HOMEWORK 4

1. (a) With $E = \mu \cdot B$ and considering the spin up and spin down cases, we have:

$$\begin{aligned}\Delta E &= \mu_B B\cos(0) - \mu_B B\cos(\pi) \\ &= 2\mu_B B \\ &= 2\frac{e\hbar}{2m_e}B \\ &= 2.24 \times 10^{-23}\text{ J} = 1.40 \times 10^{-4}\text{ eV}\end{aligned}$$

   (b) The energy required for the electron to flip spin is equal to the ΔE found in the last question. Using the equation $E = \frac{hc}{\lambda}$, we have:

$$\begin{aligned}\lambda &= \frac{hc}{\Delta E} \\ &= 8.94 \times 10^{-3}\text{ m}\end{aligned}$$

   The wavelength which causes the electrons' spin to flip is 8.94 mm

2. We have:

$$\Delta E = 2\mu_B B = \frac{hc}{\lambda}$$

   Thus,

$$B = \frac{hc}{\lambda}\frac{1}{2\mu_B} = .0511\text{ T}$$

3. For the states with $\ell = 4$ and $s = 1/2$, The state with the largest possible j and largest possible m_j has $j = 9/2$ and $m_j = 9/2$.

   (a) We have:

$$\begin{aligned}\vec{J} &= \vec{L} + \vec{S} \\ |\vec{J}| &= \sqrt{j(j+1)}\hbar = \sqrt{99}\hbar/2 \\ |\vec{L}| &= \sqrt{\ell(\ell+1)}\hbar = \sqrt{120}\hbar \\ |\vec{S}| &= \sqrt{s(s+1)}\hbar = \sqrt{3}\hbar/2\end{aligned}$$

   Now, apply the law of cosines

$$\begin{aligned}J^2 &= L^2 + S^2 - 2LS\cos(\pi - \theta) \\ \frac{99}{4} &= 20 + \frac{3}{4} + 2\sqrt{15}\cos\theta \\ \theta &= \cos^{-1}\left(\frac{2}{\sqrt{15}}\right) = 1.03\text{ rad or }58.9^\circ\end{aligned}$$

   (b) $\vec{\mu}_\ell$ is antiparallel to $\vec{L}$ and $\vec{\mu}_s$ is antiparallel to $\vec{S}$, thus the angle is the same as in part (a), $\theta = 58.9^\circ$.

(c)

$$\cos\theta = \frac{J_z}{|\vec{J}|} = \frac{9}{\sqrt{99}}$$

Thus, $\theta = 25.24^\circ$.

4. (a) For the *3d* level, we know that $n = 3$, $\ell = 2$, and $s = 1/2$.
   Therefore, $j = \frac{3}{2}, \frac{5}{2}$ and $m_j = -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$
   For $j = \frac{3}{2}$, we have $m_j = \frac{-3}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{3}{2}$, i.e., 4 degenerate states. Also, the spectroscopic notation:

$$3^2D_{3/2}$$

   For $j = \frac{5}{2}$, we have $m_j = \frac{-5}{2}, \frac{-3}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, i.e., 6 degenerate states. Also, the spectroscopic notation:

$$3^2D_{5/2}$$

   (b) It is impossible to have this state because the maximum possible value for ℓ is $\ell = n - 1$, and since $n = 2$ in this state, it is impossible for $\ell = 2$, i.e., the D subshell.

   (c) The ground state of the H atom is not split into two sub-levels by spin-orbit coupling because $\ell = 0$, and there is only one electron, thus there is no internal magnetic field present to interact with the electron's magnetic moment.

5. (a)

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2}$$

$$\begin{aligned}
E_I &= \frac{e^2}{4\pi\epsilon_0}\frac{1}{2a_0} \\
&= \frac{e^2}{4\pi\epsilon_0}\frac{m_e^2 e^2}{8\pi\epsilon_0\hbar^2} \\
&= \frac{1}{2}\frac{e^4}{16\pi^2\epsilon_0^2\hbar^2 c^2} m_e^2 c^2 \\
&= \frac{1}{2}\left(\frac{e^2}{4\pi\epsilon_0\hbar c}\right)^2 m_e^2 c^2 \\
&= \frac{1}{2}\alpha^2 m_e^2 c^2
\end{aligned}$$

   (b) We have the relativistic energy:

$$E = \sqrt{(\mu c^2)^2 + (pc)^2}$$

   And we know that the rest energy, E_0:

$$E_0 = \mu c^2$$

Thus we can find Kinetic energy by subtracting the rest energy from the total relativistic energy.

$$\begin{aligned}
\text{KE} &= \sqrt{\left(\mu c^2\right)^2 + \left(pc\right)^2} - \mu c^2 \\
&= \sqrt{\mu^2 c^4 + \frac{p^2 c^2 \mu^2 c^4}{\mu^2 c^4}} - \mu c^2 \\
&= \mu c^2 \sqrt{1 + \left(\frac{pc}{\mu c^2}\right)^2} - \mu c^2 \\
&\approx \mu c^2 \left[1 + \frac{1}{2}\left(\frac{pc}{\mu c^2}\right)^2 - \frac{1}{8}\left(\frac{pc}{\mu c^2}\right)^4\right] - \mu c^2 && \text{(using a Taylor series expansion)} \\
&= \mu c^2 \left[\frac{1}{2}\left(\frac{pc}{\mu c^2}\right)^2 - \frac{1}{8}\left(\frac{pc}{\mu c^2}\right)^4\right] \\
&= \frac{p^2}{2\mu} - \frac{1}{8}\frac{p^4}{\mu^3 c^2} \\
&= -\frac{p^4}{8\mu^3 c^2} && \text{(subtracting classical } KE = p^2/2\mu\text{)}
\end{aligned}$$

Thus, we have that the lowest order relativistic correction to the classical kinetic energy is

$$\Delta\text{KE} = \frac{p^4}{8\mu^3 c^2}.$$

(c) Since $j = \ell + s$, there are two possibilities for j:

$$j = \ell - \frac{1}{2} \quad \text{and} \quad j = \ell + \frac{1}{2}$$

Thus,

$$\ell = j + \frac{1}{2} \quad \text{and} \quad \ell = j - \frac{1}{2}$$

Start with the $\ell = j - 1/2$ case:

$$\begin{aligned}
f(j,\ell) &= \frac{j\left(j+1\right) - 3\ell\left(\ell+1\right) - \frac{3}{4}}{\ell\left(\ell+\frac{1}{2}\right)\left(l+1\right)} \\
&= \frac{j\left(j+1\right) - 3\left(j-\frac{1}{2}\right)\left(j+\frac{1}{2}\right) - \frac{3}{4}}{\left(j-\frac{1}{2}\right)\left(j\right)\left(j+\frac{1}{2}\right)} \\
&= \frac{j^2 + j - 3j^2}{\left(j-\frac{1}{2}\right)\left(j\right)\left(j+\frac{1}{2}\right)} \\
&= \frac{j\left(1-2j\right)}{\left(j-\frac{1}{2}\right)\left(j\right)\left(j+\frac{1}{2}\right)} \\
&= \frac{1-2j}{\left(j-\frac{1}{2}\right)\left(j+\frac{1}{2}\right)} \\
&= \frac{-2\left(j-\frac{1}{2}\right)}{\left(j-\frac{1}{2}\right)\left(j+\frac{1}{2}\right)} \\
&= -\frac{2}{\left(j+\frac{1}{2}\right)}
\end{aligned}$$

Now, start with the $\ell = j + 1/2$ case:

$$
\begin{aligned}
f(j,\ell) &= \frac{j\left(j+1\right) - 3\ell\left(\ell+1\right) - \frac{3}{4}}{\ell\left(\ell+\frac{1}{2}\right)\left(l+1\right)} \\
&= \frac{j\left(j+1\right) - 3\left(j+\frac{1}{2}\right)\left(j+\frac{3}{2}\right) - \frac{3}{4}}{\left(j+\frac{1}{2}\right)\left(j+1\right)\left(j+\frac{3}{2}\right)} \\
&= \frac{j\left(j+1\right) - 3j^2 - 6j - 3}{\left(j+\frac{1}{2}\right)\left(j+1\right)\left(j+\frac{3}{2}\right)} \\
&= \frac{-2j^2 - 5j - 3}{\left(j+\frac{1}{2}\right)\left(j+1\right)\left(j+\frac{3}{2}\right)} \\
&= \frac{-2\left(j^2+\frac{5}{2}j+\frac{3}{2}\right)}{\left(j+\frac{1}{2}\right)\left(j^2+\frac{5}{2}j+\frac{3}{2}\right)} \\
&= -\frac{2}{j+\frac{1}{2}}
\end{aligned}
$$

Hence, we have seen for both case that $f(j,\ell)$ is actually independent of ℓ and equal to

$$
f(n,\ell) = -\frac{2}{j+\frac{1}{2}}.
$$