

Question : Let  $N(t)_{t \geq 0}$  be a Poisson process with rate 1. Consider the following statements.

- (a)  $pr(N(3) = 3 | N(5) = 5) = {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2$   
 (b) If  $S_5$  denotes the time of occurrence of the 5<sup>th</sup> event for the above Poisson process, then  $E(S_5 | N(5) = 3) = 7$

Which of the above statements is/are true?

- (i) only (a)  
 (ii) only (b)  
 (iii) Both (a) and (b)  
 (iv) Neither (a) and (b)

**Solution:**

- (a) Using the Poisson probability formula,

$$pr(N(t) = k) = Po(t; k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad (1)$$

here  $\lambda$  is 1

$$pr(N(t) = k) = \frac{(t)^k e^{-t}}{k!} \quad (2)$$

$$(3)$$

$X$  and  $Y$  are independent Poisson random variables, then  $X + Y$  is also Poisson

$$pr(X = k, X + Y = n) = pr(X = k, Y = n - k) \quad (4)$$

$$= \frac{t_1^k}{k!} e^{-t_1} \frac{t_2^{n-k}}{(n-k)!} e^{-t_2} \quad (5)$$

$$= e^{-(t_1+t_2)} \left( \frac{(t_1+t_2)^n}{n!} \right) {}^nC_k \left( \frac{t_1}{t_1+t_2} \right)^k \left( \frac{t_2}{t_1+t_2} \right)^{n-k} \quad (6)$$

$$pr(X + Y = n) = e^{-(t_1+t_2)} \left( \frac{(t_1+t_2)^n}{n!} \right) \quad (7)$$

From conditional probability,

$$pr(X = k | X + Y = n) = \frac{pr(X = k, X + Y = n)}{pr(X + Y = n)} \quad (8)$$

$$= {}^nC_k \left( \frac{t_1}{t_1+t_2} \right)^k \left( \frac{t_2}{t_1+t_2} \right)^{n-k} \quad (9)$$

From given question,

$$pr(N(3) = 3 | N(5) = 5) = {}^5C_3 \left( \frac{3}{2+3} \right)^3 \left( \frac{2}{2+3} \right)^2 \quad (10)$$

$$= {}^5C_3 \left( \frac{3}{5} \right)^3 \left( \frac{2}{5} \right)^2 \quad (11)$$

Hence statement (a) is true.

(b) The expected value  $E(S_n)$  of the time at which the  $n_{th}$  event occurs in a Poisson process with rate  $\lambda$  is

$$E(S_2) = \frac{n}{\lambda} \quad (12)$$

The conditional expectation  $E(S_n|N(t) = x)$  represents the expected time at which the  $n^{th}$  event occurs given that exactly  $x$  events have occurred in the first  $t$  units of time in a Poisson process with rate  $\lambda$  is given by.

$$E(S_n|N(t) = x) = t + E(S_{n-x}) \quad (13)$$

By the law of total expectation,

$$E(S_n|N(t) = x) = E(E(S_n|N(t) = x, N(t))) \quad (14)$$

$$= E(E(S_x + S_{n-x}|N(t) = x, N(t) = x)) \quad (15)$$

$$= E(E(S_x + S_{n-x}|N(t) = x)) \quad (16)$$

$$= E(t + E(S_{n-x})) \quad (17)$$

$$= E(t) + E(E(S_{n-x})) \quad (18)$$

$$= t + E(S_{n-x}) \quad (19)$$

From above result,

$$E(S_5|N(5) = 3) = 5 + E(S_2) \quad (20)$$

$$= 5 + 2 \quad (21)$$

$$= 7 \quad (22)$$

Hence statement (b) is true.

Both (a) and (b) are true.