Question : Let $N(t)_{t\geq 0}$ be a Poisson process with rate 1. Consider the following statements.

(a) $\Pr(N(3) = 3|N(5) = 5) = {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2$

(b) If S_5 denotes the time of occurrence of the 5^{th} event for the above Poisson process,then $E(S_5|N(5) = 3) = 7$

Which of the above statements is/are true?

(i) only (a)
(ii) only (b)
(iii) Both (a) and (b)
(iv) Neither (a) and (b)

Solution:

Parameter	Values	Description
X	$N(t_1)$	poisson
Y	$N(t_2)$	random
$X+Y$	$N(t_1+t_2)$	variables

TABLE 4
TABLE 1

(a) Using the Poisson probability formula,

$$\Pr(N(t) = k) = Po(t; k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \tag{1}$$

here λ is 1

$$pr(N(t) = k) = \frac{(t)^k e^{-t}}{k!} \tag{2}$$

$$\tag{3}$$

X and Y are independent Poisson random variables, then $X+Y$ is also Poisson

$$\Pr(X = k, X+Y = n) = pr(X = k, Y = n-k) \tag{4}$$

$$= \frac{(t_1)^k}{k!} e^{-t_1} \frac{(t_2)^{n-k}}{(n-k)!} e^{-t_2} \tag{5}$$

$$= e^{-(t_1+t_2)} \left(\frac{(t_1+t_2)^n}{n!}\right) {}^nC_k \left(\frac{t_1}{t_1+t_2}\right)^k \left(\frac{t_2}{t_1+t_2}\right)^{n-k} \tag{6}$$

$$\Pr(X+Y = n) = e^{-(t_1+t_2)} \left(\frac{(t_1+t_2)^n}{n!}\right) \tag{7}$$

From conditional probability, from the equations (6) and (7)

$$\Pr(X = k|X+Y = n) = \frac{\Pr(X = k, Y = n-k)}{\Pr(X+Y = n)} \tag{8}$$

$$= {}^nC_k \left(\frac{t_1}{t_1+t_2}\right)^k \left(\frac{t_2}{t_1+t_2}\right)^{n-k} \tag{9}$$

For the given question,

Parameter	Values
t_1	3
t_1	5

TABLE (a)
TABLE 1

$$\Pr(N(3) = 3|N(5) = 5) = {}^5C_3\left(\frac{3}{2+3}\right)^3\left(\frac{2}{2+3}\right)^2 \tag{10}$$

$$= {}^5C_3\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2 \tag{11}$$

Hence statement (a) is true.

(b) The expected value $E(S_n)$ of the time at which the n_{th} event occurs in a Poisson process with rate λ is

$$E(S_n) = \frac{n}{\lambda} \tag{12}$$

PDF of gamma function is $\frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$

$$E(S_n) = \int_0^\infty t f(t)\, dt \tag{13}$$

$$= \int_0^\infty \frac{\lambda^n t^n e^{-\lambda t}}{(n-1)!}\, dt \tag{14}$$

$$= \frac{\lambda^n}{(n-1)!}\int_0^\infty \frac{t^n e^{-\lambda t}}{}dt \tag{15}$$

$$= \frac{\lambda^n}{(n-1)!}\frac{n!}{\lambda^{n+1}} \tag{16}$$

$$= \frac{n}{\lambda} \tag{17}$$

The conditional expectation $E(S_n|N(t) = x)$ represents the expected time at which the n^{th} event occurs at exactly x events have occurred in the first t units of time in a Poisson process with rate λ is given by.

$$E(S_n|N(t) = x) = t + E(S_{n-x}) \tag{18}$$

from (17) and (18)
By the law of total expectation,

$$E(S_n|N(t) = x) = E(E(S_n|N(t) = x, N(t))) \tag{19}$$

$$= E(E(S_x + S_{n-x}|N(t) = x, N(t) = x)) \tag{20}$$

$$= E(E(S_x + S_{n-x}|N(t) = x)) \tag{21}$$

$$= E(t + E(S_{n-x})) \tag{22}$$

$$= E(t) + E(E(S_{n-x})) \tag{23}$$

$$= t + E(S_{n-x}) \tag{24}$$

From above result,

$$E(S_5|N(5) = 3) = 5 + E(S_2) \tag{25}$$

$$= 5 + 2 \tag{26}$$

$$= 7 \tag{27}$$

Hence statement (b) is true.
Both (a) and (b) are true.