Question : Let $N(t)_{t\geq 0}$ be a Poisson process with rate 1. Consider the following statements.

(a) $pr\left(N(3) = 3|N(5) = 5\right) = {}^5C_3\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2$
(b) If S_5 denotes the time of occurrence of the 5^{th} event for the above Poisson process,then $E(S_5|N(5) = 3) = 7$

Which of the above statements is/are true?

(i) only (a)
(ii) only (b)
(iii) Both (a) and (b)
(iv) Neither (a) and (b)

Solution:

(a) Using the Poisson probability formula,

$$pr\left(N(t) = k\right) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \tag{1}$$

here λ is 1

$$pr\left(N(t) = k\right) = \frac{(t)^k e^{-t}}{k!} \tag{2}$$

$$pr\left(N(3) = 3\right) = \frac{(3)^3 e^{-3}}{3!} \tag{3}$$

$$pr\left(N(5) = 5\right) = \frac{(5)^5 e^{-5}}{5!} \tag{4}$$

$X = Y + Z$
Conditional pmf Y given $X = x_o$,
X and Y are equivalent events

$$pr\left(Y = y|X = x_o\right) = \frac{pr\left(Y = y, X = x_o\right)}{pr\left(X = x_o\right)} \tag{5}$$

$$= \frac{pr\left(Y = y, Z = x_o - y\right)}{pr\left(X = x_o\right)} \tag{6}$$

Y and Z are independent,

$$pr\left(Y = y, Z = x_o - y\right) = pr\left(Y = y\right) pr\left(Z = x_o - y\right) \tag{7}$$

$X, Y$and Z in poissions distribution,

$$pr\left(Y = y\right) = \frac{(\lambda)^y e^{-\lambda}}{y!} \tag{8}$$

$$pr\left(X = x_o\right) = \frac{(\lambda)^{x_o} e^{-\lambda}}{x_o!} \tag{9}$$

$$pr\left(Z = x_o - y\right) = \frac{(\lambda)^{x_o - y} e^{-\lambda}}{(x_o - y)!} \tag{10}$$

From conditional probability,

$$pr\left(N(3) = 3|N(5) = 5\right) = \frac{\frac{(3)^3 e^{-3}}{3!}\frac{(2)^2 e^{-2}}{2!}}{\frac{(5)^5 e^{-5}}{5!}} \tag{11}$$

$$= \frac{(3)^3(2)^2}{(5)^5}\frac{5!}{3!2!} \tag{12}$$

$$= {}^5C_3\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2 \tag{13}$$

Hence statement (a) is true.

(b) The expected value $E(S_n)$ of the time at which the n_{th} event occurs in a Poisson process with rate λ is

$$E(S_2) = \frac{n}{\lambda} \tag{14}$$

The conditional expectation $E(S_n|N(t) = x)$ represents the expected time at which the n^{th} event occurs given that exactly x events have occurred in the first t units of time in a Poisson process with rate λ is given by.

$$E(S_n|N(t) = x) = t + E(S_{n-x}) \tag{15}$$

By the law of total expectation,

$$E(S_n|N(t) = x) = E(E(S_n|N(t) = x, N(t))) \tag{16}$$

$$= E(E(S_x + S_{n-x}|N(t) = x, N(t) = x)) \tag{17}$$

$$= E(E(S_x + S_{n-x}|N(t) = x)) \tag{18}$$

$$= E(t + E(S_{n-x})) \tag{19}$$

$$= E(t) + E(E(S_{n-x})) \tag{20}$$

$$= t + E(S_{n-x}) \tag{21}$$

From above result,

$$E(S_5|N(5) = 3) = 5 + E(S_2) \tag{22}$$

$$= 5 + 2 \tag{23}$$

$$= 7 \tag{24}$$

Hence statement (b) is true.

Both (a) and (b) are true.