HOMEWORK 2

1. For the Time-Independent Schrodinger Equation in 3-D, we have:

$$-\frac{\hbar}{2\mu}\nabla^2 + U\left(x,y,z\right)\Psi\left(x,y,z\right) = E\Psi\left(x,y,z\right)$$

Plugging in the potential $U\left(x,y,z\right) = \frac{1}{2}\mu\omega^2\left(x^2+y^2+z^2\right)$, we have:

$$-\frac{\hbar}{2\mu}\nabla^2 + \frac{1}{2}\mu\omega^2\left(x^2+y^2+z^2\right)\Psi\left(x,y,z\right) = E\Psi\left(x,y,z\right)$$

For using the separation of variables method, we must split the Schrödinger equation into parts which depend only on x, y, and z.

First, we expand the "del" operator into its separate terms:

$$-\frac{\hbar}{2\mu}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) + \frac{1}{2}\mu\omega^2\left(x^2+y^2+z^2\right)\Psi\left(x,y,z\right) = E\Psi\left(x,y,z\right)$$

2.

3.

4.

5.

$$\begin{aligned}
\hat{L}^2 Y_1^{-1}\left(\theta,\varphi\right) &= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{-i\varphi}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{-i\varphi}\right] \\
&= -\hbar^2\frac{1}{2}\sqrt{\frac{3}{2\pi}}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\cos\theta e^{-i\varphi} - \frac{1}{\sin^2\theta}\sin\theta e^{-i\varphi}\right] \\
&= -\hbar^2\frac{1}{2}\sqrt{\frac{3}{2\pi}}\left[\frac{\cos^2\theta-\sin^2\theta}{\sin\theta}e^{-i\varphi} - \frac{1}{\sin\theta}e^{-i\varphi}\right] \\
&= -\hbar^2\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{-i\varphi}\left[\frac{1-2\sin^2\theta}{\sin\theta} - \frac{1}{\sin\theta}\right] \\
&= \hbar^2\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{-i\varphi}2\sin\theta \\
&= 2\hbar^2\ Y_1^{-1}\left(\theta,\varphi\right)
\end{aligned}$$

Thus, $2\hbar^2$ is the eigenvalue.

$$\begin{aligned}
\hat{L}_z Y_1^{-1}\left(\theta,\varphi\right) &= \frac{\hbar}{i}\frac{\partial}{\partial\varphi}\left[\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{-i\varphi}\right] \\
&= -i\frac{\hbar}{i}\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{-i\varphi} \\
&= -\hbar\ Y_1^{-1}\left(\theta,\varphi\right)
\end{aligned}$$

Thus, $-\hbar$ is the eigenvalue.

$$\begin{aligned}
\hat{L}^2 Y_2^2(\theta,\varphi) &= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\varphi}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\varphi}\right] \\
&= -\hbar^2\frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{2i\varphi}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}2\sin^2\theta\cos\theta-\frac{4\sin^2\theta}{\sin^2\theta}\right] \\
&= -\hbar^2\frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{2i\varphi}\left[\frac{1}{\sin\theta}\left(4\cos^2\theta\sin\theta-\sin^3\theta\right)-4\right] \\
&= -\hbar^2\frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{2i\varphi}\left[4\cos^2\theta-\sin^2\theta-4\right] \\
&= -\hbar^2\frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{2i\varphi}\left[4-4\sin^2\theta-\sin^2\theta-4\right] \\
&= 5\hbar^2\ Y_2^2(\theta,\varphi)
\end{aligned}$$

Thus, the eigenvalue is $5\hbar^2$.

$$\begin{aligned}
\hat{L}_z Y_2^2(\theta,\varphi) &= \frac{\hbar}{i}\frac{\partial}{\partial\varphi}\left[\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\varphi}\right] \\
&= 2\hbar\ \hat{L}_z Y_2^2(\theta,\varphi)
\end{aligned}$$

Thus, the eigenvalue is $2\hbar$.