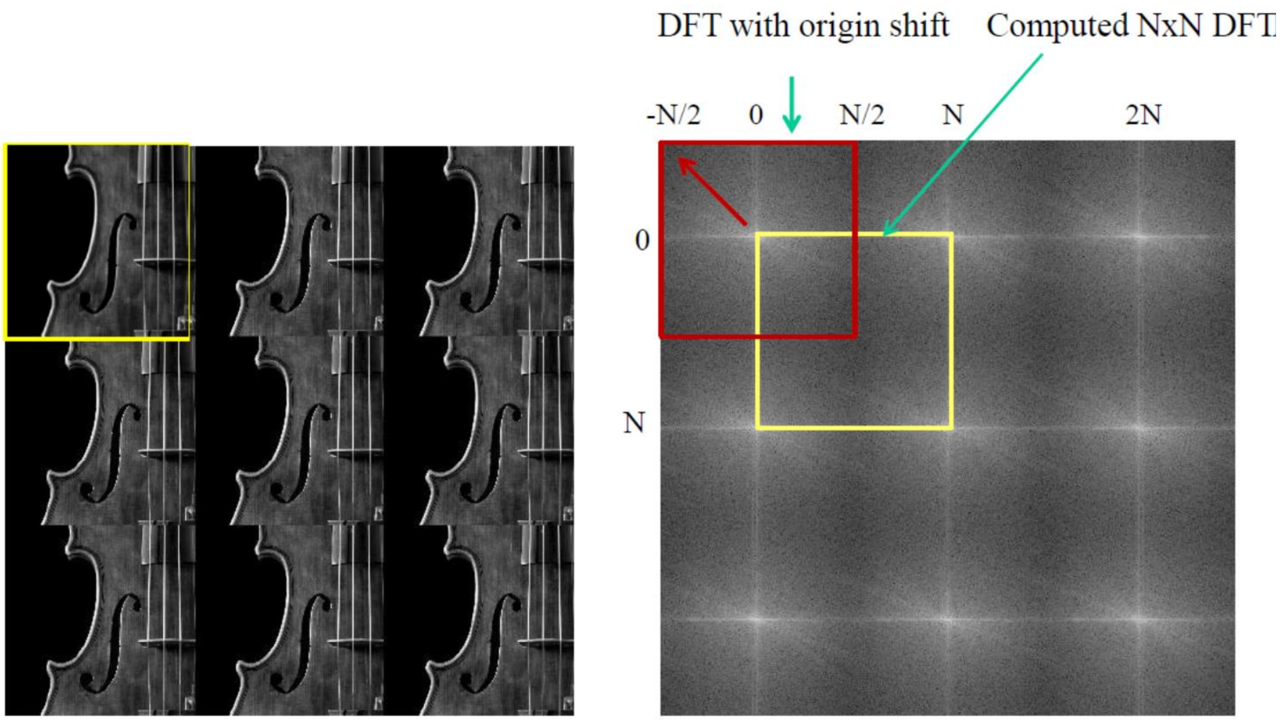
Discrete Fourier Transform (DFT) – 2D



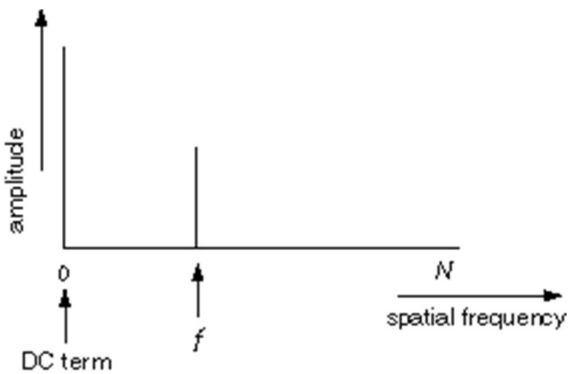
Shifting origin



Shifting origin

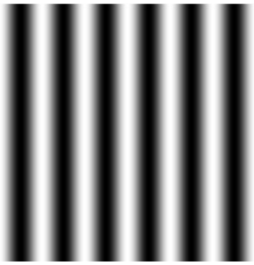


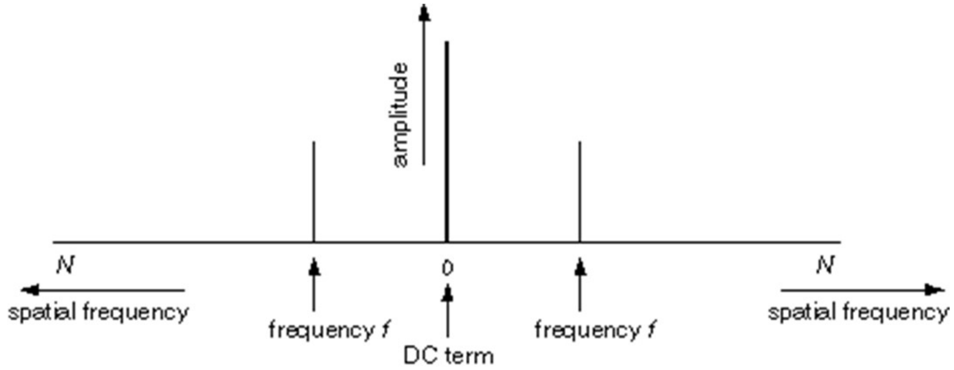
DFT Example



Courtesy: http://cns-alumni.bu.edu/~slehar/fourier/fourier.html

DFT Example





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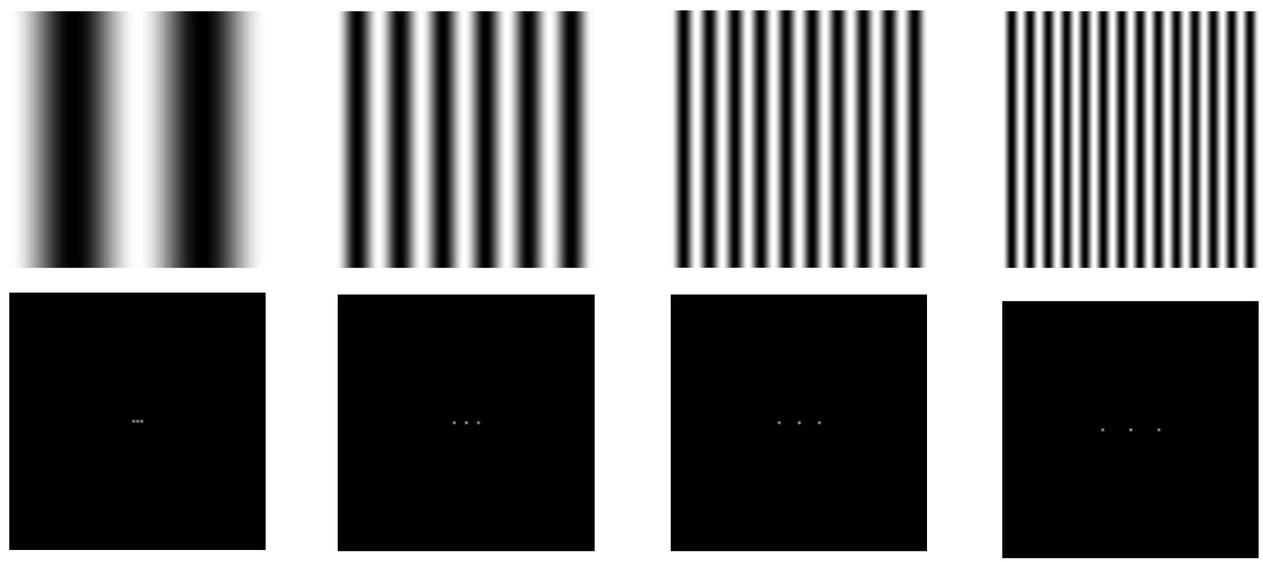
DFT Example



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DFT Example

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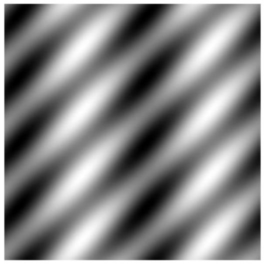
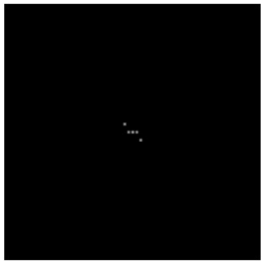
DFT Example





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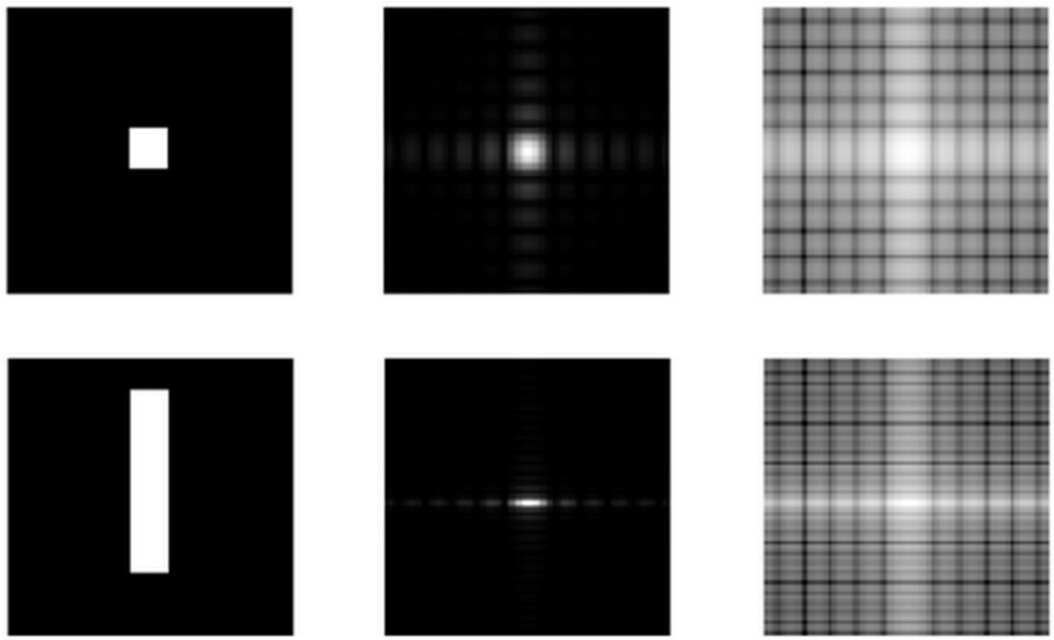
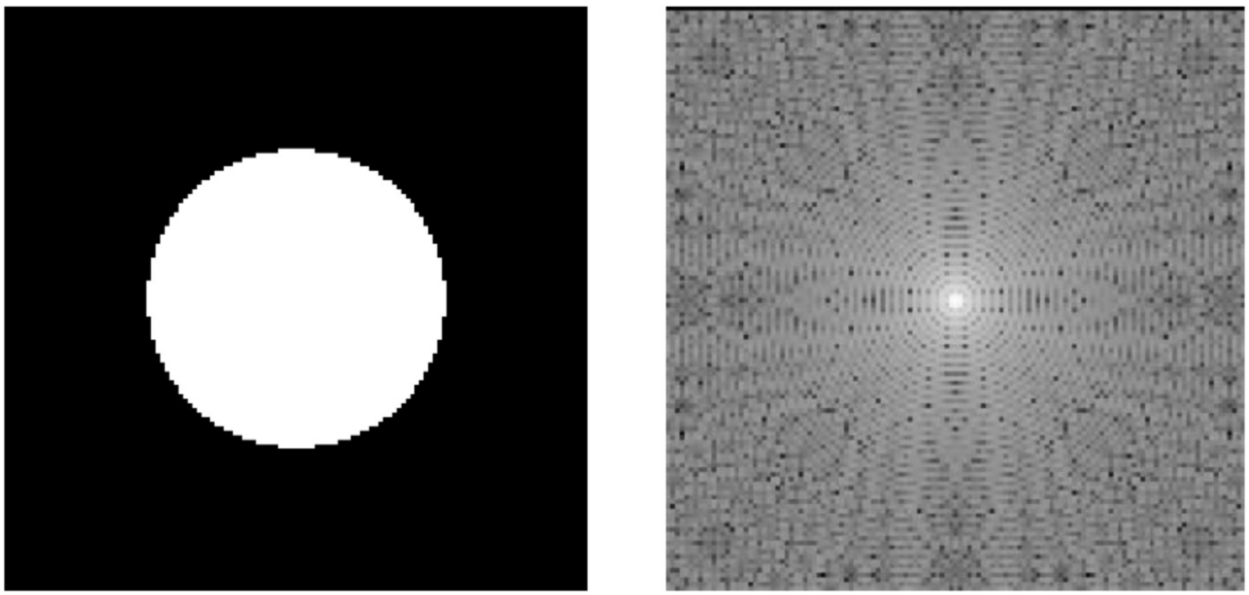
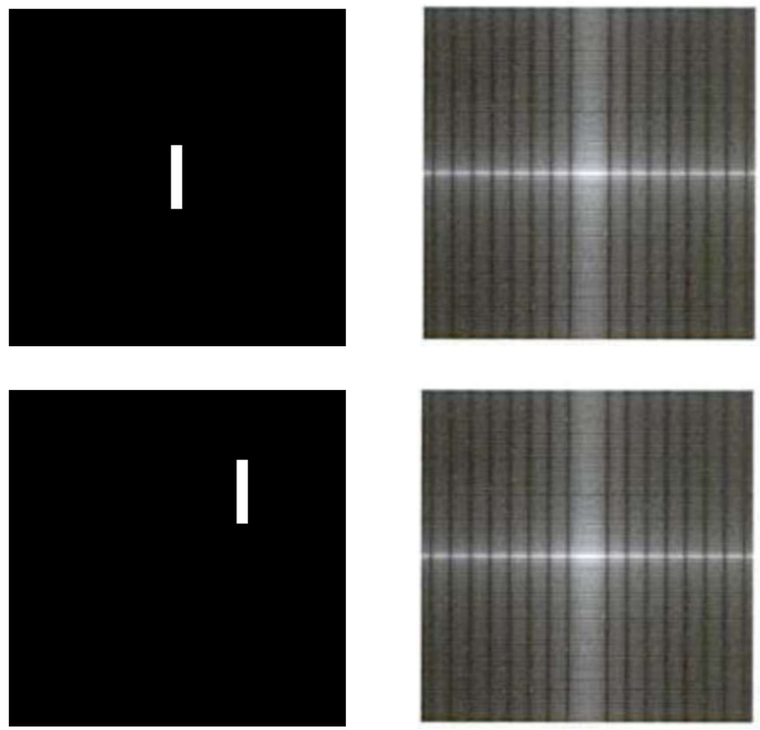
DFT Example



DFT Example



DFT Example



DFT Example

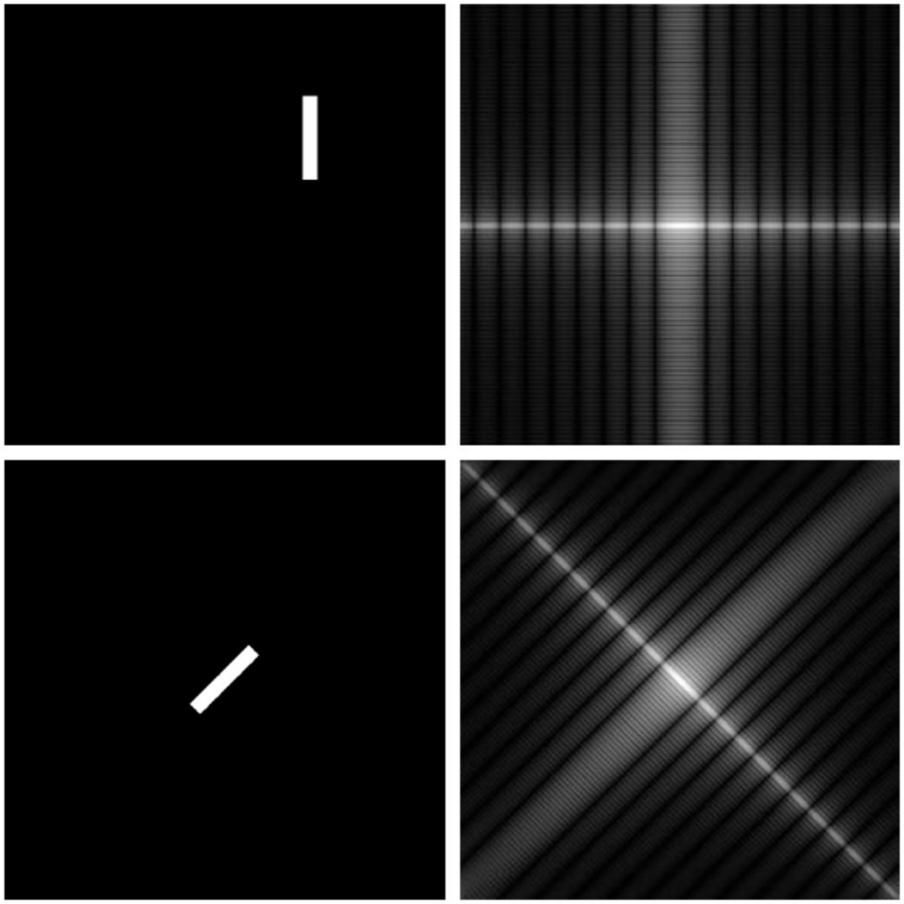
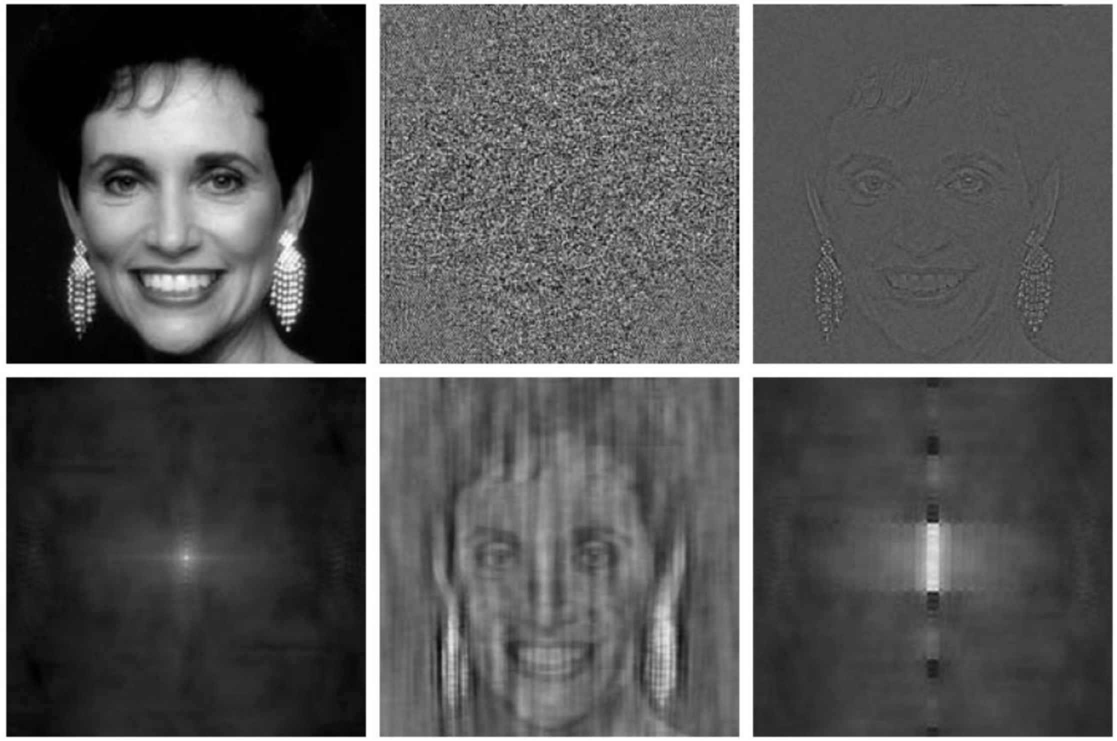


FIGURE 4.25 (a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum. (c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).



a b c
d e f

FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.

Properties of Exponential Function

When its domain is extended from the real line to the complex plane, the exponential function retains the following properties:

- $e^{z+w} = e^z e^w$
- $e^0 = 1$
- $e^z \neq 0$
- $\dfrac{\mathrm{d}}{\mathrm{d}z} e^z = e^z$
- $(e^z)^n = e^{nz}, n \in \mathbb{Z}$

for all $w, z \in \mathbb{C}$.