Chapter 3 Vectors

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Philosophy is written in this grand book, the universe which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles and other geometric figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth.[1]

Galileo Galilee

3.1 Vector Analysis

3.1.1 Introduction to Vectors

Certain physical quantities such as mass or the absolute temperature at some point in space only have magnitude. A single number can represent each of these quantities, with appropriate units, which are called **scalar** quantities. There are, however, other physical quantities that have both magnitude and direction. Force is an example of a quantity that has both direction and magnitude (strength). Three numbers are needed to represent the magnitude and direction of a vector quantity in a three dimensional space. These quantities are called **vector** quantities. Vector quantities also satisfy two distinct operations, vector addition and multiplication of a vector by a scalar. We can add two forces together and the sum of the forces must satisfy the rule for vector addition. We can multiply a force by a scalar thus increasing or decreasing its strength. Position, displacement, velocity, acceleration, force, and momentum are all physical quantities that can be represented mathematically by vectors. The set of vectors and the two operations form what is called a **vector space**. There are many types of vector spaces but we shall restrict our attention to the very familiar type of vector space in three dimensions that most students have encountered in their mathematical courses. We shall begin our discussion by defining what we mean by a vector in three dimensional space, and the rules for the operations of vector addition and multiplication of a vector by a scalar.

3.1.2 Properties of Vectors

A vector is a quantity that has both direction and magnitude. Let a vector be denoted by the symbol $\vec{\mathbf{A}}$. The magnitude of $\vec{\mathbf{A}}$ is $|\vec{\mathbf{A}}| \equiv A$. We can represent vectors as geometric objects using arrows. The length of the arrow corresponds to the magnitude of the vector. The arrow points in the direction of the vector (Figure 3.1).

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[1] Galileo Galilei, *The Assayer*, tr. Stillman Drake (1957), *Discoveries and Opinions of Galileo* pp. 237-8.

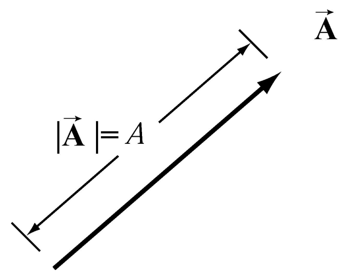

Figure 3.1 Vectors as arrows.

There are two defining operations for vectors:

(1) Vector Addition:

Vectors can be added. Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors. We define a new vector, $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$, the "vector addition" of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, by a geometric construction. Draw the arrow that represents $\vec{\mathbf{A}}$. Place the tail of the arrow that represents $\vec{\mathbf{B}}$ at the tip of the arrow for $\vec{\mathbf{A}}$ as shown in Figure 3.2a. The arrow that starts at the tail of $\vec{\mathbf{A}}$ and goes to the tip of $\vec{\mathbf{B}}$ is defined to be the "vector addition" $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$. There is an equivalent construction for the law of vector addition. The vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ can be drawn with their tails at the same point. The two vectors form the sides of a parallelogram. The diagonal of the parallelogram corresponds to the vector $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$, as shown in Figure 3.2b.

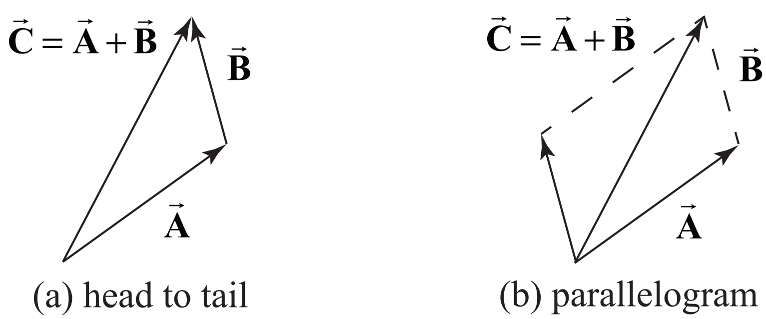

Figure 3.2a **Figure 3.2b**

Vector addition satisfies the following four properties:

(i) Commutativity:

The order of adding vectors does not matter;

$$\vec{\mathbf{A}} + \vec{\mathbf{B}} = \vec{\mathbf{B}} + \vec{\mathbf{A}}\,. \tag{3.1.1}$$

Our geometric definition for vector addition satisfies the commutative property (3.1.1). We can understand this geometrically because in the head to tail representation for the addition of vectors, it doesn't matter which vector you begin with, the sum is the same vector, as seen in Figure 3.3.

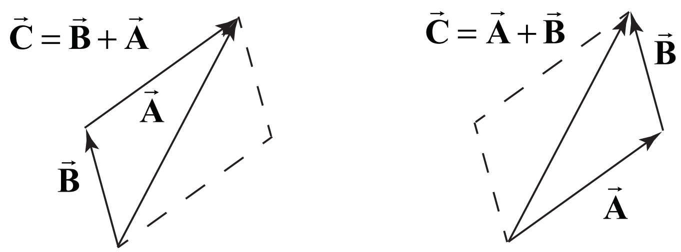

Figure 3.3 Commutative property of vector addition.

(ii) Associativity:

When adding three vectors, it doesn't matter which two you start with

$$(\vec{\mathbf{A}} + \vec{\mathbf{B}}) + \vec{\mathbf{C}} = \vec{\mathbf{A}} + (\vec{\mathbf{B}} + \vec{\mathbf{C}}). \tag{3.1.2}$$

In Figure 3.4a, we add $(\vec{\mathbf{B}} + \vec{\mathbf{C}}) + \vec{\mathbf{A}}$, and use commutativity to get $\vec{\mathbf{A}} + (\vec{\mathbf{B}} + \vec{\mathbf{C}})$. In figure, we add $(\vec{\mathbf{A}} + \vec{\mathbf{B}}) + \vec{\mathbf{C}}$ to arrive at the same vector as in Figure 3.4a.

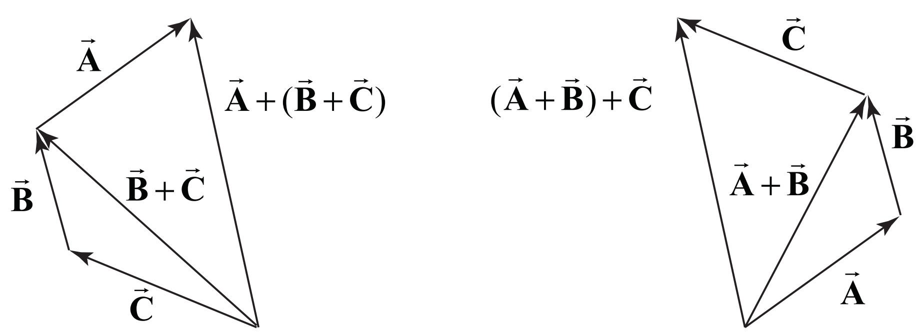

Figure 3.4a Associative law.

(iii) Identity Element for Vector Addition:

There is a unique vector, $\vec{\mathbf{0}}$, that acts as an identity element for vector addition. For all vectors $\vec{\mathbf{A}}$,

$$\vec{\mathbf{A}} + \vec{\mathbf{0}} = \vec{\mathbf{0}} + \vec{\mathbf{A}} = \vec{\mathbf{A}}. \tag{3.1.3}$$

(iv) Inverse Element for Vector Addition:

For every vector $\vec{\mathbf{A}}$, there is a unique inverse vector $-\vec{\mathbf{A}}$ such that

$$\vec{\mathbf{A}} + (-\vec{\mathbf{A}}) = \vec{\mathbf{0}}. \tag{3.1.4}$$

The vector $-\vec{\mathbf{A}}$ has the same magnitude as $\vec{\mathbf{A}}$, $|\vec{\mathbf{A}}| = |-\vec{\mathbf{A}}| = A$, but they point in opposite directions (Figure 3.5).

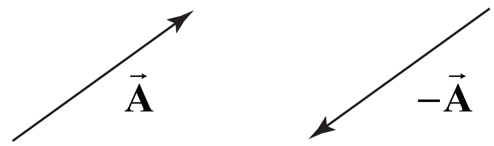

Figure 3.5 Additive inverse

(2) Scalar Multiplication of Vectors:

Vectors can be multiplied by real numbers. Let $\vec{\mathbf{A}}$ be a vector. Let c be a real positive number. Then the multiplication of $\vec{\mathbf{A}}$ by c is a new vector, which we denote by the symbol $c\vec{\mathbf{A}}$. The magnitude of $c\vec{\mathbf{A}}$ is c times the magnitude of $\vec{\mathbf{A}}$ (Figure 3.6a),

$$\left|c\vec{\mathbf{A}}\right| = c\left|\vec{\mathbf{A}}\right|. \tag{3.1.5}$$

Let $c > 0$, then the direction of $c\vec{\mathbf{A}}$ is the same as the direction of $\vec{\mathbf{A}}$. However, the direction of $-c\vec{\mathbf{A}}$ is opposite of $\vec{\mathbf{A}}$ (Figure 3.6).

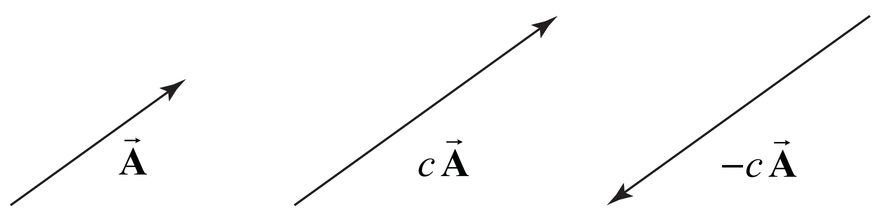

Figure 3.6 Multiplication of vector $\vec{\mathbf{A}}$ by $c > 0$, and $-c < 0$.

Scalar multiplication of vectors satisfies the following properties:

(i) Associative Law for Scalar Multiplication:

The order of multiplying numbers is doesn't matter. Let b and c be real numbers. Then

$$b(c\vec{\mathbf{A}}) = (bc)\vec{\mathbf{A}} = (cb\vec{\mathbf{A}}) = c(b\vec{\mathbf{A}}). \tag{3.1.6}$$

(ii) Distributive Law for Vector Addition:

Vectors satisfy a distributive law for vector addition. Let c be a real number. Then

$$c(\vec{\mathbf{A}}+\vec{\mathbf{B}})=c\,\vec{\mathbf{A}}+c\,\vec{\mathbf{B}}\,. \tag{3.1.7}$$

Figure 3.7 illustrates this property.

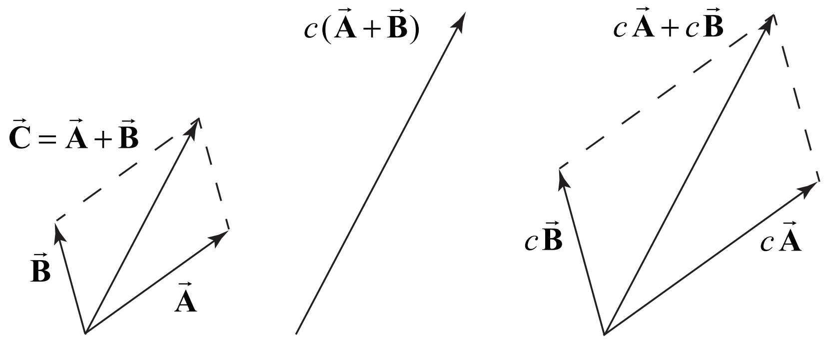

Figure 3.7 Distributive Law for vector addition.

(iii) Distributive Law for Scalar Addition:

Vectors also satisfy a distributive law for scalar addition. Let b and c be real numbers. Then

$$(b+c)\,\vec{\mathbf{A}}=b\,\vec{\mathbf{A}}+c\,\vec{\mathbf{A}} \tag{3.1.8}$$

Our geometric definition of vector addition and scalar multiplication satisfies this condition as seen in Figure 3.8.

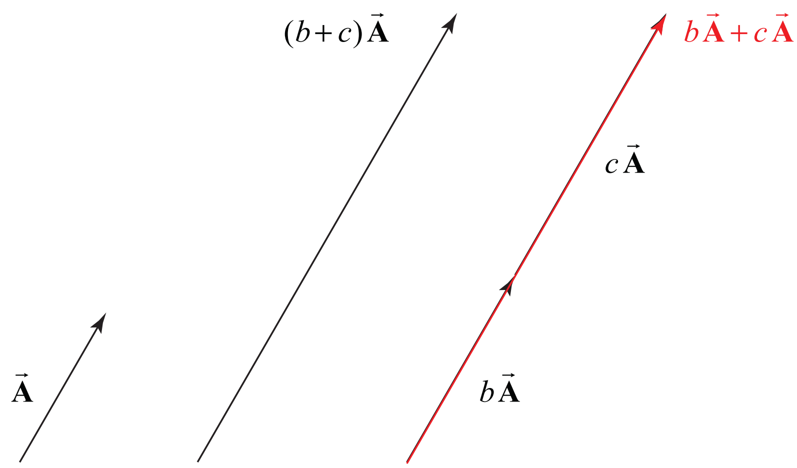

Figure 3.8 Distributive law for scalar multiplication.

(iv) Identity Element for Scalar Multiplication:

The number 1 acts as an identity element for multiplication,

$$1\,\vec{\mathbf{A}}=\vec{\mathbf{A}}\,. \tag{3.1.9}$$

Unit vector:

Dividing a vector by its magnitude results in a vector of unit length which we denote with a caret symbol

$$\hat{\mathbf{A}} = \frac{\vec{\mathbf{A}}}{|\vec{\mathbf{A}}|}. \tag{3.1.10}$$

Note that $|\hat{\mathbf{A}}| = |\vec{\mathbf{A}}| / |\vec{\mathbf{A}}| = 1$.

3.2 Coordinate Systems

Physics involve the study of phenomena that we observe in the world. In order to connect the phenomena to mathematics we begin by introducing the concept of a coordinate system. A coordinate system consists of four basic elements:

(1) Choice of origin

(2) Choice of axes

(3) Choice of positive direction for each axis

(4) Choice of unit vectors at every point in space

There are three commonly used coordinate systems: Cartesian, cylindrical and spherical. In this chapter we will describe a Cartesian coordinate system and a cylindrical coordinate system.

3.2.1 Cartesian Coordinate System

Cartesian coordinates consist of a set of mutually perpendicular axes, which intersect at a common point, the origin O. We live in a three-dimensional spatial world; for that reason, the most common system we will use has three axes.

(1) Choice of Origin: Choose an origin O at any point that is most convenient.

(2) Choice of Axes: The simplest set of axes is known as the Cartesian axes, x-axis, y-axis, and the z-axis, that are at right angles with respect to each other. Then each point P in space can be assigned a triplet of values (x_P, y_P, z_P), the Cartesian coordinates of the point P. The ranges of these values are: $-\infty < x_P < +\infty$, $-\infty < y_P < +\infty$, $-\infty < z_P < +\infty$.

(3) Choice of Positive Direction: Our third choice is an assignment of positive direction for each coordinate axis. We shall denote this choice by the symbol + along the positive axis. In physics problems we are free to choose our axes and positive directions any way that we decide best fits a given problem. Problems that are very difficult using the conventional choices may turn out to be much easier to solve by making a thoughtful choice of axes.

(4) Choice of Unit Vectors: We now associate to each point P in space, a set of three **unit vectors** $(\hat{\mathbf{i}}_P, \hat{\mathbf{j}}_P, \hat{\mathbf{k}}_P)$. A unit vector has magnitude one: $\left|\hat{\mathbf{i}}_P\right| = 1$, $\left|\hat{\mathbf{j}}_P\right| = 1$, and $\left|\hat{\mathbf{k}}_P\right| = 1$. We assign the direction of $\hat{\mathbf{i}}_P$ to point in the direction of the increasing x-coordinate at the point P. We define the directions for $\hat{\mathbf{j}}_P$ and $\hat{\mathbf{k}}_P$ in the direction of the increasing y-coordinate and z-coordinate respectively, (Figure 3.10). If we choose a different point S, and define a similar set of unit vectors $(\hat{\mathbf{i}}_S, \hat{\mathbf{j}}_S, \hat{\mathbf{k}}_S)$, the unit vectors at S and P satisfy the equalities

$$\hat{\mathbf{i}}_S = \hat{\mathbf{i}}_P,\ \hat{\mathbf{j}}_S = \hat{\mathbf{j}}_P, \text{and } \hat{\mathbf{k}}_S = \hat{\mathbf{k}}_P\ , \tag{3.2.1}$$

because vectors are equal if they have the same direction and magnitude regardless of where they are located in space.

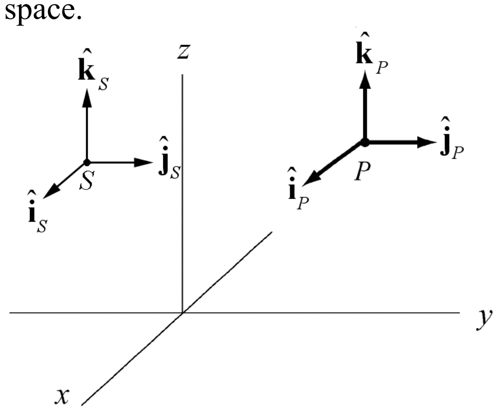

Figure 3.10 Choice of unit vectors at points P and S.

A Cartesian coordinate system is the only coordinate system in which Eq. (3.2.1) holds for all pair of points. We therefore drop the reference to the point P and use $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ to represent the unit vectors in a Cartesian coordinate system (Figure 3.11).

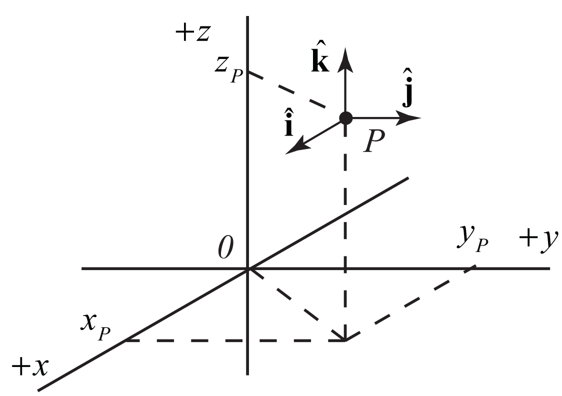

Figure 3.11 Unit vectors in a Cartesian coordinate system

3.2.2 Cylindrical Coordinate System

Many physical objects demonstrate some type of symmetry. For example if you rotate a uniform cylinder about the longitudinal axis (symmetry axis), the cylinder appears unchanged. The operation of rotating the cylinder is called a symmetry operation, and the object undergoing the operation, the cylinder, is exactly the same as before the operation was performed. This symmetry property of cylinders suggests a coordinate system, called a cylindrical coordinate system, that makes the symmetrical property under rotations transparent.

First choose an origin O and axis through O, which we call the z-axis. The ***cylindrical coordinates*** for a point P are the three numbers (r,θ,z) (Figure 3.12). The number z represents the familiar coordinate of the point P along the z-axis. The nonnegative number r represents the distance from the z-axis to the point P. The points in space corresponding to a constant positive value of r lie on a circular cylinder. The locus of points corresponding to $r=0$ is the z-axis. In the plane $z=0$, define a reference ray through O, which we shall refer to as the positive x-axis. Draw a line through the point P that is parallel to the z-axis. Let D denote the point of intersection between that line PD and the plane $z=0$. Draw a ray OD from the origin to the point D. Let θ denote the directed angle from the reference ray to the ray OD. The angle θ is positive when measured counterclockwise and negative when measured clockwise.

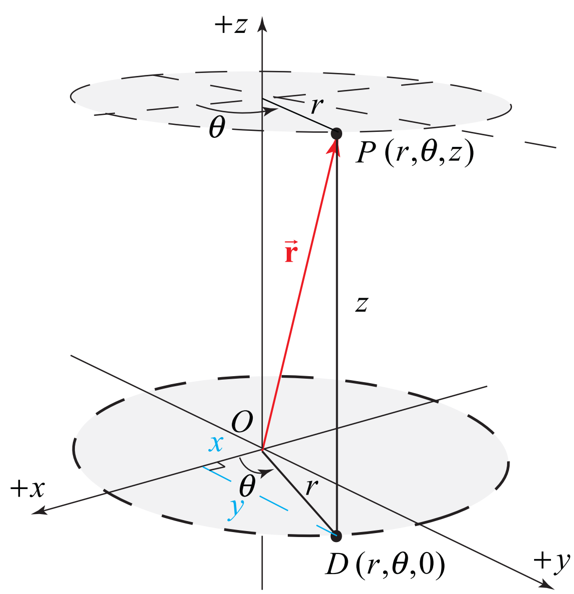

Figure 3.12 Cylindrical Coordinates

The coordinates (r,θ) are called ***polar coordinates***. The coordinate transformations between (r,θ) and the Cartesian coordinates (x,y) are given by

$$x = r\cos\theta\,, \tag{3.2.2}$$
$$y = r\sin\theta\,. \tag{3.2.3}$$

Conversely, if we are given the Cartesian coordinates (x,y), the coordinates (r,θ) can be determined from the coordinate transformations

$$r = +(x^2 + y^2)^{1/2}\,, \tag{3.2.4}$$
$$\theta = \tan^{-1}(y/x)\,. \tag{3.2.5}$$

We choose a set of unit vectors $(\hat{\mathbf{r}}_P, \hat{\boldsymbol{\theta}}_P, \hat{\mathbf{k}}_P)$ at the point P as follows. We choose $\hat{\mathbf{k}}_P$ to point in the direction of increasing z. We choose $\hat{\mathbf{r}}_P$ to point in the direction of increasing r, directed radially away from the z-axis. We choose $\hat{\boldsymbol{\theta}}_P$ to point in the direction of increasing θ. This unit vector points in the counterclockwise direction, tangent to the circle (Figure 3.13a). One crucial difference between cylindrical coordinates and Cartesian coordinates involves the choice of unit vectors. Suppose we consider a different point S in the plane. The unit vectors $(\hat{\mathbf{r}}_S, \hat{\boldsymbol{\theta}}_S, \hat{\mathbf{k}}_S)$ at the point S are also shown in Figure 3.13. Note that $\hat{\mathbf{r}}_P \neq \hat{\mathbf{r}}_S$ and $\hat{\boldsymbol{\theta}}_P \neq \hat{\boldsymbol{\theta}}_S$ because their direction differ. We shall drop the subscripts denoting the points at which the unit vectors are defined at and simple refer to the set of unit vectors at a point as $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{k}})$, with the understanding that the directions of the set $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}})$ depend on the location of the point in question.

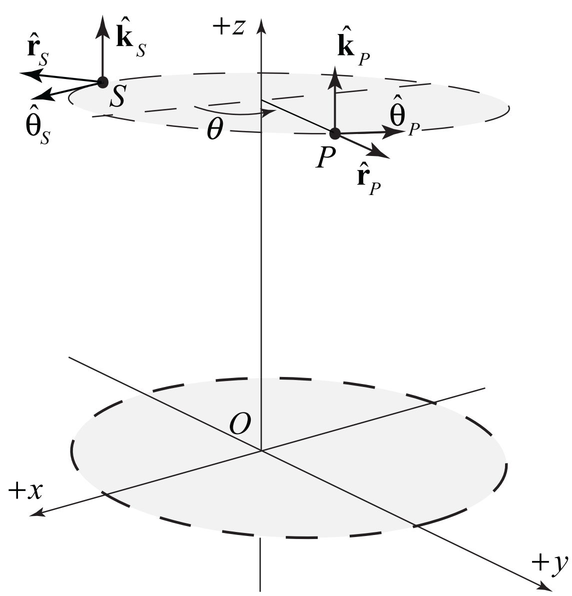

Figure 3.13a Unit vectors at two different points in cylindrical coordinates.

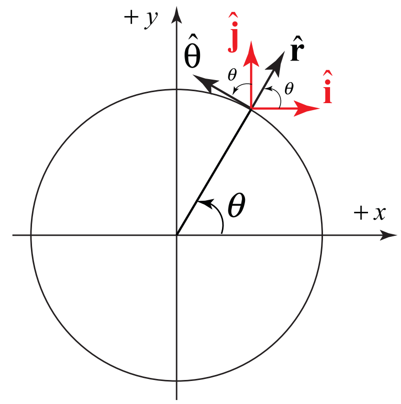

Figure 3.13b Unit vectors in polar coordinates and Cartesian coordinates.

The unit vectors $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}})$ at the point P also are related to the Cartesian unit vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}})$ by the transformations

$$\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}, \tag{3.2.6}$$

$$\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}. \tag{3.2.7}$$

Similarly the inverse transformations are given by

$$\hat{\mathbf{i}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}, \tag{3.2.8}$$

$$\hat{\mathbf{j}} = \sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}. \tag{3.2.9}$$

A cylindrical coordinate system is also a useful choice to describe the motion of an object moving in a circle about a central point. Consider a vertical axis passing perpendicular to the plane of motion passing through that central point. Then any rotation about this vertical axis leaves circles unchanged.

3.3 Vectors

3.3.1 The Use of Vectors in Physics

From the last section we have three important ideas about vectors, (1) vectors can exist at any point P in space, (2) vectors have direction and magnitude, and (3) any two vectors that have the same direction and magnitude are equal no matter where in space they are located. When we apply vectors to physical quantities it's nice to keep in the back of our minds all these formal properties. However from the physicist's point of view, we are interested in representing physical quantities such as displacement, velocity, acceleration, force, impulse, and momentum as vectors. We can't add force to velocity or subtract momentum from force. We must always understand the physical context for the vector quantity. Thus, instead of approaching vectors as formal mathematical objects we shall instead consider the following essential properties that enable us to represent physical quantities as vectors.

3.3.2 Vectors in Cartesian Coordinates

(1) Vector Decomposition: Choose a coordinate system with an origin, axes, and unit vectors. We can decompose a vector into component vectors along each coordinate axis (Figure 3.14).

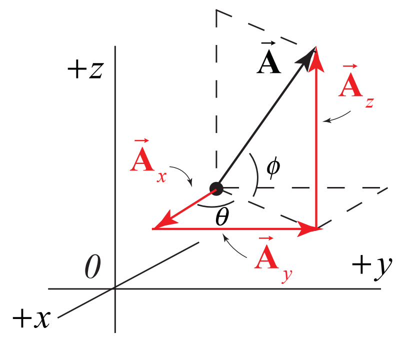

Figure 3.14 Component vectors in Cartesian coordinates.

A vector $\vec{\mathbf{A}}$ at P can be decomposed into the vector sum,

$$\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y + \vec{\mathbf{A}}_z, \tag{3.3.1}$$

where $\vec{\mathbf{A}}_x$ is the x-component vector pointing in the positive or negative x-direction, $\vec{\mathbf{A}}_y$ is the y-component vector pointing in the positive or negative y-direction, and $\vec{\mathbf{A}}_z$ is the z-component vector pointing in the positive or negative z-direction.

(2) Vector Components: Once we have defined unit vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$, we then define the **components** of a vector. Recall our vector decomposition, $\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y + \vec{\mathbf{A}}_z$. We define the x-component vector, $\vec{\mathbf{A}}_x$, as

$$\vec{\mathbf{A}}_x = A_x \,\hat{\mathbf{i}}. \tag{3.3.2}$$

In this expression the term A_x, (without the arrow above) is called the x-component of the vector $\vec{\mathbf{A}}$. The x-component A_x can be positive, zero, or negative. It is not the magnitude of $\vec{\mathbf{A}}_x$ which is given by $(A_x^{\,2})^{1/2}$. The x-component A_x is a scalar quantity and the x-component vector, $\vec{\mathbf{A}}_x$ is a vector. In a similar fashion we define the y-component, A_y, and the z-component, A_z, of the vector $\vec{\mathbf{A}}$ according to

$$\vec{\mathbf{A}}_y = A_y \,\hat{\mathbf{j}}, \quad \vec{\mathbf{A}}_z = A_z \,\hat{\mathbf{k}}. \tag{3.3.3}$$

A vector $\vec{\mathbf{A}}$ is represented by its three components (A_x, A_y, A_z). Thus we need three numbers to describe a vector in three-dimensional space. We write the vector $\vec{\mathbf{A}}$ as

$$\vec{\mathbf{A}} = A_x\ \hat{\mathbf{i}} + A_y\ \hat{\mathbf{j}} + A_z\ \hat{\mathbf{k}}\ . \tag{3.3.4}$$

(3) Magnitude: Using the Pythagorean theorem, the magnitude of $\vec{\mathbf{A}}$ is,

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}\ . \tag{3.3.5}$$

(4) Direction: Let's consider a vector $\vec{\mathbf{A}} = (A_x, A_y, 0)$. Because the z-component is zero, the vector $\vec{\mathbf{A}}$ lies in the x-y plane. Let θ denote the angle that the vector $\vec{\mathbf{A}}$ makes in the counterclockwise direction with the positive x-axis (Figure 3.15).

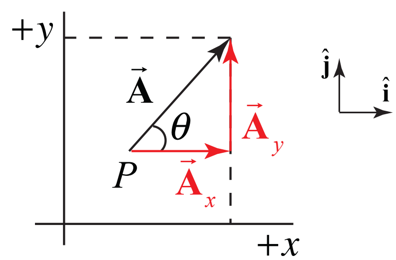

Figure 3.15 Components of a vector in the xy-plane.

Then the x-component and y-component are

$$A_x = A\cos(\theta), \quad A_y = A\sin(\theta)\ . \tag{3.3.6}$$

We now write a vector in the xy-plane as

$$\vec{\mathbf{A}} = A\cos(\theta)\ \hat{\mathbf{i}} + A\sin(\theta)\ \hat{\mathbf{j}} \tag{3.3.7}$$

Once the components of a vector are known, the tangent of the angle θ can be determined by

$$\frac{A_y}{A_x} = \frac{A\sin(\theta)}{A\cos(\theta)} = \tan(\theta)\ , \tag{3.3.8}$$

and hence the angle θ is given by

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right). \tag{3.3.9}$$

Clearly, the direction of the vector depends on the sign of A_x and A_y. For example, if both $A_x > 0$ and $A_y > 0$, then $0 < \theta < \pi/2$. If $A_x < 0$ and $A_y > 0$ then $\pi/2 < \theta < \pi$. If $A_x < 0$ and $A_y < 0$ then $\pi < \theta < 3\pi/2$. If $A_x > 0$ and $A_y < 0$, then $3\pi/2 < \theta < 2\pi$. Note that $\tan(\theta)$ is a double valued function because

$$\frac{-A_y}{-A_x} = \frac{A_y}{A_x}, \text{and } \frac{A_y}{-A_x} = \frac{-A_y}{A_x}. \tag{3.3.10}$$

(5) Unit Vectors: Unit vector in the direction of $\vec{\mathbf{A}}$: Let $\vec{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}$. Let $\hat{\mathbf{A}}$ denote a unit vector in the direction of $\vec{\mathbf{A}}$. Then

$$\hat{\mathbf{A}} = \frac{\vec{\mathbf{A}}}{\left|\vec{\mathbf{A}}\right|} = \frac{A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}}{(A_x^{\,2} + A_y^{\,2} + A_z^{\,2})^{1/2}}. \tag{3.3.11}$$

(6) Vector Addition: Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors in the *x*-*y* plane. Let θ_A and θ_B denote the angles that the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ make (in the counterclockwise direction) with the positive x-axis. Then

$$\vec{\mathbf{A}} = A\cos(\theta_A)\,\hat{\mathbf{i}} + A\sin(\theta_A)\,\hat{\mathbf{j}}, \tag{3.3.12}$$

$$\vec{\mathbf{B}} = B\cos(\theta_B)\,\hat{\mathbf{i}} + B\sin(\theta_B)\,\hat{\mathbf{j}} \tag{3.3.13}$$

In Figure 3.16, the vector addition $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ is shown. Let θ_C denote the angle that the vector $\vec{\mathbf{C}}$ makes with the positive *x*-axis.

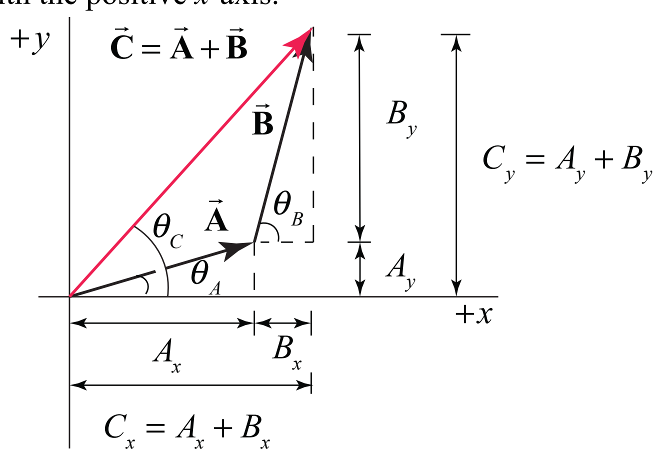

Figure 3.16 Vector addition using components.

From Figure 3.16, the components of $\vec{\mathbf{C}}$ are

$$C_x = A_x + B_x, \quad C_y = A_y + B_y. \tag{3.3.14}$$

In terms of magnitudes and angles, we have

$$\begin{aligned} C_x &= C\cos(\theta_C) = A\cos(\theta_A) + B\cos(\theta_B) \\ C_y &= C\sin(\theta_C) = A\sin(\theta_A) + B\sin(\theta_B). \end{aligned} \tag{3.3.15}$$

We can write the vector $\vec{\mathbf{C}}$ as

$$\vec{\mathbf{C}} = (A_x + B_x)\,\hat{\mathbf{i}} + (A_y + B_y)\,\hat{\mathbf{j}} = C\cos(\theta_C)\,\hat{\mathbf{i}} + C\sin(\theta_C)\,\hat{\mathbf{j}}, \tag{3.3.16}$$

Example 3.1 Vector Addition

Given two vectors, $\vec{\mathbf{A}} = 2\,\hat{\mathbf{i}} + -3\,\hat{\mathbf{j}} + 7\,\hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = 5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, find: (a) $\left|\vec{\mathbf{A}}\right|$; (b) $\left|\vec{\mathbf{B}}\right|$; (c) $\vec{\mathbf{A}} + \vec{\mathbf{B}}$; (d) $\vec{\mathbf{A}} - \vec{\mathbf{B}}$; (e) a unit vector $\hat{\mathbf{A}}$ pointing in the direction of $\vec{\mathbf{A}}$; (f) a unit vector $\hat{\mathbf{B}}$ pointing in the direction of $\vec{\mathbf{B}}$.

Solution:

(a) $\left|\vec{\mathbf{A}}\right| = \left(2^2 + (-3)^2 + 7^2\right)^{1/2} = \sqrt{62} = 7.87$. (b) $\left|\vec{\mathbf{B}}\right| = \left(5^2 + 1^2 + 2^2\right)^{1/2} = \sqrt{30} = 5.48$.

(c)

$$\begin{aligned} \vec{\mathbf{A}} + \vec{\mathbf{B}} &= (A_x + B_x)\,\hat{\mathbf{i}} + (A_y + B_y)\,\hat{\mathbf{j}} + (A_z + B_z)\,\hat{\mathbf{k}} \\ &= (2+5)\,\hat{\mathbf{i}} + (-3+1)\,\hat{\mathbf{j}} + (7+2)\,\hat{\mathbf{k}} \\ &= 7\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} + 9\,\hat{\mathbf{k}}. \end{aligned}$$

(d)

$$\begin{aligned} \vec{\mathbf{A}} - \vec{\mathbf{B}} &= (A_x - B_x)\,\hat{\mathbf{i}} + (A_y - B_y)\,\hat{\mathbf{j}} + (A_z - B_z)\,\hat{\mathbf{k}} \\ &= (2-5)\,\hat{\mathbf{i}} + (-3-1)\,\hat{\mathbf{j}} + (7-2)\,\hat{\mathbf{k}} \\ &= -3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}} + 5\,\hat{\mathbf{k}}. \end{aligned}$$

(e) A unit vector $\hat{\mathbf{A}}$ in the direction of $\vec{\mathbf{A}}$ can be found by dividing the vector $\vec{\mathbf{A}}$ by the magnitude of $\vec{\mathbf{A}}$. Therefore

$$\hat{\mathbf{A}} = \vec{\mathbf{A}} / \left|\vec{\mathbf{A}}\right| = \left(2\,\hat{\mathbf{i}} + -3\,\hat{\mathbf{j}} + 7\,\hat{\mathbf{k}}\right) / \sqrt{62}.$$

(f) In a similar fashion, $\hat{\mathbf{B}} = \vec{\mathbf{B}} / \left|\vec{\mathbf{B}}\right| = \left(5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}\right) / \sqrt{30}$.

Example 3.2 Sinking Sailboat

A Coast Guard ship is located 35 km away from a checkpoint in a direction 52° north of west. A distressed sailboat located in still water 24 km from the same checkpoint in a direction 18° south of east is about to sink. Draw a diagram indicating the position of both ships. In what direction and how far must the Coast Guard ship travel to reach the sailboat?

Solution: The diagram of the set-up is Figure 3.17.

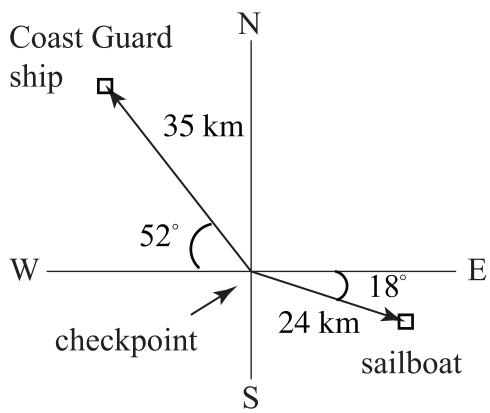

Figure 3.17 Example 3.2

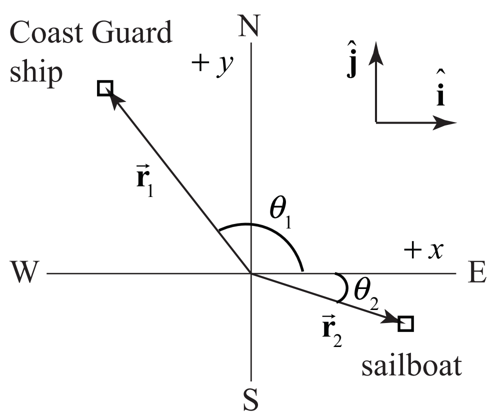

Figure 3.18 Coordinate system for sailboat and ship

Choose the checkpoint as the origin of a Cartesian coordinate system with the positive x-axis in the East direction and the positive y–axis in the North direction. Choose the corresponding unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ as shown in Figure 3.18. The Coast Guard ship is then a distance $r_1 = 35\text{ km}$ at an angle $\theta_1 = 180^\circ - 52^\circ = 128^\circ$ from the positive x-axis, and the sailboat is at a distance $r_2 = 24\text{ km}$ at an angle $\theta_2 = -18^\circ$ from the positive x-axis. The position of the Coast Guard ship is then

$$\vec{\mathbf{r}}_1 = r_1(\cos\theta_1\,\hat{\mathbf{i}} + \sin\theta_1\,\hat{\mathbf{j}})$$
$$\vec{\mathbf{r}} = -21.5\,\text{km}\,\hat{\mathbf{i}} + 27.6\,\text{km}\,\hat{\mathbf{j}}$$

and the position of the sailboat is

$$\vec{\mathbf{r}}_2 = r_2(\cos\theta_2\,\hat{\mathbf{i}} + \sin\theta_2\,\hat{\mathbf{j}})$$
$$\vec{\mathbf{r}}_2 = 22.8\,\text{km}\,\hat{\mathbf{i}} - 7.4\,\text{km}\,\hat{\mathbf{j}}.$$

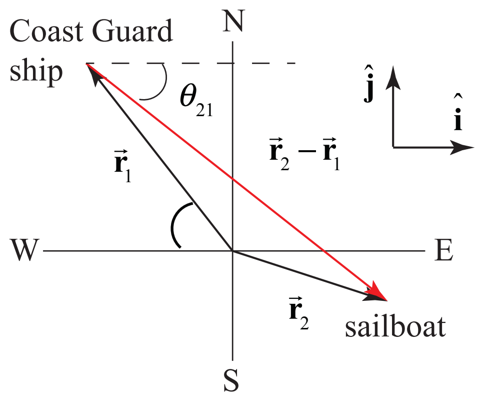

Figure 3.19 Relative position vector from ship to sailboat

The relative position vector from the Coast Guard ship to the sailboat is (Figure 3.19)

$$\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1 = (22.8\,\text{km}\,\hat{\mathbf{i}} - 7.4\,\text{km}\,\hat{\mathbf{j}}) - (-21.5\,\text{km}\,\hat{\mathbf{i}} + 27.6\,\text{km}\,\hat{\mathbf{j}})$$
$$\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1 = 44.4\,\text{km}\,\hat{\mathbf{i}} - 35.0\,\text{km}\,\hat{\mathbf{j}}.$$

The distance between the ship and the sailboat is

$$\left|\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1\right| = ((44.4\,\text{km})^2 + (-35.0\,\text{km})^2)^{1/2} = 56.5\,\text{km}$$

The rescue ship’s heading would be the inverse tangent of the ratio of the y- and x-components of the relative position vector,

$$\theta_{21} = \tan^{-1}(-35.0\,\text{km}/44.4\,\text{km}) = -38.3^\circ.$$

or 38.3° South of East.

Example 3.3 Vector Addition

Two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, such that $|\vec{\mathbf{B}}| = 2|\vec{\mathbf{A}}|$, have a resultant $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ of magnitude 26.5. The vector $\vec{\mathbf{C}}$ makes an angle $\theta_C = 41^\circ$ with respect to vector $\vec{\mathbf{A}}$. Find the magnitude of each vector and the angle between vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

Solution: We begin by making a sketch of the three vectors, choosing $\vec{\mathbf{A}}$ to point in the positive x-direction (Figure 3.20).

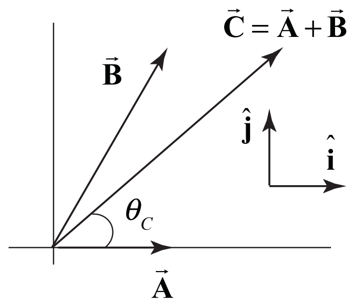

Figure 3.20 Choice of coordinates system for Example 3.3

Denote the magnitude of $\vec{\mathbf{C}}$ by $C \equiv |\vec{\mathbf{C}}| = \sqrt{(C_x)^2 + (C_y)^2} = 26.5$. The components of $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ are given by

$$C_x = A_x + B_x = C\cos\theta_C = (26.5)\cos(41^\circ) = 20 \qquad (3.3.17)$$

$$C_y = B_y = C\sin\theta_C = (26.5)\sin(41^\circ) = 17.4\,. \qquad (3.3.18)$$

From the condition that $|\vec{\mathbf{B}}| = 2|\vec{\mathbf{A}}|$, the square of their magnitudes satisfy

$$(B_x)^2 + (B_y)^2 = 4(A_x)^2\,. \qquad (3.3.19)$$

Using Eqs. (3.3.17) and (3.3.18), Eq. (3.3.19)becomes

$$(C_x - A_x)^2 + (C_y)^2 = 4(A_x)^2$$

$$(C_x)^2 - 2C_xA_x + (A_x)^2 + (C_y)^2 = 4(A_x)^2\,.$$

This is a quadratic equation

$$0 = 3(A_x)^2 + 2C_xA_x - C^2$$

which we solve for the component A_x:

$$A_x = \frac{-2C_x \pm \sqrt{(2C_x)^2 + (4)(3)(C^2)}}{6} = \frac{-2(20) \pm \sqrt{(40))^2 + (4)(3)(26.5)^2}}{6} = 10.0,$$

where we choose the positive square root because we originally chose $A_x > 0$. The components of $\vec{\mathbf{B}}$ are then given by Eqs. (3.3.17) and (3.3.18):

$$B_x = C_x - A_x = 20.0 - 10.0 = 10.0$$
$$B_y = 17.4.$$

The magnitude of $|\vec{\mathbf{B}}| = \sqrt{(B_x)^2 + (B_y)^2} = 20.0$ which is equal to two times the magnitude of $|\vec{\mathbf{A}}| = 10.0$. The angle between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is given by

$$\theta = \sin^{-1}(B_y / |\vec{\mathbf{B}}|) = \sin^{-1}(17.4 / 20.0 \text{ N}) = 60^\circ.$$

Example 3.4 Vector Description of a Point on a Line

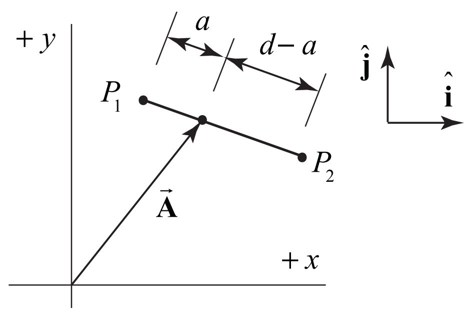

Figure 3.21 Example 3.4

Consider two points, P_1 with coordinates (x_1, y_1) and P_2 with coordinates (x_1, y_1), that are separated by distance d. Find a vector $\vec{\mathbf{A}}$ from the origin to the point on the line connecting P_1 and P_2 that is located a distance a from the point P_1 (Figure 3.21).

Solution: Let $\vec{\mathbf{r}}_1 = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}}$ be the position vector of P_1 and $\vec{\mathbf{r}}_2 = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}}$ the position vector of P_2. Let $\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$ be the vector from P_2 to P_1 (Figure 3.22a). The unit vector pointing from P_2 to P_1 is given by $\hat{\mathbf{r}}_{21} = (\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2)/|\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2| = (\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2)/d$, where $d = ((x_2 - x_1)^2 + (y_2 - y_1)^2)^{1/2}$

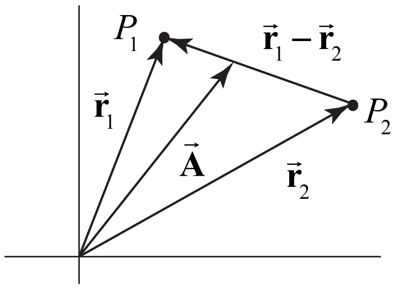

Figure 3.22a: Relative position vector

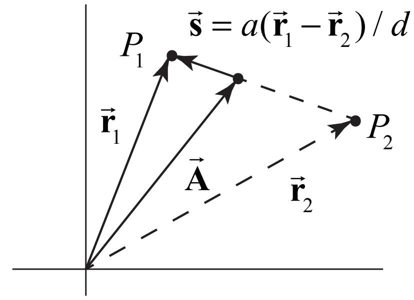

Figure 3.22b: Relative position vector

The vector $\vec{\mathbf{s}}$ in Figure 3.22b connects $\vec{\mathbf{A}}$ to the point at $\vec{\mathbf{r}}_1$, points in the direction of $\hat{\mathbf{r}}_{12}$, and has length a. Therefore $\vec{\mathbf{s}} = a\hat{\mathbf{r}}_{21} = a(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2)/d$. The vector $\vec{\mathbf{r}}_1 = \vec{\mathbf{A}} + \vec{\mathbf{s}}$. Therefore

$$\vec{\mathbf{A}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{s}} = \vec{\mathbf{r}}_1 - a(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2)/d = (1 - a/d)\vec{\mathbf{r}}_1 + (a/d)\vec{\mathbf{r}}_2$$
$$\vec{\mathbf{A}} = (1 - a/d)(x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}}) + (a/d)(x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}})$$
$$\vec{\mathbf{A}} = \left(x_1 + \frac{a(x_2 - x_1)}{((x_2 - x_1)^2 + (y_2 - y_1)^2)^{1/2}}\right)\hat{\mathbf{i}} + \left(y_1 + \frac{a(y_2 - y_1)}{((x_2 - x_1)^2 + (y_2 - y_1)^2)^{1/2}}\right)\hat{\mathbf{j}}.$$

3.3.2 Transformation of Vectors in Rotated Coordinate Systems

Consider two Cartesian coordinate systems S and S' such that the (x', y') coordinate axes in S' are rotated by an angle θ with respect to the (x, y) coordinate axes in S, (Figure 3.23).

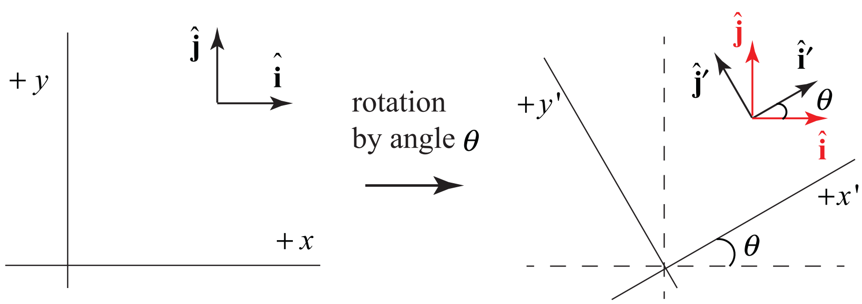

Figure 3.23 Rotated coordinate systems

The components of the unit vector $\hat{\mathbf{i}}'$ in the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ direction are given by $i'_x = |\hat{\mathbf{i}}'|\cos\theta = \cos\theta$ and $i'_y = |\hat{\mathbf{i}}'|\sin\theta = \sin\theta$. Therefore

$$\hat{\mathbf{i}}' = i'_x\,\hat{\mathbf{i}} + i'_y\,\hat{\mathbf{j}} = \hat{\mathbf{i}}\cos\theta + \hat{\mathbf{j}}\sin\theta. \qquad (3.3.20)$$

A similar argument holds for the components of the unit vector $\hat{\mathbf{j}}'$. The components of $\hat{\mathbf{j}}'$ in the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ direction are given by $j'_x = -|\hat{\mathbf{j}}'|\sin\theta = -\sin\theta$ and $j'_y = |\hat{\mathbf{j}}'|\cos\theta = \cos\theta$. Therefore

$$\hat{\mathbf{j}}' = j'_x\,\hat{\mathbf{i}} + j'_y\,\hat{\mathbf{j}} = \hat{\mathbf{j}}\cos\theta - \hat{\mathbf{i}}\sin\theta\,. \tag{3.3.21}$$

Conversely, from Figure 3.23 and similar vector decomposition arguments, the components of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ in S' are given by

$$\hat{\mathbf{i}} = \hat{\mathbf{i}}'\cos\theta - \hat{\mathbf{j}}'\sin\theta\,, \tag{3.3.22}$$

$$\hat{\mathbf{j}} = \hat{\mathbf{i}}'\sin\theta + \hat{\mathbf{j}}'\cos\theta\,. \tag{3.3.23}$$

Consider a fixed vector $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ with components (x, y) in coordinate system S. In coordinate system S', the vector is given by $\vec{\mathbf{r}} = x'\,\hat{\mathbf{i}}' + y'\hat{\mathbf{j}}'$, where (x', y') are the components in S', (Figure 3.24).

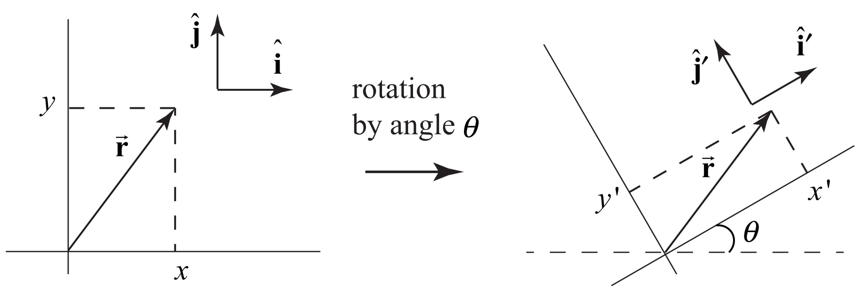

Figure 3.24 Transformation of vector components

Using the Eqs. (3.3.20) and (3.3.21), we have that

$$\begin{aligned}\vec{\mathbf{r}} &= x\,\hat{\mathbf{i}} + y\hat{\mathbf{j}} = x(\hat{\mathbf{i}}'\cos\theta - \hat{\mathbf{j}}'\sin\theta) + y(\hat{\mathbf{j}}'\cos\theta + \hat{\mathbf{i}}'\sin\theta)\\ \vec{\mathbf{r}} &= (x\cos\theta + y\sin\theta)\hat{\mathbf{i}}' + (x\sin\theta - y\cos\theta)\hat{\mathbf{j}}'.\end{aligned} \tag{3.3.24}$$

Therefore the components of the vector transform according to

$$x' = x\cos\theta + y\sin\theta\;, \tag{3.3.25}$$

$$y' = x\sin\theta - y\cos\theta\;. \tag{3.3.26}$$

We now consider an alternate approach to understanding the transformation laws for the components of the position vector of a fixed point in space. In coordinate system S, suppose the position vector $\vec{\mathbf{r}}$ has length $r = |\vec{\mathbf{r}}|$ and makes an angle ϕ with respect to the positive x-axis (Figure 3.25).

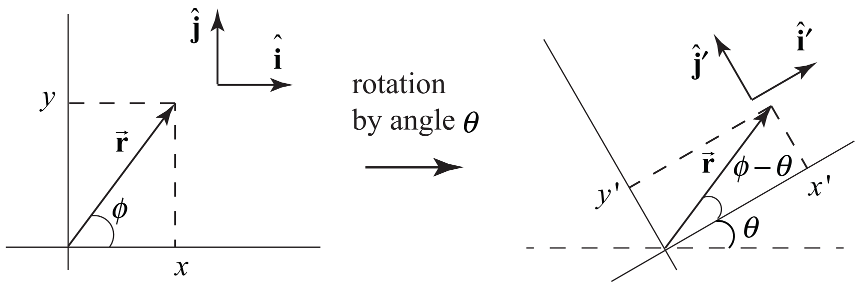

Figure 3.25 Transformation of vector components of the position vector

Then the components of $\vec{\mathbf{r}}$ in S are given by

$$x = r\cos\phi \ , \tag{3.3.27}$$

$$y = r\sin\phi \ . \tag{3.3.28}$$

In coordinate system S', the components of $\vec{\mathbf{r}}$ are given by

$$x' = r\cos(\phi-\theta) \ , \tag{3.3.29}$$

$$y' = r\sin(\phi-\theta) \ . \tag{3.3.30}$$

Apply the addition of angle trigonometric identities to Eqs. (3.3.29) and (3.3.30) yielding

$$x' = r\cos(\phi-\theta) = r\cos\phi\cos\theta + r\sin\phi\sin\theta = x\cos\theta + y\sin\theta \ , \tag{3.3.31}$$

$$y' = r\sin(\phi-\theta) = r\sin\phi\cos\theta - r\cos\phi\sin\theta = y\cos\theta - x\sin\theta \ , \tag{3.3.32}$$

in agreement with Eqs. (3.3.25) and (3.3.26).

Example 3.5 Vector Decomposition in Rotated Coordinate Systems

With respect to a given Cartesian coordinate system S, a vector $\vec{\mathbf{A}}$ has components $A_x = 5$, $A_y = -3$, $A_z = 0$. Consider a second coordinate system S' such that the (x', y') coordinate axes in S' are rotated by an angle $\theta = 60^\circ$ with respect to the (x, y) coordinate axes in S, (Figure 3.26). (a) What are the components $A_{x'}$ and $A_{y'}$ of vector $\vec{\mathbf{A}}$ in coordinate system S'? (b) Calculate the magnitude of the vector using the (A_x, A_y) components and using the $(A_{x'}, A_{y'})$ components. Does your result agree with what you expect?

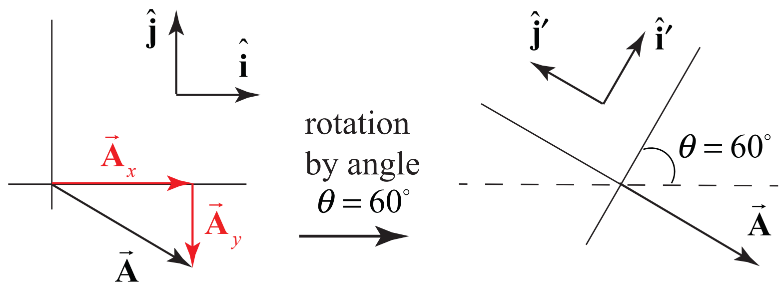

Figure 3.26 Example 3.4

Solution: a) We begin by considering the vector decomposition of $\vec{\mathbf{A}}$ with respect to the coordinate system S,

$$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}}. \tag{3.3.33}$$

Now we can use our results for the transformation of unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ in terms of $\hat{\mathbf{i}}'$ and $\hat{\mathbf{j}}'$, (Eqs. (3.3.22) and (3.3.23)) in order decompose the vector $\vec{\mathbf{A}}$ in coordinate system S'

$$\begin{aligned}\vec{\mathbf{A}} &= A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} = A_x(\cos\theta \hat{\mathbf{i}}' - \sin\theta \hat{\mathbf{j}}') + A_y(\sin\theta \hat{\mathbf{i}}' + \cos\theta \hat{\mathbf{j}}') \\ &= (A_x \cos\theta + A_y \sin\theta)\hat{\mathbf{i}}' + (-A_x \sin\theta + A_y \cos\theta)\hat{\mathbf{j}}' \\ &= A_{x'} \hat{\mathbf{i}} + A_{y'} \hat{\mathbf{j}},\end{aligned} \tag{3.3.34}$$

where

$$A_{x'} = A_x \cos\theta + A_y \sin\theta \tag{3.3.35}$$

$$A_{y'} = -A_x \sin\theta + A_y \cos\theta. \tag{3.3.36}$$

We now use the given information that $A_x = 5$, $A_y = -3$, and $\theta = 60^{\circ}$ to solve for the components of $\vec{\mathbf{A}}$ in coordinate system S'

$$A_{x'} = A_x \cos\theta + A_y \sin\theta = (1/2)(5 - 3\sqrt{3}),$$

$$A_{y'} = -A_x \sin\theta + A_y \cos\theta = (1/2)(-5\sqrt{3} - 3).$$

b) The magnitude can be calculated in either coordinate system

$$\left|\vec{\mathbf{A}}\right| = \sqrt{(A_x)^2 + (A_y)^2} = \sqrt{(5)^2 + (-3)^2} = \sqrt{34}$$

$$\left|\vec{\mathbf{A}}\right| = \sqrt{(A_{x'})^2 + (A_{y'})^2} = \sqrt{\left((1/2)(5 - 3\sqrt{3})\right)^2 + \left((1/2)(-5\sqrt{3} - 3)\right)^2} = \sqrt{34}.$$

This result agrees with what I expect because the length of vector $\vec{\mathbf{A}}$ is independent of the choice of coordinate system.

3.4 Vector Product (Cross Product)

Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors. Because any two non-parallel vectors form a plane, we denote the angle θ to be the angle between the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ as shown in Figure 3.27. The **magnitude of the vector product** $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is defined to be product of the magnitude of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with the sine of the angle θ between the two vectors,

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|\sin(\theta). \qquad (3.3.37)$$

The angle θ between the vectors is limited to the values $0 \le \theta \le \pi$ ensuring that $\sin(\theta) \ge 0$.

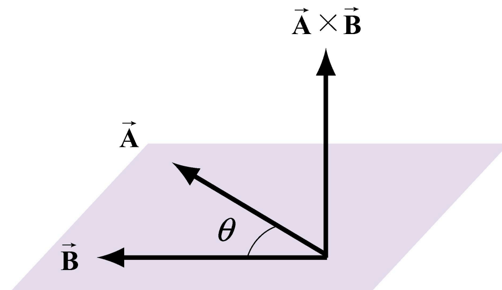

Figure 3.27 Vector product geometry.

The direction of the vector product is defined as follows. The vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ form a plane. Consider the direction perpendicular to this plane. There are two possibilities: we shall choose one of these two (the one shown in Figure 3.27) for the direction of the vector product $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ using a convention that is commonly called the "**right-hand rule**".

3.4.1 Right-hand Rule for the Direction of Vector Product

The first step is to redraw the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ so that the tails are touching. Then draw an arc starting from the vector $\vec{\mathbf{A}}$ and finishing on the vector $\vec{\mathbf{B}}$. Curl your right fingers the same way as the arc. Your right thumb points in the direction of the vector product $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ (Figure 3.28).

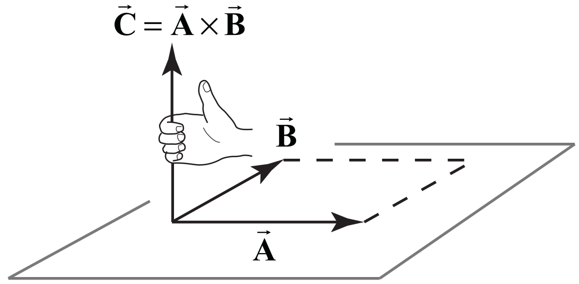

Figure 3.28 Right-Hand Rule.

You should remember that the direction of the vector product $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ is perpendicular to the plane formed by $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. We can give a geometric interpretation to the magnitude of the vector product by writing the magnitude as

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\right|\left(\left|\vec{\mathbf{B}}\right|\sin\theta\right). \tag{3.3.38}$$

The vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ form a parallelogram. The area of the parallelogram is equal to the height times the base, which is the magnitude of the vector product. In Figure 3.29, two different representations of the height and base of a parallelogram are illustrated. As depicted in Figure 3.29(a), the term $\left|\vec{\mathbf{B}}\right|\sin\theta$ is the projection of the vector $\vec{\mathbf{B}}$ in the direction perpendicular to the vector $\vec{\mathbf{B}}$. We could also write the magnitude of the vector product as

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \left(\left|\vec{\mathbf{A}}\right|\sin\theta\right)\left|\vec{\mathbf{B}}\right|. \tag{3.3.39}$$

The term $\left|\vec{\mathbf{A}}\right|\sin\theta$ is the projection of the vector $\vec{\mathbf{A}}$ in the direction perpendicular to the vector $\vec{\mathbf{B}}$ as shown in Figure 3.29(b). The vector product of two vectors that are parallel (or anti-parallel) to each other is zero because the angle between the vectors is 0 (or π) and $\sin(0)=0$ (or $\sin(\pi)=0$). Geometrically, two parallel vectors do not have a unique component perpendicular to their common direction.

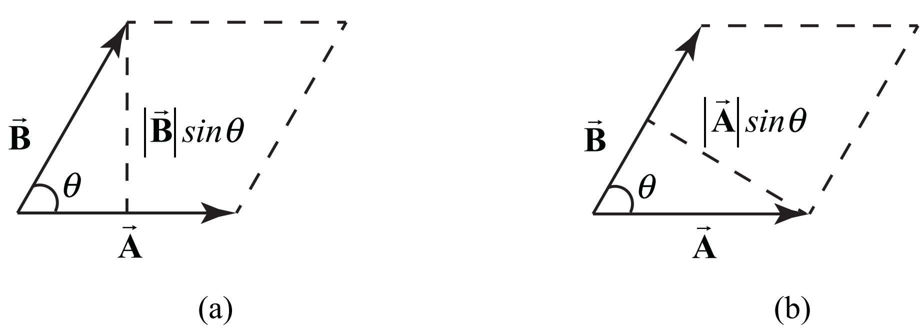

Figure 3.29 Projection of (a) $\vec{\mathbf{B}}$ perpendicular to $\vec{\mathbf{A}}$, (b) of $\vec{\mathbf{A}}$ perpendicular to $\vec{\mathbf{B}}$

3.4.2 Properties of the Vector Product

(1) The vector product is anti-commutative because changing the order of the vectors changes the direction of the vector product by the right hand rule:

$$\vec{\mathbf{A}}\times\vec{\mathbf{B}} = -\vec{\mathbf{B}}\times\vec{\mathbf{A}}. \tag{3.3.40}$$

(2) The vector product between a vector $c\vec{\mathbf{A}}$ where c is a scalar and a vector $\vec{\mathbf{B}}$ is

$$c\vec{\mathbf{A}}\times\vec{\mathbf{B}} = c(\vec{\mathbf{A}}\times\vec{\mathbf{B}}). \tag{3.3.41}$$

Similarly,

$$\vec{\mathbf{A}}\times c\vec{\mathbf{B}} = c(\vec{\mathbf{A}}\times\vec{\mathbf{B}}). \tag{3.3.42}$$

(3) The vector product between the sum of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with a vector $\vec{\mathbf{C}}$ is

$$(\vec{\mathbf{A}}+\vec{\mathbf{B}})\times\vec{\mathbf{C}} = \vec{\mathbf{A}}\times\vec{\mathbf{C}}+\vec{\mathbf{B}}\times\vec{\mathbf{C}} \tag{3.3.43}$$

Similarly,

$$\vec{\mathbf{A}}\times(\vec{\mathbf{B}}+\vec{\mathbf{C}}) = \vec{\mathbf{A}}\times\vec{\mathbf{B}}+\vec{\mathbf{A}}\times\vec{\mathbf{C}}. \tag{3.3.44}$$

3.4.3 Vector Decomposition and the Vector Product: Cartesian Coordinates

We first calculate that the magnitude of vector product of the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$:

$$|\hat{\mathbf{i}}\times\hat{\mathbf{j}}| = |\hat{\mathbf{i}}||\hat{\mathbf{j}}|\sin(\pi/2) = 1, \tag{3.3.45}$$

because the unit vectors have magnitude $|\hat{\mathbf{i}}| = |\hat{\mathbf{j}}| = 1$ and $\sin(\pi/2) = 1$. By the right hand rule, the direction of $\hat{\mathbf{i}}\times\hat{\mathbf{j}}$ is in the $+\hat{\mathbf{k}}$ as shown in Figure 3.30. Thus $\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}$.

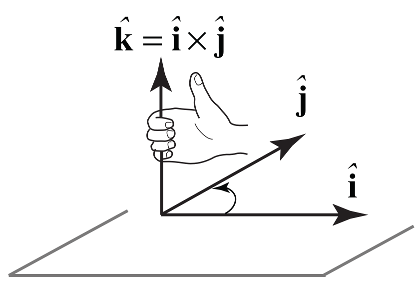

Figure 3.30 Vector product of $\hat{\mathbf{i}}\times\hat{\mathbf{j}}$

We note that the same rule applies for the unit vectors in the *y* and *z* directions,

$$\hat{\mathbf{j}}\times\hat{\mathbf{k}}=\hat{\mathbf{i}},\quad \hat{\mathbf{k}}\times\hat{\mathbf{i}}=\hat{\mathbf{j}}. \tag{3.3.46}$$

By the anti-commutatively property (1) of the vector product,

$$\hat{\mathbf{j}}\times\hat{\mathbf{i}}=-\hat{\mathbf{k}},\quad \hat{\mathbf{i}}\times\hat{\mathbf{k}}=-\hat{\mathbf{j}} \tag{3.3.47}$$

The vector product of the unit vector $\hat{\mathbf{i}}$ with itself is zero because the two unit vectors are parallel to each other, ($\sin(0)=0$),

$$|\hat{\mathbf{i}}\times\hat{\mathbf{i}}|=|\hat{\mathbf{i}}||\hat{\mathbf{i}}|\sin(0)=0. \tag{3.3.48}$$

The vector product of the unit vector $\hat{\mathbf{j}}$ with itself and the unit vector $\hat{\mathbf{k}}$ with itself are also zero for the same reason,

$$\left|\hat{\mathbf{j}}\times\hat{\mathbf{j}}\right|=0,\quad \left|\hat{\mathbf{k}}\times\hat{\mathbf{k}}\right|=0. \tag{3.3.49}$$

With these properties in mind we can now develop an algebraic expression for the vector product in terms of components. Let's choose a Cartesian coordinate system with the vector $\vec{\mathbf{B}}$ pointing along the positive *x*-axis with positive *x*-component B_x. Then the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ can be written as

$$\vec{\mathbf{A}}=A_x\,\hat{\mathbf{i}}+A_y\,\hat{\mathbf{j}}+A_z\,\hat{\mathbf{k}} \tag{3.3.50}$$

$$\vec{\mathbf{B}}=B_x\,\hat{\mathbf{i}}, \tag{3.3.51}$$

respectively. The vector product in vector components is

$$\vec{\mathbf{A}}\times\vec{\mathbf{B}}=(A_x\,\hat{\mathbf{i}}+A_y\,\hat{\mathbf{j}}+A_z\,\hat{\mathbf{k}})\times B_x\,\hat{\mathbf{i}}. \tag{3.3.52}$$

This becomes,

$$\begin{aligned}\vec{\mathbf{A}}\times\vec{\mathbf{B}}&=(A_x\,\hat{\mathbf{i}}\times B_x\,\hat{\mathbf{i}})+(A_y\,\hat{\mathbf{j}}\times B_x\,\hat{\mathbf{i}})+(A_z\,\hat{\mathbf{k}}\times B_x\,\hat{\mathbf{i}})\\&=A_xB_x\,(\hat{\mathbf{i}}\times\hat{\mathbf{i}})+A_yB_x\,(\hat{\mathbf{j}}\times\hat{\mathbf{i}})+A_zB_x\,(\hat{\mathbf{k}}\times\hat{\mathbf{i}})\\&=-A_yB_x\,\hat{\mathbf{k}}+A_zB_x\,\hat{\mathbf{j}}\end{aligned}. \tag{3.3.53}$$

The vector component expression for the vector product easily generalizes for arbitrary vectors

$$\vec{\mathbf{A}}=A_x\,\hat{\mathbf{i}}+A_y\,\hat{\mathbf{j}}+A_z\,\hat{\mathbf{k}} \tag{3.3.54}$$

$$\vec{\mathbf{B}}=B_x\,\hat{\mathbf{i}}+B_y\,\hat{\mathbf{j}}+B_z\,\hat{\mathbf{k}}, \tag{3.3.55}$$

to yield

$$\vec{\mathbf{A}}\times\vec{\mathbf{B}}=(A_yB_z-A_zB_y)\,\hat{\mathbf{i}}+(A_zB_x-A_xB_z)\,\hat{\mathbf{j}}+(A_xB_y-A_yB_x)\,\hat{\mathbf{k}}\,. \tag{3.3.56}$$

3.4.4 Vector Decomposition and the Vector Product: Cylindrical Coordinates

Recall the cylindrical coordinate system, which we show in Figure 3.31. We have chosen two directions, radial and tangential in the plane, and a perpendicular direction to the plane.

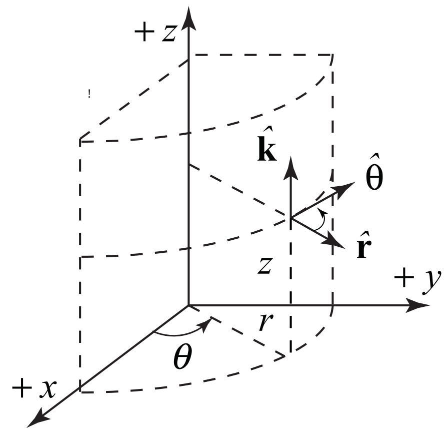

Figure 3.31 Cylindrical coordinates

The unit vectors are at right angles to each other and so using the right hand rule, the vector product of the unit vectors are given by the relations

$$\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}=\hat{\mathbf{k}} \tag{3.3.57}$$
$$\hat{\boldsymbol{\theta}}\times\hat{\mathbf{k}}=\hat{\mathbf{r}} \tag{3.3.58}$$
$$\hat{\mathbf{k}}\times\hat{\mathbf{r}}=\hat{\boldsymbol{\theta}}\,. \tag{3.3.59}$$

Because the vector product satisfies $\vec{\mathbf{A}}\times\vec{\mathbf{B}}=-\vec{\mathbf{B}}\times\vec{\mathbf{A}}$, we also have that

$$\hat{\boldsymbol{\theta}}\times\hat{\mathbf{r}}=-\hat{\mathbf{k}} \tag{3.3.60}$$
$$\hat{\mathbf{k}}\times\hat{\boldsymbol{\theta}}=-\hat{\mathbf{r}} \tag{3.3.61}$$
$$\hat{\mathbf{r}}\times\hat{\mathbf{k}}=-\hat{\boldsymbol{\theta}}\,. \tag{3.3.62}$$

Finally

$$\hat{\mathbf{r}}\times\hat{\mathbf{r}}=\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\theta}}=\hat{\mathbf{k}}\times\hat{\mathbf{k}}=\vec{\mathbf{0}}\,. \tag{3.3.63}$$

Example 3.6 Vector Products

Given two vectors, $\vec{\mathbf{A}}=2\,\hat{\mathbf{i}}+-3\,\hat{\mathbf{j}}+7\,\hat{\mathbf{k}}$ and $\vec{\mathbf{B}}=5\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, find $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$.

Solution:

$$\begin{aligned}\vec{\mathbf{A}}\times\vec{\mathbf{B}} &= (A_yB_z - A_zB_y)\,\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\,\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\,\hat{\mathbf{k}}\\ &= ((-3)(2)-(7)(1))\,\hat{\mathbf{i}} + ((7)(5)-(2)(2))\,\hat{\mathbf{j}} + ((2)(1)-(-3)(5))\,\hat{\mathbf{k}}\\ &= -13\,\hat{\mathbf{i}} + 31\,\hat{\mathbf{j}} + 17\,\hat{\mathbf{k}}.\end{aligned}$$

Example 3.7 Law of Sines

For the triangle shown in Figure 3.32(a), prove the law of sines, $|\vec{\mathbf{A}}|/\sin\alpha = |\vec{\mathbf{B}}|/\sin\beta = |\vec{\mathbf{C}}|/\sin\gamma$, using the vector product.

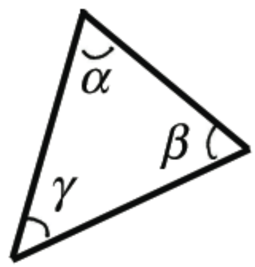

Figure 3.32(a) Example 3.6

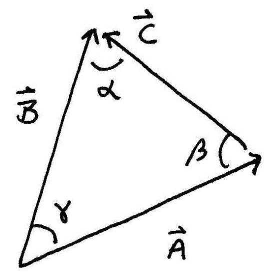

Figure 3.32(b) Vector analysis

Solution: Consider the area of a triangle formed by three vectors $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, and $\vec{\mathbf{C}}$, where $\vec{\mathbf{A}}+\vec{\mathbf{B}}+\vec{\mathbf{C}}=0$ (Figure 3.32(b)). Because $\vec{\mathbf{A}}+\vec{\mathbf{B}}+\vec{\mathbf{C}}=0$, we have that $0=\vec{\mathbf{A}}\times(\vec{\mathbf{A}}+\vec{\mathbf{B}}+\vec{\mathbf{C}})=\vec{\mathbf{A}}\times\vec{\mathbf{B}}+\vec{\mathbf{A}}\times\vec{\mathbf{C}}$. Thus $\vec{\mathbf{A}}\times\vec{\mathbf{B}}=-\vec{\mathbf{A}}\times\vec{\mathbf{C}}$ or $|\vec{\mathbf{A}}\times\vec{\mathbf{B}}|=|\vec{\mathbf{A}}\times\vec{\mathbf{C}}|$. From Figure 17.7b we see that $|\vec{\mathbf{A}}\times\vec{\mathbf{B}}|=|\vec{\mathbf{A}}||\vec{\mathbf{B}}|\sin\gamma$ and $|\vec{\mathbf{A}}\times\vec{\mathbf{C}}|=|\vec{\mathbf{A}}||\vec{\mathbf{C}}|\sin\beta$. Therefore $|\vec{\mathbf{A}}||\vec{\mathbf{B}}|\sin\gamma=|\vec{\mathbf{A}}||\vec{\mathbf{C}}|\sin\beta$, and hence $|\vec{\mathbf{B}}|/\sin\beta=|\vec{\mathbf{C}}|/\sin\gamma$. A similar argument shows that $|\vec{\mathbf{B}}|/\sin\beta=|\vec{\mathbf{A}}|/\sin\alpha$ proving the law of sines.

Example 3.8 Unit Normal

Find a unit vector perpendicular to $\vec{\mathbf{A}}=\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $\vec{\mathbf{B}}=-2\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}}$.

Solution: The vector product $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ is perpendicular to both $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. Therefore the unit vectors $\hat{\mathbf{n}}=\pm\vec{\mathbf{A}}\times\vec{\mathbf{B}}/|\vec{\mathbf{A}}\times\vec{\mathbf{B}}|$ are perpendicular to both $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. We first calculate

$$\begin{aligned}\vec{\mathbf{A}}\times\vec{\mathbf{B}} &= (A_yB_z - A_zB_y)\,\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\,\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\,\hat{\mathbf{k}}\\ &= ((1)(3)-(-1)(-1))\,\hat{\mathbf{i}} + ((-1)(2)-(1)(3))\,\hat{\mathbf{j}} + ((1)(-1)-(1)(2))\,\hat{\mathbf{k}}\\ &= 2\,\hat{\mathbf{i}} - 5\,\hat{\mathbf{j}} - 3\,\hat{\mathbf{k}}.\end{aligned}$$

We now calculate the magnitude

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = (2^2+5^2+3^2)^{1/2} = (38)^{1/2} \quad .$$

Therefore the perpendicular unit vectors are

$$\hat{\mathbf{n}} = \pm\vec{\mathbf{A}}\times\vec{\mathbf{B}}\,/\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \pm(2\,\hat{\mathbf{i}} - 5\,\hat{\mathbf{j}} - 3\,\hat{\mathbf{k}})\,/\,(38)^{1/2} \,.$$

Example 3.9 Volume of Parallelepiped

Show that the volume of a parallelepiped with edges formed by the vectors $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, and $\vec{\mathbf{C}}$ is given by $\vec{\mathbf{A}}\cdot(\vec{\mathbf{B}}\times\vec{\mathbf{C}})$.

Solution: The volume of a parallelepiped is given by area of the base times height. If the base is formed by the vectors $\vec{\mathbf{B}}$ and $\vec{\mathbf{C}}$, then the area of the base is given by the magnitude of $\vec{\mathbf{B}}\times\vec{\mathbf{C}}$. The vector $\vec{\mathbf{B}}\times\vec{\mathbf{C}} = \left|\vec{\mathbf{B}}\times\vec{\mathbf{C}}\right|\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the base (Figure 3.33).

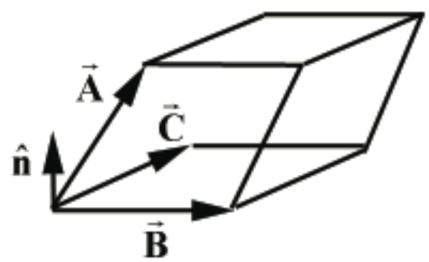

Figure 3.33 Example 3.9

The projection of the vector $\vec{\mathbf{A}}$ along the direction $\hat{\mathbf{n}}$ gives the height of the parallelepiped. This projection is given by taking the dot product of $\vec{\mathbf{A}}$ with a unit vector and is equal to $\vec{\mathbf{A}}\cdot\hat{\mathbf{n}} = height$. Therefore

$$\vec{\mathbf{A}}\cdot(\vec{\mathbf{B}}\times\vec{\mathbf{C}}) = \vec{\mathbf{A}}\cdot(\left|\vec{\mathbf{B}}\times\vec{\mathbf{C}}\right|)\hat{\mathbf{n}} = (\left|\vec{\mathbf{B}}\times\vec{\mathbf{C}}\right|)\vec{\mathbf{A}}\cdot\hat{\mathbf{n}} = (area)(height) = (volume)\,.$$

Example 3.10 Vector Decomposition

Let $\vec{\mathbf{A}}$ be an arbitrary vector and let $\hat{\mathbf{n}}$ be a unit vector in some fixed direction. Show that $\vec{\mathbf{A}} = (\vec{\mathbf{A}}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + (\hat{\mathbf{n}}\times\vec{\mathbf{A}})\times\hat{\mathbf{n}}$.

Solution: Let $\vec{\mathbf{A}} = A_{\parallel}\hat{\mathbf{n}} + A_{\perp}\hat{\mathbf{e}}$ where $A_{\parallel}$ is the component $\vec{\mathbf{A}}$ in the direction of $\hat{\mathbf{n}}$, $\hat{\mathbf{e}}$ is the direction of the projection of $\vec{\mathbf{A}}$ in a plane perpendicular to $\hat{\mathbf{n}}$, and $A_{\perp}$ is the component of $\vec{\mathbf{A}}$ in the direction of $\hat{\mathbf{e}}$. Because $\hat{\mathbf{e}}\cdot\hat{\mathbf{n}}=0$, we have that $\vec{\mathbf{A}}\cdot\hat{\mathbf{n}} = A_{\parallel}$. Note that

$$\hat{\mathbf{n}}\times\vec{\mathbf{A}} = \hat{\mathbf{n}}\times(A_{\parallel}\hat{\mathbf{n}} + A_{\perp}\hat{\mathbf{e}}) = \hat{\mathbf{n}}\times A_{\perp}\hat{\mathbf{e}} = A_{\perp}(\hat{\mathbf{n}}\times\hat{\mathbf{e}}).$$

The unit vector $\hat{\mathbf{n}}\times\hat{\mathbf{e}}$ lies in the plane perpendicular to $\hat{\mathbf{n}}$ and is also perpendicular to $\hat{\mathbf{e}}$. Therefore $(\hat{\mathbf{n}}\times\hat{\mathbf{e}})\times\hat{\mathbf{n}}$ is also a unit vector that is parallel to $\hat{\mathbf{e}}$ (by the right hand rule. So $(\hat{\mathbf{n}}\times\vec{\mathbf{A}})\times\hat{\mathbf{n}} = A_{\perp}\hat{\mathbf{e}}$. Thus

$$\vec{\mathbf{A}} = A_{\parallel}\hat{\mathbf{n}} + A_{\perp}\hat{\mathbf{e}} = (\vec{\mathbf{A}}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + (\hat{\mathbf{n}}\times\vec{\mathbf{A}})\times\hat{\mathbf{n}}.$$