Question : Let $N(t)_{t\geq 0}$ be a Poisson process with rate 1. Consider the following statements.

(a) $\Pr(N(3) = 3|N(5) = 5) = {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2$

(b) If S_5 denotes the time of occurrence of the 5^{th} event for the above Poisson process,then $E(S_5|N(5) = 3) = 7$

Which of the above statements is/are true?

(i) only (a)
(ii) only (b)
(iii) Both (a) and (b)
(iv) Neither (a) and (b)

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Solution:
(b) S_n is the sum of n rv's, each with the density function

$$f_X(t) = \lambda e^{-\lambda t} \tag{1}$$

Poisson probability formula,

$$\Pr(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \tag{2}$$

$$\Pr(N(t) = 0) = e^{-\lambda t} \tag{3}$$

X is the time of the first event in a Poission process
$N(t)$ is the number of events that occur in the time $[0, t]$
$\Pr(N(t) = 0)$ is the probability that there are no events in the time interval $[0, t]$
In the poission process with intensity λ, the time of the first event is denoted by $X.N(t)$ is total number of events that occurred by time t

$$N(t) = I(t - X) \tag{4}$$

$I(t)$ is the indicator function

$$I(t) = \begin{cases} 1 & t \geq 0 \\ 0 & Otherwise \end{cases} \tag{5}$$

CDF of X is

$$F_X(t) = \Pr(X \leq t) \tag{6}$$

$$= 1 - \Pr(N(t) = 0) \tag{7}$$

$$= 1 - e^{-\lambda t} \tag{8}$$

PDF of X is

$$f_X(t) = \frac{d}{dt} F_X(t) = \lambda e^{-\lambda t} \tag{9}$$

PDF of sum of n independent and identically distributed exponential random variables, can obtained by convolving $f_X(t)$

$$f_{S_n}(t) = f_X(t) * f_X(t) * ... * f_X(t) \tag{10}$$

so we get PDF of S_n as

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \tag{11}$$

The expected value $E(S_n)$ of the time at which the n_{th} event occurs in a Poisson process with rate λ is

$$\begin{aligned}
E(S_n) &= \int_0^\infty t f(t)\, dt && (12)\\
&= \int_0^\infty \frac{\lambda^n t^n e^{-\lambda t}}{(n-1)!}\, dt && (13)\\
&= \frac{\lambda^n}{(n-1)!} \int_0^\infty t^n e^{-\lambda t}\, dt && (14)\\
&= \frac{\lambda^n}{(n-1)!} \frac{n!}{\lambda^{n+1}} && (15)\\
&= \frac{n}{\lambda} && (16)
\end{aligned}$$

The conditional expectation $E(S_n|N(t) = x)$ represents the expected time at which the n^{th} event occurs at exactly x events have occurred in the first t units of time in a Poisson process with rate λ is given by. By the law of total expectation,

$$\begin{aligned}
E(S_n|N(t) = x) &= E(E(S_n|N(t) = x, N(t))) && (17)\\
&= E(E(S_x + S_{n-x}|N(t) = x, N(t) = x)) && (18)\\
&= E(E(S_x + S_{n-x}|N(t) = x)) && (19)\\
&= E(t + E(S_{n-x})) && (20)\\
&= E(t) + E(E(S_{n-x})) && (21)\\
&= t + E(S_{n-x}) && (22)
\end{aligned}$$

From (22) and (**??**)

$$\begin{aligned}
E(S_5|N(5) = 3) &= 5 + E(S_2) && (23)\\
&= 5 + 2 && (24)\\
&= 7 && (25)
\end{aligned}$$

Hence statement (b) is true.