Question : Let $N(t)_{t\geq 0}$ be a Poisson process with rate 1. Consider the following statements.

(a) $\Pr(N(3) = 3|N(5) = 5) = {}^5C_3\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2$

(b) If S_5 denotes the time of occurrence of the 5^{th} event for the above Poisson process,then $E(S_5|N(5) = 3) = 7$

Which of the above statements is/are true?

(i) only (a)
(ii) only (b)
(iii) Both (a) and (b)
(iv) Neither (a) and (b)

(GATE ST 2023)

Solution:

(b) A random variable X said to have a Poisson distribution $Po(t)$ with parameter $t \geq 0$,

$$\Pr(X = k) = \Pr(N(t) = k) = Po(t; k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \tag{1}$$

$$\Pr(X = 0) = \Pr(N(t) = 0) = e^{-\lambda t} \tag{2}$$

Let τ_1 denote the time interval to the first arrival from any fixed point t_0
$N(t)$ is the number of events that occur in the time $[0, t]$
$\Pr(N(t) = 0)$ is the probability that there are no events in the time interval $[0, t]$
CDF of X is

$$\begin{aligned} F_{\tau_1}(t) &= \Pr(\tau_1 \leq t) = \Pr(X(t) > 0) && (3)\\ &= 1 - \Pr(N(t_0, t_0 + t) = 0) && (4)\\ &= 1 - e^{-\lambda t} && (5) \end{aligned}$$

PDF of τ_1 is

$$f_{\tau_1}(t) = \frac{dF_{\tau_1}(t)}{dt} = \lambda e^{-\lambda t}, t \geq 0 \tag{6}$$

S_n represent the n^{th} random arrival point of Poission process

$$S_n = \sum_{i=1}^{n} \tau_i \tag{7}$$

Then,

$$\begin{aligned} F_{S_n}(t) &= \Pr(X(t) < n) && (8)\\ &= 1 - \Pr(X(t) \geq n) && (9)\\ &= 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!} && (10) \end{aligned}$$

so we get PDF of S_n as

$$\begin{aligned} f_{S_n}(t) &= \frac{dF_{t_n}(t)}{dt} && (11)\\ &= \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} && (12) \end{aligned}$$

The expected value $E(S_n)$ of the time at which the n_{th} event occurs in a Poisson process with rate λ is

$$\begin{aligned}
E(S_n) &= \int_0^\infty t f(t)\, dt && (13)\\
&= \int_0^\infty \frac{\lambda^n t^n e^{-\lambda t}}{(n-1)!}\, dt && (14)\\
&= \frac{\lambda^n}{(n-1)!} \int_0^\infty t^n e^{-\lambda t}\, dt && (15)\\
&= \frac{\lambda^n}{(n-1)!} \frac{n!}{\lambda^{n+1}} && (16)\\
&= \frac{n}{\lambda} && (17)
\end{aligned}$$

The conditional expectation $E(S_n|N(t) = x)$ represents the expected time at which the n^{th} event occurs at exactly x events have occurred in the first t units of time in a Poisson process with rate λ is given by. By the law of total expectation,

$$\begin{aligned}
E(S_n|N(t) = x) &= E(E(S_n|N(t) = x, N(t))) && (18)\\
&= E(E(S_x + S_{n-x}|N(t) = x, N(t) = x)) && (19)\\
&= E(E(S_x + S_{n-x}|N(t) = x)) && (20)\\
&= E(t + E(S_{n-x})) && (21)\\
&= E(t) + E(E(S_{n-x})) && (22)\\
&= t + E(S_{n-x}) && (23)
\end{aligned}$$

From (17) and (23)

$$\begin{aligned}
E(S_5|N(5) = 3) &= 5 + E(S_2) && (24)\\
&= 5 + 2 && (25)\\
&= 7 && (26)
\end{aligned}$$

Hence statement (b) is true.