

Question : Let $N(t)_{t \geq 0}$ be a Poisson process with rate 1. Consider the following statements.

- (a) $\Pr(N(3) = 3 | N(5) = 5) = {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2$
 (b) If S_5 denotes the time of occurrence of the 5th event for the above Poisson process, then $E(S_5 | N(5) = 3) = 7$

Which of the above statements is/are true?

- (i) only (a)
 (ii) only (b)
 (iii) Both (a) and (b)
 (iv) Neither (a) and (b)

Solution:

Parameter	Values	Description
X	$N(t_1)$	poisson
Y	$N(t_2)$	random
$X + Y$	$N(t_1 + t_2)$	variables

TABLE 4

TABLE 1

(a) Using the Poisson probability formula,

$$\Pr(N(t) = k) = Po(t; k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad (1)$$

here λ is 1

$$\Pr(N(t) = k) = \frac{(t)^k e^{-t}}{k!} \quad (2)$$

$$(3)$$

X and Y are independent Poisson random variables, then $X + Y$ is also Poisson

$$\Pr(X = k, X + Y = n) = \Pr(X = k, Y = n - k) \quad (4)$$

$$= \frac{(t_1)^k}{k!} e^{-t_1} \frac{(t_2)^{n-k}}{(n-k)!} e^{-t_2} \quad (5)$$

$$= e^{-(t_1+t_2)} \left(\frac{(t_1 + t_2)^n}{n!} \right) {}^nC_k \left(\frac{t_1}{t_1 + t_2} \right)^k \left(\frac{t_2}{t_1 + t_2} \right)^{n-k} \quad (6)$$

$$\Pr(X + Y = n) = e^{-(t_1+t_2)} \left(\frac{(t_1 + t_2)^n}{n!} \right) \quad (7)$$

From conditional probability, from the equations (6) and (7)

$$\Pr(X = k | X + Y = n) = \frac{\Pr(X = k, Y = n - k)}{\Pr(X + Y = n)} \quad (8)$$

$$= {}^nC_k \left(\frac{t_1}{t_1 + t_2} \right)^k \left(\frac{t_2}{t_1 + t_2} \right)^{n-k} \quad (9)$$

For the given question,

Parameter	Values
t_1	3
t_1	5

TABLE (a)

TABLE 1

$$\Pr(N(3) = 3|N(5) = 5) = {}^5C_3 \left(\frac{3}{2+3} \right)^3 \left(\frac{2}{2+3} \right)^2 \quad (10)$$

$$= {}^5C_3 \left(\frac{3}{5} \right)^3 \left(\frac{2}{5} \right)^2 \quad (11)$$

Hence statement (a) is true.