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Question : Let $N(t)_{t\geq0}$ be a Poisson process with rate 1. Consider the following statements.

(a) $\Pr(N(3) = 3|N(5) = 5) = {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2$

(b) If S_5 denotes the time of occurrence of the 5th event for the above Poisson process,then $E(S_5|N(5) = 3) = 7$

Which of the above statements is/are true?

(i) only (a)

(ii) only (b)

(iii) Both (a) and (b)

(iv) Neither (a) and (b)

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Solution:

(b)

Parameter	Description
X	poisson random variable
τ_1	time interval to the first arrival
S_n	n^{th} random arrival point of Poission process

TABLE 4

TABLE 1

A random variable X said to have a Poisson distribution $Po(t)$ with parameter $t \geq 0$,

$$\Pr(N(t) = k) = Po(t; k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \tag{1}$$

$$\Pr(N(t) = 0) = e^{-\lambda t} \tag{2}$$

Let τ_1 denote the time interval to the first arrival from any fixed point t_0

$N(t)$ is the number of events that occur in the time $[0, t]$

$\Pr(N(t) = 0)$ is the probability that there are no events in the time interval $[0, t]$

CDF of X is

$$F_{\tau_1}(t) = \Pr(\tau_1 \leq t) \tag{3}$$

$$= 1 - \Pr(N(t_0, t_0 + t) = 0) \tag{4}$$

$$= 1 - e^{-\lambda t} \tag{5}$$

PDF of τ_1 is

$$f_{\tau_1}(t) = \frac{dF_{\tau_1}(t)}{dt} = \lambda e^{-\lambda t}, t \geq 0 \tag{6}$$

S_n represent the nth random arrival point of Poission process

$$S_n = \sum_{i=1}^{n} \tau_i \tag{7}$$

Then,

$$F_{S_n}(t) = \Pr(X < n) \tag{8}$$

$$= 1 - \Pr(X \geq n) \tag{9}$$

$$= 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \tag{10}$$

so we get PDF of S_n as

$$f_{S_n}(t) = \frac{dF_{t_n}(t)}{dt} \tag{11}$$

$$= \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \tag{12}$$

The expected value $E(S_n)$ of the time at which the n_{th} event occurs in a Poisson process with rate λ is

$$E(S_n) = \int_0^\infty t f(t) \, dt \tag{13}$$

$$= \int_0^\infty \frac{\lambda^n t^n e^{-\lambda t}}{(n-1)!} \, dt \tag{14}$$

$$= \frac{\lambda^n}{(n-1)!} \int_0^\infty t^n e^{-\lambda t} \, dt \tag{15}$$

$$= \frac{\lambda^n}{(n-1)!} \frac{n!}{\lambda^{n+1}} \tag{16}$$

$$= \frac{n}{\lambda} \tag{17}$$

The conditional expectation $E(S_n|N(t) = x)$ represents the expected time at which the nth event occurs at exactly x events have occurred in the first t units of time in a Poisson process with rate λ is given by. By the law of total expectation,

$$E(S_n|N(t) = x) = E(E(S_n|N(t) = x, N(t))) \tag{18}$$

$$= E(E(S_x + S_{n-x}|N(t) = x, N(t) = x)) \tag{19}$$

$$= E(E(S_x + S_{n-x}|N(t) = x)) \tag{20}$$

$$= E(t + E(S_{n-x})) \tag{21}$$

$$= E(t) + E(E(S_{n-x})) \tag{22}$$

$$= t + E(S_{n-x}) \tag{23}$$

From (17) and (23)

$$E(S_5|N(5) = 3) = 5 + E(S_2) \tag{24}$$

$$= 5 + 2 \tag{25}$$

$$= 7 \tag{26}$$

Hence statement (b) is true.