

Question : Let  $N(t)_{t \geq 0}$  be a Poisson process with rate 1. Consider the following statements.

(a)  $\Pr(N(3) = 3 | N(5) = 5) = {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2$

(b) If  $S_5$  denotes the time of occurrence of the 5<sup>th</sup> event for the above Poisson process, then  $E(S_5 | N(5) = 3) = 7$

Which of the above statements is/are true?

- (i) only (a)
- (ii) only (b)
- (iii) Both (a) and (b)
- (iv) Neither (a) and (b)

( GATE ST 2023)

**Solution:**

(b)

Parameter	Description
$X$	poisson random variable
$\tau_1$	time interval to the first arrival
$S_n$	$n^{\text{th}}$ random arrival point of Poission process

TABLE 4

TABLE 1

A random variable  $X$  said to have a Poisson distribution  $Po(t)$  with parameter  $t \geq 0$ ,

$$\Pr(N(t) = k) = Po(t; k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad (1)$$

Let  $\tau_1$  denote the time interval to the first arrival from any fixed point  $t_0$

$N(t)$  is the number of events that occur in the time  $[0, t]$

$\Pr(N(t) = 0)$  is the probability that there are no events in the time interval  $[0, t]$

Distribution function of  $\tau_1$  is:

$$F_{\tau_1}(t) = \Pr(\tau_1 \leq t) \quad (2)$$

$$= 1 - \Pr(N(t_0, t_0 + t) = 0) \quad (3)$$

$$= 1 - e^{-\lambda t} \quad (4)$$

PDF of  $\tau_1$  is

$$f_{\tau_1}(t) = \frac{dF_{\tau_1}(t)}{dt} = \lambda e^{-\lambda t}, t \geq 0 \quad (5)$$

$S_n$  represent the  $n^{\text{th}}$  random arrival point of Poission process

$$S_n = \sum_{i=1}^n \tau_i \quad (6)$$

Then,

$$F_{S_n}(t) = \Pr(X < n) \quad (7)$$

$$= 1 - \Pr(X \geq n) \quad (8)$$

$$= 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad (9)$$

so we get PDF of  $S_n$  as

$$f_{S_n}(t) = \frac{dF_{t_n}(t)}{dt} \quad (10)$$

$$= \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \quad (11)$$

The expected value  $E(S_n)$  of the time at which the  $n_{th}$  event occurs in a Poisson process with rate  $\lambda$  is

$$E(S_n) = \int_0^\infty t f(t) dt \quad (12)$$

$$= \int_0^\infty \frac{\lambda^n t^n e^{-\lambda t}}{(n-1)!} dt \quad (13)$$

$$= \frac{\lambda^n}{(n-1)!} \int_0^\infty t^n e^{-\lambda t} dt \quad (14)$$

$$= \frac{\lambda^n}{(n-1)!} \frac{n!}{\lambda^{n+1}} \quad (15)$$

$$= \frac{n}{\lambda} \quad (16)$$

The conditional expectation  $E(S_n|N(t) = x)$  represents the expected time at which the  $n^{th}$  event occurs at exactly  $x$  events have occurred in the first  $t$  units of time in a Poisson process with rate  $\lambda$  is given by.

By the law of total expectation,

$$E(S_n|N(t) = x) = E(E(S_n|N(t) = x, N(t))) \quad (17)$$

$$= E(E(S_x + S_{n-x}|N(t) = x, N(t) = x)) \quad (18)$$

$$= E(E(S_x + S_{n-x}|N(t) = x)) \quad (19)$$

$$= E(t + E(S_{n-x})) \quad (20)$$

$$= E(t) + E(E(S_{n-x})) \quad (21)$$

$$= t + E(S_{n-x}) \quad (22)$$

From (16) and (22)

$$E(S_5|N(5) = 3) = 5 + E(S_2) \quad (23)$$

$$= 5 + 2 \quad (24)$$

$$= 7 \quad (25)$$

Hence statement (b) is true.