Physics 40—Quantum Physics of Matter

HW 1—Due April 3, 2024

1. **Euler identity practice!** Use the Euler identity $e^{i\theta} = \cos\theta + i\sin\theta$ and/or your knowlege of complex numbers in general to show the following.

   (a) Derive the relations
   $$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$
   and
   $$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}).$$
   Then write out them out 1000 times each. (JK! But you really should memorize them.)

   (b) Show that the probability density $P(x)$ of any quantum state is independent of a global complex phase factor; *i.e.*, if $\psi(x) = \chi(x)e^{i\varphi}$ then $P(x)$ depends only on χ, not φ.

   (c) Using your newly-memorized results from (a) above, derive the trigonometric identity
   $$\sin\theta\,\sin\varphi = \frac{1}{2}\left[\cos(\theta - \varphi) - \cos(\theta + \varphi)\right].$$

2. **Orthonormality of eigenstates** — As mentioned in class, eigenstates of the Hamiltonian can always be chosen to be orthonormal. In Dirac notation,
   $$\langle\psi_n|\psi_m\rangle = \delta_{nm},$$
   where δ_{nm} is the Kronecker delta introduced in class.

   (a) Show that the eigenstates
   $$\psi_n = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$
   of the infinite square well (taken to be between $x = 0$ and $x = a$) are normalized.

   (b) Show that the same square well eigenstates are orthogonal if $n \neq m$. For this step, the trig identity from 1(c) will be useful.

   (c) Show that the quantum harmonic oscillator eigenstates $\psi_0(x)$ and $\psi_2(x)$, where
   $$\psi_n(x) = \left(\frac{1}{x_0^2\pi}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(x/x_0)e^{-x^2/(2x_0^2)}$$
   are orthogonal (don't bother w/ showing they're normalized). Use the convention $H_0(\xi) = 1$ and $H_2(\xi) = 4\xi^2 - 2$. (*Hint:* Make it easy on yourself. Define $\xi = x/x_0$ and do everything terms of ξ. Also, do all the complicated normalization constants on ψ_n even matter? You only want to show orthogonality...)

3. Consider the superposition state

$$|\Psi(x,t)\rangle = \frac{1}{\sqrt{2}}(e^{-E_1 t/\hbar}|\psi_1(x)\rangle + e^{-E_2 t/\hbar}|\psi_2(x)\rangle)$$

where ψ_1 and ψ_2 are energy eigenstates of the infinite square well of width a as in Problem 2(a) and E_1 and E_2 are the associated energies.

(a) Show that $|\Psi(x,t)\rangle$ is normalized. (*Hint:* if you like, make life easy by using Dirac notation and the results of Problems 2(a) and 2(b).)

(b) Find the expectation value of the energy in $|\Psi(x,t)\rangle$ given by $\langle\Psi(x,t)|\hat{H}|\Psi(x,t)\rangle$. (*Hint*: if you like, use Dirac notation again, and remember that $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$.)

(c) **5 pts extra credit:** Show that the expectation value of position $\langle\Psi(x,t)|\hat{x}|\Psi(x,t)\rangle$ is time-dependent and given by

$$\langle\Psi(x,t)|\hat{x}|\Psi(x,t)\rangle = \frac{a}{2}\left[1 - \frac{16}{9\pi^2}\cos(\frac{3\pi^2\hbar}{2ma^2}t)\right].$$

You'll need to evaluate three integrals. Two can be dealt with without actually evaluating them by using a symmetry argument! For the third, you can use the result of Problem 1(c) and integration by parts.

4. Do Eisberg & Resnick Problems 6-30 and 6-31 (page 231), applying your knowledge of harmonic oscillators to molecular vibrations. Please keep track of your units!

Handy Integrals

$$\int_{-\infty}^{\infty} e^{-\xi^2}\, d\xi = \sqrt{\pi}$$

and

$$\int_{-\infty}^{\infty} \xi^2 e^{-\xi^2}\, d\xi = \frac{\sqrt{\pi}}{2}$$