Question : Let $N(t)_{t\geq 0}$ be a Poisson process with rate 1. Consider the following statements.

(a) $P(N(3)=3|N(5)=5) = {}^5C_3\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2$

(b) If S_5 denotes the time of occurrence of the 5^{th} event for the above Poisson process,then $E(S_5|N(5)=3)=7$

Which of the above statements is/are true?

(i) only (a)
(ii) only (b)
(iii) Both (a) and (b)
(iv) Neither (a) and (b)

Solution:

(a) Using the Poisson probability formula,

$$P(N(t)=k) = \frac{k!(\lambda t)^k e^{-\lambda t}}{k!} \tag{1}$$

$$\tag{2}$$

here λ is 1

$$P(N(t)=k) = \frac{k!(t)^k e^{-t}}{k!} \tag{3}$$

$$P(N(3)=3) = \frac{3!(3)^3 e^{-3}}{3!} \tag{4}$$

$$P(N(5)=5) = \frac{5!(5)^5 e^{-5}}{5!} \tag{5}$$

From conditional probability,

$$P(N(3)=3|N(5)=5) = \frac{\frac{3!(3)^3e^{-3}}{3!}\frac{2!(2)^2e^{-2}}{2!}}{\frac{5!(5)^5e^{-5}}{5!}} \tag{6}$$

$$= \frac{(3)^3(2)^2}{(5)^5}\frac{5!}{3!2!} \tag{7}$$

$$= {}^5C_3\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2 \tag{8}$$

Hence statement (a) is true.

(b) The expected value $E(S_n)$ of the time at which the n_{th} event occurs in a Poisson process with rate λ is

$$E(S_2) = \frac{n}{\lambda} \tag{9}$$

The conditional expectation $E(S_n|N(t)=x)$ represents the expected time at which the n^{th} event occurs given that exactly x events have occurred in the first t units of time in a Poisson process with rate λ is given by.

$$E(S_n|N(t)=x) = t + E(S_{n-x}) \tag{10}$$

From the above statement,

$$E(S_5|N(5)=3) = 5 + E(S_2) \tag{11}$$

$$= 5+2 \tag{12}$$

$$= 7 \tag{13}$$

Hence statement (b) is true.