

HOMEWORK 1

1. (a)

$$\begin{aligned}
 \cos \theta &= \frac{1}{2} (e^{i\pi} + e^{-i\pi}) \\
 &= \frac{1}{2} (\cos \theta + i \sin \theta + \cos (-\theta) + i \sin (-\theta)) \\
 &= \frac{1}{2} (\cos \theta + i \sin \theta + \cos \theta - i \sin \theta) && \text{(because cos is even and sin is odd)} \\
 &= \frac{1}{2} (2 \cos \theta) \\
 &= \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \sin \theta &= \frac{1}{2i} (e^{i\theta} - e^{-i\theta}) \\
 &= \frac{1}{2i} (\cos \theta + i \sin \theta - \cos \theta + i \sin \theta) && \text{(using trick from last derivation)} \\
 &= \frac{1}{2i} (2i \sin \theta) \\
 &= \sin \theta
 \end{aligned}$$

(b) The probability density, $P(x)$ is defined:

$$P(x) = \psi^*(x) \psi(x)$$

So we have:

$$\begin{aligned}
 P(x) &= \chi(x) e^{-i\varphi} \chi(x) e^{i\varphi} \\
 &= [\chi(x)]^2
 \end{aligned}$$

Hence we see that the probability density is independent of the global complex phase factor.

(c)

$$\begin{aligned}
 &\sin \theta \sin \varphi \\
 &= \frac{1}{2i} \frac{1}{2i} (e^{i\theta} - e^{-i\theta}) (e^{i\varphi} - e^{-i\varphi}) \\
 &= -\frac{1}{4} [e^{i\theta+i\varphi} - e^{i\theta-i\varphi} - e^{i\varphi-i\theta} + e^{-i\theta-i\varphi}] \\
 &= -\frac{1}{4} [2 \cos (\theta + \varphi) - 2 \cos (\theta - \varphi)] \\
 &= \frac{1}{2} [\cos (\theta - \varphi) - \cos (\theta + \varphi)]
 \end{aligned}$$

2. (a) To show that the eigenstates of the infinite square well have been normalized between $x = 0$ and $x = a$, we must show that its inner product is one over that region:

$$\begin{aligned}
& \int_0^a |\psi_n|^2 dx \\
&= \frac{2}{a} \int_0^a \sin^2 \left(\frac{n\pi x}{a} \right) dx \\
&= \frac{2}{a} \left[\frac{x}{2} - \frac{a \sin \left(\frac{2n\pi x}{a} \right)}{4n\pi} \right]_0^a \\
&= \frac{2}{a} \cdot \frac{a}{2} \\
&= 1
\end{aligned}$$

Hence the eigenstates are normalized.

(b)

$$\begin{aligned}
& \int_0^a \psi_n \psi_m dx \\
&= \frac{2}{a} \int_0^a \sin \left(\frac{n\pi x}{a} \right) \sin \left(\frac{m\pi x}{a} \right) dx \\
&= a \int_0^a \left[\cos \left(\frac{(n-m)\pi}{a} x \right) - \cos \left(\frac{(n+m)\pi}{a} x \right) \right] dx \quad (\text{using the trig identity from 1c}) \\
&= a \left[\frac{a}{(n-m)\pi} \sin \left(\frac{(n-m)\pi}{a} x \right) - \frac{a}{(n+m)\pi} \sin \left(\frac{(n+m)\pi}{a} x \right) \right]_0^a \\
&= 0 \quad (\text{because m and n are integers})
\end{aligned}$$

(c) To show that the eigenstates for the QHM are orthogonal, ψ_0 and ψ_2 , we must show that the expression $\int_{-\infty}^{\infty} \psi_0 \psi_2 dx = 0$.

First substitute $H_0(\xi) = 1$ and $H_2(\xi) = 4\xi^2 - 2$ and put everything in terms of ξ . We also drop all normalization constants for clarity since they do not matter when showing orthogonality:

$$\psi_0(x) = e^{\frac{1}{2}\xi^2} \qquad \psi_2(x) = (4\xi - 2) e^{-\frac{1}{2}\xi}$$

$$\begin{aligned}
& \int_{-\infty}^{\infty} \psi_0 \psi_2 dx \\
&= \int_{-\infty}^{\infty} e^{\frac{1}{2}\xi^2} (4\xi - 2) e^{-\frac{1}{2}\xi} d\xi \\
&= \int_{-\infty}^{\infty} [4\xi^2 e^{-\xi^2} - 2e^{-\xi^2}] d\xi \\
&= 2\sqrt{\pi} - 2\sqrt{\pi} \quad (\text{using "Handy Integrals"}) \\
&= 0
\end{aligned}$$

Hence, we have shown that the two eigenstates are orthogonal.

3. (a) To show that $|\Psi(x, t)\rangle$ is normalized, we must show:

$$\langle \Psi(x, t) | \Psi(x, t) \rangle = 1$$

We start:

$$\langle \Psi(x, t) | \Psi(x, t) \rangle = \frac{1}{2} \left[\langle \psi_1 | \psi_1 \rangle + \langle \psi_2 | \psi_2 \rangle + \langle \psi_2 | \psi_1 \rangle e^{(E_2 - E_1) \frac{it}{\hbar}} + \langle \psi_1 | \psi_2 \rangle e^{(E_1 - E_2) \frac{it}{\hbar}} \right]$$

Since we have shown in (2a, 2b) that the eigenstates of the infinite square well, $\langle \psi_n | \psi_m \rangle = \delta_{nm}$ where δ_{nm} is the Kronecker delta, the right hand side simplifies:

$$\begin{aligned} &= \frac{1}{2} [1 + 1 + 0 + 0] \\ &= 1 \end{aligned}$$

Hence, we have shown that the superposition state is normalized.

(b)

$$\begin{aligned} \langle \Psi | \hat{H} | \Psi \rangle &= \frac{1}{2} \left(\langle \psi_1 | \hat{H} | \psi_1 \rangle + \langle \psi_2 | \hat{H} | \psi_2 \rangle + \langle \psi_2 | \hat{H} | \psi_1 \rangle e^{(E_2 - E_1) \frac{it}{\hbar}} + \langle \psi_1 | \hat{H} | \psi_2 \rangle e^{(E_1 - E_2) \frac{it}{\hbar}} \right) \\ &= \frac{1}{2} \left(\langle \psi_1 | E_1 | \psi_1 \rangle + \langle \psi_2 | E_2 | \psi_2 \rangle + \langle \psi_2 | E_1 | \psi_1 \rangle e^{(E_2 - E_1) \frac{it}{\hbar}} + \langle \psi_1 | E_2 | \psi_2 \rangle e^{(E_1 - E_2) \frac{it}{\hbar}} \right) \\ &= \frac{1}{2} \left(E_1 \langle \psi_1 | \psi_1 \rangle + E_2 \langle \psi_2 | \psi_2 \rangle + E_1 \langle \psi_2 | \psi_1 \rangle e^{(E_2 - E_1) \frac{it}{\hbar}} + E_2 \langle \psi_1 | \psi_2 \rangle e^{(E_1 - E_2) \frac{it}{\hbar}} \right) \\ &= \frac{1}{2} (E_1 + E_2) \quad (\text{applying simplification as in (a)}) \end{aligned}$$

4. • (6-30)

The zero-point energy is given by:

$$\begin{aligned} E_0 &= \frac{1}{2} \hbar \omega \\ &= \frac{1}{2} \hbar \sqrt{\frac{C}{m}} \\ &= \frac{1}{2} (1.05 \times 10^{-34}) \sqrt{\frac{10^3}{4.1 \times 10^{-26}}} \quad (\text{plugging in numbers}) \\ &\approx 8.20 \times 10^{-21} \text{ J} \quad \text{or} \quad 0.051 \text{ eV} \end{aligned}$$

- (6-31)

(a) To get the discrete energy levels for the SHO, we have the equation:

$$E_n = \left(n + \frac{1}{2} \right) \omega$$

Thus,

$$E_{\text{photon}} = \Delta E = E_1 - E_0 \approx 0.154 \text{ eV} - 0.051 \text{ eV} = 0.103 \text{ eV}$$

(b) We expect that the energy emitted is equal to the difference in energy between the ground

state and first energy level, or 0.103 eV.

(c) For the photon,

$$E_{\text{photon}} = \hbar\omega_{\text{photon}}$$

And, we also know that

$$E_{\text{photon}} = \Delta E = \hbar\omega \quad (\text{where } \omega \text{ is the classical oscillation frequency})$$

Thus,

$$\omega_{\text{photon}} = \omega$$

We get frequency, f :

$$f = \frac{E_{\text{photon}}}{h} = \frac{0.102 \cdot 1.6 \times 10^{-19}}{6.626 \times 10^{-34}} = 2.46 \times 10^{13} \text{ Hz}$$

This frequency ($\lambda \approx 12000 \text{ nm}$) corresponds with infrared light.