Physics 40—Quantum Physics of Matter

HW 5—Due May 2, 2024

1. This problem is based on the wave function given in Example 9.2 of ER.
   (a) ER problem 9-4: verify that the expanded form of the three-particle eigenfunction of Example 9-2 is antisymmetric with respect to an exchange of the labels of any two particles.
   (b) ER problem 9-5: verify that the expanded form of the three-particle eigenfunction of Example 9-2 is identically equal to zero if two particles are in the same space and spin quantum state.

2. (After Serway 9.17) Eight identical, noninteracting particles with mass equal to the electron mass m_e are placed in a cubical box with side length $a = 2$ Å. Find the lowest energy of the system (in electron volts) and list the quantum numbers of all occupied states if (a) the particles are fermions and (b) the particles are bosons.

3. Consider two non-interacting indistinguishable particles of equal mass m in an infinite square well with width a described by a wave function $\psi(x_1, x_2)$ that could be either symmetric or antisymmetric:

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}} \left[\psi_n(x_1)\psi_m(x_2) \pm \psi_n(x_2)\psi_m(x_1)\right].$$

   Here, $\psi_n(x)$ is the nth single particle energy eigenstate. Consider the case in which one particle is in its ground state, and the other in its first excited state.

   (a) Show that the joint probability density $\mathcal{P}(x_1, x_2) = |\psi(x_1, x_2)|^2$ of finding one particle at x_1 and the other at x_2 is given by

$$\mathcal{P}(x_1, x_2) = \frac{1}{2} \left[|\psi_n(x_1)|^2|\psi_m(x_2)|^2 + |\psi_n(x_2)|^2|\psi_m(x_1)|^2 \pm 2\psi_n(x_1)\psi_m(x_2)\psi_n(x_2)\psi_m(x_1)\right]$$

   (b) Calculate the joint probability

$$P_{\text{left}} = \int_0^{a/2}\int_0^{a/2} \mathcal{P}(x_1, x_2)\, dx_1 dx_2$$

   of finding both particles in the left side of the well if they are in a *symmetric* total wave function.

   Note: All told, I count a total of 6 separate integrations in this problem. Fortunately, with proper use of symmetry, you only need to explicitly calculate one. Remember that the particles are identical. Also, what important symmetry property *must* all the $\psi_n(x)$ possess? (*Hint:* see Lecture 3.)

(c) Find the joint probability of finding both particles in the left side of the well if they are in an *antisymmetric* total wave function. (Very short answer, no new integration needed.)

4. Consider two identical particles of mass m confined to a one-dimensional harmonic oscillator potential and interacting via a repulsive "rubber band" interaction with Hamiltonian $\hat{H} = \hat{H}_0 + \hat{W}$ where

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 + \frac{1}{2}m\omega_0^2(x_1^2 + x_2^2)$$

is the two-particle harmonic oscillator Hamiltonian and

$$\hat{W} = -\frac{1}{2}m\omega_1(x_1 - x_2)^2$$

is the rubber band interaction. Assume the particles occupy a symmetric or antisymmetric wave function as in Problem 3 with one particle in the ground state and the other in the first excited state, except that in this case the $\psi_n(x)$ are the energy eigenstates of the harmonic oscillator (see below).

The goal of this problem is to calculate the direct integral

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi_n(x_1)|^2 \, \hat{W} \, |\psi_m(x_2)|^2 \; dx_1 dx_2$$

and the exchange integral

$$J = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_n(x_1)^*\psi_m(x_2) \, \hat{W} \, \psi_n(x_2)^*\psi_m(x_1) \; dx_1 dx_2$$

without actually evaluating any integrals at all. Here's how to do this:

(a) First, use a simple symmetry argument to show that any integral of the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi_1^n \xi_2^m e^{-(\xi_1^2+\xi_2^2)} \; d\xi_1 d\xi_2$$

where n, m are positive integers vanishes (equals zero) unless both n and m are even.

(b) Now use the result

$$\int_{-\infty}^{\infty} e^{-a\xi^2} \; d\xi = \sqrt{\frac{\pi}{a}}$$

and the trick I showed you in Lecture 9 to calculate $\int_{-\infty}^{\infty} \xi^2 e^{-\xi^2} \; d\xi$ and $\int_{-\infty}^{\infty} \xi^4 e^{-\xi^2} \; d\xi$.

(c) Now find I and J, and show that particles in a symmetric spatial state as usual have a higher energy than ones in an antisymmetric state (see Lecture 17).

Harmonic Oscillator Single Particle Eigenstates

Ground state:

$$\psi_0 = \left(\frac{1}{\pi x_0^2}\right)^{1/4} \frac{1}{\sqrt{2}}\, e^{-\frac{1}{2}\xi^2}$$

First excited state:

$$\psi_0 = \left(\frac{1}{\pi x_0^2}\right)^{1/4} \frac{1}{\sqrt{2}}\, \xi e^{-\frac{1}{2}\xi^2}$$

Here $x_0 = \sqrt{\frac{\hbar}{m\omega_0}}$ and $\xi = \frac{x}{x_0}$.