HOMEWORK 1

1. (a)

$$
\begin{aligned}
&\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)\\
=&\frac{1}{2}\left(\cos\theta+i\sin\theta+\frac{1}{\cos\theta+i\sin\theta}\right)\\
=&\frac{1}{2}\left(\frac{\cos^2\theta+i\sin\theta\cos\theta}{\cos\theta+i\sin\theta}+\frac{i\sin\theta\cos\theta-\sin^2\theta}{\cos\theta i\sin\theta}+\frac{1}{\cos\theta+i\sin\theta}\right)\\
=&\frac{1}{2}\left(\frac{2\cos^2\theta+2i\cos\theta\sin\theta}{\cos\theta+i\sin\theta}\right)\\
=&\frac{2\cos\theta}{2}\left(\frac{\cos\theta+i\sin\theta}{\cos\theta+i\sin\theta}\right)\\
=&\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)\\
=&\frac{1}{2i}\left(\cos\theta+i\sin\theta-\frac{1}{\cos\theta+i\sin\theta}\right)\\
=&\frac{1}{2}\left(\frac{\cos^2\theta+i\sin\theta\cos\theta}{\cos\theta+i\sin\theta}+\frac{i\sin\theta\cos\theta-\sin^2\theta}{\cos\theta i\sin\theta}-\frac{1}{\cos\theta+i\sin\theta}\right)\\
=&\frac{1}{2i}\left(\frac{-2\sin^2\theta+2i\sin\theta\cos\theta}{\cos\theta+i\sin\theta}\right)\\
=&\sin\theta\left(\frac{-\sin\theta+i\cos\theta}{-\sin\theta+i\cos\theta}\right)\\
=&\sin\theta
\end{aligned}
$$

(b) The probability density, $P(x)$ is defined:

$$P(x)=\psi^*(x)\,\psi(x)$$

So we have:

$$
\begin{aligned}
P(x)&=\chi(x)e^{-i\varphi}\chi(x)e^{i\varphi}\\
&=[\chi(x)]^2
\end{aligned}
$$

Hence we see that the probability density is independent of the global complex phase factor.

(c)

$$
\begin{aligned}
&\sin\theta\sin\varphi \\
=&\frac{1}{2i}\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)\left(e^{i\varphi}-e^{-i\varphi}\right) \\
=&-\frac{1}{4}\left[e^{i\theta+i\varphi}-e^{i\theta-i\varphi}-e^{i\varphi-i\theta}+e^{-i\theta-i\varphi}\right] \\
=&-\frac{1}{4}\left[2\cos\left(\theta+\varphi\right)-2\cos\left(\theta-\varphi\right)\right] \\
=&\frac{1}{2}\left[\cos\left(\theta-\varphi\right)-\cos\left(\theta+\varphi\right)\right]
\end{aligned}
$$

2. (a) To show that the eigenstates of the infinite square well have been normalized between $x = 0$ and $x = a$, we must show that its inner product is one over that region:

$$
\begin{aligned}
&\int_0^a \left|\psi_n\right|^2 dx \\
=&\frac{2}{a}\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx \\
=&\frac{2}{a}\left[\frac{x}{2}-\frac{a\sin\left(\frac{2n\pi x}{a}\right)}{4n\pi}\right]_0^a \\
=&\frac{2}{a}\cdot\frac{a}{2} \\
=&1
\end{aligned}
$$

Hence the eigenstates are normalized.

(b)

$$
\begin{aligned}
&\int_0^a \psi_n\psi_m dx \\
&= \frac{2}{a}\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx \\
&= a\int_0^a \left[\cos\left(\frac{(n-m)\pi}{a}x\right)-\cos\left(\frac{(n+m)\pi}{a}x\right)\right]dx && \text{(using the trig identity from 1c)} \\
&= a\left[\frac{a}{(n-m)\pi}\sin\left(\frac{(n-m)\pi}{a}x\right)-\frac{a}{(n+m)\pi}\sin\left(\frac{(n+m)\pi}{a}x\right)\right]_0^a \\
&= 0 && \text{(because m and n are integers)}
\end{aligned}
$$

(c) To show that the eigenstates for the QHM are orthogonal, ψ_0 and ψ_2, we must show that the expression $\int_{-\infty}^{\infty}\psi_0\psi_2 dx = 0$.
First substitute $H_0(\xi) = 1$ and $H_2(\xi) = 4\xi^2 - 2$ and put everything in terms of ξ. We also drop all normalization constants for clarity since they do not matter when showing orthogonality:

$$\psi_0(x) = e^{\frac{1}{2}\xi^2} \qquad \psi_2(x) = (4\xi - 2)\,e^{-\frac{1}{2}\xi}$$

$$\begin{aligned}
&\int_{-\infty}^{\infty} \psi_0 \psi_2 dx \\
&= \int_{-\infty}^{\infty} e^{\frac{1}{2}\xi^2} \left(4\xi - 2\right) e^{-\frac{1}{2}\xi} d\xi \\
&= \int_{-\infty}^{\infty} \left[4\xi^2 e^{-\xi^2} - 2e^{-\xi^2}\right] d\xi \\
&= 2\sqrt{\pi} - 2\sqrt{\pi} && \text{(using "Handy Integrals")} \\
&= 0
\end{aligned}$$

Hence, we have shown that the two eigenstates are orthogonal.

3. (a) To show that $|\Psi(x,t)\rangle$ is normalized, we must show:

$$\langle \Psi(x,t) | \Psi(x,t) \rangle = 1$$

We start:

$$\langle \Psi(x,t) | \Psi(x,t) \rangle = \frac{1}{2}\left[\langle \psi_1|\psi_1\rangle + \langle \psi_2|\psi_2\rangle + \langle \psi_2|\psi_1\rangle e^{(E_2-E_1)\frac{it}{\hbar}} + \langle \psi_1|\psi_2\rangle e^{(E_1-E_2)\frac{it}{\hbar}}\right]$$

Since we have shown in *(2a, 2b)* that the eigenstates of the infinite square well, $\langle \psi_n|\psi_m\rangle = \delta_{nm}$ where δ_{nm} is the Kronecker delta, the right hand side simplifies:

$$\begin{aligned}
&= \frac{1}{2}\left[1 + 1 + 0 + 0\right] \\
&= 1
\end{aligned}$$

Hence, we have shown that the superposition state is normalized.

(b)

$$\begin{aligned}
\langle \Psi|\hat{H}|\Psi\rangle &= \frac{1}{2}\left(\langle \psi_1|\hat{H}|\psi_1\rangle + \langle \psi_2|\hat{H}|\psi_2\rangle + \langle \psi_2|\hat{H}|\psi_1\rangle e^{(E_2-E_1)\frac{it}{\hbar}} + \langle \psi_1|\hat{H}|\psi_2\rangle e^{(E_1-E_2)\frac{it}{\hbar}}\right) \\
&= \frac{1}{2}\left(\langle \psi_1|E_1|\psi_1\rangle + \langle \psi_2|E_2|\psi_2\rangle + \langle \psi_2|E_1|\psi_1\rangle e^{(E_2-E_1)\frac{it}{\hbar}} + \langle \psi_1|E_2|\psi_2\rangle e^{(E_1-E_2)\frac{it}{\hbar}}\right) \\
&= \frac{1}{2}\left(E_1 \langle \psi_1|\psi_1\rangle + E_2 \langle \psi_2|\psi_2\rangle + E_1 \langle \psi_2|\psi_1\rangle e^{(E_2-E_1)\frac{it}{\hbar}} + E_2 \langle \psi_1|\psi_2\rangle e^{(E_1-E_2)\frac{it}{\hbar}}\right) \\
&= \frac{1}{2}\left(E_1 + E_2\right) \qquad \text{(applying simplification as in (a))}
\end{aligned}$$

4. • *(6-30)*

The zero-point energy is given by:

$$\begin{aligned} E_0 &= \frac{1}{2}\hbar\omega \\ &= \frac{1}{2}\hbar\sqrt{\frac{C}{m}} \\ &= \frac{1}{2}\left(1.05 \times 10^{-34}\right)\sqrt{\frac{10^3}{4.1 \times 10^{-26}}} && \text{(plugging in numbers)} \\ &\approx 8.20 \times 10^{-21}\ \text{J} \quad \text{or} \quad 0.051\ \text{eV} \end{aligned}$$

- *(6-31)*

  (a) To get the discrete energy levels for the SHO, we have the equation:

  $$E_n = \left(n + \frac{1}{2}\right)\omega$$

  Thus,

  $$E_{photon} = \Delta E = E_1 - E_0 \approx 0.154\ \text{eV} - 0.051\ \text{eV} = 0.103\ \text{eV}$$

  (b) We expect that the energy emitted is equal to the difference in energy between the ground state and first energy level, or 0.103 eV.

  (c) For the photon,

  $$E_{photon} = \hbar\omega_{photon}$$

  And, we also know that

  $$E_{photon} = \Delta E = \hbar\omega \qquad \text{(where } \omega \text{ is the classical oscillation frequency)}$$

  Thus,

  $$\omega_{photon} = \omega$$

  We get frequency, f:

  $$f = \frac{E_{photon}}{h} = \frac{0.102 \cdot 1.6 \times 10^{-19}}{6.626 \times 10^{-34}} = 2.46 \times 10^{13}\ \text{Hz}$$

  This frequency ($\lambda \approx 12000$ nm) corresponds with infrared light.