

Question : Let  $N(t)_{t \geq 0}$  be a Poisson process with rate 1. Consider the following statements.

- (a)  $pr(N(3) = 3 | N(5) = 5) = {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2$   
 (b) If  $S_5$  denotes the time of occurrence of the 5<sup>th</sup> event for the above Poisson process, then  $E(S_5 | N(5) = 3) = 7$

Which of the above statements is/are true?

- (i) only (a)  
 (ii) only (b)  
 (iii) Both (a) and (b)  
 (iv) Neither (a) and (b)

**Solution:**

- (a) Using the Poisson probability formula,

$$pr(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad (1)$$

here  $\lambda$  is 1

$$pr(N(t) = k) = \frac{(t)^k e^{-t}}{k!} \quad (2)$$

$$pr(N(3) = 3) = \frac{(3)^3 e^{-3}}{3!} \quad (3)$$

$$pr(N(5) = 5) = \frac{(5)^5 e^{-5}}{5!} \quad (4)$$

Conditional pmf Y given  $x = x_o$ ,  
 X and Y are equivalent events

$$pr(Y = y | X = x_o) = \frac{pr(Y = y, X = x_o)}{pr(X = x_o)} \quad (5)$$

$$= \frac{pr(Y = y, Z = x_o - y)}{pr(X = x_o)} \quad (6)$$

Y and Z are independent,

$$pr(Y = y, Z = x_o - y) = pr(Y = y) pr(Z = x_o - y) \quad (7)$$

From conditional probability,

$$pr(N(3) = 3 | N(5) = 5) = \frac{\frac{(3)^3 e^{-3}}{3!} \frac{(2)^2 e^{-2}}{2!}}{\frac{(5)^5 e^{-5}}{5!}} \quad (8)$$

$$= \frac{(3)^3 (2)^2}{(5)^5} \frac{5!}{3!2!} \quad (9)$$

$$= {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2 \quad (10)$$

Hence statement (a) is true.

- (b) The expected value  $E(S_n)$  of the time at which the  $n$ <sup>th</sup> event occurs in a Poisson process with rate  $\lambda$  is

$$E(S_2) = \frac{n}{\lambda} \quad (11)$$

The conditional expectation  $E(S_n | N(t) = x)$  represents the expected time at which the  $n$ <sup>th</sup> event occurs given that exactly  $x$  events have occurred in the first  $t$  units of time in a Poisson process with rate  $\lambda$  is given by.

$$E(S_n | N(t) = x) = t + E(S_{n-x}) \quad (12)$$

By the law of total expectation,

$$E(S_n | N(t) = x) = E(E(S_n | N(t) = x, N(t))) \quad (13)$$

$$= E(E(S_x + S_{n-x} | N(t) = x, N(t) = x)) \quad (14)$$

$$= E(E(S_x + S_{n-x} | N(t) = x)) \quad (15)$$

$$= E(t + E(S_{n-x})) \quad (16)$$

$$= E(t) + E(E(S_{n-x})) \quad (17)$$

$$= t + E(S_{n-x}) \quad (18)$$

From above result,

$$E(S_5 | N(5) = 3) = 5 + E(S_2) \quad (19)$$

$$= 5 + 2 \quad (20)$$

$$= 7 \quad (21)$$

Hence statement (b) is true.

Both (a) and (b) are true.