

Chapter 23 Simple Harmonic Motion

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Chapter 23 Simple Harmonic Motion

...Indeed it is not in the nature of a simple pendulum to provide equal and reliable measurements of time, since the wide lateral excursions often made may be observed to be slower than more narrow ones; however, we have been led in a different direction by geometry, from which we have found a means of suspending the pendulum, with which we were previously unacquainted, and by giving close attention to a line with a certain curvature, the time of the swing can be chosen equal to some calculated value and is seen clearly in practice to be in wonderful agreement with that ratio. As we have checked the lapses of time measured by these clocks after making repeated land and sea trials, the effects of motion are seen to have been avoided, so sure and reliable are the measurements; now it can be seen that both astronomical studies and the art of navigation will be greatly helped by them...¹

Christian Huygens

23.1 Introduction: Periodic Motion

There are two basic ways to measure time: by duration or periodic motion. Early clocks measured duration by calibrating the burning of incense or wax, or the flow of water or sand from a container. Our calendar consists of years determined by the motion of the sun; months determined by the motion of the moon; days by the rotation of the earth; hours by the motion of cyclic motion of gear trains; and seconds by the oscillations of springs or pendulums. In modern times a second is defined by a specific number of vibrations of radiation, corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.

Sundials calibrate the motion of the sun through the sky, including seasonal corrections. A clock escapement is a device that can transform continuous movement into discrete movements of a gear train. The early escapements used oscillatory motion to stop and start the turning of a weight-driven rotating drum. Soon, complicated escapements were regulated by pendulums, the theory of which was first developed by the physicist Christian Huygens in the mid 17th century. The accuracy of clocks was increased and the size reduced by the discovery of the oscillatory properties of springs by Robert Hooke. By the middle of the 18th century, the technology of timekeeping advanced to the point that William Harrison developed timekeeping devices that were accurate to one second in a century.

23.1.1 Simple Harmonic Motion: Quantitative

¹ Christian Huygens, *The Pendulum Clock or The Motion of Pendulums Adapted to Clocks By Geometrical Demonstrations*, tr. Ian Bruce, p. 1.

One of the most important examples of periodic motion is ***simple harmonic motion*** (SHM), in which some physical quantity varies sinusoidally. Suppose a function of time has the form of a sine wave function,

$$y(t) = A \sin(2\pi t / T) \quad (23.1.1)$$

where $A > 0$ is the ***amplitude*** (maximum value). The function $y(t)$ varies between A and $-A$, because a sine function varies between +1 and -1. A plot of $y(t)$ vs. time is shown in Figure 23.1.

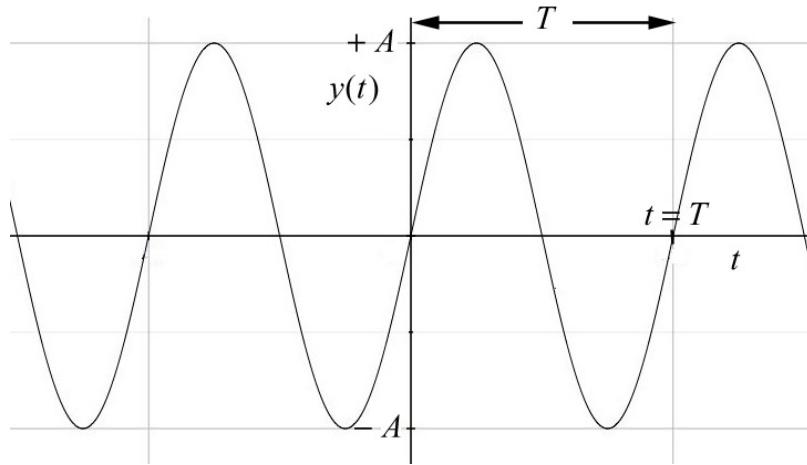


Figure 23.1 Sinusoidal function of time

The sine function is periodic in time. This means that the value of the function at time t will be exactly the same at a later time $t' = t + T$, where T is the ***period***. That the sine function satisfies the periodic condition can be seen from

$$y(t+T) = A \sin\left[\frac{2\pi}{T}(t+T)\right] = A \sin\left[\frac{2\pi}{T}t + 2\pi\right] = A \sin\left[\frac{2\pi}{T}t\right] = y(t). \quad (23.1.2)$$

The ***frequency***, f , is defined to be

$$f \equiv 1/T. \quad (23.1.3)$$

The SI unit of frequency is inverse seconds, $\left[s^{-1}\right]$, or hertz [Hz]. The ***angular frequency*** of oscillation is defined to be

$$\omega_0 \equiv 2\pi / T = 2\pi f, \quad (23.1.4)$$

and is measured in radians per second. (The angular frequency of oscillation is denoted by ω_0 to distinguish from the angular speed $\omega = |d\theta / dt|$.) One oscillation per second, 1 Hz, corresponds to an angular frequency of $2\pi \text{ rad} \cdot \text{s}^{-1}$. (Unfortunately, the same

symbol ω is used for angular speed in circular motion. For uniform circular motion the angular speed is equal to the angular frequency but for non-uniform motion the angular speed is not constant. The angular frequency for simple harmonic motion is a constant by definition.) We therefore have several different mathematical representations for sinusoidal motion

$$y(t) = A \sin(2\pi t / T) = A \sin(2\pi f t) = A \sin(\omega_0 t) . \quad (23.1.5)$$

23.2 Simple Harmonic Motion: Analytic

Our first example of a system that demonstrates simple harmonic motion is a spring-object system on a frictionless surface, shown in Figure 23.2

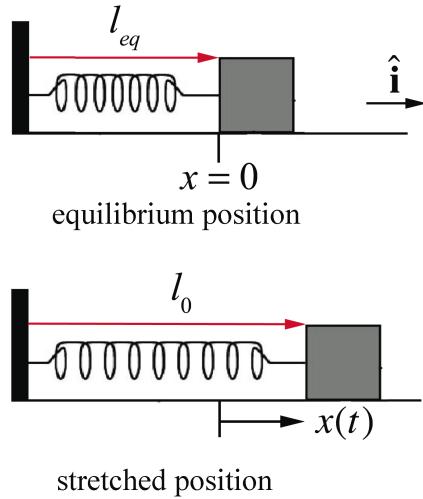


Figure 23.2 Spring-object system

The object is attached to one end of a spring. The other end of the spring is attached to a wall at the left in Figure 23.2. Assume that the object undergoes one-dimensional motion. The spring has a spring constant k and equilibrium length l_{eq} . Choose the origin at the equilibrium position and choose the positive x -direction to the right in the Figure 23.2. In the figure, $x > 0$ corresponds to an extended spring, and $x < 0$ to a compressed spring. Define $x(t)$ to be the position of the object with respect to the equilibrium position. The force acting on the spring is a linear restoring force, $F_x = -k x$ (Figure 23.3). The initial conditions are as follows. The spring is initially stretched a distance l_0 and given some initial speed v_0 to the right away from the equilibrium position. The initial position of the stretched spring from the equilibrium position (our choice of origin) is $x_0 = (l_0 - l_{eq}) > 0$ and its initial x -component of the velocity is $v_{x,0} = v_0 > 0$.

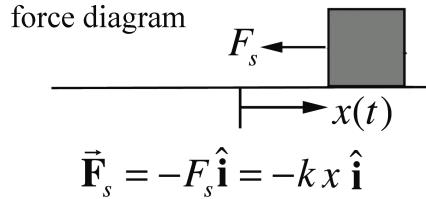


Figure 23.3 Free-body force diagram for spring-object system

Newton's Second law in the x -direction becomes

$$-kx = m \frac{d^2x}{dt^2}. \quad (23.2.1)$$

This equation of motion, Eq. (23.2.1), is called the **simple harmonic oscillator equation** (SHO). Because the spring force depends on the distance x , the acceleration is not constant. Eq. (23.2.1) is a second order linear differential equation, in which the second derivative of the dependent variable is proportional to the negative of the dependent variable,

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x. \quad (23.2.2)$$

In this case, the constant of proportionality is k/m ,

Eq. (23.2.2) can be solved from energy considerations or other advanced techniques but instead we shall first guess the solution and then verify that the guess satisfies the SHO differential equation (see Appendix 22.3.A for a derivation of the solution).

We are looking for a position function $x(t)$ such that the second time derivative position function is proportional to the negative of the position function. Since the sine and cosine functions both satisfy this property, we make a preliminary *ansatz* (educated guess) that our position function is given by

$$x(t) = A \cos((2\pi/T)t) = A \cos(\omega_0 t), \quad (23.2.3)$$

where ω_0 is the angular frequency (as of yet, undetermined).

We shall now find the condition that the angular frequency ω_0 must satisfy in order to insure that the function in Eq. (23.2.3) solves the simple harmonic oscillator equation, Eq. (23.2.1). The first and second derivatives of the position function are given by

$$\begin{aligned}\frac{dx}{dt} &= -\omega_0 A \sin(\omega_0 t) \\ \frac{d^2x}{dt^2} &= -\omega_0^2 A \cos(\omega_0 t) = -\omega_0^2 x.\end{aligned}\tag{23.2.4}$$

Substitute the second derivative, the second expression in Eq. (23.2.4), and the position function, Equation (23.2.3), into the SHO Equation (23.2.1), yielding

$$-\omega_0^2 A \cos(\omega_0 t) = -\frac{k}{m} A \cos(\omega_0 t).\tag{23.2.5}$$

Eq. (23.2.5) is valid for all times provided that

$$\omega_0 = \sqrt{\frac{k}{m}}.\tag{23.2.6}$$

The period of oscillation is then

$$T = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{m}{k}}.\tag{23.2.7}$$

One possible solution for the position of the block is

$$x(t) = A \cos\left(\sqrt{\frac{k}{m}} t\right),\tag{23.2.8}$$

and therefore by differentiation, the x -component of the velocity of the block is

$$v_x(t) = -\sqrt{\frac{k}{m}} A \sin\left(\sqrt{\frac{k}{m}} t\right).\tag{23.2.9}$$

Note that at $t = 0$, the position of the object is $x_0 \equiv x(t = 0) = A$ since $\cos(0) = 1$ and the velocity is $v_{x,0} \equiv v_x(t = 0) = 0$ since $\sin(0) = 0$. The solution in (23.2.8) describes an object that is released from rest at an initial position $A = x_0$ but does not satisfy the initial velocity condition, $v_x(t = 0) = v_{x,0} \neq 0$. We can try a sine function as another possible solution,

$$x(t) = B \sin\left(\sqrt{\frac{k}{m}} t\right).\tag{23.2.10}$$

This function also satisfies the simple harmonic oscillator equation because

$$\frac{d^2x}{dt^2} = -\frac{k}{m} B \sin\left(\sqrt{\frac{k}{m}} t\right) = -\omega_0^2 x, \quad (23.2.11)$$

where $\omega_0 = \sqrt{k/m}$. The x -component of the velocity associated with Eq. (23.2.10) is

$$v_x(t) = \frac{dx}{dt} = \sqrt{\frac{k}{m}} B \cos\left(\sqrt{\frac{k}{m}} t\right). \quad (23.2.12)$$

The proposed solution in Eq. (23.2.10) has initial conditions $x_0 \equiv x(t=0) = 0$ and $v_{x,0} \equiv v_x(t=0) = (\sqrt{k/m})B$, thus $B = v_{x,0} / \sqrt{k/m}$. This solution describes an object that is initially at the equilibrium position but has an initial non-zero x -component of the velocity, $v_{x,0} \neq 0$.

23.2.1 General Solution of Simple Harmonic Oscillator Equation

Suppose $x_1(t)$ and $x_2(t)$ are both solutions of the simple harmonic oscillator equation,

$$\begin{aligned} \frac{d^2}{dt^2} x_1(t) &= -\frac{k}{m} x_1(t) \\ \frac{d^2}{dt^2} x_2(t) &= -\frac{k}{m} x_2(t). \end{aligned} \quad (23.2.13)$$

Then the sum $x(t) = x_1(t) + x_2(t)$ of the two solutions is also a solution. To see this, consider

$$\frac{d^2x(t)}{dt^2} = \frac{d^2}{dt^2}(x_1(t) + x_2(t)) = \frac{d^2x_1(t)}{dt^2} + \frac{d^2x_2(t)}{dt^2}. \quad (23.2.14)$$

Using the fact that $x_1(t)$ and $x_2(t)$ both solve the simple harmonic oscillator equation (23.2.13), we see that

$$\begin{aligned} \frac{d^2}{dt^2} x(t) &= -\frac{k}{m} x_1(t) + -\frac{k}{m} x_2(t) = -\frac{k}{m} (x_1(t) + x_2(t)) \\ &= -\frac{k}{m} x(t). \end{aligned} \quad (23.2.15)$$

Thus the *linear combination* $x(t) = x_1(t) + x_2(t)$ is also a solution of the SHO equation, Eq. (23.2.1). Therefore the sum of the sine and cosine solutions is the *general solution*,

$$x(t) = C \cos(\omega_0 t) + D \sin(\omega_0 t), \quad (23.2.16)$$

where the constant coefficients C and D depend on a given set of initial conditions $x_0 \equiv x(t = 0)$ and $v_{x,0} \equiv v_x(t = 0)$ where x_0 and $v_{x,0}$ are constants. For this general solution, the x -component of the velocity of the object at time t is then obtained by differentiating the position function,

$$v_x(t) = \frac{dx}{dt} = -\omega_0 C \sin(\omega_0 t) + \omega_0 D \cos(\omega_0 t). \quad (23.2.17)$$

To find the constants C and D , substitute $t = 0$ into the Eqs. (23.2.16) and (23.2.17). Because $\cos(0) = 1$ and $\sin(0) = 0$, the initial position at time $t = 0$ is

$$x_0 \equiv x(t = 0) = C. \quad (23.2.18)$$

The x -component of the velocity at time $t = 0$ is

$$v_{x,0} = v_x(t = 0) = -\omega_0 C \sin(0) + \omega_0 D \cos(0) = \omega_0 D. \quad (23.2.19)$$

Thus

$$C = x_0 \text{ and } D = \frac{v_{x,0}}{\omega_0}. \quad (23.2.20)$$

The position of the object-spring system is then given by

$$x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}t\right) + \frac{v_{x,0}}{\sqrt{k/m}} \sin\left(\sqrt{\frac{k}{m}}t\right) \quad (23.2.21)$$

and the x -component of the velocity of the object-spring system is

$$v_x(t) = -\sqrt{\frac{k}{m}}x_0 \sin\left(\sqrt{\frac{k}{m}}t\right) + v_{x,0} \cos\left(\sqrt{\frac{k}{m}}t\right). \quad (23.2.22)$$

Although we had previously specified $x_0 > 0$ and $v_{x,0} > 0$, Eq. (23.2.21) is seen to be a valid solution of the SHO equation for any values of x_0 and $v_{x,0}$.

Example 23.1: Phase and Amplitude

Show that $x(t) = C \cos \omega_0 t + D \sin \omega_0 t = A \cos(\omega_0 t + \phi)$, where $A = (C^2 + D^2)^{1/2} > 0$, and $\phi = \tan^{-1}(-D/C)$.

Solution: Use the identity $A\cos(\omega_0 t + \phi) = A\cos(\omega_0 t)\cos(\phi) - A\sin(\omega_0 t)\sin(\phi)$. Thus $C\cos(\omega_0 t) + D\sin(\omega_0 t) = A\cos(\omega_0 t)\cos(\phi) - A\sin(\omega_0 t)\sin(\phi)$. Comparing coefficients we see that $C = A\cos\phi$ and $D = -A\sin\phi$. Therefore

$$(C^2 + D^2)^{1/2} = A^2(\cos^2\phi + \sin^2\phi) = A^2.$$

We choose the positive square root to ensure that $A > 0$, and thus

$$A = (C^2 + D^2)^{1/2} \quad (23.2.23)$$

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{-D/A}{C/A} = -\frac{D}{C},$$

$$\phi = \tan^{-1}(-D/C). \quad (23.2.24)$$

Thus the position as a function of time can be written as

$$x(t) = A\cos(\omega_0 t + \phi). \quad (23.2.25)$$

In Eq. (23.2.25) the quantity $\omega_0 t + \phi$ is called the **phase**, and ϕ is called the **phase constant**. Because $\cos(\omega_0 t + \phi)$ varies between +1 and -1, and $A > 0$, A is the amplitude defined earlier. We now substitute Eq. (23.2.20) into Eq. (23.2.23) and find that the amplitude of the motion described in Equation (23.2.21), that is, the maximum value of $x(t)$, and the phase are given by

$$A = \sqrt{x_0^2 + (v_{x,0}/\omega_0)^2}. \quad (23.2.26)$$

$$\phi = \tan^{-1}(-v_{x,0}/\omega_0 x_0). \quad (23.2.27)$$

A plot of $x(t)$ vs. t is shown in Figure 23.4a with the values $A = 3$, $T = \pi$, and $\phi = \pi/4$. Note that $x(t) = A\cos(\omega_0 t + \phi)$ takes on its maximum value when $\cos(\omega_0 t + \phi) = 1$. This occurs when $\omega_0 t + \phi = 2\pi n$ where $n = 0, \pm 1, \pm 2, \dots$. The maximum value associated with $n = 0$ occurs when $\omega_0 t + \phi = 0$ or $t = -\phi/\omega_0$. For the case shown in Figure 23.4a where $\phi = \pi/4$, this maximum occurs at the instant $t = -T/8$. Let's plot $x(t) = A\cos(\omega_0 t + \phi)$ vs. t for $\phi = 0$ (Figure 23.4b). For $\phi > 0$, Figure 23.4a shows the plot $x(t) = A\cos(\omega_0 t + \phi)$ vs. t . Notice that when $\phi > 0$, $x(t)$ is shifted to the left compared with the case $\phi = 0$ (compare Figures 23.4a with 23.4b). The function $x(t) = A\cos(\omega_0 t + \phi)$ with $\phi > 0$ reaches its maximum value at an earlier time than the function $x(t) = A\cos(\omega_0 t)$. The difference in phases for these two cases is $(\omega_0 t + \phi) - \omega_0 t = \phi$ and ϕ is sometimes referred to as the *phase shift*. When $\phi < 0$, the

function $x(t) = A\cos(\omega_0 t + \phi)$ reaches its maximum value at a later time $t = T / 8$ than the function $x(t) = A\cos(\omega_0 t)$ as shown in Figure 23.4c.

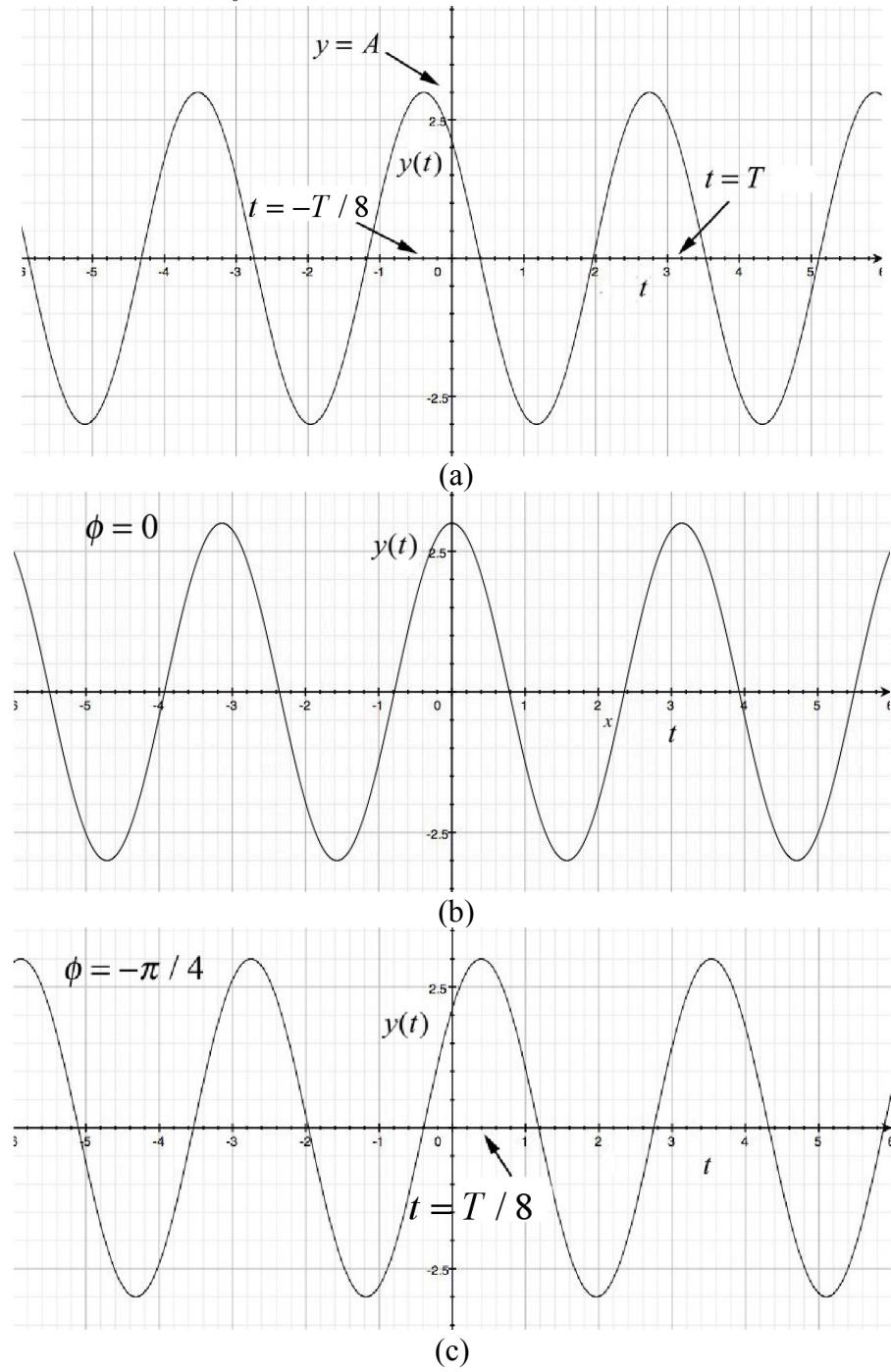


Figure 23.4 Phase shift of $x(t) = A\cos(\omega_0 t + \phi)$ (a) to the left by $\phi = \pi / 4$, (b) no shift $\phi = 0$, (c) to the right $\phi = -\pi / 4$

Example 23.2: Block-Spring System

A block of mass m is attached to a spring with spring constant k and is free to slide along a horizontal frictionless surface. At $t = 0$, the block-spring system is stretched an amount $x_0 > 0$ from the equilibrium position and is released from rest, $v_{x,0} = 0$. What is the period of oscillation of the block? What is the velocity of the block when it first comes back to the equilibrium position?

Solution: The position of the block can be determined from Eq. (23.2.21) by substituting the initial conditions $x_0 > 0$, and $v_{x,0} = 0$ yielding

$$x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}} t\right), \quad (23.2.28)$$

and the x -component of its velocity is given by Eq. (23.2.22),

$$v_x(t) = -\sqrt{\frac{k}{m}} x_0 \sin\left(\sqrt{\frac{k}{m}} t\right). \quad (23.2.29)$$

The angular frequency of oscillation is $\omega_0 = \sqrt{k/m}$ and the period is given by Eq. (23.2.7),

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}. \quad (23.2.30)$$

The block first reaches equilibrium when the position function first reaches zero. This occurs at time t_1 satisfying

$$\sqrt{\frac{k}{m}} t_1 = \frac{\pi}{2}, \quad t_1 = \frac{\pi}{2} \sqrt{\frac{m}{k}} = \frac{T}{4}. \quad (23.2.31)$$

The x -component of the velocity at time t_1 is then

$$v_x(t_1) = -\sqrt{\frac{k}{m}} x_0 \sin\left(\sqrt{\frac{k}{m}} t_1\right) = -\sqrt{\frac{k}{m}} x_0 \sin(\pi/2) = -\sqrt{\frac{k}{m}} x_0 = -\omega_0 x_0 \quad (23.2.32)$$

Note that the block is moving in the negative x -direction at time t_1 ; the block has moved from a positive initial position to the equilibrium position (Figure 23.4(b)).

23.3 Energy and the Simple Harmonic Oscillator

Let's consider the block-spring system of Example 23.2 in which the block is initially stretched an amount $x_0 > 0$ from the equilibrium position and is released from rest, $v_{x,0} = 0$. We shall consider three states: state 1, the initial state; state 2, at an arbitrary time in which the position and velocity are non-zero; and state 3, when the object first comes back to the equilibrium position. We shall show that the mechanical energy has the same value for each of these states and is constant throughout the motion. Choose the equilibrium position for the zero point of the potential energy.

State 1: all the energy is stored in the object-spring potential energy, $U_1 = (1/2)kx_0^2$. The object is released from rest so the kinetic energy is zero, $K_1 = 0$. The total mechanical energy is then

$$E_1 = U_1 = \frac{1}{2}kx_0^2. \quad (23.3.1)$$

State 2: at some time t , the position and x -component of the velocity of the object are given by

$$\begin{aligned} x(t) &= x_0 \cos\left(\sqrt{\frac{k}{m}}t\right) \\ v_x(t) &= -\sqrt{\frac{k}{m}}x_0 \sin\left(\sqrt{\frac{k}{m}}t\right). \end{aligned} \quad (23.3.2)$$

The kinetic energy is

$$K_2 = \frac{1}{2}mv^2 = \frac{1}{2}kx_0^2 \sin^2\left(\sqrt{\frac{k}{m}}t\right), \quad (23.3.3)$$

and the potential energy is

$$U_2 = \frac{1}{2}kx^2 = \frac{1}{2}kx_0^2 \cos^2\left(\sqrt{\frac{k}{m}}t\right). \quad (23.3.4)$$

The mechanical energy is the sum of the kinetic and potential energies

$$\begin{aligned} E_2 &= K_2 + U_2 = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 \\ &= \frac{1}{2}kx_0^2 \left(\cos^2\left(\sqrt{\frac{k}{m}}t\right) + \sin^2\left(\sqrt{\frac{k}{m}}t\right) \right) \\ &= \frac{1}{2}kx_0^2, \end{aligned} \quad (23.3.5)$$

where we used the identity that $\cos^2 \omega_0 t + \sin^2 \omega_0 t = 1$, and that $\omega_0 = \sqrt{k/m}$ (Eq. (23.2.6)).

The mechanical energy in state 2 is equal to the initial potential energy in state 1, so the mechanical energy is constant. This should come as no surprise; we isolated the object-spring system so that there is no external work performed on the system and no internal non-conservative forces doing work.

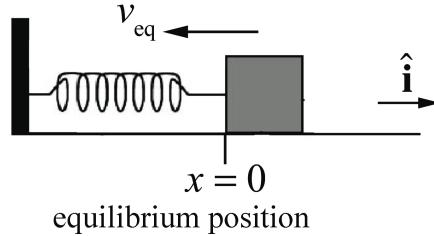


Figure 23.5 State 3 at equilibrium and in motion

State 3: now the object is at the equilibrium position so the potential energy is zero, $U_3 = 0$, and the mechanical energy is in the form of kinetic energy (Figure 23.5).

$$E_3 = K_3 = \frac{1}{2} m v_{\text{eq}}^2. \quad (23.3.6)$$

Because the system is closed, mechanical energy is constant,

$$E_1 = E_3. \quad (23.3.7)$$

Therefore the initial stored potential energy is released as kinetic energy,

$$\frac{1}{2} k x_0^2 = \frac{1}{2} m v_{\text{eq}}^2, \quad (23.3.8)$$

and the x -component of velocity at the equilibrium position is given by

$$v_{x,\text{eq}} = \pm \sqrt{\frac{k}{m}} x_0. \quad (23.3.9)$$

Note that the plus-minus sign indicates that when the block is at equilibrium, there are two possible motions: in the positive x -direction or the negative x -direction. If we take $x_0 > 0$, then the block starts moving towards the origin, and $v_{x,\text{eq}}$ will be negative the first time the block moves through the equilibrium position.

We can show more generally that the mechanical energy is constant at all times as follows. The mechanical energy at an arbitrary time is given by

$$E = K + U = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 . \quad (23.3.10)$$

Differentiate Eq. (23.3.10)

$$\frac{dE}{dt} = mv_x \frac{dv_x}{dt} + kx \frac{dx}{dt} = v_x \left(m \frac{d^2x}{dt^2} + kx \right) . \quad (23.3.11)$$

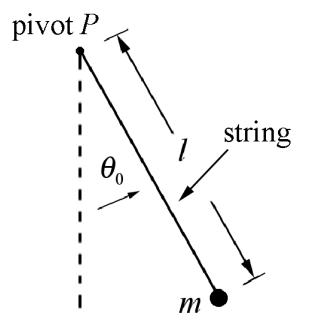
Now substitute the simple harmonic oscillator equation of motion, (Eq. (23.2.1)) into Eq. (23.3.11) yielding

$$\frac{dE}{dt} = 0 , \quad (23.3.12)$$

demonstrating that the mechanical energy is a constant of the motion.

23.3.1 Simple Pendulum: Force Approach

A pendulum consists of an object hanging from the end of a string or rigid rod pivoted about the point P . The object is pulled to one side and allowed to oscillate. If the object has negligible size and the string or rod is massless, then the pendulum is called a **simple pendulum**. Consider a simple pendulum consisting of a massless string of length l and a point-like object of mass m attached to one end, called the *bob*. Suppose the string is fixed at the other end and is initially pulled out at an angle θ_0 from the vertical and released from rest (Figure 23.6). Neglect any dissipation due to air resistance or frictional forces acting at the pivot.



object released from rest

Figure 23.6 Simple pendulum

Let's choose polar coordinates for the pendulum as shown in Figure 23.7a along with the free-body force diagram for the suspended object (Figure 23.7b). The angle θ is defined with respect to the equilibrium position. When $\theta > 0$, the bob is has moved to the right, and when $\theta < 0$, the bob has moved to the left. The object will move in a circular arc centered at the pivot point. The forces on the object are the tension in the string

$\vec{T} = -T \hat{\mathbf{r}}$ and gravity $m\vec{g}$. The gravitation force on the object has $\hat{\mathbf{r}}$ - and $\hat{\theta}$ -components given by

$$m\vec{g} = mg(\cos\theta \hat{\mathbf{r}} - \sin\theta \hat{\theta}). \quad (23.3.13)$$

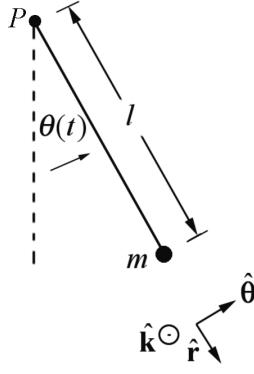


Figure 23.7 (a) Coordinate system

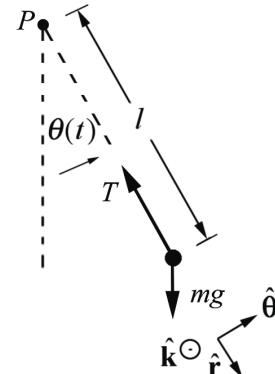


Figure 23.7 (b) free-body force diagram

Our concern is with the tangential component of the gravitational force,

$$F_\theta = -mg \sin\theta. \quad (23.3.14)$$

The sign in Eq. (23.3.14) is crucial; the tangential force tends to restore the pendulum to the equilibrium value $\theta = 0$. If $\theta > 0$, $F_\theta < 0$ and if $\theta < 0$, $F_\theta > 0$, where we are that because the string is flexible, the angle θ is restricted to the range $-\pi/2 < \theta < \pi/2$. (For angles $|\theta| > \pi/2$, the string would go slack.) In both instances the tangential component of the force is directed towards the equilibrium position. The tangential component of acceleration is

$$a_\theta = l\alpha = l \frac{d^2\theta}{dt^2}. \quad (23.3.15)$$

Newton's Second Law, $F_\theta = ma_\theta$, yields

$$-mgl \sin\theta = ml^2 \frac{d^2\theta}{dt^2}. \quad (23.3.16)$$

We can rewrite this equation is the form

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin\theta. \quad (23.3.17)$$

This is not the simple harmonic oscillator equation although it still describes periodic motion. In the limit of small oscillations, $\sin\theta \approx \theta$, Eq. (23.3.17) becomes

$$\frac{d^2\theta}{dt^2} \equiv -\frac{g}{l}\theta. \quad (23.3.18)$$

This equation is similar to the object-spring simple harmonic oscillator differential equation

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x. \quad (23.3.19)$$

By comparison with Eq. (23.2.6) the angular frequency of oscillation for the pendulum is approximately

$$\omega_0 \approx \sqrt{\frac{g}{l}}, \quad (23.3.20)$$

with period

$$T = \frac{2\pi}{\omega_0} \approx 2\pi\sqrt{\frac{l}{g}}. \quad (23.3.21)$$

The solutions to Eq. (23.3.18) can be modeled after Eq. (23.2.21). With the initial conditions that the pendulum is released from rest, $\frac{d\theta}{dt}(t=0)=0$, at a small angle $\theta(t=0)=\theta_0$, the angle the string makes with the vertical as a function of time is given by

$$\theta(t) = \theta_0 \cos(\omega_0 t) = \theta_0 \cos\left(\frac{2\pi}{T}t\right) = \theta_0 \cos\left(\sqrt{\frac{g}{l}}t\right). \quad (23.3.22)$$

The z -component of the angular velocity of the bob is

$$\omega_z(t) = \frac{d\theta}{dt}(t) = -\sqrt{\frac{g}{l}}\theta_0 \sin\left(\sqrt{\frac{g}{l}}t\right). \quad (23.3.23)$$

Keep in mind that the component of the angular velocity $\omega_z = d\theta / dt$ changes with time in an oscillatory manner (sinusoidally in the limit of small oscillations). The angular frequency ω_0 is a parameter that describes the system. The z -component of the angular velocity $\omega_z(t)$, besides being time-dependent, depends on the amplitude of oscillation θ_0 . In the limit of small oscillations, ω_0 does not depend on the amplitude of oscillation.

The fact that the period is independent of the mass of the object follows algebraically from the fact that the mass appears on both sides of Newton's Second Law and hence cancels. Consider also the argument that is attributed to Galileo: if a pendulum, consisting of two identical masses joined together, were set to oscillate, the two halves would not exert forces on each other. So, if the pendulum were split into two pieces, the

pieces would oscillate the same as if they were one piece. This argument can be extended to simple pendula of arbitrary masses.

23.3.2 Simple Pendulum: Energy Approach

We can use energy methods to find the differential equation describing the time evolution of the angle θ . When the string is at an angle θ with respect to the vertical, the gravitational potential energy (relative to a choice of zero potential energy at the bottom of the swing where $\theta = 0$ as shown in Figure 23.8) is given by

$$U = mgl(1 - \cos\theta) \quad (23.3.24)$$

The θ -component of the velocity of the object is given by $v_\theta = l(d\theta/dt)$ so the kinetic energy is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2. \quad (23.3.25)$$

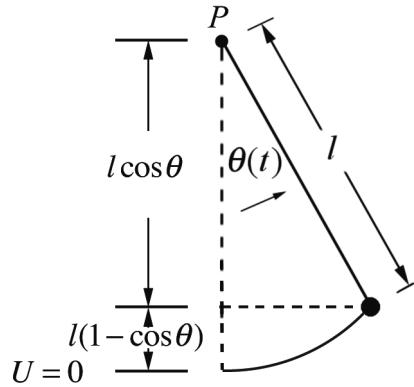


Figure 23.8 Energy diagram for simple pendulum

The mechanical energy of the system is then

$$E = K + U = \frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2 + mgl(1 - \cos\theta). \quad (23.3.26)$$

Because we assumed that there is no non-conservative work (i.e. no air resistance or frictional forces acting at the pivot), the energy is constant, hence

$$\begin{aligned} 0 &= \frac{dE}{dt} = \frac{1}{2}m2l^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + mgl\sin\theta\frac{d\theta}{dt} \\ &= ml^2\frac{d\theta}{dt}\left(\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta\right). \end{aligned} \quad (23.3.27)$$

There are two solutions to this equation; the first one $d\theta/dt = 0$ is the equilibrium solution. That the z -component of the angular velocity is zero means the suspended object is not moving. The second solution is the one we are interested in

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin \theta = 0, \quad (23.3.28)$$

which is the same differential equation (Eq. (23.3.16)) that we found using the force method.

We can find the time t_1 that the object first reaches the bottom of the circular arc by setting $\theta(t_1) = 0$ in Eq. (23.3.22)

$$0 = \theta_0 \cos\left(\sqrt{\frac{g}{l}} t_1\right). \quad (23.3.29)$$

This zero occurs when the argument of the cosine satisfies

$$\sqrt{\frac{g}{l}} t_1 = \frac{\pi}{2}. \quad (23.3.30)$$

The z -component of the angular velocity at time t_1 is therefore

$$\frac{d\theta}{dt}(t_1) = -\sqrt{\frac{g}{l}} \theta_0 \sin\left(\sqrt{\frac{g}{l}} t_1\right) = -\sqrt{\frac{g}{l}} \theta_0 \sin\left(\frac{\pi}{2}\right) = -\sqrt{\frac{g}{l}} \theta_0. \quad (23.3.31)$$

Note that the negative sign means that the bob is moving in the negative $\hat{\theta}$ -direction when it first reaches the bottom of the arc. The θ -component of the velocity at time t_1 is therefore

$$v_\theta(t_1) \equiv v_1 = l \frac{d\theta}{dt}(t_1) = -l \sqrt{\frac{g}{l}} \theta_0 \sin\left(\sqrt{\frac{g}{l}} t_1\right) = -\sqrt{lg} \theta_0 \sin\left(\frac{\pi}{2}\right) = -\sqrt{lg} \theta_0. \quad (23.3.32)$$

We can also find the components of both the velocity and angular velocity using energy methods. When we release the bob from rest, the energy is only potential energy

$$E = U_0 = mgl(1 - \cos \theta_0) \cong mgl \frac{\theta_0^2}{2}, \quad (23.3.33)$$

where we used the approximation that $\cos \theta_0 \cong 1 - \theta_0^2 / 2$. When the bob is at the bottom of the arc, the only contribution to the mechanical energy is the kinetic energy given by

$$K_1 = \frac{1}{2}mv_1^2. \quad (23.3.34)$$

Because the energy is constant, we have that $U_0 = K_1$ or

$$mgl \frac{\theta_0^2}{2} = \frac{1}{2}mv_1^2. \quad (23.3.35)$$

We can solve for the θ -component of the velocity at the bottom of the arc

$$v_{\theta,1} = \pm \sqrt{gl} \theta_0. \quad (23.3.36)$$

The two possible solutions correspond to the different directions that the motion of the bob can have when at the bottom. The z -component of the angular velocity is then

$$\frac{d\theta}{dt}(t_1) = \frac{v_1}{l} = \pm \sqrt{\frac{g}{l}} \theta_0, \quad (23.3.37)$$

in agreement with our previous calculation.

If we do not make the small angle approximation, we can still use energy techniques to find the θ -component of the velocity at the bottom of the arc by equating the energies at the two positions

$$mgl(1 - \cos \theta_0) = \frac{1}{2}mv_1^2, \quad (23.3.38)$$

$$v_{\theta,1} = \pm \sqrt{2gl(1 - \cos \theta_0)}. \quad (23.3.39)$$

23.4 Worked Examples

Example 23.3: Rolling Without Slipping Oscillating Cylinder

Attach a solid cylinder of mass M and radius R to a horizontal massless spring with spring constant k so that it can roll without slipping along a horizontal surface. At time t , the center of mass of the cylinder is moving with speed V_{cm} and the spring is compressed a distance x from its equilibrium length. What is the period of simple harmonic motion for the center of mass of the cylinder?

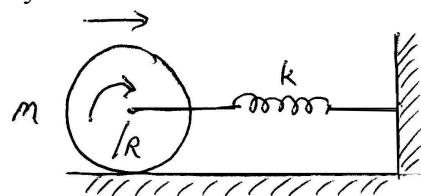


Figure 23.9 Example 23.3

Solution: At time t , the energy of the rolling cylinder and spring system is

$$E = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}kx^2 . \quad (23.4.1)$$

where x is the amount the spring has compressed, $I_{cm} = (1/2)MR^2$, and because it is rolling without slipping

$$\frac{d\theta}{dt} = \frac{V_{cm}}{R} . \quad (23.4.2)$$

Therefore the energy is

$$E = \frac{1}{2}MV_{cm}^2 + \frac{1}{4}MR^2\left(\frac{V_{cm}}{R}\right)^2 + \frac{1}{2}kx^2 = \frac{3}{4}MV_{cm}^2 + \frac{1}{2}kx^2 . \quad (23.4.3)$$

The energy is constant (no non-conservative force is doing work on the system) so

$$0 = \frac{dE}{dt} = \frac{3}{4}2MV_{cm}\frac{dV_{cm}}{dt} + \frac{1}{2}k2x\frac{dx}{dt} = V_{cm}\left(\frac{3}{2}M\frac{d^2x}{dt^2} + kx\right) \quad (23.4.4)$$

Because V_{cm} is non-zero most of the time, the displacement of the spring satisfies a simple harmonic oscillator equation

$$\frac{d^2x}{dt^2} + \frac{2k}{3M}x = 0 . \quad (23.4.5)$$

Hence the period is

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{3M}{2k}} . \quad (23.4.6)$$

Example 23.4: U-Tube

A U-tube open at both ends is filled with an incompressible fluid of density ρ . The cross-sectional area A of the tube is uniform and the total length of the fluid in the tube is L . A piston is used to depress the height of the liquid column on one side by a distance x_0 , (raising the other side by the same distance) and then is quickly removed (Figure 23.10). What is the angular frequency of the ensuing simple harmonic motion? Neglect any resistive forces and at the walls of the U-tube.

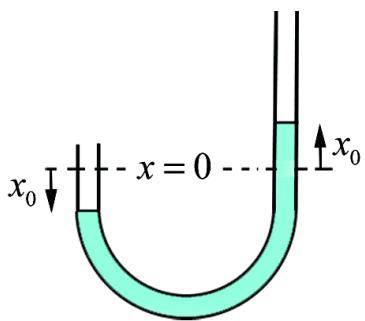


Figure 23.10 Example 23.4

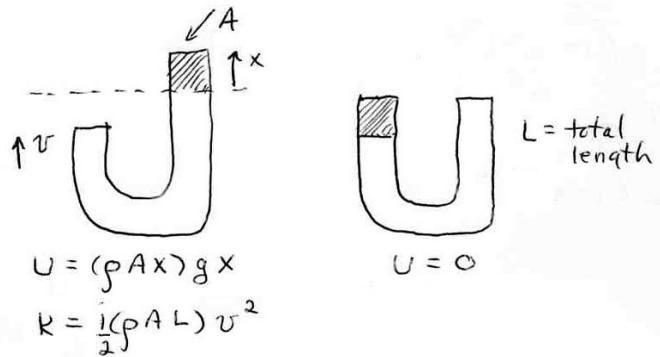


Figure 23.11 Energy diagram for water

Solution: We shall use conservation of energy. First choose as a zero for gravitational potential energy in the configuration where the water levels are equal on both sides of the tube. When the piston on one side depresses the fluid, it rises on the other. At a given instant in time when a portion of the fluid of mass $\Delta m = \rho Ax$ is a height x above the equilibrium height (Figure 23.11), the potential energy of the fluid is given by

$$U = \Delta mgx = (\rho Ax)gx = \rho Agx^2. \quad (23.4.7)$$

At that same instant the entire fluid of length L and mass $m = \rho AL$ is moving with speed v , so the kinetic energy is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}\rho ALv^2. \quad (23.4.8)$$

Thus the total energy is

$$E = K + U = \frac{1}{2}\rho ALv^2 + \rho Agx^2. \quad (23.4.9)$$

By neglecting resistive force, the mechanical energy of the fluid is constant. Therefore

$$0 = \frac{dE}{dt} = \rho ALv \frac{dv}{dt} + 2\rho Agx \frac{dx}{dt}. \quad (23.4.10)$$

If we just consider the top of the fluid above the equilibrium position on the right arm in Figure 23.13, we rewrite Eq. (23.4.10) as

$$0 = \frac{dE}{dt} = \rho ALv_x \frac{dv_x}{dt} + 2\rho Agx \frac{dx}{dt}, \quad (23.4.11)$$

where $v_x = dx / dt$. We now rewrite the energy condition using $dv_x / dt = d^2x / dt^2$ as

$$0 = v_x \rho A \left(L \frac{d^2 x}{dt^2} + 2gx \right). \quad (23.4.12)$$

This condition is satisfied when $v_x = 0$, i.e. the equilibrium condition or when

$$0 = L \frac{d^2 x}{dt^2} + 2gx. \quad (23.4.13)$$

This last condition can be written as

$$\frac{d^2 x}{dt^2} = -\frac{2g}{L}x. \quad (23.4.14)$$

This last equation is the simple harmonic oscillator equation. Using the same mathematical techniques as we used for the spring-block system, the solution for the height of the fluid above the equilibrium position is given by

$$x(t) = B \cos(\omega_0 t) + C \sin(\omega_0 t), \quad (23.4.15)$$

where

$$\omega_0 = \sqrt{\frac{2g}{L}} \quad (23.4.16)$$

is the angular frequency of oscillation. The x -component of the velocity of the fluid on the right-hand side of the U-tube is given by

$$v_x(t) = \frac{dx(t)}{dt} = -\omega_0 B \sin(\omega_0 t) + \omega_0 C \cos(\omega_0 t). \quad (23.4.17)$$

The coefficients B and C are determined by the initial conditions. At $t = 0$, the height of the fluid is $x(t = 0) = B = x_0$. At $t = 0$, the speed is zero so $v_x(t = 0) = \omega_0 C = 0$, hence $C = 0$. The height of the fluid above the equilibrium position on the right hand-side of the U-tube as a function of time is thus

$$x(t) = x_0 \cos\left(\sqrt{\frac{2g}{L}}t\right). \quad (23.4.18)$$

23.5 Damped Oscillatory Motion

Let's now consider our spring-block system moving on a horizontal frictionless surface but now the block is attached to a damper that resists the motion of the block due to viscous friction. This damper, commonly called a *dashpot*, is shown in Figure 23.13. The viscous force arises when objects move through fluids at speeds slow enough so that there is no turbulence. When the viscous force opposes the motion and is proportional to the velocity, so that

$$\vec{f}_{\text{vis}} = -b\vec{v}, \quad (23.5.1)$$

the dashpot is referred to as a *linear dashpot*. The constant of proportionality b depends on the properties of the dashpot.

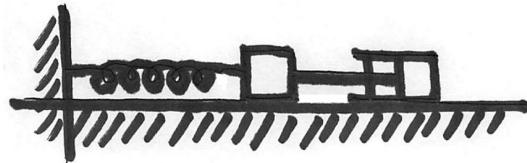


Figure 23.12 Spring-block system connected to a linear dashpot

Choose the origin at the equilibrium position and choose the positive x -direction to the right in the Figure 23.13. Define $x(t)$ to be the position of the object with respect to the equilibrium position. The x -component of the total force acting on the spring is the sum of the linear restoring spring force, and the viscous friction force (Figure 23.13),

$$F_x = -kx - b \frac{dx}{dt} \quad (23.5.2)$$

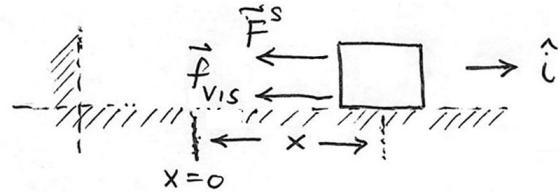


Figure 23.13 Free-body force diagram for spring-object system with linear dashpot

Newton's Second law in the x -direction becomes

$$-kx - b \frac{dx}{dt} = m \frac{d^2x}{dt^2}. \quad (23.5.3)$$

We can rewrite Eq. (23.5.3) as

$$\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = 0. \quad (23.5.4)$$

When $(b/m)^2 < 4k/m$, the oscillator is called *underdamped*, and the solution to Eq. (23.5.4) is given by

$$x(t) = x_m e^{-\alpha t} \cos(\gamma t + \phi) \quad (23.5.5)$$

where $\gamma = (k/m - (b/2m)^2)^{1/2}$ is the angular frequency of oscillation, $\alpha = b/2m$ is a parameter that measures the exponential decay of the oscillations, x_m is a constant and ϕ is the phase constant. Recall the undamped oscillator has angular frequency $\omega_0 = (k/m)^{1/2}$, so the angular frequency of the underdamped oscillator can be expressed as

$$\gamma = (\omega_0^2 - \alpha^2)^{1/2}. \quad (23.5.6)$$

In Appendix 23B: Complex Numbers, we introduce complex numbers and use them to solve Eq.(23.5.4) in Appendix 23C: Solution to the Underdamped Simple Harmonic Oscillator Equation.

The x -component of the velocity of the object is given by

$$v_x(t) = dx/dt = (-\gamma x_m \sin(\gamma t + \phi) - \alpha x_m \cos(\gamma t + \phi)) e^{-\alpha t}. \quad (23.5.7)$$

The position and the x -component of the velocity of the object oscillate but the amplitudes of the oscillations decay exponentially. In Figure 23.14, the position is plotted as a function of time for the underdamped system for the special case $\phi = 0$. For that case

$$x(t) = x_m e^{-\alpha t} \cos(\gamma t). \quad (23.5.8)$$

and

$$v_x(t) = dx/dt = (-\gamma x_m \sin(\gamma t) - \alpha x_m \cos(\gamma t)) e^{-\alpha t}. \quad (23.5.9)$$

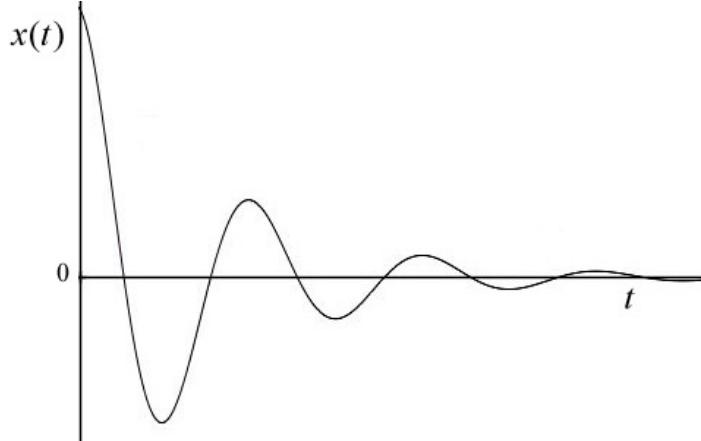


Figure 23.14 Plot of position $x(t)$ of object for underdamped oscillator with $\phi = 0$

Because the coefficient of exponential decay $\alpha = b/2m$ is proportional to the b , we see that the position will decay more rapidly if the viscous force increases. We can introduce a *time constant*

$$\tau = 1/\alpha = 2m/b. \quad (23.5.10)$$

When $t = \tau$, the position is

$$x(t = \tau) = x_m \cos(\gamma\tau)e^{-1}. \quad (23.5.11)$$

The envelope of exponential decay has now decreased by a factor of e^{-1} , i.e. the amplitude can be at most $x_m e^{-1}$. During this time interval $[0, \tau]$, the position has undergone a number of oscillations. The total number of radians associated with those oscillations is given by

$$\gamma\tau = (k/m - (b/2m)^2)^{1/2} (2m/b). \quad (23.5.12)$$

The closest integral number of cycles is then

$$n = [\gamma\tau / 2\pi] = [(k/m - (b/2m)^2)^{1/2} (m/\pi b)]. \quad (23.5.13)$$

If the system is very weakly damped, such that $(b/m)^2 \ll 4k/m$, then we can approximate the number of cycles by

$$n = [\gamma\tau / 2\pi] \approx [(k/m)^{1/2} (m/\pi b)] = [\omega_0 (m/\pi b)], \quad (23.5.14)$$

where $\omega_0 = (k/m)^{1/2}$ is the angular frequency of the undamped oscillator.

We define the *quality*, Q , of this oscillating system to be proportional to the number of integral cycles it takes for the exponential envelope of the position function to fall off by a factor of e^{-1} . The constant of proportionality is chosen to be π . Thus

$$Q = n\pi. \quad (23.5.15)$$

For the weakly damped case, we have that

$$Q \approx \omega_0 (m/b). \quad (23.5.16)$$

23.5.1 Energy in the Underdamped Oscillator

For the underdamped oscillator, $(b/m)^2 < 4k/m$, $\gamma = (k/m - (b/2m)^2)^{1/2}$, and $\alpha = b/2m$. Let's choose $t = 0$ such that the phase shift is zero $\phi = 0$. The stored energy in the system will decay due to the energy loss due to dissipation. The mechanical energy stored in the potential and kinetic energies is then given by

$$E = \frac{1}{2} kx^2 + \frac{1}{2} mv^2. \quad (23.5.17)$$

where the position and the x -component of the velocity are given by Eqs. (23.5.8) and (23.5.9). The mechanical energy is then

$$E = \frac{1}{2}kx_m^2 \cos^2(\gamma t)e^{-2\alpha t} + \frac{1}{2}m(-\gamma x_m \sin(\gamma t) - \alpha x_m \cos(\gamma t))^2 e^{-2\alpha t}. \quad (23.5.18)$$

Expanding this expression yields

$$E = \frac{1}{2}(k + m\alpha^2)x_m^2 \cos^2(\gamma t)e^{-2\alpha t} + m\gamma\alpha x_m^2 \sin(\gamma t) \cos(\gamma t) e^{-2\alpha t} + \frac{1}{2}m\gamma^2 x_m^2 \sin^2(\gamma t) e^{-2\alpha t} \quad (23.5.19)$$

The kinetic energy, potential energy, and mechanical energy are shown in Figure 23.15.

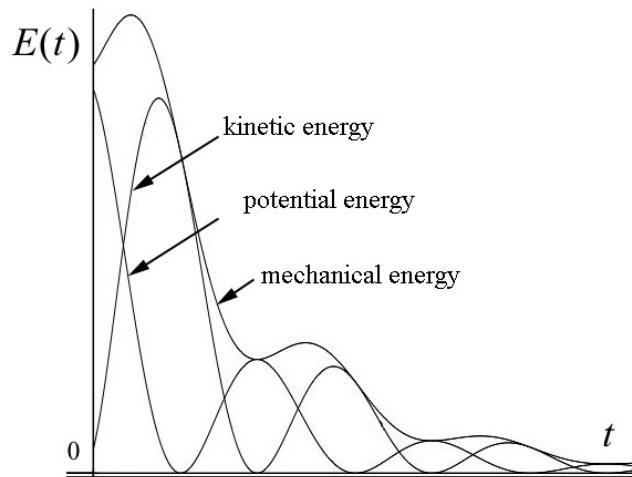


Figure 23.15 Kinetic, potential and mechanical energy for the underdamped oscillator

The stored energy at time $t = 0$ is

$$E(t = 0) = \frac{1}{2}(k + m\alpha^2)x_m^2 \quad (23.5.20)$$

The mechanical energy at the conclusion of one cycle, with $\gamma T = 2\pi$, is

$$E(t = T) = \frac{1}{2}(k + m\alpha^2)x_m^2 e^{-2\alpha T} \quad (23.5.21)$$

The change in the mechanical energy for one cycle is then

$$E(t = T) - E(t = 0) = -\frac{1}{2}(k + m\alpha^2)x_m^2(1 - e^{-2\alpha T}). \quad (23.5.22)$$

Recall that $\alpha^2 = b^2/4m^2$. Therefore

$$E(t=T) - E(t=0) = -\frac{1}{2}(k + b^2/4m)x_m^2(1 - e^{-2\alpha T}). \quad (23.5.23)$$

We can show (although the calculation is lengthy) that the energy dissipated by the viscous force over one cycle is given by the integral

$$E_{\text{dis}} = \int_0^T \vec{\mathbf{F}}_{\text{vis}} \cdot \vec{\mathbf{v}} dt = -\left(k + \frac{b^2}{4m}\right) \frac{x_m^2}{2} (1 - e^{-2\alpha t}). \quad (23.5.24)$$

By comparison with Eq. (23.5.23), the change in the mechanical energy in the underdamped oscillator during one cycle is equal to the energy dissipated due to the viscous force during one cycle.

23.6 Forced Damped Oscillator

Let's drive our damped spring-object system by a sinusoidal force. Suppose that the x -component of the driving force is given by

$$F_x(t) = F_0 \cos(\omega t), \quad (23.6.1)$$

where F_0 is called the *amplitude* (maximum value) and ω is the *driving angular frequency*. The force varies between F_0 and $-F_0$ because the cosine function varies between +1 and -1. Define $x(t)$ to be the position of the object with respect to the equilibrium position. The x -component of the force acting on the object is now the sum

$$F_x = F_0 \cos(\omega t) - kx - b \frac{dx}{dt}. \quad (23.6.2)$$

Newton's Second law in the x -direction becomes

$$F_0 \cos(\omega t) - kx - b \frac{dx}{dt} = m \frac{d^2x}{dt^2}. \quad (23.6.3)$$

We can rewrite Eq. (23.6.3) as

$$F_0 \cos(\omega t) = m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx. \quad (23.6.4)$$

We derive the solution to Eq. (23.6.4) in Appendix 23E: Solution to the forced Damped Oscillator Equation. The solution to is given by the function

$$x(t) = x_0 \cos(\omega t + \phi) , \quad (23.6.5)$$

where the amplitude x_0 is a function of the driving angular frequency ω and is given by

$$x_0(\omega) = \frac{F_0 / m}{\left((b/m)^2 \omega^2 + (\omega_0^2 - \omega^2)^2\right)^{1/2}} . \quad (23.6.6)$$

The phase constant ϕ is also a function of the driving angular frequency ω and is given by

$$\phi(\omega) = \tan^{-1} \left(\frac{(b/m)\omega}{\omega^2 - \omega_0^2} \right) . \quad (23.6.7)$$

In Eqs. (23.6.6) and (23.6.7)

$$\omega_0 = \sqrt{\frac{k}{m}} \quad (23.6.8)$$

is the natural angular frequency associated with the undriven undamped oscillator. The x -component of the velocity can be found by differentiating Eq. (23.6.5),

$$v_x(t) = \frac{dx}{dt}(t) = -\omega x_0 \sin(\omega t + \phi) , \quad (23.6.9)$$

where the amplitude $x_0(\omega)$ is given by Eq. (23.6.6) and the phase constant $\phi(\omega)$ is given by Eq. (23.6.7).

23.6.1 Resonance

When $b/m \ll 2\omega_0$ we say that the oscillator is *lightly damped*. For a lightly-damped driven oscillator, after a transitory period, the position of the object will oscillate with the same angular frequency as the driving force. The plot of amplitude $x_0(\omega)$ vs. driving angular frequency ω for a lightly damped forced oscillator is shown in Figure 23.16. If the angular frequency is increased from zero, the amplitude of the $x_0(\omega)$ will increase until it reaches a maximum when the angular frequency of the driving force is the same as the natural angular frequency, ω_0 , associated with the undamped oscillator. This is called **resonance**. When the driving angular frequency is increased above the natural angular frequency the amplitude of the position oscillations diminishes.

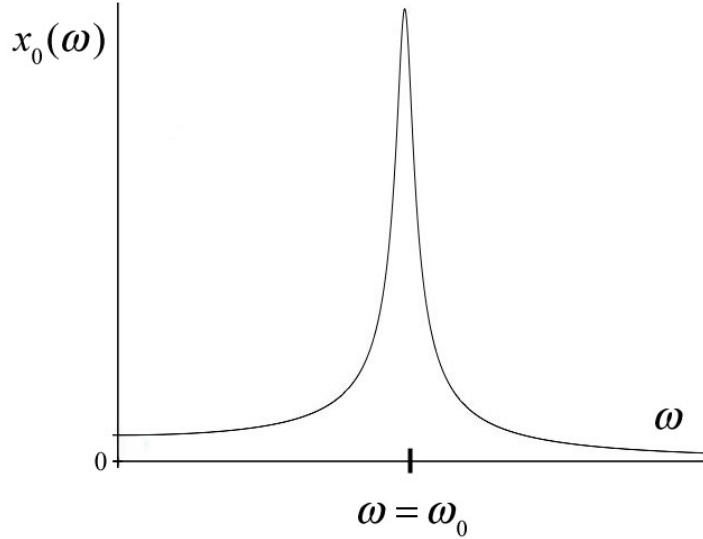


Figure 23.16 Plot of amplitude $x_0(\omega)$ vs. driving angular frequency ω for a lightly damped oscillator with $b/m \ll 2\omega_0$

We can find the angular frequency such that the amplitude $x_0(\omega)$ is at a maximum by setting the derivative of Eq. (23.6.6) equal to zero,

$$0 = \frac{d}{dt}x_0(\omega) = -\frac{F_0(2\omega)}{2m} \frac{((b/m)^2 - 2(\omega_0^2 - \omega^2))}{((b/m)^2\omega^2 + (\omega_0^2 - \omega^2)^2)^{3/2}} . \quad (23.6.10)$$

This vanishes when

$$\omega = (\omega_0^2 - (b/m)^2/2)^{1/2} . \quad (23.6.11)$$

For the lightly-damped oscillator, $\omega_0 \gg (1/2)b/m$, and so the maximum value of the amplitude occurs when

$$\omega \approx \omega_0 = (k/m)^{1/2} . \quad (23.6.12)$$

The amplitude at resonance is then

$$x_0(\omega = \omega_0) = \frac{F_0}{b\omega_0} \text{ (lightly damped)} . \quad (23.6.13)$$

The plot of phase constant $\phi(\omega)$ vs. driving angular frequency ω for a lightly damped forced oscillator is shown in Figure 23.17.

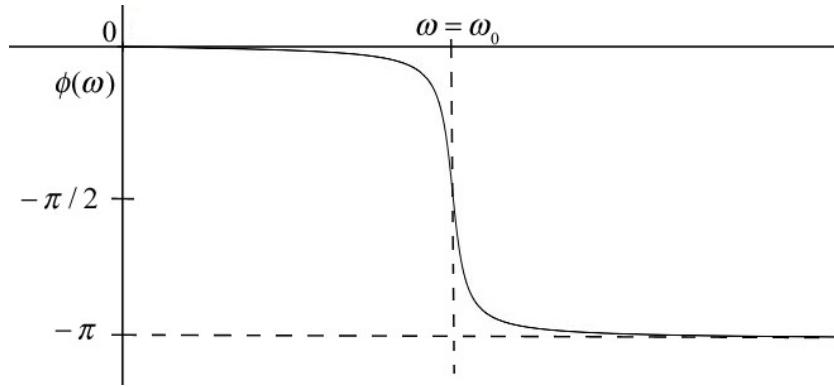


Figure 23.17 Plot of phase constant $\phi(\omega)$ vs. driving angular frequency ω for a lightly damped oscillator with $b/m \ll 2\omega_0$

The phase constant at resonance is zero,

$$\phi(\omega = \omega_0) = 0 . \quad (23.6.14)$$

At resonance, the x -component of the velocity is given by

$$v_x(t) = \frac{dx}{dt}(t) = -\frac{F_0}{b} \sin(\omega_0 t) \quad (\text{lightly damped}) . \quad (23.6.15)$$

When the oscillator is not lightly damped ($b/m \approx \omega_0$), the resonance peak is shifted to the left of $\omega = \omega_0$ as shown in the plot of amplitude vs. angular frequency in Figure 23.18. The corresponding plot of phase constant vs. angular frequency for the non-lightly damped oscillator is shown in Figure 23.19.

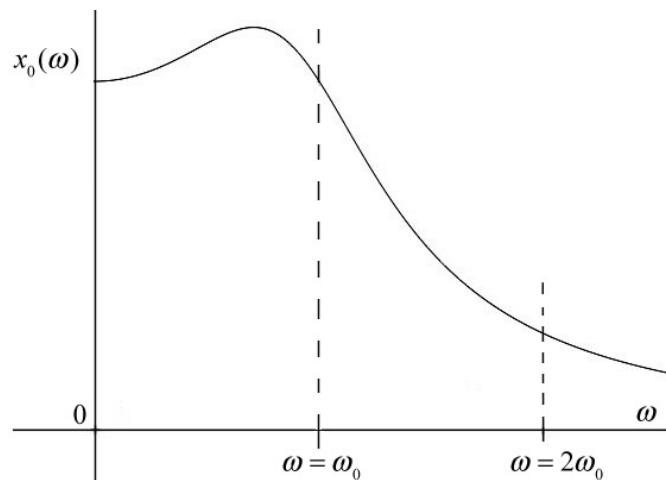


Figure 23.18 Plot of amplitude vs. angular frequency for lightly-damped driven oscillator where $b/m \approx \omega_0$

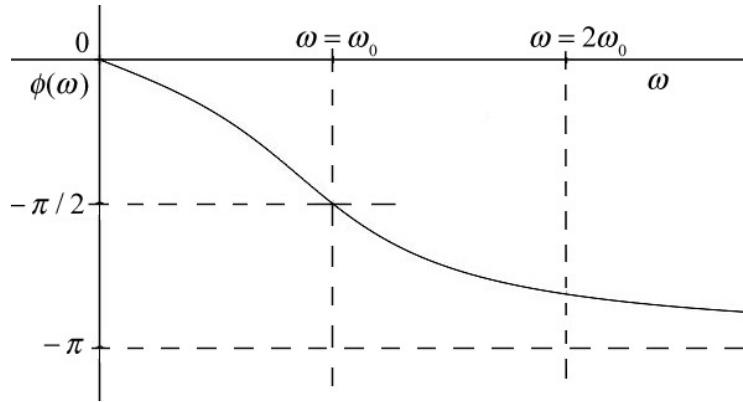


Figure 23.19 Plot of phase constant vs. angular frequency for lightly-damped driven oscillator where $b/m \approx \omega_0$

23.6.2 Mechanical Energy

The kinetic energy for the driven damped oscillator is given by

$$K(t) = \frac{1}{2}mv^2(t) = \frac{1}{2}m\omega^2x_0^2 \sin^2(\omega t + \phi) . \quad (23.6.16)$$

The potential energy is given by

$$U(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kx_0^2 \cos^2(\omega t + \phi) . \quad (23.6.17)$$

The mechanical energy is then

$$E(t) = \frac{1}{2}mv^2(t) + \frac{1}{2}kx^2(t) = \frac{1}{2}m\omega^2x_0^2 \sin^2(\omega t + \phi) + \frac{1}{2}kx_0^2 \cos^2(\omega t + \phi) . \quad (23.6.18)$$

Example 23.5: Time-Averaged Mechanical Energy

The period of one cycle is given by $T = 2\pi/\omega$. Show that

$$(i) \quad \frac{1}{T} \int_0^T \sin^2(\omega t + \phi) dt = \frac{1}{2} , \quad (23.6.19)$$

$$(ii) \quad \frac{1}{T} \int_0^T \cos^2(\omega t + \phi) dt = \frac{1}{2} , \quad (23.6.20)$$

$$(iii) \quad \frac{1}{T} \int_0^T \sin(\omega t) \cos(\omega t) dt = 0 . \quad (23.6.21)$$

Solution: (i) We use the trigonometric identity

$$\sin^2(\omega t + \phi) = \frac{1}{2}(1 - \cos(2(\omega t + \phi))) \quad (23.6.22)$$

to rewrite the integral in Eq. (23.6.19) as

$$\frac{1}{T} \int_0^T \sin^2(\omega t + \phi) dt = \frac{1}{2T} \int_0^T (1 - \cos(2(\omega t + \phi))) dt \quad (23.6.23)$$

Integration yields

$$\begin{aligned} \frac{1}{2T} \int_0^T (1 - \cos(2(\omega t + \phi))) dt &= \frac{1}{2} - \left(\frac{\sin(2(\omega t + \phi))}{2\omega} \right) \Big|_{T=0}^{T=2\pi/\omega} \\ &= \frac{1}{2} - \left(\frac{\sin(4\pi + 2\phi)}{2\omega} - \frac{\sin(2\phi)}{2\omega} \right) = \frac{1}{2}, \end{aligned} \quad (23.6.24)$$

where we used the trigonometric identity that

$$\sin(4\pi + 2\phi) = \sin(4\pi)\cos(2\phi) + \sin(2\phi)\cos(4\pi) = \sin(2\phi), \quad (23.6.25)$$

proving Eq. (23.6.19).

(ii) We use a similar argument starting with the trigonometric identity that

$$\cos^2(\omega t + \phi) = \frac{1}{2}(1 + \cos(2(\omega t + \phi))). \quad (23.6.26)$$

Then

$$\frac{1}{T} \int_0^T \cos^2(\omega t + \phi) dt = \frac{1}{2T} \int_0^T (1 + \cos(2(\omega t + \phi))) dt. \quad (23.6.27)$$

Integration yields

$$\begin{aligned} \frac{1}{2T} \int_0^T (1 + \cos(2(\omega t + \phi))) dt &= \frac{1}{2} + \left(\frac{\sin(2(\omega t + \phi))}{2\omega} \right) \Big|_{T=0}^{T=2\pi/\omega} \\ &= \frac{1}{2} + \left(\frac{\sin(4\pi + 2\phi)}{2\omega} - \frac{\sin(2\phi)}{2\omega} \right) = \frac{1}{2}. \end{aligned} \quad (23.6.28)$$

(iii) We first use the trigonometric identity that

$$\sin(\omega t)\cos(\omega t) = \frac{1}{2}\sin(2\omega t) . \quad (23.6.29)$$

Then

$$\begin{aligned} \frac{1}{T} \int_0^T \sin(\omega t)\cos(\omega t) dt &= \frac{1}{T} \int_0^T \frac{1}{2}\sin(2\omega t) dt \\ &= -\frac{1}{T} \left. \frac{\cos(2\omega t)}{2\omega} \right|_0^T = -\frac{1}{2\omega T}(1-1) = 0. \end{aligned} \quad (23.6.30)$$

The values of the integrals in Example 23.5 are called the **time-averaged values**. We denote the time-average value of a function $f(t)$ over one period by

$$\langle f \rangle \equiv \frac{1}{T} \int_0^T f(t) dt . \quad (23.6.31)$$

In particular, the time-average kinetic energy as a function of the angular frequency is given by

$$\langle K(\omega) \rangle = \frac{1}{4} m \omega^2 x_0^2 . \quad (23.6.32)$$

The time-averaged potential energy as a function of the angular frequency is given by

$$\langle U(\omega) \rangle = \frac{1}{4} k x_0^2 . \quad (23.6.33)$$

The time-averaged value of the mechanical energy as a function of the angular frequency is given by

$$\langle E(\omega) \rangle = \frac{1}{4} m \omega^2 x_0^2 + \frac{1}{4} k x_0^2 = \frac{1}{4} (m \omega^2 + k) x_0^2 . \quad (23.6.34)$$

We now substitute Eq. (23.6.6) for the amplitude into Eq. (23.6.34) yielding

$$\langle E(\omega) \rangle = \frac{F_0^2}{4m} \frac{(\omega_0^2 + \omega^2)}{\left((b/m)^2 \omega^2 + (\omega_0^2 - \omega^2)^2\right)} . \quad (23.6.35)$$

A plot of the time-averaged energy versus angular frequency for the lightly-damped case ($b/m \ll 2\omega_0$) is shown in Figure 23.20.

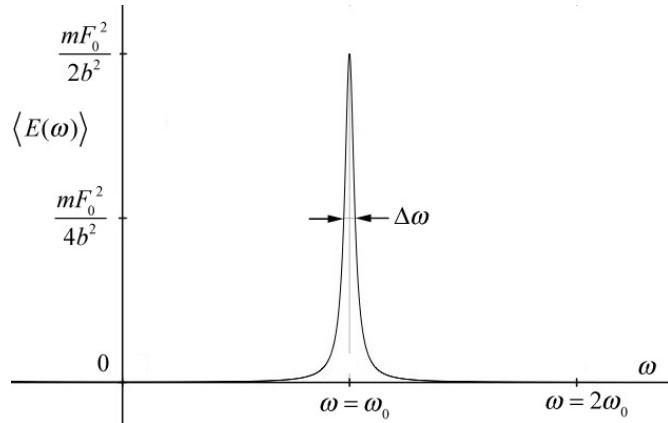


Figure 23.20 Plot of the time-averaged energy versus angular frequency for the lightly-damped case ($b / m \ll 2\omega_0$)

We can simplify the expression for the time-averaged energy for the lightly-damped case by observing that the time-averaged energy is nearly zero everywhere except where $\omega = \omega_0$, (see Figure 23.20). We first substitute $\omega = \omega_0$ everywhere in Eq. (23.6.35) except the term $\omega_0^2 - \omega^2$ that appears in the denominator, yielding

$$\langle E(\omega) \rangle = \frac{F_0^2}{2m} \frac{(\omega_0^2)}{\left((b/m)^2 \omega_0^2 + (\omega_0^2 - \omega^2)^2\right)}. \quad (23.6.36)$$

We can approximate the term

$$\omega_0^2 - \omega^2 = (\omega_0 - \omega)(\omega_0 + \omega) \approx 2\omega_0(\omega_0 - \omega) \quad (23.6.37)$$

Then Eq. (23.6.36) becomes

$$\langle E(\omega) \rangle = \frac{F_0^2}{2m} \frac{1}{\left((b/m)^2 + 4(\omega_0 - \omega)^2\right)} \quad (\text{lightly damped}). \quad (23.6.38)$$

The right-hand expression of Eq. (23.6.38) takes on its maximum value when the denominator has its minimum value. By inspection, this occurs when $\omega = \omega_0$. Alternatively, to find the maximum value, we set the derivative of Eq. (23.6.35) equal to zero and solve for ω ,

$$0 = \frac{d}{d\omega} \langle E(\omega) \rangle = \frac{d}{d\omega} \frac{F_0^2}{2m} \frac{1}{((b/m)^2 + 4(\omega_0 - \omega)^2)} \\ = \frac{4F_0^2}{m} \frac{(\omega_0 - \omega)}{\left((b/m)^2 + 4(\omega_0 - \omega)^2\right)^2}. \quad (23.6.39)$$

The maximum occurs when occurs at $\omega = \omega_0$ and has the value

$$\langle E(\omega_0) \rangle = \frac{mF_0^2}{2b^2} \quad (\text{underdamped}). \quad (23.6.40)$$

23.6.3 The Time-averaged Power

The time-averaged power delivered by the driving force is given by the expression

$$\langle P(\omega) \rangle = \frac{1}{T} \int_0^T F_x v_x dt = -\frac{1}{T} \int_0^T \frac{F_0^2 \omega \cos(\omega t) \sin(\omega t + \phi)}{m((b/m)^2 \omega^2 + (\omega_0^2 - \omega^2)^2)^{1/2}} dt, \quad (23.6.41)$$

where we used Eq. (23.6.1) for the driving force, and Eq. (23.6.9) for the x -component of the velocity of the object. We use the trigonometric identity

$$\sin(\omega t + \phi) = \sin(\omega t) \cos(\phi) + \cos(\omega t) \sin(\phi) \quad (23.6.42)$$

to rewrite the integral in Eq. (23.6.41) as two integrals

$$\langle P(\omega) \rangle = -\frac{1}{T} \int_0^T \frac{F_0^2 \omega \cos(\omega t) \sin(\omega t) \cos(\phi)}{m((b/m)^2 \omega^2 + (\omega_0^2 - \omega^2)^2)^{1/2}} dt \\ - \frac{1}{T} \int_0^T \frac{F_0^2 \omega \cos^2(\omega t) \sin(\phi)}{m((b/m)^2 \omega^2 + (\omega_0^2 - \omega^2)^2)^{1/2}} dt. \quad (23.6.43)$$

Using the time-averaged results from Example 23.5, we see that the first term in Eq. (23.6.43) is zero and the second term becomes

$$\langle P(\omega) \rangle = \frac{F_0^2 \omega \sin(\phi)}{2m((b/m)^2 \omega^2 + (\omega_0^2 - \omega^2)^2)^{1/2}} \quad (23.6.44)$$

For the underdamped driven oscillator, we make the same approximations in Eq. (23.6.44) that we made for the time-averaged energy. In the term in the numerator and the

term on the left in the denominator, we set $\omega \approx \omega_0$, and we use Eq. (23.6.37) in the term on the right in the denominator yielding

$$\langle P(\omega) \rangle = \frac{F_0^2 \sin(\phi)}{2m((b/m)^2 + 2(\omega_0 - \omega))^{1/2}} \quad (\text{underdamped}) . \quad (23.6.45)$$

The time-averaged power dissipated by the resistive force is given by

$$\begin{aligned} \langle P_{\text{dis}}(\omega) \rangle &= \frac{1}{T} \int_0^T (F_x)_{\text{dis}} v_x dt = -\frac{1}{T} \int_0^T b v_x^2 dt = \frac{1}{T} \int_0^T \frac{F_0^2 \omega^2 \sin^2(\omega t + \phi) dt}{m^2 ((b/m)^2 \omega^2 + (\omega_0^2 - \omega^2)^2)} \\ &= \frac{F_0^2 \omega^2 dt}{2m^2 ((b/m)^2 \omega^2 + (\omega_0^2 - \omega^2)^2)}. \end{aligned} \quad (23.6.46)$$

where we used Eq. (23.5.1) for the dissipative force, Eq. (23.6.9) for the x -component of the velocity of the object, and Eq. (23.6.19) for the time-averaging.

23.6.4 Quality Factor

The plot of the time-averaged energy vs. the driving angular frequency for the underdamped oscillator has a width, $\Delta\omega$ (Figure 23.20). One way to characterize this width is to define $\Delta\omega = \omega_+ - \omega_-$, where ω_{\pm} are the values of the angular frequency such that time-averaged energy is equal to one half its maximum value

$$\langle E(\omega_{\pm}) \rangle = \frac{1}{2} \langle E(\omega_0) \rangle = \frac{m F_0^2}{4b^2}. \quad (23.6.47)$$

The quantity $\Delta\omega$ is called the *line width at half energy maximum* also known as the *resonance width*. We can now solve for ω_{\pm} by setting

$$\langle E(\omega_{\pm}) \rangle = \frac{F_0^2}{2m} \frac{1}{((b/m)^2 + 4(\omega_0 - \omega_{\pm})^2)} = \frac{m F_0^2}{4b^2}, \quad (23.6.48)$$

yielding the condition that

$$(b/m)^2 = 4(\omega_0 - \omega_{\pm})^2. \quad (23.6.49)$$

Taking square roots of Eq. (23.6.49) yields

$$\mp(b/2m) = \omega_0 - \omega_{\pm}. \quad (23.6.50)$$

Therefore

$$\omega_{\pm} = \omega_0 \pm (b/2m). \quad (23.6.51)$$

The half-width is then

$$\Delta\omega = \omega_+ - \omega_- = (\omega_0 + (b/2m)) - (\omega_0 - (b/2m)) = b/m. \quad (23.6.52)$$

We define the quality Q of the resonance as the ratio of the resonant angular frequency to the line width,

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{b/m}. \quad (23.6.53)$$

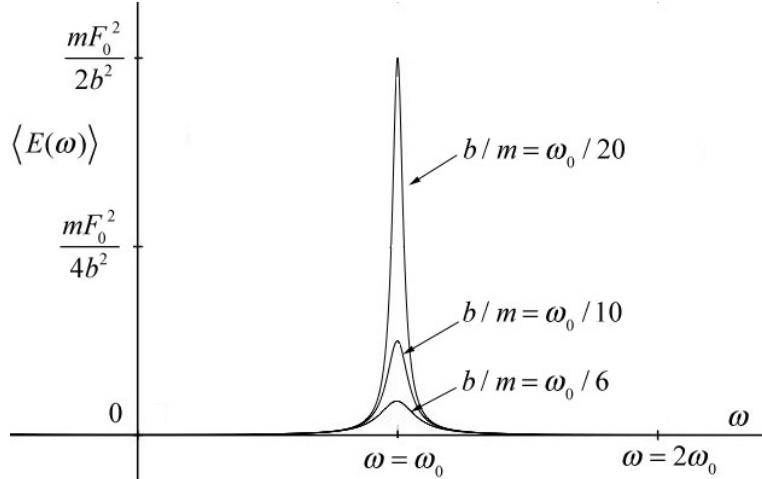


Figure 23.21 Plot of time-averaged energy vs. angular frequency for different values of b/m

In Figure 23.21 we plot the time-averaged energy vs. angular frequency for several different values of the quality factor $Q = 10, 5, \text{ and } 3$. Recall that this was the same result that we had for the quality of the free oscillations of the damped oscillator, Eq. (23.5.16) (because we chose the factor π in Eq. (23.5.16)).

23.7 Small Oscillations

Any object moving subject to a force associated with a potential energy function that is quadratic will undergo simple harmonic motion,

$$U(x) = U_0 + \frac{1}{2}k(x - x_{eq})^2. \quad (23.7.1)$$

where k is a “spring constant”, x_{eq} is the equilibrium position, and the constant U_0 just depends on the choice of reference point x_{ref} for zero potential energy, $U(x_{ref}) = 0$,

$$0 = U(x_{ref}) = U_0 + \frac{1}{2}k(x_{ref} - x_{eq})^2. \quad (23.7.2)$$

Therefore the constant is

$$U_0 = -\frac{1}{2}k(x_{ref} - x_{eq})^2. \quad (23.7.3)$$

The minimum of the potential x_0 corresponds to the point where the x -component of the force is zero,

$$\left. \frac{dU}{dx} \right|_{x=x_0} = 2k(x_0 - x_{eq}) = 0 \Rightarrow x_0 = x_{eq}, \quad (23.7.4)$$

corresponding to the equilibrium position. Therefore the constant is $U(x_0) = U_0$ and we rewrite our potential function as

$$U(x) = U(x_0) + \frac{1}{2}k(x - x_0)^2. \quad (23.7.5)$$

Now suppose that a potential energy function is not quadratic but still has a minimum at x_0 . For example, consider the potential energy function

$$U(x) = -U_1 \left(\left(\frac{x}{x_1} \right)^3 - \left(\frac{x}{x_1} \right)^2 \right), \quad (23.7.6)$$

(Figure 23.22), which has a stable minimum at x_0 .

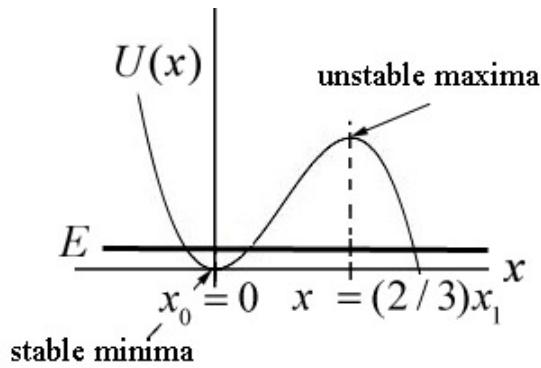


Figure 23.22 Potential energy function with stable minima and unstable maxima

When the energy of the system is very close to the value of the potential energy at the minimum $U(x_0)$, we shall show that the system will undergo small oscillations about the

minimum value x_0 . We shall use the Taylor formula to approximate the potential function as a polynomial. We shall show that near the minimum x_0 , we can approximate the potential function by a quadratic function similar to Eq. (23.7.5) and show that the system undergoes simple harmonic motion for small oscillations about the minimum x_0 .

We begin by expanding the potential energy function about the minimum point using the Taylor formula

$$U(x) = U(x_0) + \frac{dU}{dx} \Big|_{x=x_0} (x - x_0) + \frac{1}{2!} \frac{d^2U}{dx^2} \Big|_{x=x_0} (x - x_0)^2 + \frac{1}{3!} \frac{d^3U}{dx^3} \Big|_{x=x_0} (x - x_0)^3 + \dots \quad (23.7.7)$$

where $\frac{1}{3!} \frac{d^3U}{dx^3} \Big|_{x=x_0} (x - x_0)^3$ is a third order term in that it is proportional to $(x - x_0)^3$, and $\frac{d^3U}{dx^3} \Big|_{x=x_0}$, $\frac{d^2U}{dx^2} \Big|_{x=x_0}$, and $\frac{dU}{dx} \Big|_{x=x_0}$ are constants. If x_0 is the minimum of the potential energy, then the linear term is zero, because

$$\frac{dU}{dx} \Big|_{x=x_0} = 0 \quad (23.7.8)$$

and so Eq. ((23.7.7)) becomes

$$U(x) \approx U(x_0) + \frac{1}{2} \frac{d^2U}{dx^2} \Big|_{x=x_0} (x - x_0)^2 + \frac{1}{3!} \frac{d^3U}{dx^3} \Big|_{x=x_0} (x - x_0)^3 + \dots \quad (23.7.9)$$

For small displacements from the equilibrium point such that $|x - x_0|$ is sufficiently small, the third order term and higher order terms are very small and can be ignored. Then the potential energy function is approximately a quadratic function,

$$U(x) \approx U(x_0) + \frac{1}{2} \frac{d^2U}{dx^2} \Big|_{x=x_0} (x - x_0)^2 = U(x_0) + \frac{1}{2} k_{\text{eff}} (x - x_0)^2 \quad (23.7.10)$$

where we define k_{eff} , the ***effective spring constant***, by

$$k_{\text{eff}} \equiv \frac{d^2U}{dx^2} \Big|_{x=x_0} . \quad (23.7.11)$$

Because the potential energy function is now approximated by a quadratic function, the system will undergo simple harmonic motion for small displacements from the minimum with a force given by

$$F_x = -\frac{dU}{dx} = -k_{\text{eff}}(x - x_0). \quad (23.7.12)$$

At $x = x_0$, the force is zero

$$F_x(x_0) = \frac{dU}{dx}(x_0) = 0. \quad (23.7.13)$$

We can determine the period of oscillation by substituting Eq. (23.7.12) into Newton's Second Law

$$-k_{\text{eff}}(x - x_0) = m_{\text{eff}} \frac{d^2x}{dt^2} \quad (23.7.14)$$

where m_{eff} is the **effective mass**. For a two-particle system, the effective mass is the reduced mass of the system.

$$m_{\text{eff}} = \frac{m_1 m_2}{m_1 + m_2} \equiv \mu_{\text{red}}, \quad (23.7.15)$$

Eq. (23.7.14) has the same form as the spring-object ideal oscillator. Therefore the angular frequency of small oscillations is given by

$$\omega_0 = \sqrt{\frac{k_{\text{eff}}}{m_{\text{eff}}}} = \sqrt{\frac{\left. \frac{d^2U}{dx^2} \right|_{x=x_0}}{m_{\text{eff}}}}. \quad (23.7.16)$$

Example 23.6: Quartic Potential

A system with effective mass m has a potential energy given by

$$U(x) = U_0 \left(-2 \left(\frac{x}{x_0} \right)^2 + \left(\frac{x}{x_0} \right)^4 \right), \quad (23.7.17)$$

where U_0 and x_0 are positive constants and $U(0) = 0$. (a) Find the points where the force on the particle is zero. Classify these points as stable or unstable. Calculate the value of $U(x)/U_0$ at these equilibrium points. (b) If the particle is given a small displacement from an equilibrium point, find the angular frequency of small oscillation.

Solution: (a) A plot of $U(x)/U_0$ as a function of x/x_0 is shown in Figure 23.23.

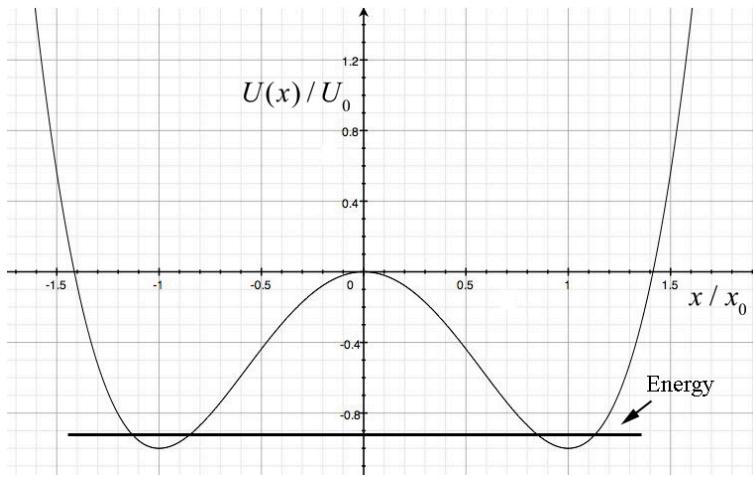


Figure 22.23 Plot of $U(x) / U_0$ as a function of x / x_0

The force on the particle is zero at the minimum of the potential energy,

$$\begin{aligned} 0 &= \frac{dU}{dx} = U_0 \left(-4\left(\frac{1}{x_0}\right)^2 x + 4\left(\frac{1}{x_0}\right)^4 x^3 \right) \\ &= -4U_0 x \left(\frac{1}{x_0} \right)^2 \left(1 - \left(\frac{x}{x_0} \right)^2 \right) \Rightarrow x^2 = x_0^2 \text{ and } x = 0. \end{aligned} \quad (23.7.18)$$

The equilibrium points are at $x = \pm x_0$ which are stable and $x = 0$ which is unstable. The second derivative of the potential energy is given by

$$\frac{d^2U}{dx^2} = U_0 \left(-4\left(\frac{1}{x_0}\right)^2 + 12\left(\frac{1}{x_0}\right)^4 x^2 \right). \quad (23.7.19)$$

If the particle is given a small displacement from $x = x_0$ then

$$\left. \frac{d^2U}{dx^2} \right|_{x=x_0} = U_0 \left(-4\left(\frac{1}{x_0}\right)^2 + 12\left(\frac{1}{x_0}\right)^4 x_0^2 \right) = U_0 \frac{8}{x_0^2}. \quad (23.7.20)$$

(b) The angular frequency of small oscillations is given by

$$\omega_0 = \sqrt{\left. \frac{d^2U}{dx^2} \right|_{x=x_0} / m} = \sqrt{\frac{8U_0}{mx_0^2}}. \quad (23.7.21)$$

Example 23.7: Lennard-Jones 6-12 Potential

A commonly used potential energy function to describe the interaction between two atoms is the Lennard-Jones 6-12 potential

$$U(r) = U_0 \left[(r_0 / r)^{12} - 2(r_0 / r)^6 \right]; \quad r > 0 \quad , \quad (23.7.22)$$

where r is the distance between the atoms. Find the angular frequency of small oscillations about the stable equilibrium position for two identical atoms bound to each other by the Lennard-Jones interaction. Let m denote the effective mass of the system of two atoms.

Solution: The equilibrium points are found by setting the first derivative of the potential energy equal to zero,

$$0 = \frac{dU}{dr} = U_0 \left[-12r_0^{12}r^{-13} + 12r_0^6r^{-7} \right] = U_0 12r_0^6r^{-7} \left[-\left(\frac{r_0}{r} \right)^6 + 1 \right]. \quad (23.7.23)$$

The equilibrium point occurs when $r = r_0$. The second derivative of the potential energy function is

$$\frac{d^2U}{dr^2} = U_0 \left[+(12)(13)r_0^{12}r^{-14} - (12)(7)r_0^6r^{-8} \right]. \quad (23.7.24)$$

Evaluating this at $r = r_0$ yields

$$\left. \frac{d^2U}{dr^2} \right|_{r=r_0} = 72U_0 r_0^{-2}. \quad (23.7.25)$$

The angular frequency of small oscillation is therefore

$$\omega_0 = \sqrt{\left. \frac{d^2U}{dr^2} \right|_{r=r_0} / m} = \sqrt{72U_0 / mr_0^2}. \quad (23.7.26)$$

Appendix 23A: Solution to Simple Harmonic Oscillator Equation

In our analysis of the solution of the simple harmonic oscillator equation of motion, Equation (23.2.1),

$$-kx = m \frac{d^2x}{dt^2}, \quad (23.A.1)$$

we assumed that the solution was a linear combination of sinusoidal functions,

$$x(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t), \quad (23.A.2)$$

where $\omega_0 = \sqrt{k/m}$. We shall now derive Eq. (23.A.2).

Assume that the mechanical energy of the spring-object system is given by the constant E . Choose the reference point for potential energy to be the unstretched position of the spring. Let x denote the amount the spring has been compressed ($x < 0$) or stretched ($x > 0$) from equilibrium at time t and denote the amount the spring has been compressed or stretched from equilibrium at time $t = 0$ by $x(t = 0) \equiv x_0$. Let $v_x = dx/dt$ denote the x -component of the velocity at time t and denote the x -component of the velocity at time $t = 0$ by $v_x(t = 0) \equiv v_{x,0}$. The constancy of the mechanical energy is then expressed as

$$E = K + U = \frac{1}{2}kx^2 + \frac{1}{2}mv^2. \quad (23.A.3)$$

We can solve Eq. (23.A.3) for the square of the x -component of the velocity,

$$v_x^2 = \frac{2E}{m} - \frac{k}{m}x^2 = \frac{2E}{m} \left(1 - \frac{k}{2E}x^2\right). \quad (23.A.4)$$

Taking square roots, we have

$$\frac{dx}{dt} = \sqrt{\frac{2E}{m}} \sqrt{1 - \frac{k}{2E}x^2}. \quad (23.A.5)$$

(why we take the positive square root will be explained below).

Let $a_1 \equiv \sqrt{2E/m}$ and $a_2 \equiv k/2E$. It's worth noting that a_1 has dimensions of velocity and w has dimensions of [length] $^{-2}$. Eq. (23.A.5) is separable,

$$\begin{aligned}\frac{dx}{dt} &= a_1 \sqrt{1 - a_2 x^2} \\ \frac{dx}{\sqrt{1 - a_2 x^2}} &= a_1 dt.\end{aligned}\tag{23.A.6}$$

We now integrate Eq. (23.A.6),

$$\int \frac{dx}{\sqrt{1 - a_1 x^2}} = \int a_1 dt. \tag{23.A.7}$$

The integral on the left in Eq. (23.A.7) is well known, and a derivation is presented here. We make a change of variables $\cos\theta = \sqrt{a_2} x$ with the differentials $d\theta$ and dx related by $-\sin\theta d\theta = \sqrt{a_2} dx$. The integration variable is

$$\theta = \cos^{-1}(\sqrt{a_2} x). \tag{23.A.8}$$

Eq. (23.A.7) then becomes

$$\int \frac{-\sin\theta d\theta}{\sqrt{1 - \cos^2\theta}} = \int \sqrt{a_2} a_1 dt. \tag{23.A.9}$$

This is a good point at which to check the dimensions. The term on the left in Eq. (23.A.9) is dimensionless, and the product $\sqrt{a_2} a_1$ on the right has dimensions of inverse time, $[\text{length}]^{-1} [\text{length} \cdot \text{time}^{-1}] = [\text{time}^{-1}]$, so $\sqrt{a_2} a_1 dt$ is dimensionless. Using the trigonometric identity $\sqrt{1 - \cos^2\theta} = \sin\theta$, Eq. (23.A.9) reduces to

$$\int d\theta = - \int \sqrt{a_2} a_1 dt. \tag{23.A.10}$$

Although at this point in the derivation we don't know that $\sqrt{a_2} a_1$, which has dimensions of frequency, is the angular frequency of oscillation, we'll use some foresight and make the identification

$$\omega_0 \equiv \sqrt{a_2} a_1 = \sqrt{\frac{k}{2E}} \sqrt{\frac{2E}{m}} = \sqrt{\frac{k}{m}}, \tag{23.A.11}$$

and Eq. (23.A.10) becomes

$$\int_{\theta=\theta_0}^{\theta} d\theta = - \int_{t=0}^t \omega_0 dt. \tag{23.A.12}$$

After integration we have

$$\theta - \theta_0 = -\omega_0 t, \quad (23.A.13)$$

where $\theta_0 \equiv -\phi$ is the constant of integration. Because $\theta = \cos^{-1}(\sqrt{a_2} x(t))$, Eq. (23.A.13) becomes

$$\cos^{-1}(\sqrt{a_2} x(t)) = -(\omega_0 t + \phi). \quad (23.A.14)$$

Take the cosine of each side of Eq. (23.A.14), yielding

$$x(t) = \frac{1}{\sqrt{a_2}} \cos(-(\omega_0 t + \phi)) = \sqrt{\frac{2E}{k}} \cos(\omega_0 t + \phi). \quad (23.A.15)$$

At $t = 0$,

$$x_0 \equiv x(t = 0) = \sqrt{\frac{2E}{k}} \cos \phi. \quad (23.A.16)$$

The x -component of the velocity as a function of time is then

$$v_x(t) = \frac{dx(t)}{dt} = -\omega_0 \sqrt{\frac{2E}{k}} \sin(\omega_0 t + \phi). \quad (23.A.17)$$

At $t = 0$,

$$v_{x,0} \equiv v_x(t = 0) = -\omega_0 \sqrt{\frac{2E}{k}} \sin \phi. \quad (23.A.18)$$

We can determine the constant ϕ by dividing the expressions in Eqs. (23.A.18) and (23.A.16),

$$-\frac{v_{x,0}}{\omega_0 x_0} = \tan \phi. \quad (23.A.19)$$

Thus the constant ϕ can be determined by the initial conditions and the angular frequency of oscillation,

$$\phi = \tan^{-1} \left(-\frac{v_{x,0}}{\omega_0 x_0} \right). \quad (23.A.20)$$

Use the identity

$$\cos(\omega_0 t + \phi) = \cos(\omega_0 t) \cos(\phi) - \sin(\omega_0 t) \sin(\phi) \quad (23.A.21)$$

to expand Eq. (23.A.15) yielding

$$x(t) = \sqrt{\frac{2E}{k}} \cos(\omega_0 t) \cos(\phi) - \sqrt{\frac{2E}{k}} \sin(\omega_0 t) \sin(\phi), \quad (23.A.22)$$

and substituting Eqs. (23.A.16) and (23.A.18) into Eq. (23.A.22) yields

$$x(t) = x_0 \cos \omega_0 t + \frac{v_{x,0}}{\omega_0} \sin \omega_0 t, \quad (23.A.23)$$

agreeing with Eq. (23.2.21).

So, what about the missing \pm that should have been in Eq. (23.A.5)? Strictly speaking, we would need to redo the derivation for the block moving in different directions. Mathematically, this would mean replacing ϕ by $\pi - \phi$ (or $\phi - \pi$) when the block's velocity changes direction. Changing from the positive square root to the negative *and* changing ϕ to $\pi - \phi$ have the collective action of reproducing Eq. (23.A.23).

Appendix 23B: Complex Numbers

A complex number z can be written as a sum of a real number x and a purely imaginary number iy where $i = \sqrt{-1}$,

$$z = x + iy. \quad (23.B.1)$$

The complex number can be represented as a point in the x-y plane as shown in Figure 23B.1.

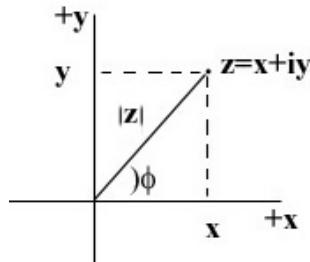


Figure 23B.1 Complex numbers

The complex conjugate \bar{z} of a complex number z is defined to be

$$\bar{z} = x - iy. \quad (23.B.2)$$

The modulus of a complex number is

$$|z| = (z\bar{z})^{1/2} = ((x+iy)(x-iy))^{1/2} = (x^2 + y^2)^{1/2}. \quad (23.B.3)$$

where we used the fact that $i^2 = -1$. The modulus $|z|$ represents the length of the ray from the origin to the complex number z in Figure 23B.1. Let ϕ denote the angle that the ray with the positive x -axis in Figure 23B.1. Then

$$x = |z| \cos \phi, \quad (23.B.4)$$

$$y = |z| \sin \phi. \quad (23.B.5)$$

Hence the angle ϕ is given by

$$\phi = \tan^{-1}(y/x). \quad (23.B.6)$$

The inverse of a complex number is then

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x - iy}{(x^2 + y^2)}. \quad (23.B.7)$$

The modulus of the inverse is the inverse of the modulus;

$$\left| \frac{1}{z} \right| = \frac{1}{(x^2 + y^2)^{1/2}} = \frac{1}{|z|}. \quad (23.B.8)$$

The sum of two complex numbers, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, is the complex number

$$z_3 = z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) = x_3 + iy_3, \quad (23.B.9)$$

where $x_3 = x_1 + x_2$, $y_3 = y_1 + y_2$. We can represent this by the vector sum in Figure 23B.2,

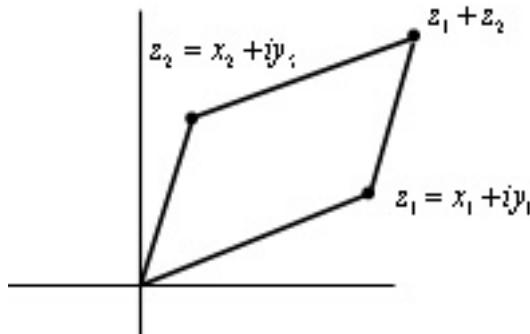


Figure 23B.2 Sum of two complex numbers

The product of two complex numbers is given by

$$z_3 = z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) = x_3 + iy_3, \quad (23.B.10)$$

where $x_3 = x_1 x_2 - y_1 y_2$, and $y_3 = x_1 y_2 + x_2 y_1$.

One of the most important identities in mathematics is the Euler formula,

$$e^{i\phi} = \cos\phi + i\sin\phi. \quad (23.B.11)$$

This identity follows from the power series representations for the exponential, sine, and cosine functions,

$$e^{i\phi} = \sum_{n=0}^{\infty} \frac{1}{n!} (i\phi)^n = 1 + i\phi - \frac{\phi^2}{2} - i\frac{\phi^3}{3!} + \frac{\phi^4}{4!} + i\frac{\phi^5}{5!} \dots, \quad (23.B.12)$$

$$\cos\phi = 1 - \frac{\phi^2}{2} + \frac{\phi^4}{4!} - \dots, \quad (23.B.13)$$

$$\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \dots. \quad (23.B.14)$$

We define two projection operators. The first one takes the complex number $e^{i\phi}$ and gives its real part,

$$\operatorname{Re} e^{i\phi} = \cos\phi. \quad (23.B.15)$$

The second operator takes the complex number $e^{i\phi}$ and gives its imaginary part, which is the real number

$$\operatorname{Im} e^{i\phi} = \sin\phi. \quad (23.B.16)$$

A complex number $z = x + iy$ can also be represented as the product of a modulus $|z|$ and a phase factor $e^{i\phi}$,

$$z = |z| e^{i\phi}. \quad (23.B.17)$$

The inverse of a complex number is then

$$\frac{1}{z} = \frac{1}{|z| e^{i\phi}} = \frac{1}{|z|} e^{-i\phi}, \quad (23.B.18)$$

where we used the fact that

$$\frac{1}{e^{i\phi}} = e^{-i\phi}. \quad (23.B.19)$$

In terms of modulus and phase, the sum of two complex numbers, $z_1 = |z_1| e^{i\phi_1}$ and $z_2 = |z_2| e^{i\phi_2}$, is

$$z_1 + z_2 = |z_1| e^{i\phi_1} + |z_2| e^{i\phi_2}. \quad (23.B.20)$$

A special case of this result is when the phase angles are equal, $\phi_1 = \phi_2$, then the sum $z_1 + z_2$ has the same phase factor $e^{i\phi_1}$ as z_1 and z_2 ,

$$z_1 + z_2 = |z_1| e^{i\phi_1} + |z_2| e^{i\phi_1} = (|z_1| + |z_2|) e^{i\phi_1}. \quad (23.B.21)$$

The product of two complex numbers, $z_1 = |z_1| e^{i\phi_1}$, and $z_2 = |z_2| e^{i\phi_2}$ is

$$z_1 z_2 = |z_1| e^{i\phi_1} |z_2| e^{i\phi_2} = |z_1| |z_2| e^{i(\phi_1 + \phi_2)}. \quad (23.B.22)$$

When the phases are equal, the product does not have the same factor as z_1 and z_2 ,

$$z_1 z_2 = |z_1| e^{i\phi_1} |z_2| e^{i\phi_1} = |z_1| |z_2| e^{i2\phi_1}. \quad (23.B.23)$$

Appendix 23C: Solution to the Underdamped Simple Harmonic Oscillator

Consider the underdamped simple harmonic oscillator equation (Eq. (23.5.4)),

$$\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = 0. \quad (23.C.1)$$

When $(b/m)^2 < 4k/m$, we show that the equation has a solution of the form

$$x(t) = x_m e^{-\alpha t} \cos(\gamma t + \phi). \quad (23.C.2)$$

Solution: Let's suppose the function $x(t)$ has the form

$$x(t) = A \operatorname{Re}(e^{zt}) \quad (23.C.3)$$

where z is a number (possibly complex) and A is a real number. Then

$$\frac{dx}{dt} = zAe^{zt} \quad (23.C.4)$$

$$\frac{d^2x}{dt^2} = z^2 A e^{zt} \quad (23.C.5)$$

We now substitute Eqs. (23.C.3), (23.C.4), and (23.C.5), into Eq. (23.C.1) resulting in

$$z^2 Ae^{zt} + \frac{b}{m} z Ae^{zt} + \frac{k}{m} Ae^{zt} = 0. \quad (23.C.6)$$

Collecting terms in Eq. (23.C.6) yields

$$\left(z^2 + \frac{b}{m} z + \frac{k}{m} \right) Ae^{zt} = 0 \quad (23.C.7)$$

The condition for the solution is that

$$z^2 + \frac{b}{m} z + \frac{k}{m} = 0. \quad (23.C.8)$$

This quadratic equation has solutions

$$z = \frac{-(b/m) \pm ((b/m)^2 - 4k/m)^{1/2}}{2}. \quad (23.C.9)$$

When $(b/m)^2 < 4k/m$, the oscillator is called *underdamped*, and we have two solutions for z , however the solutions are complex numbers. Let

$$\gamma = (k/m - (b/2m)^2)^{1/2}; \quad (23.C.10)$$

and

$$\alpha = b/2m. \quad (23.C.11)$$

Recall that the imaginary number $i = \sqrt{-1}$. The two solutions are then $z_1 = -\alpha + i\gamma t$ and $z_2 = -\alpha - i\gamma t$. Because our system is linear, our general solution is a linear combination of these two solutions,

$$x(t) = A_1 e^{-\alpha+i\gamma t} + A_2 e^{-\alpha-i\gamma t} = (A_1 e^{i\gamma t} + A_2 e^{-i\gamma t}) e^{-\alpha t}, \quad (23.C.12)$$

where A_1 and A_2 are constants. We shall transform this expression into a more familiar equation involving sine and cosine functions with help from the Euler formula,

$$e^{\pm i\gamma t} = \cos(\gamma t) \pm i \sin(\gamma t). \quad (23.C.13)$$

Therefore we can rewrite our solution as

$$x(t) = (A_1(\cos(\gamma t) + i \sin(\gamma t)) + A_2(\cos(\gamma t) - i \sin(\gamma t))) e^{-\alpha t}. \quad (23.C.14)$$

A little rearrangement yields

$$x(t) = ((A_1 + A_2)\cos(\gamma t) + i(A_1 - A_2)\sin(\gamma t))e^{-\alpha t}. \quad (23.C.15)$$

Define two new constants $C = A_1 + A_2$ and $D = i(A_1 - A_2)$. Then our solution looks like

$$x(t) = (C \cos(\gamma t) + D \sin(\gamma t))e^{-\alpha t}. \quad (23.C.16)$$

Recall from Example 23.5 that we can rewrite

$$C \cos(\gamma t) + D \sin(\gamma t) = x_m \cos(\gamma t + \phi) \quad (23.C.17)$$

,

where

$$x_m = (C^2 + D^2)^{1/2}, \text{ and } \phi = \tan^{-1}(D/C).$$

Then our general solution for the underdamped case (Eq. (23.C.16)) can be written as

$$x(t) = x_m e^{-\alpha t} \cos(\gamma t + \phi). \quad (23.C.18)$$

There are two other possible cases which we shall not analyze: when $(b/m)^2 > 4k/m$, a case referred to as *overdamped*, and when $(b/m)^2 = 4k/m$, a case referred to as *critically damped*.

Appendix 23D: Solution to the Forced Damped Oscillator Equation

We shall now use complex numbers to solve the differential equation

$$F_0 \cos(\omega t) = m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx. \quad (23.D.1)$$

We begin by assuming a solution of the form

$$x(t) = x_0 \cos(\omega t + \phi). \quad (23.D.2)$$

where the amplitude x_0 and the phase constant ϕ need to be determined. We begin by defining the complex function

$$z(t) = x_0 e^{i(\omega t + \phi)}. \quad (23.D.3)$$

Our desired solution can be found by taking the real projection

$$x(t) = \operatorname{Re}(z(t)) = x_0 \cos(\omega t + \phi) . \quad (23.D.4)$$

Our differential equation can now be written as

$$F_0 e^{i\omega t} = m \frac{d^2 z}{dt^2} + b \frac{dz}{dt} + kz . \quad (23.D.5)$$

We take the first and second derivatives of Eq. (23.D.3),

$$\frac{dz}{dt}(t) = i\omega x_0 e^{i(\omega t + \phi)} = i\omega z . \quad (23.D.6)$$

$$\frac{d^2 z}{dt^2}(t) = -\omega^2 x_0 e^{i(\omega t + \phi)} = -\omega^2 z . \quad (23.D.7)$$

We substitute Eqs. (23.D.3), (23.D.6), and (23.D.7) into Eq. (23.D.5) yielding

$$F_0 e^{i\omega t} = (-\omega^2 m + bi\omega + k)z = (-\omega^2 m + bi\omega + k)x_0 e^{i(\omega t + \phi)} . \quad (23.D.8)$$

We divide Eq. (23.D.8) through by $e^{i\omega t}$ and collect terms using yielding

$$x_0 e^{i\phi} = \frac{F_0 / m}{((\omega_0^2 - \omega^2) + i(b/m)\omega)} . \quad (23.D.9)$$

where we have used $\omega_0^2 = k/m$. Introduce the complex number

$$z_1 = (\omega_0^2 - \omega^2) + i(b/m)\omega . \quad (23.D.10)$$

Then Eq. (23.D.9) can be written as

$$x_0 e^{i\phi} = \frac{F_0}{my} . \quad (23.D.11)$$

Multiply the numerator and denominator of Eq. (23.D.11) by the complex conjugate $\bar{z}_1 = (\omega_0^2 - \omega^2) - i(b/m)\omega$ yielding

$$x_0 e^{i\phi} = \frac{F_0 \bar{z}_1}{m z_1 \bar{z}_1} = \frac{F_0}{m} \frac{((\omega_0^2 - \omega^2) - i(b/m)\omega)}{((\omega_0^2 - \omega^2)^2 + (b/m)^2 \omega^2)} \equiv u + iv . \quad (23.D.12)$$

where

$$u = \frac{F_0}{m} \frac{(\omega_0^2 - \omega^2)}{((\omega_0^2 - \omega^2)^2 + (b/m)^2 \omega^2)} , \quad (23.D.13)$$

$$v = -\frac{F_0}{m} \frac{(b/m)\omega}{((\omega_0^2 - \omega^2)^2 + (b/m)^2\omega^2)} . \quad (23.D.14)$$

Therefore the modulus x_0 is given by

$$x_0 = (u^2 + v^2)^{1/2} = \frac{F_0 / m}{((\omega_0^2 - \omega^2)^2 + (b/m)^2\omega^2)} , \quad (23.D.15)$$

and the phase is given by

$$\phi = \tan^{-1}(v/u) = \frac{-(b/m)\omega}{(\omega_0^2 - \omega^2)} . \quad (23.D.16)$$