Chapter 6 Circular Motion

Chapter 6 Central Motion

And the seasons they go round and round
And the painted ponies go up and down
We're captive on the carousel of time
We can't return we can only look
Behind from where we came
And go round and round and round
In the circle game [1]

Joni Mitchell

6.1 Introduction

We shall now investigate a special class of motions, motion in a plane about a central point, a motion we shall refer to as central motion, the most outstanding case of which is circular motion. Special cases often dominate our study of physics, and circular motion about a central point is certainly no exception. There are many instances of central motion about a point; a bicycle rider on a circular track, a ball spun around by a string, and the rotation of a spinning wheel are just a few examples. Various planetary models described the motion of planets in circles before any understanding of gravitation. The motion of the moon around the earth is nearly circular. The motions of the planets around the sun are nearly circular. Our sun moves in nearly a circular orbit about the center of our galaxy, 50,000 light years from a massive black hole at the center of the galaxy. When Newton solved the two-body under a gravitational central force, he discovered that the orbits can be circular, elliptical, parabolic or hyperbolic. All of these orbits still display central force motion about the center of mass of the two-body system. Another example of central force motion is the scattering of particles by a Coulombic central force, for example Rutherford scattering of an alpha particle (two protons and two neutrons bound together into a particle identical to a helium nucleus) against an atomic nucleus such as a gold nucleus.

We shall begin by describing the kinematics of circular motion, the position, velocity, and acceleration, as a special case of two-dimensional motion. We will see that unlike linear motion, where velocity and acceleration are directed along the line of motion, in circular motion the direction of velocity is always tangent to the circle. This means that as the object moves in a circle, the direction of the velocity is always changing. When we examine this motion, we shall see that the direction of the change of the velocity is towards the center of the circle. This means that there is a non-zero component of the acceleration directed radially inward, which is called the *centripetal acceleration*. If our object is increasing its speed or slowing down, there is also a non-zero *tangential acceleration* in the direction of motion. But when the object is moving at a constant speed in a circle then only the centripetal acceleration is non-zero.

[1] Joni Mitchell, *The Circle Game,* Siquomb Publishing Company.

In 1666, twenty years before Newton published his *Principia*, he realized that the moon is always "falling" towards the center of the earth; otherwise, by the First Law, it would continue in some linear trajectory rather than follow a circular orbit. Therefore there must be a *centripetal force*, a radial force pointing inward, producing this centripetal acceleration.

In all of these instances, when an object is constrained to move in a circle, there must exist a force $\vec{\mathbf{F}}$ acting on the object directed towards the center. Because Newton's Second Law is a vector equality, the radial component of the Second Law is

$$F_r = m a_r . \tag{6.1.1}$$

6.2 Circular Motion: Velocity and Angular Velocity

We begin our description of circular motion by choosing polar coordinates. In Figure 6.1 we sketch the position vector $\vec{\mathbf{r}}(t)$ of the object moving in a circular orbit of radius r.

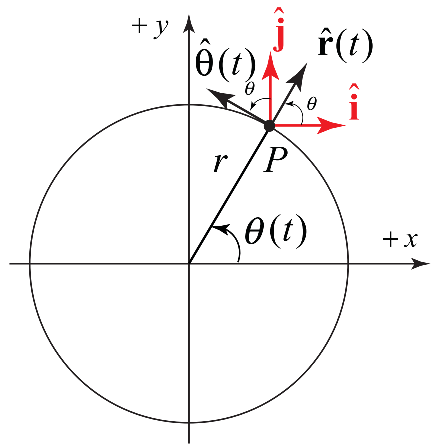

Figure 6.1 A circular orbit with unit vectors.

At time t, the particle is located at the point P with coordinates $(r,\theta(t))$ and position vector given by

$$\vec{\mathbf{r}}(t) = r\,\hat{\mathbf{r}}(t) . \tag{6.2.1}$$

At the point P, consider two sets of unit vectors ($\hat{\mathbf{r}}(t)$, $\hat{\boldsymbol{\theta}}(t)$) and ($\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$), as shown in Figure 6.1. The vector decomposition expression for $\hat{\mathbf{r}}(t)$ and $\hat{\boldsymbol{\theta}}(t)$ in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ is given by

$$\hat{\mathbf{r}}(t) = \cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}} , \tag{6.2.2}$$

$$\hat{\boldsymbol{\theta}}(t) = -\sin\theta(t)\,\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}} . \tag{6.2.3}$$

Before we calculate the velocity, we shall calculate the time derivatives of Eqs. (6.2.2) and (6.2.3). Let's first begin with $d\hat{\mathbf{r}}(t)/dt$:

$$\frac{d\hat{\mathbf{r}}(t)}{dt} = \frac{d}{dt}(\cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}}) = (-\sin\theta(t)\frac{d\theta(t)}{dt}\,\hat{\mathbf{i}} + \cos\theta(t)\frac{d\theta(t)}{dt}\,\hat{\mathbf{j}})$$
$$= \frac{d\theta(t)}{dt}(-\sin\theta(t)\,\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}) = \frac{d\theta(t)}{dt}\hat{\boldsymbol{\theta}}(t) \quad ; \tag{6.2.4}$$

where we used the chain rule to calculate that

$$\frac{d}{dt}\cos\theta(t) = -\sin\theta(t)\frac{d\theta(t)}{dt} \ , \tag{6.2.5}$$

$$\frac{d}{dt}\sin\theta(t) = \cos\theta(t)\frac{d\theta(t)}{dt} \ . \tag{6.2.6}$$

The calculation for $d\hat{\boldsymbol{\theta}}(t)/dt$ is similar:

$$\frac{d\hat{\boldsymbol{\theta}}(t)}{dt} = \frac{d}{dt}(-\sin\theta(t)\,\hat{\mathbf{i}} + \cos\theta(t\,\hat{\mathbf{j}}) = (-\cos\theta(t)\frac{d\theta(t)}{dt}\,\hat{\mathbf{i}} - \sin(t)\frac{d\theta(t)}{dt}\,\hat{\mathbf{j}})$$
$$= \frac{d\theta(t)}{dt}(-\cos\theta(t)\,\hat{\mathbf{i}} - \sin\theta(t)\,\hat{\mathbf{j}}) = -\frac{d\theta(t)}{dt}\hat{\mathbf{r}}(t) \quad . \tag{6.2.7}$$

The velocity vector is then

$$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{r}}(t)}{dt} = r\frac{d\hat{\mathbf{r}}}{dt} = r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}(t) = v_\theta\,\hat{\boldsymbol{\theta}}(t), \tag{6.2.8}$$

where the $\hat{\boldsymbol{\theta}}$-component of the velocity is given by

$$v_\theta = r\frac{d\theta}{dt}, \tag{6.2.9}$$

a quantity we shall refer to as the *tangential component of the velocity*. Denote the magnitude of the velocity by $v \equiv \left|\vec{\mathbf{v}}\right|$, The angular speed is the magnitude of the rate of change of angle with respect to time, which we denote by the Greek letter ω,

$$\omega \equiv \left|\frac{d\theta}{dt}\right|. \tag{6.2.10}$$

6.2.1 Geometric Derivation of the Velocity for Circular Motion

Consider a particle undergoing circular motion. At time t, the position of the particle is $\vec{\mathbf{r}}(t)$. During the time interval Δt, the particle moves to the position $\vec{\mathbf{r}}(t+\Delta t)$ with a displacement $\Delta\vec{\mathbf{r}}$.

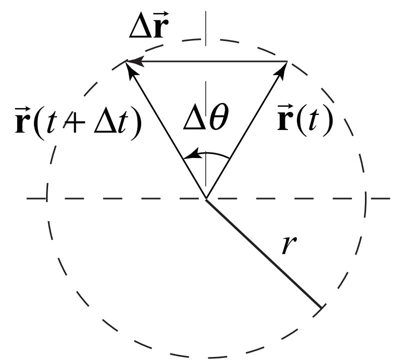

Figure 6.2 Displacement vector for circular motion

The magnitude of the displacement, $|\Delta\vec{\mathbf{r}}|$, is represented by the length of the horizontal vector $\Delta\vec{\mathbf{r}}$ joining the heads of the displacement vectors in Figure 6.2 and is given by

$$|\Delta\vec{\mathbf{r}}| = 2r\sin(\Delta\theta/2). \tag{6.2.11}$$

When the angle $\Delta\theta$ is small, we can approximate

$$\sin(\Delta\theta/2) \cong \Delta\theta/2. \tag{6.2.12}$$

This is called the *small angle approximation*, where the angle $\Delta\theta$ (and hence $\Delta\theta/2$) is measured in radians. This fact follows from an infinite power series expansion for the sine function given by

$$\sin\left(\frac{\Delta\theta}{2}\right) = \frac{\Delta\theta}{2} - \frac{1}{3!}\left(\frac{\Delta\theta}{2}\right)^3 + \frac{1}{5!}\left(\frac{\Delta\theta}{2}\right)^5 - \cdots. \tag{6.2.13}$$

When the angle $\Delta\theta/2$ is small, only the first term in the infinite series contributes, as successive terms in the expansion become much smaller. For example, when $\Delta\theta/2 = \pi/30 \cong 0.1$, corresponding to 6°, $(\Delta\theta/2)^3/3! \cong 1.9\times10^{-4}$; this term in the power series is three orders of magnitude smaller than the first and can be safely ignored for small angles.

Using the small angle approximation, the magnitude of the displacement is

$$|\Delta\vec{\mathbf{r}}| \cong r\,\Delta\theta. \tag{6.2.14}$$

This result should not be too surprising since in the limit as $\Delta\theta$ approaches zero, the length of the chord approaches the arc length $r\,\Delta\theta$.

The magnitude of the velocity, v, is proportional to the rate of change of the magnitude of the angle with respect to time,

$$v \equiv \left|\vec{\mathbf{v}}(t)\right| = \lim_{\Delta t\to 0}\frac{\left|\Delta\vec{\mathbf{r}}\right|}{\Delta t} = \lim_{\Delta t\to 0}\frac{r\left|\Delta\theta\right|}{\Delta t} = r\lim_{\Delta t\to 0}\frac{\left|\Delta\theta\right|}{\Delta t} = r\left|\frac{d\theta}{dt}\right| = r\omega\ . \tag{6.2.15}$$

The direction of the velocity can be determined by considering that in the limit as $\Delta t \to 0$ (note that $\Delta\theta \to 0$), the direction of the displacement $\Delta\vec{\mathbf{r}}$ approaches the direction of the tangent to the circle at the position of the particle at time t (Figure 6.3).

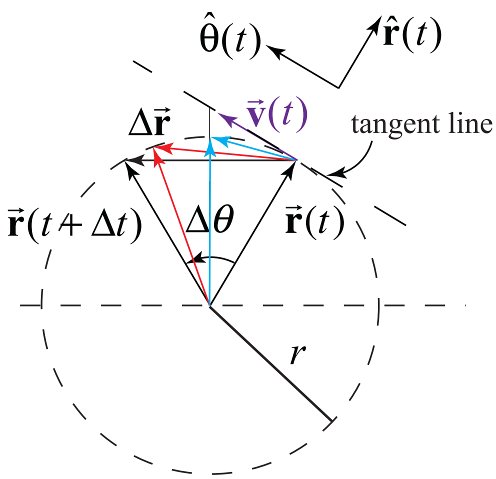

Figure 6.3 Direction of the displacement approaches the direction of the tangent line

Thus, in the limit $\Delta t \to 0$, $\Delta\vec{\mathbf{r}} \perp \vec{\mathbf{r}}$, and so the direction of the velocity $\vec{\mathbf{v}}(t)$ at time t is perpendicular to the position vector $\vec{\mathbf{r}}(t)$ and tangent to the circular orbit in the $+\hat{\boldsymbol{\theta}}$-direction for the case shown in Figure 6.3.

6.3 Circular Motion: Tangential and Radial Acceleration

When the motion of an object is described in polar coordinates, the acceleration has two components, the tangential component a_θ, and the radial component, a_r. We can write the acceleration vector as

$$\vec{\mathbf{a}} = a_r\,\hat{\mathbf{r}}(t) + a_\theta\,\hat{\boldsymbol{\theta}}(t)\ . \tag{6.3.1}$$

Keep in mind that as the object moves in a circle, the unit vectors $\hat{\mathbf{r}}(t)$ and $\hat{\boldsymbol{\theta}}(t)$ change direction and hence are not constant in time.

We will begin by calculating the tangential component of the acceleration for circular motion. Suppose that the tangential velocity $v_\theta = r\,d\theta / dt$ is changing in magnitude due to the presence of some tangential force; we shall now consider that $d\theta / dt$ is changing in time, (the magnitude of the velocity is changing in time). Recall that in polar coordinates the velocity vector Eq. (6.2.8) can be written as

$$\vec{\mathbf{v}}(t) = r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}(t)\,. \tag{6.3.2}$$

We now use the product rule to determine the acceleration.

$$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}(t)}{dt} = r\frac{d^2\theta(t)}{dt^2}\hat{\boldsymbol{\theta}}(t) + r\frac{d\theta(t)}{dt}\frac{d\hat{\boldsymbol{\theta}}(t)}{dt}\,. \tag{6.3.3}$$

Recall from Eq. (6.2.3) that $\hat{\boldsymbol{\theta}}(t) = -\sin\theta(t)\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}$. So we can rewrite Eq. (6.3.3) as

$$\vec{\mathbf{a}}(t) = r\frac{d^2\theta(t)}{dt^2}\hat{\boldsymbol{\theta}}(t) + r\frac{d\theta(t)}{dt}\frac{d}{dt}(-\sin\theta(t)\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}})\,. \tag{6.3.4}$$

We again use the chain rule (Eqs. (6.2.5) and (6.2.6)) and find that

$$\vec{\mathbf{a}}(t) = r\frac{d^2\theta(t)}{dt^2}\hat{\boldsymbol{\theta}}(t) + r\frac{d\theta(t)}{dt}\left(-\cos\theta(t)\frac{d\theta(t)}{dt}\hat{\mathbf{i}} - \sin\theta(t)\frac{d\theta(t)}{dt}\,\hat{\mathbf{j}}\right). \tag{6.3.5}$$

Recall that $\omega \equiv d\theta / dt$, and from Eq. (6.2.2), $\hat{\mathbf{r}}(t) = \cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}}$, therefore the acceleration becomes

$$\vec{\mathbf{a}}(t) = r\frac{d^2\theta(t)}{dt^2}\hat{\boldsymbol{\theta}}(t) - r\left(\frac{d\theta(t)}{dt}\right)^2\hat{\mathbf{r}}(t)\,. \tag{6.3.6}$$

The *tangential component of the acceleration* is then

$$a_\theta = r\frac{d^2\theta(t)}{dt^2}\,. \tag{6.3.7}$$

The *radial component of the acceleration* is given by

$$a_r = -r\left(\frac{d\theta(t)}{dt}\right)^2 = -r\omega^2 < 0\ . \tag{6.3.8}$$

Because $a_r < 0$, that radial vector component $\vec{\mathbf{a}}_r(t) = -r\omega^2\,\hat{\mathbf{r}}(t)$ is always directed towards the center of the circular orbit.

Example 6.1 Circular Motion Kinematics

A particle is moving in a circle of radius R. At $t = 0$, it is located on the x-axis. The angle the particle makes with the positive x-axis is given by $\theta(t) = At^3 - Bt$, where A and B are positive constants. Determine (a) the velocity vector, and (b) the acceleration vector. Express your answer in polar coordinates. At what time is the centripetal acceleration zero?

Solution:

The derivatives of the angle function $\theta(t) = At^3 - Bt$ are $d\theta / dt = 3At^2 - B$ and $d^2\theta / dt^2 = 6At$. Therefore the velocity vector is given by

$$\vec{\mathbf{v}}(t) = R\frac{d^2\theta(t)}{dt}\,\hat{\boldsymbol{\theta}}(t) = R(3At^2 - Bt)\hat{\boldsymbol{\theta}}(t).$$

The acceleration is given by

$$\begin{aligned}\vec{\mathbf{a}}(t) &= R\frac{d^2\theta(t)}{dt^2}\,\hat{\boldsymbol{\theta}}(t) - R\left(\frac{d\theta(t)}{dt}\right)^2 \hat{\mathbf{r}}(t) \\ &= R(6At)\,\hat{\boldsymbol{\theta}}(t) - R\left(3At^2 - B\right)^2 \hat{\mathbf{r}}(t)\end{aligned}.$$

The centripetal acceleration is zero at time $t = t_1$ when

$$3At_1^2 - B = 0 \Rightarrow t_1 = \sqrt{B / 3A}\,.$$

6.4 Period and Frequency for Uniform Circular Motion

If the object is constrained to move in a circle and the total tangential force acting on the object is zero, $F_\theta^{\text{total}} = 0$, then (Newton's Second Law), the tangential acceleration is zero,

$$a_\theta = 0\,. \tag{6.4.1}$$

This means that the magnitude of the velocity (the speed) remains constant. This motion is known as *uniform circular motion*. The acceleration is then given by only the acceleration radial component vector

$$\vec{\mathbf{a}}_r(t) = -r\omega^2(t)\,\hat{\mathbf{r}}(t) \quad \text{uniform circular motion}\,. \tag{6.4.2}$$

Because the speed $v = r|\omega|$ is constant, the amount of time that the object takes to complete one circular orbit of radius r is also constant. This time interval, T, is called the *period***.** In one period the object travels a distance $s = vT$ equal to the circumference, $s = 2\pi r$; thus

$$s = 2\pi r = vT\,. \tag{6.4.3}$$

The period T is then given by

$$T = \frac{2\pi r}{v} = \frac{2\pi r}{r\omega} = \frac{2\pi}{\omega}\,. \tag{6.4.4}$$

The *frequency* f is defined to be the reciprocal of the period,

$$f = \frac{1}{T} = \frac{\omega}{2\pi}. \tag{6.4.5}$$

The SI unit of frequency is the inverse second, which is defined as the hertz, $\left[\mathrm{s}^{-1}\right] \equiv [\mathrm{Hz}]$.

The magnitude of the radial component of the acceleration can be expressed in several equivalent forms since both the magnitudes of the velocity and angular velocity are related by $v = r\omega$. Thus we have several alternative forms for the magnitude of the centripetal acceleration. The first is that in Equation (6.5.3). The second is in terms of the radius and the angular velocity,

$$|a_r| = r\omega^2\,. \tag{6.4.6}$$

The third form expresses the magnitude of the centripetal acceleration in terms of the speed and radius,

$$|a_r| = \frac{v^2}{r}. \tag{6.4.7}$$

Recall that the magnitude of the angular velocity is related to the frequency by $\omega = 2\pi f$, so we have a fourth alternate expression for the magnitude of the centripetal acceleration in terms of the radius and frequency,

$$|a_r| = 4\pi^2 r f^2\,. \tag{6.4.8}$$

A fifth form commonly encountered uses the fact that the frequency and period are related by $f = 1/T = \omega/2\pi$. Thus we have the fourth expression for the centripetal acceleration in terms of radius and period,

$$|a_r| = \frac{4\pi^2 r}{T^2}. \tag{6.4.9}$$

Other forms, such as $4\pi^2 r^2 f \,/\, T$ or $2\pi r\omega f$, while valid, are uncommon.

Often we decide which expression to use based on information that describes the orbit. A convenient measure might be the orbit's radius. We may also independently know the period, or the frequency, or the angular velocity, or the speed. If we know one, we can calculate the other three but it is important to understand the meaning of each quantity.

6.4.1 Geometric Interpretation for Radial Acceleration for Uniform Circular Motion

An object traveling in a circular orbit is always accelerating towards the center. Any radial inward acceleration is called *centripetal acceleration*. Recall that the direction of the velocity is always tangent to the circle. Therefore the direction of the velocity is constantly changing because the object is moving in a circle, as can be seen in Figure 6.4. Because the velocity changes direction, the object has a nonzero acceleration.

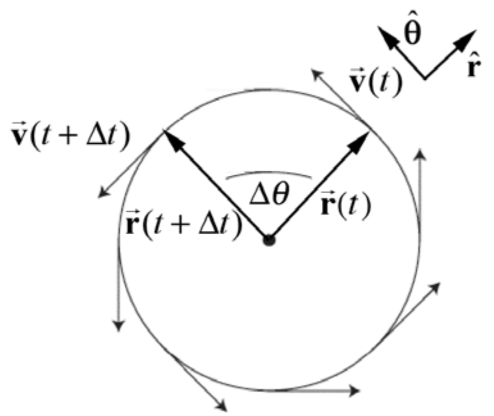

Figure 6.4 Direction of the velocity for circular motion.

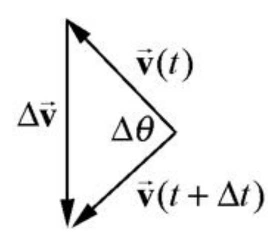

Figure 6.5 Change in velocity vector.

The calculation of the magnitude and direction of the acceleration is very similar to the calculation for the magnitude and direction of the velocity for circular motion, but the change in velocity vector, $\Delta\vec{\mathbf{v}}$, is more complicated to visualize. The change in velocity $\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}(t+\Delta t) - \vec{\mathbf{v}}(t)$ is depicted in Figure 6.5. The velocity vectors have been given a common point for the tails, so that the change in velocity, $\Delta\vec{\mathbf{v}}$, can be visualized.

The length $\left|\Delta\vec{\mathbf{v}}\right|$ of the vertical vector can be calculated in exactly the same way as the displacement $\left|\Delta\vec{\mathbf{r}}\right|$. The magnitude of the change in velocity is

$$\left|\Delta\vec{\mathbf{v}}\right| = 2v\sin(\Delta\theta/2). \tag{6.5.1}$$

We can use the small angle approximation $\sin(\Delta\theta/2) \cong \Delta\theta/2$ to approximate the magnitude of the change of velocity,

$$\left|\Delta\vec{\mathbf{v}}\right| \cong v\left|\Delta\theta\right|. \tag{6.5.2}$$

The magnitude of the radial acceleration is given by

$$\left|a_r\right| = \lim_{\Delta t\to 0}\frac{\left|\Delta\vec{\mathbf{v}}\right|}{\Delta t} = \lim_{\Delta t\to 0}\frac{v\left|\Delta\theta\right|}{\Delta t} = v\lim_{\Delta t\to 0}\frac{\left|\Delta\theta\right|}{\Delta t} = v\left|\frac{d\theta}{dt}\right| = v\left|\omega\right|. \tag{6.5.3}$$

The direction of the radial acceleration is determined by the same method as the direction of the velocity; in the limit $\Delta\theta \to 0$, $\Delta\vec{\mathbf{v}} \perp \vec{\mathbf{v}}$, and so the direction of the acceleration radial component vector $\vec{\mathbf{a}}_r(t)$ at time t is perpendicular to position vector $\vec{\mathbf{v}}(t)$ and directed inward, in the $-\hat{\mathbf{r}}$-direction.

6.5 Angular Velocity and Angular Acceleration

6.5.1. Angular Velocity

We shall always choose a right-handed cylindrical coordinate system. If the positive z-axis points up, then we choose θ to be increasing in the counterclockwise direction as shown in Figures 6.6.

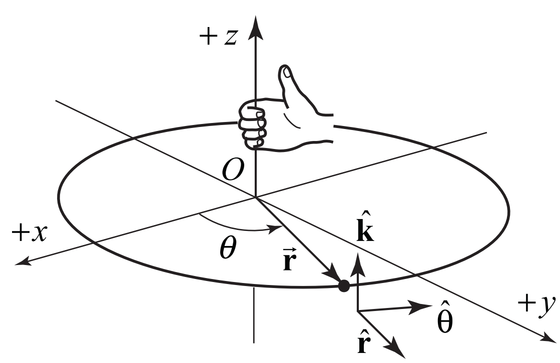

Figure 6.6 Right handed coordinate system

For a point object undergoing circular motion about the z-axis, the angular velocity vector $\vec{\boldsymbol{\omega}}$ is directed along the z-axis with z-component equal to the time derivative of the angle θ,

$$\vec{\boldsymbol{\omega}} = \frac{d\theta}{dt}\,\hat{\mathbf{k}} = \omega_z\,\hat{\mathbf{k}}\,. \tag{6.5.4}$$

The SI units of angular velocity are $[\mathrm{rad}\cdot\mathrm{s}^{-1}]$. Note that the angular speed is just the magnitude of the z-component of the angular velocity,

$$\omega \equiv \left|\omega_z\right| = \left|\frac{d\theta}{dt}\right|. \tag{6.5.5}$$

If the velocity of the object is in the $+\hat{\boldsymbol{\theta}}$-direction, (rotating in the counterclockwise direction in Figure 6.7(a)), then the z-component of the angular velocity is positive, $\omega_z = d\theta / dt > 0$. The angular velocity vector then points in the $+\hat{\mathbf{k}}$-direction as shown in Figure 6.7(a). If the velocity of the object is in the $-\hat{\boldsymbol{\theta}}$-direction, (rotating in the clockwise direction in Figure 6.7(b)), then the z-component of the angular velocity angular velocity is negative, $\omega_z = d\theta / dt < 0$. The angular velocity vector then points in the $-\hat{\mathbf{k}}$-direction as shown in Figure 6.7(b).

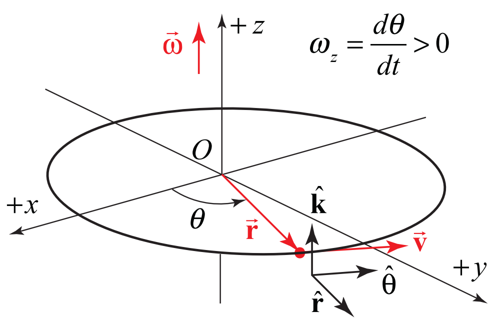

Figure 6.7(a) Angular velocity vector vector for motion with $d\theta / dt > 0$.

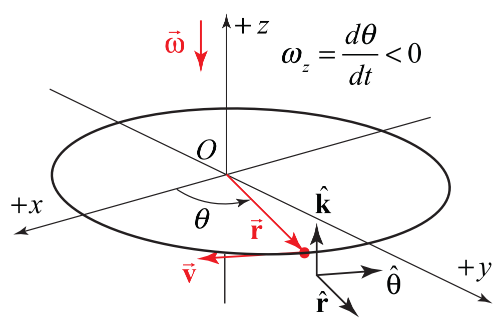

Figure 6.7(b) Angular velocity for motion with $d\theta / dt < 0$.

The velocity and angular velocity are related by

$$\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}} = \frac{d\theta}{dt}\,\hat{\mathbf{k}} \times r\hat{\mathbf{r}} = r\frac{d\theta}{dt}\,\hat{\boldsymbol{\theta}}\,. \tag{6.5.6}$$

Example 6.2 Angular Velocity

A particle is moving in a circle of radius R. At $t = 0$, it is located on the x-axis. The angle the particle makes with the positive x-axis is given by $\theta(t) = At - Bt^3$, where A and B are positive constants. Determine (a) the angular velocity vector, and (b) the velocity vector. Express your answer in polar coordinates. (c) At what time, $t = t_1$, is the angular velocity zero? (d) What is the direction of the angular velocity for (i) $t < t_1$, and (ii) $t > t_1$?

Solution: The derivative of $\theta(t) = At - Bt^3$ is

$$\frac{d\theta(t)}{dt} = A - 3Bt^2 .$$

Therefore the angular velocity vector is given by

$$\vec{\boldsymbol{\omega}}(t) = \frac{d\theta(t)}{dt}\hat{\mathbf{k}} = (A - 3Bt^2)\hat{\mathbf{k}} .$$

The velocity is given by

$$\vec{\mathbf{v}}(t) = R\frac{d\theta(t)}{dt}\hat{\boldsymbol{\theta}}(t) = R(A - 3Bt^2)\hat{\boldsymbol{\theta}}(t).$$

The angular velocity is zero at time $t = t_1$ when

$$A - 3Bt_1^2 = 0 \Rightarrow t_1 = \sqrt{A/3B} .$$

For $t < t_1$, $\dfrac{d\theta(t)}{dt} = A - 3Bt_1^2 > 0$ hence $\vec{\boldsymbol{\omega}}(t)$ points in the positive $\hat{\mathbf{k}}$-direction.

For $t > t_1$, $\dfrac{d\theta(t)}{dt} = A - 3Bt_1^2 < 0$ hence $\vec{\boldsymbol{\omega}}(t)$ points in the negative $\hat{\mathbf{k}}$-direction.

6.5.2 Angular Acceleration

In a similar fashion, for a point object undergoing circular motion about the fixed z-axis, the angular acceleration is defined as

$$\vec{\boldsymbol{\alpha}} = \frac{d^2\theta}{dt^2}\hat{\mathbf{k}} = \alpha_z\hat{\mathbf{k}} . \tag{6.5.7}$$

The SI units of angular acceleration are $[\mathrm{rad}\cdot\mathrm{s}^{-2}]$. The magnitude of the angular acceleration is denoted by the Greek symbol alpha,

$$\alpha \equiv |\vec{\alpha}| = \left|\frac{d^2\theta}{dt^2}\right| . \tag{6.5.8}$$

There are four special cases to consider for the direction of the angular velocity. Let's first consider the two types of motion with $\vec{\alpha}$ pointing in the $+\hat{\mathbf{k}}$-direction: (i) if the object is rotating counterclockwise and speeding up then both $d\theta / dt > 0$ and $d^2\theta / dt^2 > 0$ (Figure 6.8(a), (ii) if the object is rotating clockwise and slowing down then $d\theta / dt < 0$ but $d^2\theta / dt^2 > 0$ (Figure 6.8(b). There are two corresponding cases in which $\vec{\alpha}$ pointing in the $-\hat{\mathbf{k}}$-direction: (iii) if the object is rotating counterclockwise and slowing down then $d\theta / dt > 0$ but $d^2\theta / dt^2 < 0$ (Figure 6.9(a), (iv) if the object is rotating clockwise and speeding up then both $d\theta / dt < 0$ and $d^2\theta / dt^2 < 0$ (Figure 6.9(b).

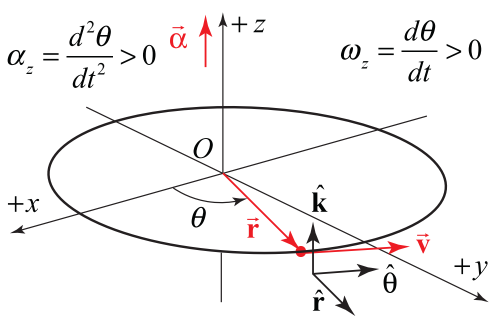

Figure 6.8(a) Angular acceleration vector vector for motion with $d\theta / dt > 0$, and $d^2\theta / dt^2 > 0$.

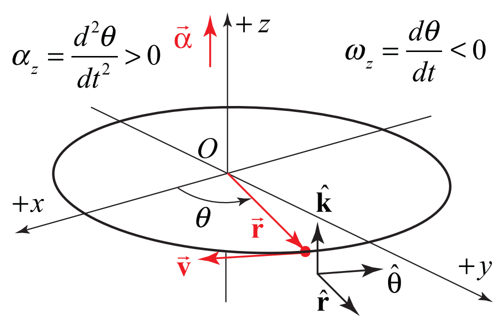

Figure 6.8(b) Angular velocity for motion with $d\theta / dt < 0$, and $d^2\theta / dt^2 > 0$.

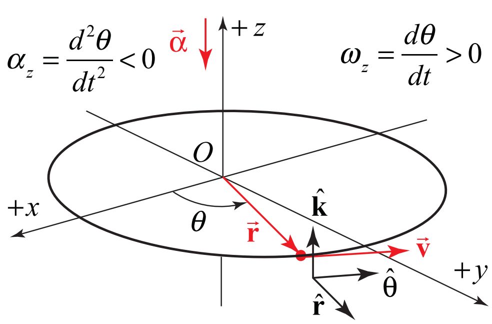

Figure 6.9(a) Angular acceleration vector vector for motion with $d\theta / dt > 0$, and $d^2\theta / dt^2 < 0$.

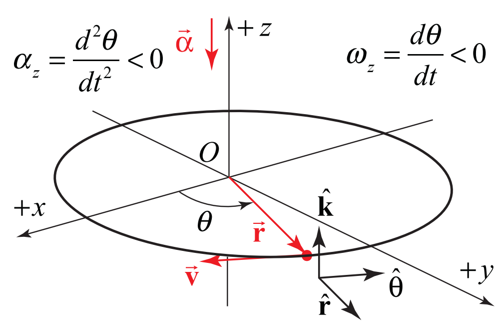

Figure 6.9(b) Angular velocity for motion with $d\theta / dt < 0$, and $d^2\theta / dt^2 < 0$.

Example 6.3 Integration and Circular Motion Kinematics

A point-like object is constrained to travel in a circle. The z-component of the angular acceleration of the object for the time interval $[0, t_1]$ is given by the function

$$\alpha_z(t) = \begin{cases} b\left(1 - \dfrac{t}{t_1}\right); & 0 \le t \le t_1 \\ 0;\ t > t_1 \end{cases},$$

where b is a positive constant with units $\mathrm{rad} \cdot \mathrm{s}^{-3}$.

a) Determine an expression for the angular velocity of the object at $t = t_1$.

b) Through what angle has the object rotated at time $t = t_1$?

Solution:

a) The angular velocity at time $t = t_1$ is given by

$$\omega_z(t_1) - \omega_z(t=0) = \int_{t'=0}^{t'=t_1} \alpha_z(t')\,dt' = \int_{t'=0}^{t'=t_1} b\left(1 - \frac{t'}{t_1}\right) dt' = b\left(t_1 - \frac{t_1^{\,2}}{2t_1}\right) = \frac{bt_1}{2}$$

b) In order to find the angle $\theta(t_1) - \theta(t=0)$ that the object has rotated through at time $t = t_1$, you first need to find $\omega_z(t)$ by integrating the z-component of the angular acceleration

$$\omega_z(t) - \omega_z(t=0) = \int_{t'=0}^{t'=t} \alpha_z(t')\,dt' = \int_{t'=0}^{t'=t} b\left(1 - \frac{t'}{t_1}\right) dt' = b\left(t - \frac{t^2}{2t_1}\right).$$

Because it started from rest, $\omega_z(t=0) = 0$, hence $\omega_z(t) = b\left(t - \dfrac{t^2}{2t_1}\right);\ 0 \le t \le t_1$.

Then integrate $\omega_z(t)$ between $t = 0$ and $t = t_1$ to find that

$$\theta(t_1) - \theta(t=0) = \int_{t'=0}^{t'=t_1} \omega_z(t')\,dt' = \int_{t'=0}^{t'=t_1} b\left(t' - \frac{t'^2}{2t_1}\right) dt' = b\left(\frac{t_1^{\,2}}{2} - \frac{t_1^{\,3}}{6t_1}\right) = \frac{bt_1^{\,2}}{3}.$$

6.5 Non-circular Central Planar Motion

Let's now consider central motion in a plane that is non-circular. In Figure 6.10, we show the spiral motion of a moving particle. In polar coordinates, the key point is that the time derivative dr/dt of the position function r is no longer zero. The second derivative d^2r/dt^2 also may or may not be zero. In the following calculation we will drop all explicit references to the time dependence of the various quantities. The position vector is still given by Eq. (6.2.1), which we shall repeat below

$$\vec{\mathbf{r}} = r\,\hat{\mathbf{r}}\,. \tag{6.5.9}$$

Because $dr/dt \neq 0$, when we differentiate Eq. (6.5.9), we need to use the product rule

$$\vec{\mathbf{v}} = \frac{d\,\vec{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\,\hat{\mathbf{r}}}{dt}\,. \tag{6.5.10}$$

Substituting Eq. (6.2.4) into Eq. (6.5.10)

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} = v_r\,\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}. \tag{6.5.11}$$

The velocity is no longer tangential but now has a radial component as well

$$v_r = \frac{dr}{dt}. \tag{6.5.12}$$

In order to determine the acceleration, we now differentiate Eq. (6.5.11), again using the product rule, which is now a little more involved:

$$\vec{\mathbf{a}} = \frac{d\,\vec{\mathbf{v}}}{dt} = \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\,\hat{\mathbf{r}}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{dt}. \tag{6.5.13}$$

Now substitute Eqs. (6.2.4) and (6.2.7) for the time derivatives of the unit vectors in Eq. (6.5.13), and after collecting terms yields

$$\begin{aligned}\vec{\mathbf{a}} &= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{\mathbf{r}} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\hat{\boldsymbol{\theta}} \\ &= a_r\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}}\end{aligned}. \tag{6.5.14}$$

The radial and tangential components of the acceleration are now more complicated than then in the case of circular motion due to the non-zero derivatives of dr/dt and d^2r/dt^2. The radial component is

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2. \tag{6.5.15}$$

and the tangential component is

$$a_\theta = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}. \tag{6.5.16}$$

The firs term in the tangential component of the acceleration, $2(dr/dt)(d\theta/dt)$ has a special name, the *coriolis acceleration*,

$$a_{cor} = 2\frac{dr}{dt}\frac{d\theta}{dt}. \tag{6.5.17}$$

Example 6.4 Spiral Motion

A particle moves outward along a spiral starting from the origin at $t = 0$. Its trajectory is given by $r = b\theta$, where b is a positive constant with units $[\text{m}\cdot\text{rad}^{-1}]$. θ increases in time according to $\theta = ct^2$, where $c > 0$ is a positive constant (with units $[\text{rad}\cdot\text{s}^{-2}]$).

a) Determine the acceleration as a function of time.
b) Determine the time at which the radial acceleration is zero.
c) What is the angle when the radial acceleration is zero?
d) Determine the time at which the radial and tangential accelerations have equal magnitude.

Solution:

a) The position coordinate as a function of time is given by $r = b\theta = bct^2$. The acceleration is given by Eq. (6.5.14). In order to calculate the acceleration, we need to calculate the four derivatives $dr/dt = 2bct$, $d^2r/dt^2 = 2bc$, $d\theta/dt = 2ct$, and $d^2\theta/dt^2 = 2c$. The acceleration is then

$$\vec{\mathbf{a}} = \left(2bc - 4bc^3t^4\right)\hat{\mathbf{r}} + \left(8bc^2t^2 + 2bc^2t^2\right)\hat{\boldsymbol{\theta}} = \left(2bc - 4bc^3t^4\right)\hat{\mathbf{r}} + 10bc^2t^2\,\hat{\boldsymbol{\theta}}.$$

b) The radial acceleration is zero when

$$t_1 = \left(\frac{1}{2c^2}\right)^{1/4}.$$

c) The angle when the radial acceleration is zero is

$$\theta_1 = ct_1^{\,2} = \sqrt{2}/2.$$

d) The radial and tangential accelerations have equal magnitude when after some algebra

$$(2bc - 4bc^3t^4) = 10bc^2t^2 \Rightarrow 0 = t^4 + (5/2c)t^2 - (1/2c^2).$$

This equation has as only positive solution for t^2:

$$t_2^{\,2} = \frac{-(5/2c) \pm ((5/2c)^2 + 2c^2)^{1/2}}{2} = \frac{\sqrt{33} - 5}{4c}.$$

Therefore the magnitudes of the two components are equal when

$$t_2 = \sqrt{\frac{\sqrt{33}-5}{4c}}\ .$$