Probability: Assignment 1

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CC22RESCH11003

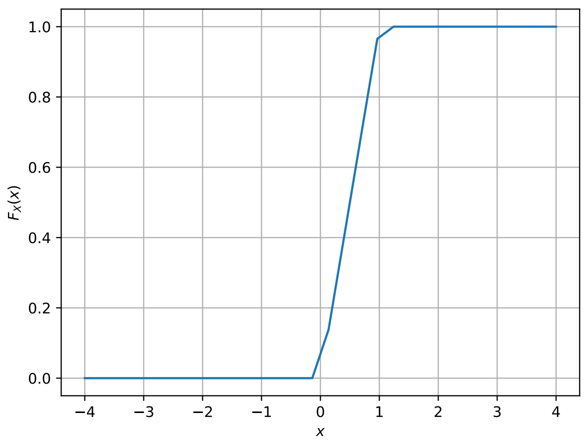

Fig. 1.2. The CDF of U

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
codes/exrand.c
codes/coeffs.h
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.2.1}$$

Solution: The following code plots Fig. 1.2

```
codes/cdf_plot.py
```

1.3 Find a theoretical expression for $F_U(x)$.
Solution: The Probaility Density Function for Uniform Distribution is given by:

$$f(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{Otherwise} \end{cases} \tag{1.3.1}$$

The Cumulative Distribution Function is defined by

$$F_U(x) = \int_{-\infty}^{x} f(x)dx$$

So, we can split in for three intervals, as

$$F_U(x) = \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{\infty} f(x)dx$$

Accordingly, we have three cases as follows:
Case 1:

$$\int_{-\infty}^{0} f(x)dx$$

if $x < 0$ then,$f(x) = 0$ and but $\lim_{x\to\infty} F(x) = 0$
Case 2:

$$\int_{0}^{1} f(x)dx$$

if $0 \leq x \leq 1$ then,

$$\int_{0}^{1} f(x)dx = \frac{1}{1-0}\int_{0}^{1} dx = \left[\frac{x}{1-0}\right]_0^1 = 1$$

Case 3:

$$\int_{1}^{\infty} f(x)dx$$

if $x > b$ then, $f(x) = 0$ but $\lim_{x\to\infty} F(x) = 1$
Thus, The Cumulative Distribution for uniform random variable is given as follows:

$$F_U(x) = \Pr(U \leq x) = \begin{cases} 0 & \text{for x} < 0 \\ \frac{x-0}{1-0} & x \in [0, 1] \\ 1 & \text{for x} > 1 \end{cases} \tag{1.3.2}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.4.1}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.4.2}$$

Write a C program to find the mean and variance of U.
Solution: The C Program is executed in the following

```
codes/exrand.c
codes/coeffs.h
```

The mean value is **0.50** and variance value is **0.083**

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.5.1}$$

Solution: Mean is given by $E[U] = \int_0^1 x dF_U(x)$ Accordingly,

$$E[U] = \int_0^1 x dF_U(x)$$
$$= \int_0^1 x dx$$
$$= \left[\frac{x^2}{2}\right]_0^1$$
$$= \frac{1}{2} = 0.50$$

Now, for $E[U^2]$, we have,

$$E[U^2] = \int_0^1 x^2 dF_U(x)$$
$$= \int_0^1 x^2 dx$$
$$= \left[\frac{x^3}{3}\right]_0^1$$
$$= \frac{1}{3}$$

Now, we know from **1.4** that,

$$\text{var}\left[U\right] = E\left[U - E\left[U\right]\right]^2$$
$$= E[U^2 - 2UE[U] + (E[U])^2]$$
$$= E[U^2] - 2E[U]E[U] + E[E[U]^2]$$
$$= E[U^2] - 2E[U]^2 + E[U]^2$$
$$= E[U^2] - E[U]^2$$

Substituting the Values for $E[U]$ and $E[U^2]$ in the above equation, we get,

$$\text{var}[U] = \frac{1}{3} - (\frac{1}{2})^2$$
$$\frac{1}{3} - \frac{1}{4} = \frac{1}{12} = 0.083$$

Hence Proved.