Question : Let $N(t)_{t\geq 0}$ be a Poisson process with rate 1. Consider the following statements.

(a) $pr\,(N(3) = 3|N(5) = 5) = {}^5C_3\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2$

(b) If S_5 denotes the time of occurrence of the 5^{th} event for the above Poisson process,then $E(S_5|N(5) = 3) = 7$

Which of the above statements is/are true?

(i) only (a)
(ii) only (b)
(iii) Both (a) and (b)
(iv) Neither (a) and (b)

Solution:

(a) Using the Poisson probability formula,

$$pr\,(N(t) = k) = Po(t;k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \tag{1}$$

here λ is 1

$$pr\,(N(t) = k) = \frac{(t)^k e^{-t}}{k!} \tag{2}$$

$$\tag{3}$$

X and Y are independent Poisson random variables, then $X + Y$ is also Poisson

$$\begin{aligned} pr\,(X = k, X+Y = n) &= pr\,(X = k, Y = n-k) && (4)\\ &= \frac{t_1^k}{k!}e^{-t_1}\frac{t_2^{n-k}}{(n-k)!}e^{-t_2} && (5)\\ &= e^{-(t_1+t_2)}\left(\frac{(t_1+t_2)^n}{n!}\right){}^nC_k\left(\frac{(t_1)}{t_1+t_2}\right)^k\left(\frac{(t_2)}{t_1+t_2}\right)^{n-k} && (6)\end{aligned}$$

$$pr\,(X+Y = n) = e^{-(t_1+t_2)}\left(\frac{(t_1+t_2)^n}{n!}\right) \tag{7}$$

From conditional probability,

$$\begin{aligned} pr\,(X = k|X+Y = n) &= \frac{pr\,(X = k, Y = n-k)}{pr\,(X+Y = n)} && (8)\\ &= {}^nC_k\left(\frac{(t_1)}{t_1+t_2}\right)^k\left(\frac{(t_2)}{t_1+t_2}\right)^{n-k} && (9)\end{aligned}$$

For the given question,

$$\begin{aligned} pr\,(N(3) = 3|N(5) = 5) &= {}^5C_3\left(\frac{3}{2+3}\right)^3\left(\frac{2}{2+3}\right)^2 && (10)\\ &= {}^5C_3\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2 && (11)\end{aligned}$$

Hence statement (a) is true.

(b) The expected value $E(S_n)$ of the time at which the n_{th} event occurs in a Poisson process with rate λ is

$$E(S_2) = \frac{n}{\lambda} \tag{12}$$

The conditional expectation $E(S_n|N(t) = x)$ represents the expected time at which the n^{th} event occurs at exactly x events have occurred in the first t units of time in a Poisson process with rate λ is given by.

$$E(S_n|N(t) = x) = t + E(S_{n-x}) \tag{13}$$

By the law of total expectation,

$$\begin{aligned}
E(S_n|N(t) = x) &= E(E(S_n|N(t) = x, N(t))) &(14)\\
&= E(E(S_x + S_{n-x}|N(t) = x, N(t) = x)) &(15)\\
&= E(E(S_x + S_{n-x}|N(t) = x)) &(16)\\
&= E(t + E(S_{n-x})) &(17)\\
&= E(t) + E(E(S_{n-x})) &(18)\\
&= t + E(S_{n-x}) &(19)
\end{aligned}$$

From above result,

$$\begin{aligned}
E(S_5|N(5) = 3) &= 5 + E(S_2) &(20)\\
&= 5 + 2 &(21)\\
&= 7 &(22)
\end{aligned}$$

Hence statement (b) is true.

Both (a) and (b) are true.