Multiple Linear Regression (MLR)

This section is partly adapted from notes by Prof. Carlos M. Carvalho
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Linear Regression

Note: I will use typical **statistics notation:**

coefficients are called βs, the dependent variable is Y, and estimates are indicated by "hats".

(Simple) Linear Regression: single independent variable X.

Fit the "best" straight line.

$$Y = \beta_0 + \beta_1 X + \epsilon$$

Multiple linear Regression: more than one independent variables.

Fit the "best" hyperplane.

Cost: Mean (or sum) squared error. (error = difference between Y and $\hat{Y}$.)

Thus Best fit = LEAST SQUARES solution

Training error vs. expected error for future samples.

The MLR Model

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_p X_p + \epsilon$$

Key General Issues

- What are the model assumptions? Are they valid?
- How do you estimate the parameters from data?
  - (i) cost function (true or surrogate?)
  - (ii) optimization method
- How do you evaluate your model?
  - (i) training/validation/test/scoring error
  - (ii) performance measures

Assumptions behind the MLR Model

(i) The conditional mean of Y is linear in the X_j variables.

(ii) The error term (deviations from line)

- are normally distributed
- independent from each other
- identically distributed (i.e., they have constant variance)

$$Y|X_1 \ldots X_p \sim N(\beta_0 + \beta_1 X_1 \ldots + \beta_p X_p, \sigma^2)$$

Then minimizing Mean Squared Error (MSE) on the training data yields the Maximum Likelihood Estimate (MLE) solution of the assumed *probabilistic (aka generative) model.*

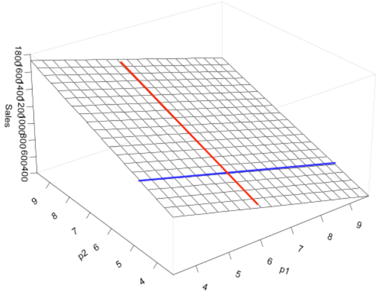
Q: What do the βs mean?

MLR On Sales Data

Consider sales of a product as predicted by price of this product (P1) and the price of a competing product (P2).

$$Sales = \beta_0 + \beta_1 P1 + \beta_2 P2 + \epsilon \ ; Thus \ \beta_j = \frac{\partial E[Y|X_1, \ldots, X_p]}{\partial X_j}$$

Holding all other variables constant, β_j is the average change in Y per unit change in X_j.



Q: Will your sales go up if you reduce the price?

Least Squares

Model: $Sales_i = \beta_0 + \beta_1 P1_i + \beta_2 P2_i + \epsilon_i$, $\epsilon \sim N(0, \sigma^2)$

Regression Statistics	
Multiple R	0.99
R Square	0.99
Adjusted R Square	0.99
Standard Error	28.42
Observations	100.00

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	2.00	6004047.24	3002023.62	3717.29	0.00
Residual	97.00	78335.60	807.58		
Total	99.00	6082382.84			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	115.72	8.55	13.54	0.00	98.75	132.68
p1	-97.66	2.67	-36.60	0.00	-102.95	-92.36
p2	108.80	1.41	77.20	0.00	106.00	111.60

$b_0 = \hat{\beta}_0 = 115.72$, $b_1 = \hat{\beta}_1 = -97.66$, $b_2 = \hat{\beta}_2 = 108.80$,

$s = \hat{\sigma} = 28.42$

Why will your predictions be uncertain? Where is uncertainty quantified?

Plug-in Prediction in MLR

Suppose that by using advanced corporate espionage tactics, I discover that my competitor will charge \$10 the next quarter. After some marketing analysis I decided to charge \$8. How much will I sell?

Our model is

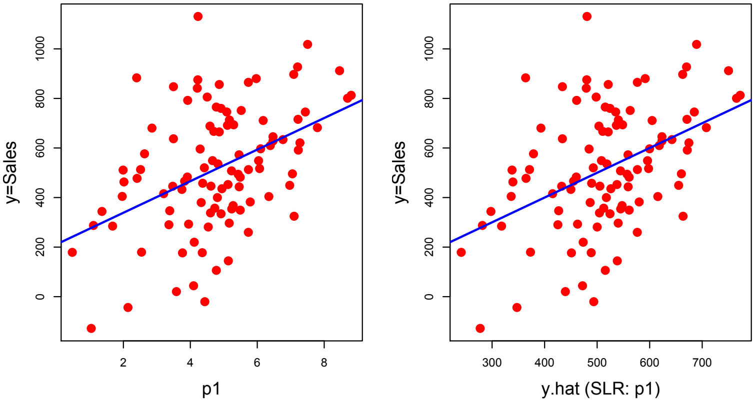
$$Sales = \beta_0 + \beta_1 P1 + \beta_2 P2 + \epsilon$$

Our estimates are $b_0 = 115$, $b_1 = -97$, $b_2 = 109$ and $s = 28$; i.e., $\epsilon \sim N(0, 28^2)$

Q: How will you estimate of sales when P1=8, P2=10 (95% confidence interval)?

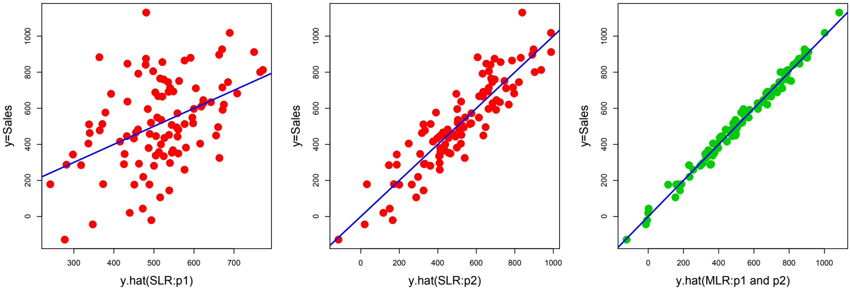
Fitted Values in MLR

With just $P1$...



- Left plot: *Sales* vs $P1$ (something odd?)
- Right plot: Sales vs. $\hat{y}$ (only $P1$ as a regressor)

Fitted Values in MLR

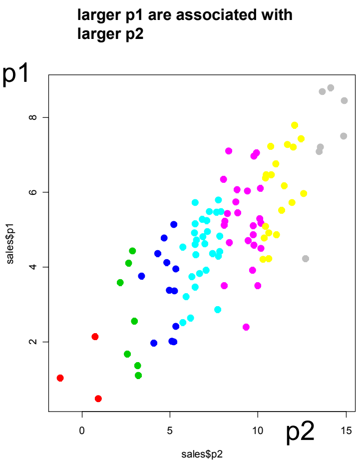


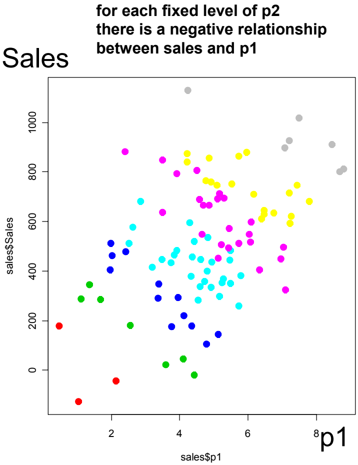
- First plot: *Sales* regressed on $P1$ alone..
- Second plot: *Sales* regressed on $P2$ alone...
- Third plot: Sales regressed on $P1$ and $P2$

Also look at residuals

Solving the Puzzle

- Let's look at a subset of points where $P1$ varies and $P2$ is held approximately constant...





Understanding Multiple Regression: Summary

- There are two, very important things we need to understand about the MLR model:

    1. How dependencies between the X's affect our interpretation of a multiple regression;
    2. How dependencies between the X's inflate standard errors (aka multicolinearity)
    3. Correlation does not imply causation
    4. Succinct models with the "right" predictors are superior

Understanding Multiple Regression

A relationship between y and x (or x's) may be driven by variables "lurking" in the background which are related to your current x's.
correlation is NOT (necessarily) causation
Any time a report says two variables are related and there's a suggestion of a "causal" relationship, ask yourself whether or not other variables might be the real reason for the effect. Multiple regression allows us to control for all important variables by including them into the regression.

- Example: "Once we control for weight, height and beers are NOT related"!!
- Why is it better to model beer vs. weight rather than beer vs. both height and weight?

Understanding Multiple Regression: Collinearity

- Previous Example: relationship amongst the X's can affect our interpretation of a multiple regression
- These dependencies also inflate the standard errors for the regression coefficients, and hence our uncertainty about them.
- In simple linear regression our uncertainty about b_1 is measured by

$$s_{b_1}^2 = \frac{s^2}{(n-1)s_x^2} = \frac{s^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

- The more variation in X (the larger s_x^2) the more "we know" about β_1... ie, $(b_1 - \beta_1)$ is smaller.

Understanding Multiple Regression: Collinearity

- In Multiple Regression we seek to relate the variation in Y to the variation in an X holding the other X's fixed. So, we need to see how much each X varies on its own.
- in MLR, the standard errors are defined by the following formula:

$$s_{b_j}^2 = \frac{s^2}{\text{variation in } X_j \text{ not associated with other } X\text{'s}}$$

- How do we measure the bottom part of the equation? We regress X_j on all the other X's and compute the residual sum of squares (call it SSE_j) so that $s_{b_j}^2 = \frac{s^2}{SSE_j}$
- SSE_j is (much) lower if independent variables are (highly) correlated!

More Decisions

- How many X's do you have and what are they?
  - Bank Example: dummy coding and interaction effects
  - What if number of (potential) predictors is very large (p vs. n)
- Outliers in X or in Y
- Transformation of Variables (look at residuals!)
  - Non-constant residuals may suggest log transform