Question: Let $N(t)_{t\geq 0}$ be a Poisson process with rate 1. Consider the following statements.

- (a) $\Pr(N(3) = 3|N(5) = 5) = {}^5C_3\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2$ (b) If S_5 denotes the time of occurrence of the 5^{th} event for the above Poisson process,then $E(S_5|N(5) =$ 3) = 7

Which of the above statements is/are true?

- (i) only (a)
- (ii) only (b)
- (iii) Both (a) and (b)
- (iv) Neither (a) and (b)

Solution:

Parameter	Values	Description
X	$N(t_1)$	poisson
Y	$N(t_2)$	random
X + Y	$N(t_1 + t_2)$	variables

TABLE 4 Table 1

(a) Using the Poisson probability formula,

$$\Pr(N(t) = k) = Po(t; k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$
(1)

here λ is 1

$$pr(N(t) = k) = \frac{(t)^k e^{-t}}{k!}$$
 (2)

(3)

X and Y are independent Poisson random variables, then X + Y is also Poisson

$$Pr(X = k, X + Y = n) = pr(X = k, Y = n - k)$$
(4)

$$=\frac{(t_1)^k}{k!}e^{-t_1}\frac{(t_2)^{n-k}}{(n-k)!}e^{-t_2}$$
(5)

$$= e^{-(t_1+t_2)} \left(\frac{(t_1+t_2)^n}{n!} \right)^n C_k \left(\frac{t_1}{t_1+t_2} \right)^k \left(\frac{t_2}{t_1+t_2} \right)^{n-k}$$
 (6)

$$\Pr(X + Y = n) = e^{-(t_1 + t_2)} \left(\frac{(t_1 + t_2)^n}{n!} \right)$$
 (7)

From conditional probability, from the equations (6) and (7)

$$\Pr(X = k | X + Y = n) = \frac{\Pr(X = k, Y = n - k)}{\Pr(X + Y = n)}$$
(8)

$$= {}^{n}C_{k} \left(\frac{t_{1}}{t_{1} + t_{2}}\right)^{k} \left(\frac{t_{2}}{t_{1} + t_{2}}\right)^{n-k} \tag{9}$$

For the given question,

Parameter	Values	
t_1	3	
t_1	5	
TABLE (a)		
Table 1		

$$\Pr(N(3) = 3|N(5) = 5) = {}^{5}C_{3} \left(\frac{3}{2+3}\right)^{3} \left(\frac{2}{2+3}\right)^{2}$$
 (10)

$$= {}^{5}C_{3} \left(\frac{3}{5}\right)^{3} \left(\frac{2}{5}\right)^{2} \tag{11}$$

Hence statement (a) is true.

(b) The expected value $E(S_n)$ of the time at which the n_{th} event occurs in a Poisson process with rate λ is

$$E(S_n) = \frac{n}{\lambda} \tag{12}$$

PDF of gamma function is $\frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$

$$E(S_n) = \int_0^\infty t f(t) dt \tag{13}$$

$$= \int_0^\infty \frac{\lambda^n t^n e^{-\lambda t}}{(n-1)!} dt \tag{14}$$

$$= \frac{\lambda^n}{(n-1)!} \int_0^\infty t^n e^{-\lambda t} dt$$
 (15)

$$= \frac{\lambda^n}{(n-1)!} \frac{n!}{\lambda^{n+1}}$$

$$= \frac{n}{\lambda}$$
(16)

$$=\frac{n}{\lambda}\tag{17}$$

The conditional expectation $E(S_n|N(t)=x)$ represents the expected time at which the n^{th} event occurs at exactly x events have occurred in the first t units of time in a Poisson process with rate λ is given by.

$$E(S_n|N(t) = x) = t + E(S_{n-x})$$
(18)

from (17) and (18)

By the law of total expectation,

$$E(S_n|N(t) = x) = E(E(S_n|N(t) = x, N(t)))$$
(19)

$$= E(E(S_x + S_{n-x}|N(t) = x, N(t) = x))$$
(20)

$$= E(E(S_x + S_{n-x}|N(t) = x))$$
(21)

$$= E(t + E(S_{n-x})) \tag{22}$$

$$= E(t) + E(E(S_{n-x}))$$
 (23)

$$= t + E(S_{n-x}) \tag{24}$$

From above result,

$$E(S_5|N(5) = 3) = 5 + E(S_2)$$
(25)

$$=5+2\tag{26}$$

$$=7$$

Hence statement (b) is true.

Both (a) and (b) are true.