

Question : Let $N(t)_{t \geq 0}$ be a Poisson process with rate 1. Consider the following statements.

- (a) $P(N(3) = 3 | N(5) = 5) = {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2$
 (b) If S_5 denotes the time of occurrence of the 5th event for the above Poisson process, then $E(S_5 | N(5) = 3) = 7$

Which of the above statements is/are true?

- (i) only (a)
 (ii) only (b)
 (iii) Both (a) and (b)
 (iv) Neither (a) and (b)

Solution:

- (a) Using the Poisson probability formula,

$$P(N(t) = k) = \frac{k!(\lambda t)^k e^{-\lambda t}}{k!} \quad (1)$$

(2)

here λ is 1

$$P(N(t) = k) = \frac{k!(t)^k e^{-t}}{k!} \quad (3)$$

$$P(N(3) = 3) = \frac{3!(3)^3 e^{-3}}{3!} \quad (4)$$

$$P(N(5) = 5) = \frac{5!(5)^5 e^{-5}}{5!} \quad (5)$$

From conditional probability,

$$P(N(3) = 3 | N(5) = 5) = \frac{\frac{3!(3)^3 e^{-3}}{3!} \frac{2!(2)^2 e^{-2}}{2!}}{\frac{5!(5)^5 e^{-5}}{5!}} \quad (6)$$

$$= \frac{(3)^3 (2)^2}{(5)^5} \frac{5!}{3!2!} \quad (7)$$

$$= {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2 \quad (8)$$

Hence statement (a) is true.

- (b) The expected value $E(S_n)$ of the time at which the n th event occurs in a Poisson process with rate λ is

$$E(S_2) = \frac{n}{\lambda} \quad (9)$$

The conditional expectation $E(S_n | N(t) = x)$ represents the expected time at which the n th event occurs given that exactly x events have occurred in the first t units of time in a Poisson process with rate λ is given by.

$$E(S_n | N(t) = x) = t + E(S_{n-x}) \quad (10)$$

From the above statement,

$$E(S_5|N(5) = 3) = 5 + E(S_2) \quad (11)$$

$$= 5 + 2 \quad (12)$$

$$= 7 \quad (13)$$

Hence statement (b) is true.