Problem 9.1

a) We know the comoving wave number at horizon from eq 9.63 will be:

$$k_{eq} = a_{eq} H = H_0 \sqrt{2 \frac{\Omega_m^2}{\Omega_r}} = 0.02\, h\, \text{Mpc}^{-1}$$

and $\frac{1}{k_{eq}}$ is the actual comoving horizon.

with equation 9.32 we know $c_s = \frac{c}{\sqrt{3}}$

so really we only have to multiply $a_{eq} H$ by a factor of $\frac{1}{\sqrt{3}}$

$$\frac{1}{k_{eq,cs}} = \frac{1}{\sqrt{3}\, a_{eq} H} = \frac{50}{\sqrt{3}}\ \text{Mpc}\, h^{-1} = 28.9\ \text{Mpc}\, h^{-1}$$

b) So we know the linear growth function:

$$D(a) = \frac{\delta(a)}{\delta(1)} = \frac{a\, g(a)}{g(1)}$$

We know that $g(a) = 1$ for an einstein-de-sitter universe, so we can say $g(a) = 1$ and $g(1) = 1$:

$$D(a) = a = \frac{\delta(a)}{\delta(1)} \qquad a = \frac{1}{z} \qquad z = 3500 \text{ at matter radiation equality}$$

$$\frac{1}{3500} = \frac{\delta(a)}{2\times10^{-5}} \Rightarrow \delta(a) = \frac{2\times10^{-5}}{3500} = \boxed{5.71\times10^{-9}}$$

Problem 9.4

$$\frac{dn}{d\ln M} d\ln M = \frac{\rho_{M,0}}{M} \left| \frac{dF(M)}{d\ln M} \right| d\ln M. \tag{9.90}$$

After taking the derivative of Eq. (9.87) analytically, and including the miraculous factor of 2, we get (see Problem 9.4) the Press-Schechter mass function:

$$\frac{dn}{d\ln M} = \sqrt{\frac{2}{\pi}} \frac{\rho_{M,0}}{M} \frac{\delta_c}{\sigma} \left| \frac{d\ln\sigma}{d\ln M} \right| e^{-\delta_c^2/(2\sigma^2)}. \tag{9.91}$$

$$F(M) \equiv \int_{\delta_c}^{\infty} P(\delta) d\delta = \frac{1}{\sqrt{2\pi}\sigma(M)} \int_{\delta_c}^{\infty} e^{-\delta^2/(2\sigma(M)^2)} d\delta$$
$$\equiv \frac{1}{2} \text{erfc}\left(\frac{\nu_c}{\sqrt{2}}\right), \tag{9.87}$$

where erfc is the complementary error function (see e.g. Wikipedia) and

$$\nu_c \equiv \frac{\delta_c}{\sigma(M)} \tag{9.88}$$

$$\frac{d}{dz} \text{erf}\, z = \frac{2}{\sqrt{\pi}} e^{-z^2}.$$

$$F(M) = \frac{1}{2}\,\mathrm{erc}\left(\frac{\nu_c}{\sqrt{2}}\right) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2}\,\sigma(M)}\right)$$

$$\frac{\partial F(M)}{d\sigma(M)} = \frac{1}{2}\,\frac{2}{\sqrt{\pi}}\, e^{-\frac{\delta_c^2}{2\sigma^2}}\left(-\frac{\delta_c}{\sqrt{2}\,\sigma^2(M)}\right)$$

$$d\ln a = \frac{1}{a}\,da \Rightarrow d\sigma(M) = \sigma(M)\, d\ln\sigma$$

$$\frac{dF(M)}{\sigma\, d\ln\sigma} = \frac{1}{\sqrt{\pi}}\, e^{-\frac{\delta_c^2}{2\sigma^2}}\left(-\frac{\delta_c}{\sqrt{2}\,\sigma^2}\right)$$

$$dF(M) = -\frac{1}{\sqrt{2\pi}}\,\frac{\delta_c}{\sigma}\, e^{-\frac{\delta_c^2}{2\sigma^2}}\, d\ln\sigma$$

now taking 4.40 $\frac{dn}{d\ln M}d\ln M = \frac{\rho_{M,0}}{M}\left|\frac{dF(M)}{d\ln M}\right| d\ln M.$ and substituting $dF(M)$

$$\frac{dn}{d\ln M} = \frac{\rho_{M,0}}{M}\left|\frac{-\frac{1}{\sqrt{2\pi}}\frac{\delta_c}{\sigma}e^{-\frac{\delta_c^2}{2\sigma^2}}\, d\ln\sigma}{d\ln M}\right|$$

$$\frac{dn}{d\ln M} = \frac{\rho_{M,0}}{M}\,\frac{1}{\sqrt{2\pi}}\,\frac{\delta_c}{\sigma}\left|\frac{d\ln\sigma}{d\ln M}\right| e^{-\frac{\delta_c^2}{2\sigma^2}}$$

and introducing a magic factor of two like the book suggests:

$$\frac{dn}{d\ln M} = \frac{\rho_{M,0}}{M}\sqrt{\frac{2}{\pi}}\,\frac{\delta_c}{\sigma}\left|\frac{d\ln\sigma}{d\ln M}\right| e^{-\frac{\delta_c^2}{2\sigma^2}}$$

Additional Problem #2

Looking at 4.3.54 and 4.3.55 in the Baumann book

$$\partial_\eta(\delta\rho) = -3\mathcal{H}(\delta\rho + \delta P) + 3\Phi'(\bar\rho + \bar P) - \partial_i q^i$$ } Continuity equation

(4.3.40)

remember from lecture!

$$\partial_\eta(\delta\rho) = \partial_\eta(\bar\rho\cdot\overset{\text{overdensity}}{\delta}) = \partial_\eta(\bar\rho)\cdot\delta + \bar\rho\,\partial_\eta(\delta) = \bar\rho'\cdot\delta + \bar\rho\,\delta'$$

Remember $\bar\rho' = -3H(\bar\rho + \bar P)$ for radiation $P_r = \frac{1}{3}\rho_r \Rightarrow \bar\rho' = -3H\left(\frac{4}{3}\rho_r\right) = -4H\rho_r$

$$\partial_\eta(\delta p) = -4H\bar{\rho}_r \cdot \delta + \bar{\rho}\delta'$$

$$\partial_\eta(\delta\rho) = -3\mathcal{H}\,(\delta\rho + \delta p) + 3\phi'(\bar{\rho}+\bar{p}) - \partial_i q^i$$

$(4/3\,\bar{\rho})$

$$-4H\bar{\rho}_r \cdot \delta + \bar{\rho}\delta' = -3\mathcal{H}\,(\delta\rho + \delta p) + 3\phi'(\bar{\rho}+\bar{p}) - \partial_i q^i$$

$\longrightarrow q^i = (\bar{\rho} + \bar{p})V_i \Rightarrow \frac{4}{3}\bar{\rho}V_i$

$$\cancel{-4H\bar{\rho}_r \cdot \delta} + \bar{\rho}\delta' = \cancel{-3\mathcal{H}(\bar{\rho}\cdot\delta)} \cancel{- \mathcal{H}\,\delta p} + 4\phi'\bar{\rho} - \frac{4}{3}\bar{\rho}\,\partial_i V^i$$

$$\cancel{\bar{\rho}}\,\delta' = 4\phi'\cancel{\bar{\rho}} - \frac{4}{3}\cancel{\bar{\rho}}\,\nabla V$$

$$\underline{\delta' = 4\phi' - \frac{4}{3}\nabla V}$$