12.3 Rocket Propulsion

A rocket at time $t = 0$ is moving with speed $v_{r,0}$ in the positive x-direction. The rocket burns fuel that is then ejected backward with velocity $\vec{\mathbf{u}} = -u\hat{\mathbf{i}}$ relative to the rocket, where $u > 0$ is the relative speed of the ejected fuel. This exhaust velocity is independent of the velocity of the rocket. The rocket must exert a force to accelerate the ejected fuel backwards and therefore by Newton's Third law, the fuel exerts a force that is equal in magnitude but opposite in direction resulting in propelling the rocket forward. The rocket velocity is a function of time, $\vec{\mathbf{v}}_r(t) = v_{r,x}(t)\hat{\mathbf{i}}$, and the x-component increases at a rate $dv_{r,x} / dt$. Because fuel is leaving the rocket, the mass of the rocket is also a function of time, $m_r(t)$, and is decreasing at a rate dm_r / dt. We shall use the momentum principle, Eq. **Error! Reference source not found.**, to determine a differential equation that relates $dv_{r,x} / dt$, dm_r / dt, u, $v_{r,x}(t)$, and $F_{ext,x}$, an equation known as the rocket equation.

Let $t = t_i$ denote the instant the rocket begins to burn fuel and let $t = t_f$ denote the instant the rocket has finished burning fuel. At some arbitrary time t during this process, the rocket has velocity $\vec{\mathbf{v}}_r(t) = v_{r,x}(t)\hat{\mathbf{i}}$, with the mass of the rocket denoted by $m_r(t) \equiv m_r$. During the time interval $[t, t + \Delta t]$, with Δt taken to be a small interval (we shall eventually consider the limit that $\Delta t \to 0$), a small amount of fuel of mass Δm_f (in the limit that $\Delta t \to 0$, $\Delta m_f \to 0$) is ejected backwards with speed u relative to the rocket. The fuel was initially traveling at the speed of the rocket and so undergoes a change in momentum. The rocket recoils forward, undergoing a change in momentum. In order to keep track of all momentum changes, we define our system to be the rocket (including all the fuel that is not burned during the time interval Δt) and the small amount of fuel that is ejected during the interval Δt. At time t, the fuel has not yet been ejected so it is still inside the rocket. Figure 12.14 represents a momentum diagram at time t for our system relative to a fixed inertial reference frame in which the rocket at time t is moving with speed $v_{r,x}(t)$.

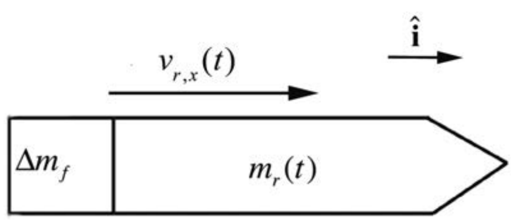

Figure 12.14 Momentum diagram for system at time t

The x-component of the momentum of the system at time t is therefore

$$p_{sys,x}(t) = (m_r(t) + \Delta m_f) v_{r,x}(t). \qquad (0.0.1)$$

During the interval $[t, t + \Delta t]$ the fuel is ejected backwards relative to the rocket with speed u. The rocket recoils forward with an increased x-component of the velocity $v_{r,x}(t + \Delta t) = v_{r,x}(t) + \Delta v_{r,x}$, where $\Delta v_{r,x}$ represents the increase the rocket's x-component of the velocity. As usual let's assume that the fuel element, with mass Δm_f, has left the rocket at the end of the time interval, so that the x-component of the velocity of the fuel is $v_{f,x} = v_{r,x} + \Delta v_{r,x} - u$. The momentum diagram of the system at time $t + \Delta t$ is shown in Figure 12.15.

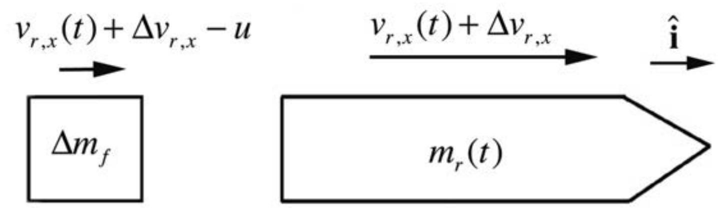

Figure 12.15 Momentum diagram for system at time $t + \Delta t$

The x-component of the momentum of the system at time $t + \Delta t$ is therefore

$$p_{sys,x}(t + \Delta t) = m_r(t)(v_{r,x}(t) + \Delta v_{r,x}) + \Delta m_f(v_{r,x}(t) + \Delta v_{r,x} - u). \qquad (0.0.2)$$

In Figure 12.16, we show the diagram depicting the change in the x-component of the momentum of the system consisting of the ejected fuel and rocket.

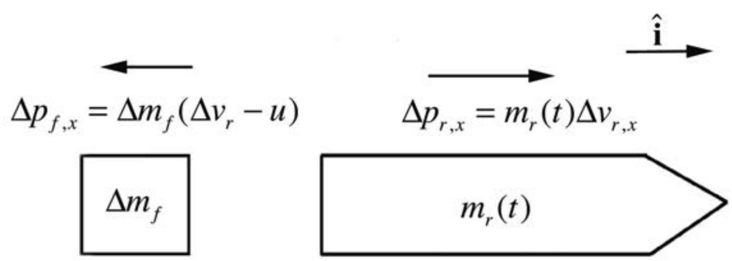

Figure 12.16 Change in momentum for system during time interval $[t, t + \Delta t]$

Therefore the change in the x-component of the momentum of the system is given by

$$\Delta p_{sys,x} = \Delta p_{r,x} + \Delta p_{f,x} = m_r(t)\Delta v_{r,x} + \Delta m_f(\Delta v_{r,x} - u). \qquad (0.0.3)$$

We again note that $\Delta p_{f,x} = \Delta m_f(\Delta v_{r,x} - u) \simeq -\Delta m_f u$, and we show the modified diagram for the change in the x-component of the momentum of the system in Figure 12.17.

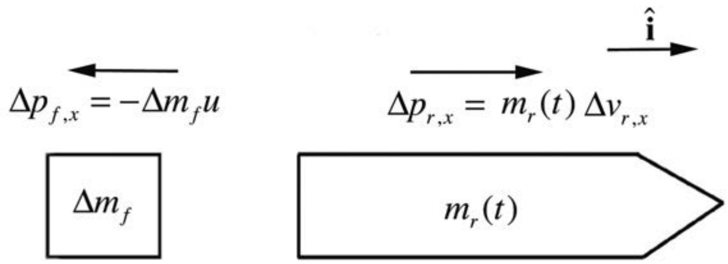

Figure 12.17 Modified change in momentum for system during time interval $[t, t+\Delta t]$

We can now apply Newton's Second Law in the form of the momentum principle (Eq. **Error! Reference source not found.**), for the system consisting of the rocket and exhaust fuel,

$$F_{ext,x} = \lim_{\Delta t \to 0} \frac{p_{sys,x}(t+\Delta t) - p_{sys,x}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta p_{sys,x}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta p_{r,x}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta p_{f,x}}{\Delta t}. \quad (0.0.4)$$

From our diagram depicting the change in the x-component of the momentum of the system, we have that

$$F_{ext,x} = \lim_{\Delta t \to 0} \frac{m_r(t)\Delta v_{r,x}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_f(\Delta v_{r,x} - u)}{\Delta t}. \quad (0.0.5)$$

We note that $\Delta m_f \Delta v_{r,x}$ is a second order differential, therefore

$$\lim_{\Delta t \to 0} \frac{\Delta m_f \Delta v_{r,x}}{\Delta t} = 0. \quad (0.0.6)$$

We also note that

$$\frac{dv_{r,x}}{dt} \equiv \lim_{\Delta t \to 0} \frac{\Delta v_{r,x}}{\Delta t}, \quad (0.0.7)$$

and

$$\frac{dm_f}{dt} \equiv \lim_{\Delta t \to 0} \frac{\Delta m_f}{\Delta t}. \quad (0.0.8)$$

Therefore Eq. (0.0.5) becomes

$$F_{ext,x} = m_r(t)\frac{dv_{r,x}}{dt} - \frac{dm_f}{dt}u. \quad (0.0.9)$$

We can rewrite Eq. (0.0.9) as

$$F_{ext,x} + \frac{dm_f}{dt}u = m_r(t)\frac{dv_{r,x}}{dt}. \quad (0.0.10)$$

The second term on the left-hand-side of Eq. (0.0.10) is called the ***thrust***

$$F_{thrust,x} \equiv \frac{dm_f}{dt} u \,. \qquad (0.0.11)$$

Note that this is not an extra force but the result of the forward recoil due to the ejection of the fuel. Because we are burning fuel at a positive rate $dm_f / dt > 0$ and the speed $u > 0$, the direction of the thrust is in the positive x-direction.

The rate of decrease of the mass of the rocket, dm_r / dt, is equal to the negative of the rate of increase of the exhaust fuel

$$\frac{dm_r}{dt} = -\frac{dm_f}{dt} \,. \qquad (0.0.12)$$

Therefore substituting Eq. (0.0.12) into Eq. (0.0.9), we find that the differential equation describing the motion of the rocket and exhaust fuel is given by

$$F_{ext,x} - \frac{dm_r}{dt} u = m_r(t) \frac{dv_{r,x}}{dt} \,. \qquad (0.0.13)$$

Eq. (0.0.13) is called the ***rocket equation***.

12.3.1 Rocket Equation in Gravity-free Space

We shall first consider the case in which there are no external forces acting on the system, then Eq. (0.0.13) becomes

$$-\frac{dm_r}{dt} u = m_r(t) \frac{dv_{r,x}}{dt} \,. \qquad (0.0.14)$$

In order to solve this equation, we separate the variable quantities $v_{r,x}(t)$ and $m_r(t)$

$$\frac{dv_{r,x}}{dt} = -\frac{u}{m_r(t)} \frac{dm_r}{dt} \,. \qquad (0.0.15)$$

We now multiply both sides by dt and integrate with respect to time between the initial time t_i when the ejection of the burned fuel began and the final time t_f when the process stopped.

$$\int_{t'=t_i}^{t'=t_f} \frac{dv_{r,x}}{dt'} dt' = -\int_{t'=t_i}^{t'=t_f} \frac{u}{m_r(t)} \frac{dm_r}{dt'} dt' \,. \qquad (0.0.16)$$

We can rewrite the integrands and endpoints as

$$\int_{v'_{r,x}=v_{r,x,i}}^{v'_{r,x}=v_{r,x,f}} dv'_{r,x} = -\int_{m'_r=m_{r,i}}^{m'_r=m_{r,f}} \frac{u}{m'_r}\,dm'_r \,. \tag{0.0.17}$$

Performing the integration and substituting in the values at the endpoints gives

$$v_{r,x,f} - v_{r,x,i} = -u\ln\left(\frac{m_{r,f}}{m_{r,i}}\right). \tag{0.0.18}$$

Because the rocket is losing fuel, $m_{r,f} < m_{r,i}$, we can rewrite Eq. (0.0.18) as

$$v_{r,x,f} - v_{r,x,i} = u\ln\left(\frac{m_{r,i}}{m_{r,f}}\right). \tag{0.0.19}$$

We note $\ln(m_{r,i} / m_{r,f}) > 1$. Therefore $v_{r,x,f} > v_{r,x,i}$, as we expect.

After a slight rearrangement of Eq. (0.0.19), we have an expression for the velocity of the rocket as a function of the mass m_r of the rocket

$$v_{r,x,f} = v_{r,x,i} + u\ln\left(\frac{m_{r,i}}{m_{r,f}}\right). \tag{0.0.20}$$

Let's examine our result. First, let's suppose that all the fuel was burned and ejected. Then $m_{r,f} \equiv m_{r,d}$ is the final dry mass of the rocket (empty of fuel). The ratio

$$R = \frac{m_{r,i}}{m_{r,d}} \tag{0.0.21}$$

is the ratio of the initial mass of the rocket (including the mass of the fuel) to the final dry mass of the rocket (empty of fuel). The final velocity of the rocket is then

$$v_{r,x,f} = v_{r,x,i} + u\ln R \,. \tag{0.0.22}$$

This is why multistage rockets are used. You need a big container to store the fuel. Once all the fuel is burned in the first stage, the stage is disconnected from the rocket. During the next stage the dry mass of the rocket is much less and so R is larger than the single stage, so the next burn stage will produce a larger final speed then if the same amount of fuel were burned with just one stage (more dry mass of the rocket). In general rockets do not burn fuel at a constant rate but if we assume that the burning rate is constant where

$$b = \frac{dm_f}{dt} = -\frac{dm_r}{dt} \tag{0.0.23}$$

then we can integrate Eq. (0.0.23)

$$\int_{m_r'=m_{r,i}}^{m_r'=m_r(t)} dm_r' = -b\int_{t'=t_i}^{t'=t} dt' \tag{0.0.24}$$

and find an equation that describes how the mass of the rocket changes in time

$$m_r(t) = m_{r,i} - b(t - t_i)\,. \tag{0.0.25}$$

For this special case, if we set $t_f = t$ in Eq. (0.0.20), then the velocity of the rocket as a function of time is given by

$$v_{r,x,f} = v_{r,x,i} + u\ln\left(\frac{m_{r,i}}{m_{r,i} - bt}\right). \tag{0.0.26}$$

Example 12.4 Single-Stage Rocket

Before a rocket begins to burn fuel, the rocket has a mass of $m_{r,i} = 2.81\times10^7\,\text{kg}$, of which the mass of the fuel is $m_{f,i} = 2.46\times10^7\,\text{kg}$. The fuel is burned at a constant rate with total burn time is $510\,\text{s}$ and ejected at a speed $u = 3000\text{ m/s}$ relative to the rocket. If the rocket starts from rest in empty space, what is the final speed of the rocket after all the fuel has been burned?

Solution: The dry mass of the rocket is $m_{r,d} \equiv m_{r,i} - m_{f,i} = 0.35\times10^7\,\text{kg}$, hence $R = m_{r,i} / m_{r,d} = 8.03$. The final speed of the rocket after all the fuel has burned is

$$v_{r,f} = \Delta v_r = u\ln R = 6250\text{ m/s}\ . \tag{0.0.27}$$

Example 12.5 Two-Stage Rocket

Now suppose that the same rocket in Example 12.4 burns the fuel in two stages ejecting the fuel in each stage at the same relative speed. In stage one, the available fuel to burn is $m_{f,1,i} = 2.03\times10^7\,\text{kg}$ with burn time $150\,\text{s}$. Then the empty fuel tank and accessories from stage one are disconnected from the rest of the rocket. These disconnected parts have a mass $m = 1.4\times10^6\,\text{kg}$. All the remaining fuel with mass is burned during the second stage with burn time of $360\,\text{s}$. What is the final speed of the rocket after all the fuel has been burned?

Solution: The mass of the rocket after all the fuel in the first stage is burned is $m_{r,1,d} = m_{r,1,i} - m_{f,1,i} = 0.78\times10^7\,\text{kg}$ and $R_1 = m_{r,1,i} / m_{r,1,d} = 3.60$. The change in speed after the first stage is complete is

$$\Delta v_{r,1} = u\ln R_1 = 3840\ \text{m/s}. \qquad (0.0.28)$$

After the empty fuel tank and accessories from stage one are disconnected from the rest of the rocket, the remaining mass of the rocket is $m_{r,2,d} = 2.1\times10^6\,\text{kg}$. The remaining fuel has mass $m_{f,2,i} = 4.3\times10^6\,\text{kg}$. The mass of the rocket plus the unburned fuel at the beginning of the second stage is $m_{r,2,i} = 6.4\times10^6\,kg$. Then $R_2 = m_{r,2,i} / m_{r,2,d} = 3.05$. Therefore the rocket increases its speed during the second stage by an amount

$$\Delta v_{r,2} = u\ln R_2 = 3340\ \text{m/s}. \qquad (0.0.29)$$

The final speed of the rocket is the sum of the change in speeds due to each stage,

$$v_f = \Delta v_r = u\ln R_1 + u\ln R_2 = u\ln(R_1R_2) = 7190\ \text{m/s}, \qquad (0.0.30)$$

which is greater than if the fuel were burned in one stage. Plots of the speed of the rocket as a function time for both one-stage and two-stage burns are shown Figure 12.18.

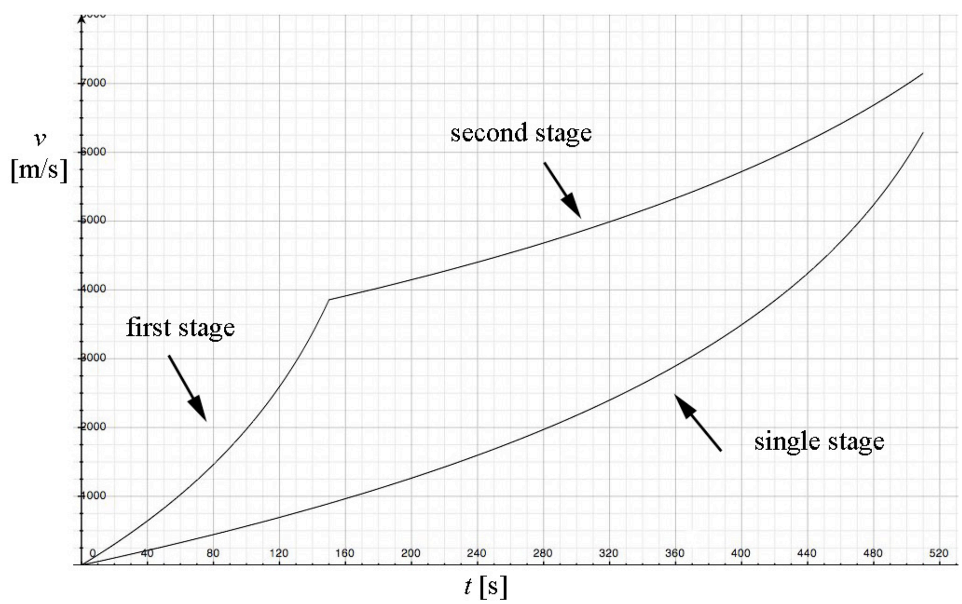

Figure 12.18 Plots of speed of rocket for both one-stage burn and two-stage burn

12.3.2 Rocket in a Constant Gravitational Field:

Now suppose that the rocket takes off from rest at time $t = 0$ in a constant gravitational field then the external force is

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = m_r \vec{\mathbf{g}}\,. \tag{0.0.31}$$

Choose the positive x-axis in the upward direction then $F_{ext,x}(t) = -m_r(t)g$. Then the rocket equation (Eq. (0.0.13) becomes

$$-m_r(t)g - \frac{dm_r}{dt}u = m_r(t)\frac{dv_{r,x}}{dt}\,. \tag{0.0.32}$$

Multiply both sides of Eq. (0.0.32) by dt, and divide both sides by $m_r(t)$. Then Eq. (0.0.32) can be written as

$$dv_{r,x} = -gdt - \frac{dm_r}{m_r(t)}u\,. \tag{0.0.33}$$

We now integrate both sides

$$\int_{v_{r,x,i}=0}^{v_{r,x}(t)} dv'_{r,x} = -u\int_{m_{r,i}}^{m_r(t)} \frac{dm'_r}{m'_r} - g\int_0^t dt'\,, \tag{0.0.34}$$

where $m_{r,i}$ is the initial mass of the rocket and the fuel. Integration yields

$$v_{r,x}(t) = -u\ln\left(\frac{m_r(t)}{m_{r,i}}\right) - gt = u\ln\left(\frac{m_{r,i}}{m_r(t)}\right) - gt\,. \tag{0.0.35}$$

After all the fuel is burned at $t = t_f$, the mass of the rocket is equal to the dry mass $m_{r,f} = m_{r,d}$ and so

$$v_{r,x}(t_f) = u\ln R - gt_f\,. \tag{0.0.36}$$

The first term on the right hand side is independent of the burn time. However the second term depends on the burn time. The shorter the burn time, the smaller the negative contribution from the third turn, and hence the larger the final speed. So the rocket engine should burn the fuel as fast as possible in order to obtain the maximum possible speed.