Physics 40—Quantum Physics of Matter

HW 2—Due April 10, 2024

1. Consider an *isotropic* 3D harmonic oscillator, *i.e.* one for which the potential $U(x, y, z) = \frac{1}{2}\mu\omega^2(x^2 + y^2 + z^2)$ where μ is the mass. This is the harmonic oscillator equivalent of the 3D box.

   (a) Using separation of variables as for the 3D box, show that the eigenfunctions of the time-independent Shrödinger equation for the 3D isotropic harmonic oscillator can be written as

   $$\psi_{pqs}(x, y, z) = A_{pqs} H_p(x/\ell_{\text{zp}}) H_q(y/\ell_{\text{zp}}) H_s(z/\ell_{\text{zp}}) e^{-(x^2+y^2+z^2)/2\ell_{\text{zp}}^2}$$

   and the associated energies are given by

   $$E_n = \left(n + \frac{3}{2}\right)\hbar\omega.$$

   Here,

   $$\ell_{\text{zp}} = \sqrt{\frac{\hbar}{\mu\omega}}$$

   the quantum numbers $p, q, s, n = 0, 1, 2, ...$ are non-negative integers, the H_n are Hermite polynomials, and A_{pqs} is a normalization constant you don't need to determine. You need only show that the Schrödinger equation can be rewritten so that it clearly splits into parts that depend only on x, y or z respectively. Then simply use previously shown results for the 1D harmonic oscillator.

   (b) Due to the high degree of symmetry in this problem, there are numerous degeneracies. Determine the number of degenerate states that occur for the energies corresponding to $n = 0$, 1 and 2, and explicitly list the triples of quantum numbers (p, q, s) that give rise to each.

2. Continuing with the isotropic 3D harmonic oscillator, determine the following time dependent wave functions $\Psi(x, y, z, t)$:

   (a) An example of a superposition state with the lowest possible energy that is remains a stationary state. (Note that there are several combinations of eigenstates you can pick! Choose whichever you like). Explicitly explain how you know that this is a stationary state.

   (b) An example of a superposition state with the lowest possible energy that is *not* a stationary state. Again, there are multiple options; choose whichever you like, but use different coefficients than I did in the first problem set. Show that the probability distribution for your nonstationary $\Psi(x, y, z, t)$ is time dependent. (*Hint*: give shorthand names to the spatial parts of your wavefuntion like ψ_{pqs} unless you *really* feel like writing out that part explicitly.)

(c) If your superposition state is not normalized, then normalize it; in any case, verify its normalization. Calculate the expectation value of the energy $\langle E \rangle$ for your state, showing that it is in fact independent of time. (*Hint*: use Dirac notation for the spatial parts of your wavefunction like $|\psi_{pqs}\rangle$. You may assume your $|\psi_{pqs}\rangle$ are orthonormal.)

3. Finally, show that the isotropic 3D harmonic oscillator can also be treated as a central potential problem, *i.e.*, show that the potential U depends only on the radius r in spherical coordinates, and not the angular variables θ and φ. Starting with a Hamiltonian of the form

$$\hat{H} = -\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\hat{L}^2}{2\mu r^2} + U(r)$$

propose an appropriate separable form for the wave function $\psi(r, \theta, \varphi)$ and write a differential equation for the radial part of ψ only. Use results from lecture liberally! The point is to refer to known results as much as possible while still making your case, as is actually done in practice.

4. The function $e^{im\varphi}$ is a solution to the equation for $\Phi(\varphi)$

$$\frac{d^2\Phi}{d\,\varphi^2} = -m^2\,\Phi$$

arising from the time independent Schrödinger equation in the central potential problem. As discussed in class, it's also an eigenfunction of the operator $\hat{L}_z$

(a) The function $\Phi(\varphi) = \cos(m\,\varphi)$ is also a solution of the equation for Φ. Is it also an eigenfunction of $\hat{L}_z$ operator? Briefly explain your answer.

(b) Finally, could a wave function with $\Phi(\varphi) = \cos(m\,\varphi)$ as its φ dependence be a stationary state? Again, explain your answer.

5. Explicitly verify that the spherical harmonics

$$Y_1^{-1}(\theta, \varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{-i\varphi}$$

and

$$Y_2^{2}(\theta, \varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\varphi}$$

are eigenstates of the operators $\hat{L}^2$ and $\hat{L}_z$ (shown below) and determine their associated eigenvalues.

Angular Momentum Operators

$$\hat{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right]$$

$$\hat{L}_z = \frac{\hbar}{i} \frac{\partial}{\partial\varphi}$$