

Question : Let $N(t)_{t \geq 0}$ be a Poisson process with rate 1. Consider the following statements.

(a) $\Pr(N(3) = 3 | N(5) = 5) = {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2$

(b) If S_5 denotes the time of occurrence of the 5th event for the above Poisson process, then $E(S_5 | N(5) = 3) = 7$

Which of the above statements is/are true?

- (i) only (a)
- (ii) only (b)
- (iii) Both (a) and (b)
- (iv) Neither (a) and (b)

(GATE ST 2023)

Solution:

(b) S_n is the sum of n rv's, each with the density function

$$f_X(t) = \lambda e^{-\lambda t} \quad (1)$$

A random variable X said to have a Poisson distribution $Po(t)$ with parameter $t \geq 0$,

$$\Pr(X = k) = \Pr(N(t) = k) = Po(t; k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad (2)$$

$$\Pr(X = 0) = \Pr(N(t) = 0) = e^{-\lambda t} \quad (3)$$

X is the time of the first event in a Poisson process

$N(t)$ is the number of events that occur in the time $[0, t]$

$\Pr(N(t) = 0)$ is the probability that there are no events in the time interval $[0, t]$

CDF of X is

$$F_X(t) = \Pr(X \leq t) \quad (4)$$

$$= 1 - \Pr(N(t) = 0) \quad (5)$$

$$= 1 - e^{-\lambda t} \quad (6)$$

PDF of X is

$$f_X(t) = \frac{d}{dt} F_X(t) = \lambda e^{-\lambda t} \quad (7)$$

PDF of sum of n independent and identically distributed exponential random variables, can obtained by convolving $f_X(t)$

$$f_{S_n}(t) = f_X(t) * f_X(t) * \dots * f_X(t) \quad (8)$$

so we get PDF of S_n as

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \quad (9)$$

The expected value $E(S_n)$ of the time at which the n_{th} event occurs in a Poisson process with rate λ is

$$E(S_n) = \int_0^{\infty} t f(t) dt \quad (10)$$

$$= \int_0^{\infty} \frac{\lambda^n t^n e^{-\lambda t}}{(n-1)!} dt \quad (11)$$

$$= \frac{\lambda^n}{(n-1)!} \int_0^{\infty} t^n e^{-\lambda t} dt \quad (12)$$

$$= \frac{\lambda^n}{(n-1)!} \frac{n!}{\lambda^{n+1}} \quad (13)$$

$$= \frac{n}{\lambda} \quad (14)$$

The conditional expectation $E(S_n|N(t) = x)$ represents the expected time at which the n^{th} event occurs at exactly x events have occurred in the first t units of time in a Poisson process with rate λ is given by.

By the law of total expectation,

$$E(S_n|N(t) = x) = E(E(S_n|N(t) = x, N(t))) \quad (15)$$

$$= E(E(S_x + S_{n-x}|N(t) = x, N(t) = x)) \quad (16)$$

$$= E(E(S_x + S_{n-x}|N(t) = x)) \quad (17)$$

$$= E(t + E(S_{n-x})) \quad (18)$$

$$= E(t) + E(E(S_{n-x})) \quad (19)$$

$$= t + E(S_{n-x}) \quad (20)$$

From (14) and (20)

$$E(S_5|N(5) = 3) = 5 + E(S_2) \quad (21)$$

$$= 5 + 2 \quad (22)$$

$$= 7 \quad (23)$$

Hence statement (b) is true.