

Question : Let $N(t)_{t \geq 0}$ be a Poisson process with rate 1. Consider the following statements.

- (a) $\Pr(N(3) = 3 | N(5) = 5) = {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2$
 (b) If S_5 denotes the time of occurrence of the 5th event for the above Poisson process, then $E(S_5 | N(5) = 3) = 7$

Which of the above statements is/are true?

- (i) only (a)
 (ii) only (b)
 (iii) Both (a) and (b)
 (iv) Neither (a) and (b)

Solution:

Parameter	Values	Description
X	$N(t_1)$	poisson
Y	$N(t_2)$	random
$X + Y$	$N(t_1 + t_2)$	variables

TABLE 4

TABLE 1

(a) Using the Poisson probability formula,

$$\Pr(N(t) = k) = Po(t; k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad (1)$$

here λ is 1

$$pr(N(t) = k) = \frac{(t)^k e^{-t}}{k!} \quad (2)$$

$$(3)$$

X and Y are independent Poisson random variables, then $X + Y$ is also Poisson

$$\Pr(X = k, X + Y = n) = pr(X = k, Y = n - k) \quad (4)$$

$$= \frac{(t_1)^k}{k!} e^{-t_1} \frac{(t_2)^{n-k}}{(n-k)!} e^{-t_2} \quad (5)$$

$$= e^{-(t_1+t_2)} \left(\frac{(t_1 + t_2)^n}{n!} \right) {}^nC_k \left(\frac{t_1}{t_1 + t_2} \right)^k \left(\frac{t_2}{t_1 + t_2} \right)^{n-k} \quad (6)$$

$$\Pr(X + Y = n) = e^{-(t_1+t_2)} \left(\frac{(t_1 + t_2)^n}{n!} \right) \quad (7)$$

From conditional probability, from the equations (6) and (7)

$$\Pr(X = k | X + Y = n) = \frac{\Pr(X = k, Y = n - k)}{\Pr(X + Y = n)} \quad (8)$$

$$= {}^nC_k \left(\frac{t_1}{t_1 + t_2} \right)^k \left(\frac{t_2}{t_1 + t_2} \right)^{n-k} \quad (9)$$

For the given question,

Parameter	Values
t_1	3
t_1	5

TABLE (a)
TABLE 1

$$\Pr(N(3) = 3 | N(5) = 5) = {}^5C_3 \left(\frac{3}{2+3} \right)^3 \left(\frac{2}{2+3} \right)^2 \quad (10)$$

$$= {}^5C_3 \left(\frac{3}{5} \right)^3 \left(\frac{2}{5} \right)^2 \quad (11)$$

Hence statement (a) is true.

(b) The expected value $E(S_n)$ of the time at which the n_{th} event occurs in a Poisson process with rate λ is

$$E(S_n) = \frac{n}{\lambda} \quad (12)$$

PDF of gamma function is $\frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$

$$E(S_n) = \int_0^\infty t f(t) dt \quad (13)$$

$$= \int_0^\infty \frac{\lambda^n t^n e^{-\lambda t}}{(n-1)!} dt \quad (14)$$

$$= \frac{\lambda^n}{(n-1)!} \int_0^\infty t^n e^{-\lambda t} dt \quad (15)$$

$$= \frac{\lambda^n}{(n-1)!} \frac{n!}{\lambda^{n+1}} \quad (16)$$

$$= \frac{n}{\lambda} \quad (17)$$

The conditional expectation $E(S_n | N(t) = x)$ represents the expected time at which the n^{th} event occurs at exactly x events have occurred in the first t units of time in a Poisson process with rate λ is given by.

By the law of total expectation,

$$E(S_n | N(t) = x) = E(E(S_n | N(t) = x, N(t))) \quad (18)$$

$$= E(E(S_x + S_{n-x} | N(t) = x, N(t) = x)) \quad (19)$$

$$= E(E(S_x + S_{n-x} | N(t) = x)) \quad (20)$$

$$= E(t + E(S_{n-x})) \quad (21)$$

$$= E(t) + E(E(S_{n-x})) \quad (22)$$

$$= t + E(S_{n-x}) \quad (23)$$

From (17) and (23)

$$E(S_5 | N(5) = 3) = 5 + E(S_2) \quad (24)$$

$$= 5 + 2 \quad (25)$$

$$= 7 \quad (26)$$

Hence statement (b) is true.

Both (a) and (b) are true.