Q1) Solve the following problem using Bala's Algorithm

maximize $5x_1 + 2x_2 + 12x_3 + 10x_4$

s.t $3x_1 + 6x_2 + 5x_3 + 5x_4 \leq 12$

$4x_1 + 9x_2 - 2x_3 + x_4 \leq 25$

All x_i's are binary.

Sol:

To get to standard form of Bala's Algorithm we replace $x_i = 1 - y_i$

Then the problem becomes

$\max 5(1-y_1) + 2(1-y_2) + 12(1-y_3) + 10(1-y_4)$

$\Rightarrow \max 5 - 5y_1 + 2 - 2y_2 + 12 - 12y_3 + 10 - 10y_4$

$\Rightarrow \max -5y_1 - 2y_2 - 12y_3 - 10y_4 + 29$

$\Rightarrow \min 5y_1 + 2y_2 + 12y_3 + 10y_4 - 29$

Given constraints become

$\rightarrow 3x_1 + 6x_2 + 5x_3 + 5x_4 \leq 12$

$3(1-y_1) + 6(1-y_2) + 5(1-y_3) + 5(1-y_4) \leq 12$

$-3y_1 - 6y_2 - 5y_3 - 5y_4 + 19 \leq 12$

$3y_1 + 6y_2 + 5y_3 + 5y_4 - 19 \geq -12$

$\therefore 3y_1 + 6y_2 + 5y_3 + 5y_4 \geq 7 \rightarrow (1)$

$\rightarrow 4x_1 + 9x_2 - 2x_3 + x_4 \leq 25$

$4(1-y_1) + 9(1-y_2) - 2(1-y_3) + (1-y_4) \leq 25$

$-4y_1 - 9y_2 + 2y_3 - y_4 + 12 \leq 25$

$4y_1 + 9y_2 - 2y_3 + y_4 - 12 \geq -25$

$4y_1 + 9y_2 - 2y_3 + y_4 \geq -13 \rightarrow (2)$

So the modified problem is

$$\min \; 2y_2 + 5y_1 + 10y_4 + 12y_3$$

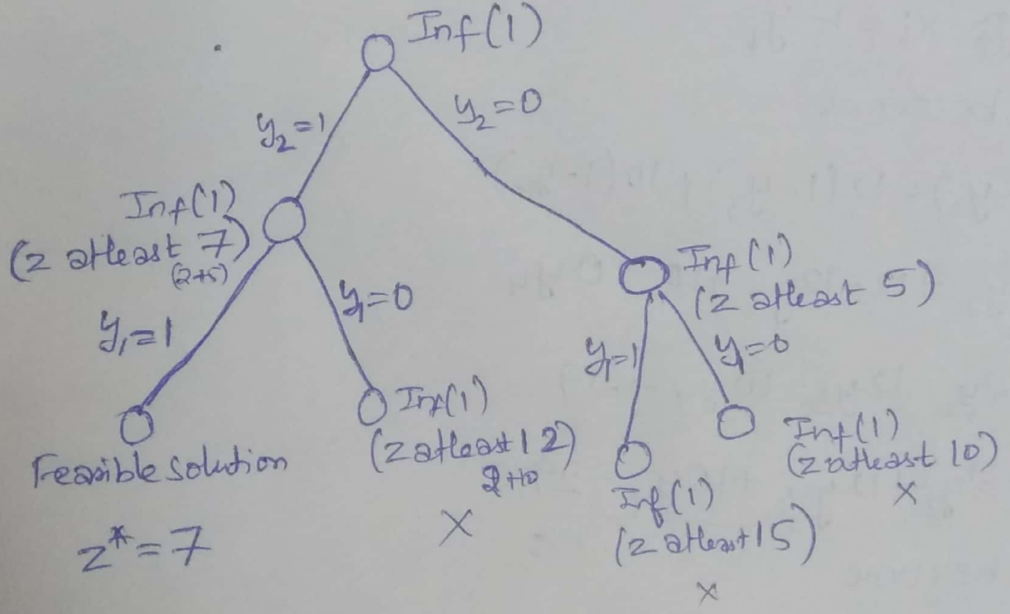
with constraints

$C_1.\quad 6y_2 + 3y_1 + 5y_4 + 5y_3 \geq 7$

$C_2.\quad 9y_2 + 4y_1 + y_4 - 2y_3 \geq -13$

Applying Bala's Algorithm.

First, considering all $y_i = 0$, C_1 is not satisfied.



∴ For min objective we get

$Z^*_{min} = 7$ when $y_1=1, y_2=1, y_3=0, y_4=0$

So we need to consider

$x_1=0, x_2=0, x_3=1, x_4=1$

∴ Original objective becomes

$\underline{\underline{z^* = 22}}$ when $\underline{\underline{x_1=0, x_2=0, x_3=1, x_4=1}}$

Q2) Vertex cover of graph.

Given an undirected graph $G=(V,E)$

We need to find set $S \subseteq V$

Vertex cover formulated as Integer Linear Program Problem.

$$OPT_{ILP} = \min_{x_i} \sum_{i \in V} x_i$$

$$\text{s.t} \quad x_i \in \{0,1\} \ \forall i \in V$$

$$x_i + x_j \geq 1 \ \forall \{i,j\} \in E$$

LP relaxed problem is

$$OPT_{LP} = \sum_{i \in V} x_i$$

$$\text{s.t} \quad 0 \leq x_i \leq 1 \ \forall i \in V$$

$$x_i + x_j \geq 1 \ \forall \{i,j\} \in E$$

Rounding procedure used is

$$x_i^* = \begin{cases} 1 & \text{if } x_i \geq \frac{1}{2} \\ 0 & \text{if } x_i < \frac{1}{2} \end{cases}$$

And $OPT_{ROUND} = \sum_{i \in V} x_i^*$

(a) $OPT_{LP} \leq OPT_{ILP}$

In ILP we have more strict constraint than in LP

So, a valid solution for ILP is also a valid solution for relaxed LP.

For LP, we have more choices & obviously when we have more possibilities, we get better solution. It could only get better or maybe stay the same, but it will never be worse.

We can also show this with an example, when graph has 3 vertices and all are connected, 1 solution for ILP is $(1,1,0)$ where $OPT_{ILP} = 2$

If we do LP relaxation we may get a solution like $(0.7, 0.7, 0.3)$

then $OPT_{LP} = 1.7$.

$\therefore \underline{OPT_{LP} \leq OPT_{ILP}}$

(b) $S^* = \{i \in V \mid x_i^* = 1\}$ still produces a valid set cover.

For every $(i,j) \in E$

we have a condition $x_i + x_j \geq 1$

Hence there will exist at least one vertex which is greater than $\frac{1}{2}$ ie, $x_i \geq \frac{1}{2}$

After rounding procedure,
it will become $x_i^* = 1$

$\therefore$ we will get a feasible solution

(c) Approximation factor.

Maximum value of $\frac{OPT_{ROUND}}{OPT_{ILP}}$

$\left(x_i \geq \frac{1}{2} \Rightarrow 2x_i \geq 1\right)$

$$OPT_{ROUND} = \sum_{i \in V} x_i^*$$

$$\leq \sum_{i \in V} 2x_i$$

$$= 2 \cdot \sum_{i \in V} x_i$$

$$= 2 \cdot OPT_{LP}$$

$$\leq 2 \cdot OPT_{ILP} \quad (\because \text{from (a)})$$

$$\therefore \frac{OPT_{ROUND}}{OPT_{ILP}} \leq 2$$

Q3) Given system of linear equation

$$2x_1+2x_2+7x_3=7$$
$$3x_1+(\lambda+9)x_2+(\lambda+11)x_3=10$$
$$3x_1+(\lambda+4)x_2+11x_3=10$$

~~Q~~ Find λ when equations have unique solution.

Sol:

System of linear equations have unique solution when

$\det|A| \neq 0$

$$\therefore \begin{vmatrix} 2 & 2 & 7 \\ 3 & (\lambda+9) & (\lambda+11) \\ 3 & (\lambda+4) & 11 \end{vmatrix} \neq 0$$

$$\Rightarrow 2(\cancel{11\lambda}+99-\lambda^2-4\lambda-\cancel{11\lambda}-44)-2(33-3\lambda-33)+7(\cancel{3\lambda}+12-\cancel{3\lambda}-27)\neq 0$$

$$\Rightarrow -2\lambda^2-8\lambda+110+6\lambda-105 \neq 0$$

$$\Rightarrow -2\lambda^2-2\lambda+5 \neq 0$$

The solutions of quadratic equation $2\lambda^2-2\lambda+5=0$ are

$$x=\frac{-1\pm\sqrt{11}}{2} = 1.158, -2.158$$

$\therefore$ values of λ when equations have unique solution is

$$\lambda \in R-\{1.158,-2.158\}$$

Q4) Solve the following LP using Simplex Tableau Method.

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minimize $z = 6x_1 + 7x_2 - 3x_3$

subject to $5x_1 - 6x_2 - x_3 \leq -9$

$-2x_1 + x_2 + 4x_3 = 3$

$13x_1 - 8x_3 \leq 0$

$x_1, x_2, x_3 \geq 0$

Sol: Converting to standard form.

C1. $5x_1 - 6x_2 - x_3 \leq -9$

Here constant $= -9 < 0$ ∴ Multiply by -1 to make it > 0

$-5x_1 + 6x_2 + x_3 \geq 9$

We have this of $\geq$ type ∴ subtract surplus variable x_4.

& add artificial variable a_1

C2: It is equality '=' constraint. so we ~~subtract~~ add artificial variable a_2

C3: We add surplus variable x_5

∴ Problem becomes.

min $z = 6x_1 + 7x_2 - 3x_3 + 0.x_4 + 0.x_5 + Ma_1 + Ma_2$

s.t

$-5x_1 + 6x_2 + x_3 - x_4 + a_1 = 9$

$-2x_1 + x_2 + 4x_3 + a_2 = 3$

$13x_1 - 8x_3 + x_5 = 0$

& $x_1, x_2, x_3, x_4, x_5, a_1, a_2 \geq 0$

Initial Table : Iteration-1

			x_1	x_2	x_3	x_4	x_5	a_1	a_2		
Coefficient	Basic variables	Constants coeff.	6	7	-3	0	0	M	M	$\}$ c_j	Min Ratio
M	a_1	9	-5	(6)	1	-1	0	1	0		$\frac{9}{6} = 1.5$
M	a_2	3	-2	1	4	0	0	0	1		$\frac{3}{1} = 3$
0	x_5	0	13	0	-8	0	1	0	0		—
	z	z= 12M	-7M	7M	5M	-M	0	M	M		
	$z-c_j$		-7M-6	7M-7	5M+3	-M	0	0	0		

Incoming variable = x_2

Outgoing variable = a_1

Key element = 6

Iteration-2

$R_1(\text{new}) = R_1(\text{old})/6$

$R_2(\text{new}) = R_2(\text{old}) - R_1(\text{new})$

$R_3(\text{new}) = R_3(\text{old})$

			x_1	x_2	x_3	x_4	x_5	a_1	a_2	
Coefficients	Basic variable	Constants	6	7	-3	0	0	M	M	Min ratio
7	x_2	$9/6$	$-5/6$	1	$1/6$	$-1/6$	0	$1/6$	0	$\frac{9}{6} \times \frac{6}{1} = 9$
M	a_2	$3 - \frac{9}{6} = \frac{9}{6}$	$-2 + \frac{5}{6} = -\frac{7}{6}$	0	$4 - \frac{1}{6} = \frac{23}{6}$	$\frac{1}{6}$	0	$-\frac{1}{6}$	1	$\frac{9}{6} \times \frac{6}{23} = \frac{9}{23} = 0.39$
0	x_5	0	13	0	-8	0	1	0	0	
	z	$z = 1.5M + 10.5$	$\frac{-35}{6} - \frac{7M}{6}$	7	$\frac{7}{6} + \frac{23M}{6}$	$-\frac{7}{6} + \frac{M}{6}$	0	$\frac{7}{6} - \frac{M}{6}$	M	
	$z - c_j$		$\frac{-35}{6} - \frac{7M}{6} - 6$	0	$\frac{7}{6} + \frac{23M}{6} - 3$	$-\frac{7}{6} + \frac{M}{6}$	0	$\frac{7}{6} - \frac{7M}{6}$	0	

$\uparrow$

Incoming variable $= x_3$

Outgoing variable $= a_2$

Key element $= \frac{23}{6} = 3.833$

Iteration - 3

$R_2(\text{new}) = R_2(\text{old}) * \left(\frac{6}{23}\right)$

$R_1(\text{new}) = R_1(\text{old}) - \frac{1}{6} R_2(\text{new})$

$R_3(\text{new}) = R_1(\text{old}) + 8\, R_2(\text{new})$

			x_1	x_2	x_3	x_4	x_5	a_1	a_2
	Basic variable	Constants	6	7	-3	0	0	M	M
7	x_2	$\frac{33}{23}$	$\frac{-108}{23 \times 6}$	1	0	$-4/23$	0	$4/23$	$\frac{-1}{23}$
-3	x_3	$\frac{9}{23}$	$\frac{-7}{23}$	0	1	$1/23$	0	$-1/23$	$6/23$
0	x_5	$\frac{72}{23}$	$\frac{243}{23}$	0	0	$8/23$	1	$-8/23$	$48/23$
~~Z_j~~	Z	$Z = \frac{204}{23} = 8.8696$	$-\frac{315}{69} = -4.56$	7	-3	$-\frac{31}{23} = -1.347$	0	$31/23 = 1.347$	$-25/23 = -1.088$
	$Z_j - C_j$		-10.56	0	0	-1.347	0	$-M + 1.347$	$-M - 1.087$

Here, all $Z_j - C_j \leq 0$ ∴ We reached optimal solution

∴ Min $Z = 8.8696$ when $x_1 = 0,\ x_2 = 1.43,\ x_3 = 0.39$

Q5) Show the calculations in the Cholesky factorization

$$\begin{bmatrix} 4 & 6 & 2 & -6 \\ 6 & 34 & 3 & -9 \\ 2 & 3 & 2 & -1 \\ -6 & -9 & -1 & 8 \end{bmatrix}$$

Sol:

To apply Cholesky factorization, the matrix need to be positive ~~semi~~ definite and symmetric

The given matrix is symmetric.

To find whether a matrix is positive definite, we need to find eigenvalues. A matrix is positive definite if all the eigen values are positive.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} (4-\lambda) & 6 & 2 & -6 \\ 6 & (34-\lambda) & 3 & -9 \\ 2 & 3 & (2-\lambda) & -1 \\ -6 & -9 & -1 & (8-\lambda) \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)\begin{vmatrix} (34-\lambda) & 3 & -9 \\ 3 & (2-\lambda) & -1 \\ -9 & -1 & (8-\lambda) \end{vmatrix} - 6\begin{vmatrix} 6 & 3 & -9 \\ 2 & (2-\lambda) & -1 \\ -6 & -1 & (8-\lambda) \end{vmatrix} + 2\begin{vmatrix} 6 & (34-\lambda) & 9 \\ 2 & 3 & -1 \\ -6 & -9 & (8-\lambda) \end{vmatrix}$$

$$+ 6\begin{vmatrix} 6 & (34-\lambda) & 3 \\ 2 & 3 & (2-\lambda) \\ -6 & -9 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda) \times (330 - 265\lambda + 44\lambda^2 - \lambda^3) - 6(-30 + 6\lambda^2)$$
$$+ 2(-250 + 60\lambda - 2\lambda^2) + 6(250 + 160\lambda - 6\lambda^2) = 0$$

$$\Rightarrow \lambda^2 - 48\lambda^3 + 365\lambda^2 - 310\lambda - 500 = 0$$

$$f(\lambda) = \underbrace{\lambda^4 - 48\lambda^3}_{1} + \underbrace{365\lambda^2 - 310\lambda}_{2,3} - 500 = 0$$

We need to find the signs of this equation to obtain the sign of eigen values.

∴ Applying Descarte's Rule of signs

No of positive real roots $\leq$ No of sign changes in $f(\lambda)$
≤ 3

No of negative real roots $\leq$ No of sign changes in $f(-\lambda)$

$$f(-\lambda) = \lambda^4 + 48\lambda^3 + 365\lambda^2 + \underbrace{310\lambda - 500}_{1} = 0$$

∴ No of negative real roots ≤ 1

Here we got no of negative & positive real roots ≤ 4

∴ We have all real roots.

∴ We have 3 positive roots and 1 negative root

∴ Matrix is not positive definite because we have 1 negative eigen value.

∴ We cannot apply Cholesky decomposition

Cholesky

```
A = [[4,6,2,-6],[6,34,3,-9],[2,3,2,-1],[-6,-9,-1,8]]
```

```
L = np.linalg.cholesky(A)
```

```
---------------------------------------------------------------------------
LinAlgError                               Traceback (most recent call last)
<ipython-input-6-a8e4a4945d52> in <module>()
----> 1 L = np.linalg.cholesky(A)

<__array_function__ internals> in cholesky(*args, **kwargs)

~/anaconda3/lib/python3.7/site-packages/numpy/linalg/linalg.py in cholesky(a)
    753     t, result_t = _commonType(a)
    754     signature = 'D->D' if isComplexType(t) else 'd->d'
--> 755     r = gufunc(a, signature=signature, extobj=extobj)
    756     return wrap(r.astype(result_t, copy=False))
    757 

~/anaconda3/lib/python3.7/site-packages/numpy/linalg/linalg.py in _raise_linalgerror_nonposdef(err, flag)
     98 
     99 def _raise_linalgerror_nonposdef(err, flag):
--> 100     raise LinAlgError("Matrix is not positive definite")
    101 
    102 def _raise_linalgerror_eigenvalues_nonconvergence(err, flag):

LinAlgError: Matrix is not positive definite
```

Q6) Show the SVD factorization of:

$$\begin{bmatrix} 2 & 5 & 6 \\ 0 & 1 & 5 \\ 3 & 9 & 4 \end{bmatrix}$$

Sol:

SVD Factorization

$A = UDV^T$ $\quad$ U, V – orthonormal vectors, D – Diagonal matrix

Method

1. V = eigenvectors(A^TA)

If eigenvectors of A^TA – v_1, v_2, v_3 $\quad$ $V = [v_1 \ v_2 \ v_3]$

2. D = eigenvalues(A^TA) as diagonal.

$$D = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$ where $\sigma_2 = \sqrt{\lambda_2}$ where λ – eigenvalue of A^TA.

3. $U = \left[\frac{1}{\sigma_1} A v_1 \quad \frac{1}{\sigma_2} A v_2 \quad \frac{1}{\sigma_3} A v_3\right]$

Calculation

$$A^TA = \begin{bmatrix} 2 & 0 & 3 \\ 5 & 1 & 9 \\ 6 & 5 & 4 \end{bmatrix} \begin{bmatrix} 2 & 5 & 6 \\ 0 & 1 & 5 \\ 3 & 9 & 4 \end{bmatrix} = \begin{bmatrix} 4+9 & 10+27 & 12+12 \\ 10+27 & 25+1+81 & 30+5+36 \\ 12+12 & 30+5+36 & 36+25+16 \end{bmatrix} = \begin{bmatrix} 13 & 37 & 24 \\ 37 & 107 & 71 \\ 24 & 71 & 77 \end{bmatrix}$$

Finding eigenvalues and eigen vectors of A^TA.

$|A^TA - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} (13-\lambda) & 37 & 24 \\ 37 & (107-\lambda) & 71 \\ 24 & 71 & (77-\lambda) \end{vmatrix} = 0$$

$\Rightarrow (13-\lambda)\left((107-\lambda)(77-\lambda) - 71\times71\right) - 37\left[37(77-\lambda) - 71\times24\right] + 24\left[37\times71 - 24(107-\lambda)\right] = 0$

$\Rightarrow \lambda^3 - 197\lambda^2 + 365\lambda - 625 = 0$

Roots of this equation found using Newton's method

$\lambda_3 = 0.1731, \ \lambda_2 = 20.4758, \ \lambda_1 = 176.3511$

Eigen vector for $\lambda_1 = 176.3511$

$$A'A - \lambda I = \begin{bmatrix} 13 & 37 & 24 \\ 37 & 107 & 71 \\ 24 & 71 & 77 \end{bmatrix} - 176.3511 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -163.3511 & 37 & 24 \\ 37 & -69.3511 & 71 \\ 24 & 71 & -99.3511 \end{bmatrix}$$

$R_1 \leftarrow R_1 \div -163.3511$

$$\begin{bmatrix} 1 & -0.2265 & -0.1469 \\ 37 & -69.3511 & 71 \\ 24 & 71 & -99.3511 \end{bmatrix}$$

$R_2 \leftarrow R_2 - 37 \times R_1$

$$\begin{bmatrix} 1 & -0.2265 & -0.1469 \\ 0 & -60.9704 & 76.4361 \\ 24 & 71 & -99.3511 \end{bmatrix}$$

$R_3 \leftarrow R_3 - 24 \times R_1$

$$\begin{bmatrix} 1 & -0.2265 & -0.1469 \\ 0 & -60.9704 & 76.4361 \\ 0 & 76.4361 & -95.825 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & -0.2265 & -0.1469 \\ 0 & 76.4361 & -95.825 \\ 0 & -60.9704 & 76.4361 \end{bmatrix}$$

$R_2 \leftarrow R_2 \div 76.4361$

$$\begin{bmatrix} 1 & -0.2265 & -0.1469 \\ 0 & 1 & -1.2537 \\ 0 & -60.9704 & 76.4361 \end{bmatrix}$$

$R_1 \leftarrow R_1 \div 0.2265 \times R_2$

$$= \begin{bmatrix} 1 & 0 & -0.4309 \\ 0 & 1 & -1.2537 \\ 0 & -60.9704 & 76.4361 \end{bmatrix}$$

$R_3 \leftarrow R_3 + 60.9704 \times R_2$

$$\begin{bmatrix} 1 & 0 & -0.4309 \\ 0 & 1 & -1.2537 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The eigen vector for $\lambda_1 = 176.3511$

$$v_1 = \begin{bmatrix} 0.4309 \\ 1.2537 \\ 1 \end{bmatrix}$$

Normalizing we get $v_1 = \begin{bmatrix} 0.2595 \\ 0.755 \\ 0.6022 \end{bmatrix}$

($\because$ They are orthonormal vectors)

By following similar procedure, we get:

Eigen vector after normalizing for $\lambda_2 = 20.4758$

$$v_2 = \begin{bmatrix} -0.273 \\ -0.7038 \\ 1 \end{bmatrix} \xrightarrow{\text{normalizing}} \begin{bmatrix} -0.2179 \\ -0.5617 \\ 0.7981 \end{bmatrix}$$

Eigen vector for $\lambda_3 = 0.1731$

$$v_3 = \begin{bmatrix} 50.1343 \\ -18.0289 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.9408 \\ -0.3383 \\ 0.0188 \end{bmatrix}$$

$$\therefore V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 0.2595 & -0.2179 & 0.9408 \\ 0.755 & -0.5617 & -0.3383 \\ 0.6022 & 0.7981 & 0.0188 \end{bmatrix}$$

$$D = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} \sqrt{176.3571} & 0 & 0 \\ 0 & \sqrt{20.4758} & 0 \\ 0 & 0 & \sqrt{0.1731} \end{bmatrix} = \begin{bmatrix} 13.2797 & 0 & 0 \\ 0 & 4.525 & 0 \\ 0 & 0 & 0.416 \end{bmatrix}$$

Finding U using formula $u_i = \frac{1}{\sigma_i} A \cdot v_i$

$$u_1 = \frac{1}{176.351} \begin{bmatrix} 2 & 5 & 6 \\ 0 & 1 & 5 \\ 3 & 9 & 4 \end{bmatrix}_{3\times3} \begin{bmatrix} 0.2595 \\ 0.755 \\ 0.6022 \end{bmatrix}_{3\times1} = \begin{bmatrix} 0.5954 \\ 0.2836 \\ 0.7517 \end{bmatrix}$$

$$u_2 = \frac{1}{20.4758} \begin{bmatrix} 2 & 5 & 6 \\ 0 & 1 & 5 \\ 3 & 9 & 4 \end{bmatrix} \begin{bmatrix} -0.2179 \\ -0.5617 \\ 0.7981 \end{bmatrix} = \begin{bmatrix} 0.3413 \\ 0.7577 \\ -0.5562 \end{bmatrix}$$

$$u_3 = \frac{1}{0.1731} \begin{bmatrix} 2 & 5 & 6 \\ 0 & 1 & 5 \\ 3 & 9 & 4 \end{bmatrix} \begin{bmatrix} 0.9408 \\ -0.3383 \\ 0.0188 \end{bmatrix} = \begin{bmatrix} 0.7281 \\ -0.5872 \\ -0.3536 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 0.5954 & 0.3413 & 0.7281 \\ 0.2836 & 0.7577 & -0.5872 \\ 0.7517 & -0.5562 & -0.3536 \end{bmatrix}$$

3.6 - SVD

```
A = [[2,5,6],[0,1,5],[3,9,4]]
```

```
U, D, Vh = np.linalg.svd(A, full_matrices=True)
```

```
U
```

```
array([[-0.59543443, -0.34125688, -0.72732495],
       [-0.28359671, -0.75774694,  0.58770101],
       [-0.75168527,  0.55620438,  0.35440929]])
```

```
D
```

```
array([13.27972522,  4.52502073,  0.41603546])
```

```
Vh
```

```
array([[-0.25948765, -0.7549807 , -0.60222114],
       [ 0.21792151,  0.56172296, -0.79810872],
       [-0.94083812,  0.3383363 , -0.01876637]])
```

```
All the values are same.Only signs have been reversed
```

Q7) If $D = D_{3\times3}$ is the diagonal matrix with entries d_1, d_2, d_3 what is $D^{-1}AD$? what are its eigen values in following case?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Sol: D is diagonal matrix $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$

D^{-1} - inverse of diagonal matrix (when no entries are zero) $D^{-1} = \begin{bmatrix} 1/d_1 & 0 & 0 \\ 0 & 1/d_2 & 0 \\ 0 & 0 & 1/d_3 \end{bmatrix}$

$D^{-1}AD$

$$\Rightarrow \begin{bmatrix} 1/d_1 & 0 & 0 \\ 0 & 1/d_2 & 0 \\ 0 & 0 & 1/d_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{1}{d_1} & \frac{1}{d_1} & \frac{1}{d_1} \\ \frac{1}{d_2} & \frac{1}{d_2} & \frac{1}{d_2} \\ \frac{1}{d_3} & \frac{1}{d_3} & \frac{1}{d_3} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & d_2/d_1 & d_3/d_1 \\ d_1/d_2 & 1 & d_3/d_2 \\ d_1/d_3 & d_2/d_3 & 1 \end{bmatrix}$$

$$\therefore D^{-1}AD = \begin{bmatrix} 1 & d_2/d_1 & d_3/d_1 \\ d_1/d_2 & 1 & d_3/d_2 \\ d_1/d_3 & d_2/d_3 & 1 \end{bmatrix}$$

Finding Eigen values:

$$\begin{vmatrix} 1-\lambda & d_2/d_1 & d_3/d_1 \\ d_1/d_2 & 1-\lambda & d_3/d_2 \\ d_1/d_3 & d_2/d_3 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(1-\lambda)^2 - \frac{d_3}{d_2}\times\frac{d_2}{d_3}\right] - \frac{d_2}{d_1}\left((1-\lambda)\frac{d_1}{d_2} - \frac{d_3}{d_2}\times\frac{d_1}{d_3}\right) + \frac{d_3}{d_1}\left(\frac{d_1}{d_2}\times\frac{d_2}{d_3} - (1-\lambda)\frac{d_1}{d_3}\right) = 0$$

$$\Rightarrow (1-\lambda)(\cancel{1} - 2\lambda + \lambda^2 - \cancel{1}) + 2\lambda = 0$$

$$\Rightarrow -2\cancel{\lambda} + \lambda^2 + 2\lambda^2 - \lambda^3 + \cancel{2\lambda} = 0$$

$$\Rightarrow -\lambda^3 + 3\lambda^2 = 0$$

$$\Rightarrow \lambda = 0, 0, 3$$

$\therefore$ Eigen values of $D^{-1}AD = 0, 0, 3$

Q8) Find solution of $Ax=b$ with the smallest value of $\|x\|_2 - c^Tx$

Problem
$$\min \|x\|_2 - c^Tx$$
$$\text{s.t } Ax=b$$

Here $c - n\times 1$, $A - m\times n$, $b - m\times 1$ & A is right invertible.

Sol:
$$\min \|x\|_2 - c^Tx$$
$$\text{s.t } Ax=b$$

Solving using Lagrain method.

$$L(x,\lambda) = x^Tx - c^Tx + \lambda^T(Ax-b)$$

$$\frac{\partial L}{\partial x} = 0$$

$$\Rightarrow 2x - c^* + A^T\lambda = 0$$

$$\Rightarrow x = \frac{-A^T\lambda}{2} + c^*$$

Substituting it in $Ax=b$

$$A\left(\frac{-A^T\lambda}{2} + c^*\right) = b$$

$$\Rightarrow \frac{-AA^T\lambda}{2} + Ac^* = b$$

$$\Rightarrow -AA^T\lambda = 2(b - Ac^*)$$

$$\Rightarrow \lambda = -2(AA^T)^{-1}(b - Ac^*)$$

[$\because$ Here $\underset{m\times n}{A}$ is right invertible
$\therefore A^*A^T$ is invertible $- m\times m$]

Substituting λ in x

$$\Rightarrow x = \frac{-A^T(-2(AA^T)^{-1}(b - Ac^*))}{2} + c^*$$

$$\cancel{\Rightarrow x = A^T(AA^T)^{-1}(b-Ac^T) + c^T}$$

$$\Rightarrow x = \underset{n\times m}{A^T}\underset{m\times m}{(AA^T)^{-1}}(\underset{m\times 1}{b} - \underset{m\times n}{A}\underset{n\times 1}{c^*}) + \underset{n\times 1}{c^*}$$

Q9) Write down dual of problem.

minimize $z = 6x_1 + 7x_2 - 3x_3$

s.t. $5x_1 - 6x_2 - x_3 \le -9$

$-2x_1 + x_2 + 4x_3 = 3$

$13x_1 - 8x_3 \le 0$

$x_1, x_2, x_3 \ge 0$

Sol: Converting problem to standard form for minimization.

$C_1: \; 5x_1 - 6x_2 - x_3 \le -9 \Rightarrow -5x_1 + 6x_2 + x_3 \ge 9$

$C_2: \; -2x_1 + x_2 + 4x_3 = 3$

$\Rightarrow -2x_1 + x_2 + 4x_3 \ge 3$ and $-2x_1 + x_2 + 4x_3 \le 3$

$\Downarrow$

$2x_1 - x_2 - 4x_3 \ge -3$

$C_3: \; 13x_1 - 8x_3 \le 0 \Rightarrow -13x_1 + 8x_3 \ge 0$

Writing Primal form in Matrix form | Dual form

Min $C^T X$

$AX \ge b$

$X \ge 0$

Max $C'^T Y$

$A' Y \le b'$

$Y \ge 0$

$$C = \begin{bmatrix} 6 & 7 & -3 \end{bmatrix}^T \xrightarrow[\text{Dual form}]{\text{Converting to}} C' = \begin{bmatrix} 9 & 3 & -3 & 0 \end{bmatrix}^T$$

$$A = \begin{bmatrix} -5 & 6 & 1 \\ -2 & 1 & 4 \\ 2 & -1 & -4 \\ -13 & 0 & 8 \end{bmatrix} \qquad A' = \begin{bmatrix} -5 & -2 & 2 & -13 \\ 6 & 1 & -1 & 0 \\ 1 & 4 & -4 & 8 \end{bmatrix}$$

$$b = \begin{bmatrix} 9 \\ 3 \\ -3 \\ 0 \end{bmatrix} \qquad b' = \begin{bmatrix} 6 \\ 7 \\ -3 \end{bmatrix}$$

Number of variables in DUAL
= Number of constraints in PRIMAL = 4

No. of constraints in DUAL
= No. of variables in PRIMAL = 3

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Equations in Dual form

maximize $z = 9y_1 + 3y_2 - 3y_3$

s.t $-5y_1 - 2y_2 + 2y_3 - 13y_4 \leq 6$

$6y_1 + y_2 - y_3 \leq 7$

$y_1 + 4y_2 - 4y_3 + 8y_4 \leq -3$

$y_1, y_2, y_3, y_4 \geq 0$