Question : Let $N(t)_{t \geq 0}$ be a Poisson process with rate 1. Consider the following statements.

(a) $pr\,(N(3) = 3|N(5) = 5) = {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2$

(b) If S_5 denotes the time of occurrence of the 5th event for the above Poisson process, then $E(S_5|N(5) = 3) = 7$

Which of the above statements is/are true?

 (i) only (a)
 (ii) only (b)
(iii) Both (a) and (b)
(iv) Neither (a) and (b)

Solution:

Parameter	Values	Description
X	$N(t_1)$	poisson random
Y	$N(t_2)$	variables
$X + Y$	$N(t_1 + t_2)$	variables

TABLE 4
TABLE 1

(a) Using the Poisson probability formula,

$$pr\,(N(t) = k) = Po(t; k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \tag{1}$$

here λ is 1

$$pr\,(N(t) = k) = \frac{(t)^k e^{-t}}{k!} \tag{2}$$

$$\tag{3}$$

X and Y are independent Poisson random variables, then $X + Y$ is also Poisson

$$pr\,(X = k, X + Y = n) = pr\,(X = k, Y = n - k) \tag{4}$$

$$= \frac{t_1^k}{k!} e^{-t_1} \frac{t_2^{n-k}}{(n-k)!} e^{-t_2} \tag{5}$$

$$= e^{-(t_1 + t_2)} \left(\frac{(t_1 + t_2)^n}{n!}\right) {}^nC_k \left(\frac{(t_1)}{t_1 + t_2}\right)^k \left(\frac{(t_2)}{t_1 + t_2}\right)^{n-k} \tag{6}$$

$$pr\,(X + Y = n) = e^{-(t_1 + t_2)} \left(\frac{(t_1 + t_2)^n}{n!}\right) \tag{7}$$

From conditional probability,

$$pr\,(X = k|X + Y = n) = \frac{pr\,(X = k, Y = n - k)}{pr\,(X + Y = n)} \tag{8}$$

$$= {}^nC_k \left(\frac{(t_1)}{t_1 + t_2}\right)^k \left(\frac{(t_2)}{t_1 + t_2}\right)^{n-k} \tag{9}$$

For the given question,

$$pr\,(N(3) = 3|N(5) = 5) = {}^5C_3 \left(\frac{3}{2+3}\right)^3 \left(\frac{2}{2+3}\right)^2 \tag{10}$$

$$= {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2 \tag{11}$$

Hence statement (a) is true.

(b) The expected value $E(S_n)$ of the time at which the n_{th} event occurs in a Poisson process with rate λ is

$$E(S_2) = \frac{n}{\lambda} \tag{12}$$

The conditional expectation $E(S_n|N(t) = x)$ represents the expected time at which the nth event occurs at exactly x events have occurred in the first t units of time in a Poisson process with rate λ is given by.

$$E(S_n|N(t) = x) = t + E(S_{n-x}) \tag{13}$$

By the law of total expectation,

$$E(S_n|N(t) = x) = E(E(S_n|N(t) = x, N(t))) \tag{14}$$

$$= E(E(S_x + S_{n-x}|N(t) = x, N(t) = x)) \tag{15}$$

$$= E(E(S_x + S_{n-x}|N(t) = x)) \tag{16}$$

$$= E(t + E(S_{n-x})) \tag{17}$$

$$= E(t) + E(E(S_{n-x})) \tag{18}$$

$$= t + E(S_{n-x}) \tag{19}$$

From above result,

$$E(S_5|N(5) = 3) = 5 + E(S_2) \tag{20}$$

$$= 5 + 2 \tag{21}$$

$$= 7 \tag{22}$$

Hence statement (b) is true.

Both (a) and (b) are true.