Question : Let $N(t)_{t\geq 0}$ be a Poisson process with rate 1. Consider the following statements.

(a) $\Pr\left(N(3) = 3|N(5) = 5\right) = {}^5C_3\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2$

(b) If S_5 denotes the time of occurrence of the 5^{th} event for the above Poisson process,then $E(S_5|N(5) = 3) = 7$

Which of the above statements is/are true?

(i) only (a)
(ii) only (b)
(iii) Both (a) and (b)
(iv) Neither (a) and (b)

(GATE ST 2023)

Solution:
(b)

Parameter	Description
X	poisson random variable
τ_1	time interval to the first arrival
S_n	n^{th} random arrival point of Poission process
$N(t)$	Number of events that occur in the time $[0, t]$
λ	rate of Poission process

TABLE 4
TABLE 1

A random variable X said to have a Poisson distribution $Po(t)$ with parameter $t \geq 0$,

$$\Pr\left(X = k\right) = \Pr\left(N(t) = k\right) = Po(t; k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \tag{1}$$

Let τ_1 denote the time interval to the first arrival from any fixed point t_0
$N(t)$ is the number of events that occur in the time $[0, t]$
Distribution function of τ_1 is:

$$F_{\tau_1}(t) = \Pr\left(\tau_1 \leq t\right) \tag{2}$$
$$= 1 - \Pr\left(N(t_0, t_0 + t) = 0\right) \tag{3}$$
$$= 1 - e^{-\lambda t} \tag{4}$$

PDF of τ_1 is

$$f_{\tau_1}(t) = \frac{dF_{\tau_1}(t)}{dt} = \lambda e^{-\lambda t}, t \geq 0 \tag{5}$$

S_n represent the n^{th} random arrival point of Poission process

$$S_n = \sum_{i=1}^{n} \tau_i \tag{6}$$

Then,

$$F_{S_n}(t) = \Pr\left(X \geq n\right) \tag{7}$$
$$= 1 - \Pr\left(X < n\right) \tag{8}$$
$$= 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \tag{9}$$

so we get PDF of S_n as

$$f_{S_n}(t) = \frac{dF_{t_n}(t)}{dt} \tag{10}$$
$$= \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \tag{11}$$

The expected value $E(S_n)$ of the time at which the n_{th} event occurs in a Poisson process with rate λ is

$$E(S_n) = \int_0^\infty t f(t)\, dt \tag{12}$$
$$= \int_0^\infty \frac{\lambda^n t^n e^{-\lambda t}}{(n-1)!}\, dt \tag{13}$$
$$= \frac{\lambda^n}{(n-1)!} \int_0^\infty t^n e^{-\lambda t}\, dt \tag{14}$$
$$= \frac{\lambda^n}{(n-1)!} \frac{n!}{\lambda^{n+1}} \tag{15}$$
$$= \frac{n}{\lambda} \tag{16}$$

The conditional expectation $E(S_n|N(t) = x)$ represents the expected time at which the n^{th} event occurs at exactly x events have occurred in the first t units of time in a Poisson process with rate λ is given by. By the law of total expectation,

$$E(S_n|N(t) = x) = E(E(S_n|N(t) = x, N(t))) \tag{17}$$
$$= E(E(S_x + S_{n-x}|N(t) = x, N(t) = x)) \tag{18}$$
$$= E(E(S_x + S_{n-x}|N(t) = x)) \tag{19}$$
$$= E(t + E(S_{n-x})) \tag{20}$$
$$= E(t) + E(E(S_{n-x})) \tag{21}$$
$$= t + E(S_{n-x}) \tag{22}$$

From (16) and (22)

$$E(S_5|N(5) = 3) = 5 + E(S_2) \tag{23}$$
$$= 5 + 2 \tag{24}$$
$$= 7 \tag{25}$$

Hence statement (b) is true.