Probability: Assignment 2

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Contents

1 Central Limit Theorem

1.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{1.1.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution:

```
Assignment 2/codes/gxrand.c
Assignment 2/codes/coeffs.h
Assignment 2/codes/gau.dat
```

1.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 1.2. The CDF defined for a continuous random variable is given as:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

The cumulative distribution function $F_X(x)$ of a random variable has the following important properties:

1 Every CDF F_X is non decreasing and right continuous i.e., $\lim_{x\to-\infty} F_X(x) = 0$ and $\lim_{x\to\infty} F_X(x) = 1$

2 For all real numbers a and b with continuous random variable X, then the function f_x is equal to the derivative of F_X, such that $F_X(b) - F_X(a) = P(a < x < b) = \int_a^b f_X(x)dx$

3 If X is a completely discrete random variable, then it takes the values x_1, x_2, x_3,.. with probability $p_i = p(x_i)$, and the CDF of X will be discontinuous at the points x_i:

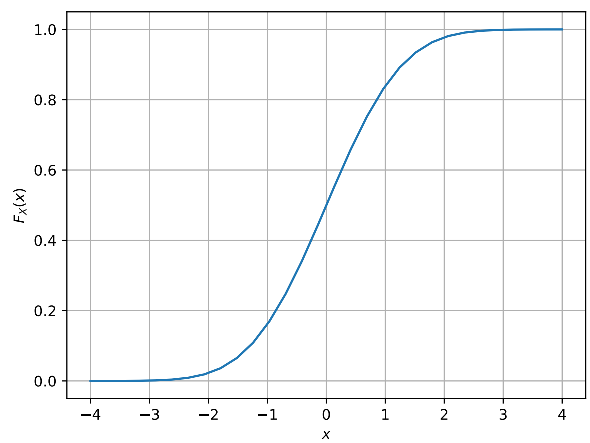

Fig. 1.2. The CDF of X

$F_X(x) = P(X \le x) = \sum_{x_i \le x} P(X = x_i) = \sum_{x_i - x \le x} p(x_i)$

1.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{1.3.1}$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 1.3 using the code below.

```
Assignment 2/codes/pdf_plot.py
```

The PDF is given by

$$P(a \le x \le b) = \int_a^b f(x)dx$$

Let X be the continuous random variable with density function $f(X)$, and the probability density function should satisfy the following conditions:

1 For a continuous random variable that takes some value between certain limits, say a and b, the PDF is calculated by finding the area under its curve and the X-axis within the

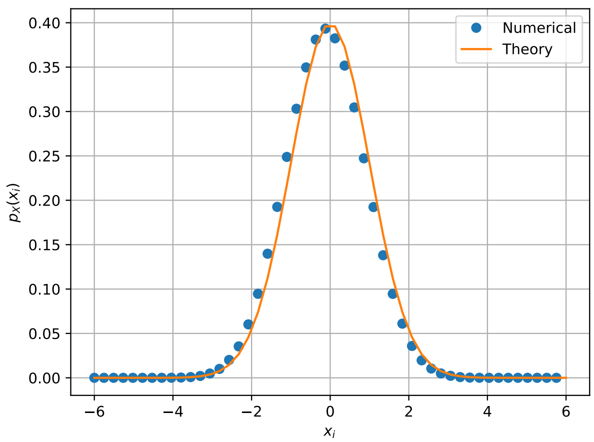

Fig. 1.3. The PDF of X

lower limit (a) and upper limit (b). Thus, the PDF is given by $\int_a^b f(x)dx$.

2 The probability density function is non-negative for all the possible values, i.e. $f(x) \geq 0$, for all x.

3 The area between the density curve and horizontal X-axis is equal to 1, i.e. $\int_{-\infty}^{\infty} f(x)dx = 1$

4 Due to the property of continuous random variables, the density function curve is continued for all over the given range. Also, this defines itself over a range of continuous values or the domain of the variable.

1.4 Find the mean and variance of X by writing a C program.

Solution:

```
Assignment 2/codes/gxrand.c
```

The value of mean is 0.000 and variance Value: 1.000.

1.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{1.5.1}$$

repeat the above exercise theoretically.

Solution: The expected value (Mean) of a standard normal random variable X is derived as:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(\frac{-x^2}{2}\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} x \exp\left(\frac{-x^2}{2}\right) dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x \exp\left(\frac{-x^2}{2}\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{-x^2}{2}\right)_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left(\frac{-x^2}{2}\right)_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}}[-1+0] + \frac{1}{\sqrt{2\pi}}[0+1]$$

$$= \frac{1}{\sqrt{2\pi}} - \frac{1}{\sqrt{2\pi}}$$

$$= 0$$

The variance can be calculated by $Var[X] = E[X^2] - E[X]^2$

So we have,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(\frac{-x^2}{2}\right) dx$$

$$= \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} x^2 \exp\left(\frac{-x^2}{2}\right) dx\right] + \left]\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x^2 \exp\left(\frac{-x^2}{2}\right)\right.$$

Upon integrating by parts the first part and second part in the above equation, we get

$$= \frac{1}{\sqrt{2\pi}} \left\{\left[-x \exp\left(\frac{-x^2}{2}\right)\right]_{-\infty}^{0} + \int_{-\infty}^{0} \exp\left(\frac{-x^2}{2}\right) dx\right\} + \frac{1}{\sqrt{2\pi}}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{(0-0) + \int_{-\infty}^{0} \exp\left(\frac{-x^2}{2}\right) dx + (0-0) + \int_{0}^{\infty} \exp\right.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2}\right) dx$$

The probability density fucntion over the entire region i

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2}\right) dx$$

$$= 1$$

From above, we know that

$E[X]^2 = 0^2 = 0$

$\mathrm{Var}[X] = E[X^2] - E[X]^2 = 1 - 0 = 1$

Hence Proved.