OM-S20-24: More General Constrained Optimization

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Keywords

- Lagrangians
- KKT Conditions
- Lagrangian Duality
- Convex Optimization
- Conditions for Optimality

Example

Consider an LP

$$\max\ 2x_1 + x_2$$

subject to:

$$x_1 + x_2 \le 5; x_1 \le 3; x_2 \le 4; x_1, x_2 \ge 0$$

and Dual

$$\min 5y_1 + 3y_2 + 4y_3$$

$$y_1 + y_2 \le 2; y_1 + y_3 \le 1; y_1, y_2 \ge 0$$

How did we find the dual variables or "coefficients" y_i?. If

$$2x_1 + x_2 = y_1(x_1 + x_2) + y_2(x_1) + y_3(x_2)$$

$$y_1 = 1 \text{ and } y_2 = 1 \text{ and } y_3 = 0$$

Or at the point of optimality

- Constraints 1 and 2 (or P) are tight and corresponding y_i are non-zero.
- Constraint 3 is not-tight and corresponding y_i is zero.

Non-linear extension

Consider a similar situation in the nonlinear setting too.
(*Graphical Explanation*)

More General Problem Setting

$$\min f(x)$$

subject to

$$g_j(x) \leq 0$$

or written as equality as:

$$g_j(x) + s_j^2 = 0$$

for some positive slack variable s_j.
Corresponding Lagrangian

$$\mathcal{L} = f(x) + \sum_j \lambda_j (g_j(x) + s_j^2)$$

KKT Conditions

Necessary condition for optimality obtained by differentiating and equating to zero wrt x_i, λ_j and s_j:

$$\frac{\partial f}{\partial x_i} + \sum_j \lambda_j \frac{\partial g_j}{\partial x_i} \quad \forall i \tag{1}$$

$$\nabla f + \lambda^T \nabla g_j = 0 \tag{2}$$

$$g_j(x) + s_j = 0 \text{ or } g_j(x) \leq 0 \quad \forall j \tag{3}$$

$$\lambda_j s_j = 0 \text{ or } \lambda_j g_j(x) = 0 \quad \forall j \tag{4}$$

Also $\lambda_j \geq 0$
(1 and 2) are the optimality constraints (3) is the feasibility constraint and (4) is the complementary slackness constraint
These are the Karush-Kun-Tucker (KKT) conditions.

Example 1

$$\min \; x_1^2 - 4x_1 + x_2^2 - 6x_2$$

Subject to:

$$x_1 + x_2 \leq 3$$

$$-2x_1 + x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

Example 1(cont.)

KKT conditions yield:

$$2x_1 - 4 + \lambda_1 - 2\lambda_2 = 0$$

$$2x_2 - 6 + \lambda_1 + \lambda_2 = 0$$

$$x_1 + x_2 \leq 3$$

$$-2x_1 + x_2 \leq 2$$

$$\lambda_1(x_1 + x_2 - 3) = 0$$

$$\lambda_2(-2x_1 + x_2 - 2) = 0$$

$$\lambda_1 \geq 0; \lambda_2 \geq 0$$

$$x_1^* = 1 \text{ and } x_2^* = 2; \lambda_1 = 2; \lambda_2 = 0$$

Example 2

$$\min -x_2$$

subject to:

$$x_1^2 + x_2^2 - 4 \leq 0$$
$$-x_1^2 + x_2 \leq 0$$

KKT conditions lead to:

$$x_1\lambda_1 - x_2\lambda_2 = 0$$
$$-1 + 2x_2\lambda_1 + \lambda_2 = 0$$
$$x_1^2 + x_2^2 \leq 4$$
$$-x_1^2 + x_2 \leq 0$$
$$\lambda_1(x_1^2 + x_2^2 - 4) = 0$$
$$\lambda_2(-x_1^2 + x_2) = 0$$
$$\lambda_1, \lambda_2 \geq 0$$

- What about $(0, 0)$? Satisfies but not the minima of interest.

Sufficiency

- KKT conditions are only “necessary” for optimality, in general.
- The conditions are sufficient for a convex optimization problems.

Lagrange Dual

$$\min f(x)$$

subject to

$$g_j(x) \leq 0$$

We know:

$$\mathcal{L}(x, \lambda) = f(x) + \sum_j \lambda j g_j(x) = f(x) + \lambda^T g$$

Consider the equivalent problems:

$$p^* = \min_x \max_{\lambda \geq 0} \mathcal{L}(x, \lambda)$$

Note:

$$f(x) = \max_{\lambda \geq 0} f(x) + \lambda^T g$$

and constraints are satisfied. $\max \lambda^T g$ is zero if $g_j \leq 0$ and ∞ otherwise. i.e., p^* is $\min_x f(x)$ with constraints satisfied.

Lagrange Dual (No proofs)

We know:

$$\mathcal{L} = f(x) + \sum_j \lambda j g_j(x)$$

Consider the two problems (Primal and dual):

$$p^* = \min_x \max_{\lambda \geq 0} \mathcal{L}(x, \lambda)$$

$$d^* = \max_{\lambda \geq 0} \min_x \mathcal{L}(x, \lambda)$$

Weak Duality:

$$d^* \leq p^*$$

What next? (Announcements)

Plans for closing:

- Shall share some videos and material for Langrangians, KKT and Lagrange Duality. See those before the next session.
- Close the discussions on Gradient based learning. (by this weekend, 19th).
- Answer the questions on these two ILMs and close (within a week i.e., 23 and 26 April).
- KKT module will also end by April 30.

What next? (Announcements)

Reviews

- With this (including todays) we have 120+ questions and we will be taking best 100.
- Two review sessions (next week) will be used for replacing any missed/lost opportunities with a valid reason. These will be optional.
- The next two sessions will be some followup/elaboration/problem solving from nonlinear constrained optimization. (no new topic planned.)