HOMEWORK 1

1. (a)

$$
\begin{aligned}
&\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)\\
=&\frac{1}{2}\left(\cos\theta+i\sin\theta+\frac{1}{\cos\theta+i\sin\theta}\right)\\
=&\frac{1}{2}\left(\frac{\cos^2\theta+i\sin\theta\cos\theta}{\cos\theta+i\sin\theta}+\frac{i\sin\theta\cos\theta-\sin^2\theta}{\cos\theta i\sin\theta}+\frac{1}{\cos\theta+i\sin\theta}\right)\\
=&\frac{1}{2}\left(\frac{2\cos^2\theta+2i\cos\theta\sin\theta}{\cos\theta+i\sin\theta}\right)\\
=&\frac{2\cos\theta}{2}\left(\frac{\cos\theta+i\sin\theta}{\cos\theta+i\sin\theta}\right)\\
=&\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)\\
=&\frac{1}{2i}\left(\cos\theta+i\sin\theta-\frac{1}{\cos\theta+i\sin\theta}\right)\\
=&\frac{1}{2}\left(\frac{\cos^2\theta+i\sin\theta\cos\theta}{\cos\theta+i\sin\theta}+\frac{i\sin\theta\cos\theta-\sin^2\theta}{\cos\theta i\sin\theta}-\frac{1}{\cos\theta+i\sin\theta}\right)\\
=&\frac{1}{2i}\left(\frac{-2\sin^2\theta+2i\sin\theta\cos\theta}{\cos\theta+i\sin\theta}\right)\\
=&\sin\theta\left(\frac{-\sin\theta+i\cos\theta}{-\sin\theta+i\cos\theta}\right)\\
=&\sin\theta
\end{aligned}
$$

(b) The probability density, $P(x)$ is defined:

$$P(x)=\psi^*(x)\,\psi(x)$$

So we have:

$$
\begin{aligned}
P(x)&=\chi(x)e^{-i\varphi}\chi(x)e^{i\varphi}\\
&=[\chi(x)]^2
\end{aligned}
$$

Hence we see that the probability density is independent of the global complex phase factor.

(c)

$$
\begin{aligned}
&\sin\theta\sin\varphi \\
=&\frac{1}{2i}\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)\left(e^{i\varphi}-e^{-i\varphi}\right) \\
=&-\frac{1}{4}\left[e^{i\theta+i\varphi}-e^{i\theta-i\varphi}-e^{i\varphi-i\theta}+e^{-i\theta-i\varphi}\right] \\
=&-\frac{1}{4}\left[2\cos\left(\theta+\varphi\right)-2\cos\left(\theta-\varphi\right)\right] \\
=&\frac{1}{2}\left[\cos\left(\theta-\varphi\right)-\cos\left(\theta+\varphi\right)\right]
\end{aligned}
$$

2. To show that the eigenstates of the infinite square well have been normalized between $x = 0$ and $x = a$, we must show that its inner product is one over that region:

$$
\begin{aligned}
&\int_0^a |\psi_n|^2\, dx \\
=&\frac{2}{a}\int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx \\
=&\frac{2}{a}\left[\frac{x}{2}-\frac{a\sin\left(\frac{2n\pi x}{a}\right)}{4n\pi}\right]_0^a \\
=&\frac{2}{a}\cdot\frac{a}{2} \\
=&1
\end{aligned}
$$

Hence the eigenstates are normalized.

3.

$$
\begin{aligned}
&\int_0^a \psi_n\psi_m dx \\
&= \frac{2}{a}\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right) dx \\
&= a\int_0^a \left[\cos\left(\frac{(n-m)\,\pi}{a}x\right)-\cos\left(\frac{(n+m)\,\pi}{a}x\right)\right] dx && \text{(Using the trig identity from 1c)} \\
&= a\left[\frac{a}{(n-m)\,\pi}\sin\left(\frac{(n-m)\,\pi}{a}x\right)-\frac{a}{(n+m)\,\pi}\sin\left(\frac{(n+m)\,\pi}{a}x\right)\right]_0^a \\
&= 0 && \text{(Because m and n are integers)}
\end{aligned}
$$