1.1. INTRODUCTION TO FUNCTIONS

Learning Objectives:

- To define a function from a set into another set and to view functions as relations
- To define a function and its domain and range
- To determine the domain and range of real valued functions of a real variable.
- To define the Sums, Differences, Products and Quotients of functions and determine their domains.
- To learn the concepts of composite functions, even and odd functions and piecewise defined functions.
  AND
- To solve the related problems

The terms *map*, *mapping*, *transformation* are also used as alternative names for the function. The choice of which word is used in a given situation is usually determined by tradition.

Suppose that to each element of a set A we assign a unique element of a set B; the collection of such assignments is called a ***function from A to B***. The set A is called the ***domain*** of the function, and the set B is called the ***co-domain***. Let f denote a function from A to B. Then we write

$$\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}$$

which is read: ***f is a function from A into B,*** or ***f maps A into B***.

Suppose $f: A \rightarrow B$ and $a \in A$. Then $f(a)$, read "f of a", will denote the unique element of B which f assigns to a. This element $\boldsymbol{f(a)}$ in B is called the ***image* of $\boldsymbol{a}$ under $\boldsymbol{f}$** or the ***value* of $\boldsymbol{f}$ at $\boldsymbol{a}$**. We also say that f *sends* or *maps* a into $f(a)$. The set of all such image values is called the ***range*** or ***image*** of f, and it is denoted by $\boldsymbol{Ran(f)}$**,** $\boldsymbol{Im(f)}$ or $\boldsymbol{f(A)}$. That is, $\boldsymbol{Im(f)} = \{\boldsymbol{f(a)} \mid \boldsymbol{a} \in \boldsymbol{A}\}$ **,** Clearly, $Im(f) \subseteq B$.

Example:

The figure below defines a function f from $A = \{a, b, c, d\}$ into $B = \{r, s, t, u\}$

From the figure,

$f(a) = s, \quad f(b) = u, \quad f(c) = r, \quad f(d) = s$

The image of f is the set $\{r, s, u\}$. We note that t does not belong to the image of f because t is not the image of any element of A under f.

Functions as Relations

Definition: A function $f: A \to B$ is a relation from $A\ to\ B$ (i.e., a subset of $A \times B$) such that each $a \epsilon A$ belongs to a unique ordered pair (a, b) in f.

The defining condition of a function, that each $a \epsilon A$ belongs to a unique pair (a, b) in f, is equivalent to the geometrical condition of each vertical line intersecting the graph in exactly one point.

Example: **Consider the relation $\boldsymbol{f}$ from $\boldsymbol{A = \{a, b, c, d\}}$ to $\boldsymbol{B = \{s, u, r\}}$:**
$\boldsymbol{f = \{(a, s), (b, u), (c, r), (d, s)\}}$**. Ascertain whether this is a function.**

Solution:

The given relation is a function $f: A \to B$ with the domain $A = \{a, b, c, d\}$ and the range = $\{s, u, r\}$, since each member of A appears as the first coordinate in exactly one ordered pair in f.

Example: **Consider the following relations on the set $\boldsymbol{A = \{1, 2, 3\}}$.**

$$\boldsymbol{f = \{(1,3), (2,3), (3,1)\}},$$
$$\boldsymbol{g = \{(1,2), (3,1)\}},$$
$$\boldsymbol{h = \{(1,3), (2,1), (1,2), (3,1)\}}$$

Ascertain whether each relation is a function.

Solution:

The relation f is a function from $A\ into\ A$, since each member of A appears as the first coordinate in exactly one ordered pair in f; here $f(1) = 3, f(2) = 3, f(3) = 1$.

The relation g is not a function from $A\ into\ A$ since $2 \epsilon A$ is not the first coordinate of any pair in g and so g does not assign any image to 2.

The relation h is not a function from $A\ into\ A$ since $1 \epsilon A$ appears as the first coordinate of two distinct ordered pairs in h, (1,3) and (1,2). If h is to be a function it cannot assign both 3 and 2 to the element $1 \epsilon A$.

Example: Which of the following relations $\boldsymbol{A = \{a, b, c, d, e\}}$ to $\boldsymbol{B = \{1, 2, 3, 4\}}$ are functions? Give reasons. If it is a function, determine its domain and range.

$$\boldsymbol{R_1 = \{(a, 2), (b, 4), (c, 1), (d, 3), (e, 4), (d, 1)\}}$$
$$\boldsymbol{R_2 = \{(a, 4), (b, 2), (c, 3), (e, 1)\}}$$
$$\boldsymbol{R_3 = \{(a, 3), (b, 2), (c, 4), (d, 1), (e, 2)\}}$$

Solution: We have, $R_1 = \{(a, 2), (b, 4), (c, 1), (d, 3), (e, 4), (d, 1)\}$.

R_1 is not a function from A into B, since $d \in A$ appears as the first coordinate of two distinct ordered pairs in R_1 i.e., $(d, 3), (d, 1) \in R_1$.

We have, $R_2 = \{(a, 4), (b, 2), (c, 3), (e, 1)\}$.

R_2 is not a function from A into B, since $d \in A$ is not the first coordinate of any ordered pair in R_2.
We have, $R_3 = \{(a,3),(b,2),(c,4),(d,1),(e,2)\}$.
R_3 is a function from A into B since, each element of A appears as the first coordinate in exactly one ordered pair in R_3.
The domain and range of R_3 respectively are $\{a,b,c,d,e\}$ and $\{1,2,3,4\}$.

Functions are used to describe the relationships between variable quantities and hence play a central role in applications. For example, an engineer may need to know how the illumination from a light source on an object is related to the distance between the object and the source.
Frequently, a function can be expressed by means of a mathematical formula.
Suppose the value of one variable quantity, called y, depends on the value of another variable quantity, called x. If the value of y is completely determined by the value of x, then we say that ***y is a function of x***.
If A is the area and r is the radius of a circle then we have $A = \pi r^2$. Thus A is a function of r. Now, the equation $A = \pi r^2$ is a *rule* that tells how to calculate a unique output value of A for each possible input value of the radius r.
The set of all possible input values for x is the ***Domain*** of the function. The set of all output values of y is the ***Range*** of the function.
Since the circles cannot have negative radii or areas, the domain and the range of these are both in the interval $[0,\infty)$, consisting of all nonnegative real numbers.
We often refer to a generic function without having any particular formula in mind.
Euler, a Swiss mathematician, gave a symbolic way to say "***y is a function of x***" by writing $y = f(x)$ ("y *equals* f *of* x")
In this notation, the symbol f represents the function. The letter x, called the **Independent variable**, represents an input value from the domain of f, and y, the **dependent variable**, represents the corresponding output value of $f(x)$ in the range of f.

Thus, a function is usually expressed in one of two ways:

1. By giving formula such as $y = x^2$ that uses a dependent variable y to denote the value of the function, or
2. By giving a formula such as $f(x) = x^2$ that defines a function symbol f to name the function.

We use the symbol $f(x)$ both for representing the function and denoting the value of the function at the point x. It is also convenient to use a single letter to denote both a function and its dependent variable. For instance, we might say that the area A of a circle of radius r is given by the function $A(r) = \pi r^2$.

Example 1:

The volume V of a ball (solid sphere) of radius r is given by the function

$$V(r) = \frac{4}{3}\pi r^3$$

The volume of a ball of radius 3 m is

$$\mathrm{V}(3) = \frac{4}{3}\pi(3)^3 = 36\pi\ m^3$$

Example 2:

Suppose that the function f is defined for all real numbers t by the formula

$$f(t) = 2(t-1)+3$$

Evaluate f at the input values $0, 2, x+2$, and $f(2)$.

Solution

$$f(0) = 2(0-1)+3 = 1$$

$$f(2) = 2(2-1)+3 = 5$$

$$f(x+2) = 2(x+2-1)+3 = 2x+5$$

$$f(f(2)) = f(5) = 2(5-1)+3 = 11$$

Functions, whose domains and ranges are sets of real numbers, are called ***real-valued functions*** of a ***real variable***.

To fully describe a function we not only specify the rule that relates the inputs and outputs, but we also specify the domain, that is, the set of allowable inputs.

The domains and ranges of real-valued functions of a real variable are sets of real numbers. We evaluate such functions by substituting particular values from the domain into the function's defining rule to calculate the corresponding values in the range.

Example 3: Consider the function $f(x)=x^3$

$i.e.$, f assigns to each real number its cube. Then the image of 2 is 8, and so we may write $f(2)=8$

Similarly, $f(-3)=-27$ $f(0)=0$

If the domain is not stated explicitly, it is assumed to be the largest set of x-values for which the formula of the function gives the real y- values. This is called the ***Natural domain*** of the function. If we want the domain to be restricted in some way, we must say so.

The domain of the function $y=x^2$ is understood to be the entire set of real numbers. The formula gives a real y-value for every real number x. If we want to restrict the domain to values of $x \geq 2$, we must write $y=x^2,\ x\geq 2$. Changing the domain to which we apply a formula usually changes the range as well. The range of $y=x^2$ is $[0,\infty)$.

The range of $y=x^2,\ x\geq 2$, is the set of all numbers obtained by squaring numbers greater than or equal to 2. That is, the range is $[4,\infty)$.

Most of the functions will have domains that are either intervals or unions of intervals. The domain and range of several functions is as follows:

Function	Domain (x)	Range (y)
$y=\sqrt{1-x^2}$	$[-1,1]$	$[0,1]$
$y=\dfrac{1}{x}$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$
$y=\sqrt{x}$	$[0,\infty)$	$[0,\infty)$
$y=\sqrt{4-x}$	$(-\infty,4]$	$[0,\infty)$

The formula $y=\sqrt{1-x^2}$ gives a real y-value for every x in the closed interval from -1 to 1. Beyond this domain, $1-x^2$ is negative and its square root is not a real number. The formula $y=\dfrac{1}{x}$ gives a real

y-value for every x except $x = 0$. We *cannot divide any number by zero*. The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. In $y = \sqrt{4 - x}$, $4 - x \geq 0$ or $x \leq 4$. The formula gives real values for all $x \leq 4$.

Sums, Differences, Products and Quotients

Functions can be added, subtracted, multiplied, and divided to produce new functions. If *f* and *g* are functions, then for every *x* that belongs to the domains of both *f* and *g*, we define the functions *f + g, f − g,* and *fg* by the formulas

$$(f+g)(x) = f(x) + g(x)$$

$$(f-g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)g(x)$$

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function $\frac{f}{g}$ by the formula $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

Functions can also be multiplied by constants: if c is a real number, then the function cf is defined for all x in the domain of f by

$$(cf)(x) = cf(x)$$

The domains of the following combined functions may be noted:

Function	Formula	Domain
f	$f(x) = \sqrt{x}$	$[0, \infty)$
g	$g(x) = \sqrt{1-x}$	$(-\infty, 1]$
$3g$	$3g(x) = 3\sqrt{1-x}$	$(-\infty, 1]$
$f+g$	$(f+g)(x) = \sqrt{x} + \sqrt{1-x}$	$[0,1] = D(f) \cap D(g)$
$f-g$	$(f-g)(x) = \sqrt{x} - \sqrt{1-x}$	$[0,1]$
$g-f$	$(g-f)(x) = \sqrt{1-x} - \sqrt{x}$	$[0,1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1-x)}$	$[0,1]$
$\frac{f}{g}$	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1-x}}$	$[0,1)$ ($x = 1$ excluded)
$\frac{g}{f}$	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1-x}{x}}$	$(0,1]$ ($x = 0$ excluded)

Composite Functions

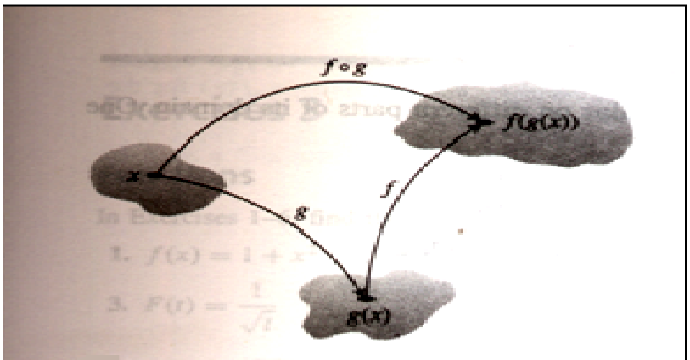
Composition is another method for combining functions.

If f and g are functions, the *composite function* $f \circ g$ ("f circle g") is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f

The definition says that two functions can be composed when the range of the first lies in the domain of the second.



The relation of $f \circ g$ to g and f.

To evaluate the composite function $(f \circ g)(x)$, we first find $g(x)$ and second find $f(g(x))$. To evaluate the composite function $(g \circ f)$, we reverse the order, finding $f(x)$ first and then $g(f(x))$. The domain of $(g \circ f)$ is the set of numbers x in the domain of f such that $f(x)$ lies in the domain of g.

The functions $f \circ g$, $g \circ f$ are usually quite different.

Example 4: *If* $f(x) = \sqrt{x}$ *and* $g(x) = x + 1$, **find**

A. $(f \circ g)(x)$

B. $(g \circ f)(x)$

C. $(f \circ f)(x)$

D. $(g \circ g)(x)$

Solution

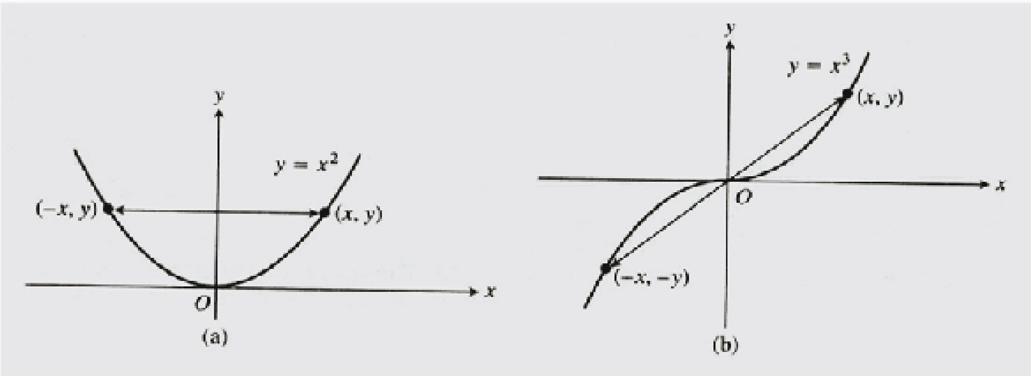
	Composite	*Domain*
A.	$f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
B.	$g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
C.	$f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{\frac{1}{4}}$	$[0, \infty)$
D.	$g(g(x)) = g(x) + 1 = (x+1) + 1 = x + 2$	$(-\infty, \infty)$

Even and Odd Functions

A function $y = f(x)$ is ***even*** if $f(-x) = f(x)$ for every number x in the domain of f. The function $f(x) = x^2$ is even because $f(-x) = (-x)^2 = x^2 = f(x)$

The graph of an even function $y = f(x)$ is symmetric about the y-axis.

Both (x, y) and $(-x, y)$ are points on the graph. Once we know the graph on one side of the y-axis, we automatically know it on the other side.



A function $y = f(x)$ is ***odd*** if $f(-x) = -f(x)$ for every number x in the domain of f. The function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

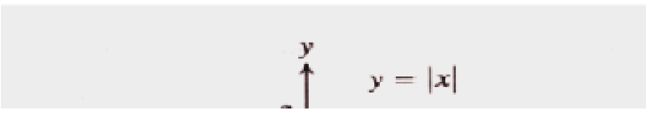
The graph of an odd function is symmetric about the origin. Both the points (x, y) and $(-x, -y)$ lie on the graph. Here again, once we know the graph of f on one side of the y −axis, we know it on both sides.

Piecewise Defined Functions

A function may be defined using different formulas on different parts of its domain. Such functions are called *piecewise defined functions*. One example is the absolute value function.

$$|x| = \begin{cases} x \ , & x \geq 0 \\ -x \ , & x < 0 \end{cases}$$

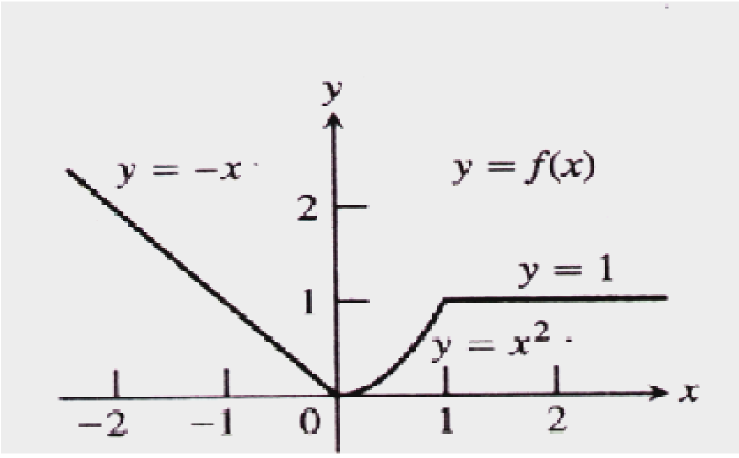
The graph of the absolute value function is shown below



Example 5: The function

$$f(x) = \begin{cases} -x \ , & x < 0 \\ x^2 \ , & 0 \le x \le 1 \\ 1 \ , & x > 1 \end{cases}$$

is defined on the entire real line but has values given by different formulas depending on the position of x.



PROBLEM SET:

IP1: If the function $f: R \to R$ defined by $f(x) = \frac{4^x}{4^x+2}$, then show that $f(1-x) = 1 - f(x)$.

Solution:

Step1: Given function $f: R \to R$ defined by

$$f(x) = \frac{4^x}{4^x+2}$$

Step2: Now $f(1-x) = \frac{4^{1-x}}{4^{1-x}+2}$

$$= \frac{4.(4^{-x})}{4.(4^{-x})+2}$$

$$= \frac{4.4^{-x}}{2.(4^{-x})(2+4^x)}$$

$$= \frac{2}{4^x+2}$$

$$= \frac{4^x+2-4^x}{4^x+2}$$

$$= 1 - \frac{4^x}{4^x+2}$$

$$= 1 - f(x)$$

$\therefore \ f(1-x) = 1 - f(x)$

Hence be proved.

P1. If the function $f: R \to R$ defined by $f(x) = \frac{3^x+3^{-x}}{2}$, and $f(x+y) + f(x-y) = \alpha f(x) f(y)$ Then $\alpha =$ _____

A. **1**

B. **2**
C. **3**
D. 4

Answer: **B**

Solution: Given that the function $f: R \to R$ is defined by

$$f(x) = \frac{3^x + 3^{-x}}{2}$$

Now,

$f(x+y) + f(x-y)$

$$= \frac{3^{(x+y)} + 3^{-(x+y)}}{2} + \frac{3^{(x-y)} + 3^{-(x-y)}}{2}$$

$$= \frac{1}{2}[\, 3^x.3^y + 3^{-x}.3^{-y} + 3^x.3^{-y} + 3^{-x}.3^y \,]$$

$$= \frac{1}{2}[\, 3^x(3^y + 3^{-y}) + 3^{-x}(3^y + 3^{-y})]$$

$$= \frac{1}{2}[\, (3^x + 3^{-x})(3^y + 3^{-y}) \,]$$

$$= \frac{3^x + 3^{-x}}{2} \,.\, 2 \,.\, \frac{3^y + 3^{-y}}{2}$$

$$= 2\left[\frac{3^x + 3^{-x}}{2} \,.\, \frac{3^y + 3^{-y}}{2}\right]$$

$$= 2.\; f(x)\,.\; f(y) = \alpha\, f(x)\, f(y)$$

$$\therefore\; \alpha = 2$$

IP2: If $\boldsymbol{f(x) = \sqrt{x^2 - 1}}$, $\boldsymbol{g(x) = \frac{1}{\sqrt{x^2 - 3x + 2}}}$ then the domain of $\boldsymbol{(f+g)}$ is

Solution:

Step 1: Given that $f(x) = \sqrt{x^2 - 1}$; $g(x) = \frac{1}{\sqrt{x^2 - 3x + 2}}$

Step 2: Domain of f is $D(f) = \{x \in \boldsymbol{R}:\; x^2 - 1 \geq 0\}$

$$= \{x \in \boldsymbol{R}:\; (x-1)(x+1) \geq 0\}$$

We have

If $(x-a)(x-b) \geq 0, a < b$ then $x \in (-\infty,\; a] \cup [\, b, \infty)$

$$D(f) = \{x \in \boldsymbol{R}:\; (x-(-1))(x-1) \geq 0\}$$

$$= (-\infty,\; -1] \cup [\,1,\; \infty\,) = \boldsymbol{R} - (-1, 1\,)$$

Step 3:

Domain of g is $D(g) = \{x \in \boldsymbol{R}: x^2 - 3x + 2 > 0\}$

$$(\because\; x^2 - 3x + 2 \neq 0)$$

$$= \{x \in \boldsymbol{R}: (x-1)(x-2) > 0\}$$

$$= (-\infty,\; 1) \cup (\,2,\; \infty\,) = R - [1,\; 2]$$

Step 4: Domain of $(f+g)$ is

$D\,(f+g) = D(f) \cap D(g)$

$$= \{R - (-1,1)\} \;\cap\; \{R - [\,1,2\,]\}$$

$$= R - \{(-1,1) \cup [\,1,2\,]\}$$

$= R - (-1,2]$

$\therefore\ D(f+g) = (-\infty,-1] \cup (2,\infty) = R - (-1,2]$.

P2: If $\boldsymbol{f(x) = \sqrt{2-x}}$ and $\boldsymbol{g(x) = \sqrt{1+x}}$ then the domain of $\boldsymbol{(f+g)}$ is

A. $(-1,\ 2]$

B. $(-1,\ 2)$

C. $[-1,\ 2)$

D. $[-1,\ 2]$

Answer: D

Solution: Given that $f(x) = \sqrt{2-x}$, $\quad g(x) = \sqrt{1+x}$.

Domain of f is $D(f) = \{x \in \boldsymbol{R}: 2-x \geq 0\}$

$= \{x \in \boldsymbol{R}: x \leq 2\}$

$= (-\infty, 2]$

Domain of g is $D(g) = \{x \in \boldsymbol{R}: 1+x \geq 0\}$

$= \{x \in \boldsymbol{R}: 1 \leq -x\}$

$= \{x \in \boldsymbol{R}:\ x \geq -1\}$

$= [-1,\ \infty)$

$\therefore$ Domain of $(f+g)$ is $D(f+g) = D(f) \cap D(g)$

$= (-\infty,\ 2] \cap [-1,\ \infty)$

$= [-1\ ,\ 2]$

IP3. Let $\boldsymbol{f: R \longrightarrow R}$ and $\boldsymbol{g: R \longrightarrow R}$ are two functions defined by

$$\boldsymbol{f(x) = \begin{cases} x+2 & , \quad x \leq -1 \\ x^2 & , -1 \leq x < 1 \\ 2-x & , \quad x \geq 1 \end{cases}}$$

$$\boldsymbol{g(x) = \begin{cases} x+4 & , \quad x < -4 \\ 3x+2 & , -4 \leq x < 4 \\ x-4 & , \quad x \geq 4 \end{cases}}$$

Then find the values of $\boldsymbol{fog(-1)\ and\ gof(-2)}$

Solution:

Step1: Given that

Let $f: \boldsymbol{R} \to \boldsymbol{R}$ and $g: \boldsymbol{R} \to \boldsymbol{R}$ are two functions defined by

$$f(x) = \begin{cases} x+2 & , \quad x \leq -1 \\ x^2 & , -1 < x < 1 \\ 2-x & , \quad x \geq 1 \end{cases}$$

$$g(x) = \begin{cases} x+4 & , \quad x < -4 \\ 3x+2 & , -4 \leq x < 4 \\ x-4 & , \quad x \geq 4 \end{cases}$$

Step2:

Now, $fog(-1) = f(g(-1))$

$= f(3(-1)+2)$

$= f(-1) = -1+2 = 1$

P3. Let us consider the functions

$$\boldsymbol{f(x)} = \begin{cases} \boldsymbol{x+2} & , \quad \boldsymbol{x} > 1 \\ \boldsymbol{2} & , \quad -\boldsymbol{1} \leq \boldsymbol{x} \leq \boldsymbol{1} \\ \boldsymbol{x-1} & , \quad -\boldsymbol{3} < x < -\boldsymbol{1} \end{cases}$$

$$\boldsymbol{g(x)} = \begin{cases} \boldsymbol{3x-2} & , \quad \boldsymbol{x} > 3 \\ \boldsymbol{x^2-2} & , \quad -\boldsymbol{2} \leq \boldsymbol{x} \leq \boldsymbol{2} \\ \boldsymbol{2x+1} & , \quad \boldsymbol{x} \leq -\boldsymbol{3} \end{cases}$$

Then the value of $\boldsymbol{fog(-1)} =$

A. 0

B. 1

C. 2

D. 3

Answer: C

Solution: Given functions are

$$f(x) = \begin{cases} x+2 & x > 1 \\ 2 & -1 \leq x \leq 1 \\ x-1 & -3 < x < -1 \end{cases}$$

$$g(x) = \begin{cases} 3x-2 & x > 3 \\ x^2-2 & -2 \leq x \leq 2 \\ 2x+1 & x \leq -3 \end{cases}$$

Now,

$$\begin{aligned} fog(-1) &= f(g(-1)) \\ &= f\left((-1)^2 - 2\right) \\ &= f(-1) = 2 \end{aligned}$$

$$fog(-1) = 2$$

P4: If $\boldsymbol{f(x) = \frac{x+1}{x-1}}$, $\boldsymbol{x \neq 1}$, then find $\boldsymbol{(f \circ f)(x)}$.

Solution:

We have, $f(x) = \frac{x+1}{x-1}$, $x \neq 1$

$$\therefore (f \circ f)(x) = f[f(x)] = f\left(\frac{x+1}{x-1}\right)\left[\because f(x) = \frac{x+1}{x-1}\right]$$

$$= \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} = \frac{2x}{2} = x$$

IP4: If $\boldsymbol{f(x) = \frac{x}{\sqrt{1-x^2}}, g(x) = \frac{x}{\sqrt{1+x^2}}}$, then show that $\boldsymbol{f \circ g = g \circ f}$.

Solution: Given, $f(x) = \frac{x}{\sqrt{1-x^2}}$ and $g(x) = \frac{x}{\sqrt{1+x^2}}$

$$(f \circ g)(x) = f[g(x)] = f\left(\frac{x}{\sqrt{1+x^2}}\right) \quad \left[\because g(x) = \frac{x}{\sqrt{1+x^2}}\right]$$

$$= \frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1-\left(\frac{x}{\sqrt{1+x^2}}\right)^2}} = \frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1-\frac{x^2}{1+x^2}}} = \frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{\frac{1+x^2-x^2}{1+x^2}}} = \frac{\frac{x}{\sqrt{1+x^2}}}{\frac{1}{\sqrt{1+x^2}}} = x$$

$$(g \circ f)(x) = g[f(x)] = g\left(\frac{x}{\sqrt{1-x^2}}\right) \quad \left[\because f(x) = \frac{x}{\sqrt{1-x^2}}\right]$$

$$= \frac{\frac{x}{\sqrt{1-x^2}}}{\sqrt{1+\left(\frac{x}{\sqrt{1-x^2}}\right)^2}} = \frac{\frac{x}{\sqrt{1-x^2}}}{\sqrt{1+\frac{x^2}{1-x^2}}} = \frac{\frac{x}{\sqrt{1-x^2}}}{\sqrt{\frac{1-x^2+x^2}{1-x^2}}} = \frac{\frac{x}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1-x^2}}} = x$$

$$\therefore f \circ g = g \circ f$$

IP5: If $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$, then find the domain of $g \circ f$.

Solution:

Given, $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$

$$(g \circ f)(x) = g[f(x)] = g(\sqrt{x}) \quad [\because f(x) = \sqrt{x}]$$

$$= x - 1 \quad \left[\begin{array}{c} \because g(x) = x^2 - 1 \\ g(\sqrt{x}) = (\sqrt{x})^2 - 1 = x - 1 \end{array}\right]$$

$\therefore (g \circ f)(x) = x - 1$

Domain of $(g \circ f)(x)$ is $[0, \infty)$.

P5: If $f(x) = \frac{1}{x}$ and $g(x) = \sqrt{x}$, then find the domain of gof.

Solution:

Given, $f(x) = \frac{1}{x}$ and $g(x) = \sqrt{x}$

$$(gof)(x) = g[f(x)] = g\left(\frac{1}{x}\right) \quad \left[\because f(x) = \frac{1}{x}\right]$$

$$= \sqrt{\frac{1}{x}} \quad \left[\begin{array}{c} \because g(x) = \sqrt{x} \\ g\left(\frac{1}{x}\right) = \sqrt{\frac{1}{x}} \end{array}\right]$$

$\therefore (gof)(x) = \frac{1}{\sqrt{x}}$

Domain of $(gof)(x)$ $is\ x > 0\ \ i.e.,\ (0, \infty)$

IP6: If $f(x) = x.\frac{a^x+1}{a^x-1}$ and $g(x) = \frac{sin^4 x + cos^4 x}{x + x^2 \tan x}$ then show that f is an even function and g is an odd function.

Solution:

Step 1: Given $f(x) = x \,.\left[\frac{a^x+1}{a^x-1}\right]$

Now, $f(-x) = (-x)\left[\frac{a^{-x}+1}{a^{-x}-1}\right]$

$$= (-x)\left[\frac{a^{-x}(1+a^x)}{a^{-x}(1-a^x)}\right]$$

$$= -x \,.\left[\frac{a^x+1}{-(a^x-1)}\right]$$

$$= x \, . \, \left[\frac{a^x+1}{a^x-1}\right] = f(x)$$

$\therefore \; f(-x) = f(x)$

Hence f is an even function.

Step 2:

Given $\quad g(x) = \frac{sin^4 x + cos^4 x}{x + x^2 \, tan\, x}$

Now, $\quad g(-x) = \frac{[\sin(-x)]^4 + [cos(-x)]^4}{x + (-x)^2 \tan(-x)} = \frac{sin^4 x + cos^4 x}{-x - x^2 \, tan\, x}$

$g(-x) = -\frac{sin^4 x + tan^4 x}{x + x^2 \, tan\, x} = -\, g(x)$

$\therefore \; g(-x) = -\, g(x)$

Hence g is an odd function.

P6: If $f(x) = \sin x + \cos x$ then $f(x)$ is

A. Even

B. Odd

C. Neither even nor odd

D. None of the above

Answer: C

Solution: Given that

$$f(x) = \sin x + \cos x$$

$$f(-x) = \sin(-x) + \cos(-x)$$

$$= -\sin x + \cos x$$

Here neither $f(-x) = -f(x)$ nor $f(-x) = f(x)$

$\therefore \; f$ is neither even nor odd.

Exercises

1. If $f(x) = \dfrac{x^2 - 3x + 1}{x - 1}$ find $f(-2) + f\left(\dfrac{1}{3}\right)$

2. Given $f(x) = x^2 - 5x + 4$, find

$$f(0),\; f(2),\; f(-3),\; f(a),\; f(-x),$$

$$f(b+1),\; f(3x),\; f(x+a),\; \frac{f(x+a) - f(x)}{a}$$

3. Given $f(x+1) = 3x + 5$ evaluate
   a. $f(-2)$
   b. $f(2x)$

4. Let f be defined by $f(x)=\dfrac{x}{2x+1},\ x\in \boldsymbol{R}$. Find
   a. $f(2)$
   b. $f(\frac{1}{5})$
   c. $f(2)\ f(\frac{1}{5})$
   d. $x\ when\ f(x)=\frac{1}{3}$
5. Find the domain of
   a. $f(x)=\sqrt{a^2-x^2}$
   b. $f(x)=\dfrac{1}{3x+2}$
   c. $f(x)=\sqrt{x}+\sqrt{2x-1}$
6. Find the domain and range of the following functions:
   a. $f(x)=\dfrac{x}{x^2-3x+2}$
   b. $f(x)=\dfrac{3}{2-x^2}$
   c. $f(x)=|x-3|$
   d. $f(x)=\dfrac{x}{1+x^2}$
   e. $f(x)=\sqrt{16-x^2}$
   f. $f(x)=\dfrac{1}{\sqrt{x-5}}$
7. Find the domain for the given functions
   a. $f(x)=\dfrac{(2x-1)(x+3)}{x+3}$
   b. $f(x)=\dfrac{4}{5-\cos x}$
8. A function *f* on the set of real numbers is defined

$$f(x)=\begin{cases}2x+1, & 0\le x<2\\ x-2, & 2\le x\le 5\end{cases}$$

Find

a. The range of f.

b. The value of x for which $f(x) = \frac{1}{2}$

c. Whether the function is many-one or one-one.

9. A function f is defined $f(x) = \begin{cases} x+1, & 1 \le x < 2 \\ 2x-1, & 2 \le x < 4 \\ 3x-10, & 4 \le x < 6 \end{cases}$

Find

a. The range of f.

b. $f(4)$

c. Whether the function is many-one or one-one.

10. Let $f: R \to R$ and $g: R \to R$ be defined by

$f(x) = x^2$ $g(x) = x+3$

Then, find $(g \circ f)(2)$ and $(f \circ g)(2)$.

11. Let the functions f and g be defined by

$f(x) = 2x+1$ and $g(x) = x^2 - 2$. Find the formula defining the composition functions: (a) $g \circ f$ (b) $f \circ g$.

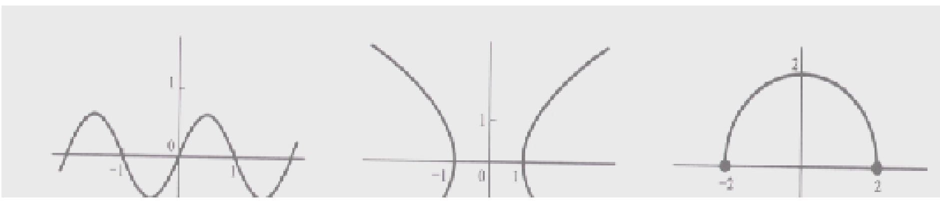
12. $f: R \to R$ be defined by $f(x) = x^2 + 2x$.

a) Find $(f \circ f)(2)$ and $(f \circ f)(3)$.

b) Find a formula for $f \circ f$.

13. If $f(x) = 3x,\ g(x) = \frac{1}{2-sinx}$ then the range of fog.

14. Determine which of the graphs below are functions from $\boldsymbol{R}$ into $\boldsymbol{R}$.



15. Determine whether the following functions are even or odd

a. $f(x) = x\left(\frac{e^x-1}{e^x+1}\right)$

b. $f(x) = \log\left(x + \sqrt{x^2+1}\right)$

1.2. Types of Functions

Learning Objectives:

- To define a one-one function, an onto function and the inverse of a function
- To study geometrical characterization of one-one and onto functions $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$ AND
- To practice the related problems

One-to-One, Onto, and Invertible Functions

A function $f: A \rightarrow B$ is said to be **one-to-one** if distinct elements in the domain A have distinct images. In other words,

$$f \text{ is one-to-one if } f(a) = f(a') \Rightarrow a = a'$$

Example: If $A = \{4,5,6\}$, $B = \{a, b, c, d\}$ and if $f: A \rightarrow B$ such that $f = \{(4, a), (5, b), (6, c)\}$ then f is one-to-one.

Example: The mapping $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$ such that $f(x) = x^2$ is not a one-to-one function, since $f(-2) = 4$ and $f(2) = 4$, that is, two distinct elements −2 and 2 have the same image 4.

If a function $f: A \rightarrow B$ is such that two or more elements of A have the same f-image in B, then the mapping is called **many-to-one mapping** or **many-to-one function**.

Example: If $A = \{a, b, c, d, e\}, B = \{1,2,3\}$, and if $f: A \rightarrow B$ is such that
$f = \{(a, 1), (b, 1), (c, 1), (d, 2), (e, 2)\}$
then the function is many-to-one.

A function $f: A \rightarrow B$ is called an **onto** function if every element of B is the image of at least one element of A. That is, for every $b \in B$ there exist at least one element $a \epsilon A$ such that $f(a) = b$. That is,

$$f \text{ is onto if } \forall\, b \in B, \exists\, a \epsilon A \text{ such that } f(a) = b$$

In such a case, we say that f maps A onto B(Here the symbol ∀ means *for every*, and ∃ means *there exist*).

If $f: A \rightarrow B$ is not an onto function, that is, some of the elements of B remain unused, then f is called an **into** function.

Example: The function $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$ defined by $f(x) = |x|$ is not onto since the negative numbers in the co-domain are not used. Similarly, $f(x) = x^2$ is also not onto.

If $f: A \to B$ is both one-to-one and onto, then f is called a ***one-to-one correspondence*** between A and B. This terminology comes from the fact that each element of A will correspond to a unique element of B and vice versa.

We also use the term ***injective*** for a one-to-one function, ***surjective*** for an onto function, and ***bijective*** for a one-to-one correspondence.

Example: For the function $f: \boldsymbol{R} \to \boldsymbol{R}$ defined by

$f(x) = e^x$, which of the following is true.

(a) onto (b) many- to-one (c) one-to-one and into

(d) many-to- one and onto

Solution: Let $x_1, x_2 \in \boldsymbol{R}$,

$f(x_1) = f(x_2) \Rightarrow e^{x_1} = e^{x_2} \Rightarrow x_1 = x_2;$

Therefore, f is one - one. Notice that e^x is always positive, i.e., the negative real numbers and zero have no preimages. Therefore, f is into.

Answer is (c).

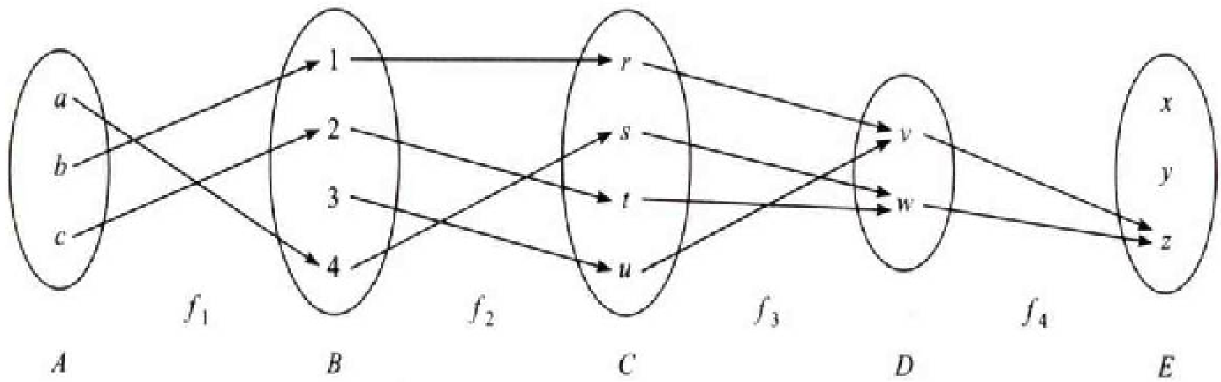
A function $f: A \to B$ is said to be ***invertible*** if its inverse relation f^{-1} is a function from B to A. Equivalently, $f: A \to B$ is **invertible** if there exists a function $f^{-1}: B \to A$, called the **inverse** of f, such that

$$f^{-1} \circ f = I_A \text{ and } f \circ f^{-1} = I_B$$

In general, an inverse function f^{-1} need not exist or, equivalently, the inverse relation f^{-1} may not be a function.

Theorem: A function $f: A \to B$ is invertible if and only if f is both one-to-one and onto.

Example: Consider functions $f_1: A \to B, f_2: B \to C$, $f_3: C \to D, f_4: D \to E$ defined in the figure below.



The function f_1 is one-to-one since no element of B is the image of more than one element of A. It is not onto since $3 \epsilon B$ is not the image of any element of A under f_1.
The function, f_2 is one-to-one and f_2 is onto, since every element of C is the image of some element B under f_2. Further, f_3 is not one-to-one but onto and f_4 is neither one-to-one nor onto.

Since f_2 is both one-to-one and onto, it is a one-to-one correspondence between A and B. Hence f_2 is invertible and ${f_2}^{-1}$ is a function from C to B.

Geometrical Characterization

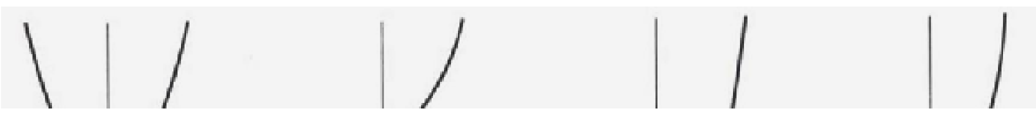
Consider a real-valued function $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$. It may be identified with its graph which is plotted in the Cartesian plane $\boldsymbol{R}^2$. The concepts of being one-to-one and onto have the following geometrical meaning.

One-to-one means that there are no two distinct pairs $(a_1, b)$and (a_2, b) in f; hence each horizontal line in $\boldsymbol{R}^2$ can intersect the graph of f in *at most one point* (Horizontal line test).
Onto means that for every $b \epsilon \boldsymbol{R}$ there is at least one point $a \epsilon \boldsymbol{R}$ such that (a, b) belongs to the graph of f; hence each horizontal line in $\boldsymbol{R}^2$ must intersect the graph of f *at least once*.

Accordingly, the function $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$ is one-to-one and onto (and therefore invertible) if and only if each horizontal line in $\boldsymbol{R}^2$ will intersect the graph of f in *exactly one point*.

Example: Consider the following four functions from $\boldsymbol{R}$ into $\boldsymbol{R}$ whose graphs are shown below.

$f_1(x) = x^2, f_2(x) = 2^x, f_3(x) = x^3 - 2x^2 - 5x + 6, \; f_4(x) = x^3$



There are horizontal lines which intersect the graph of f_1 twice and there are horizontal lines which do not intersect the graph of f_1 at all; hence f_1 is neither one-to-one nor onto.
Similarly, f_2 is one-to-one but not onto, f_3 is onto but not one-to-one, and f_4 is both one-to-one and onto.
The inverse of f_4 is the cube root function, that is,

$$f_4^{-1}(x) = \sqrt[3]{x}$$

NOTE:

Sometimes, we restrict the domain and co-domain of a function f in order to obtain an inverse function f^{-1}. For example, suppose we restrict the domain and co-domain of the function $f_1(x) = x^2$ to be the set D of nonnegative real numbers. Then f_1 is one-to-one and onto and its inverse is the square root function, that is,

$$f_1^{-1}(x) = \sqrt{x}$$

PROBLEM SET

IP1: Discuss the injection (one-one) and surjection (onto) of the function $\boldsymbol{f}: \boldsymbol{Z} \longrightarrow \boldsymbol{Z}$ defined by $\boldsymbol{f(x) = x^2 + x}$ for all $\boldsymbol{x \in Z}$.

Solution:

We have, $f(x) = x^2 + x$ for all $x \in \boldsymbol{Z}$
Notice that $f(1) = f(-2) = 2$. Thus f maps two distinct elements into an element and hence f is not a one-to-one function.
The function f is not onto, since $1 \in \boldsymbol{Z}$(co-domain) has no pre-image in $\boldsymbol{Z}$ (domain).
(If 1 has a pre-image say x, then $f(x) = 1$ this implies $x^2 + x = 1$ i.e., $x = \frac{-1 \pm \sqrt{5}}{2}$.
Notice that $x = \frac{-1 \pm \sqrt{5}}{2} \notin \boldsymbol{Z}$)
Therefore, f is neither one-one nor onto.

P1: Discuss the injection (one-one) and surjection (onto) of the function
$\boldsymbol{f}: \boldsymbol{R} \longrightarrow \boldsymbol{R}$ defined by $\boldsymbol{f(x) = x^2 + 2}$ for all $\boldsymbol{x \in R}$.

Solution:

We have, $f(x) = x^2 + 2$ for all $x \in \boldsymbol{R}$
Notice that $f(1) = f(-1) = 3$. Thus f maps two distinct elements into an element and hence f is not a one – one function.
The function f is not onto, since $1 \in \boldsymbol{R}$(co-domain) has no pre-image in $\boldsymbol{R}$(domain).
(If 1 has a pre-image say x, then $f(x) = 1$ this implies $x^2 + 2 = 1$ i.e., $x^2 = -1$ and this is not possible for any $x \in \boldsymbol{R}$.

(The above result can also be seen as follows: $f(x) = x^2 + 2 \geq 2 \ \forall \ x \in \boldsymbol{R}$.
So, negative real numbers in $\boldsymbol{R}$ (co-domain) do not have their pre-images in $\boldsymbol{R}$ (domain). Hence, f is not an onto function).
Therefore, f is neither one-one nor onto.

IP2: Discuss the injection (one-one) and surjection (onto) of the function
$$\boldsymbol{f}: \boldsymbol{Z} \longrightarrow \boldsymbol{Z} \text{ defined by } \boldsymbol{f(x) = ax + b, a \neq 0, \pm 1;\ a, b \in Z}.$$
Solution:
We have, $f(x) = ax + b, a \neq 0, \pm 1;\ a, b \in \boldsymbol{Z}$

$f(x) = f(y) \Rightarrow ax + b = ay + b \Rightarrow ax = ay \Rightarrow x = y$

$\therefore f$ is a one – one function.

Let y be an element of $\boldsymbol{Z}$(co-domain). Then
$$f(x) = y \Rightarrow ax + b = y \Rightarrow x = \frac{y-b}{a}$$
Clearly, $b + 1 \in \boldsymbol{Z}$; if $y = b + 1$, then $x = \frac{b+1-b}{a} = \frac{1}{a} \notin \boldsymbol{Z}$.
Thus, $y = b + 1 \in \boldsymbol{Z}$ does not have its pre-image in $\boldsymbol{Z}$(domain).

$\therefore f$ is not an onto function.

Hence, f is one – one but not onto.

Note:
The functions $f, g: \boldsymbol{Z} \to \boldsymbol{Z}$ defined by
$$f(x) = x + b \quad , \quad g(x) = -x + b$$
are one – one and onto.

P2: Discuss the injection (one-one) and surjection (onto) of the function
$$\boldsymbol{f}: \boldsymbol{Z} \longrightarrow \boldsymbol{Z} \text{ defined by } \boldsymbol{f(x) = 3x + 2} \text{ for all } \boldsymbol{x \in Z}.$$
Solution:
We have, $f(x) = 3x + 2 \ \forall \ x \in \boldsymbol{Z}$

$f(x) = f(y) \Rightarrow 3x + 2 = 3y + 2 \Rightarrow 3x = 3y \Rightarrow x = y$

$\therefore f$ is a one – one function.

Let y be an element of $\boldsymbol{Z}$(co-domain).
Then $f(x) = y \Rightarrow 3x + 2 = y \Rightarrow x = \frac{y-2}{3}$.
Clearly, if $y = 3$, then $x = \frac{3-2}{3} = \frac{1}{3} \notin \boldsymbol{Z}$.
Thus, $y = 3 \in \boldsymbol{Z}$ does not have its pre-image in $\boldsymbol{Z}$(domain).

$\therefore f$ is not an onto function.

Hence, f is one – one but not onto.

IP3: Prove that the function $f\colon \boldsymbol{Q} \longrightarrow \boldsymbol{Q}$ given by $f(x) = 2x - 3$ for all $x \in \boldsymbol{Q}$ is a bijection (one-one and onto).

Solution:

Injectivity(one-to-one):

Let x, y be two elements in $\boldsymbol{Q}$(domain).

Then, $f(x) = f(y) \Rightarrow 2x - 3 = 2y - 3 \Rightarrow 2x = 2y \Rightarrow x = y$

Thus, $f(x) = f(y) \Rightarrow x = y \ \ \forall \ \ x, y \in \boldsymbol{Q}$.

So, f is a one-to-one function.

Surjectivity(onto):

Let y be an element of $\boldsymbol{Q}$(co-domain).

Then, $f(x) = y \Rightarrow 2x - 3 = y \Rightarrow x = \frac{y+3}{2}$

Clearly, for all $y \in \boldsymbol{Q}$, $x = \frac{y+3}{2} \in \boldsymbol{Q}$.

Thus for all $y \in \boldsymbol{Q}$(co-domain) there exists $x \in \boldsymbol{Q}$(domain) given by $x = \frac{y+3}{2}$

such that $\quad f(x) = f\left(\frac{y+3}{2}\right) = 2\left(\frac{y+3}{2}\right) - 3 \qquad [\because f(x) = 2x - 3]$

$$= y + 3 - 3 = y$$

Thus, every element in the co-domain has its pre-image in $\boldsymbol{Q}$.

So, f is an onto function.

Hence, f is a bijection (one – one and onto).

P3: Show that the function $f\colon \boldsymbol{R} \longrightarrow \boldsymbol{R}$ defined by $f(x) = 3x^3 + 5$ for all $x \in \boldsymbol{R}$ is a bijection (one-one and onto).

Solution:

Injectivity(one-one):

Let x, y be two elements in $\boldsymbol{R}$(domain).

Then, $f(x) = f(y) \Rightarrow 3x^3 + 5 = 3y^3 + 5$

$$\Rightarrow x^3 = y^3 \Rightarrow x = y$$

Thus, $f(x) = f(y) \Rightarrow x = y \ \forall \ x, y \in \boldsymbol{R}$.

So, f is an one – one function.

Surjectivity(onto):

Let y be an element of $\boldsymbol{R}$(co-domain).

Then, $f(x) = y \Rightarrow 3x^3 + 5 = y \Rightarrow x^3 = \frac{y-5}{3} \Rightarrow x = \left(\frac{y-5}{3}\right)^{\frac{1}{3}}$

Thus, we find that for all $y \in \boldsymbol{R}$(co-domain), there exists $x = \left(\frac{y-5}{3}\right)^{\frac{1}{3}} \in \boldsymbol{R}$(domain) such that

$$f(x) = f\left(\left(\frac{y-5}{3}\right)^{\frac{1}{3}}\right) = 3\left(\left(\frac{y-5}{3}\right)^{\frac{1}{3}}\right)^3 + 5 \ \ [\because f(x) = 3x^3 + 5]$$

$$= y - 5 + 5 = y$$

This shows that every element in the co-domain has its pre-image in the domain.
So, f is an onto function.
Hence, f is a bijection (one – one and onto).

IP4: Show that the function $f: R - \{-1\} \longrightarrow R - \{1\}$ given by $f(x) = \frac{x}{x+1}$ is invertible. Also, find f^{-1}.

Solution:

In order to prove the invertibility of $f(x)$, it is sufficient to show that it is a bijection.
For any $x, y \in \boldsymbol{R} - \{-1\}$,
we have $f(x) = f(y) \Rightarrow \frac{x}{x+1} = \frac{y}{y+1} \Rightarrow xy + x = xy + y \Rightarrow x = y$

$\therefore f$ is one – one.

Let $y \in \boldsymbol{R} - \{1\}$. Then, $f(x) = y \Rightarrow \frac{x}{x+1} = y \Rightarrow x = \frac{y}{1-y}$

Clearly, $x \in \boldsymbol{R}$ for all $y \in \boldsymbol{R} - \{1\}$. Also $x \neq -1$. Because, if $x = -1 \Rightarrow \frac{y}{1-y} = -1 \Rightarrow y = -1 + y$, which is not possible.

Thus, for each $y \in \boldsymbol{R} - \{1\}$ there exists $x = \frac{y}{1-y} \in R - \{-1\}$ such that

$$f(x) = \frac{x}{x+1} = \frac{\frac{y}{1-y}}{\frac{y}{1-y}+1} = y$$

$\therefore f$ is onto.

Thus, f is both one – one and onto. Consequently it is invertible.

Now, $fof^{-1}(x) = x \quad \forall \ x \in \boldsymbol{R} - \{1\}$

$\Rightarrow f[f^{-1}(x)] = x \Rightarrow \frac{f^{-1}(x)}{f^{-1}(x)+1} = x \qquad \left[\because f(x) = \frac{x}{x+1}\right]$

$\Rightarrow f^{-1}(x) = xf^{-1}(x) + x \Rightarrow f^{-1}(x)(1-x) = x$

$\Rightarrow f^{-1}(x) = \frac{x}{1-x} \quad \forall \, x \in \boldsymbol{R} - \{1\}$

P4: Show that the function $f: R - \{0\} \longrightarrow R - \{0\}$ given by $f(x) = \frac{3}{x}$ is invertible and it is inverse of itself.

Solution:

In order to prove that f is invertible, it is sufficient to show that it is a bijection.
For any $x, y \in \boldsymbol{R} - \{0\}$, we have

$f(x) = f(y) \Rightarrow \frac{3}{x} = \frac{3}{y} \Rightarrow x = y$

$\therefore f$ is one – one.

Let $y \in \boldsymbol{R} - \{0\}$. Then, $f(x) = y \Rightarrow \frac{3}{x} = y \Rightarrow x = \frac{3}{y}$

Thus, for each $y \in \boldsymbol{R} - \{0\}$ there exists $x = \frac{3}{y} \in \boldsymbol{R} - \{0\}$

such that $f(x) = f\left(\frac{3}{y}\right) = \frac{3}{\frac{3}{y}} = y$

$\therefore f$ is onto.

Thus, f is both one – one and onto. Consequently it is invertible.

Let, $f(x) = y$. Then,

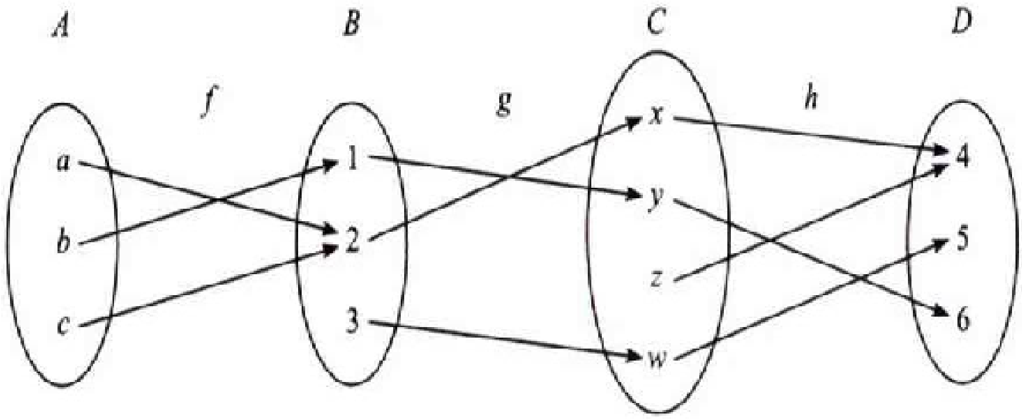
$\Rightarrow \frac{3}{x} = y \Rightarrow x = \frac{3}{y} \Rightarrow f^{-1}(y) = \frac{3}{y}$

Thus, f^{-1} is given by $f^{-1}(x) = \frac{3}{x}$ for all $x \in \boldsymbol{R} - \{0\}$.

Hence, f is inverse of itself.

Exercises:

1. Let the functions $f: A \to B, g: B \to C$ and $h: C \to D$, be defined by the figure below.



   a) Determine if each function is one-to-one.
   b) Determine if each function is onto.
   c) Determine if each function is invertible.
   d) Find the composition $hogof$

2. Prove that $f: \boldsymbol{R} \to \boldsymbol{R}$, given by $f(x) = 2x$, is one-one and onto.
3. Show that the function $f: \boldsymbol{R} \to \boldsymbol{R}$ be defined as $f(x) = x^3$, is a bijection.
4. Show that the function $f: \boldsymbol{R} \to \boldsymbol{R}$ given by $f(x) = x^3 + x$, is a bijection.
5. Show that the function $f: \boldsymbol{R} \to \boldsymbol{R}$ be defined as $f(x) = x^2$, is neither one-one nor onto.
6. Let $f: \boldsymbol{R} \to \boldsymbol{R}$ be defined by $f(x) = 2x - 3$. The function f is one-to-one and onto; hence f has an inverse function f^{-1}. Find a formula for f^{-1}.
7. If $f: \boldsymbol{R} \to \boldsymbol{R}$ is defined by $f(x) = 2x + 7$. Prove that f is a bijection. Also, find the inverse of f.
8. Find a formula for the inverse of $g(x) = \frac{2x-3}{5x-7}$
9.

1.3. Inverse functions

Learning objectives:

* To define a one-to-one function and its inverse
* To state the Horizontal Line Test for a one-to-one function.
* To find the inverse of a given one-to-one function.

  AND
* To practice the related problems.

One-to-One Functions

A function is a rule that assigns a value from its range to each point in its domain. Some functions assign the same value to more than one point. The squares of -1 and 1 are both 1; The sines of $\pi/3$ and $2\pi/3$ are both $\sqrt{3}/2$. There are functions which never assume a given value more than once. The square roots and cubes of different numbers are always different. A function that has distinct values at distinct points is called one-to-one.

Definition

A function $f(x)$ is ***one-to-one*** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

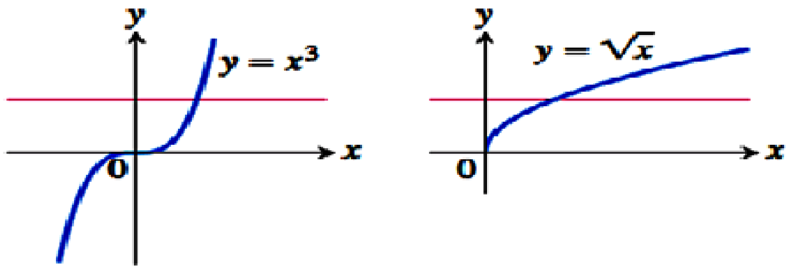
Example 1:

The function $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.

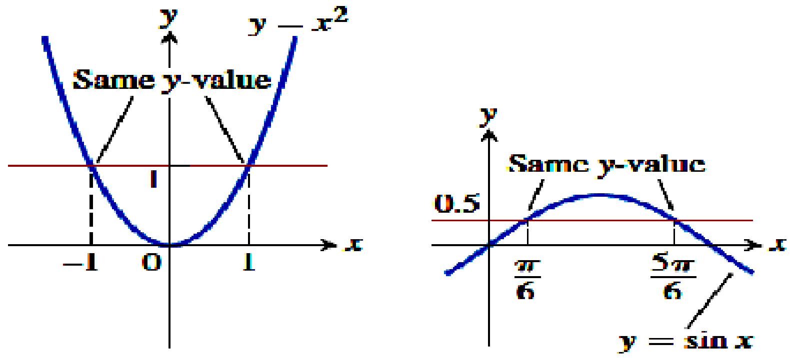
Example 2:

The function $g(x) = sinx$ is not one-to-one on the interval $[0, \pi]$ because $sin\left(\frac{\pi}{6}\right) = sin\left(\frac{5\pi}{6}\right)$. The sine is one-to-one on $[0, \pi/2]$, however, because sines of angles in the first quadrant are distinct.

The graph of a one-to-one function $y = f(x)$ can intersect a given horizontal line at most once. If it intersects the line more than once it assumes the same y –value more than once, and is therefore not one-to-one.



One-to-one: Graph meets each horizontal line at most once.



Not one-to-one: Graph meets one or more horizontal lines more than once.

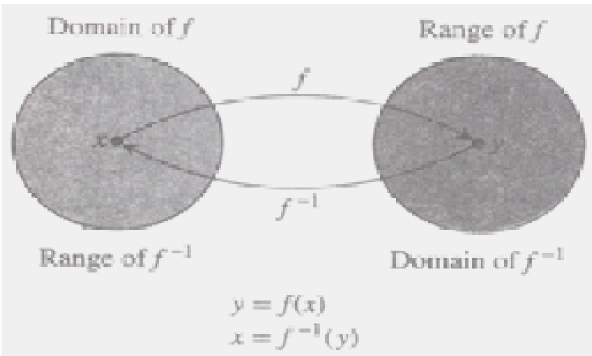
The Horizontal Line Test

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

Inverses

Since each output of a one-to-one function comes from just one input, a one-to-one function can be reversed to send the outputs back to the inputs from which they came. The function defined by reversing a one-to-one function f is called the ***inverse*** of f. The symbol for the inverse of f is f^{-1}, read "f inverse".

The result of composing f and f^{-1} in either order is the identity function, the function that assigns each number to itself.



This gives a way to test whether two functions f and g are inverses of one another. Compute $f \circ g$ and $g \circ f$. If $(f \circ g)(x) = (g \circ f)(x) = x$, then f and g are inverse of one another; otherwise they are not.

Functions f and g are an inverse pair if and only if

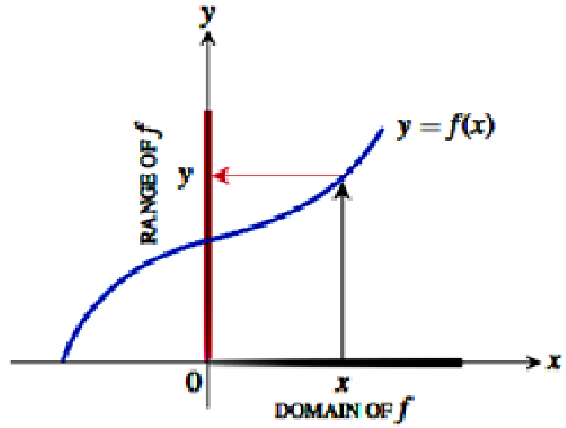
$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x.$$

In this case, $g = f^{-1}$ *and* $f = g^{-1}$.

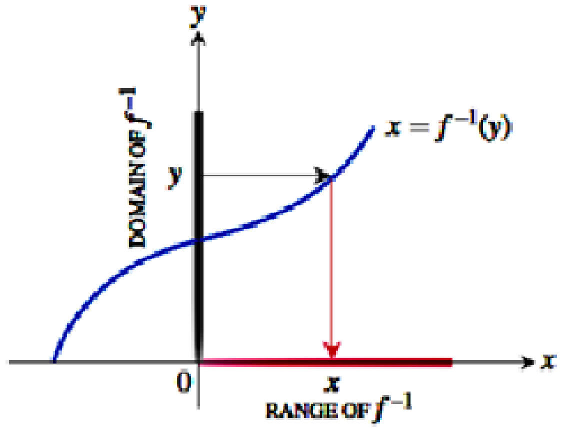
A function has an inverse if and only if it is one-to-one. This means, for example, that increasing functions have inverses, and decreasing functions have inverses. Functions with positive derivatives have inverses because they increase throughout their domains. Similarly, functions with negative derivatives have inverses because they decrease throughout their domains.

Finding Inverses

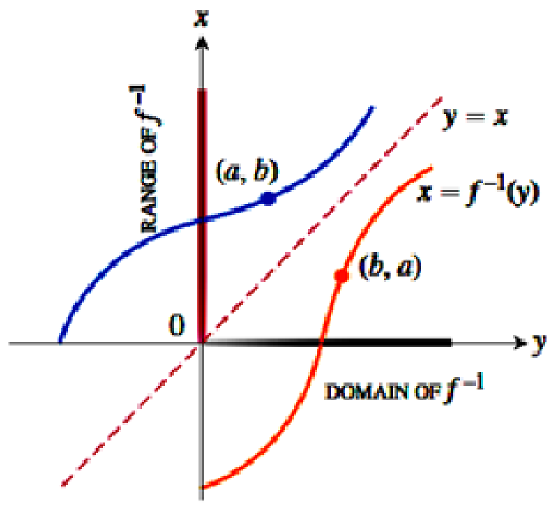
If the function is increasing, its graph rises from left to right, like the graph in figure (a) below.



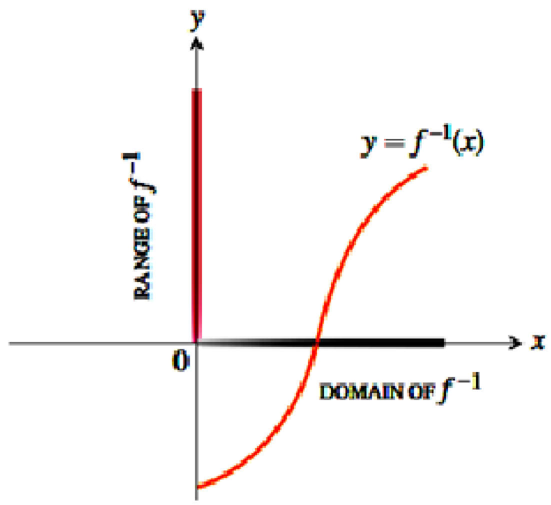
(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.



(b) The graph of f is already the graph of f^{-1}, but with x and y interchanged. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.



(c) To draw the graph of f^{-1} in the more usual way, we reflect the system in the line $y = x$.



(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

To read the graph, we start at the point x on the x −axis, go up to the graph, and then move over to the y −axis to read the value of y. If we start with y and want to find x from which it came, we reverse the process (figure b).

The graph of f is the graph of f^{-1} with the input-output pairs reversed. To display the graph in the usual way, we have to reverse the pairs by reflecting the graph in the $45°$ line $y = x$ (figure (c)) and interchanging the letters x and y (figure (d)). This puts the independent variable, now called x, on the horizontal axis and the dependent variable, now called y, on the vertical axis. The graphs of $f(x)$ and $f^{-1}(x)$ are symmetric about the line $y = x$

The procedure for expressing f^{-1} as a function of x is given below.

Solve the equation $y = f(x)$ for x in terms of y.

Interchange x and y. The resulting formula will be $y = f^{-1}(x)$.

Example 3

Find the inverse of $y = \frac{1}{2}x + 1$ expressed as a function of x.

Solution

We solve for x in terms of y.

$$y = \frac{1}{2}x + 1 \Rightarrow 2y = x + 2 \Rightarrow x = 2y - 2$$

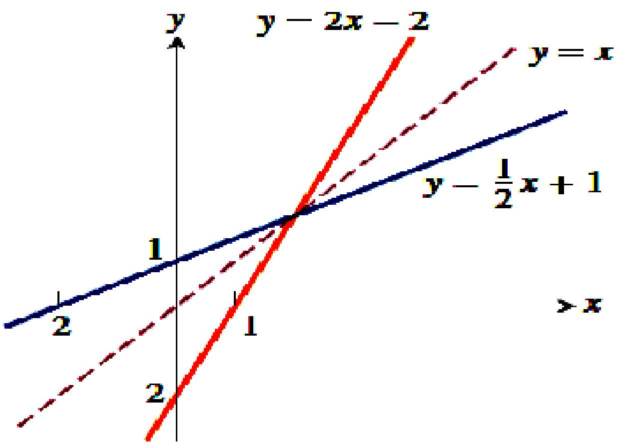
Interchange x and y: $y = 2x - 2$

The inverse of the function $f(x) = \frac{1}{2}x + 1$ is the function $f^{-1}(x) = 2x - 2$.

We verify that both composites give the identity function.

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f\left(f^{-1}(x)\right) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x$$



Example 4: Find the inverse of the function $y = x^2,\ x \geq 0$, expressed as a function of x.

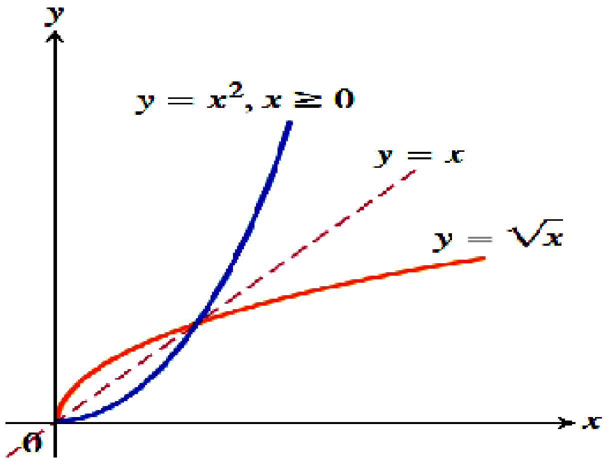
Solution: Solve for x in terms of y:

$$y = x^2$$

$$\Rightarrow \sqrt{y} = \sqrt{x^2} = |x| = x, \quad |x| = x \text{ because } x \geq 0$$

Interchange x and y: $\quad y = \sqrt{x}$

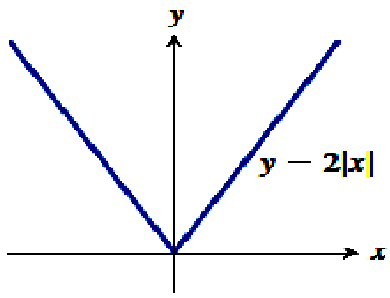
The inverse of the function $y = x^2,\ x \geq 0$, is the function $y = \sqrt{x}$.



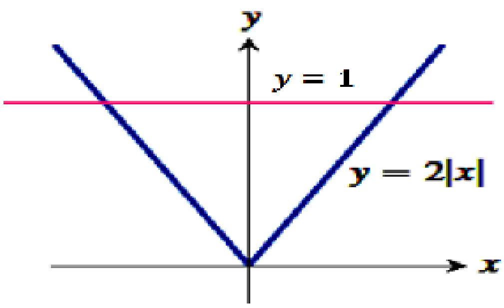
We note that, unlike the restricted function $y = x^2,\ x \geq 0$, the unrestricted function $y = x^2$ is not one-to-one and therefore has no inverse.

PROBLEM SET

IP1. Verify the graph given below is one to one or not.



Solution:

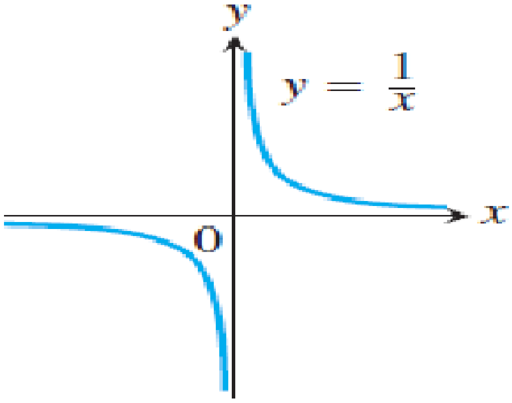


Notice that a horizontal line, for example $y = 1$ intersects the given graph at two points. Therefore, by the *Horizontal Line Test* the function $y = f(x) = 2|x|$ is not a one-to-one function.

Note:

For $x_1 = 1$, $x_2 = -1$ we have $f(x_1) = f(x_2) = 2$ i.e., distinct elements in the domain of f are mapped to the same element in the range. Thus, f is not one-to-one.

P1. Verify the graph given below is one to one or not.



Solution:

Notice that the graph intersects each horizontal line at most once. Therefore, by the *Horizontal Line Test* the function $y = f(x) = \frac{1}{x}, x \neq 0$ is a one-to-one function.

IP2. Let $f(x) = x^3 - 1$ then find a formula for f^{-1}.

Solution:

Given $y = f(x) = x^3 - 1$

Notice that

$x_1 \neq x_2 \Rightarrow {x_1}^3 \neq {x_2}^3 \Rightarrow {x_1}^3 - 1 \neq {x_2}^3 - 1 \Rightarrow f(x_1) \neq f(x_2)$

Therefore, $f(x)$ is one-to-one and its inverse exists.

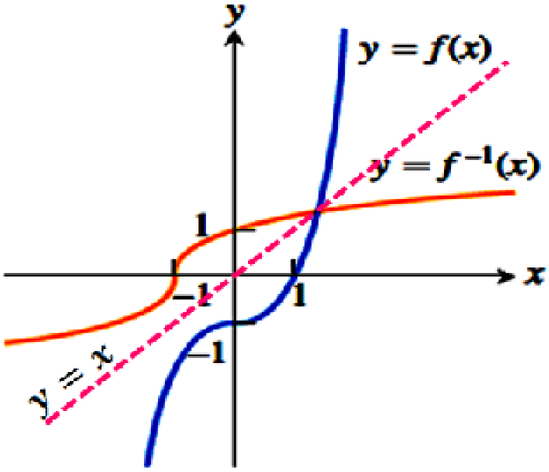
Now, solve for x in terms of y:

i.e., $y = x^3 - 1 \Rightarrow x^3 = y + 1 \Rightarrow x = \sqrt[3]{y+1}$

Interchange x and y: $\quad y = \sqrt[3]{x+1}$

Therefore, the inverse of the function $f(x) = x^3 - 1$ is the function $y = f^{-1}(x) = \sqrt[3]{x+1}$.

The related graphs of f and f^{-1} are shown below.



P2. Let $f(x) = x^2 + 1,\ x \geq 0$ then find a formula for f^{-1}.

Solution:

Given $y = f(x) = x^2 + 1,\ x \geq 0$

Notice that

$x_1 \neq x_2 \Longrightarrow {x_1}^2 \neq {x_2}^2 \Longrightarrow {x_1}^2 + 1 \neq {x_2}^2 + 1 \Longrightarrow f(x_1) \neq f(x_2)$

Thus, $f(x)$ is a one-to-one function and its inverse exists.

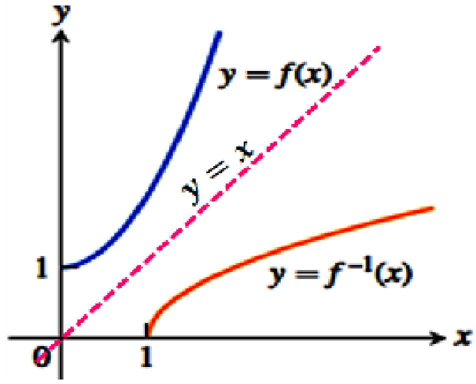
Now, solve for x in terms of y:

i.e., $y = x^2 + 1 \Rightarrow x^2 = 1 - y \Rightarrow x = \sqrt{y-1}$

Interchange x and y: $y = \sqrt{x-1}$

Therefore, the inverse of the function $f(x) = x^2 + 1$ is the function $y = f^{-1}(x) = \sqrt{x-1}$.

The related graphs of f and f^{-1} are shown below.



IP3. Let $f(x) = \frac{1}{x^2}$, $x > 0$ then find a formula for f^{-1} and identify the domain and range of f^{-1}. As a check show that

$$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$$

Solution: Given $y = f(x) = \frac{1}{x^2}$, $x > 0$

Notice that Domain of $f = (0, \infty)$; Range of $f = (0, \infty)$

$x_1 \neq x_2 \Longrightarrow \frac{1}{{x_1}^2} \neq \frac{1}{{x_2}^2} \Longrightarrow f(x_1) \neq f(x_2)$

Therefore, f is one-to-one and its inverse exists.

Now, solve for x in terms of y:

i.e., $y = \frac{1}{x^2} \Longrightarrow x^2 = \frac{1}{y} \Longrightarrow x = \frac{1}{\sqrt{y}}$

Interchange x and y: $y = \frac{1}{\sqrt{x}}$

Therefore, the inverse of the function $f(x) = \frac{1}{x^2}$ is the function $y = f^{-1}(x) = \frac{1}{\sqrt{x}}$.

Domain of f^{-1} = Range of $f = (0, \infty)$

Range of f^{-1} = Domain of $f = (0, \infty)$

$$\left(f \circ f^{-1}\right)(x) = f\left(f^{-1}(x)\right) = f\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\left(1/\sqrt{x}\right)^2} = x$$

$$\left(f^{-1} \circ f\right)(x) = f^{-1}\left(f(x)\right) = f^{-1}\left(\frac{1}{x^2}\right) = \frac{1}{\sqrt{\left(1/x^2\right)}} = x$$

$$\text{Thus, } \left(f \circ f^{-1}\right)(x) = \left(f^{-1} \circ f\right)(x) = x$$

This checks that f and f^{-1} are inverse to each to other.

P3. Let $f(x) = x^5$ then find a formula for f^{-1} and identify the domain and range of f^{-1}. As a check show that $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$

Solution: Given $y = f(x) = x^5$

Notice that Domain of $f = \mathbb{R}$; Range of $f = \mathbb{R}$

$x_1 \neq x_2 \Longrightarrow {x_1}^5 \neq {x_2}^5 \Longrightarrow f(x_1) \neq f(x_2)$

Therefore, f is one-to-one and its inverse exists.

Now, solve for x in terms of y:

i.e., $y = x^5 \Longrightarrow x = y^{\frac{1}{5}}$

Interchange x and y: $y = x^{\frac{1}{5}}$

Therefore, the inverse of the function $f(x) = x^5$ is the function $y = f^{-1}(x) = x^{\frac{1}{5}}$.

Domain of f^{-1} = range of $f = \mathbb{R}$

Range of f^{-1} = Domain of $f = \mathbb{R}$

$$\left(f \circ f^{-1}\right)(x) = f\left(f^{-1}(x)\right) = f\left(x^{1/5}\right) = \left(x^{1/5}\right)^5 = x$$

$$\left(f^{-1} \circ f\right)(x) = f^{-1}\left(f(x)\right) = f^{-1}\left(x^5\right) = \left(x^5\right)^{1/5} = x$$

$$\text{Thus, } \left(f \circ f^{-1}\right)(x) = \left(f^{-1} \circ f\right)(x) = x$$

This checks that f and f^{-1} are inverse to each to other.

IP4. If the functions f and g be defined by $f(x) = 3x - 4$, $g(x) = 2 + 3x$ for $x \in R$ respectively then $\left(g^{-1} \circ f^{-1}\right)(38) =$

Solution:

Given $f(x) = 3x - 4$.Clearly f is one-to-one.

Now, solve for x in terms of y

$y = 3x - 4 \Longrightarrow 3x = y + 4 \Longrightarrow x = \frac{y+4}{3}$

Interchange x and y : $y = \frac{x+4}{3}$

The inverse of the function $f(x) = 3x - 4$ is the function $y = f^{-1}(x) = \frac{x+4}{3}$

Again, we have $g(x) = 2 + 3x$.Clearly g is one-to-one.

Now, solve for x in terms of y

$y = 2 + 3x \Longrightarrow 3x = y - 2 \Longrightarrow x = \frac{y-2}{3}$

Interchange x and y : $\quad y = \frac{x-2}{3}$

The inverse of the function $g(x) = 2 + 3x$ is the function $y = g^{-1}(x) = \frac{x-2}{3}$

$\therefore \left(g^{-1} \circ f^{-1}\right)(38) = g^{-1}\left[\frac{38+4}{3}\right] = g^{-1}[14] = \frac{14-2}{3} = 4$

P4. Let the functions $f: R \to R$, $g: R \to R$ be defined by $f(x) = 4x - 1$ and $g(x) = x + 2$. If $(gof^{-1})\left(\frac{a+1}{4}\right) = 2$ then find the value of a .

Solution:

Given $f(x) = 4x - 1$.Clearly f is one-to-one.

Now, solve for x in terms of y

$y = 4x - 1 \Longrightarrow 4x = y + 1 \Longrightarrow x = \frac{y+1}{4}$

Interchange x and y : $\quad y = \frac{x+1}{4}$

The inverse of the function $f(x) = 4x - 1$ is the function $y = f^{-1}(x) = \frac{x+1}{4}$

Now, $(gof^{-1})\left(\frac{a+1}{4}\right) = g\left[\frac{\frac{a+1}{4}+1}{4}\right] = g\left[\frac{a+5}{16}\right] = \frac{a+5}{16} + 2$

$$= \frac{a+5+32}{16} = \frac{a+37}{16}$$

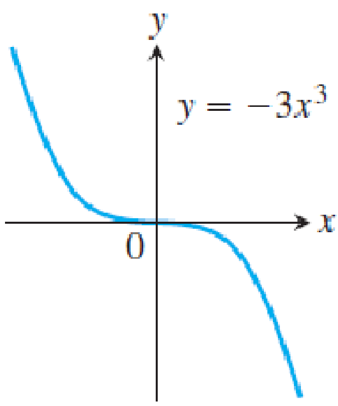
$\therefore \ (gof^{-1})\left(\frac{a+1}{4}\right) = \frac{a+37}{16} = 2$ (By hypothesis)

$$a + 37 = 32 \Longrightarrow a = -5$$

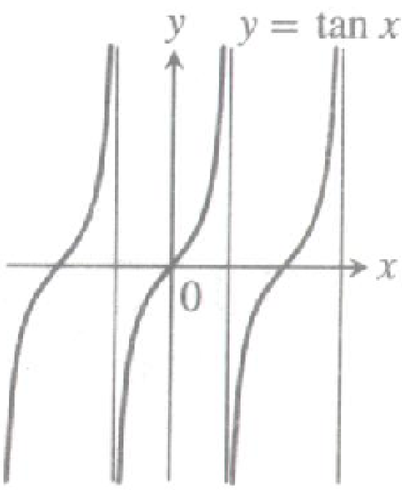
EXERCISES

1. Which of the functions graphed below are one-to-one, and which are not?

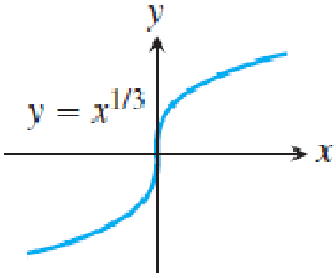
a.



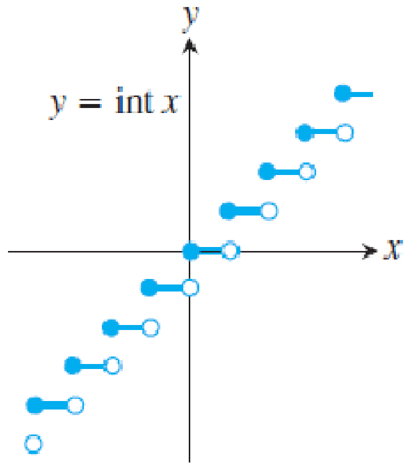
b.



c.



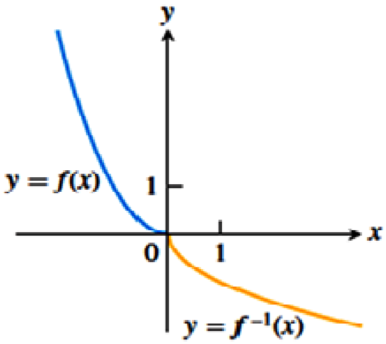
d.



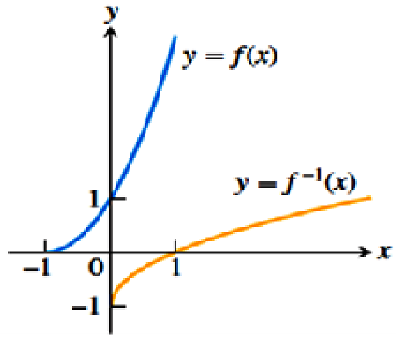
2. Graph the function $f(x)=\sqrt{1-x^2}$, $0 \le x \le 1$. What symmetry does the graph have? Show that f is its own inverse.

3. The formula for a function $y = f(x)$ and show the graphs of f and f^{-1} are given below. Find a formula for f^{-1} in each case.

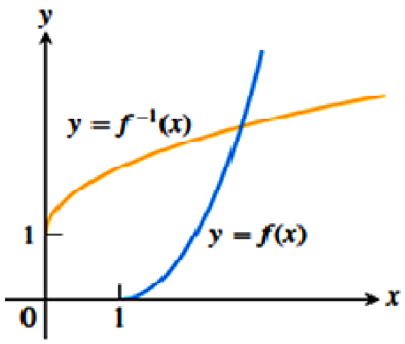
$f(x) = x^2, \quad x \le 0$



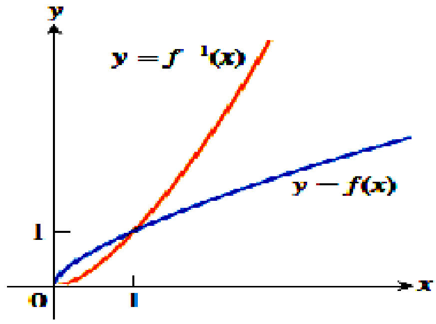
$f(x) = (x+1)^2, \quad x \ge -1$



$f(x) = x^2 - 2x + 1, \quad x \ge 1$



$f(x) = x^{2/3}, \quad x \ge 0$



4. The formula for a function $y = f(x)$ is given below. In each case, find $f^{-1}(x)$ and identify the domain and range of f^{-1}. As a check, show that $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$.
   a. $f(x) = x^4$, $x \geq 0$
   b. $f(x) = x^3 + 1$
   c. $f(x) = \frac{x}{2} - \frac{7}{2}$
   d. $f(x) = \frac{1}{x^3}$, $x \neq 0$

1.4. Exponential Functions

Learning Objectives

- To define an exponential function and to study its graph called exponential curve
  And
- To practice problems on compound interest and half life of a radioactive substance.

From the theory of indices, we have the following relations

$$a^m = a \cdot a \cdots a \ (m \text{ times}), \quad a^0 = 1, \qquad a^{-m} = \frac{1}{a^m}$$

Where m is a positive integer.

Exponents are extended to include all rational numbers by defining

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

for any rational number $\frac{m}{n}$.

For example,

$$2^4 = 16, \quad 2^{-4} = \frac{1}{2^4} = \frac{1}{16}, \; 125^{\frac{2}{3}} = 5^2 = 25$$

Example: Evaluate $2^5, \quad 3^{-4}, \; 8^{\frac{2}{3}}, \; 25^{-\frac{3}{2}}$

$$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$$

$$3^{-4}=\frac{1}{3^4}=\frac{1}{81}$$

$$8^{\frac{2}{3}}=\left(\sqrt[3]{8}\right)^2=2^2=4$$

$$25^{-\frac{3}{2}}=\frac{1}{25^{\frac{3}{2}}}=\frac{1}{\left(\sqrt{25}\right)^3}=\frac{1}{5^3}=\frac{1}{125}$$

As a further extension, the exponents can also be allowed to be real numbers. We may define an exponential function as follows.

A function of the form $f(x) = b^x$, where the base b is a positive constant, $b \neq 1$, is called an ***exponential function***. The domain and range of an exponential function are $(-\infty, \infty)$ and $(0, \infty)$ respectively. An exponential function never assumes the value 0.

Each of the following is an exponential function:

$$f(x)=2^x, \qquad y=3^x, \qquad f(x)=\left(\frac{1}{4}\right)^x$$

Example The decay of radioactive iodine-131 is described by the exponential function $A=A_0\cdot 2^{-t/8}$

where A and A_0 are measured in μg and t in days.

Find its half-life. (*The half life of a decaying substance is defined as the time it takes to decrease to half of its original amount*).

Solution: Half life is given by

$$\frac{A_0}{2}=A_0\cdot 2^{-t/8}\Rightarrow 2^{-1}=2^{\frac{-t}{8}}$$

$$\Rightarrow -1=-\frac{t}{8}$$

$$\Rightarrow t=8\ days$$

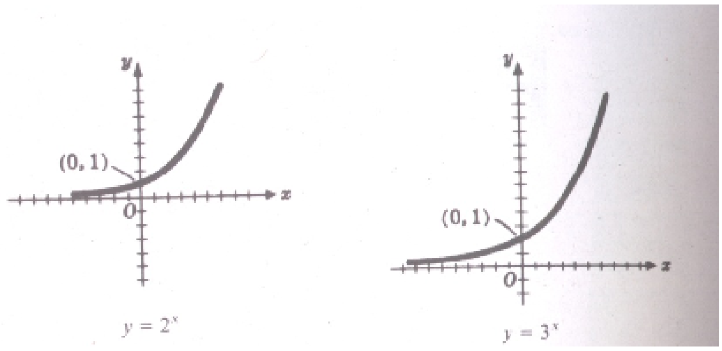
Exponential curve: The curve whose equation is

$$y=a^x\ ,\ where\ a\ is\ a\ positive\ constant,\ a\neq 1,\ x\in R.$$

Is called an ***exponential curve***. The general properties of such curves are that the curve passes through the point $(0,1)$ and that the curve lies above the x - axis x-axis is an asymptote of the curve.

Example: Sketch the graphs of $y = 2^x$, $y = 3^x$

The graphs are shown below.



The exponential equation appears frequently in the form

$y = ce^{kx}$ Where c and k are nonzero constants and $e = 2.71828\cdots$. We can solve equations that contain a variable as exponent if we can convert both sides of the equation to an expression with the same base.

Example: Solve the equation $8^x = \frac{1}{2}$

Solution:

$$8^x = \frac{1}{2} \Rightarrow \left(2^3\right)^x = \frac{1}{2} \Rightarrow 2^{3x} = 2^{-1}$$

$$\Rightarrow 3x = -1 \qquad \Rightarrow x = -\frac{1}{3}$$

Compound Interest: If P rupees is deposited in an account with an annual interest rate r, compounded n times per year, then the amount of money in the account after t years is given by the exponential equation

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

The number e, like π, is an irrational number. Like π, it can be approximated with a decimal number. Whereas π is approximately 3.1416, e is approximately 2.7183. ***The exponential function based on the number e is called the natural exponential function.***

One common application of natural exponential functions is with interest bearing accounts. The formula $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$

gives the amount of money in an account if P rupees are deposited for t years at annual interest rate r, compounded n times per year. If we let the number of compounding periods become indefinitely large (that is, we compound the interest every moment), we have an account with an interest that is compounded continuously. The amount of money in the account after t years is given by

$$A(t) = \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt}$$
$$= Pe^{rt}$$

Example1:

Suppose you deposit Rs 500 in an account with interest rate of 8% compounded continuously. Then find the amount of money in the account after 5 years.

Solution:

Since the interest is compounded continuously, we use the formula $A(t) = Pe^{rt}$.

$$A(t) = 500e^{0.08t}$$

After 5 years, this account will contain

$$A(5) = 500e^{0.08\times(5)} = 500e^{0.4} = Rs.745.91$$

Example2:

Suppose you deposit Rs 500 in an account with an annual interest rate of 8% compounded **monthly**. Find the amount of money in the account after 5 years?

Solution: We have

$$A(t) = P(1+\frac{r}{n})^{nt}$$

where $P = Rs.500, r = 0.08, n = 12, t = 5$

$$A(5) = 500\left(1+\frac{8}{12\times100}\right)^{12\times5}$$
$$= 500(\frac{451}{450})^{60}$$
$$= Rs.744.92$$

Remark: ***The effect of continuous compounding as compared with monthly compounding is an addition of Rs 0.99.***

PROBLEM SET

IP1: Mahesh place $Rs.4000$ in a bank account that earns 5% interest compounded continuously. What is the total amount (in rupees) in the account after 3 years?

Solution:

STEP1: $P = 4000, t = 3, and\ r = 0.05$.

STEP2: Since interest is compounded continuously.

$A(t) = Pe^{rt}$

STEP3: Substituting in these values results in

$A(3) = 4000e^{0.05\times 3}$

$A(3) = 4000e^{0.15}$

STEP4: $= 4000(1.161834) \quad = 4647.33$

P1: Rahul place $Rs. 4000$ in a bank account that earns 5% interest compounded quarterly. What is the total amount (in rupees) in the account after 3 years?

A. 4643.02
B. 4463.02
C. 4346.02
D. 4643.20

Answer: A

Solution: $= 4000, t = 3, and\ r = 0.05$.

Since interest is compounded quarterly,

There are 4 compounding periods per year.

Substituting in these values results in

$A = P(1 + \frac{r}{n})^{nt}$

$A = 4000(1 + \frac{0.05}{4})^{4\times 3}$

$= 4000(1.160754518)$

$= 4643.02$

IP2: The decay of radioactive iodine-131 is described by the exponential function

$$A = A_0 \cdot 2^{-t/6}$$

where A and A_0 are measured in μg and t in days.

Find it s half-life. *(The half life of a decaying substance is defined as the time it takes to decrease to half of its original amount.)*

Solution:

STEP1: Given (The decay of radioactive substance.) $A = A_0 \cdot 2^{-t/6}$

STEP2: Half life is given by

$$\frac{A_0}{2} = A_0 \cdot 2^{-t/6}$$

STEP3: $\Rightarrow \quad -1 = -\frac{t}{6}$ **STEP4**: $t = 6\ days$

$\Rightarrow \quad t = 6\ days$

P2: The decay of radioactive substance is described by the exponential function

$$A = A_0 \cdot 2^{-t/4}$$

where A and A_0 are measured in μg and t in days.

Find its half-life. ***(The half life of a decaying substance is defined as the time it takes to decrease to half of its original amount.)***

A. 2
B. 4
C. 6
D. 8

Answer: B

Solution: Half life is given by

$$\frac{A_0}{2} = A_0 \cdot 2^{-t/4} \Rightarrow 2^{-1} = 2^{-t/4}$$

$$\Rightarrow \quad -1 = -\frac{t}{4}$$

$$\Rightarrow \quad t = 4 \ days$$

IP3: A plant's mass initially increases exponentially at a rate of 5% per day. On the initial day of recorded growth, the plant has mass of $10g$. What is the mass of the plant after 30 days from the initial day?

Solution:

STEP1: Plant will grow according to the formula

$A = P.e^{rt}$, where $P = 10, r = 0.05, and\ t = 30$

we get , $A = 10.e^{0.05\times30}$

STEP2: $A = 10.e^{1.5}$

$= 10 \times 4.48$

$= 44.8\ g$

P3: A plant's mass initially increases exponentially at a rate of 5% per day. On the initial day of recorded growth, the plant has mass of $10g$. What is the mass of the plant after 4 days from the initial day?

Solution: Plant will grow according to the formula $A = P.e^{rt}$

Where $P = 10, r = 0.05, and\ t = 4$

We get

$A = 10.e^{0.05\times4}$

$A = 10.e^{0.2}$

$= 10 \times 1.22$

$= 12.2\ g$

IP4: Suppose that a certain material has a half life of 15 years, and there are $\boldsymbol{A(t)\ =\ 18(\frac{1}{2})^{t/15}}$ grams remaining after t years. Find the amount after 45 years.

Solution:

Step1: $A(t)\ =\ A_0(\frac{1}{2})^{t/k}$ (Here k is half life)

The initial amount is 18g, as we can read directly off of the formula for $A(t)$.

$A(t)\ =\ 18(\dfrac{1}{2})^{t/15}$

Step2: $A(t)\ =\ 18(\frac{1}{2})^{t/15}$

After 45 years, we have

Step3: $A(45) =\ 18\left(\frac{1}{2}\right)^{\frac{45}{15}} =\ 2.25$ grams left.

P4: Suppose that a certain material has a half life of 25 years, and there are $\boldsymbol{A(t)\ =\ 10(\frac{1}{2})^{t/25}}$ grams remaining after t years. Find the amount after 80 years.

Solution: The initial amount is 10g, as we can read directly off of the formula for $A(t)$.

$A(t)\ =\ A_0(\frac{1}{2})^{t/k}$ (Here k is half life)

$A(t)\ =\ 10(\dfrac{1}{2})^{t/25}$

after 80 years, we have

$A(80)\ =\ 10(\frac{1}{2})^{80/25} \approx\ 1.088$ grams left.

Exercises

1) Find 6^3, 7^{-2}, $4^{\frac{5}{2}}$, $27^{-\frac{4}{3}}$
2) A patient is administered a 1200 μg dose of iodine-131, whose half life is 10 days. How much iodine-131 will be in the patient's system after 20 days?
3) Sketch the graph of $y = 3^{-x}$ $\quad y = e^{2x}$ $\quad y = e^{-x^2}$
4) Suppose you deposit Rs 500 in an account with an annual interest rate of 8% compounded quarterly. Find an equation that gives the amount of money in the account after t years. Then find the amount of money in the account after 5 years.
5) Suppose you deposit Rs 1000 in an account with an annual interest rate of 12% compounded monthly.
   i) Find an equation that gives the amount of money in the account after t years.

ii) Find the amount of money in the account after 5 years.

iii) If the interest were compounded continuously, how much money would the account contain after 5 years?

6) Compute the value of an account of $\$12{,}000$ after four years if the interest rate is 7% and is compounded.

a. Monthly
b. quarterly
c. Semiannually
d. Weekly
e. continuously

7) Compute the interest earned on a CD of $\$1500$ after 18 months if the interest rate is 8% and is compounded monthly.

8) Compute the interest earned on an investment of $\$5000$ after 18 months if the interest rate is 8.25% and is compounded daily.

9) If you had $\$5{,}000$ to invest for 4 years with the goal of greatest return on your investment, would you rather invest in an account paying?

a. 7.2% Compounded quarterly,
b. 7.15% Compounded daily, or.
c. 7.1% Compounded hourly?

1.5. Logarithmic Functions

Learning objectives:

- About the logarithmic functions, common Logarithms, Natural Logarithms, Binary Logarithms and Logarithmic Equations.
- The relationship between the Exponential and Logarithmic functions.
  Solve the problems related to the above concepts.

Logarithmic Functions

Logarithms are related to exponents as follows.

Let b be a positive number and $b \neq 1$ then the logarithm of any positive number x to the baseb, written $\log_b x$ represents the exponent to which b must be raised to obtain x. That is, **If** $y = \log_b x$ **then** $x = b^y$

Accordingly,

$$log_2 8 = 3 \qquad \text{since} \qquad 2^3 = 8$$

$$log_2 64 = 6 \qquad \text{since} \qquad 2^6 = 64$$

$$log_{10} 100 = 2 \qquad \text{since} \qquad 10^2 = 100$$

$$log_{10} 0.001 = -3 \quad \text{since} \qquad 10^{-3} = 0.001$$

Furthermore, for any base b,

$$log_b 1 = 0 \quad \text{since} \qquad b^0 = 1$$

$$log_b b = 1 \quad \text{since} \qquad b^1 = b$$

The logarithm of a negative number and the logarithm of 0 *are not defined.*
Frequently, logarithms are expressed using approximate values. For example, using tables or calculators, one obtains

$$log_{10} 300 = 2.4771 \quad log_e 40 = 3.6889 \qquad (e = 2.718281\cdots)$$

as approximate answers.

The integral part of the logarithm is called the *characteristic*. The fractional part is called the *mantissa*.

Thus 2 and 3 above are the characteristic, while 0.4771 and 0.6889 are the mantissa of the logarithms of the corresponding numbers.

Three classes of logarithms are of special importance: logarithms to base 10, called ***common logarithms***; logarithms to base e, called ***natural logarithms***; and logarithms to base 2, called ***binary logarithms***.
In the initial mathematical work, it is common to use $log\, x$ to mean $log_{10} x$ and $ln\, x$ to mean $log_e x$.
In the advanced mathematical work, the term $log\, x$ is used for $log_e x$.

There are two special identities each of which is a consequence of the definition of a logarithm:

$$\boldsymbol{b^{\log_b x} = x} \qquad \text{and} \quad \boldsymbol{\log_b b^x = x}$$

The first identity simply says that we take logarithm of x first and then exponent ate; whereas the second identity says that we take exponential first and then the logarithm. Evidently, both should yield x since *exponential and logarithms are inverse to each other*. However, they can also be formally proved.

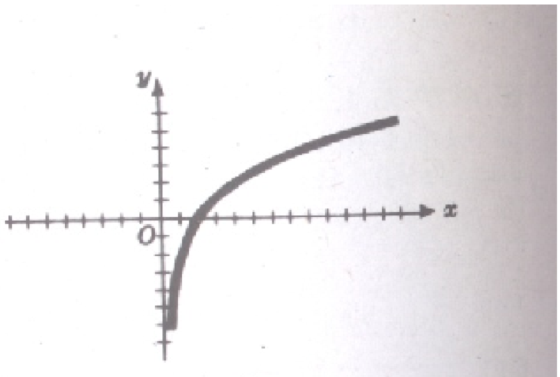
Logarithmic Curve

The curve whose equation is $y = \log_b x, \quad b > 1$

is called a ***logarithmic curve***. The general properties of this curve are that the curve passes through the point $(1,0)$ and the curve lies to the right of the y-axis and has that axis as an asymptote.

Example 1: Sketch the graph of $y = \log_2 x$.

The graph is shown below.



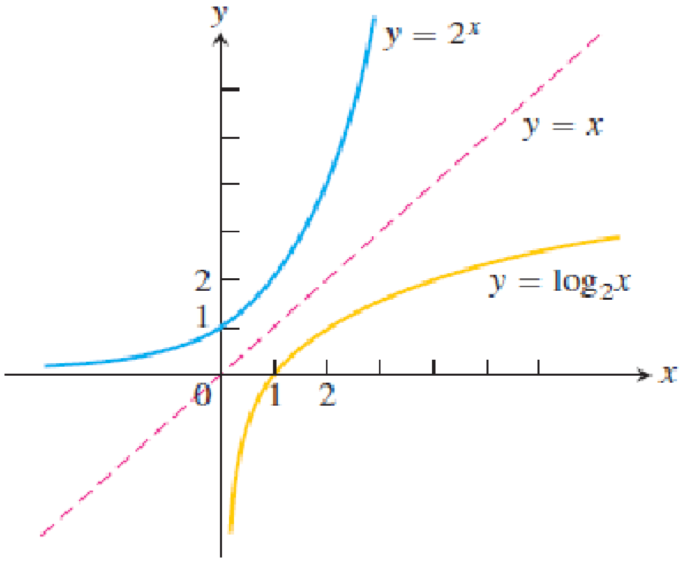
$y = \log_2 x$

Relationship between Exponential and Logarithmic Functions

The basic relationship between the exponential and the logarithmic functions

$$f(x) = b^x \text{ and } g(x) = \log_b x$$

is that they are inverses of each other; hence the graphs of these functions are related geometrically. This relationship is illustrated below where the graphs of the exponential function $f(x) = 2^x$, the logarithmic function $g(x) = \log_2 x$, and the linear function $h(x) = x$ appear on the same coordinate axis. Since $f(x) = 2^x$ and $g(x) = \log_2 x$ are inverse functions, they are symmetric with respect to the linear function $h(x) = x$ or, in other words, the line $y = x$.



From this figure, we notice an important property of the exponential and logarithmic functions. For any positive c, we have

$$g(c) < h(c) < f(c)$$

As c increases in value, the vertical distances $h(c) - g(c)$ and $f(c) - g(c)$ increase in value. Also, the logarithmic function $g(x)$ grows very slowly compared with the linear function $h(x)$, and the exponential function $f(x)$ grows very quickly compared with $h(x)$.

Logarithmic Equation

When finding the logarithm of an expression, the expression is called the ***argument*** of the logarithm. For example, in $log\ 3$, 3 is the argument, and in $\log(2x+4)$, the $2x + 4$ is the argument.

A *logarithmic equation* is one in which a variable appears in the argument of some logarithm.

Some logarithmic equations can be solved by expressing the equation in exponential form.

***Example 2*: Solve the equation** $\log_2 (x+1)^3 = 4$

***Solution*:** Writing the equation in exponential form gives

$$(x+1)^3 = 2^4$$

$$(x+1)^3 = 16, \qquad x+1 = \sqrt[3]{16}, \qquad x = -1 + \sqrt[3]{16}$$

Several logarithmic equations can be solved by using the properties of the logarithms.($we\ have\ studied\ the\ properties\ of\ logarithms\ in\ earlier\ classes$).

***Example 3*: Solve the equation** $\log(3x+2) + \log 9 = \log(x+5)$

***Solution*:**

$$\log(3x+2)+\log 9=\log(x+5)$$
$$\Rightarrow \log 9(3x+2)=\log(x+5)$$
$$\Rightarrow 9(3x+2)=(x+5)$$
$$\Rightarrow 27x+18=x+5$$
$$\Rightarrow 26x=-13$$
$$\Rightarrow x=-\frac{1}{2}$$

The solution to a logarithmic equation should be checked since it is possible to introduce extraneous solutions. In the present case, a check shows that the solution is correct.

Example 4: **Solve the equation** $\log x+\log(x+1)=\log 12$

Solution:

$$\log x+\log(x+1)=\log 12$$
$$\Rightarrow \log x(x+1)=\log 12$$
$$\Rightarrow x(x+1)=12$$
$$\Rightarrow x^2+x-12=0$$
$$\Rightarrow (x+4)(x-3)=0$$
$$\Rightarrow x=-4 \qquad \text{or} \qquad x=3$$

A check for $x=3$ shows that it is a correct solution.

Checking for $= -4$, we have

$$\log(-4)+\log(-3)=\log 12$$

Logarithm of negative numbers is not defined. Thus, -4 is an extraneous solution.

PROBLEM SET

IP1: If $\frac{\ln x}{b-c}=\frac{\ln y}{c-a}=\frac{\ln z}{a-b}$ **then** $x^{b+c}.y^{c+a}.z^{a+b}=$

Solution:

Step 1: Given that $\frac{\ln x}{b-c}=\frac{\ln y}{c-a}=\frac{\ln z}{a-b}=k\ (say)$

$\Longrightarrow\ \ln x=k(b-c)$, $\ln y=k(c-a)$, $\ln z=k(a-b)$

Step2: Now,

$x = e^{k(b-c)}$, $y = e^{k(c-a)}$, $z = e^{k(a-b)}$

Now,

$$x^{b+c}.y^{c+a}.z^{a+b} = e^{k(b-c)(b+c)}.e^{k(c-a)(c+a)}.e^{k(a-b)(a+b)}$$
$$= e^{k(b^2-c^2+c^2-a^2+a^2-b^2)}$$
$$= e^0 = 1$$

P1: If $\frac{\ln x}{b-c} = \frac{\ln y}{c-a} = \frac{\ln z}{a-b}$ then $x^a.y^b.z^c =$

Solution:

Given that $\frac{\ln x}{b-c} = \frac{\ln y}{c-a} = \frac{\ln z}{a-b} = k(say)$

$\Rightarrow$ $ln\, x = k(b-c),\ ln\, y = k(c-a),\ ln\, z = k(a-b),$

Now,

$a\, ln\, x + \ b\, ln\, y + c\ ln\, z$

$= k.[ab - ac + bc - ba + ca - cb]$

$= k(0) = 0$

$i.e.,\ a\, ln\, x + \ b\, ln\, y + c\ ln\, z = 0$

$\Rightarrow ln\, x^a + ln\, y^b + ln\, z^c = 0 \Rightarrow ln\, x^a y^b z^c = 0$

$\Rightarrow x^a.y^b.z^c = e^0$

$\Rightarrow x^a.y^b.z^c = 1$

IP2: $\log_{\sqrt{x}} \sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}} =$

Solution:

Step1: We have to compute the value of $\log_{\sqrt{x}} \sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}}$

Step2:

$$\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}} = x^{\frac{1}{2}}.\ x^{\frac{1}{4}}.x^{\frac{1}{8}}.\ x^{\frac{1}{16}}$$
$$= x^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}} = x^{\frac{15}{16}}$$

Step3: $\log_{\sqrt{x}} \sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}} = \log_{\sqrt{x}} x^{\frac{15}{16}}$

$$= \log_{\sqrt{x}}\left(\sqrt{x}\right)^{\frac{15}{8}} = \frac{15}{8}\log_{\sqrt{x}}\sqrt{x} = \frac{15}{8}$$

Step4: Hence $\log_{\sqrt{x}} \sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}} = \frac{15}{8}$

P2: The value of $\log_{\sqrt{2}}\sqrt{4\sqrt{16\sqrt{64\sqrt{256}}}} =$

Solution:

Given that $log_{\sqrt{2}}\sqrt{4\sqrt{16\sqrt{64\sqrt{256}}}}$

$$\sqrt{4\sqrt{16\sqrt{64\sqrt{256}}}} = 4^{\frac{1}{2}}.(16)^{\frac{1}{4}}.(64)^{\frac{1}{8}}.(256)^{\frac{1}{16}}$$

$$= 2^1.2^1.2^{\frac{3}{4}}.2^{\frac{1}{2}}$$

$$= 2^{2+\frac{3}{4}+\frac{1}{2}} = 2^{\frac{13}{4}} = (\sqrt{2})^{\frac{13}{2}}$$

Now, $\log_{\sqrt{2}}\sqrt{4\sqrt{16\sqrt{64\sqrt{256}}}} = \log_{\sqrt{2}}(\sqrt{2})^{\frac{13}{2}} = \frac{13}{2}$

IP3: $\log_a\left(1-\frac{1}{2}\right) + \log_a\left(1-\frac{1}{3}\right) + \log_a\left(1-\frac{1}{4}\right) + \cdots + \log_a\left(1-\frac{1}{a}\right) =$

Solution:

$$\log_a\left(1-\frac{1}{2}\right) + \log_a\left(1-\frac{1}{3}\right) + \log_a\left(1-\frac{1}{4}\right) + \cdots + \log_a\left(1-\frac{1}{a}\right)$$

$$= \log_a\left(\frac{1}{2}\right) + \log_a\left(\frac{2}{3}\right) + \log_a\left(\frac{3}{4}\right) + \cdots + \log_a\left(\frac{a-1}{a}\right)$$

$$= \log_a\left(\frac{1}{2}\ .\frac{2}{3}.\ \frac{3}{4}\dots\dots\dots\frac{a-1}{a}\right)$$

$$= \log_a\frac{1}{a} = \log_a a^{-1} = -1$$

P3: If $\log_a\left(1+\frac{1}{3}\right) + \log_a\left(1+\frac{1}{4}\right) + \log_a\left(1+\frac{1}{5}\right) + \cdots + \log_a\left(1+\frac{1}{242}\right) = 4$ then the value of $a =$

Solution:

Given that

$$\log_a\left(1+\frac{1}{3}\right) + \log_a\left(1+\frac{1}{4}\right) + \log_a\left(1+\frac{1}{5}\right) + \cdots + \log_a\left(1+\frac{1}{242}\right) = 4$$

$$\Rightarrow \log_a\left(\frac{4}{3}\right) + \log_a\left(\frac{5}{4}\right) + \log_a\left(\frac{6}{5}\right) + \cdots + \log_a\left(\frac{243}{242}\right) = 4$$

$$\Rightarrow \log_a\left(\frac{4}{3}\ .\frac{5}{4}.\ \frac{6}{5}\dots\dots\dots\frac{243}{242}\right) = 4$$

$$\Rightarrow \log_a\frac{243}{3} = 4$$

$$\Rightarrow \log_a 81 = 4$$

$$\Rightarrow \log_a 3^4 = 4$$

$$\Rightarrow 4\log_a 3 = 4$$

$$\Rightarrow \log_a 3 = 1 \Rightarrow 3 = a^1 \Rightarrow a = 3$$

IP4: If $\log_2(3^{2x-2}+7) = 2+\log_2(3^{x-1}+1)$ then the value of x is

Solution:

Step1:

Given that $\log_2(3^{2x-2}+7) = 2+\{\log_2(3^{x-1}+1)\}$

$\Longrightarrow \log_2(3^{2x-2}+7) = \log_2 4+\{\log_2(3^{x-1}+1)\}$

$\Longrightarrow \log_2(3^{2x-2}+7) = \log_2 4(3^{x-1}+1)$

$\Longrightarrow (3^{x-1})^2+7 = 4(3^{x-1}+1)$

Step2:

Put $3^{x-1} = a$ then we have

$\Longrightarrow a^2+7 = 4(a+1)$

$\Longrightarrow a^2-4a+3 = 0$

$\Longrightarrow (a-3)(a-1) = 0$

$\Longrightarrow a = 1,\ \ a = 3$

Step3:

$i.e., 3^{x-1} = 1$ and $3^{x-1} = 3$

$3^x = 3$ and $3^x = 9 = 3^2$

Hence $x = 1$ and $x = 2$

P4: If $\log_2(5.2^x+1)+\log_2 2 = \log_2(2^{1-x}+1)$ then the value of x is

Solution:

$\log_2(5.2^x+1)+\log_2 2 = \log_2(2^{1-x}+1)$

$\Longrightarrow \log_2(5.2^x+1)\,2 = \log_2(2^{1-x}+1)$

$\Longrightarrow 2(5.2^x+1) = 2^{\log_2(2^{1-x}+1)}$

$\Longrightarrow 10.2^x+2 = (2^{1-x}+1)^{\log_2 2} = 2^{1-x}+1$

$\Longrightarrow 10.2^x+2 = \frac{2^1}{2^x}+1$

Put $2^x = a$

$\Longrightarrow 10a+1 = \frac{2}{a}$

$\Longrightarrow 10a^2+a-2 = 0$

$\Longrightarrow (5a-2)(2a+1) = 0$

$\Longrightarrow a = \frac{2}{5}, -\frac{1}{2}$

$\Longrightarrow 2^x = \frac{2}{5}$ $(\because 2^x > 0)$

$\Longrightarrow x = \log_2\left(\frac{2}{5}\right)$

EXERCISES

1. Prove that $b^{\log_b x} = x$ and $\log_b b^x = x$.

2. Solve the equation $\log(3x-5)-\log 5x = 1.23$

3. Solve the equation $\log(2x-4)^3 = .648$

4. If $\log_a[1+\log_b\{1+\log_c x\}] = 0\ find\ x.$

5. Find the value of $log_x x + log_x x^3 + log_x x^5 + \cdots + log_x x^{2n-1}$

6. Find the value of

7. $\log_3\left(1+\frac{1}{3}\right) + \log_3\left(1+\frac{1}{4}\right) + \cdots + \log_3\left(1+\frac{1}{80}\right)$

8. Find the value of xyz if $\frac{log\,x}{y-z} = \frac{log\,y}{z-x} = \frac{log\,z}{x-y}$

9. If $x^2+y^2 = 6xy$, Prove that

$$2\,log(x+y) = log\,x + log\,y + 3\,log\,2.$$

10. If $(x+y)^2 = 125xy\ show\ that$

$$2\log(x+y) = 3\log 5 + \log x + \log y$$

11. If $\frac{log_2 a}{4} = \frac{log_2 b}{6} = \frac{log_2 c}{3p}$ and $a^3b^2c = 1$ find the value of p.

12. If $x = 1 + log_a bc\,, y = 1 + log_b ca\,, z = 1 + log_c ab$ then show that $xyz = xy + yz + zx$.

13. Solve $2\log_e 2 + \log_e(2x^2-6x+5) = \log_e(2x-5)$

14. If $\frac{log\,x}{log\,3} = \frac{log\,25}{log\,9}$ then the value of x is

15. Find the value of x if $log_4(x^2+x) - log_4(x+1) = 2$

16. If $2log_e(a+b) + log_e(a-b) - log_e(a^2-b^2) = log_e x$
Then find x.

1.6. Graphs of Functions

Learning Objectives

- To learn the concepts of solution, solution set and the graph of an equation in variables x and y
- To define x and y intercepts of a graph and to learn vertical line test for the graph of a function
- To study the graphs of
  - i. Absolute value function
  - ii. Greatest and least integer functions
  - iii. Power functions and
  - iv. Circles and Parabolas
- To study the concept of shifting a graph

Graphs

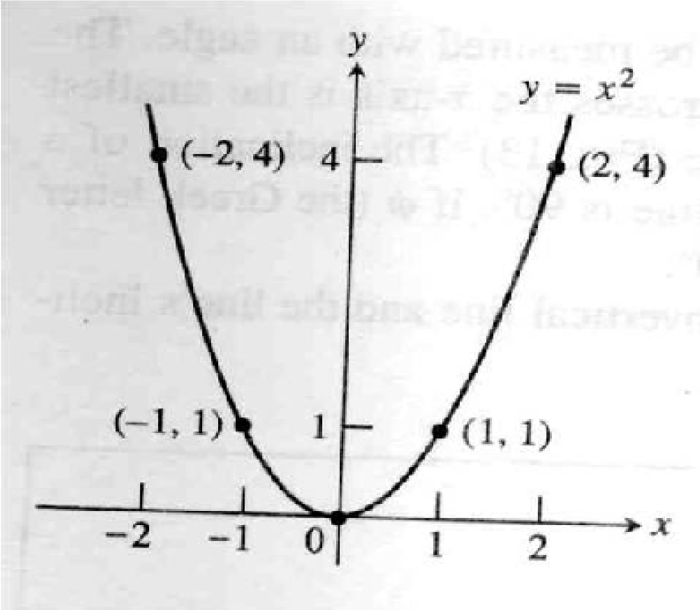
The correspondence between points in a plane and ordered pairs of real numbers will enable the visualization of algebraic equations as geometric curves, and, conversely, to represent geometric curves by algebraic equations.

Suppose that we have a xy-coordinate system and an equation involving two variables x and y, say $6x - 4y = 10$. We define a ***solution*** of such an equation to be any ordered pair of real numbers (a, b) whose coordinates satisfy the equation when we substitute $x = a$ and $y = b$. For example, the ordered pair $(3, 2)$ is a solution of the equation $6x - 4y = 10$, since the equation is satisfied by $x = 3$ and $y = 2$. However, the ordered pair $(2, 0)$ is not a solution of this equation, since the equation is not satisfied by $x = 2$ and $y = 0$.

A solution of an equation involving two variables x and y is an ordered pair of real numbers (a, b) whose coordinates satisfy the equation when a, b are substituted for x and y respectively.

The set of all solutions of an equation in x and y is called the solution set of the equation.

The graph of an equation or inequality involving the variables x and y is the set of all points $P\ (x, y)$ whose coordinates satisfy the equation or inequality.



The parabola $y = x^2$.

The above figure is the graph of the equation $y = x^2$. Some points whose coordinates satisfy this equation are $(0, 0), (1, 1), (-1, 1), (2, 4),$ and $(-2, 4)$. These points, and all others satisfying the equation, make up a smooth curve called a ***parabola***.

A graph intersects the x-axis at a point which has the form $(a, 0)$ and the y-axis at a point which has the form $(0, b)$. The number *a* is called an ***x-intercept*** of the graph and the number *b* is called an ***y-intercept***.

The vertical line Test:

A function *f* can have only one value $f(x)$ for each x in its domain, so no vertical line can intersect the graph of a function more than once. If *a* is in the domain of a function *f*, then the vertical line $x = a$ will intersect the graph of *f* in the single point $(a, f(a))$.

The graph of a function f is the graph of the equation $y = f(x)$. It consists of the points in the plane whose coordinates (x, y) are input-output pairs for f. The graph of a function can be obtained by plotting several coordinate pairs that satisfy the functional rule and joining them by a smooth curve.

Example 1:

Find intercepts of $3x + 2y = 6$

Solution

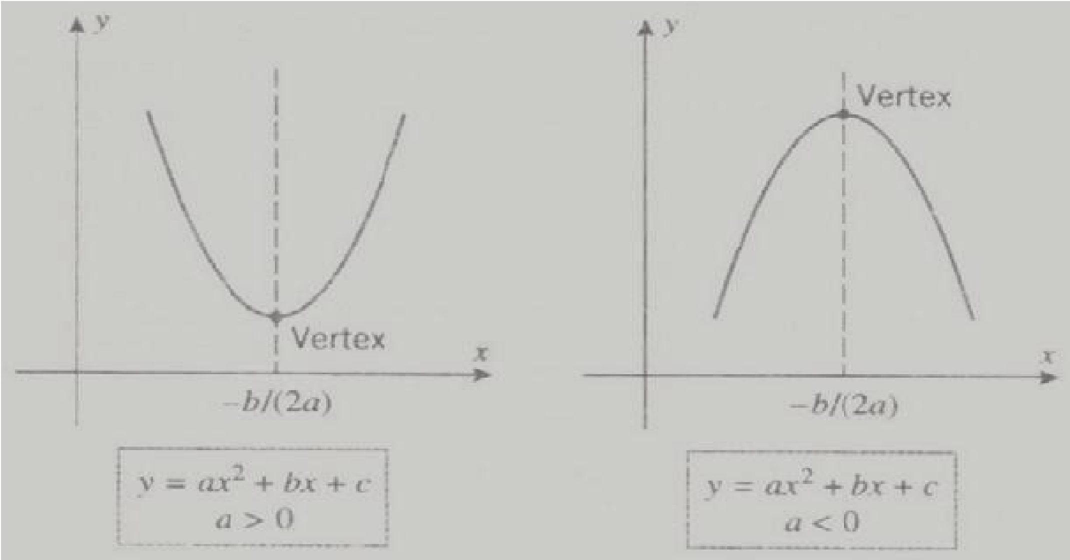
To find the x —intercept, set $y = 0$ and solve for x: $3x = 6$ or $x = 2$

To find the y-intercept, set $x = 0$ and

solve for y: $2y = 6 \; or \; y = 3$

The Graph of $y = ax^2 + bx + c$

An equation of the form $y = ax^2 + bx + c, (a \neq 0)$ is called a ***quadratic equation in x***. Depending on whether *a* is positive or negative, the graph has one of the two forms shown below.



In both cases, the parabola is symmetric about a vertical line parallel to the $y-axis$. This line of symmetry cuts the parabola at a point called the ***vertex***. The vertex is the low point on the curve if *a* > 0 and the high point if *a* < 0. The quadratic equation can be written as

$$y = a\left(x^2 + \frac{b}{a}x\right) + c = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - \frac{b^2}{4a}$$

$$= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

The *x*-coordinate of the vertex is given by $x = -\frac{b}{2a}$.

Example 2: **Sketch the graph of** $y = x^2 - 2x - 2$

Solution:

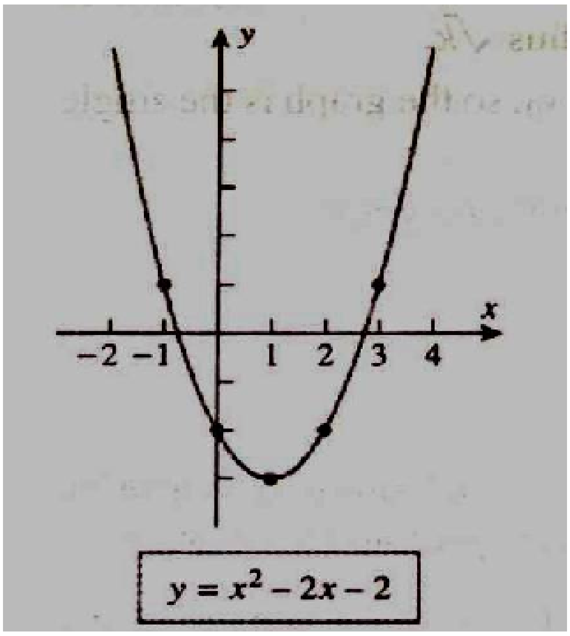
$$x = -\frac{-2}{2 \times 1} = 1, \quad y = -3, \quad \text{(coordinate s of the vertex)}$$

$$\text{y - intercept } = \text{- } 2$$

$$\text{x - intercepts } : x^2 - 2x - 2 = 0$$

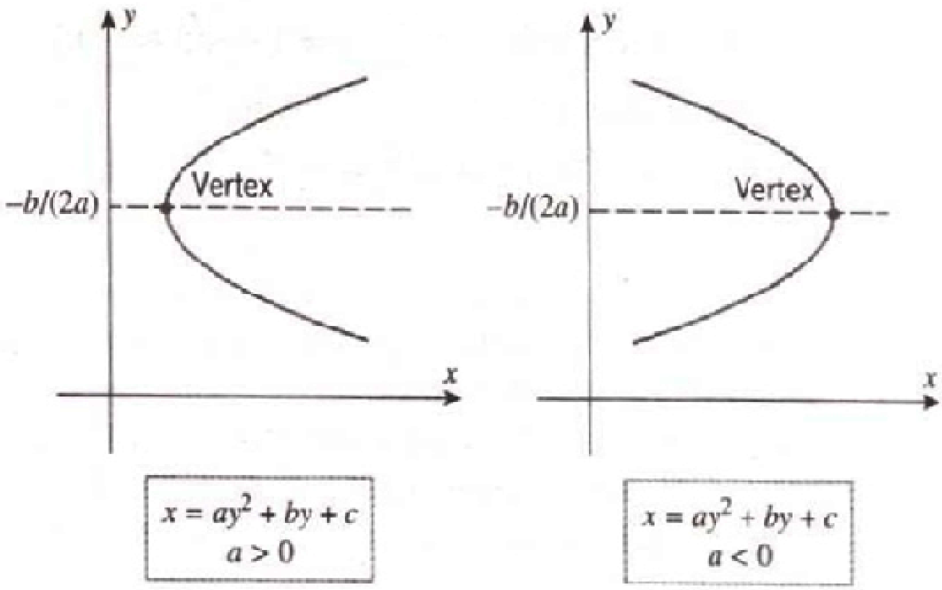
$$x = \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 \pm \sqrt{12}}{2} = 2.7, -0.7$$

The graph is shown below.



The graph of $x = ay^2 + by + c$

An equation of the form $x = ay^2 + by + c,\ (a \neq 0)$ is called a ***quadratic equation in y***. The graph of such an equation is a parabola with its line of symmetry parallel to the x-axis and its vertex at the point with y-coordinate $y = -\frac{b}{2a}$



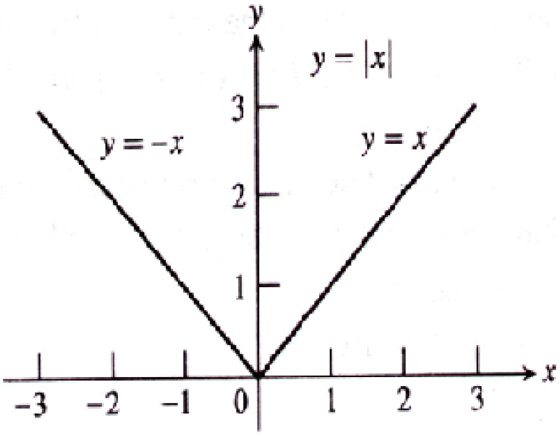
Absolute Value Function

The absolute value of the real number x, written $|x|$, is defined as x or $-x$, depending on whether x is positive or negative.

Thus,

$$|-15| = 15,\ |7| = 7,\ |-3.33| = 3.33,\ |4.44| = 4.44,\ |-0.975| = 0.975$$

We note that $|x| = |-x|$ and for $x \neq 0, |x|$ is positive. The graph of the absolute value function is shown below.



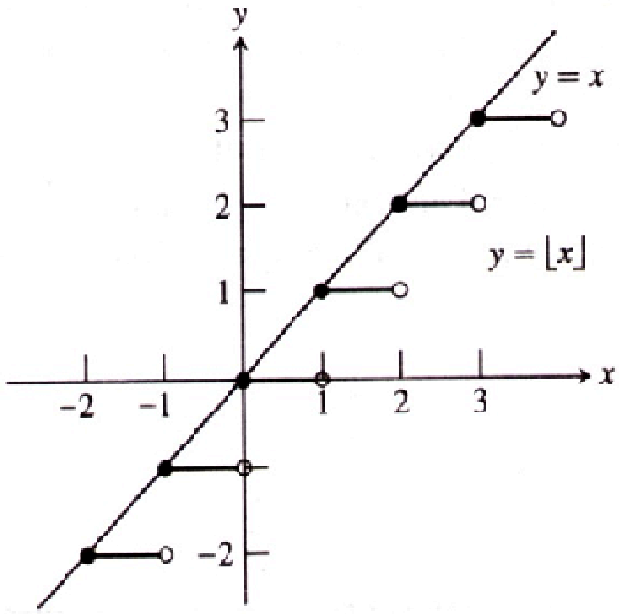
Greatest and Least Integer Functions

The function whose value, at any number x, is the greatest integer less than or equal to x is called the greatest integer function or the integer floor function. It is denoted by $\lfloor x \rfloor$. It is read as floor of x.

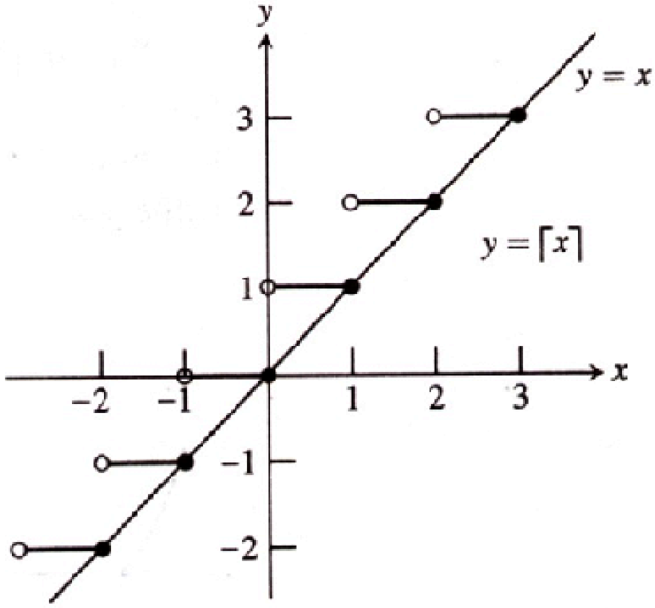
$$\lfloor 2.4 \rfloor = 2, \qquad \lfloor 1.9 \rfloor = 1, \qquad \lfloor 0 \rfloor = 0, \qquad \lfloor -1.2 \rfloor = -2$$

$$\lfloor 2 \rfloor = 2, \qquad \lfloor 0.2 \rfloor = 0, \qquad \lfloor -0.3 \rfloor = -1, \qquad \lfloor -2 \rfloor = -2$$

The graph of the greatest integer function is shown below.



The function whose value, at any number x, is the smallest integer greater than or equal to x is called the least integer function or the integer ceiling function. It is denoted by $\lceil x \rceil$. It is read as ceiling of x.



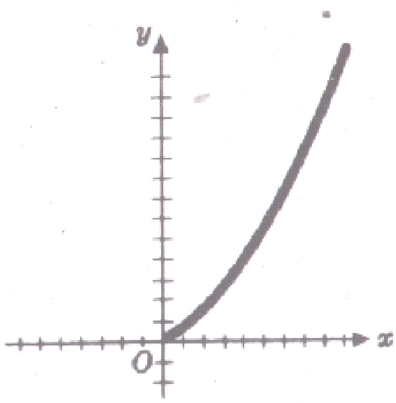
Power Functions

Power functions in x are of the form $f(x) = x^n$.

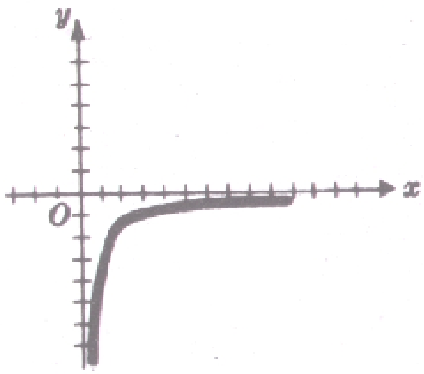
If $n > 0$, the graph of $y = x^n$ is said to be of the parabolic type. The curve is a parabola for $n = 2$. *If $n < 0$, the graph of $y = x^n$ is said to be of the hyperbolic type.* The curve is a hyperbola for $n = -1$.

Example: Sketch the graphs of $y = x^{3/2}$, $y = -x^{-\frac{3}{2}}$

We compute the values of y for selected values of x. We plot these points and draw a smooth curve through them.



$y = x^{3/2}$



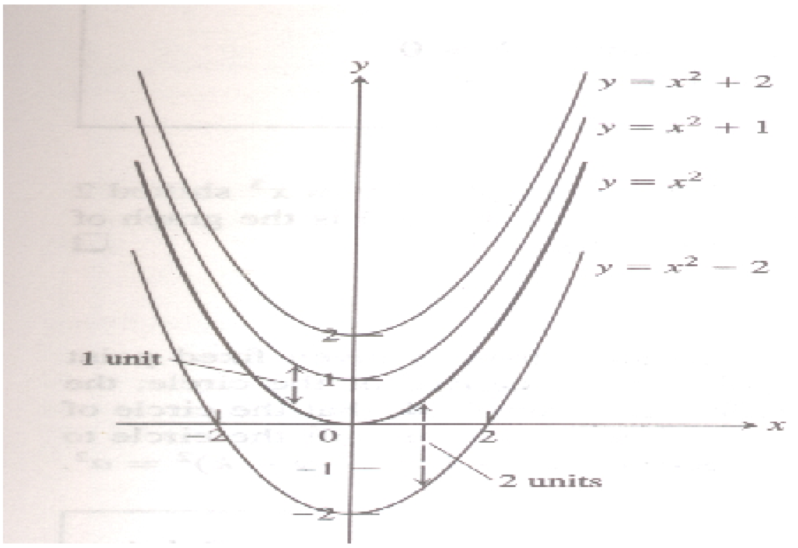
$y = -x^{-3/2}$

Shifting a Graph

The equation of a graph changes when we shift the graph up or down or to the right or left. *To shift the graph of a function $y = f(x)$ straight up, we add a positive constant to the right-hand side of the formula $y = f(x)$.*

Example

Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2+1$ shifts the graph up 1.



To shift the graph of a function $y = f(x)$ straight down, we add a negative constant to the right-hand side of the formula $y = f(x)$.

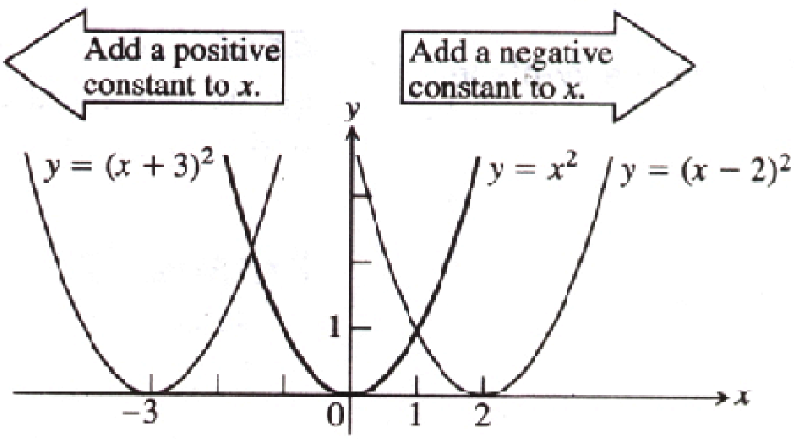
Example 3:

Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units.

To shift the graph of $y = f(x)$ to the left, we add a positive constant to x.

Example 4:

Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left.



To shift the graph of $y = f(x)$ to the right, we add a negative constant to x.

Example 5:

Adding -2 to x in $y = x^2$ to get $y = (x - 2)^2$ shifts the graph 2 units to the right.

The Shift Formulas are summarized as follows:

Vertical Shifts

$y-k=f(x)$ **Shifts the graph up k units if $k>0$**

Shifts the graph down $|k|$ units if $k<0$

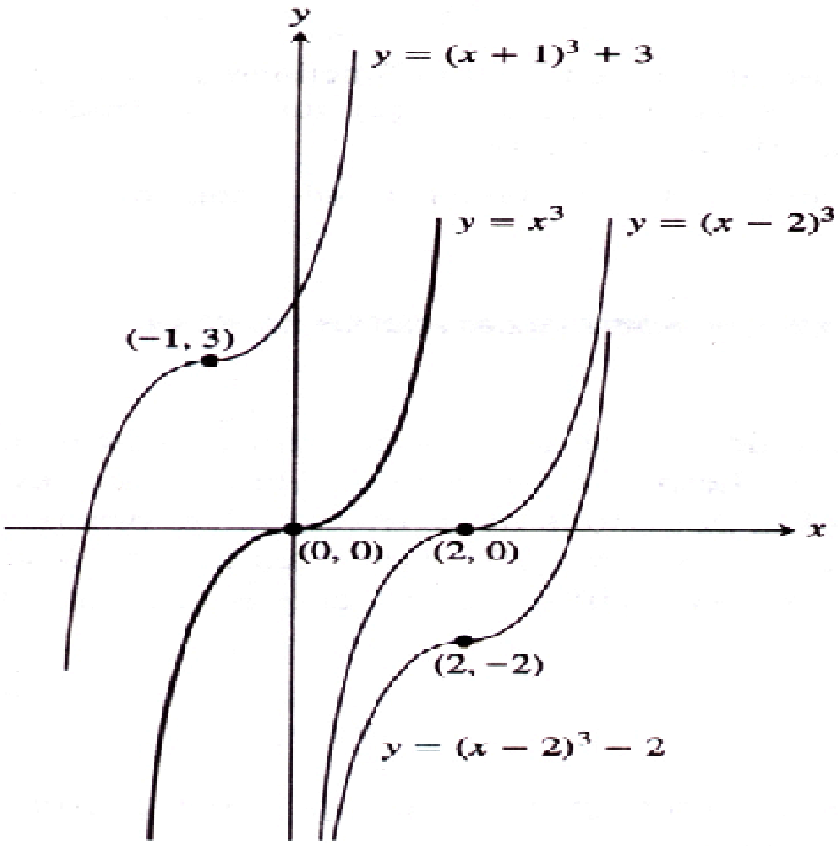
Horizantal Shifts

$y=f(x-h)$ **Shifts the graph right h units if $h>0$**

Shifts the graph left $|h|$ units if $h<0$

Example 6:

The graph of $y=(x-2)^3-2$ is the graph of $y=x^3$ shifted 2 units to the right and 2 units down. The graph of $y=(x+1)^3+3$ is the graph of $y=x^3$ shifted 1 unit to the left and 3 units up.

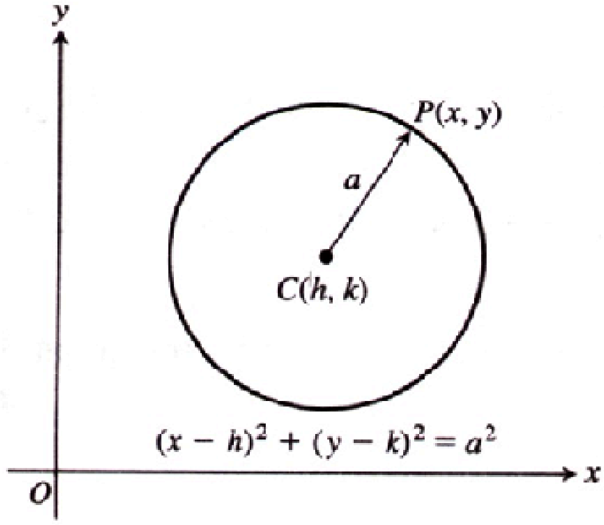


Graph of a Circle

A circle is the set of points in a plane whose distance from a given fixed point in the plane is constant.

The fixed point is the ***center*** *of the circle; the constant distance is the* ***radius***. The circle of radius *a* centered at the origin has equation $x^2+y^2=a^2$. If we shift the circle to place its center at the point (h,k), the equation of the circle is given by

$$(x-h)^2+(y-k)^2=a^2$$



Example 7:

The standard equation for the circle of radius 2 centered at $(3,4)$ is

$$(x-3)^2+(y-4)^2=(2)^2$$

$$(x-3)^2+(y-4)^2=4$$

Example 8:

Find the center and radius of the circle $(x-1)^2+(y+5)^2=3$

Solution

$$\text{The center is the point } (h,k)=(1,-5)$$

$$\text{radius } a=\sqrt{3}$$

Example 9:

Find the center and radius of the circle $x^2+y^2+4x-6y-3=0$

Solution

We convert the equation to standard form by completing the squares in x and y.

$$\left(\boldsymbol{x}^2+4\boldsymbol{x}+\left(\frac{4}{2}\right)^2+\left(\boldsymbol{y}^2-6\boldsymbol{y}+\left(\frac{-6}{2}\right)^2\right)\right)=3+\left(\frac{4}{2}\right)^2+\left(\frac{-6}{2}\right)^2$$

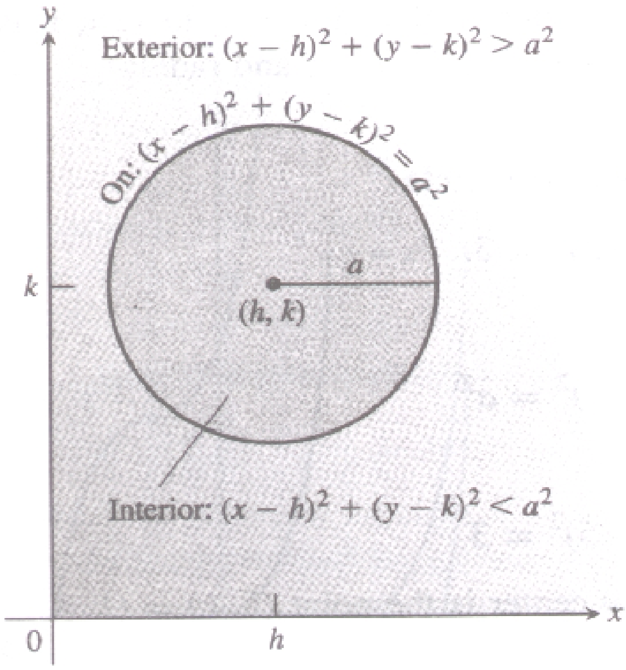
$$(\boldsymbol{x}+2)^2+(\boldsymbol{y}-3)^2=3+4+9=16$$

$$\boldsymbol{center},(\boldsymbol{h},\boldsymbol{k})=(-2,3);\quad \boldsymbol{radius},\ \boldsymbol{a}=4$$

The points that lie inside the circle $(x-h)^2+(y-k)^2=a^2$ are the points whose distance less than a units from (h,k). They satisfy the inequality $(x-h)^2+(y-k)^2<a^2$. They make up the region we call the **interior** of the circle.

The circle's ***exterior*** consists of the points that lie more than a units from *(h,k)*.

These points satisfy the inequality $(x-h)^2+(y-k)^2>a^2$

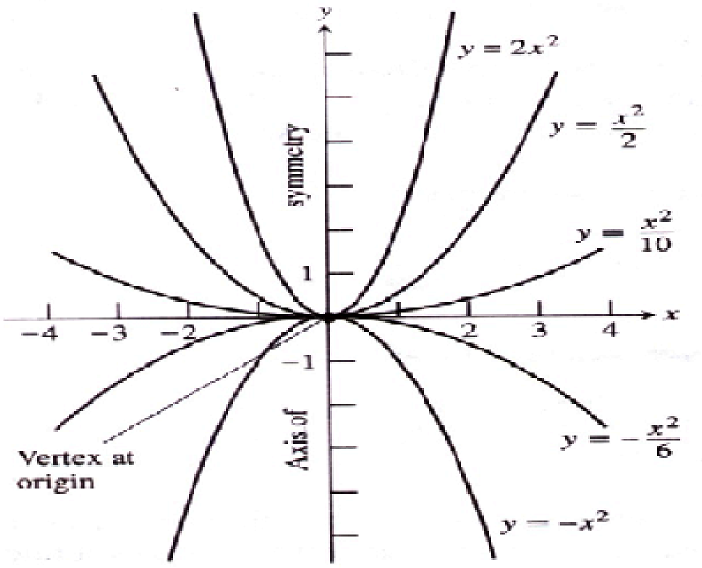


Example

Inequality	Region
$x^2+y^2<1$	**Interior of the unit circle.**
$x^2+y^2\le 1$	**Unit circle plus its interior.**
$x^2+y^2>1$	**Exterior of the unit circle.**
$x^2+y^2\ge 1$	**Unit circle plus its exterior.**

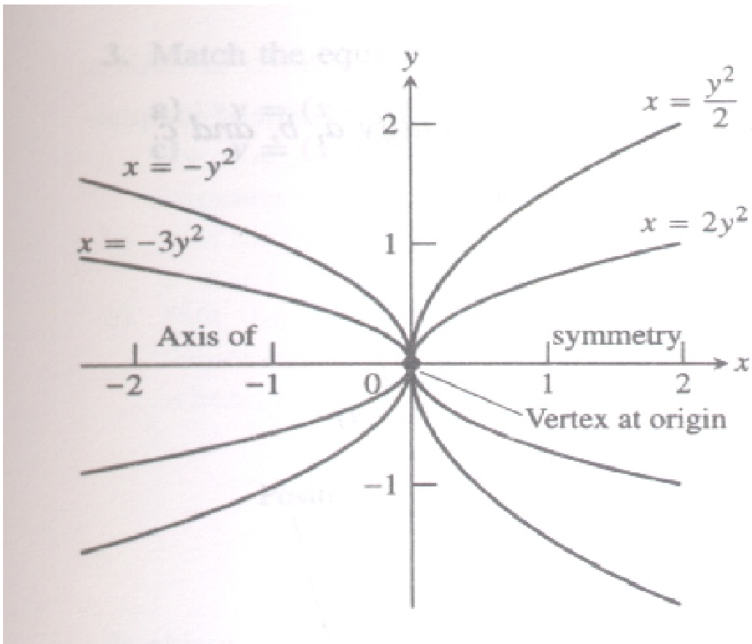
Graph of a Parabola

The graph of an equation (like $y=3x^2$ or $y=-5x^2$) *that has the form* $y=ax^2$ *is a **parabola** whose **axis** (axis of symmetry) is the y-axis.* The parabola's vertex (point where the parabola and axis cross) lies at the origin. The parabola opens upward if *a > 0* and downward if *a < 0*. The larger the value of |a|, the narrower the parabola.



If we interchange x and y in the formula $y = ax^2$, we obtain the equation $x = ay^2$

The graph of this equation is a parabola whose axis is the x −axis and whose vertex lies at the origin.

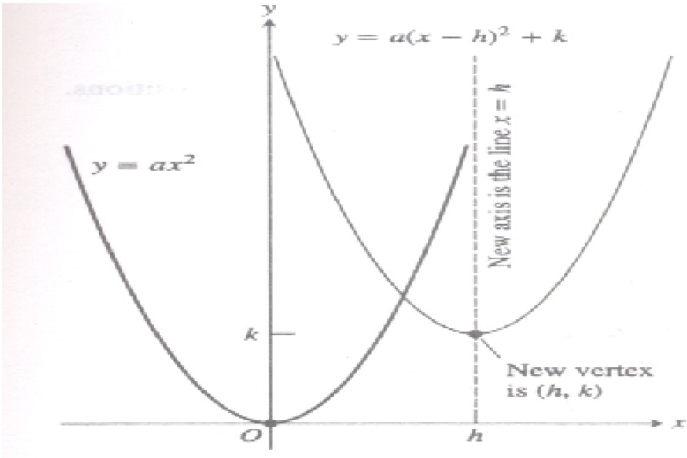


To shift the parabola $y = ax^2$ horizontally, we rewrite the equation as $y = a(x-h)^2$

To shift it vertically as well, we change the equation to

$$y - k = a(x-h)^2$$

The combined shifts place the vertex at the point (h, k) and the axis along the line $x = h$.



When the right-hand side is multiplied, the rearranged equation takes the form
$y = ax^2 + bx + c$
This tells us that *the graph of every equation of the form* $y = ax^2 + bx + c, a \neq 0$*, is the graph of* $y = ax^2$ *shifted to some position. The curve* $y = ax^2 + bx + c$ *has the same shape and orientation as the curve* $y = ax^2$.
Thus, *the graph of the equation* $y = ax^2 + bx + c, a \neq 0$*, is a parabola. The parabola opens upward if a >0 and downward if a < 0. The axis is the line* $x = -\frac{b}{2a}$
The vertex of the parabola is the point where the axis and parabola intersect. Its x-coordinate is $x = -\frac{b}{2a}$; y-coordinate is found by substituting $x = -\frac{b}{2a}$ in the parabola's equation.

Example 10:

Graph the equation $y = -\frac{1}{2}x^2 - x + 4$

Solution

$a = -\frac{1}{2}$ the graph opens down

$b = -1,\ c = 4$

The axis is the line $x = -\frac{b}{2a} = -\frac{(-1)}{2(-\frac{1}{2})} = -1$

The x - coordinate of the vertex is - 1. The y - cordinate is

$$y = -\frac{1}{2}(-1)^2 - (-1) + 4 = \frac{9}{2}$$

The vertex is $\left(-1, \dfrac{9}{2}\right)$

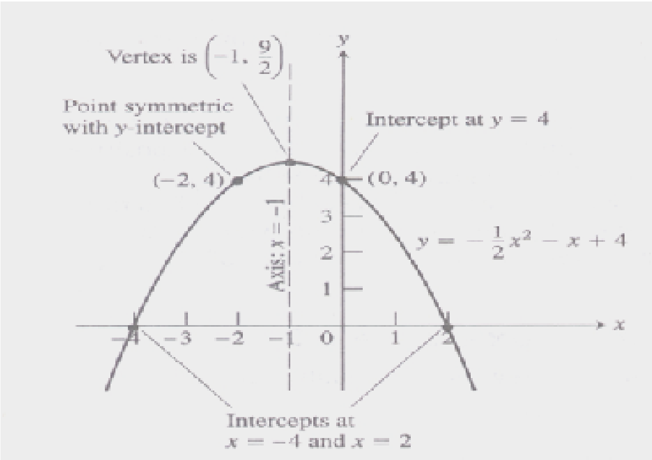
We find the x - intercepts : $-\dfrac{1}{2}x^2 - x + 4 = 0$

$$x^2 + 2x - 8 = 0$$

$$(x-2)(x+4) = 0$$

$$x = 2, \quad x = -4$$

The graph is sketched below.



PROBLEM SET

IP1: Draw the graph of the function $f: R^+ \to R$ defined by $f(x) = \sqrt{x}$

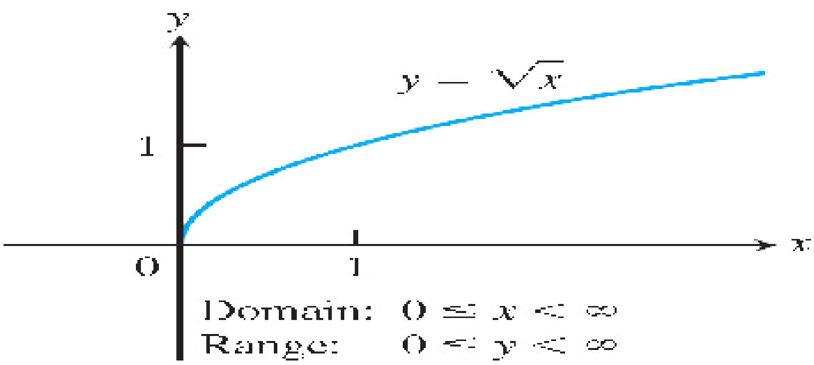
Solution:

Clearly, domain of the function f is R^+ i.e., $x \in [0, \infty)$ and its range is also $[0, \infty)$

For some values of x, we calculate the approximate values of the function f.

x	0	1	2	3	4	5	6	7	8	9
$f(x) = \sqrt{x}$	0	1	1.41	1.73	2	2.24	2.45	2.65	2.83	3

Graph:



P1: Sketch the graph of the function $f: R \to R$ defined by $f(x) = \sqrt{9 - x}$

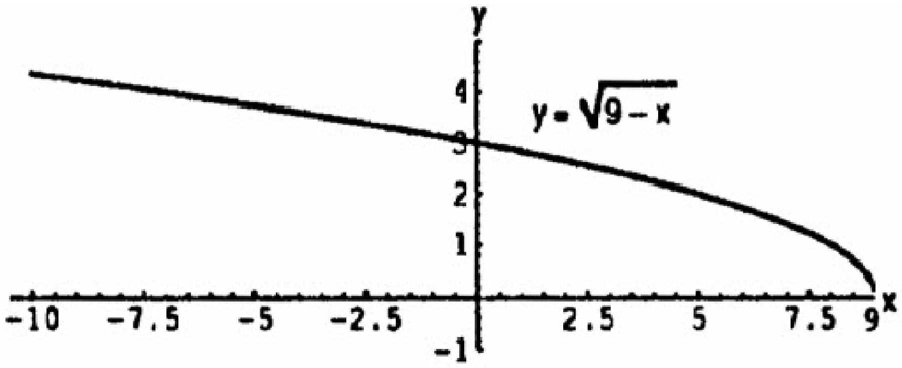
Solution: Clearly, domain of the function f is given by

$x \in R$, $9 - x \ge 0 \Rightarrow x \le 9$ i.e., $(-\infty, 9]$ and its range is $[0, \infty)$.

For some selected values of x, we get the following values for $f(x)$.

x	-16	-7	0	5	8	9
$\sqrt{9-x}$	5	4	3	2	1	0

Graph:



IP2: If any decimal value of x can be written as $x=\{x\}+\lfloor x\rfloor$ here $\lfloor x\rfloor$ is floor value of x and $\{x\}$ is decimal value of x and given $\{y\}+\lfloor z\rfloor=\mathbf{84.45}$, $\lfloor x\rfloor+\{z\}=\mathbf{94.35}$ and $\{x\}+\lfloor y\rfloor=\mathbf{105.65}$, then find $x-y+z$.

Solution :

We have , $\{y\}+\lfloor z\rfloor=84.45$ ... (1)

$\lfloor x\rfloor+\{z\}=94.35$ ... (2)

$\{x\}+\lfloor y\rfloor=105.65$ ... (3)

Step1:

$\therefore 84=\lfloor 84.45\rfloor=\lfloor\{y\}+\lfloor z\rfloor\rfloor=\lfloor z\rfloor$ ... (4)

$94=\lfloor 94.35\rfloor=\lfloor\lfloor x\rfloor+\{z\}\rfloor=\lfloor x\rfloor$ ... (5)

$105=\lfloor 105.65\rfloor=\lfloor\{x\}+\lfloor y\rfloor\rfloor=\lfloor y\rfloor$ ... (6)

Step2:

From (1) and (4), we get $\{y\}=0.45$

From (2) and (5), we get $\{z\}=0.35$

From (3) and (6), we get $\{x\}=0.65$

Step3:

Since, $x=\{x\}+\lfloor x\rfloor=0.65+94=94.65$

Like the above, we calculate

$y=\{y\}+\lfloor y\rfloor=0.45+105=105.45$

And $z=\{z\}+\lfloor z\rfloor=0.35+84=84.35$

Step4:

$$\therefore x-y+z=94.65-105.45+84.35=73.55$$

P2:

For all real numbers x, $x=\lfloor x\rfloor+\{x\}$, here $\lfloor x\rfloor$ is floor value of x and $\{x\}$ is decimal value of x (similarly for y and z). Find y, if $x+\lfloor y\rfloor+\{z\}=\mathbf{300}$;

$$\{\boldsymbol{x}\} + \boldsymbol{y} + \lfloor \boldsymbol{z} \rfloor = \mathbf{270.2};$$

and $$\lfloor \boldsymbol{x} \rfloor + \{\boldsymbol{y}\} + \boldsymbol{z} = \mathbf{258.5}$$

A. 114.35

B. 155.35

C. 155.85

D. 114.85

Answer: B

Solution:

Given: $x + \lfloor y \rfloor + \{z\} = 300$ ... (1)

$\{x\} + y + \lfloor z \rfloor = 270.2$ ... (2)

$\lfloor x \rfloor + \{y\} + z = 258.5$ .... (3)

Since, for all real numbers of x, $x = \lfloor x \rfloor + \{x\}$

Adding (1) , (2) and (3), we get

$$(x + \lfloor y \rfloor + \{z\}) + (\{x\} + y + \lfloor z \rfloor) + (\lfloor x \rfloor + \{y\} + z)$$
$$= 300 + 270.2 + 258.5$$

$$\Longrightarrow x + \lfloor y \rfloor + z - \lfloor z \rfloor + x - \lfloor x \rfloor + y + \lfloor z \rfloor + \lfloor x \rfloor + y - \lfloor y \rfloor + z = 828.7$$

$$\Longrightarrow 2(x + y + z) = 828.7$$

$$\Longrightarrow x + y + z = 414.35 \quad \text{... } (4)$$

Subtracting each of (1) , (2) , (3) from (4), we get

$\{y\} + \lfloor z \rfloor = 114.35$ ... (5)

$\lfloor x \rfloor + \{z\} = 144.15$ ... (6)

$\{x\} + \lfloor y \rfloor = 155.85$ ... (7)

$114 = \lfloor 114.35 \rfloor = \lfloor \{y\} + \lfloor z \rfloor \rfloor = \lfloor z \rfloor$ ... (8)

$144 = \lfloor 144.15 \rfloor = \lfloor \lfloor x \rfloor + \{z\} \rfloor = \lfloor x \rfloor$ ... (9)

$155 = \lfloor 155.85 \rfloor = \lfloor \{x\} + \lfloor y \rfloor \rfloor = \lfloor y \rfloor$ ... (10)

From (5) and (8), we get $\{y\} = 0.35$

From (6) and (9), we get $\{z\} = 0.15$

From (7) and (10), we get $\{x\} = 0.85$

Therefore, $y = \lfloor y \rfloor + \{y\} = 155 + 0.35 = 155.35$

IP3: Find the point $(-2, 3)$ lie which side of the circle

$$\left(\boldsymbol{x} - \frac{3}{2}\right)^2 + \left(\boldsymbol{y} - \frac{2}{3}\right)^2 = \frac{169}{9}$$

Solution:

Given circle is $\left(x - \frac{3}{2}\right)^2 + \left(y - \frac{2}{3}\right)^2 = \frac{169}{9}$ ... (1)

Step1:

Centre of the circle (1) is $\left(\frac{3}{2}, \frac{2}{3}\right)$ and $r^2 = \frac{169}{9}$

Step2:

Substituting $(-2,3)$ in the left side of the equation (1), we get $\left(-2-\frac{3}{2}\right)^2 + \left(3-\frac{2}{3}\right)^2 = \frac{49}{4}+\frac{49}{9} = 49\left(\frac{13}{36}\right)$

$$= \frac{637}{36} < \frac{169}{9}$$

$\therefore \left(x-\frac{3}{2}\right)^2 + \left(y-\frac{2}{3}\right)^2$ is less than $\frac{169}{9}$ at the point $(-2,3)$

Step3:

Therefore, the point $(-2,3)$ lie inside the circle.

P3: On which side the point $\left(\frac{1}{2},-\frac{1}{2}\right)$ lie of the circle

$$\boldsymbol{x^2+y^2-2x+12y+12=0}$$

A. Interior
B. Exterior
C. On the circle
D. None of the above

Answer: B

Solution:

Given circle : $x^2+y^2-2x+12y+12=0$

$\Longrightarrow (x^2-2x+1)+(y^2+2.6.y+36)+12-1-36=0$

$\Longrightarrow (x-1)^2+(y+6)^2=25 \quad \ldots (1)$

Given point : $\left(\frac{1}{2},-\frac{1}{2}\right)$

Substituting $\left(\frac{1}{2},-\frac{1}{2}\right)$ in the left side of the equation (1), we get

$$\left(\frac{1}{2}-1\right)^2+\left(-\frac{1}{2}+6\right)^2$$
$$=\frac{1}{4}+\frac{121}{4}$$
$$=\frac{122}{4}>25$$

$\therefore (x-1)^2+(y+6)^2>25$ at $\left(\frac{1}{2},-\frac{1}{2}\right)$

Therefore $\left(\frac{1}{2},-\frac{1}{2}\right)$ lie outside the given circle.

IP4: Sketch the graph of the function $\boldsymbol{f(y)=2y^2+1}$.

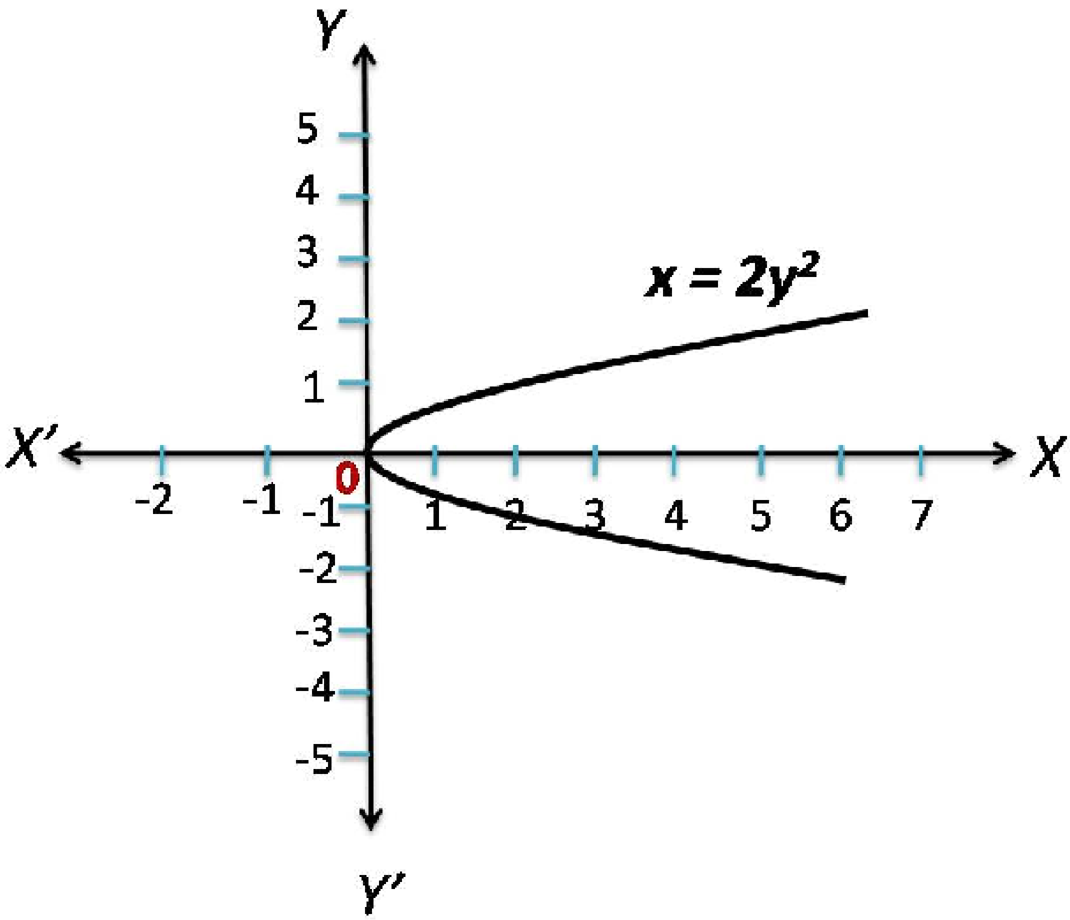
Solution:

Given function: $x=f(y)=2y^2+1$

The graph of the function, $f(y)=2y^2$ is a parabola whose axis is the $x-$axis. The parabola's vertex lies at the origin $(0,0)$ where the parabola and axis intersect each other.

The graph of the function, $x = f(y) = 2y^2 + 1$ is also a parabola whose axis is the $x-$axis and the graph is shifted 1 unit to the right on the axis. Now, the parabola's vertex lies at the point (1,0) where the parabola and axis intersect each other.

Graph of $f(y) = 2y^2$:



P4: Sketch the graph of the function, $y = x^3 - 3x^2 + 3x + 1$

Solution:

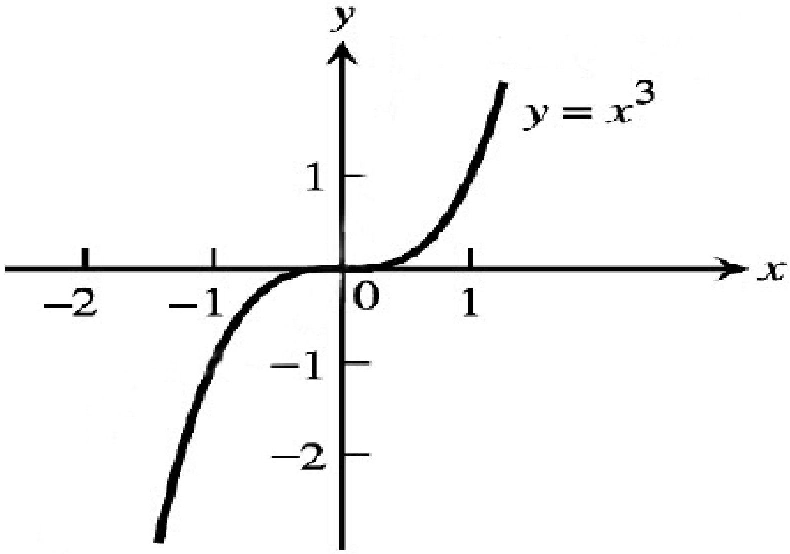
Given function: $y = x^3 - 3x^2 + 3x + 1$

$$\Rightarrow y = (x^3 - 3x^2 + 3x - 1) + 2$$

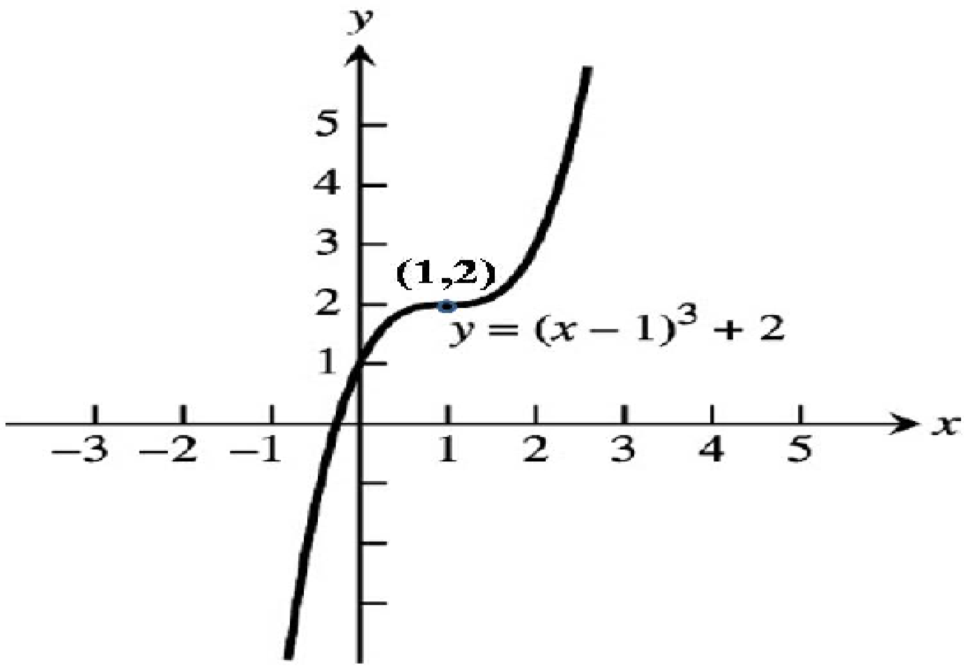
$$\Rightarrow y = (x-1)^3 + 2$$

The graph of $y = (x-1)^3 + 2$ is the graph of $y = x^3$ shifted 1 unit to the right and 2 units up.

The graph of $y = x^3$:



The graph of $y = (x+1)^3 + 2$:



Exercises:

1. Sketch the graph of $y^2 - 2y - x = 0$
2. Sketch the graph of $y = \dfrac{1}{x}$
3. Find intercepts of $x = y^2 - 2y$ and $y = \dfrac{1}{x}$
4. Sketch the graph of $y = -x^2 + 4x - 5$
5. Sketch the graph of $x = y^2 - 4y + 5$

6. Sketch the graphs of $y^2 = x^3,\ y^3 = x^2,\ y = x^{-2}$
7. Classify the given functions as even, odd, or neither.
   a. $f(x) = x^2$
   b. $f(x) = x^3$
   c. $f(x) = x^2 + 5x$
   d. Graph the above three functions
8. Find the x-and y-intercepts of the function $f(x) = -x^2 + x + 2$ and graph the function.
9. Sketch the graph of the function defined piecewise by the formula

$$f(x) = \begin{cases} 0 & , \quad x \le -1 \\ \sqrt{1-x^2}, & \quad -1 < x < 1 \\ x & , \quad x \ge 1 \end{cases}$$

10. Sketch the graph of
    a. $y = \sqrt{x-3}$
    b. $y = \sqrt{x+3}$
    c. $y = |x-3| + 2$
11. Sketch the graph of $y = x^2 - 4x + 5$
12. Sketch the graph of
    a. $f(x) = x^2 + x - 6$
    b. $g(x) = x^3 - 3x^2 - x + 3$

1.7. Exponential Equations

Learning objectives:

- To define the exponential equations.
- To solve the exponential equations using logarithms.
  AND
- To practice the related problems.

For items involved in exponential growth, the time it takes for a quantity to double is called the ***doubling time***.

For example, if you invest Rs.5000 in an account that pays 5% annual interest, compounded quarterly, you may want to know how long it will take for your money to double in value. You can find this doubling time if you can solve the equation

$$10{,}000 = 5000(1.0125)^{4t}$$

The method of converting both sides of the equation to an expression with the same base may not work for this problem.

Logarithms are very important in solving equations in which the variable appears as an exponent.

The above equation is an example of one such equation. Equations of this form are called exponential equations.

An *exponential equation* is an equation where a variable appears in one or more exponents.

If $5^x = 12$ then, so are their logarithms. Notice that both sides of the equation cannot be written as a power of the same base. Now a method in which we take the logarithm on both sides may work.

Example 1: Solve the equation. $5^n = 20$

Solution: Taking the logarithms on both sides of the equation, we have

$$\log 5^n = \log 20 \;\Rightarrow\; n \log 5 = \log 20$$

$$\Rightarrow n = \frac{\log 20}{\log 5} = \frac{1.3010}{0.6990} = 1.861$$

Example 2: **How long does it take for Rs 5000 to double if it is deposited in an account that yields 5% interest compounded once a year?**

Solution :

$$10{,}000 = 5000(1+0.05)^t \Longrightarrow 2 = (1.05)^t$$

We solve by taking the logarithm of both sides.

$$\log 2 = log(1.05)^t$$
$$= t\log 1.05$$

Dividing both sides by, $\log 1.05$ we have

$$t = \frac{\log 2}{\log 1.05} = 14.2$$

It takes a little over 14 years to double if it earns 5% interest per year, compounded once a year.

The logarithms have a property, known as ***change-of-base property***, which allows us to change from one base to another.

If a and b are both positive numbers other than 1, and

If $x > 0$ then

$$\log_a x = \frac{\log_b x}{\log_b a}$$

The logarithm on the left side has a base a , and both logarithms on the right side have a base of b. This allows us to change from base a to any other base b that is a positive number other than 1.

We prove this property as follows.

We begin by writing the identity

$$a^{\log_a x} = x$$

Taking the logarithm base b on both sides, we have

$$\log_a x \cdot \log_b a = \log_b x$$

Dividing both sides by $\log_b a$, we have the desired result.

The calculator provides for computing the logarithms with base 10 or$e$. We can use this property to find logarithms with other bases.

Example 3: Find $\log_8 24$

Solution:

We change the given expression to an equivalent expression that contains only base-10 logarithms.

$$\log_8 24 = \frac{\log 24}{\log 8}$$

$$= \frac{1.3802}{0.9031} = 1.5283$$

PROBLEM SET

IP1: If$\log_{10} 2 = 0.30103, \log_{10} 3 = 0.4771, \log_{10} 7 = 0.8450,$
$\log_{10} 5 = 0.6990$. Then find value of x in the following equation

$$\mathbf{21^x = 2^{2x+1}.5^x}$$

Solution:

Step1:

The given equation is $21^x = 2^{2x+1}.5^x$

Step2:

By taking logarithms to the base 10 on both sides

$\Rightarrow log\ 21^x = log\ 2^{2x+1}.5^x$

$\Rightarrow log(7 \times 3)^x = log\ 2^{2x+1}.5^x$

$\Rightarrow log\ 7^x + log\ 3^x = log\ 2^{2x+1} + log\ 5^x$

$\Rightarrow x\ log\ 7 + x\ log\ 3 = (2x+1)\ log\ 2 + x\ log\ 5$

$\Rightarrow x(log\ 7 + log\ 3 - 2\ log\ 2 - log\ 5) = log\ 2$

Step3:

Substituting the values of logarithms

$\Rightarrow x(0.8450 + 0.4771 - 2(0.3010) - 0.6990) = 0.3010$

$\Rightarrow x(0.0211) = 0.3010$

$\Rightarrow x = 14.2654$

P1 :

If $log_{10} 2 = 0.30103, log_{10} 3 = 0.4771, log_{10} 7 = 0.8450$ then find the value of x in the following equation $3^{1+x} = 7^{\frac{x}{2}}$

Solution:

Given that $3^{1+x} = 7^{\frac{x}{2}}$

Taking Logarithms to the base 10 on both sides

$\Rightarrow \log 3^{1+x} = \log 7^{\frac{x}{2}}$

$\Rightarrow (1+x)\log 3 = \frac{x}{2}\log 7$

$\Rightarrow 2(1+x)\log 3 = x\log 7$

$\Rightarrow 2\log 3 = x\log 7 - 2x\log 3$

$$\Longrightarrow 2\log 3 = x\,(\log 7 - 2\log 3) \qquad \ldots (1)$$

Now, by substituting the values of $\log 3$ and $\log 7$ in (1), we get

$$\Longrightarrow 2(0.4771) = x(0.8450 - 2(0.4771))$$
$$\Longrightarrow 0.9542 = x(0.8450 - 0.9542)$$
$$\Longrightarrow 0.9542 = x(-0.1092)$$
$$\Longrightarrow x = -8.7381$$

IP2: Solve the equation for x

$$\mathbf{3^{\log_3 x^2} = 5e^{\log x} - 3.\,10^{\log_{10} 2}}$$

Solution:

Step1:

The given equation is

$$\mathbf{3^{\log_3 x^2} = 5e^{\log x} - 3.\,10^{\log_{10} 2}}$$

Step2:

Using the property $\boldsymbol{a^{\log_a x} = x}$, we can reduce the equation as

$$\boldsymbol{x^2 = 5x - 3 \times 2}$$
$$\boldsymbol{x^2 = 5x - 6}$$

Step3:

Solving the quadratic equation

$$\boldsymbol{x^2 - 2x - 3x + 6 = 0}$$
$$\boldsymbol{\Longrightarrow x(x-2) - 3(x-2) = 0}$$
$$\boldsymbol{\Longrightarrow (x-2)(x-3) = 0}$$
$$\boldsymbol{\Longrightarrow x = 3, 2}$$

P2: Solve the equation for x

$$\boldsymbol{8^{\log_8 3} - e^{\log 5} = x^2 - 7^{\log_7 3x}}.$$

A. **1,3**
B. **2,3**
C. **1,2**
D. **2,2**

Answer: C

Solution: The given equation is

$$8^{\log_8 3} - e^{\log 5} = x^2 - 7^{\log_7 3x}$$
$$\Longrightarrow 3 - 5 = x^2 - 3x$$
$$\Longrightarrow x^2 - 3x + 2 = 0$$
$$\Longrightarrow x(x-2) - 1(x-2) = 0$$
$$\Longrightarrow (x-1)(x-2) = 0$$
$$\Longrightarrow x = 1,2$$

IP3: **Find the value of** $\frac{\log_9 x}{\log_3 x}$.

Solution:

Step1: Given expression is $\frac{\log_9 x}{\log_3 x}$.

Step2: We know that the property $\log_a x = \frac{\log x}{\log \mathrm{a}}$

$$\Rightarrow \frac{\log_9 x}{\log_3 \mathrm{x}} = \frac{\frac{\log x}{\log 9}}{\frac{\log x}{\log 3}}$$

$$= \frac{\log x}{\log 9} \times \frac{\log 3}{\log x}$$

$$= \frac{\log 3}{\log 3^2}$$

$$= \frac{\log 3}{2 \log 3}$$

$$= \frac{1}{2}$$

P3: **Find the value of the** $\frac{\log_2 x}{\log_8 x}$.

A. 1

B. 2

C. 3

D. 4

Answer: C

Solution: Given expression is $\frac{\log_2 x}{\log_8 x}$.

$$\Rightarrow \frac{\log_2 x}{\log_8 x} = \frac{\frac{\log x}{\log 2}}{\frac{\log x}{\log 8}}$$

$$= \frac{\log x}{\log 2} \times \frac{\log 8}{\log x}$$

$$= \frac{\log 8}{\log 2}$$

$$= \frac{\log 2^3}{\log 2}$$

$$= 3\frac{\log 2}{\log 2}$$

$$= 3$$

IP4: **Solve the equation for** x

$$x^{\left[\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}\right]} = \sqrt{2} \ .$$

Solution:

Step1: The Given equation is

$$x^{\left[\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}\right]} = \sqrt{2}$$

Step2: Taking logarithm with base x on both sides

$$\log_x x^{\left[\frac{3}{4}(\log_2 x)^2+\log_2 x-\frac{5}{4}\right]} = \log_x \sqrt{2}$$

$$\frac{3}{4}[\log_2 x]^2+\log_2 x-\frac{5}{4} = \log_x \sqrt{2}$$

Step3: Let $\log_2 x = t$ then $x = 2^t$

$$\frac{3}{4}t^2+t-\frac{5}{4} = \frac{1}{2}\log_x 2$$

$$\Longrightarrow \frac{3}{4}t^2+t-\frac{5}{4} = \frac{1}{2t}$$

$$\Longrightarrow 3t^3+4t^2-5t-2 = 0$$

$t = 1$ Satisfies the equation

Therefore, $(t-1)$is a factor of the equation

$$\Longrightarrow (t-1)(3t^2+7t+2) = 0$$

$$\Longrightarrow (t-1)(t+2)(3t+1) = 0$$

$$\Longrightarrow t = 1, -2, \frac{-1}{3}.$$

$$\Longrightarrow \log_2 x = 1, \log_2 x = -2, \log_2 x = \frac{-1}{3}$$

$$\Longrightarrow x = 2, x = 2^{-2}, x = 2^{\frac{-1}{3}}$$

$$\Longrightarrow x = 2, \frac{1}{4}, \frac{1}{\sqrt[3]{2}}.$$

P4: Solve the equation for x

$$\boldsymbol{x^{\left(\frac{2}{3}\right)\left[(\log_2 x)^2+\log_2 x-\frac{5}{4}\right]} = \sqrt{2}} \text{ .}$$

A. $\mathbf{2, 2\sqrt{2}, \frac{1}{\sqrt{2}}}$

B. $\mathbf{\frac{1}{2\sqrt{2}}, \sqrt{2}, \frac{1}{2}}$

C. $\mathbf{\frac{1}{2\sqrt{2}}, \sqrt{2}, 2}$

D. $\mathbf{\frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}, 2}$

Answer: D

Solution:

The Given equation is

$$x^{\left(\frac{2}{3}\right)\left[(\log_2 x)^2+\log_2 x-\frac{5}{4}\right]} = \sqrt{2} \qquad (1)$$

Let $\log_2 x = t \Longrightarrow x = 2^t$

Taking logarithm with base x on both sides of (1)

$$\log_x x^{\left(\frac{2}{3}\right)\left[(\log_2 x)^2+\log_2 x-\frac{5}{4}\right]} = \log_x \sqrt{2}$$

$$\Longrightarrow \frac{2}{3}\left(t^2+t-\frac{5}{4}\right) = \frac{1}{2}\log_x 2$$

$$\Longrightarrow \frac{2}{3}\left(t^2+t-\frac{5}{4}\right) = \frac{1}{2\log_2 x}$$

$$\Rightarrow \frac{2}{3}\left(t^2 + t - \frac{5}{4}\right) = \frac{1}{2t}$$
$$\Rightarrow \frac{4t}{3}\left(t^2 + t - \frac{5}{4}\right) = 1$$
$$\Rightarrow 4t^3 + 4t^2 - 5t = 3$$
$$\Rightarrow 4t^3 + 4t^2 - 5t - 3 = 0$$

$t = 1$ satisfies the equation

Therefore, $(t-1)$is a factor of the equation

$$(t-1)(4t^2 + 8t + 3) = 0$$
$$\Rightarrow (t-1)(2t+3)(2t+1) = 0$$
$$\Rightarrow \;\; t = 1, \frac{-3}{2}, \frac{-1}{2}$$
$$\Rightarrow \log_2 x = 1, \log_2 x = \frac{-3}{2}, \log_2 x = \frac{-1}{2}$$
$$\Rightarrow x = 2, 2^{\frac{-3}{2}}, 2^{\frac{-1}{2}}$$
$$\Rightarrow x = \frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}, 2.$$

Exercises

1. Solve the equation for x
$$3.2^{\log_5 x} + 2^{\log_5 x} = 64$$
2. If $a^x = b, b^y = c, c^z = a$ then find the value of xyz.
3. Solve the equation
$$3^{2x+1}.4^{4x-1} = 36.$$
4. Find the value of x from the equation
$$a^{3-x}b^{5x} = a^{3x}b^{x+5}.$$
5. Solve $25^{2x+1} = 15$.
6. If $(2.3)^x = \left(\frac{1}{0.23}\right)^y = 1000$ then find the value of $\frac{1}{x} + \frac{1}{y}$.
7. If $(3.4)^x = \left(\frac{1}{0.034}\right)^y = 10000$ then find the value of $\frac{1}{x} + \frac{1}{y}$.
8. If $8^{\log_a a^x} . 64^{\log_b b^{2x}} . 512^{\log_c c^{3x}} = 8^{56}$ then find the value of x.
9. Find the value of $(3)^{\log_2 32} + (4)^{\log_5 125} + (5)^{\log_8 64}$.
10. Prove that $(yz)^{\ln\frac{y}{z}}.(zx)^{\ln\frac{z}{x}}.(xy)^{\ln\frac{x}{y}} = 1$.
11. Find the value of $x^{(\ln y - \ln z)}.y^{(\ln z - \ln x)}.z^{(\ln x - \ln y)}$
12. Suppose you deposit Rs 500 in an account with an annual interest rate of 12% compounded monthly. How many years will it take for the account to contain Rs 1000?

13. Solve the equation $\log e + 4^{-2\log_4 x} = \frac{1}{x}\log_{10} 100$.

14. If $a^x = b^y = c^z$ and $y^2 = zx$, Then prove that

$\log_b a = \log_c b$.

15. solve the equation for x

$$4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1}$$

16. Suppose that the population in a small city is 32,000 in the beginning of 1994 and that the city council assumes that the population size t years later can be estimated by the equation

$$P = 32000e^{0.05t}$$

Approximately when will the city have a population of 50,000?

2.1. Linear Inequations in One Variable

Learning Objectives:

- To define an inequation and linear inequation
- To solve the linear inequations in one variable and to find their solution sets
- To represent the solution set of an inequation on the Real Line

AND

- To practice the related problems

Linear Inequations in One Variable

Inequations

If $x \neq 5$ then either $x < 5$ or $x > 5$.

The symbols $' > '$ and $' < '$ are known as symbols of inequalities and the statements involving inequalities $x < 5$, $x > 5$ are known as inequations.

For example, the sum of the measures of two sides of a triangle is greater than the measure of its third side. This is an inequation. If a, b, c denote the measures of the sides of a triangle, then $a + b > c$

If a person has driving license, his age must be at least 18. If x denotes the age of the person, then $x \geq 18$.

In an inequation, the algebraic expressions are related by one of the four symbols of inequalities:

$>$	Greater than
$\geq$	Greater than or equal to
$<$	Less than
$\leq$	Less than or equal to

*A mathematical expression containing one or more of the symbols $<, \leq, >, \geq$ is called an **inequation**. A linear inequation is a statement of inequality in which the highest power of the variable is one.*

Examples of inequations in one variable are

$$2x + 3 \leq 5 \ , \ 4x > 3x - 5$$
$$-3 \leq -x + 5 \ , \ 2x + 3 \geq 0$$

In general, a linear inequation can be written as

$$ax + b < 0, \qquad ax + b \leq 0, \qquad ax + b > 0, \qquad ax + b \geq 0$$

where $a, b \ \in \boldsymbol{R}$ *and* $a \neq 0$. *The direction of the inequality symbol is called the **sense** of the inequality*.

Solving Inequations

A simple linear equation has only one root (solution), but a linear inequation may have more than one root.

For example, the root of $3x - 3 = 18$ is 7, but for $3x - 3 \geq 18$ the roots are all real numbers greater than or equal to 7. Solving an inequation involves finding the set of values for the variable which satisfy the inequality.

The *solution set* of an inequation is the set of numbers that when substituted in place of the variable makes the inequation a true statement. The solution set is also known as the *truth set*.

To solve an inequation, we must isolate the variable on one side of the inequality symbol. *To isolate the variable, we use the same basic techniques used in solving equations*. The following properties hold good for inequations. The first two properties are the same as in the case of solving equations.

1. *The same quantity may be added to or subtracted from both sides of an inequation.*
2. *Both sides of an inequation may be multiplied or divided by the same **positive** quantity.*
3. *On multiplying or dividing both sides of an inequation by the same **negative** quantity, the sign of inequality is reversed.*

The solution of the inequation remains the same after these operations.

For example, the inequality $x > 5$ is true for $x = 6$. If we multiply both sides by 3, we have $3x > 15$ because $18 > 15$

On the other hand, if we multiply both sides by −3, we have, $-3x < -15$ because $-18 < -15$.

Example 1: Solve the inequation $2x + 6 < 12$.

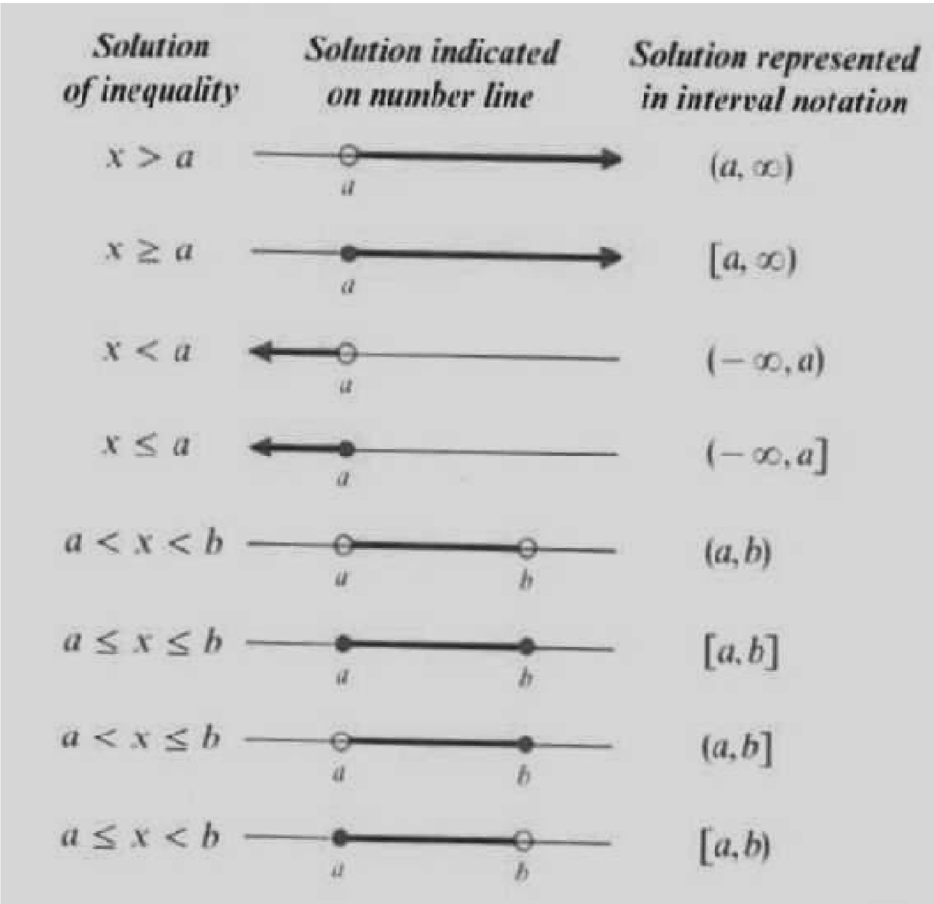
Solution:

$$2x+6<12 \Rightarrow 2x+6-6<12-6 \Rightarrow 2x<6 \Rightarrow \frac{2x}{2}<\frac{6}{2} \Rightarrow x<3$$

Any number less than 3 satisfies the inequation.

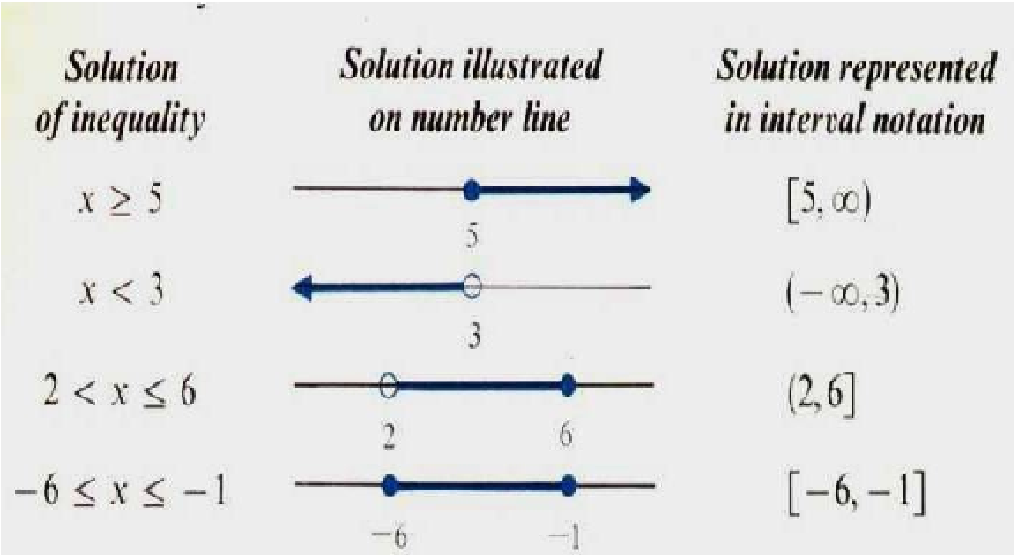
Graphing Solution Set on the Number Line

The solution set for an inequation can be graphed on the number line or written in the interval notation.



A shaded circle on the number line indicates that the end point is part of the solution, and an unshaded circle indicates that the end point is not part of the solution.

In interval notation brackets, [ ], are used to indicate that the end points are part of the solution; parenthesis, ( ), indicate that the end points are not part of the solution. The symbol ∞ is read “infinity”; it indicates that the solution continues indefinitely.

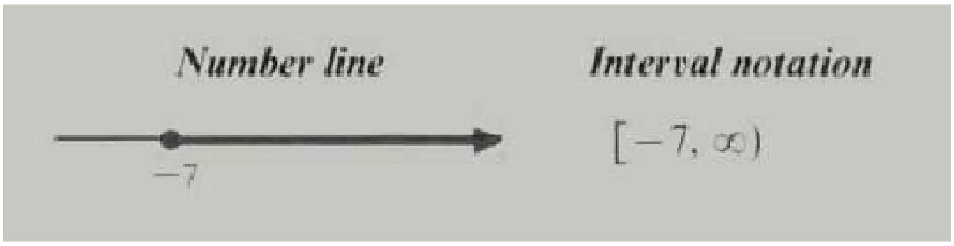


Solution of inequality	Solution illustrated on number line	Solution represented in interval notation
$x \geq 5$	5	$[5, \infty)$
$x < 3$	3	$(-\infty, 3)$
$2 < x \leq 6$	2 6	$(2, 6]$
$-6 \leq x \leq -1$	−6 −1	$[-6, -1]$

Example 2: Solve the inequality $3(x-2) \leq 5x+8$ and give the answer both on the number line and in interval notation.

Solution:

$$3(x-2) \leq 5x+8 \Rightarrow 3x-6 \leq 5x+8$$
$$\Rightarrow\ 3x-5x-6 \leq 5x-5x+8 \Rightarrow\ -2x-6 \leq 8$$
$$\Rightarrow\ -2x-6+6 \leq 8+6 \Rightarrow\ -2x \leq 14$$
$$\Rightarrow \frac{-2x}{-2} \geq \frac{14}{-2} \Rightarrow\ x \geq -7$$



Example 3: Solve the following inequalities and graph their solution sets on the real line.

$$(a)\ 2x-1 < x+3, (b)\ -\frac{x}{3} < 2x+1, (c)\ \frac{6}{x-1} \geq 5$$

Solution:

(a) $\quad 2x-1 < x+3 \Rightarrow 2x < x+4 \Rightarrow\ x < 4$

The solution set is the interval $(-\infty, 4)$

(b) $\quad -\dfrac{x}{3} < 2x+1 \Rightarrow -x < 6x+3 \Rightarrow 0 < 7x+3 \Rightarrow -3 < 7x \Rightarrow -\dfrac{3}{7} < x$

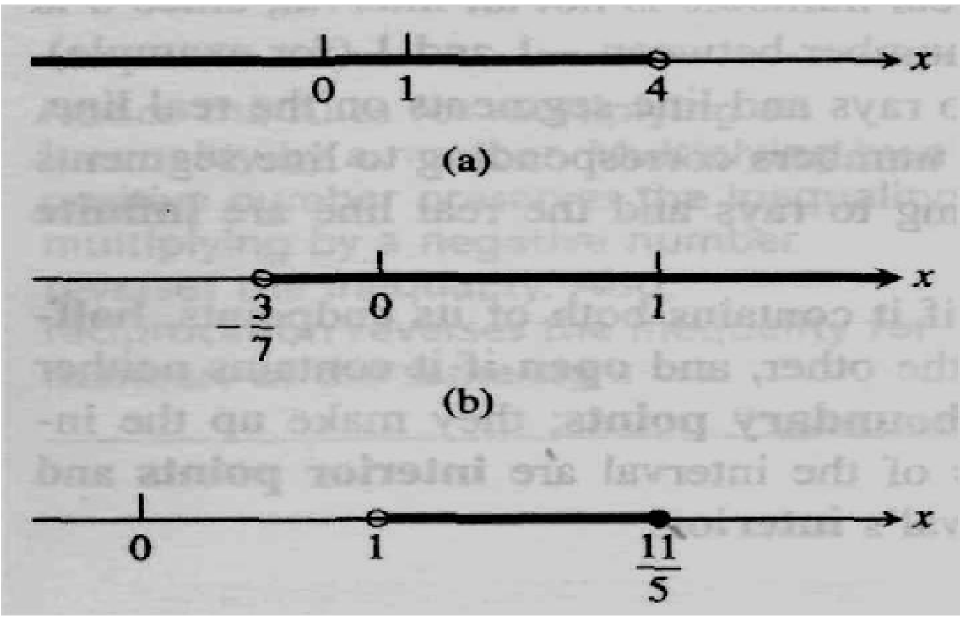
The solution set is the interval $\left(-\dfrac{3}{7}, \infty\right)$

(c) $\quad$ The inequality holds only when $x > 1$. Therefore, the inequality will be preserved if we multiply both sides by $x-1$.

$6 \geq 5(x-1) \Rightarrow 11 \geq 5x \Rightarrow x \leq \dfrac{11}{5}$

Therefore $x > 1$ and $x \leq \dfrac{11}{5} \Rightarrow 1 < x \leq \dfrac{11}{5}$

The solution set is the half - open interval $\left(1, \dfrac{11}{5}\right]$



Example 4: Solve the inequality $\quad \boldsymbol{\frac{1}{2}(4x+14) \geq 5x+4-3x-10}$

Solution:

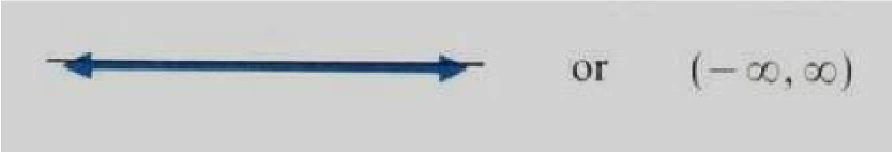
$$\frac{1}{2}(4x+14) \geq 5x+4-3x-10$$

$$\Rightarrow 2x+7 \geq 2x-6$$

$$\Rightarrow 2x-2x+7 \geq 2x-2x-6$$

$$\Rightarrow 7 \geq -6$$

Since 7 is always greater than -6, the solution is "all real numbers".



If the solution to the example above were $7 \le -6$, the answer would have been no solution, since 7 is never less than -6.

Example 5: Solve the inequality $\frac{4-2y}{3} \ge \frac{2y}{4} - 3$.

Solution: Multiply both sides of the inequality by the lowest common multiple of the denominators 12.

$$12 \times \frac{4-2y}{3} \ge 12 \times \frac{2y}{4} - 12 \times 3$$

$$\Rightarrow 16 - 8y \ge 6y - 36 \Rightarrow 16 \ge 14y - 36$$

$$\Rightarrow 52 \ge 14y \Rightarrow \frac{52}{14} \ge y$$

$$\Rightarrow \frac{26}{7} \ge y \quad \text{or} \quad y \le \frac{26}{7}$$

$$\text{The solution set is} \left(-\infty, \frac{26}{7}\right]$$

Consider the ***double inequality***

$$3 < x < 5$$

This inequality means that $3 < x$ and $x < 5$. The only numbers that are both greater than 3 and less than 5 are the numbers *between* 3 and 5. The solution set is (3, 5).

Now consider the double inequality

$$1 < x + 5 \le 7$$

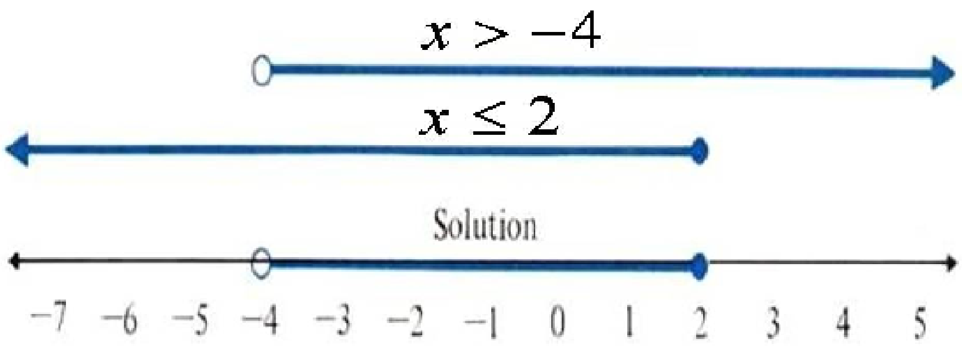
The solution to this inequality consists of those values that satisfy both the inequality $1 < x + 5$ and the inequality

$x + 5 \leq 7$.

$$1 < x + 5 \Rightarrow -4 < x \Rightarrow x > -4 \quad \text{and}$$

$$x + 5 \leq 7 \Rightarrow x \leq 2$$

The values that satisfy both inequalities simultaneously are illustrated in the figure below.



The values that satisfy both inequalities simultaneously are in the interval

$$-4 < x \leq 2.$$

This double inequality could have been solved in another way. We apply the addition, subtraction, multiplication, and division properties directly to the double inequality. We should keep in mind that whatever we do to one part we must do to all three parts. For the example above, we subtract 5 from all three parts to obtain

$$\Rightarrow 1 - 5 < x + 5 - 5 \leq 7 - 5$$

$$\Rightarrow -4 < x \leq 2 \Rightarrow x \in (-4, 2]$$

Example 6: Solve the double inequality $-3 \leq 2x - 7 < 8$

Solution: We wish to isolate the variable x. We begin by adding 7 to all three parts of the inequality.

$$-3 \leq 2x - 7 < 8$$

$$\Rightarrow -3 + 7 \leq 2x - 7 + 7 < 8 + 7$$

$$\Rightarrow 4 \leq 2x < 15$$

Now divide all three parts of the inequality by 2.

$$2 \le x < \frac{15}{2} \text{ i.e., } x \in \left[2, \frac{15}{2}\right)$$

Example 7: Solve the double inequality $-2 < \frac{4-3x}{5} < 8$.

Solution: Multiply all 3 parts by 5 to eliminate the denominator.

$$-10 < 4 - 3x < 40$$
$$\Rightarrow -10 - 4 < -3x < 4 - 4$$
$$\Rightarrow -14 < -3x < 36$$

At this point we divide all three parts of the inequality by -3. When we multiply or divide an inequality by a negative number, the sense of the inequality changes and we obtain

$$\frac{-14}{-3} > \frac{-3x}{-3} > \frac{36}{-3}$$
$$\Rightarrow \frac{14}{3} > x > -12 \quad or \quad -12 < x < \frac{14}{3} \; i.e., \; x \in \left(-12, \frac{14}{3}\right)$$

PROBLEM SET

IP1: Solve the linear inequation, $2(2x+3) - 10 \le 6(x-2)$.

Solution:

Step1: We have, $2(2x+3) - 10 \le 6(x-2)$

$\Rightarrow 4x + 6 - 10 \le 6x - 12 \Rightarrow 4x - 4 \le 6x - 12$

$\Rightarrow 4x - 6x \le -12 + 4$ [Transporting -4 to RHS and $6x$ to LHS]

$$\Rightarrow -2x \le -8 \Rightarrow \frac{-2x}{-2} \ge \frac{-8}{-2}$$

$$\Rightarrow x \ge 4 \Rightarrow x \in [4, \infty)$$

Step2: Hence, the solution set of the given linear inequation is $[4, \infty)$

P1: The solution for the inequation $\mathbf{3x + 17 \le 2(1 - x)}$ is

A. $(-\infty, 3]$
B. $(-\infty, -3]$
C. $[3, \infty)$
D. $[-3, \infty)$ **Answer:** B

Solution: We have, $3x + 17 \le 2(1 - x) \Rightarrow 3x + 17 \le 2 - 2x$

$$\Rightarrow 3x + 2x \le 2 - 17$$

[$\because$ Transporting $-2x$ to LHS and 17 to RHS]

$$\Rightarrow 5x \le -15$$

$$\Rightarrow \frac{5x}{5} \le -\frac{15}{5}$$

$$\Rightarrow x \le -3$$

$$\Rightarrow x \in (-\infty, -3]$$

Hence, the solution set of the given inequation is $(-\infty, -3]$.

IP2: Solve the linear inequation, $\mathbf{\frac{2x-3}{4} + 9 \ge 3 + \frac{4x}{3}}$.

Solution:

Step1: We have, $\frac{2x-3}{4} + 9 \ge 3 + \frac{4x}{3}$

$\Rightarrow \frac{2x-3}{4} - \frac{4x}{3} \ge 3 - 9$ [Transposing $\frac{4x}{3}$ to LHS and 9 to RHS]

$$\Rightarrow \frac{3(2x-3)-16x}{12} \ge -6 \Rightarrow \frac{6x-9-16x}{12} \ge -6$$

$$\Rightarrow \frac{-9-10x}{12} \ge -6 \Rightarrow -9 - 10x \ge -72$$

$$\Rightarrow -10x \ge -72 + 9 \Rightarrow -10x \ge -63$$

$$\Rightarrow \frac{-10x}{-10} \le \frac{-63}{-10} \Rightarrow x \le \frac{63}{10}$$

$\Rightarrow x \in \left(-\infty, \frac{63}{10}\right]$

Step2: Hence, the solution set of the given inequation is $\left(-\infty, \frac{63}{10}\right]$.

P2: The solution set of the inequation, $\frac{1}{2}\left(\frac{3}{5}x + 4\right) \geq \frac{1}{3}(x - 6)$ is

A. $[-120, \infty)$
B. $[120, \infty)$
C. $(-\infty, -120]$
D. $(-\infty, 120]$ **Answer:** D

Solution:

We have, $\frac{1}{2}\left(\frac{3}{5}x + 4\right) \geq \frac{1}{3}(x - 6)$

$\Rightarrow \frac{1}{2}\left(\frac{3x+20}{5}\right) \geq \frac{1}{3}(x - 6) \Rightarrow \frac{3x+20}{10} \geq \frac{x-6}{3}$

$\Rightarrow 3(3x + 20) \geq 10(x - 6)$ [Multiplying both sides by 30 i.e. LCM of 3 and 10]

$\Rightarrow 9x - 10x \geq -60 - 60$ [Transporting $10x$ on LHS and 60 on RHS]

$\Rightarrow -x \geq -120$

$\Rightarrow x \leq 120$ [Multiplying both sides by -1]

$\Rightarrow x \in (-\infty, 120]$

Hence, the solution set of the given linear inequation is $(-\infty, 120]$.

IP3: Solve the linear inequation, $\frac{x}{x-5} > \frac{1}{2}$

Solution:

Step1: We have, $\frac{x}{x-5} > \frac{1}{2}$

$\Rightarrow \frac{x}{x-5} - \frac{1}{2} > 0 \Rightarrow \frac{2x-(x-5)}{2(x-5)} > 0 \Rightarrow \frac{2x-x+5}{2(x-5)} > 0 \Rightarrow \frac{x+5}{x-5} > 0$

Step2: (i). If $x + 5 > 0$ and $x - 5 > 0$

$\Rightarrow x > -5$ and $x > 5$

$\Rightarrow x > 5$ satisfies the above conditions

$\Rightarrow x \in (5, \infty)$

Step3: (ii). If $x + 5 < 0$ and $x - 5 < 0$

$\Rightarrow x < -5$ and $x < 5$

$\Rightarrow x < -5$ satisfies the above conditions

$\Rightarrow x \in (-\infty, -5)$

Step4: Therefore, solution set for the given linear inequation is

$$(-\infty, -5) \cup (5, \infty)$$

P3: Solve the linear inequation, $\frac{5x+8}{4-x} < 2$

A. $(-\infty, -4) \cup (0, \infty)$
B. $(-\infty, 4) \cup (5, \infty)$
C. $(-\infty, 0) \cup (4, \infty)$
D. $(-\infty, 0] \cup (4, \infty)$ **Answer:** C

Solution:

We have, $\frac{5x+8}{4-x} < 2$

$\Rightarrow \frac{5x+8}{4-x} - 2 < 0$ [Transporting 2 to LHS]

$\Rightarrow \frac{5x+8-2(4-x)}{4-x} < 0 \Rightarrow \frac{5x+8-8+2x}{4-x} < 0$

$\Rightarrow \frac{7x}{4-x} < 0$

$\Rightarrow \frac{x}{4-x} < 0$ [Dividing by 7 on both sides of inequality]

i) $x > 0$ and $4 - x < 0$

$\Rightarrow x > 0$ and $x > 4$

$\Rightarrow x > 4$ satisfies the above conditions

$\Rightarrow x \in (4, \infty)$

ii) $x < 0$ and $4 - x > 0$

$\Rightarrow x < 0$ and $x < 4$

$\Rightarrow x < 0$ satisfies the above conditions

$\Rightarrow x \in (-\infty, 0)$

Therefore, the solution set for the given linear inequation is $(-\infty, 0) \cup (4, \infty)$

IP4: Solve, $-5 \leq \frac{2-3x}{4} \leq 9$.

Solution:

Step1: We have, $-5 \leq \frac{2-3x}{4} \leq 9$

$\Longrightarrow -5 \times 4 \leq \frac{2-3x}{4} \times 4 \leq 9 \times 4$ [Multiplying throughout by 4]

$\Longrightarrow -20 \leq 2 - 3x \leq 36$

$\Longrightarrow -20 - 2 \leq 2 - 3x - 2 \leq 36 - 2$ [Subtracting 2 throughout]

$\Longrightarrow -22 \leq -3x \leq 34$

$\Longrightarrow \frac{-22}{-3} \geq \frac{-3x}{-3} \geq \frac{34}{-3}$ [Dividing throughout by -3]

$\Longrightarrow \frac{22}{3} \geq x \geq \frac{-34}{3}$

$\Longrightarrow \frac{-34}{3} \leq x \leq \frac{22}{3}$

$\Longrightarrow x \in \left[-\frac{34}{3}, \frac{22}{3}\right]$

Step2:

Hence, the interval $\left[-\frac{34}{3}, \frac{22}{3}\right]$is the solution of the given system of inequations

P4: Solve: $-11 \leq 4x - 3 \leq 13$.

A. $[-2, 4]$
B. $[-4, 2]$
C. $[-4, -2]$
D. $[2, 4]$ **Answer:** A

Solution: We have,

$-11 \leq 4x - 3 \leq 13$

$\Rightarrow -11 + 3 \leq 4x - 3 + 3 \leq 13 + 3$ [Adding 3 throughout]

$\Rightarrow -8 \leq 4x \leq 16$

$\Rightarrow -\frac{8}{4} \leq x \leq \frac{16}{4}$ [Dividing by 4 throughout]

$\Rightarrow -2 \leq x \leq 4$

$\Rightarrow x \in [-2,4]$

Hence, the interval $[-2, 4]$ is the solution set of the given system of inequations.

Exercise:

1. Find the interval satisfying each inequality.
   a. $3 \leq x - 4 \leq 9$
   b. $-2 \leq x + 5 \leq 3$
   c. $-8 \leq 2x \leq 2$
   d. $-9 \leq -3x \leq 15$
2.
   a. Solve $x + 3 > 9$ and represent the solution set on the number line.
   b. Solve $3x + 3 < 2x - 1$ and represent the solution set on the number line.
   c. Solve the inequation $12 + 1\frac{5}{6}x \leq 5 + 3x, x \in \boldsymbol{R}$ Represent the solution on a number line.
   d. Solve $3 + 7x \leq 2x - 9$
   e. Solve $7 \leq 2 - 5x < 9$
3.
   a. Solve $-22 \leq 3x - 7 \leq 8$ and represent the solution set on the number line.
   b. Solve $2 \leq 2x - 3 \leq 5, x \in \boldsymbol{R}$ and mark it on the number line.
   c. Solve the following inequation and graph the solutions on the number line. $2x - 5 \leq 5x + 4 < 11, x \in \boldsymbol{R}$
   d. Solve the inequation $-3 \leq 3 - 2x < 9, x \in \boldsymbol{R}$. Represent the solution on a number line.
4. Find the value of x which satisfies the inequation $-2 \leq \frac{1}{2} - \frac{2x}{3} \leq 1\frac{5}{6}, x \in \boldsymbol{N}$.
5.
   a. Solve $\frac{2x-5}{x-2} \leq 1$
   b. Solve $\frac{5x-6}{x+6} < 1$
   c. Solve $\frac{4x+3}{2x-5} < 6$
   d. Solve $\frac{7x-5}{8x+3} > 4$
   e. Solve $\frac{x-1}{x+3} \geq 2$

2.2. Inequations Involving Absolute Values

Learning Objectives:

- To study the inequations involving absolute values

  AND

- To solve related problems

Inequations Involving Absolute Values

The inequation $|a| < D$ says that the distance from a to 0 is less than D. Therefore a must lie between D and $-D$. If D is any positive number, then

$$|a| < D \quad \Leftrightarrow \quad -D < a < D$$

$$|a| \leq D \quad \Leftrightarrow \quad -D \leq a \leq D$$

Example 1:

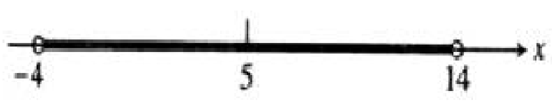
Solve the inequation $|x - 5| < 9$ **and graph the solution set on the real line.**

Solution:

$$|x-5| < 9 \Rightarrow -9 < x-5 < 9$$

$$\Rightarrow -9+5 < x < 9+5 \Rightarrow -4 < x < 14$$

The solution is the open interval $(-4,14)$



Example 2:

Solve the inequation $\left|5 - \dfrac{2}{x}\right| < 1$

Solution:

$$\left|5-\frac{2}{x}\right|<1 \Rightarrow -1<5-\frac{2}{x}<1,\ x\neq 0$$

$$\Rightarrow -6<-\frac{2}{x}<-4 \Rightarrow 3>\frac{1}{x}>2$$

$$\Rightarrow \frac{1}{3}<x<\frac{1}{2}$$

The solution set is the open interval $\left(\frac{1}{3},\frac{1}{2}\right)$

Example 3: **Solve the inequations and graph the solution set:**

$a)$ $\quad |2x-3|\leq 1$

$b)$ $\quad |2x-3|>1$

Solution:

$a)$ $\quad |2x-3|\leq 1 \Rightarrow -1\leq 2x-3\leq 1$

$$\Rightarrow 2\leq 2x\leq 4 \Rightarrow 1\leq x\leq 2$$

The solution set is the closed interval [1,2].

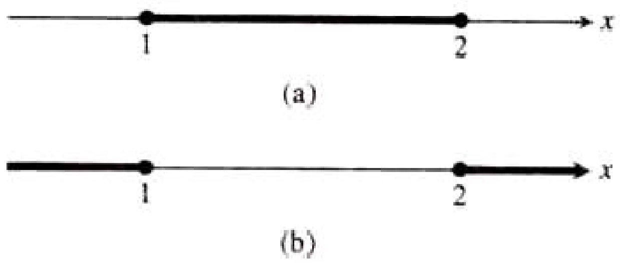
$b)$ $\quad |2x-3|>1 \Rightarrow 2x-3>1 \quad or \quad -(2x-3)>1$

$$\Rightarrow 2x-3>1 \quad or \quad 2x-3<-1$$

$$\Rightarrow x-\frac{3}{2}>\frac{1}{2} \quad or \quad x-\frac{3}{2}<-\frac{1}{2}$$

$$\Rightarrow x>2 \quad or \quad x<1$$

The solution set is $(-\infty,1)\cup(2,\infty)$



Inequations of the form $|x-a|<k$ and $|x-a|>k$ arise often, and their interpretation is as follows.

Inequation (k > 0)	*Geometric Interpretation*	*Alternative forms of the Inequation*
$\lvert x-a\rvert<k$	x is within k units of a.	$-k<x-a<k$ $a-k<x<a+k$ $x\in(a-k,a+k)$
$\lvert x-a\rvert>k$	x is more than k units away from a.	$x-a<-k\ \ or\ x-a>k$ $x<a-k\ \ or\ \ x>a+k$ $x\in(-\infty,a-k)\cup(a+k,\infty)$

PROBLEM SET

IP1: Solve: $\left|\boldsymbol{x}+\frac{\mathbf{1}}{\mathbf{3}}\right|>\frac{\mathbf{8}}{\mathbf{3}}$**.**

Solution:

We have, $\left|x+\frac{1}{3}\right|>\frac{8}{3}$ $i.e.,\left|x-(-\frac{1}{3})\right|>\frac{8}{3}$

$$\Rightarrow x\ \in\ \left(-\infty,-\frac{1}{3}-\frac{8}{3}\right)\cup\left(-\frac{1}{3}+\frac{8}{3},\infty\right)$$

$$\Rightarrow x\ \in\ (-\infty,-3)\cup\left(\frac{7}{3},\infty\right)$$

Therefore, the solution set for the given inequation is

$$(-\infty,-3)\cup\left(\frac{7}{3},\infty\right)$$

P1: Solve: $|\mathbf{4}-\boldsymbol{x}|+\mathbf{1}<3$**.**

A. $(-6,-2)$
B. $(-\infty,2)\cup(6,\infty)$
C. $(2,6)$
D. $(-\infty,-6)\cup(-2,\infty)$

Answer: C

Solution: We have, $|4 - x| + 1 < 3 \Rightarrow |4 - x| < 2$

$$\Rightarrow |x - 4| < 2$$

$$\Rightarrow x \in (4 - 2, 4 + 2)$$

$$\Rightarrow x \in (2, 6)$$

Therefore, the solution set for the given inequation is $(2, 6)$

IP2: Solve the inequation: $\left|\frac{2}{x-4}\right| > 1$.

Solution: We have, $\left|\frac{2}{x-4}\right| > 1$

$x \neq 4, |x - 4| < 2 \Rightarrow x \neq 4, x \in (4 - 2, 4 + 2)$

$$\Rightarrow x \neq 4, x \in (2, 6)$$

$$\Rightarrow x \in (2, 4) \cup (4, 6)$$

Hence, the solution set of the given inequation is $(2, 4) \cup (4, 6)$

P2: Solve the inequation: $\left|\frac{3}{x-\frac{1}{2}}\right| < 1$.

A. $\left(-\infty, -\frac{7}{2}\right) \cup \left(\frac{5}{2}, \infty\right)$
B. $\left(-\infty, -\frac{5}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$
C. $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{7}{2}, \infty\right)$
D. $\left(-\infty, -\frac{5}{2}\right) \cup \left(\frac{7}{2}, \infty\right)$

Answer: D

Solution: We have, $\left|\frac{3}{x-\frac{1}{2}}\right| < 1$

$x \neq \frac{1}{2}, \left|x - \frac{1}{2}\right| > 3 \Rightarrow x \neq \frac{1}{2}, x \in \left(-\infty, \frac{1}{2} - 3\right) \cup \left(\frac{1}{2} + 3, \infty\right)$

$$\Rightarrow x \neq \frac{1}{2}, x \in \left(-\infty, -\frac{5}{2}\right) \cup \left(\frac{7}{2}, \infty\right)$$

$$\Rightarrow x \in \left(-\infty, -\frac{5}{2}\right) \cup \left(\frac{7}{2}, \infty\right)$$

Hence, the solution set of the given inequation is $\left(-\infty, -\frac{5}{2}\right) \cup \left(\frac{7}{2}, \infty\right)$

IP3: Solve: $\frac{|x+3|+x}{x+2} > 1$.

Solution: We have, $\frac{|x+3|+x}{x+2} > 1$

Clearly, LHS of this inequation is meaningful for

$$x + 2 \neq 0 \; i.e., x \neq -2$$

(i) $\quad x + 2 > 0 \Rightarrow x > -2$

$$\frac{|x+3|+x}{x+2} > 1 \Rightarrow |x + 3| + x > x + 2$$

$$\Rightarrow |x + 3| > 2 \Rightarrow |x - (-3)| > 2$$

$$\Rightarrow x > -2 \text{ and } x \in (-\infty, -3 - 2) \cup (-3 + 2, \infty)$$

$$\Rightarrow x > -2 \text{ and } x \in (-\infty, -5) \cup (-1, \infty)$$

$$\Rightarrow x \in (-1, \infty)$$

(ii) $\quad x + 2 < 0 \Rightarrow x < -2$

$$\frac{|x+3|+x}{x+2} > 1 \Rightarrow |x + 3| + x < x + 2$$

$$\Rightarrow |x + 3| < 2 \Rightarrow |x - (-3)| < 2$$

$$\Rightarrow x \in (-3 - 2, -3 + 2)$$
$$\Rightarrow x \in (-5, -1) \text{ and } x < -2$$
$$\Rightarrow x \in (-5, -2)$$

Therefore, the solution set of the given inequation is

$$(-5, -2) \cup (-1, \infty).$$

P3: Solve: $\frac{|x+2|-x}{x} < 2$.

A. $(-\infty, 2) \cup (1, \infty)$
B. $(-\infty, 0) \cup (1, \infty)$
C. $(-\infty, -1) \cup (1, \infty)$
D. $(-\infty, 1) \cup (2, \infty)$ **Answer:** B

Solution: We have, $\frac{|x+2|-x}{x} < 2$

Clearly, LHS of this inequation is meaningful for $x \neq 0$

(i) $x > 0$

$\frac{|x+2|-x}{x} < 2 \Rightarrow |x+2| - x < 2x$

$\Rightarrow |x+2| - 3x < 0$

$\Rightarrow x + 2 - 3x < 0 \qquad (\because x > 0)$

$\Rightarrow 2 < 2x$

$\Rightarrow x > 1$

$\therefore x > 0$ and $x > 1 \Rightarrow x \in (1, \infty)$

(ii) $x < 0$

$\frac{|x+2|-x}{x} < 2 \Rightarrow |x+2| - x > 2x$

$\Rightarrow |x+2| > 3x$

$\Rightarrow x + 2 > 3x \quad or \quad -(x+2) > 3x$

$\Rightarrow 2 > 2x \quad or \quad -x - 2 > 3x$

$\Rightarrow 2x < 2 \quad or \quad 4x < -2$

$\Rightarrow x < 1 \quad or \quad x < -\frac{1}{2}$

$\Rightarrow x \in (-\infty, 1) \cup \left(-\infty, -\frac{1}{2}\right)$

$\Rightarrow x \in (-\infty, 1)$

$\therefore x < 0 \ and \ x \in (-\infty, 1) \Rightarrow x \in (-\infty, 0)$

Therefore, the solution set of the given inequation is

$$(-\infty, 0) \cup (1, \infty)$$

IP4: Solve: $|x-1| + |x-2| + |x-3| \geq 6$.

Solution:

On the LHS of the given inequation there are three terms containing modulus. By equating the expressions within the modulus to zero, we get $x = 1, 2 \ and \ 3$ as critical points. These points divide real line in four parts, $(-\infty, 1), [1,2) [2,3)$ and $[3, \infty)$.

So we consider the following four cases

Case – I: When $-\infty < x < 1$

In this case, we have $|x-1| = -(x-1)\,;|x-2| = -(x-2)$

and $|x-3| = -(x-3)$

$\therefore |x-1| + |x-2| + |x-3| \geq 6$

$\Rightarrow -(x-1) - (x-2) - (x-3) \geq 6$

$\Rightarrow -x + 1 - x + 2 - x + 3 \geq 6$

$\Rightarrow -3x + 6 \geq 6 \Rightarrow -3x + 6 - 6 \geq 6 - 6$

$\Rightarrow -3x \geq 0 \Rightarrow x \leq 0$

$\Rightarrow x \in (-\infty, 0] \qquad \ldots (i)$

Case – II: When $1 \leq x < 2$

In this case, we have $|x-1| = (x-1)\,;\,|x-2| = -(x-2)$

and $|x-3| = -(x-3)$

$\therefore |x-1| + |x-2| + |x-3| \geq 6$

$\Rightarrow (x-1) - (x-2) - (x-3) \geq 6$

$\Rightarrow x - 1 - x + 2 - x + 3 \geq 6$

$\Rightarrow -x + 4 \geq 6 \Rightarrow -x \geq 6 - 4$

$\Rightarrow -x \geq 2 \Rightarrow x \leq -2 \Rightarrow x \in (-\infty, -2)$

Therefore, $x \in [1,2)$ *and* $x \in (-\infty, -2) \Rightarrow$ The given inequation has no solution

Case – III: When $2 \leq x < 3$

In this case, we have $|x-1| = (x-1)\,;|x-2| = (x-2)$

and $|x-3| = -(x-3)$

$\therefore |x-1| + |x-2| + |x-3| \geq 6$

$\Rightarrow (x-1) + (x-2) - (x-3) \geq 6$

$\Rightarrow x - 1 + x - 2 - x + 3 \geq 6 \Rightarrow x \geq 6$

$\Rightarrow x \in [6, 0)$

Therefore, $x \in [2,3)$ and $x \in [6, 0)$ ⇒ The given inequation has no solution

Case – IV: When $x \geq 3$

In this case, we have $|x-1| = (x-1)$; $|x-2| = (x-2)$

and $|x-3| = (x-3)$

$\therefore |x-1| + |x-2| + |x-3| \geq 6$

$\Rightarrow (x-1) + (x-2) + (x-3) \geq 6$

$\Rightarrow x - 1 + x - 2 + x - 3 \geq 6$

$\Rightarrow 3x - 6 \geq 6 \Rightarrow 3x \geq 6 + 6$

$\Rightarrow 3x \geq 12 \Rightarrow x \geq 4$

$\Rightarrow x \in [4, \infty)$ ... (ii)

Combining (i) and (ii), we obtain the solution set of the given inequation

$$(-\infty, 0\,] \cup [\,4, \infty).$$

P4: Solve: $|x-1| + |x-2| \geq 4.$

E. $(-\infty, 1] \cup [2, \infty)$
F. $\left(-\infty, \frac{1}{2}\right] \cup [2, \infty)$
G. $\left(-\infty, -\frac{1}{2}\right] \cup [\frac{7}{2}, \infty)$
H. $\left(-\infty, -\frac{7}{2}\right] \cup [\frac{1}{2}, \infty)$

Answer: C

Solution:

On the LHS of the given inequation there are two terms both containing modulus. By equating the expressions within the modulus to zero, we get $x = 1, 2$ as critical points. These points divide real line in three parts, $(-\infty, 1), [1,2)$ and $[2, \infty)$.

So we consider the following three cases

Case – I: When $-\infty < x < 1$

In this case, we have $|x-1| = -(x-1)$; and $|x-2| = -(x-2)$

$\therefore |x-1|+|x-2| \geq 4$

$\Rightarrow -(x-1)-(x-2) \geq 4$

$\Rightarrow -x+1-x+2 \geq 4 \Rightarrow -2x+3 \geq 4$

$\Rightarrow -2x \geq 4-3 \Rightarrow -2x \geq 1 \Rightarrow x \leq -\frac{1}{2}$

But, $-\infty < x < 1$. Therefore, in this case the solution set of the given inequation is $(-\infty, -\frac{1}{2}]$ ... (i)

Case – II: When $1 \leq x < 2$

In this case, we have $|x-1| = (x-1)$; and $|x-2| = -(x-2)$

$\therefore |x-1|+|x-2| \geq 4$

$\Rightarrow (x-1)-(x-2) \geq 4$

$\Rightarrow x-1-x+2 \geq 4$

$\Rightarrow 1 \geq 4$, which is an absurd result.

So, the given inequation has no solution for $x \in [1,2)$.

Case – III: When $x \geq 2$

In this case, we have $|x-1| = (x-1)$; and $|x-2| = (x-2)$

$\therefore |x-1|+|x-2| \geq 4$

$\Rightarrow (x-1)+(x-2) \geq 4$

$\Rightarrow x-1+x-2 \geq 4 \Rightarrow 2x-3 \geq 4$

$\Rightarrow 2x \geq 4+3 \Rightarrow 2x \geq 7 \Rightarrow x \geq \frac{7}{2}$

But, $x \geq 2$.

Therefore, the solution set of the given inequation in this case is $[\frac{7}{2}, \infty)$ ... (ii)

Combining (i) and (ii), we obtain the solution set of the given inequation $(-\infty, -\frac{1}{2}] \cup [\frac{7}{2}, \infty)$

Exercise:

1. Rewrite without the absolute value sign:

$a)\ |x| < 3,\qquad b)\ |x-2| < 5$

2. Solve the following inequations

$a)\qquad (x-2)^2 \geq 4$

$b)\qquad x^2 + 2x - 8 \leq 0$

$c)\qquad |x^2 - 7x + 12| > x^2 - 7x + 12$

3. Solve the following inequations

$a)\qquad |x-3| < 4$

$b)\qquad |x+4| \geq 2$

$c)\qquad \left|\frac{1}{|2x-3|}\right| > 5$

4. Solve the following inequations

$a)\qquad \frac{|x-1|}{x+2} < 1$

$b)\qquad \frac{|x-2|}{x-2} > 0$

$c)\qquad \left|\frac{2x-1}{x-1}\right| > 2$

2.3. Systems of Linear Inequations

Learning Objectives:

- To study the linear inequation in two variables and to represent the solution set graphically

  AND

- To solve a system of linear inequations graphically and to find the solution region

Systems of Linear Inequations

Linear Inequations in Two Variables

A linear inequation in two variables is any expression that can be put in the form

$$ax + by < c$$

where a, b, and c are real numbers (a and b not both zero). The inequality symbol can be any one of the following four: $<, \leq, >, \geq$.

Some examples of linear inequations are

$$2x + 3y < 6 \qquad y \geq 2x + 1 \qquad x - y \leq 0$$

Although not all of these examples have the form $ax + by < c$, each one can be put in that form.

The solution set for a linear inequation in two variables is a section of the coordinate plane. The ***boundary*** for the section is found by replacing the inequality symbol with an equal sign and graphing the resulting equation. The ***boundary is included*** in the solution set and is represented by a ***solid line***, if the inequality symbol is $\leq$ or $\geq$. The ***boundary is not included***, indicated by a ***broken line***, if the symbol is $<$ or $>$.

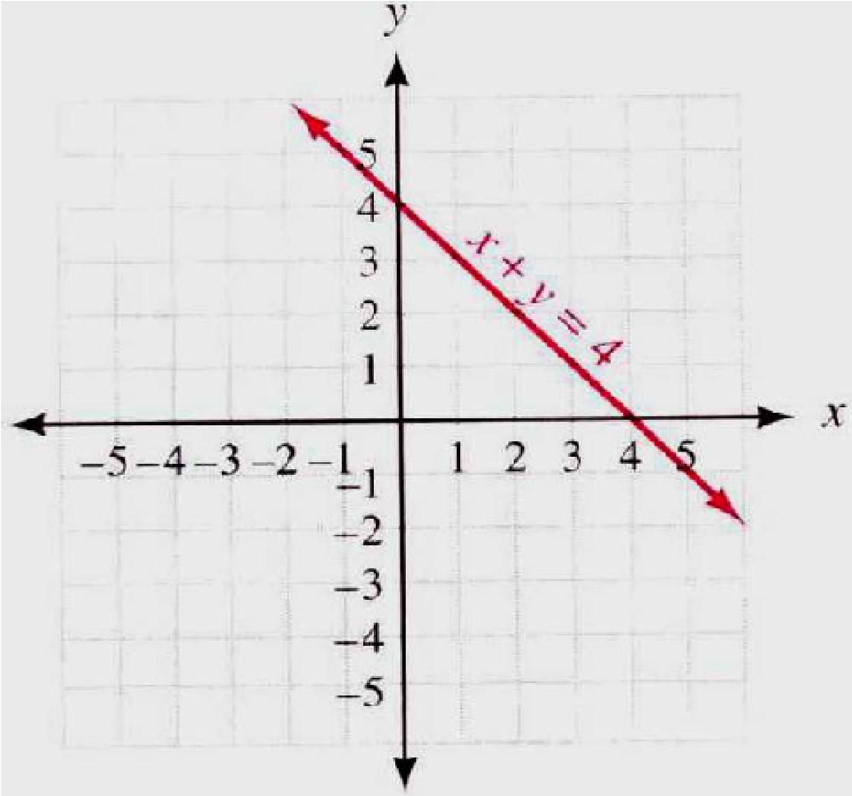
Example 1

Graph the solution set for $x + y \leq 4$.

Solution

The boundary for the required graph is the graph of $x + y = 4$. The boundary is included in the solution set because the inequality symbol is $\leq$.

Figure below is the graph of the boundary.



The boundary separates the coordinate plane into two regions: the region above the boundary and the region below it. The solution set for $x + y \leq 4$ is one of these two regions along with the boundary.

To find the correct region, we simply choose any convenient point that is not on the boundary. We then substitute the coordinates of the point into the given inequation $x + y \leq 4$. If the point we have chosen satisfies the inequation, then it is a member of the solution set, and we can assume that all points on the same side of the boundary as the chosen point are also in the solution set.

If the coordinates of our chosen point do not satisfy the given inequality, then the solution set lies on the other side of the boundary.

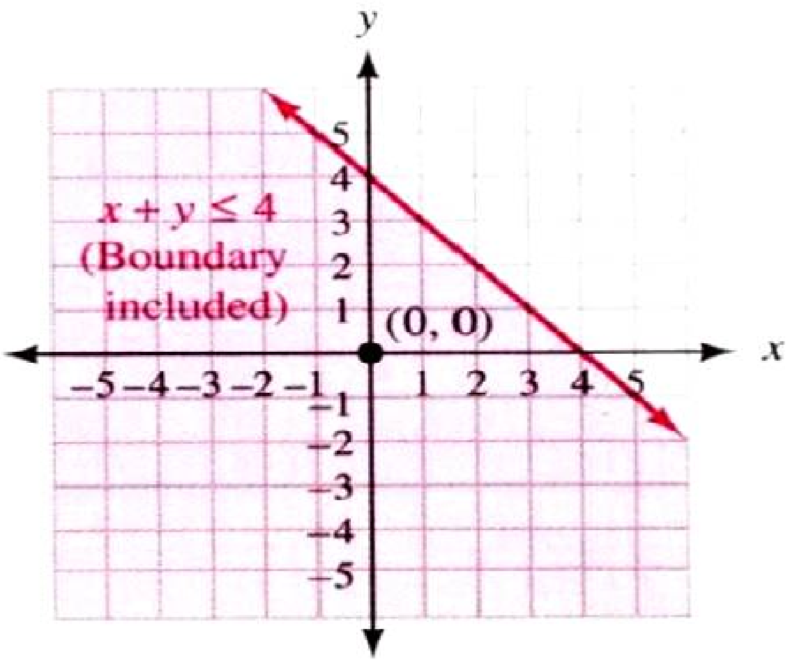
In this example, a convenient point that is not on the boundary is the origin. Substituting $(0,0)$ into the given inequality gives us

$$0 + 0 \leq 4$$

which is a true statement.

Because the origin is a solution to the inequality $x + y \leq 4$ and the origin is below the boundary, all other points below the boundary are also solutions.

Figure below is the graph of $x + y \leq 4$.

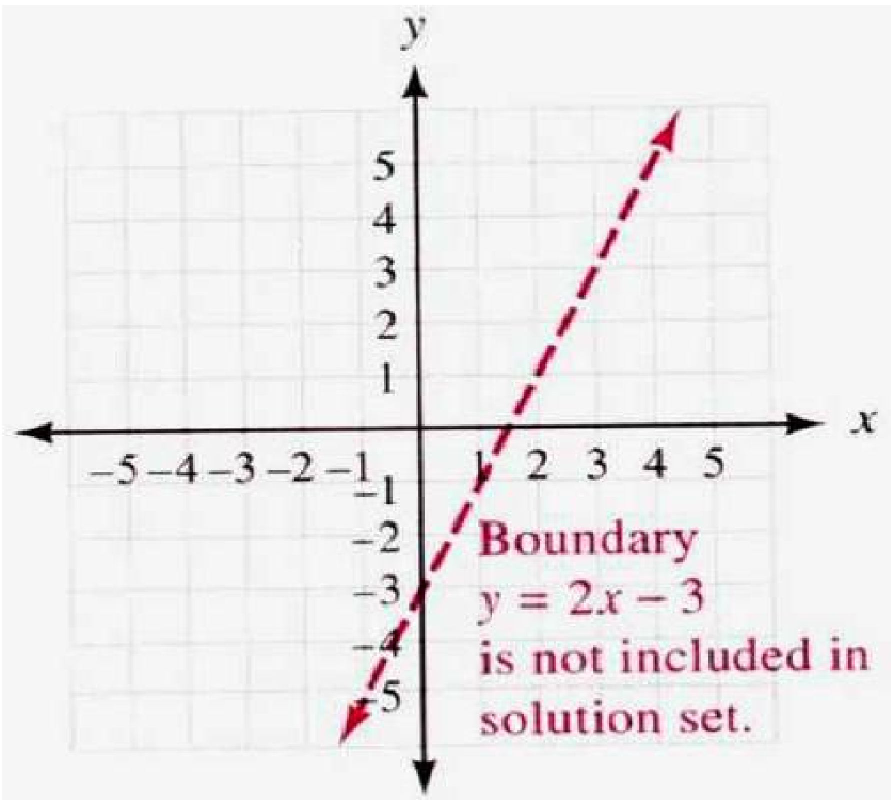


We note that the region above the boundary is described by the inequality $x + y \geq 4$.

Example 2 : Graph the solution set for $y < 2x - 3$.

Solution

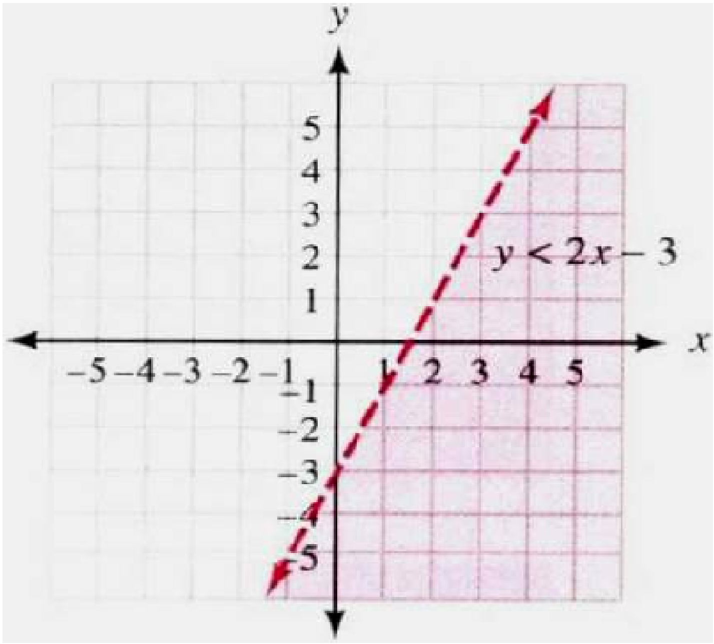
The boundary is the graph of $y = 2x - 3$, a line with slope 2 and y-intercept -3. The boundary is not included because the inequality symbol is $<$. We therefore use a broken line to represent the boundary.



We choose the origin as test point which results in

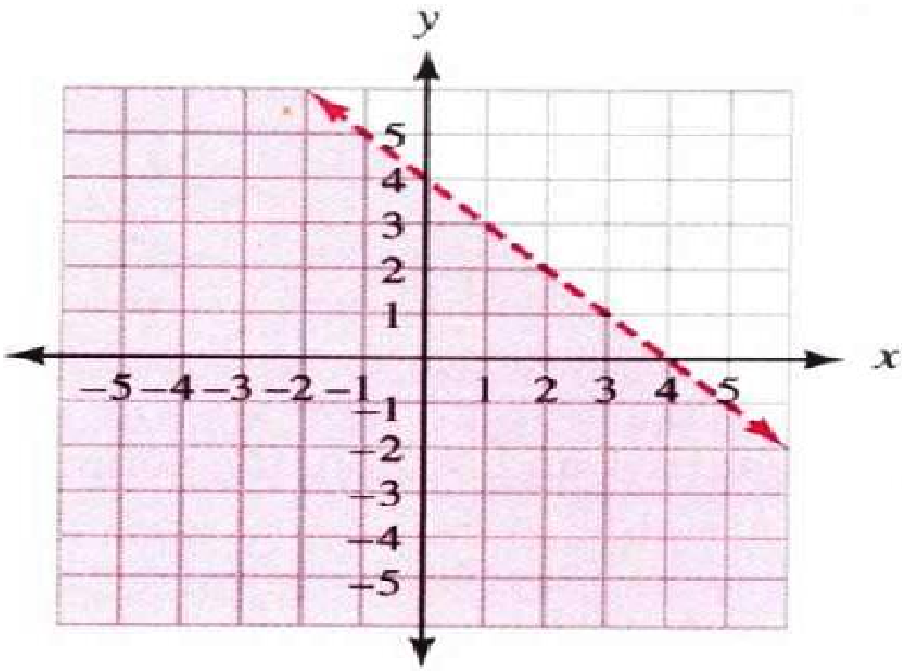
$$0 < 2(0) - 3$$
$$i.e.,\ 0 < -3$$

a false statement. Therefore the solution set must lie on the non-origin side of the boundary.



System of Two Linear Inequations

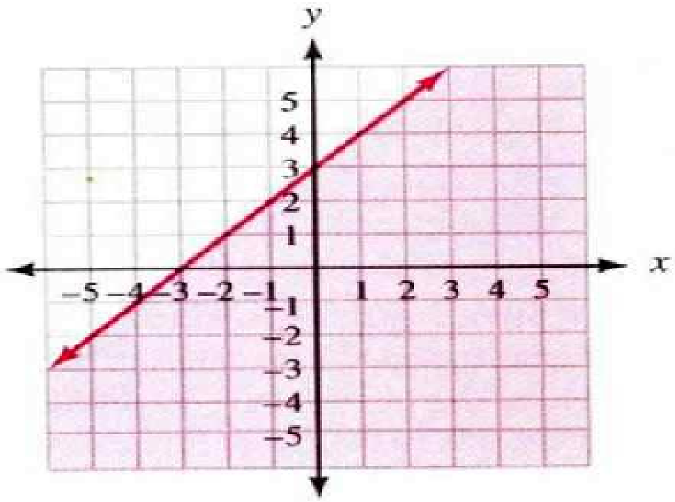
Figure below shows the graph of the inequation $x + y < 4$.



We note that the boundary is not included in the solution set, and is therefore drawn as a broken line.

Figure below shows the graph of $-x + y \leq 3$.

We note that the boundary is drawn with a solid line, because it is a part of the solution set.

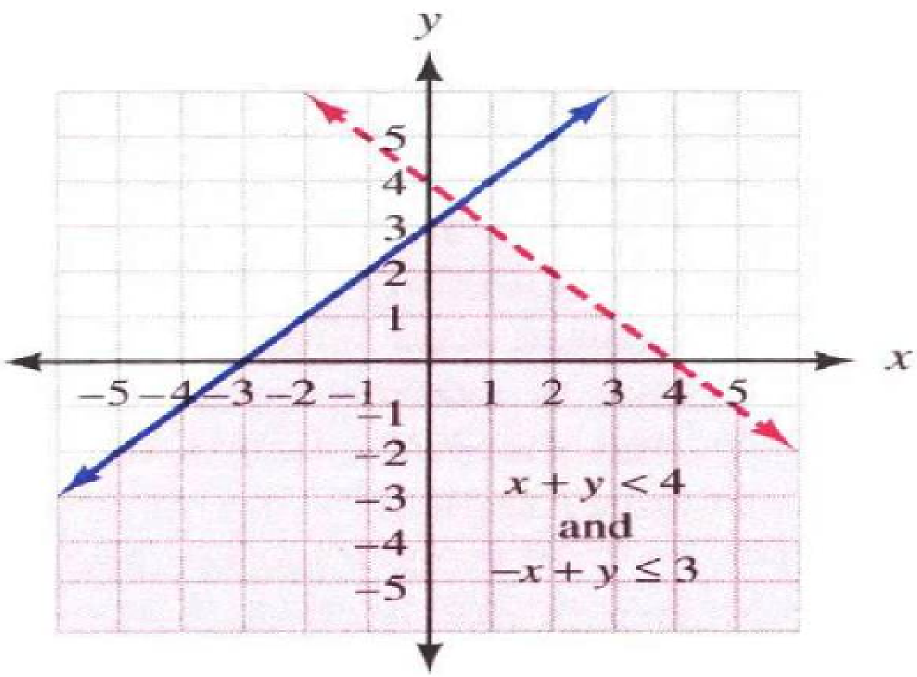


Now, we form the system of inequations with the two inequations

$$x + y < 4 \;\&\; -x + y \leq 3$$

and seek the solution of this system of linear inequations.

The solution set will be all the points common to both solution sets shown in the two figures above. It is the intersection of the two solution sets. The graph of the solution set is shown in the figure below.

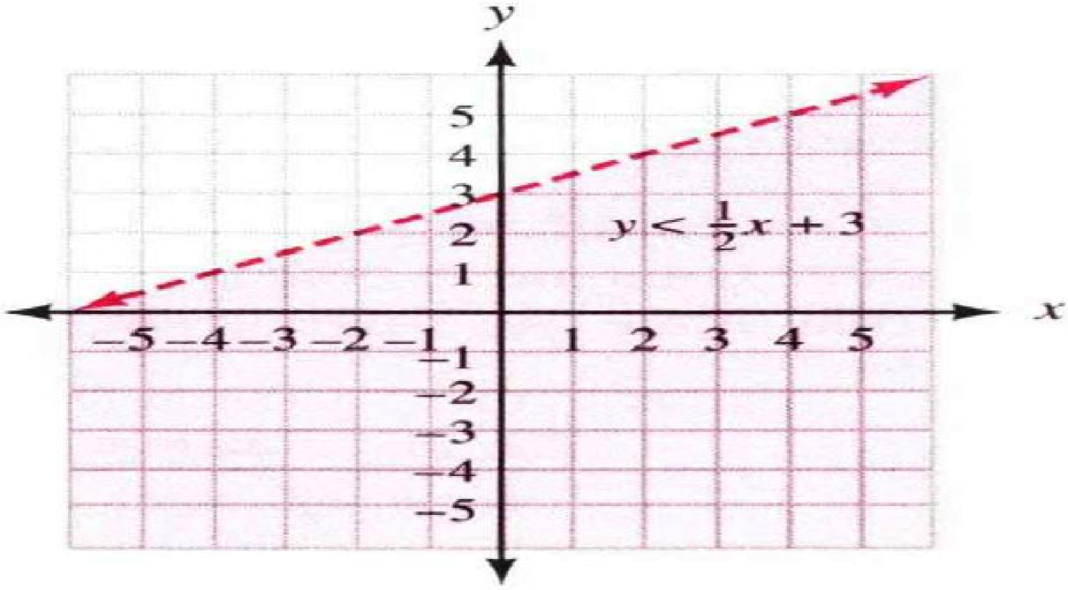


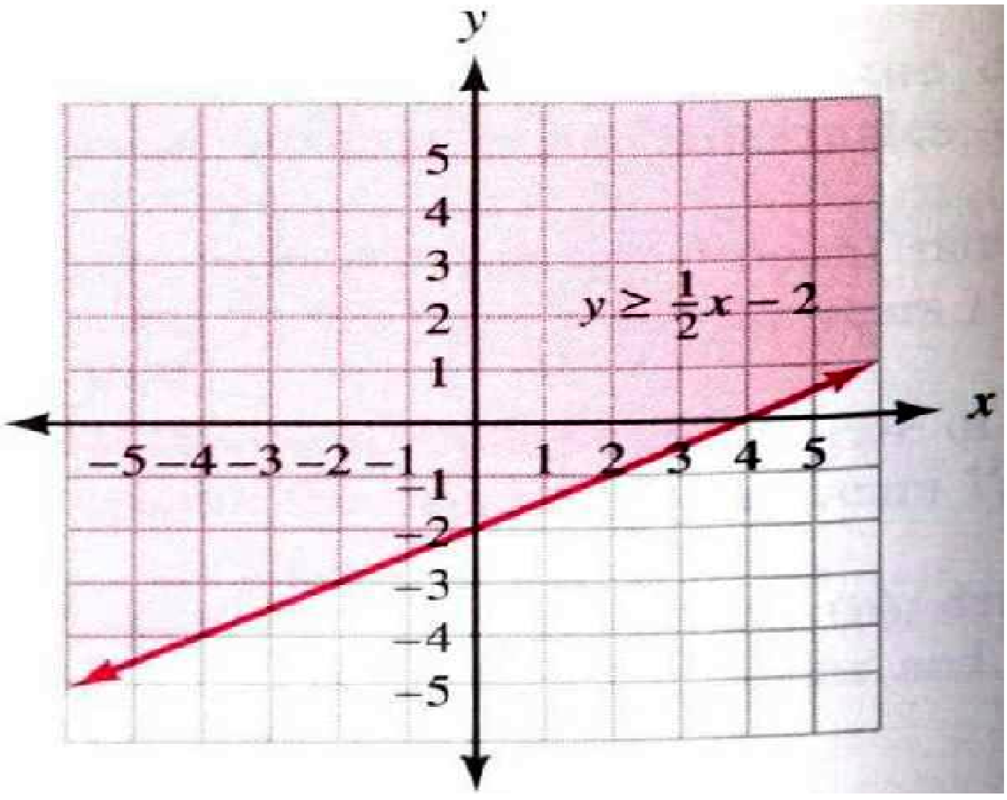
Thus: To find the solution set to the system, we graph each of the inequations on the same coordinate system. The solution set is the region that is common to all the regions graphed.

Example 3: Graph the solution to the system of linear inequations

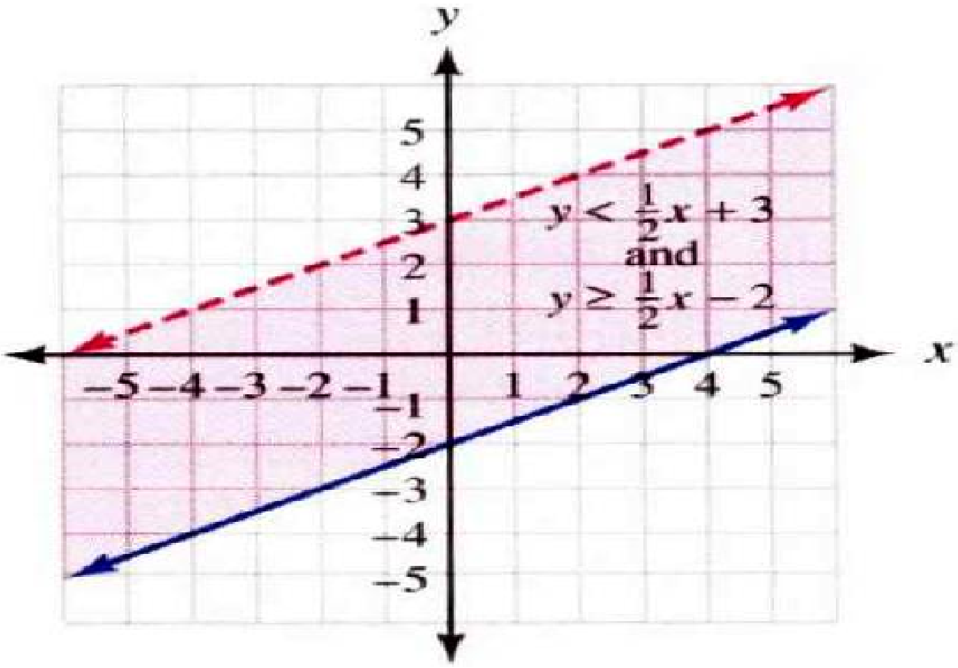
$$y < \frac{1}{2}x + 3, \quad y \geq \frac{1}{2}x - 2$$

Solution: The two figures below show the solution set for each of the inequations separately.





The figure below is the solution set to the system of inequations. It is the region consisting of points whose coordinates satisfy both inequations.

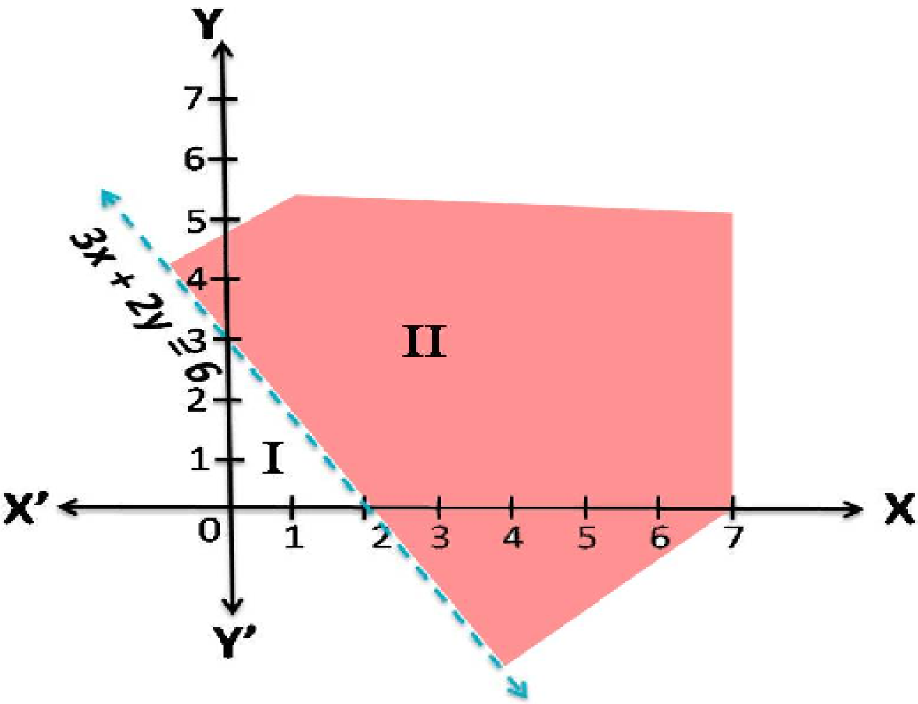


PROBLEM SET

IP1: Solve $3x + 2y > 6$ graphically.

Solution:

Graph of $3x + 2y = 6$ is given as dotted line in the figure given below.



This line divides the xy - plane in two half planes I and II. We select a point (not on the line), say $(0,0)$, which lies in the half plane I and determine if this point satisfies the given inequation.

We note that

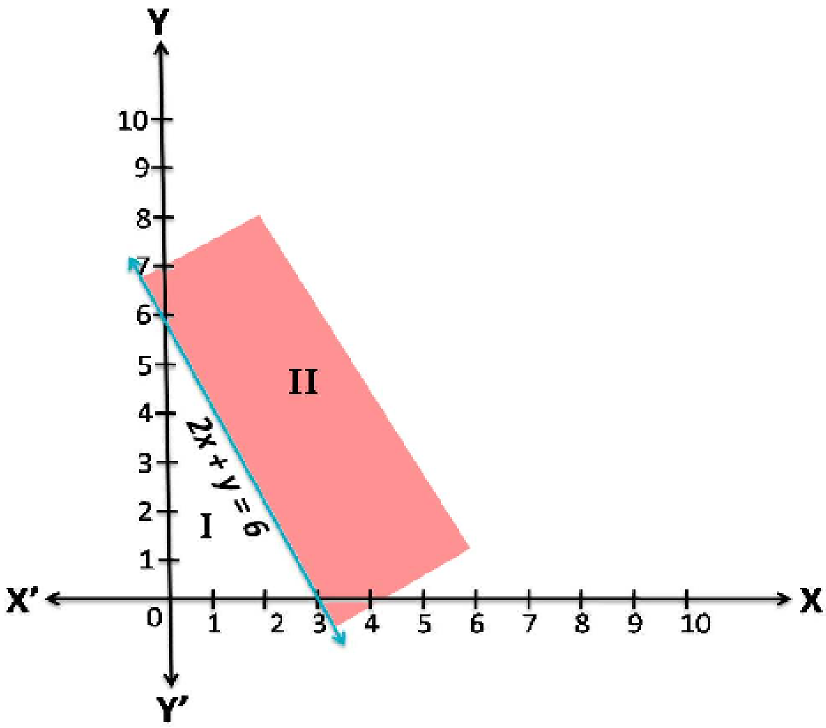
$$3(0) + 2(0) > 6 \Rightarrow 0 > 6, \text{ which is false.}$$

Hence, the half plane I is not the solution region of the given inequation. Clearly, any point on the line does not satisfy the given inequation. In other words, the shaded half plane II excluding the points on the line is the solution region of the inequation.

P1: Solve $2x + y \geq 6$ graphically.

Solution:

Graph of $2x + y = 6$ is given as straight line in the figure given below.



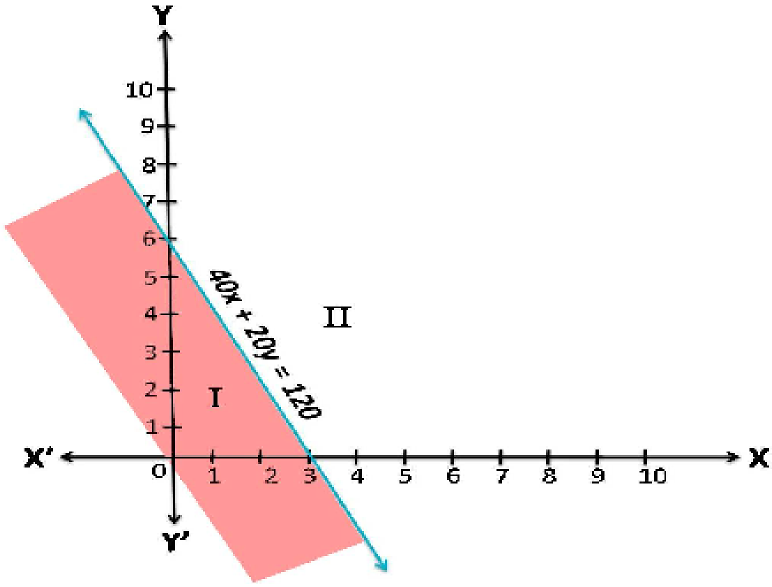
This line divides the xy - plane in two half planes I and II. We select a point (not on the line), say $(0,0)$, which lies in half plane I and determine if this point satisfies the given inequality. We note that

$$2(0) + 0 \geq 6 \Rightarrow 0 \geq 6, \text{ which is false.}$$

Hence, the half plane I is not the solution region of the given inequation. Clearly, any point on the line satisfy the given inequation. In other words, the shaded half plane II including the points on the line is the solution region of the inequation.

IP2: Solve $40x + 20y \le 120$ graphically.

Solution: The graph of $40x + 20y = 120$ is a straight line as shown in the figure.



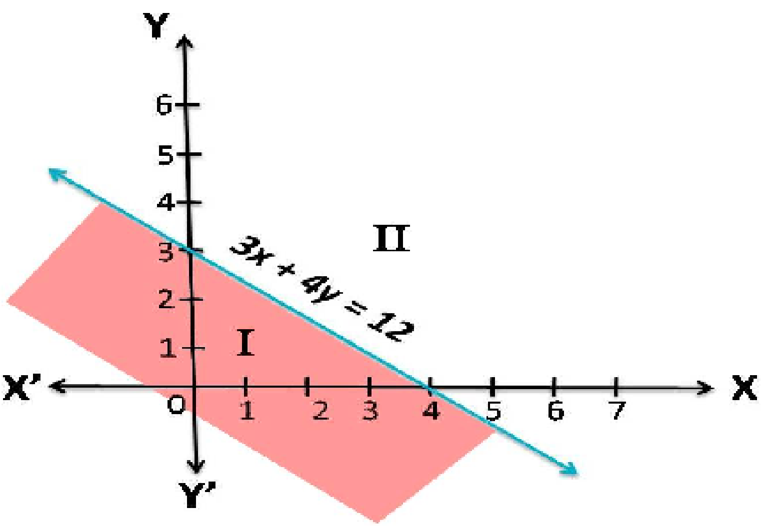
This line divides the xy - plane in two half planes I and II. We select a point (not on the line), say $(0,0)$, which lies in the half plane I and determine if this point satisfies the given inequation, we note that

$$40(0) + 20(0) \le 120 \Rightarrow 0 \le 120, \text{ which is true.}$$

Hence, the half plane I is the solution region of the given inequation. Clearly, any point on the line satisfies the given inequation. In other words, the shaded half plane I including the points on the line is the solution region of the inequation.

P2: Solve $3x + 4y \le 12$ graphically.

Solution: The graph of $3x + 4y = 12$ is a straight line shown in the figure.



This line divides the xy - plane in two half planes I and II. We select a point (not on the line), say $(0,0)$, which lies the half plane I and determine if this point satisfies the given inequation, we note that

$$3(0) + 4(0) \leq 12 \Rightarrow 0 \leq 12, \text{ which is true.}$$

Hence, the half plane I is the solution region of the given inequation. Clearly, any point on the line satisfies the given inequation. In other words, the shaded half plane I including the points on the line is the solution region of the inequation.

IP3: Solve the following system of inequations

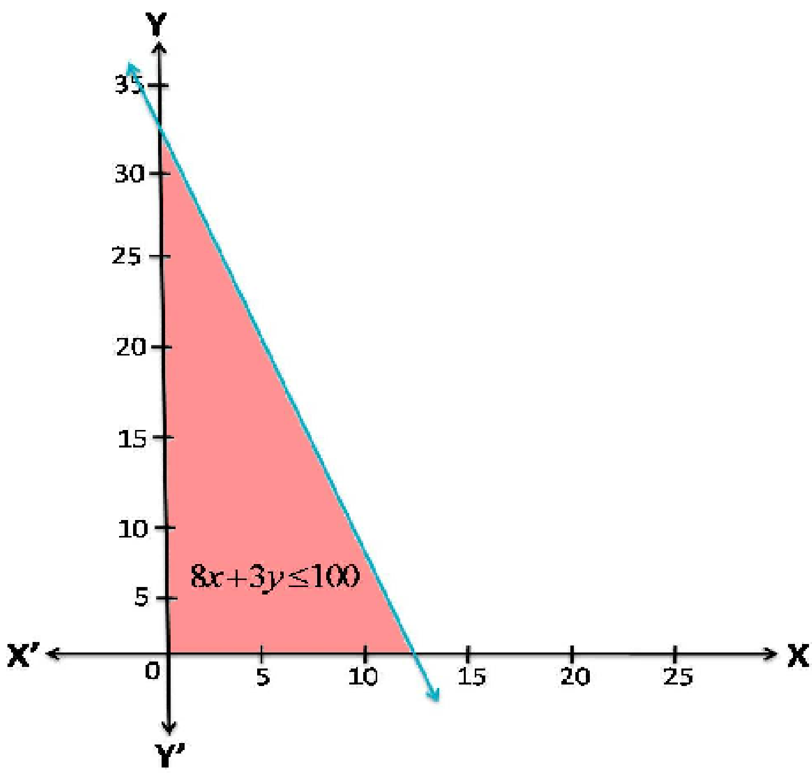
$\mathbf{8x + 3y \leq 100, x \geq 0, y \geq 0}$ **graphically.**

Solution:

We draw the graph of the line $8x + 3y = 100$

The inequation $8x + 3y \leq 100$ represents the shaded region below the line, including the points on the line $8x + 3y = 100$.

Since $x \geq 0, y \geq 0$, every point in the shaded region in the first quadrant, including the points on the line and the axes, represents the solution of the given system of inequations.



P3: Solve the following system of inequations graphically

$\mathbf{5x + 4y \le 40, x \ge 2, y \ge 3.}$

Solution: We first draw the graphs of the lines

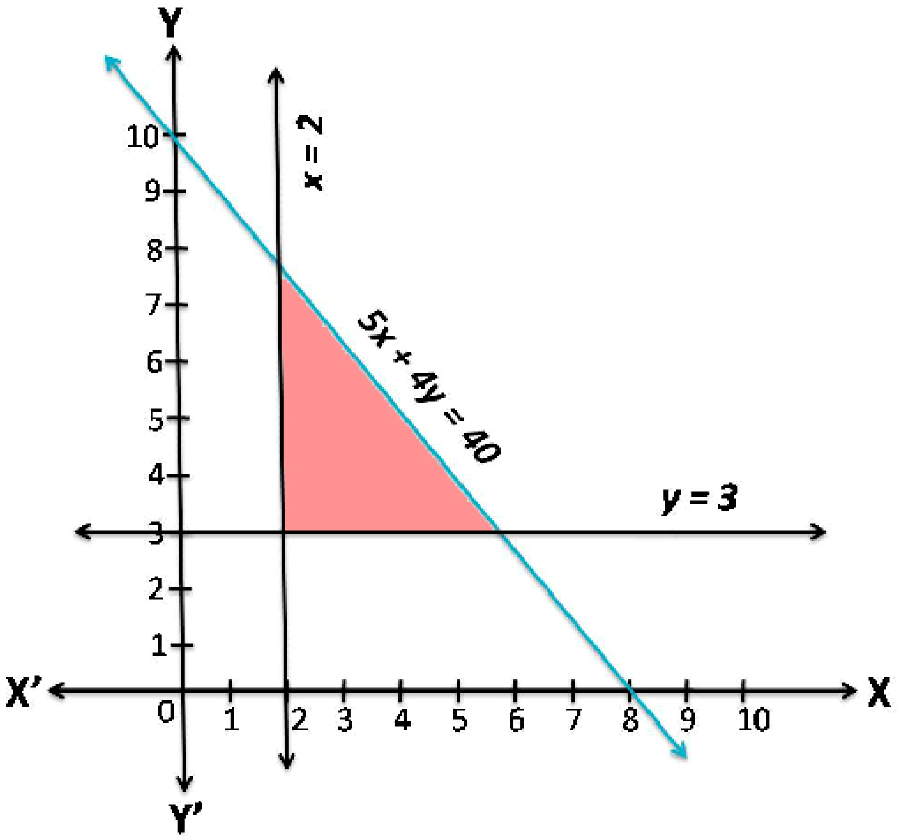
$$5x + 4y = 40, \qquad \dots(1)$$

$$x = 2 \qquad \dots(2)$$

and $y = 3$ $\dots(3)$

Then we note that the inequation (1) represents the region below the line $5x + 4y = 40$, including the points on the line, the inequation (2) represents the region right of line $x = 2$, including the points on the line and the inequation (3) represents the region above the line $y = 3$, including the points on the line.

The intersection of these three regions is the shaded region shown in the figure and it is the solution set of the given system of linear inequations.



IP4:

Solve the following system of inequations graphically

$\mathbf{3x + 4y \le 60, x + 3y \le 30, x \ge 0, y \ge 0.}$

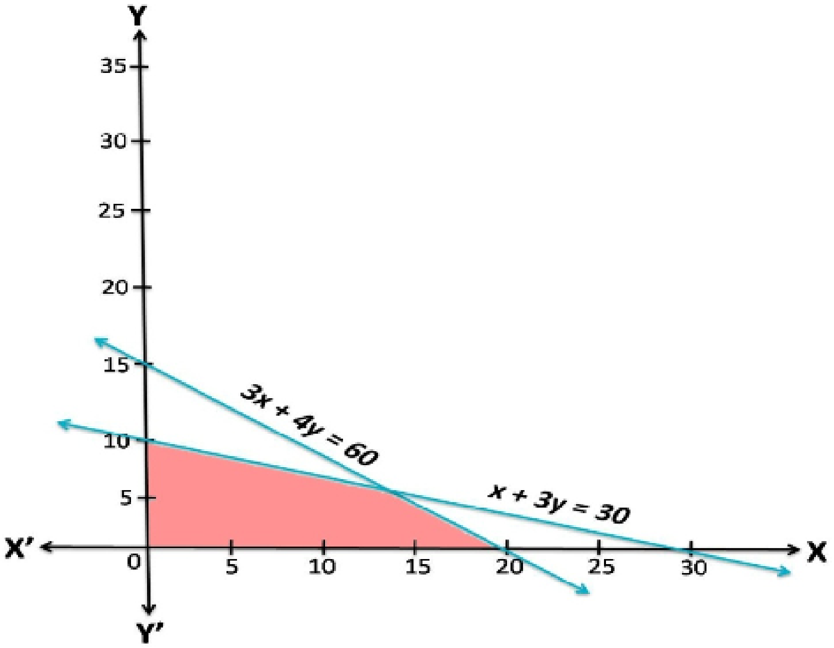
Solution:

We first draw the graphs of the lines

$$3x + 4y = 60, \qquad \dots(1)$$

and $$x + 3y = 30 \qquad \dots(2)$$

The inequations (1) and (2) represent the region below the two lines, including the points on the respective lines.



Since $x \geq 0, y \geq 0$, every point in the shaded region in the first quadrant represent a solution of the given system of inequations.

P4: Solve the following system of inequations graphically

$\boldsymbol{x + 2y \leq 8, 2x + y \leq 8, x \geq 0, y \geq 0.}$

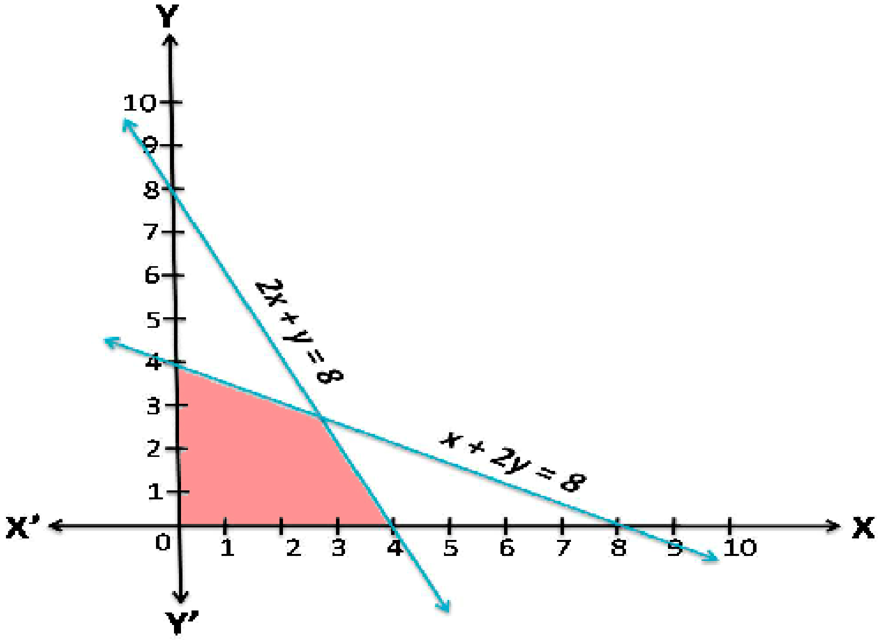
Solution:

We first draw the graphs of the lines

$$x + 2y = 8, \qquad \dots(1)$$

and $$2x + y = 8 \qquad \dots(2)$$

The inequations (1) and (2) represent the region below the two lines, including the points on the respective lines.



Since $x \geq 0, y \geq 0$, every point in the shaded region in the first quadrant represent a solution of the given system of inequations.

Exercise:

Note: The student is expected to solve at least TWO problems from each question. The institute has to supply the graph sheets.

1. Represent the solution set of the following inequations of two variables in the coordinate plane.
    a. $x + y < 5$
    b. $3y < 6 - 2x$
    c. $-3x + 2y \leq 6$
    d. $x + 2y \geq 6$
    e. $y + 8 \geq 2x$
    f. $2x - 5y + 10 > 0$

2. Solve the following systems of linear inequations graphically
    a. $x + y \geq 4, 2x - y > 0$
    b. $2x - y > 1, x - 2y < -1$
    c. $x + y \leq 6, x + y \geq 4$
    d. $2x + y \geq 8, x + 2y \geq 10$
    e. $2x + 3y \leq 6, 3x + 2y \leq 6, x \geq 0, y \geq 0$
    f. $2x + 3y \leq 6, x + 4y \leq 4, x \geq 0, y \geq 0$
    g. $3x + 4y \geq 12, x - 2y \leq 3, x \geq 0, y \geq 1$

3.1. Sequences and Series

Learning objectives

- To define the concepts of
    - A sequence of real numbers
    - A monotonic sequence
    - An alternating series
- To define a sequence through a recursion formula
- To define the convergence of the sequence of real numbers
- To study the concepts of series, finite and infinite series, partial sum of the series and the convergence of the series
- To study the summation notation of the finite series

  AND
- To study the related problems

A ***sequence*** of real numbers is a function from $\boldsymbol{N}$, the set of natural numbers, into $\boldsymbol{R}$, the set of real numbers.

That is, a sequence of real numbers assigns a unique real number for each natural number $n = 1,2,3, \dots$. The real numbers so obtained are called the ***terms*** of the sequence.

It is customary to denote the real number assigned to $n \in \boldsymbol{N}$ in a sequence by a symbol a_n. Thus, if $f : \boldsymbol{N} \longrightarrow \boldsymbol{R}$ is a sequence, we denote the image f at n by a_n, instead of $f(n)$. We denote this sequence by $\{a_n\}_{n=1}^{\infty}$. For the sequence $\{a_n\}_{n=1}^{\infty}$; $a_1, a_2, a_3, \dots$ are called respectively first term, second term, the third term … .

The n^{th} term of a sequence is called the ***General Term*** of the sequence. If we are given the formula for the general term a_n we can find any other term in the sequence.

Example

Find the first four terms of the sequence whose general term is given by $a_n = \frac{(-1)^{n+1}}{2n-1}$

Solution: Given $a_n = \frac{(-1)^{n+1}}{2n-1}$

$$a_1 = \frac{(-1)^2}{2-1} = 1 \quad , \quad a_2 = \frac{(-1)^3}{4-1} = -\frac{1}{3},$$

$$a_3 = \frac{(-1)^4}{6-1} = \frac{1}{5} \quad , \quad a_4 = \frac{(-1)^5}{8-1} = -\frac{1}{7}$$

Recursion formula

A sequence may also be described by specifying its few terms and a formula which expresses the n^{th} term in terms of its preceding terms. This formula is called a ***recursion formula*** because each term is written recursively in terms of the terms that precede it.

Fibonacci sequence

The sequence $\{a_n\}_{n=1}$ defined by

$$a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-2}\,,\ n \geq 3 \ \text{(recursion formula)}$$

is called *Fibonacci sequence*.

The sequence $\{a_n\}_{n=1}^{\infty}$ is given by

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots, \dots$$

Finding the General term

We now find a formula for the general term of a given sequence from the first few terms. It is not always automatic and it takes a while to recognize a pattern.

Example

Find the general term of the sequence.

$$2, \frac{3}{8}, \frac{4}{27}, \frac{5}{64}, \dots, \dots$$

Solution: Observe that the denominators of the terms are all perfect cubes of $1,2,3,4, \dots$ respectively and the numerators are all one more than the base of the cubes in the denominators.

Thus,

$$a_1 = \frac{2}{1} = \frac{1+1}{1^3}, a_2 = \frac{3}{8} = \frac{2+1}{2^3}, a_3 = \frac{4}{27} = \frac{3+1}{3^3}, a_4 = \frac{5}{64} = \frac{4+1}{4^3}$$

Observing the pattern, we recognize the general term to be

$$a_n = \frac{n+1}{n^3}$$

Note: Sequences following specific patterns are called ***progressions***. Some examples of progressions are ***Arithmetic Progressions, Geometric Progressions*** and ***Harmonic Progressions***.

Monotonic Sequences

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be

(i) ***Monotonically increasing*** (or ***monotonically non-decreasing***) sequence if $a_n \leq a_{n+1}, \forall\, n \in \boldsymbol{N}$.

(ii) ***Monotonically decreasing*** (or ***monotonically non-increasing***) sequence if $a_n \geq a_{n+1}\,, \forall\, n \in \boldsymbol{N}$.

A sequence which is either monotonically increasing or monotonically decreasing is called a ***monotonic sequence***.

Example

(i) The sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = n^2, n \in \boldsymbol{N}$, i.e., $1^2, 2^2, 3^2, 4^2, \dots$ is a monotonically increasing sequence since $n^2 < (n+1)^2\ \forall\, n \in \boldsymbol{N}$,
i.e., $a_n < a_{n+1}\ \forall\, n \in \boldsymbol{N}$.

(ii) The sequence $\{b_n\}_{n=1}^{\infty}$, where $b_n = \frac{1}{n}, n \in \boldsymbol{N},$ i.e., $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ is a monotonically decreasing sequence since $\frac{1}{n} \geq \frac{1}{n+1}$ $\forall\, n \in \boldsymbol{N},$

i.e., $b_n > b_{n+1}\ \forall\, n \in \boldsymbol{N}$.

(iii) The sequence $\{c_n\}_{n=1}^{\infty}$, where $c_n = \begin{cases} 1 & , \quad n \text{ is odd} \\ 0 & , \quad n \text{ is even} \end{cases}$

i.e., $1,0,1,0,1,0, \ldots$ is neither monotonically increasing nor monotonically decreasing.

Alternating sequence

A sequence is said to be an ***alternating sequence*** if the terms in the sequence are alternate in sign.

Example

The sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = \begin{cases} 1 & , \quad n \text{ is odd} \\ -1 & , \quad n \text{ is even} \end{cases}$

i.e., $1, -1, 1, -1, 1, -1, \ldots$ is an alternating sequence.

Convergence of a sequence

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be ***convergent*** if there is a real number l such that $\lim\limits_{n \to \infty} a_n = l$. A sequence is said to be ***divergent*** if it is not convergent.

(i) The sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = \frac{1}{n}$ is a convergent sequence,

since $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \frac{1}{n} = 0$

That is the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ converges to 0

Example

Show that the sequence $2, \frac{5}{4}, \frac{10}{9}, \frac{17}{16}, \frac{26}{25} \ldots$ converges to 1

Solution: The given sequence can be written as

$$1 + 1, 1 + \frac{1}{4}, 1 + \frac{1}{9}, 1 + \frac{1}{16}, 1 + \frac{1}{25}, \ldots$$

$$\text{i.e., } 1 + \frac{1}{1^2}, 1 + \frac{1}{2^2}, 1 + \frac{1}{3^2}, 1 + \frac{1}{4^2}, 1 + \frac{1}{5^2}, \ldots$$

From the above pattern, the general term of the sequence $\{a_n\}_{n=1}^{\infty}$ is given by $a_n = 1 + \frac{1}{n^2}$.

Notice that $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \left(1 + \frac{1}{n^2}\right) = \lim\limits_{n \to \infty} 1 + \lim\limits_{n \to \infty} \frac{1}{n^2} = 1.$

Series

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. The expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is called an ***infinite series*** of simply ***series***.

It is symbolically expressed in ***summation notation*** or ***sigma notation*** as

$$\sum_{n=1}^{\infty} a_n \quad or \quad simply \quad \sum a_n$$

The letter n is called the ***index of the summation***.

The real numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ are called the first, second, third, …,n^{th},… terms respectively of the series. The n^{th} term of the series is called the ***general term*** of the series.

Example

(i) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ or simply $\sum_{n=1}^{\infty} \frac{1}{n}$ is an infinite series.

Finite series

If the sequence $\{a_n\}_{n=1}^{\infty}$ is such that $a_n = 0, \forall\, n > m$, then the expression

$a_1 + a_2 + \cdots + a_m$ or $\sum_{n=1}^{m} a_n$ is called a ***finite series***.

i.e., if the number of the terms in a series is finite or limited then the series is a finite series.

Example

$1^2 + 2^2 + 3^2 + \cdots + 99^2$ or $\sum_{n=1}^{99} n^2$ is a finite series.

Note: A series always mean an infinite series, unless stated otherwise.

Partial sum of a series

The n^{th} partial sum of a series $\sum_{n=1}^{\infty} a_n$ is denoted by s_n and it is the sum of first n terms of the series,

$$\text{i.e., } s_n = a_1 + a_2 + \cdots + a_m = \sum_{k=1}^{n} a_k .$$

Convergence of series

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers with partial sums $s_n = a_1 + a_2 + \cdots + a_n$.

If the sequence $\{s_n\}$ converges to s (i.e $\lim_{n \to \infty} s_n = s$), then we say that the series $\sum_{n=1}^{\infty} a_n$ converges to s. The real number s is called the ***sum of the series*** and we write $\sum_{n=1}^{\infty} a_n = s$. If $\lim_{n \to \infty} s_n$ does not exist (i.e., the sequence $\{s_n\}_{n=1}^{\infty}$ diverges) then we say that the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example

Show that the series $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$ converges to 1.

Solution: We observe that the general term of the series is given by $a_n=\frac{1}{n(n+1)}$.

Notice that $a_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$. Now,

$s_n = n^{\text{th}}$ partial sum $= a_1+a_2+a_3+\cdots+a_n$

$=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}$

$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{n+1}$

Note that $\lim_{n\to\infty} s_n=\lim_{n\to\infty} s_n\left(1-\frac{1}{n+1}\right)=1$

Thus, $\{s_n\}$ converges to 1. Therefore the given series converges to 1 and

$$\sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)}\right)=1.$$

You will learn more details on the convergence and divergence of sequences and series of real numbers in higher classes.

PROBLEM SET

IP1. Find the $\mathbf{11^{th}}$ term of the following sequence

$$\mathbf{2, 1, 3, 4, 7, 11, 18, \ldots.}$$

Solution:

Let $a_1=2$, $a_2=1$. Observing the given sequence notice that

$$a_n=a_{n-1}+a_{n-2}\quad (n\geq 2)$$
$$a_8=a_7+a_6=18+11=29$$
$$a_9=a_8+a_7=29+18=47$$
$$a_{10}=a_9+a_8=47+29=76$$
$$a_{11}=a_{10}+a_9=76+47=123$$

Therefore, the 11$^{\text{th}}$ term of the given sequence is 123.

P1. Find the general term of the following sequence

$$\mathbf{1, 5, 14, 30, 55, \ldots}$$

Solution:

The given sequence can written as

$5=2^2+1$, $14=3^2+5$, $30=4^2+14$, $55=5^2+30, \ldots$

Observing the above pattern, the general term of the sequence is

$$a_1=1, a_n=n^2+a_{n-1}, n\geq 2$$

IP2. Which of the following is a monotonically increasing/monotonically decreasing sequence?

i) $\{(-1)^n\}_{n=1}^{\infty}$

ii) $\{3n+2\}_{n=1}^{\infty}$

Solution:

i) The given sequence is $\{(-1)^n\}_{n=1}^{\infty}$

i.e., $(-1)^1, (-1)^2, (-1)^3, (-1)^4, (-1)^5, \dots$ i.e., $-1, 1, -1, 1, -1, \dots$

Notice that $a_n = \begin{cases} -1, & n \text{ is odd} \\ 1, & n \text{ is even} \end{cases}$

and neither $a_n \le a_{n+1}$ nor $a_n \ge a_{n+1}$, $\forall n \in \boldsymbol{N}$

Therefore, it is neither a monotonically increasing nor a monotonically decreasing sequence.

Note that it is an alternating sequence since the terms in the sequence are alternate in sign.

ii) The given sequence is $\{3n+2\}_{n=1}^{\infty}$

i.e., $3(1)+2, 3(2)+2, 3(3)+2, 3(4)+2, 3(5)+2, \dots$

i.e., $5, 8, 11, 14, 17, \dots$

Notice that $a_n = 3n+2, a_{n+1} = 3(n+1)+2 = 3n+5$ and $a_n < a_{n+1}$, $\forall n \in \boldsymbol{N}$

Therefore, the given sequence is a monotonically increasing sequence

P2. Which of the following is a monotonically increasing/ a monotonically decreasing sequence?

i) $\{2^n\}_{n=1}^{\infty}$

ii) $\left\{\frac{1}{2^{n-2}}\right\}_{n=1}^{\infty}$

Solution:

a) The given sequence is $\{2^n\}_{n=1}^{\infty}$

i.e., $2, 2^2, 2^3, 2^4, 2^5, 2^6, \dots$ i.e., $2, 4, 8, 16, 32, 64, \dots.$

Notice that $a_n = 2^n$, $a_{n+1} = 2^{n+1}$ and $a_n < a_{n+1}$, $\forall n \in \boldsymbol{N}$

Therefore, the given sequence is a monotonically increasing sequence

b) The given sequence is $\left\{\frac{1}{2^{n-2}}\right\}_{n=1}^{\infty}$

i.e., $\frac{1}{2^{1-2}}, \frac{1}{2^{2-2}}, \frac{1}{2^{3-2}}, \frac{1}{2^{4-2}}, \frac{1}{2^{5-2}}, \frac{1}{2^{6-2}}, \dots$

i.e., $\frac{1}{2^{-1}}, \frac{1}{2^0}, \frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \dots$ i.e., $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$

Notice that $a_n = \frac{1}{2^{n-2}}$, $a_{n+1} = \frac{1}{2^{n-1}}$ and $a_n > a_{n+1}$, $\forall n \in \boldsymbol{N}$

Therefore, the given sequence is a monotonically decreasing sequence

IP3. Show that the sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = \frac{2n^2-5n+6}{3n^2+7}$ converges to $\frac{2}{3}$

Solution:

Given $a_n = \frac{2n^2-5n+6}{3n^2+7}$

Now, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{2n^2-5n+6}{3n^2+7}$

$$= \lim_{n\to\infty} \frac{n^2\left(2-\frac{5}{n}+\frac{6}{n^2}\right)}{n^2\left(3+\frac{7}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{\left(2-\frac{5}{n}+\frac{6}{n^2}\right)}{\left(3+\frac{7}{n^2}\right)}$$

$$= \frac{(2-0+0)}{(3+0)} = \frac{2}{3}$$

Therefore, the given sequence converges to $\frac{2}{3}$

P3. Show that the sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = \frac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}}$ converges to – 2

Solution:

Given $a_n = \frac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}}$

Now, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}}$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}} \times \frac{\sqrt{n^2-1}+\sqrt{n^2+n}}{\sqrt{n^2-1}+\sqrt{n^2+n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^2-1}+\sqrt{n^2+n}}{n^2-1-n^2-n}$$

$$= \lim_{n\to\infty} \frac{n\sqrt{1-\frac{1}{n^2}}+(n)\sqrt{1+\frac{1}{n}}}{-n\left(1+\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n^2}}+\sqrt{1+\frac{1}{n}}}{-\left(1+\frac{1}{n}\right)}$$

$$= \frac{1+1}{-1} = -2$$

Therefore, the given sequence converges to -2

IP4. Find the n^{th} partial sum of the series $\sum\limits_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)}$ and hence find the sum of the series.

Solution:

The given series is $\sum\limits_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)}$

i.e., $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots$

Let s_n be the n^{th} partial sum

i.e., s_n = sum of the first n terms of the series

$$= \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots + \frac{1}{(3n-2)(3n+1)}$$

$$= \frac{1}{3}\left[\left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{10}\right) + \cdots + \left(\frac{1}{3n-2} - \frac{1}{3n+1}\right)\right]$$

$$= \frac{1}{3}\left[1 - \frac{1}{3n+1}\right]$$

Note that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{3}\left[1 - \frac{1}{3n+1}\right] = \frac{1}{3}$

The given series converges and converges to $\frac{1}{4}$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)} = \frac{1}{3}$$

Thus, the sum of the series is $\frac{1}{3}$.

P4. Find the n^{th} partial sum of the series $\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n+1)}$ and hence find the sum of the series.

Solution:

The given series is $\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n+1)}$

i.e., $\frac{1}{1.5} + \frac{1}{5.9} + \frac{1}{9.13} + \cdots$

Let s_n be the n^{th} partial sum

i.e., s_n = sum of the first n terms of the series

$$= \frac{1}{1.5} + \frac{1}{5.9} + \frac{1}{9.13} + \cdots + \frac{1}{(4n-3)(4n+1)}$$

$$= \frac{1}{4}\left[\left(1 - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{13}\right) + \cdots + \left(\frac{1}{4n-3} - \frac{1}{4n+1}\right)\right]$$

$$= \frac{1}{4}\left[1 - \frac{1}{4n+1}\right]$$

Note that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{4}\left[1 - \frac{1}{4n+1}\right] = \frac{1}{4}$

The given series converges and converges to $\frac{1}{4}$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n+1)} = \frac{1}{4}$$

Thus, the sum of the series is $\frac{1}{4}$.

Exercises

1. Write the first ten terms of the following sequences
   i) $a_1 = 1, a_{n+1} = \frac{a_n}{n+1}$
   ii) $a_1 = 2, a_2 = -1, a_{n+2} = \frac{a_{n+1}}{a_n}$
2. Find the general formula of the following sequences
   i) $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$
   ii) $0, 1, \ 1, 2, 2, 3, 3, 4, \ldots.$
3. Which of the following sequences is monotonically increasing/ monotonically decreasing sequences.
   i) $\left\{\frac{2n-7}{3n+2}\right\}$
   ii) $\left\{\frac{n}{n^2+1}\right\}$
   iii) $a_1 = \frac{3}{2}, a_{n+1} = 2 - \frac{1}{a_n}$
4. Prove that the sequence $\left\{\frac{n^3-2n+1}{n^3-2n^2-1}\right\}_{n=1}^{\infty}$ converges to 1
5. Prove that the sequence $\left\{\frac{(3n-1)(n^4-n)}{(n^2+2)(n^3+1)}\right\}_{n=1}^{\infty}$ converges to 3
6. Find the n^{th} partial sum of the series $\sum_{n=1}^{\infty} \frac{2n+1}{1^2+2^2+3^2+\ldots+n^2}$ and hence find the sum of the series.
7. Find the n^{th} partial sum of the series $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$ and hence find the sum of the series.

3.2 Arithmetical Progression

Learning objectives:

- To study the Arithmetic progression and sum of n terms of an Arithmetic progression.
- To study the Arithmetic Mean and insertion of n Arithmetic means between two given numbers.
- To solve the related problems.

Quantities are said to be in ***Arithmetical Progression*** when they increase or decrease by a common difference.

Each of the following series forms an arithmetical progression.

$3,7,11,15,\dots,\dots$

$8,2,-4,-10,\dots,\dots$

$a, a+d, a+2d, a+3d,\dots,\dots$

The common difference is found by subtracting any term of the series from that which follows it. In the first of the above examples the common difference is 4; in the second it is -6; in the third it is d.

If we examine the series

$$a, a+d, a+2d, a+3d,\dots,\dots$$

notice that in any term the coefficient of d is always less by one than the number of the term in the series.

Thus, the 3rd term is $a+2d$, 6th term is $a+5d$ and generally, the p^{th} term is $a+(p-1)d$ and it is denoted by T_p

If n be the number of terms, and if l denote the last (n^{th} term), we have

$$l = a + (n-1)d$$

Sum of Terms

Let a denote the first term, d the common difference, and n the number of terms of an A.P. Also let l denote the last term, and $\boldsymbol{S_n}$ the required sum of first n terms of the A.P then

$S_n = a + (a+d) + (a+2d) + \cdots + (l-2d) + (l-d) + l$

If we write the series, in the reverse order,

$S_n = l + (l-d) + (l-2d) + \cdots + (a+2d) + (a+d) + a$

Adding together these two series,

$$2S_n = (a+l) + (a+l) + (a+l) + \cdots + n\ terms = n(a+l)$$

Therefore, $\boldsymbol{S_n = \frac{n}{2}(a + l)}$ and $l = a + (n-1)d$

Thus, $\boldsymbol{S_n = \frac{n}{2}[2a + (n-1)d]}$

Example

Find the sum of the series $\boldsymbol{5\frac{1}{2}, 6\frac{3}{4}, 8}$, ..., to 17 terms.

Solution: The common difference is $\boldsymbol{1\frac{1}{4}}$; hence

$$S_{17} = \frac{17}{2}\left(2 \times \frac{11}{2} + 16 \times \frac{5}{4}\right) = 263\frac{1}{2}$$

Example

The first term of a series is 5, the last 45, and the sum 400; find the number of terms, and the common difference.

***Solution**:* If $\boldsymbol{n}$ be the number of terms, then

$$400 = \frac{n}{2}(5 + 45) \Longrightarrow n = 16$$

If d be the common difference,

$$45 = T_{16} = 5 + 15d \Longrightarrow d = 2\frac{2}{3}$$

If any two terms of an arithmetical progression be given, then they provide sufficient data to set up two simultaneous equations to determine the first term and common difference. Thus the series can be completely determined.

Example

The 54^{th} and 4^{th} terms of an A.P. are $\boldsymbol{-61}$ and $\boldsymbol{64}$; find the 23^{rd} term.

Solution: Let a be the first term, and d the common difference, then we have

$$-61 = a + 53d \ ; \quad 64 = a + 3d$$

Solving, we get $d = -\frac{5}{2}$, $a = 71\frac{1}{2}$

Therefore, the 23^{rd} term is $= a + 22d = 71\frac{1}{2} + 22\left(-\frac{5}{2}\right) = 16\frac{1}{2}$

Arithmetic Mean

When three quantities are in arithmetical progression, the middle one is said to be the ***Arithmetic Mean*** of the other two.

Thus, a is the arithmetic mean between $a - d$ and $a + d$.

We propose to find the arithmetic mean between two given quantities a and b.

Let A be the arithmetic mean between a and b. Then since a, A, b are in A.P. we have

$$b - A = A - a$$

each being equal to the common difference. Hence

$$A = \frac{a+b}{2}$$

Between two given quantities it is always possible to insert any number of terms such that the whole series thus formed shall be in A.P. The terms thus inserted are called the *arithmetic means*.

Let a and b be the given quantities, n the number of means. Including the extremes the number of terms will be $n+2$; so that we have to find a series of n + 2 terms in A.P., of which a is the first and b is the last.

Let d be the common difference; then

$$b = T_{n+2} = a + (n+1)d \Rightarrow d = \frac{b-a}{n+1}$$

The required means are

$$a + \frac{b-a}{n+1}, a + \frac{2(b-a)}{n+1}, \dots, a + \frac{n(b-a)}{n+1}$$

Example

Insert 20 arithmetic means between 4 and 67.

Solution: Including the extremes, the number of terms will be $\mathbf{22}$; so that we have to find a series of $\mathbf{22}$ terms in A.P., of which $\mathbf{4}$ is the first and $\mathbf{67}$ is the last.

Let d be the common difference; then

$$67 = 4 + 21d \Rightarrow d = 3$$

The series is $4, 7, 10, \dots, 61, 64, 67$; and the required arithmetic means are $7, 10, \dots, 61, 64$.

Example

The sum of three numbers in A.P. is 27, and the sum of their squares is 293; find them.

Solution: Let $\boldsymbol{a}$ be the middle number, $\boldsymbol{d}$ the common difference; then the three numbers are $\boldsymbol{a-d}, \boldsymbol{a}, \boldsymbol{a+d}$. Hence

$$\boldsymbol{a - d + a + a + d} = 27 \Rightarrow a = 9$$

The three numbers are $9 - d, 9, 9 + d$. So,

$$(9-d)^2 + 9^2 + (9+d)^2 = 293 \Rightarrow d = \pm 5$$

Therefore, the required numbers are $4, 9, 14$.

Example

Find the sum of the first $\boldsymbol{p}$ terms of an A.P whose $\boldsymbol{n^{th}}$ term is $\mathbf{3}\boldsymbol{n} - \mathbf{1}$.

Solution: We obtain the first term and the last term of the required series by putting $n=1$ and $n=p$ respectively.

First term = $3(1)-1=2$ and Last term = $3p-1$

Therefore, $sum = \frac{p}{2}(first\ term + last\ term) = \frac{p}{2}(3p+1)$

The sum of the n terms of a series is $S = \frac{n}{2}[2a + (n-1)d]$

When s, a, d are given, the value of n can be determined from the equation

$$2s = 2an + n^2 d - nd \Rightarrow dn^2 + (2a - d)n - 2s = 0$$

which is a quadratic equation in n.

Example

How many terms of the series $-9, -6, -3, \cdots$ must be taken so that the sum may be 66?

Solution: $3n^2 - 21n - 132 = 0 \Rightarrow n^2 - 7n - 44 = 0$

$$\Rightarrow (n - 11)(n + 4) = 0 \Rightarrow n = 11, -4$$

If we take 11 terms of the series, we have

$$-9, -6, -3, 0, 3, 6, 9, 12, 15, 18, 21;$$

the sum of which is 66. The answer is 11.

Remark

However, if we begin at the last of these terms and count backwards four terms,

$$12, 15, 18, 21$$

the sum is also 66. Although -4 is not an answer to our question, it is intimately connected with that to which the positive solution applies.

Note:

When the value of n is fractional there is no exact number of terms which correspond to such a solution.

Example

How many terms of the series $26, 21, 16, \ldots$ must be taken to amount to 74?

Solution: $-5n^2 + 57n - 148 = 0 \Rightarrow 5n^2 - 57n + 148 = 0$

$$\Rightarrow (5n - 37)(n - 4) = 0 \Rightarrow n = \frac{37}{5}, 4$$

Thus the number of terms is 4. By summing the terms of the series, we will find that the sum of 7 terms is greater, while the sum of the 8 terms is less than 74.

PROBLEM SET:

IP1. If sum of the five consecutive terms of an A.P. is 25 and their product is 945, then find the terms of the A.P.

Solution: Let $a - 2d, a - d, a, a + d, a + 2d$ be three consecutive terms

By the hypothesis, we have

$$a - 2d + a - d + a + a + d + a + 2d = 25 \Rightarrow a = 5$$

Now,

$$(a - 2d)(a - d)a(a + d)(a + 2d) = 945$$

$$\Rightarrow (a^2 - 4d^2)(a^2 - 4d^2)a = 945$$

$$\Rightarrow (25 - 4d^2)(25 - d^2)5 = 945$$

$$\Rightarrow 4d^4 - 125d^2 + 436 = 0$$

$\Longrightarrow (d^2-4)(4d^2-109)=0$

$\Longrightarrow d=\pm 2 \;\; or \;\; d=\sqrt{\frac{109}{4}}$ (Neglected)

If $a=5, d=2$, then the terms of the A.P. are $1,3,5,7,9,\ldots$

If $a=5, d=-2$, then the terms of the A.P. are $9,7,5,3,1,\ldots$

P1. If sum of the five consecutive terms of an A.P. is 30 and their product is 3840, then find the terms in A.P?

Solution: Let $a-2d, a-d, a, \; a+d, a+2d$ be three consecutive terms.

By the hypothesis, we have

$a-2d+a-d+a+a+d+a+2d=30 \Rightarrow a=6$

Now, $(a-2d)(a-d)a(a+d)(a+2d)=3840$

$\Longrightarrow (a^2-4d^2)(a^2-4d^2)a=3840$

$\Longrightarrow (36-4d^2)(36-d^2)6=3840$

$\Longrightarrow d^4-45d^2+164=0$

$\Longrightarrow (d^2-4)(d^2-41)=0$

$\Longrightarrow d=\pm 2 \;\; or \;\; d=\sqrt{41}$ (Neglected)

If $a=6, d=2$, then the terms in A.P are $2,4,6,8,10,\ldots$

If $a=6, d=-2$, then the terms in A.P are $10,8,6,4,2,\ldots$

IP2. If $(\boldsymbol{b}-\boldsymbol{c})^2, \; (\boldsymbol{c}-\boldsymbol{a})^2, (\boldsymbol{a}-\boldsymbol{b})^2$ **are in A.P. then prove that** $\frac{2}{c-a}=\frac{1}{a-b}+\frac{1}{b-c}$

Solution: We know that if x,y,z are in A.P., then $y-x=z-y$

Given $(b-c)^2, \; (c-a)^2, (a-b)^2$ are in A.P

Now,

$(c-a)^2-(b-c)^2 = (a-b)^2-(c-a)^2$

$\Longrightarrow c^2+a^2-2ac-b^2-c^2+2bc=a^2+b^2-2ab-c^2-a^2+2ac$

$\Longrightarrow a^2-b^2+2bc-2ac=b^2-c^2+2ac-2ab$

$\Longrightarrow (a-b).(a+b)+2c(b-a)=(b+c).(b-c)+2a(c-b)$

$\Longrightarrow (b-a)(2c-b-a)=(c-b)(2a-b-c)$

$\Longrightarrow (b-a)\{(c-a)+(c-b)\}=(c-b)\{(a-b)+(a-c)\}$

$\Longrightarrow \{(b-a)(c-a)+(b-a)(c-b)\}$

$=\{(c-b)(a-b)+(c-b)(a-c)\}$

Dividing throughout by $(a-b),(b-c),(c-a)$, we get

$\frac{1}{c-a}-\frac{1}{b-c}=\frac{1}{c-a}-\frac{1}{a-b} \Longrightarrow \frac{2}{c-a}=\frac{1}{a-b}+\frac{1}{b-c}$

P2: If $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ **are in A.P. then show that** $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ **are also in A.P**

Solution: We know that if x,y,z are in A.P then $y-x=z-y$

Given $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ are in A.P

$\Longrightarrow \frac{1}{\sqrt{c}+\sqrt{a}}-\frac{1}{\sqrt{b}+\sqrt{c}}=\frac{1}{\sqrt{a}+\sqrt{b}}-\frac{1}{\sqrt{c}+\sqrt{a}}$

$$\Longrightarrow \frac{\sqrt{b}-\sqrt{a}}{(\sqrt{c}+\sqrt{a}).(\sqrt{b}+\sqrt{c})} = \frac{\sqrt{c}-\sqrt{b}}{(\sqrt{a}+\sqrt{b}).(\sqrt{c}+\sqrt{a})}$$

$$\Longrightarrow \frac{\sqrt{b}-\sqrt{a}}{(\sqrt{b}+\sqrt{c})} = \frac{\sqrt{c}-\sqrt{b}}{(\sqrt{a}+\sqrt{b})}$$

$$\Longrightarrow (\sqrt{b}-\sqrt{a}).(\sqrt{b}+\sqrt{a}) = (\sqrt{c}-\sqrt{b}).(\sqrt{c}+\sqrt{b})$$

$$\Longrightarrow b-a = c-b$$

Therefore, a, b, c are also in A.P

Hence proved

IP3. In an A.P. the first term is 2 and if the sum of the first five terms is equal to one-fourth of the sum of the next five terms, then find the 20th term of the series.

Solution: Let a be the first term and d be the common difference of the A.P.

Given $a = 2$

Now, the first five term of the A.P. are

$$2,\ 2+d,\ 2+2d,\ 2+3d,\ 2+4d$$

We know that sum of the n terms of the A.P. is

$$S = \frac{n}{2}[2a + (n-1)d]$$

Therefore, the sum of the first five terms of the A.P is

$$S_1 = \frac{5}{2}[4 + (5-1)d] = 10(1+d)$$

Now, the next five terms of the A.P. are

$$2+5d,\ 2+6d,\ 2+7d,\ 2+8d,\ 2+9d$$

Therefore, the sum of the next six terms is

$$S_2 = \frac{5}{2}[2(2+5d) + (5-1)d] = 5(2+7d)$$

By the hypothesis, we have $S_1 = \frac{1}{4} \times S_2$

$$\Longrightarrow 10(1+d) = \frac{1}{4} \times 5(2+7d) \Longrightarrow d = -6$$

Now, $T_{20} = a + (20-1)d = 2 + 19(-6)$

$$T_{20} = -112$$

P3: In an A.P. the first term is 3 and if the sum of the first six terms is equal to one-six of the sum of the next six terms, then find the 14th term of the A.P.

Solution: Let a be the first term and d be the common difference of the A.P. \

Given $a = 3$

Now, the first six term of the A.P. are

$$3,\ 3+d,\ 3+2d,\ 3+3d,\ 3+4d, 3+5d$$

We know that sum of the n terms of the A.P. is

$$S = \frac{n}{2}[2a + (n-1)d]$$

Therefore, the sum of the first six terms of the A.P is

$$S_1 = \frac{6}{2}[6 + (6-1)d] = 3(6+5d)$$

Now, the next six terms of the A.P. are

$3+6d, 3+7d, 3+8d, 3+9d, 3+10d, 3+11d$

Therefore, the sum of the next six terms is

$S_2 = \frac{6}{2}[2(3+6d)+(6-1)d] = 3(6+17d)$

By the hypothesis, we have $S_1 = \frac{1}{6} \times S_2$

$\Rightarrow 3(6+5d) = \frac{1}{6} \times 3(6+17d) \Rightarrow d = \frac{-30}{13}$

Now, $T_{14} = a + (14-1)\left(\frac{-30}{13}\right) = 3 + (-30)$

$T_{14} = -27$

IP4. A man borrows Rs.1200 at the total interest Rs.168.He repays the entire amount in 12 installments; each installment is being less than the preceding one by Rs.20. Find the first installment that he repaid.

Solution: Given The total amount to be repaid is $= Rs.(1200+168) = Rs.1368$

Let a be the first installment and the difference between the two successive installments is -20

We know that sum of the n terms of the A.P. is

$$S = \frac{n}{2}[2a+(n-1)d]$$

By the hypothesis $n = 12$, $d = -20$ and $S = 1368$

$$\Rightarrow S = \frac{12}{2}[2a+(12-1)(-20)]$$

$$\Rightarrow 1368 = 6(2a-220)$$

$$\Rightarrow 228 = 2a - 220$$

$$\Rightarrow a = 224$$

Therefore, the first installment that he has to repaid is $Rs.224$

P4: A man borrows Rs.3000 at the total interest Rs.1060. He repays the entire amount in 14 installments such that each installment is being less than the preceding one by Rs.40. Find the first installment that he repaid.

Solution: Given The total amount to be repaid is $= Rs.(3000+1060)$

$= Rs.4060$

Let a be the first installment and the difference between the two successive installments is -40

We know that sum of the terms in the given series is

$$S = \frac{n}{2}[2a+(n-1)d]$$

By the hypothesis $n = 14$, $d = -40$ and $S = 4060$

$$\Rightarrow S = \frac{14}{2}[2a+(14-1)(-40)]$$

$$\Rightarrow 4060 = 7(2a-520) \Rightarrow 580 = 2a - 520$$

$$\Rightarrow a = 550$$

Therefore, the first installment that he has to repaid is $Rs.550$

EXECRISES

1. If a, b, c be the $p^{\text{th}}, q^{\text{th}}$, and r^{th} terms respectively of an A.P., prove that $a(q-r)+b(r-p)+c(p-q)=0$
2. Find four numbers in A.P. such that their sum is 50 and the greatest of them is 4 times the least.
3. The sum of four integers in A.P. is 24 and their product is 945. Find the numbers.
4. If a, b, c are in A.P., prove that $b+c, c+a, a+b$ are also in A.P.
5. If a^2, b^2, c^2 are in A.P. prove that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.
6. How many terms of the series $54 + 51 + 48 + 45 + \ldots$ must be taken to make 513? Explain the double answer.
7. The interior angles of a polygon are in A.P. The smallest angle is $120°$ and the common difference is $5°$. Find the number of sides of the polygon.
8. Find the sum of all even numbers between 101 and 999.
9. The sum of the first p terms of an A.P. is equal to the sum of its first q terms, prove that the sum of its first $(p+q)$ terms is zero.
10. Find the sum of all three digits Natural numbers, which are divisible by 7.(Ans: 70336)
11. If 7 times the 7^{th} term of an A.P. is equal to 11 times the 11^{th} term, then the 18^{th} term is (Ans: 0)
12. If the sum of the n terms of two Arithmetic progressions are in the ratio $(3n+8):(7n+5)$ then find the ratio of their 12^{th} terms. (Ans: 7:16)
13. If n arithmetic means are inserted between 20 and 80 such that the ratio of first mean to the last mean is $1:3$, then the value of n (Ans: 11)
14. Between 1 and 31 if m arithmetic means are inserted so that the ratio of the 7^{th} and $(m-1)^{th}$ means is $5:9$, then find the value of m. (Ans: 14)

3.3. Geometric Progression

Learning Objectives:

- To Study the concept of Geometric Progression(G.P)
- To derive a formula for the n^{th} term and the sum of the n terms of a G.P
- To Study the Geometric mean and insertion of Geometric means between two given numbers.
  AND
- To study the related problems.

Definition

Quantities are said to be in ***Geometrical Progression (G.P.** in short)* when they increase or decrease by a *constant factor*.

Each of the following series forms a geometrical progression:

$$3, 6, 12, 24, \dots, \dots$$

$$1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \dots, \dots$$

$$a, ar, ar^2, ar^3, \dots, \dots$$

The constant factor is also called the *common ratio*, and it is found by dividing any term by that which immediately precedes it.

In the first of the above examples the common ratio is 2 ; in the second it is $-\frac{1}{3}$; in the third it is r .

If we examine the series

$$a, ar, ar^2, ar^3, \dots \dots$$

notice that in any term the index of r is always less by one than the number of the terms in the series. Thus, the 3$^{\text{rd}}$ term is ar^2; the 6$^{\text{th}}$ term is ar^5; and, generally, the p^{th} term is ar^{p-1}.

If n be the number of terms, and if l denotes the last (n^{th} term denoted by T_n), then we have

$$T_n = l = ar^{n-1}$$

Example

Find the (i) sixth term, (ii) n^{th} term of the following sequence $2, 6, 18, \dots, \dots$

Solution: Here $\boldsymbol{a = 2, r = \frac{6}{2} = 3}$

$$6^{\text{th}} \text{ term} = ar^5 = 2.3^5 = 486$$

$$T_n = n^{th} \text{ term} = 2(3^{n-1})$$

Geometric Mean

When three quantities are in geometrical progression, the middle one is called the ***Geometric Mean*** of the other two.

We propose to find the geometric mean between two given quantities a and b.
Let G be the geometric mean between a and b. Then, since a, G, b are in G.P, we have

$$\frac{b}{G} = \frac{G}{a}$$

each being equal to the common ratio. Hence $G^2 = ab$

$$\therefore G = \sqrt{ab}$$

Insertion of Geometric Means

Between two given quantities it is always possible to insert any number of terms such that the whole series thus formed shall be in G.P. The terms thus inserted are called the *Geometric Means*.

Let a and b be the given quantities, n the number of means. Including the extremes the number of terms will be $n+2$; so that we have to find a sequences of $n+2$ terms in G.P., of which a is the first and b is the last.

Let r be the common ratio; then

$$b = T_{n+2} = ar^{n+1} \Longrightarrow r^{n+1} = \frac{b}{a} \Longrightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

The required means are $a, ar, ar^2, ar^3, \dots , ar^n$, where $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

Example: Insert four geometric means between 160 and 5.

Solution: We have to find 6 terms in G.P. of which 160 is the first and 5 the sixth. Let $\boldsymbol{r}$ be the common ratio; then

$$5 = T_6 = 160r^5 \Longrightarrow r^5 = \frac{1}{32} \Longrightarrow r = \frac{1}{2}$$

The geometric means inserted between 160 and 5 are $80, 40, 20, 10$.

Sum of $\boldsymbol{n}$ Terms of a G.P

Let a be the first term, r the common ratio, n the number of terms, and S_n be the sum of n terms. Then

$$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$$

Multiply every term by r, we get

$$rS_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

Hence by subtraction,

$$rS_n - S_n = ar^n - a \Longrightarrow (r-1)S_n = a(r^n - 1)$$

Therefore, we have

$$S_n = \begin{cases} \frac{a(r^n-1)}{(r-1)}, & r > 1 \\ \frac{a(1-r^n)}{(1-r)}, & r < 1 \end{cases}$$

Since $ar^{n-1} = l$, we may write the sum as

$$S_n = \frac{rl-a}{r-1}$$ which is sometimes useful.

Example: Find the sum the series $-\frac{2}{3}, -1, \frac{3}{2}, \ldots,$ to 7 terms.

Solution: The common ratio is $-\dfrac{3}{2}$. Hence the sum is

$$S_7 = \frac{\frac{2}{3}\left[1-\left(-\frac{3}{2}\right)^7\right]}{1-\left(-\frac{3}{2}\right)} = \frac{463}{96}$$

Example

Find three numbers in G.P. whose sum is 19, and whose product is 216.

Solution: We denote the numbers by $\boldsymbol{\frac{a}{r}, a, ar}$; then

$$\frac{a}{r}.a.ar = 216 \Rightarrow a^3 = 216 \Rightarrow a = 6$$

The numbers are $\frac{6}{r}, 6, 6r$. Therefore,

$$\frac{6}{r} + 6 + 6r = 19 \Rightarrow \frac{6}{r} + 6r = 13$$

$$\Rightarrow 6r^2 - 13r + 6 = 0 \Rightarrow r = \frac{3}{2}, \frac{2}{3}$$

Required numbers are $4, 6, 9$.

PROBLEM SET:

IP1. How many terms of the G.P $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots, \ldots$ are needed to give the sum $\frac{1275}{256}$?

Solution: Let n be the number of terms needed to get the sum $\frac{1275}{256}$

Given G.P is $\quad \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots, \ldots$

Here $a = \frac{5}{2}, r = \frac{1}{2} < 1$ and $S = \frac{1275}{256}$

$$\therefore\ S = \frac{a(1-r^n)}{1-r}$$

$$\Rightarrow \frac{1275}{256} = \frac{\frac{5}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} \Rightarrow \frac{1275}{256} = 5\left(1-\frac{1}{2^n}\right)$$

$$\Rightarrow \frac{255}{256} = 1 - \frac{1}{2^n} \Rightarrow \frac{1}{2^n} = 1 - \frac{255}{256}$$

$$\Rightarrow \frac{1}{2^n} = \frac{1}{256} \Rightarrow 2^n = 256 \Rightarrow 2^n = 2^8 \Rightarrow n = 8$$

Therefore, the 8 terms are needed to get the sum $\frac{1275}{256}$

P1. How many terms of the G.P $3, \frac{3}{2}, \frac{3}{4}, \ldots, \ldots$ are needed to give the sum $\frac{3069}{512}$?

Solution: Let n be the number of terms needed to get the sum $\frac{3069}{512}$

Given sequence of G.P is $\quad 3, \frac{3}{2}, \frac{3}{4}, \ldots \ldots$

Let a be the first term and r be the common ratio of the given G.P.

Here $a = 3, r = \frac{1}{2} < 1$ and $S = \frac{3069}{512}$

$$\therefore S = \frac{a(1 - r^n)}{1 - r}$$

$\Rightarrow \frac{3069}{512} = \frac{3\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} \Rightarrow \frac{3069}{512} = 6\left(1 - \frac{1}{2^n}\right)$

$\Rightarrow \frac{3069}{3072} = 1 - \frac{1}{2^n} \Rightarrow \frac{1}{2^n} = 1 - \frac{3069}{3072}$

$\Rightarrow \frac{1}{2^n} = \frac{3}{3072} \Rightarrow \frac{1}{2^n} = \frac{1}{1024}$

$\Rightarrow 2^n = 1024 \Rightarrow 2^n = 2^{10} \Rightarrow n = 10.$

Therefore, 10 terms are needed to get sum $\frac{3069}{512}$ in the given G.P.

IP2: Find a G.P for which the sum of the second term and third term is 12 and the 8^{th} term is 81 times the 4^{th} term.

Solution: Let a be the first term and r be the common ratio of the given G.P., then we have $a, ar, ar^2, ar^3, ar^4, \dots, \dots$

Given $ar + ar^2 = 12$ … (1) and

8^{th} term $= 81 \times 4^{th}$ term

$\Longrightarrow ar^7 = 81ar^3 \Longrightarrow r^4 = 81 \Rightarrow r = \pm 3$

If $r = 3$, then from (1) we have $a = 1$

Therefore, G.P: $1, 3, 9, 27, \dots, \dots$

If $r = -3$, then from (1) we have $a = 2$

Therefore, G.P: $2, -6, 18, -24, \dots, \dots$

P2: Find a G.P for which the sum of the first two terms is -4 and the fifth term is 4 times the third term.

Solution: Let a be the first term and r be the common ratio of the given G.P. Then the G.P is $a, ar, ar^2, ar^3, ar^4, \dots, \dots$

Given $a + ar = -4$ … (1) and

5^{th} term $= 4 \times 3^{rd}$ term

i.e., $ar^4 = 4ar^2$

$\Rightarrow r^2 = 4 \Rightarrow r = \pm 2$

If $r = 2$, then from (1), we have $a = -\frac{4}{3}$

Therefore, G.P: $-\frac{4}{3}, -\frac{8}{3}, -\frac{16}{3}, -\frac{32}{3}, \dots, \dots$

If $r = -2$, then from (1), we have $a = 4$

Therefore, G.P: $4, -8, 16, -32, \dots, \dots$

IP3. If the arithmetic mean (A.M) between two numbers is 34 and their geometric mean (G.M) is 16, then find the two numbers.

Solution: Let the numbers be a and b, then we have

$$\text{A.M} = \frac{a+b}{2} \Rightarrow 34 = \frac{a+b}{2} \Rightarrow a+b = 68 \qquad \dots (1)$$

$$\text{and G.M} = \sqrt{ab} = 16 \Rightarrow ab = 256 \qquad \dots (2)$$

Now, $a-b = \pm\sqrt{(a+b)^2 - 4ab} = \pm\sqrt{(68)^2 - 4(256)}$

$= \pm\sqrt{1624 - 1024} = \pm\sqrt{3600} = \pm 60$

$\Rightarrow a - b = \pm 60$

That is $a - b = 60$ ... (3) and $a - b = -60$ ... (4)

Solving (1) and (3), we get $a = 64$ and $b = 4$

Solving (1) and (4), we get $a = 4$ and $b = 64$

P3. If the ratio of arithmetic mean (A.M) to geometric mean (G.M) of two numbers is $5 : 4$ and sum of these numbers is 100, then find the two numbers.

Solution: Let the two numbers be a and b . Then we have

$$a + b = 100 \qquad \dots (1)$$

$$\text{A.M} = \frac{a+b}{2} \text{ and G.M} = \sqrt{ab}$$

Given, $\frac{A.M}{G.M} = \frac{5}{4} \Rightarrow \frac{a+b}{2\sqrt{ab}} = \frac{5}{4}$

$\Rightarrow 4(a+b) = 10\sqrt{ab}$

$\Rightarrow 400 = 10\sqrt{ab}$ (from (1))

$\Rightarrow \sqrt{ab} = 40 \Rightarrow ab = 1600$

Now, $a-b = \pm\sqrt{(a+b)^2 - 4ab} = \pm\sqrt{(100)^2 - 4(1600)}$

$= \pm\sqrt{10000 - 6400} = \pm\sqrt{3600} = \pm 60$

$\Rightarrow a - b = \pm 60$

That is $a - b = 60$ ... (2) and $a - b = -60$ ... (3)

Solving (1) and (2), we get $a = 80$ and $b = 20$

Solving (1) and (3), we get $a = 20$ and $b = 80$

IP4: A person writes a letter to four of his friends. He asks each of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50paise to mail one letter, determine the amount spent on the postage when the 8^{th} set of letters is mailed.

Solution: As per question, the number of letters posted is in G.P

i.e., $4, 16, 64, 256, \dots$

Here first term, $a = 4$ and common ration, $r = 4 > 1$.

Number of letters mailed up to 8th set$(n = 8)$of letters

$\therefore S_n = \frac{a(r^n-1)}{r-1}$

$\Longrightarrow S_8 = \frac{4(4^8-1)}{4-1} = \frac{4}{3} \times 65535 = 87380$

Therefore, the amount spent on the postage as 50 paise per letter is

$= \frac{50}{100} \times 87380 = Rs. 43690$

P4: A person writes a letter to five of his friends. He asks each of them to copy the letter and mail to three different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 25paise to mail one letter, determine the amount spent on the postage when the 10th set of letters is mailed.

Solution: As per question, the number of letters posted is in G.P

i.e., $5, 15, 45, 135, \dots, \dots$

Here first term, $a = 5$ and common ration, $r = 3 > 1$.

Number of letters mailed up to 10th set$(n = 10)$of letters

$\therefore S_n = \frac{a(r^n-1)}{r-1}$

$\Rightarrow S_{10} = \frac{5(3^{10}-1)}{3-1} = \frac{5}{2} \times 59048 = 147620$

Therefore, the amount spent on the postage as 25 paise per letter $= \frac{25}{100} \times$ $147620 = Rs. 36905$

Exercises:

1. Write the G.P., whose 4th term is 54 and the 7th term is 1458.
2. Which term of the series $8 + 1.6 + 0.32 + \cdots\cdots$ is 0.00256?
3. If a, b, c are three consecutive terms of an A.P., and x, y, z are three consecutive terms of a G.P., the prove that $x^{b-c} y^{c-a} z^{a-b} = 1$
4. Insert 3 real geometric means between 3 and 3/16.
5. The sum of the first three terms of G.P is 7 and the sum of their squares is 21. Determine the first five terms of the G.P.
6. Which term of the following sequences:
   a. $2, 2\sqrt{2}, 4, \dots \dots$ is 128?
   b. $\sqrt{3}, 3, 3\sqrt{3}, \dots \dots$ is 729?
   c. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots \dots$ is $\frac{1}{19683}$?
7. The 4th term of a G.P is square of its second term, and the first term is -3. Determine its 7th term.

8. The 5^{th}, 8^{th} and 11^{th} terms of a G.P are p, q and s respectively. Show that $q^2 = ps$.
9. Find the sum to indicated number of terms in each of the geometric progressions given below.
    a. 0.15, 0.015, 0.0015, …… 20 terms
    b. $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \dots \dots n$ terms
    c. $1, -a, a^2, -a^3, \dots \dots n$ terms (if $a \neq -1$)
10. The sum of first three terms of a G.P is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.
11. How many terms of G.P. $3, 3^2, 3^3, \dots \dots$ are needed to give the sum 120?

12. The sum of first three terms of a G.P is 16 and the sum of the next three terms is 128. Determine the first term, the common ratio and the sum to n terms of the G.P.
13. Given a G.P. with $a = 729$ and 7^{th} term 64, determine S_7.
14. If the 4^{th}, 10^{th} and 16^{th} terms of a G.P. are x, y and z, respectively. Prove that x, y, z are in G.P.
15. Find the sum of the products of the corresponding terms of the sequences $2, 4, 8, 16, 32$ and $128, 32, 8, 2, \frac{1}{2}$.
16. Find four numbers forming a geometric progression in which the third term is greater than the first term by 9, and the second term is greater than the 4^{th} by 18.
17. Insert two numbers between 3 and 81 so that the resulting sequence is G.P.

3.4. Harmonic Progression

Learning objectives:

- To introduce the concept of Harmonic Progression and Harmonic mean.
- To derive a relation among AM,GM and Harmonic mean
  AND
- To practice the related problems.

A series is said to be in Harmonic progression when the reciprocals of the terms of the series are in arithmetical progression.
The following series are examples of H.P.

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots, \qquad \frac{2}{3}+\frac{2}{5}+\frac{2}{7}+\frac{2}{9}+\cdots$$

$$\frac{1}{a}+\frac{1}{a+d}+\frac{1}{a+2d}+\frac{1}{a+3d}+\cdots$$

If a, b, c are in H.P., then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. Therefore

$$d=\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b} \Rightarrow \frac{a-b}{ab}=\frac{b-c}{bc} \Rightarrow \frac{a-b}{b-c}=\frac{a}{c}$$

We can also define the harmonic progression as follows.

Three quantities a, b, c are said to be in harmonic progression when $\frac{a}{c}=\frac{a-b}{b-c}$.
Any number of quantities are said to be in harmonic progression when every three consecutive terms are in harmonic progression.

There is no general formula for the sum of a number of quantities in harmonic progression. Problems in H.P. are generally solved by inverting the terms, and making use of the properties of the corresponding A.P.

Example

Find the first term of a H.P. whose second term is 3/5 and the third term is $\frac{1}{2}$.

Solution

Let a be the first term. Then $a, \frac{3}{5}, \frac{1}{2}$ are in H.P. Hence,

$$\frac{a}{(1/2)}=\frac{a-(3/5)}{(3/5)-(1/2)} \Rightarrow a-\frac{3}{5}=\frac{a}{5}$$

$$\Rightarrow 5a-3=a \Rightarrow 4a=3 \Rightarrow \frac{3}{4}$$

Example: Find the 8th term of the series $\frac{2}{13}, \frac{1}{6}, \frac{2}{11}, \ldots$

Solution

The reciprocal of the terms $\frac{13}{2}, 6, \frac{11}{2}, \ldots$ are in A.P. The common difference is $d = -\frac{1}{2}$.

$$8^{th} \text{ term} = \frac{13}{2} + 7 \times \left(-\frac{1}{2}\right) = 3$$

The 8^{th}term of the given H.P. is $\frac{1}{3}$.

We can find the Harmonic Mean between two given quantities as follows.

If a and b are the two given quantities, and H their harmonic mean, then $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$, are in A.P. Therefore

$$\frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H} \Rightarrow \frac{2}{H} = \frac{1}{a} + \frac{1}{b} \Rightarrow H = \frac{2ab}{a+b}.$$

Example: Insert 40 harmonic means between 7 and 1/6.

Solution: We have to insert 40 AM's between 7 and 1/6 therefore,6 is term 42^{nd} term of an A.P., whose first term is 1/7; let d be the common difference; then

$$6 = \frac{1}{7} + 41d \Rightarrow d = \frac{1}{7}$$

Thus the arithmetic means are $\frac{2}{7}, \frac{3}{7}, \ldots \frac{41}{7}$.

Therefore, the 40 harmonic means between 7 and $\frac{1}{6}$ are $\frac{7}{2}, \frac{7}{3} \ldots, \frac{7}{41}$.

Relation among arithmetic, geometric and harmonic means

If A, G, H be the arithmetic, geometric, and harmonic means between a and b, we have proved $A = \frac{a+b}{2}$; $G = \sqrt{ab}$; $H = \frac{2ab}{a+b}$

Therefore, $$AH = \frac{a+b}{2} . \frac{2ab}{a+b} = ab = G^2$$

$$G = \sqrt{AH}$$

That is, G is the geometric mean between A and H.

From these results, we see that

$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a - 2\sqrt{ab} + b}{2} = \left(\frac{\sqrt{a} - \sqrt{b}}{\sqrt{2}}\right)^2$$

Which is positive if a and b are positive; therefore the arithmetic mean of any two positive quantities is greater than their geometric mean.

From the equation $G^2 = AH$, we see that G is intermediate between A and H. Since $A > G$ and G is intermediate between A and H, we have

$$A > G > H$$

In words, the arithmetic, geometric, and harmonic means between any two positive quantities are in descending order of magnitude.

PROBLEM SET:

IP1. Find the harmonic mean between 5 and 20.

Solution:

Step1: Let us take a, b as $5{,}20$ respectively.

Step2: We know that Harmonic mean $H = \frac{2ab}{a+b}$

Step3: Harmonic mean $H = \frac{2.5.20}{25} = 8.$

P1. Find the harmonic mean between 3 and 15.

Solution: Let a, b are $3, 15$ respectively.

We know that Harmonic mean $H = \frac{2ab}{a+b}$

Harmonic mean $H = \frac{2.3.15}{18} = 5.$

IP2. If $\boldsymbol{\frac{a^{n+1}+b^{n+1}}{a^n+b^n}}$ is A.M. between a and b then find the value of $\boldsymbol{n}$

Solution:

STEP1: A.M of a and b is $= \frac{a+b}{2}$

STEP2: From the problem $\frac{a+b}{2} = \frac{a^{n+1}+b^{n+1}}{a^n+b^n}$

STEP3: By cross multiplication you will get

$$a^{n+1} + b^{n+1} = ab^n + ba^n$$

$$a^n(a-b) = b^n(a-b)$$

STEP4: $\left(\frac{a}{b}\right)^n = 1$ it is equal to

$$\left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)^0$$

Then $\quad n = 0$

P2. If $\boldsymbol{\frac{a^{n+1}+b^{n+1}}{a^n+b^n}}$ is H.M. between a and b then find the value of $\boldsymbol{n}$.

Solution:

H.M of a and b is $= \frac{2ab}{a+b}$

From the problem $\frac{2ab}{a+b} = \frac{a^{n+1}+b^{n+1}}{a^n+b^n}$

By cross multiplication you will get

$$\left(\frac{a}{b}\right)^{n+1} = 1 \text{ it is equal to}$$

$$\left(\frac{a}{b}\right)^{n+1} = \left(\frac{a}{b}\right)^0$$

Then $\quad n + 1 = 0$

$$n = -1$$

IP3. If $\frac{1}{a}+\frac{1}{c}+\frac{1}{a-b}+\frac{1}{c-b}=0$, prove that a, b, c are in H.P.

Solution:

Step1: Here $\left(\frac{1}{a}+\frac{1}{c-b}\right)+\left(\frac{1}{c}+\frac{1}{a-b}\right)=0$

Step2:

$$\left(\frac{a+c-b}{a(c-b)}\right)+\left(\frac{c+a-b}{c(a-b)}\right)=0$$

Step3:

$$(a+c-b)\left(\frac{1}{a(c-b)}+\frac{1}{c(a-b)}\right)$$

Step4:

$$(a+c-b)\left(\frac{c(a-b)+a(c-b)}{ac(c-b)(a-b)}\right)=0 \text{ ----- (1)}$$

Step5:

From (1)

$$\frac{c(a-b)+a(c-b)}{ac(c-b)(a-b)}=0$$

Then

$$c(a-b)+a(c-b)=0$$

$$2ac-bc-ba=0$$

Step6:

$$b=\frac{2ac}{a+c}$$

Then a, b and c are in H.P.

P3. If a, b and c are in H.P. then $\frac{1}{b-a}+\frac{1}{b-c}=$

Solution:

If a, b and c are in H.P. then $b=\frac{2ac}{a+c}$

$$\frac{1}{b-a}=\frac{1}{\frac{2ac}{a+c}-a}=\frac{1}{a}\left(\frac{a+c}{c-a}\right) \text{ ----- (1)}$$

$$\frac{1}{b-c}=\frac{1}{\frac{2ac}{a+c}-c}=-\frac{1}{c}\left(\frac{a+c}{c-a}\right) \text{----- (2)}$$

From (1) and (2)

$$\frac{1}{b-a}+\frac{1}{b-c}=\left(\frac{1}{a}-\frac{1}{c}\right)\left(\frac{a+c}{c-a}\right)$$

Then $\frac{1}{b-a}+\frac{1}{b-c}=\frac{1}{c}+\frac{1}{a}$

IP4. If the GM between a and b be m times their HM then $\frac{m+\sqrt{m^2-1}}{m-\sqrt{m^2-1}}=$

Solution:

Step1: $GM=\sqrt{ab}$

Step2: $HM=\frac{2ab}{a+b}$

Step3: $GM = m.HM$

$$\sqrt{ab} = m\left(\frac{2ab}{a+b}\right)$$

Step4: $m = \frac{(a+b)}{2\sqrt{ab}}, m^2 - 1 = \frac{(a-b)^2}{4ab}$

Step5:

$$\frac{m+\sqrt{m^2-1}}{m-\sqrt{m^2-1}} = \frac{\frac{(a+b)}{2\sqrt{ab}}+\frac{(a-b)}{2\sqrt{ab}}}{\frac{(a+b)}{2\sqrt{ab}}-\frac{(a-b)}{2\sqrt{ab}}} = \frac{2a}{2b} = \frac{a}{b}$$

P4. If the arithmetic mean between a and b is m times their HM then $\frac{\sqrt{m}+\sqrt{m-1}}{\sqrt{m}-\sqrt{m-1}} =$

Solution: We have

$$AM = \frac{a+b}{2} \;;\quad HM = \frac{2ab}{a+b}$$

From problem

AM=m.HM

$$\frac{a+b}{2} = m\left(\frac{2ab}{a+b}\right)$$

$$m = \frac{(a+b)^2}{4ab}, m - 1 = \frac{(a-b)^2}{4ab}$$

$$\frac{\sqrt{m}+\sqrt{m-1}}{\sqrt{m}-\sqrt{m-1}} = \frac{\frac{(a+b)}{\sqrt{4ab}}+\frac{(a-b)}{\sqrt{4ab}}}{\frac{(a+b)}{\sqrt{4ab}}-\frac{(a-b)}{\sqrt{4ab}}} = \frac{2a}{2b} = \frac{a}{b}$$

EXERCISES

1. Find the 100^{th} term of the sequence $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots.$
2.
   a. Find the harmonic mean between 2 and 3.
   b. Insert three harmonic means between 5 and 6.
3. If $x, 1, z$ are in AP and $x, 2, z$ are in GP then prove that $x, 4, z$ will be in HP.
4. a, b, c, d be four numbers of which the first three are in AP and the last three are in HP then prove that ad = bc.
5. If $a^x = b^y = c^z = d^u$ and a, b, c and d are in GP……, then prove that x, y, z, u are in HP.
6. If the m^{th} of a H.P. is n and the n^{th} term is m, show that r^{th} term is $\frac{mn}{r}$. Hence, find $(m+n)^{th}$ term.
7. If H be the harmonic mean between x and$y$, then prove that $\frac{H+x}{H-x} + \frac{H+y}{H-y} = 2.$
8. Prove that, if $\sqrt{b}+\sqrt{c}, \sqrt{c}+\sqrt{a}, \sqrt{a}+\sqrt{b}$ are in HP., then a, b, c are in AP.
9. If $\cos(x-y)$, $cosx$ and $\cos(x+y)$are in HP.., then value of $cosx.sec\left(\frac{y}{2}\right)$is
10. If a, b, c are in A.P.., $a^2, b^2, c^2$are in HP.., then prove that either $a = b = c$ or $a, b, -\frac{c}{2}$ from a G.P.

3.5. Sums of Natural Numbers

Learning Objectives:

- To derive the formulae for
  - The sum of first n natural numbers
  - The sum of squares of first n natural numbers
  - The sum of cubes of first n natural numbers

  AND
- To solve problems related to the above formulae

The numbers 1, 2, 3 … are referred to as the natural numbers and these are in A.P with first term 1 and common difference 1. The n^{th} term of the series is n.

The sum of the first n terms is

$$S = \frac{n}{2}\{2.1 + (n-1)1\} = \frac{n}{2}(n+1)$$

This sum is denoted by $\sum n$ and $\sum n = \frac{n(n+1)}{2}$

The sum of the squares of the first n natural numbers:

Let the sum be denoted by S ; then

$$S = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

We have the identity $n^3 - (n-1)^3 = 3n^2 - 3n + 1$

Changing n to $n-1$, we get another identity

$$(n-1)^3 - (n-2)^3 = 3(n-1)^2 - 3(n-1) + 1$$

Similarly, $(n-2)^3 - (n-3)^3 = 3(n-2)^2 - 3(n-2) + 1$ $\vdots$

$$3^3 - 2^3 = 3.3^2 - 3.3 + 1$$

$$2^3 - 1^3 = 3.2^2 - 3.2 + 1$$

$$1^3 - 0^3 = 3.1^2 - 3.1 + 1$$

By addition, we have

$$n^3 = 3(1^2 + 2^2 + 3^2 + \cdots + n^2) - 3(1 + 2 + 3 + \cdots + n) + n$$

$$= 3S - 3\sum n + n$$

$$3S = n^3 - n + \frac{3n(n+1)}{2} = \frac{n(2n^2 - 2 + 3n + 3)}{2} = \frac{n(2n^2 + 3n + 1)}{2}$$

$$= \frac{n(n+1)(2n+1)}{2}$$

Therefore, $\boldsymbol{S = \frac{n(n+1)(2n+1)}{6}}$

It is denoted by $\sum n^2$

The sum of the cubes of the first n natural numbers:

Let the sum be denoted by S; then

$$S = 1^3 + 2^3 + 3^3 + \cdots + n^3$$

We consider the following identities:

$$n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1$$

$(n-1)^4-(n-2)^4=4(n-1)^3-6(n-1)^2+4(n-1)-1$

$(n-2)^4-(n-3)^4=4(n-2)^3-6(n-2)^2+4(n-2)-1$

.....

$$3^4-2^4=4.3^3-6.3^2+4.3-1$$

$$2^4-1^4=4.2^3-6.2^2+4.2-1$$

$$1^4-0^4=4.1^3-6.1^2+4.1-1$$

Hence, by addition

$$n^4=4S-6(1^2+2^2+3^2+\cdots+n^2)+4(1+2+3+\cdots+n)-n$$

$$=4S-6\sum n^2+4\sum n-n$$

$$4S=n^4+n+6\sum n^2-4\sum n$$

$$=n^4+n+n(n+1)(2n+1)-2n(n+1)$$

$$=n(n^3+1)+n(n+1)(2n+1)-2n(n+1)$$

$$=n(n+1)(n^2-n+1+2n+1-2)$$

$$=n(n+1)(n^2+n)$$

Therefore, $\boldsymbol{S}=\frac{\boldsymbol{n^2(n+1)^2}}{\boldsymbol{4}}=\left[\frac{\boldsymbol{n(n+1)}}{\boldsymbol{2}}\right]^{\boldsymbol{2}}$

It is denoted by $\sum n^3$. Notice that $S=(\sum n)^2$

Thus, the sum of the cubes of the first n natural numbers is equal to the square of the sum of the first n natural numbers.

Example*:* Sum the series $1.2+2.3+3.4+\cdots$ to n terms.

Solution: The n^{th} term = $n(n+1)=n^2+n$. The sum is given by

$$S=\sum n^2+\sum n=\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)}{2}\left[\frac{2n+1}{3}+1\right]\quad=\frac{n(n+1)(n+2)}{3}$$

Example: Sum to n terms the series whose n^{th} term is $2^{n-1}+8n^3-6n^2$

Solution: Let the sum be denoted by S; then

$$S=\sum 2^{n-1}+8\sum n^3-6\sum n^2$$

$$=\underbrace{(1+2+2^2+\cdots+2^{n-1})}_{G.P}+8\sum n^3-6\sum n^2$$

$$=\frac{2^n-1}{2-1}+\frac{8n^2(n+1)^2}{4}-\frac{6n(n+1)(2n+1)}{6}$$

$$=2^n-1+n(n+1)[2n(n+1)-(2n+1)]$$

$$=2^n-1+n(n+1)(2n^2-1)$$

PROBLEM SET:

IP1: Find the sum of n terms of the series $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdots$

Solution: Given $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdots$

The n^{th} term of the given series is $T_n=\frac{1}{(3n-2)(3n+1)}$

Let S_n be the sum of first n terms of the series, then

$$S_n=\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdots+\frac{1}{(3n-2)(3n+1)}$$

$$\therefore S_n=\frac{1}{3}\left[\left(\frac{1}{1}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{10}\right)+\cdots+\left(\frac{1}{3n-2}-\frac{1}{3n+1}\right)\right]$$

$$=\frac{1}{3}\left[\frac{1}{1}-\frac{1}{3n+1}\right]=\frac{n}{3n+1}$$

Therefore, $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$

P1: Find the sum of n terms of the series $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\cdots$

Solution: Given series is

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\cdots$$

By observing the series, we can write n^{th} term as

$$T_n=\frac{1}{(3n-1)(3n+2))}$$

Let S_n be the sum of first n terms, then

$$S_n=\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\cdots+\frac{1}{(3n-1)(3n+2))}$$

$$=\frac{1}{3}\left[\left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{8}\right)+\left(\frac{1}{8}-\frac{1}{11}\right)+\cdots+\left(\frac{1}{3n-1}-\frac{1}{3n+2}\right)\right]$$

$$=\frac{1}{3}\left[\frac{1}{2}-\frac{1}{3n+2}\right]=\frac{n}{6n+4}$$

IP2: Find the sum of the n terms of the series $2^3+5^3+8^3+\cdots$

Solution: Given series is $2^3+5^3+8^3+\cdots$

The n^{th} term of the given series is $T_n=(3n-1)^3$

Let S_n be the sum of first n terms of the series, then

$$S_n=2^3+5^3+8^3+\cdots+(3n-1)^3$$

$$\therefore S_n=\sum(3n-1)^3=\sum(27n^3-27n^2+9n-1)$$

$$=27\sum n^3-27\sum n^2+9\sum n-\sum 1$$

$$=27\frac{n^2(n+1)^2}{4}-27\frac{n(n+1)(2n+1)}{6}+9\frac{n(n+1)}{2}-n$$

$$=\frac{n}{4}[27n(n^2+2n+1)-18(2n^2+3n+1)+18(n+1)-4]$$

$$=\frac{n}{4}(27n^3+54n^2+27n-36n^2-54n-18+18n+18-4)$$

$$=\frac{n}{4}(27n^3+18n^2-9n-4)$$

Therefore, $2^3+5^3+8^3+\cdots+(3n-1)^3=\frac{n}{4}(27n^3+18n^2-9n-4)$

P2: Find the sum of the n terms of the series $\mathbf{1^3 + 3^3 + 5^3 + \cdots}$

Solution: Given series is $1^3 + 3^3 + 5^3 + \cdots$

By observing the series, we can write n^{th} term as $T_n = (2n-1)^3$

Let S_n be the sum of first n terms of the series, then

$$S_n = 1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3$$

$$\therefore S_n = \sum(2n-1)^3 = \sum(8n^3 - 12n^2 + 6n - 1)$$

$$= 8\sum n^3 - 12\sum n^2 + 6\sum n - \sum 1$$

$$= 8\frac{n^2(n+1)^2}{4} - 12\frac{n(n+1)(2n+1)}{6} + 6\frac{n(n+1)}{2} - n$$

$$= 2n^2(n^2+2n+1) - 2n(2n^2+3n+1) + 3n(n+1) - n$$

$$= n(2n^3 + 4n^2 + 2n - 4n^2 - 6n - 2 + 3n + 3 - 1)$$

$$= n(2n^3 - n) = n^2(2n^2 - 1)$$

IP3: Find the sum of the n terms of the series

$$\mathbf{1.2.3 + 2.3.4 + 3.4.5 + \cdots}$$

Solution: Given series is $1.2.3 + 2.3.4 + 3.4.5 + \cdots$

The n^{th} term of the given series is $T_n = n(n+1)(n+2)$

Let S_n be the sum of first n terms of the series, then

$$S_n = 1.2.3 + 2.3.4 + 3.4.5 + \cdots + n(n+1)(n+2)$$

$$\therefore S_n = \sum n(n+1)(n+2) = \sum n(n^2+3n+2)$$

$$= \sum(n^3 + 3n^2 + 2n) = \sum n^3 + 3\sum n^2 + 2\sum n$$

$$= \frac{n^2(n+1)^2}{4} + 3\frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2}$$

$$= \frac{1}{4}n^2(n^2+2n+1) + \frac{1}{2}n(2n^2+3n+1) + n(n+1)$$

$$= \frac{n}{4}(n^3 + 2n^2 + n + 4n^2 + 6n + 2 + 4n + 4)$$

$$= \frac{n}{4}(n^3 + 6n^2 + 11n + 6)$$

Therefore,

$$1.2.3 + 2.3.4 + 3.4.5 + \cdots + n(n+1)(n+2) = \frac{n}{4}(n^3 + 6n^2 + 11n + 6)$$

P3: Find the sum of the n terms of the series

$$\mathbf{1.1.2 + 3.2.3 + 5.3.4 + \cdots}$$

Solution: Given series is $1.1.2 + 3.2.3 + 5.3.4 + \cdots$

By observing the series, we can write n^{th} term as

$$T_n = (2n-1)n(n+1)$$

Let S_n be the sum of first n terms of the series, then

$$S_n = 1.1.2 + 3.2.3 + 5.3.4 + \cdots + (2n-1)n(n+1)$$

$$\Rightarrow S_n = \sum(2n-1)n(n+1) = \sum(2n^3 + n^2 - n)$$

$$= 2\sum n^3 + \sum n^2 - \sum n$$

$$= 2\frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)}{2}\left[n(n+1) + \frac{(2n+1)}{3} - 1\right]$$
$$= \frac{n(n+1)}{6}[3n^2 + 5n - 2] = \frac{n(n+1)(n+2)(3n-1)}{6}$$

Therefore,

$$1.1.2 + 3.2.3 + 5.3.4 + \cdots + (2n-1)n(n+1)$$
$$= \frac{n(n+1)(n+2)(3n-1)}{6}$$

IP4: Find the sum of the n terms of the series

$$\mathbf{3.1^2 + 5.3^2 + 7.5^2 + \cdots}$$

Solution: Given series is $3.1^2 + 5.3^2 + 7.5^2 + \cdots$

The n^{th} term of the given series is $T_n = (2n+1)(2n-1)^2$

Let S_n be the sum of first n terms of the series, then

$$S_n = 3.1^2 + 5.3^2 + 7.5^2 + \cdots + (2n+1)(2n-1)^2$$
$$= \sum(2n+1)(2n-1)^2 = \sum(2n+1)(4n^2 - 4n + 1)$$
$$= \sum(8n^3 - 4n^2 - 2n + 1) = 8\sum n^3 - 4\sum n^2 - 2\sum n + \sum 1$$
$$= 8\frac{n^2(n+1)^2}{4} - 2\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} + n$$
$$= \frac{n}{3}[6n(n^2 + 2n + 1) - (2n^2 + 3n + 1) - 3(n+1) + 3]$$
$$= \frac{n}{3}[6n^3 + 10n^2 - 1]$$

Therefore,

$$3.1^2 + 5.3^2 + 7.5^2 + \cdots + (2n+1)(2n-1)^2$$
$$= \frac{n}{3}[6n^3 + 10n^2 - 1]$$

P4: Find the sum of the n terms of the series

$$\mathbf{2^2.1 + 3^2.2 + 4^2.3 + \cdots}$$

Solution: Given series is $2^2.1 + 3^2.2 + 4^2.3 + \cdots$

The n^{th} term of the given series is $T_n = (n+1)^2 n$

Let S_n be the sum of the first n terms of the series, then

$$S_n = 2^2.1 + 3^2.2 + 4^2.3 + \cdots + (n+1)^2 n$$
$$= \sum(n+1)^2 n = \sum(n^3 + 2n^2 + n) = \sum n^3 + 2\sum n^2 + \sum n$$
$$= \frac{n^2(n+1)^2}{4} + 2\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)}{12}[3n(n+1) + 4(2n+1) + 6]$$
$$= \frac{n(n+1)}{12}[3n^2 + 11n + 10] = \frac{n(n+1)(n+2)(3n+5)}{12}$$

Therefore,

$$2^2.1 + 3^2.2 + 4^2.3 + \cdots + (n+1)^2 n = \frac{n(n+1)(n+2)(3n+5)}{12}$$

Exercises:

1. Find the n^{th} term and then the sum of the n terms of the series
   $3.5 + 4.7 + 5.9 + \cdots$
2. Find the n^{th} term and then the sum of the n terms of the series
   $1.4.7 + 2.5.8 + 3.6.9 + \cdots$
3. Find the n^{th} term and then the sum of the n terms of the series
   $1^2 + 3^2 + 5^2 + \cdots$
4. Find the n^{th} term and then obtain the sum of n terms of the series
   $4 + 11 + 22 + 37 + 56 + \cdots$
5. Find the n^{th} term and then the sum of the n terms of the series
   $1.2 + 2.3 + 3.4 + \cdots$
6. Find the n^{th} term and then the sum of the n terms of the series
   $1.3 + 3.5 + 5.7 + \cdots$
7. Find the n^{th} term and then the sum of the n terms of the series
   $\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \cdots$
8. Find the n^{th} term and then the sum of the n terms of the series
   $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots$

3.6. Principle of Mathematical Induction

Learning objectives:

- To study the Principle of mathematical induction and its applications.
  AND
- To practice the related problems.

Many Statements, like

$$P(n): 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

can be shown to hold for every positive integer n by applying an axiom called the ***mathematical induction principle***. A proof that uses this axiom is called a *proof by **mathematical induction*** **or a *proof by induction*.**

The following are the steps in proving a statement $P(n)$ by induction:

Step 1: Check that the statement $P(n)$ holds for $n = 1$

Step 2: Prove that if the statement $P(n)$ holds for any positive integer, $n = k, k \geq 1$, then it also holds for the next integer $n = k + 1$.

Once these steps are completed, the statement $P(n)$ holds for all positive integers n. By step 1 it holds for $n = 1$. By step 2 it holds for $n = 2$, and therefore by step 2 also for $n = 3$, and by step 2 again for $n = 4$, and so on.

Example 1: Show that for every positive integer n,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Solution: We accomplish the proof by induction. We have the statement

$$P(n): 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Step 1: The statement $P(n)$ holds for $n = 1$ because $1 = \frac{1(1+1)}{2}$

Step 2: If $P(n)$ holds for $n = k, k \geq 1$, we propose to show that it also holds for $n = k + 1$.

Suppose that $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$

Now, $(1 + 2 + 3 + \cdots + k) + (k + 1)$

$$= \frac{k(k+1)}{2} + (k + 1)$$

$$= \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2} = \frac{(k+1)\big((k+1)+1\big)}{2}$$

The last expression is $\frac{n(n+1)}{2}$ for $n = k + 1$

The mathematical induction principle now guarantees the statement

$$P(n): 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for all positive integers n.

We note that all we have to do is to carry out steps 1 and 2. The mathematical induction principle then guarantees the validity of the statement we proposed to prove.

Example 2: **Show that $x^n - 1$ is divisible by $x - 1$ for all positive integers n.**

Solution: We have the statement

$P(n)$: $x^n - 1$ is divisible by $x - 1$

Step 1: $P(n)$ holds for $n = 1$ because $x - 1$ divides $x - 1$

Step 2: If $P(n)$ holds for $n = k$, $k \geq 1$, we propose to show that it also holds for $n = k + 1$.

Suppose that $x^k - 1$ is divisible by $x - 1$. Then

$$\frac{x^{k+1}-1}{x-1} = \frac{x^{k+1}-x^k+x^k-1}{x-1} = \frac{x^k(x-1)+x^k-1}{x-1} = x^k + \frac{x^k-1}{x-1}$$

Since $x^k - 1$ is divisible by $x - 1$, $x^{k+1} - 1$ is also divisible by $x - 1$.

The mathematical induction principle now guarantees that $P(n)$ is true for positive integers n. That is, $x^n - 1$ is divisible by $x - 1$ for all positive integral values of n.

Note:

Instead of starting at $n = 1$, some induction arguments start at another integer. The steps for such an argument are as follows.

Step 1: Check that the statement $P(n)$ holds for $n = n_1$ (the first appropriate integer).

Step 2: Prove that if the statement $P(n)$ holds for any integer $n = k \geq n_1$, then it also hold for $n = k + 1$.

Once these steps are completed, the mathematical induction principle guarantees the statement for all positive integers $n \geq n_1$.

Example 3: Show that $n! > 3^n$ if n is large enough.

Solution: To know how large n large enough, we experiment:

n	1	2	3	4	5	6	7
$n!$	1	2	6	24	120	720	5040
3^n	3	9	27	81	243	729	2187

It looks as if $n! > 3^n$ for $n \geq 7$.

We now prove $n! > 3^n$ for all positive integers $n \geq 7$ by the principle of mathematical induction.

We have the statement

$P(n)$: $n! \geq 3^n, n \geq 7$

We take $n_1 = 7$ in step1 and try for step 2.

Suppose that $P(n)$ is true for any positive integer k, $k \geq 7$

i.e., $k! > 3^k$ for some positive integer $k \geq 7$. Then

$(k + 1)! = (k + 1)(k!) > (k + 1)3^k > 7.3^k > 3^{k+1}$

Thus, for $k \geq 7$,

$$k! > 3^k \Longrightarrow (k+1)! > 3^{k+1}$$

The mathematical induction principle now guarantees $P(n)$ is true for positive integers $n \geq 7$.

i.e., $n! > 3^n$ for all $n \in \boldsymbol{N}, n \geq 7$

PROBLEM SET:

IP1: Prove that for all positive integers $\boldsymbol{n}$,

$$\frac{\mathbf{1}}{\mathbf{1.2.3}} + \frac{\mathbf{1}}{\mathbf{2.3.4}} + \cdots + \frac{\mathbf{1}}{\boldsymbol{n}(\boldsymbol{n}+\mathbf{1})(\boldsymbol{n}+\mathbf{2})} = \frac{\boldsymbol{n}(\boldsymbol{n}+\mathbf{3})}{\mathbf{4}(\boldsymbol{n}+\mathbf{1})(\boldsymbol{n}+\mathbf{2})}$$

Solution: We have the statement

$$P(n): \frac{1}{1.2.3} + \frac{1}{2.3.4} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

$P(n)$ holds for $n = 1$ because

$$\frac{1}{1(1+1)(1+2)} = \frac{1(1+3)}{4(1+1)(1+2)} = \frac{1}{6}$$

Suppose that $P(n)$ holds for $n = k, k \geq 1$.

i.e., $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \cdots + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)}$

We now propose to show that it also holds for $n = k + 1$.

i.e., $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \cdots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$

$$= \frac{k(k+3)}{4(k+2)(k+1)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)} = \frac{k^2+6k+9k+4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)} = \frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

Thus, $P\ (n)$ is true for $n = k + 1$

Therefore, by the principle of mathematical induction, the statement $P(n)$ is true for all positive integers n.

P1: For all $\boldsymbol{n} \in \boldsymbol{N}$, prove that

$$\mathbf{1}^2 + \mathbf{2}^2 + \mathbf{3}^2 + \mathbf{4}^2 + \cdots + \boldsymbol{n}^2 = \frac{\boldsymbol{n}(\boldsymbol{n}+\mathbf{1})(\mathbf{2}\boldsymbol{n}+\mathbf{1})}{\mathbf{6}}$$

Solution: We have the statement

$$P(n) \ = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$P(n)$ holds for $n = 1$ because

$$1 = \frac{1(1+1)(2.1+1)}{6} = \frac{1.2.3}{6} = 1$$

Suppose that $P(n)$ holds for $n = k, k \geq 1$

i.e., $1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$

We now propose to show that it also holds for $n = k + 1$.

i.e., $1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 + (k+1)^2$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+1+1)\{2(k+1)+1\}}{6}$$

Thus, $P\ (n)$ is true for $n = k + 1$

Therefore, by the principle of mathematical induction, the statement $P(n)$ is true for all positive integers n.

IP2: For all $\boldsymbol{n} \in \boldsymbol{N}$, $\boldsymbol{P(n)}$: $\boldsymbol{10^n + 3.4^{n+2} + 5}$ is divisible by 9

Solution: We have the statement

$P(n)$: $10^n + 3.4^{n+2} + 5$

$P(n)$ holds for $n = 1$ because

$10^1 + 3.4^{1+2} + 5 = 10 + 3.4^3 + 5 = 10 + 3.64 + 5$

$= 10 + 192 + 5 = 207 = 9 \times 23$, which is divisible by 9

Suppose that $P(n)$ holds for $n = m, m \geq 1$

i.e., $10^m + 3.4^{m+2} + 5 = 9k$, where k is an integer

We now propose to show that it also holds for $n = m + 1$.

i.e., $10^{m+1} + 3.4^{m+1+2} + 5 = 10.10^m + 3.4^{m+3} + 5$

$= 10(9k - 3.4^{m+2} - 5) + 3.4^{m+3} + 5$

$= 90k - 30\ .4^{m+2} - 50 + 3.4^{m+3} + 5$

$= 90k - 30.4^{m+2} + 12.4^{m+2} - 45$

$= 90k - 18.4^{m+2} - 45 = 9\{10k - 2.4^{m+2} - 5\}$

$= 9 \times k_1$, where k_1 is an integer.

$\Longrightarrow$ 9 divides $P(n)$ when $n = m + 1$

Therefore, by the principle of mathematical induction, the statement $P(n)$ is true for all positive integers n.

P2. Prove that the binomial theorem for a positive integral index $\boldsymbol{n}$. Solution: Binomial theorem for the integral index $\boldsymbol{n}$:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k,$$

where the binomial coefficient $\binom{n}{k}$ (also denoted by ${}^{n}C_{k}$) is expressed in terms of factorial functions by

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad 0 \le k \le n$$

We have the statement

$$P(n): (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k, \text{ where} \binom{n}{k} = \frac{n!}{k!(n-k)!}, 0 \le k \le n$$

i.e., $(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n} y^n$

$P(n)$ holds for $n = 1$ because

$$(x+y)^1 = x^1 + y^1 = x + y$$

Suppose that $P(n)$ holds for $n = m, m \ge 1$

i.e., $(x+y)^m = \sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^k$

We now propose to show that it also holds for $n = m + 1$.

Now, $(x+y)^{m+1} = (x+y).(x+y)^m$

$$= (x+y) \sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^k$$

(since the binomial expansion is valid for $n = m$)

$$= \sum_{k=0}^{m} \binom{m}{k} x^{m+1-k} y^k + \sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^{k+1}$$

Put $j = k - 1$

$$= \sum_{k=0}^{m} \binom{m}{k} x^{m+1-k} y^k + \sum_{j=1}^{m+1} \binom{m}{j-1} x^{m+1-j} y^j$$

We now separate the first term of the first sum $\binom{m}{0} x^{m+1}$ and write it as $\binom{m+1}{0} x^{m+1}$. Similarly, the last term of the second summation can be replaced by $\binom{m+1}{m+1} y^{m+1}$.

The remaining terms of each of the two summations are now written together, with the summation index denoted by k in both terms. Thus,

$(x+y)^{m+1}$

$$= \binom{m+1}{0} x^{(m+1)} + \sum_{k=1}^{m} \left(\binom{m}{k} + \binom{m}{k-1} \right) x^{(m+1)-k} y^k + \binom{m+1}{m+1} y^{m+1}$$

$$= \binom{m+1}{0} x^{(m+1)} + \sum_{k=1}^{m} \binom{m+1}{k} x^{(m+1)-k} y^k + \binom{m+1}{m+1} y^{m+1}$$

(since $\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}$ by Pascal's identity)

$$= \sum_{k=0}^{m+1} \binom{m+1}{k} x^{(m+1)-k} y^k$$

Thus it has been shown that if the binomial expansion is assumed to be true for $n = m$, then it is true for $n = m + 1$.

Therefore, by the principle of mathematical induction, the statement $P(n)$ is true for all positive integers n.

IP3. For all $\boldsymbol{n} \in \boldsymbol{N}$, prove that

$$\mathbf{1.3 + 2.3^2 + 3.3^3 + \cdots + \boldsymbol{n}.3^{\boldsymbol{n}} = \frac{(2\boldsymbol{n}-1)3^{\boldsymbol{n}+1} + 3}{4}}$$

Solution: We have the statement

$$P(n): 1.3 + 2.3^2 + \cdots + n.3^n = \frac{(2n-1)3^{n+1}+3}{4}$$

$P(n)$ holds for $n = 1$ because $1.3 = \frac{(2-1)3^2+3}{4} = 3$

Suppose that $P(n)$ holds for $n = m, m \geq 1$

i.e., $1.3 + 2.3^2 + \cdots + m.3^m = \frac{(2m-1)3^{m+1}+3}{4}$

We now propose to show that it also holds for $n = k + 1$.

i.e., $1.3 + 2.3^2 + \cdots + m.3^m + (m+1)3^{m+1}$

$= \frac{(2m-1)3^{m+1}+3}{4} + (m+1)3^{m+1}$

$= \frac{1}{4}[(2m-1)3^{m+1} + 3 + (4m+4)3^{m+1}]$

$= \frac{1}{4}[3^{m+1}(2m-1+4m+4) + 3]$

$= \frac{1}{4}[3^{m+1}(6m+3) + 3]$

$= \frac{1}{4}[3^{m+2}(2m+1) + 3]$

$= \frac{1}{4}[\{2(m+1)-1\}3^{m+2} + 3]$

Thus, $P\ (n)$ is true for $n = m + 1$

Therefore, by the principle of mathematical induction, the statement $P(n)$ is true for all positive integers n.

P3: Prove that $9^{n+1} - 8n - 9$ is divisible by 64 for all $n \in N$

Solution: We have the statement

$P(n): 9^{n+1} - 8n - 9$

$P(n)$ holds for $n = 1$ because

$9^{n+1} - 8n - 9 = 9^{1+1} - 8(1) - 9 = 81 - 17 = 64$, which is divisible by 64.

Suppose that $P(n)$ holds for $n = m, m \geq 1$

i.e., $9^{m+1} - 8m - 9 = 64k$, where k is an integer

We now propose to show that it also holds for $n = m + 1$.

i.e., $9^{(m+1)+1} - 8(m+1) - 9$

$= 9^{m+1}.9 - 8m - 8 - 9$

$= 9(64k + 8m + 9) - 8m - 17$

$= 9 \times 64k + 72m + 81 - 8m - 17$

$= 9 \times 64k + 64m + 64$

$= 64(9k + m + 1)$

$= 64 \times k_1$ where k_1 is an integer.

Thus, $P\ (n)$ is true for $n = k + 1$

Therefore, by the principle of mathematical induction, the statement $P(n)$ is true for all positive integers n.

IP4. Use principle of mathematical induction to prove that $2 \cdot 4^{(2n+1)} + 3^{(3n+1)}$ is divisible by 11, $\forall\ n \in N$.

Solution: We have the statement

$P(n): 2 \cdot 4^{(2n+1)} + 3^{(3n+1)}$

$P(n)$ holds for $n = 1$ because

$2.4^{(2\cdot1+1)} + 3^{(3\cdot1+1)} = 2(4^3) + 3^4 = 209 = 11 \times 19$, which is divisible by 11.

Suppose that $P(n)$ holds for $n = k, k \geq 1$

i.e., $2 \cdot 4^{(2\cdot k+1)} + 3^{(3\cdot k+1)}$ is divisible by 11

Let $2 \cdot 4^{(2\cdot k+1)} + 3^{(3\cdot k+1)} = 11t$, for some integer t ... (1)

We now propose to show that it also holds for $n = k + 1$.

i.e., we show that $2 \cdot 4^{2k+3} + 3^{3k+4}$ is divisible by 11.

From (1), we have $2 \cdot 4^{(2k+1)} + 3^{(3k+1)} = 11t$

$\Rightarrow 2 \cdot 4^{(2k+1)} = 11t - 3^{(3k+1)}$

$\Rightarrow 2 \cdot 4^{(2k+1)} \cdot 4^2 = \left(11t - 3^{(3k+1)}\right) \cdot 4^2$

$\Rightarrow 2 \cdot 4^{(2k+3)} + 3^{(3k+4)} = \left(11t - 3^{(3k+1)}\right) \cdot 16 + 3^{(3k+4)}$

$= 11t \cdot 16 - 3^{(3k+1)} \cdot 16 + 3^{(3k+4)}$

$= 11 \cdot t \cdot 16 + 3^{(3k+1)}[3^3 - 16]$

$= 11 \cdot t \cdot 16 + 3^{(3k+1)}[11]$

$= 11\left[16t + 3^{(3k+1)}\right]$, where $16t + 3^{(3k+1)}$ is an integer

$\Longrightarrow 2 \cdot 4^{(2k+3)} + 3^{(3k+4)}$ is divisible by 11.

Thus, $P\ (n)$ is true for $n = k + 1$

Therefore, by the principle of mathematical induction, the statement $P(n)$ is true for all positive integers n.

P4: For all $\boldsymbol{n} \in \boldsymbol{N}$, prove that

$$\mathbf{1 + 2 + 3 + \cdots + \boldsymbol{n} < \frac{1}{8}(2\boldsymbol{n} + 1)^2}$$

Solution: We have the statement

$P(n): 1 + 2 + 3 + \cdots \ldots \ldots . +n < \frac{1}{8}(2n + 1)^2$

$P(n)$ holds for $n = 1$ because

$$1 < \frac{1}{8}(2(1) + 1)^2 = \frac{9}{8} \Longrightarrow 1 < 1\frac{8}{3}$$

Suppose that $P(n)$ holds for $n = k, k \geq 1$

i.e., $1 + 2 + 3 + \cdots + k < \frac{1}{8}(2k + 1)^2$

Suppose that $P(n)$ holds for $n = k, k \geq 1$

i.e., $1 + 2 + 3 + \cdots \ldots \ldots . +k + (k + 1)$

$$< \frac{1}{8}(2k + 1)^2 + (k + 1)$$

$$< \frac{1}{8}[(2k + 1)^2 + 8(k + 1)]$$

$$< \frac{1}{8}[4k^2 + 1 + 4k + 8k + 8]$$

$$< \frac{1}{8}[4k^2 + 12k + 9] = \frac{1}{8}(2k + 3)^2 = \frac{1}{8}(2(k + 1) + 1)^2$$

Thus, $P\ (n)$ is true for $n = k + 1$

Therefore, by the principle of mathematical induction, the statement $P(n)$ is true for all positive integers n.

Exercises

Use Principle of Mathematical Induction and prove that

a) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all $n \in \boldsymbol{N}$

b) $5^n + 2.(11^n)$ is a multiple of 3 for all $n \in \boldsymbol{N}$.

c) $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \ldots . \left\{1 - \frac{1}{(n+1)^2}\right\} = \frac{n+2}{2n+2}$ for all $n \in \boldsymbol{N}$.

d) $n^7 - 7n^5 + 14n^3 - 8n$ is divisible by 840 for all $n \in \boldsymbol{N}$.

e) $2^2 + (2^2 + 4^2) + (2^2 + 4^2 + 6^2) + \cdots n\ terms = \frac{n(n+1)^2(n+2)}{2}$ for all $n \in \boldsymbol{N}$

f) $1.3 + 2.3^2 + 3.3^3 + \cdots + n.3^n = \frac{(2n-1)3^{n+1}+3}{4}$ for all $n \in \boldsymbol{N}$

g) $sin\theta + sin2\theta + \cdots + \sin(n\theta) = \frac{sin\left(\frac{n+1}{2}\right)\theta . sin\left(\frac{n\theta}{2}\right)}{\sin(\theta/2)}$ for all $n \in \boldsymbol{N}$

h) $\int_0^{\pi/2} cos^n x . cos(nx) dx = \frac{\pi}{2^{n+1}}$ for all $n \in \boldsymbol{N}$.

i) $\int_0^{\pi/2} \frac{sin^2 nx}{sinx}\, dx = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$ for all $n \in \boldsymbol{N}$.

j) $\sum_{r=0}^{n} r . n_{C_r} = n\ .\ 2^{n-1}$ for all $n \in \boldsymbol{N}$.

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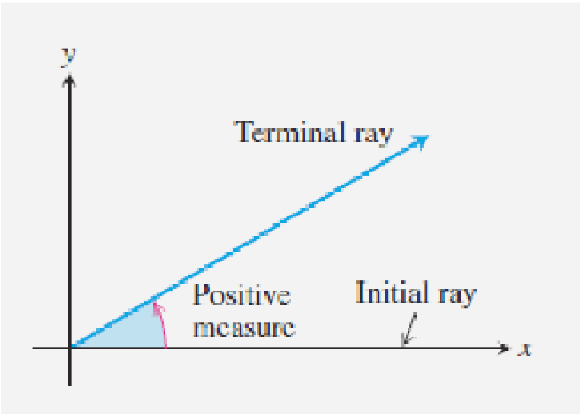
4.1. Angles and Coordinate Lines

Learning objectives:

- To understand the plane angle and measures of an angle
- To convert radians to degrees and degrees to radians
- To understand the concept of arc length and the area of a sector and AND
- To practice problems related to the above concepts

Plane Angle:

Angles in the plane can be generated by rotating a ray about its endpoint. The starting position of the ray is called the initial ray (initial side) of the angle, the final position is called the terminal ray (terminal side) of the angle, and the point at which the initial and terminal rays meet is called the vertex of the angle.



An angle in the xy-plane is said to be in ***standard position*** if its vertex lies at the origin and its initial ray lies along the positive $x - axis$. Angles measured counterclockwise from the positive x −axis are assigned **positive measures**; angles measured clockwise are assigned **negative measures**.

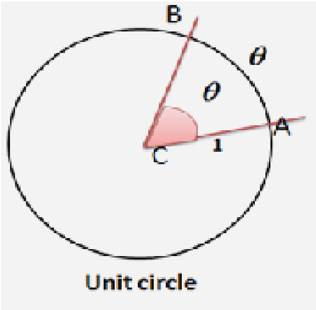
Measures of Angles

There are **two standard** measures for describing the size of an angle: **degree measure** and **radian measure**.

In degree measure, **one degree (written 1^o) is the measure of an angle generated by $\frac{1}{360}$ of one revolution**. Thus, there are 360^o in an angle of one revolution, 180° degrees in an angle of one-half revolution, 90° in an angle of one-quarter revolution (a right angle), and so forth.

One degree is divided into sixty equal parts, called **minutes**, and one minute is divided into sixty equal parts, called **seconds**. Thus, one minute (written $1'$) is $1/60$ of a degree, and one second (written 1″) is $1/60$ of a minute.

In radian measure, angles are measured by the length of the arc that the angle subtends on a circle of radius 1 with its vertex at the center (unit circle).
Let ACB be a central angle in a unit circle.



The radian measure θ of angle ACB is defined to be the length of the circular arc AB. Since the circumference of the circle is 2π and one complete revolution of a circle is 360^{o}, the relation between radians and degrees is given by the equation

$$\pi \text{ radians } = 180^{\circ}$$

So, $$1\ rad = \left(\frac{180}{\pi}\right)^{\circ} \doteq 57^{\circ} (57^{\circ}.17' \ \ approx.)$$

$$1^{\circ} = \frac{\pi}{180} \text{ rad} \doteq 0.02 \text{ rad}$$

Example1:

Convert 45^{o} to radians: $45\dfrac{\pi}{180} = \dfrac{\pi}{4}$ rad

Convert $\dfrac{\pi}{6}$ rad to degrees: $\dfrac{\pi}{6}\dfrac{180}{\pi} = 30^{\circ}$

Arc Length

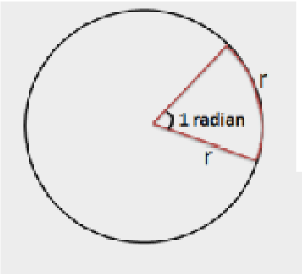
There is a relationship between the length s of an arc AB on a circle of radius r and the radian measure θ of the angle the arc subtends at the circle's center C.

If we draw a unit circle with the same center C, the arc $A'B'$ cut by the angle will have length θ, by the definition of radian measure.

There is a theorem from plane geometry which states that for two concentric circles, the ratio of the arc lengths subtended by a central angle is equal to the ratio of the corresponding radii. Therefore, from the circular sectors ACB and $A'CB'$, we have

$$\frac{s}{\theta} = \frac{r}{1}$$

Thus, the arc length is given by $s = r\theta$ where θ **is in radian measure.**



From this formula, a radian may also be defined as the measure of the central angle subtended by an arc of a circle equal to the radius of the circle.

Example 2: Consider a circle of radius 8.

a. Find the central angle subtended by an arc of length 2π on the circle.

b. Find the length of an arc subtending a central angle of $3\pi/4$.

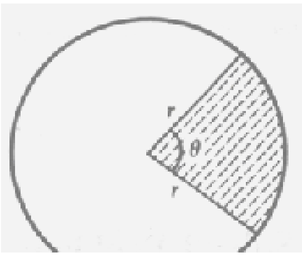
Solution:

a. $$\theta = \frac{s}{r} = \frac{2\pi}{8} = \frac{\pi}{4}$$

b. $$s = r\theta = 8 \times \frac{3\pi}{4} = 6\pi$$

Area of a Sector

The shaded region in the figure below is called a ***sector*** of the circle, whose radius is r; θ is the central angle subtend by the arc of the sector.



There is a theorem from plane geometry that the ratio of the area S of the sector to the area A of the entire circle is the same as the ratio of the central angle of the sector to the central angle of the entire circle.

Thus, $$\frac{S}{A} = \frac{\theta}{2\pi} \Rightarrow S = \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \theta$$

Thus, the area of the sector is given by the formula

$$S = \frac{1}{2} r^2 \theta \ , \text{ where } \theta \text{ is in radian measure}$$

Example 3: For a circle of radius $18\ cm$, the area of the sector intercepted by a central angle of 50° is

$$S = \frac{1}{2} r^2 \theta = \frac{1}{2} \times 18^2 \times 50 \times \frac{\pi}{180} = 45\pi\ cm^2$$

PROBLEM SET

IP1: Find the radius of the circle in which a central angle of $30°$ intercepts an arc of length $22\ cm$ (use $\pi\ = \frac{22}{7}$).

Solution:

STEP1: Arc length s = 22 cm

STEP2: Central angle $\theta = 30° = 30.\frac{\pi}{180} = \frac{\pi}{6}$

STEP3: We know that Arc length $s = r\theta$

$$r = \frac{s}{\theta} = \frac{22}{\frac{\pi}{6}} = \frac{22}{\frac{22}{42}} = 42\ cm$$

P1: Find the radius of the circle in which a central angle of $60°$ intercepts an arc of length $37.4\ cm$ (use $\pi\ = \frac{22}{7}$).

Solution:

Arc length s = 37.4 cm

Central angle $\theta = 60^0 = 60 \cdot \frac{\pi}{180} = \frac{\pi}{3}$

We know that Arc length $s = r\theta$

$$r = \frac{s}{\theta} = \frac{37.4}{\frac{\pi}{3}} = \frac{37.4}{\frac{22}{21}} = 35.7\ cm$$

IP2: If the arcs of the same lengths in two circles subtend angles $30°$and $60°$at the centre, find the ratio of their radii.

Solution:

STEP1: Let r_1, r_2 radii of two circles.

Given that $\theta_1 = 30° = 30\ .\frac{\pi}{180} = \frac{\pi}{6}$ rad

STEP2: $\theta_2 = 60° = 60\ .\frac{\pi}{180} = \frac{\pi}{3}\ rad$

STEP3:

Let s be the length of each of the arc.

Then $s\ = r_1\theta_1 = r_2\theta_2$

STEP4:

$$r_1 \frac{\pi}{6} = r_2 \frac{\pi}{3}$$

$$r_1 : r_2 = 2 : 1$$

P2: If the arcs of the same lengths in two circles subtend angles $65°$ and $110°$ at the centre, find the ratio of their radii.

Solution:

Let r_1, r_2 radii of two circles.

Given that $\theta_1 = 65° = 65 .\frac{\pi}{180} = \frac{13\pi}{36}$

$\theta_2 = 110° = 110 .\frac{\pi}{180} = \frac{11\pi}{18}$

Let s be the length of each of the arc.

Then $s = r_1\theta_1 = r_2\theta_2$

$\Rightarrow r_1 \frac{13\pi}{36} = r_2 \frac{11\pi}{18}$

$\Rightarrow r_1 : r_2 = 22 : 13$

IP3: The minute hand of a watch is $6\ cm$ long. How far does its tip move in 40 minutes? (Use $\pi = 3.14$).

Solution:

STEP1: We know that minute hand of watch completes one revolution in 60 minutes. Then in 40 minutes it completes $\frac{2}{3}$ revolution.

STEP2: So $\theta = \frac{2}{3}. 360^0 = \frac{4\pi}{3}$ radians

STEP3: $r = 6\ cm$

STEP4:

$$s = r\theta = (6)\,.\frac{4\pi}{3} = (6).\frac{4(3.14)}{3} = 25.12\ cm$$

P3: The minute hand of a watch is $1.5\ cm$ long. How far does its tip move in 40 minutes? (Use π = 3.14).

Solution:

We know that minute hand of watch completes one revolution in 60 minutes.

Then in 40 minutes it completes $\frac{2}{3}$ revolution.

So $\theta = \frac{2}{3}. 360^0 = \frac{4\pi}{3}$ radians and $r = 1.5\ cm$

$$s = r\theta = (1.5)\,.\frac{4\pi}{3} = (1.5).\frac{4(3.14)}{3} = 6.28\ cm$$

IP4: For a circle of radius $9\ cm$, find the area of the sector intercepted by a central angle of $40°$.

Solution:

STEP1: Radius of the circle r = 9 cm

Central angle $\theta = 40^0$

STEP2: Area of the sector $S = \frac{1}{2}r^2\theta$

STEP3: $$S = \frac{1}{2} \times 9^2 \times 40 \times \frac{\pi}{180} = 9\pi \text{ cm}^2$$

P4: For a circle of radius $6\ cm$, find the area of the sector intercepted by a central angle of $30°$.

Solution:

Radius of the circle $r = 6\ cm$

Central angle $\theta = 30^0$

Area of the sector $S = \frac{1}{2}r^2\theta$

$$S = \frac{1}{2} \times 6^2 \times 30 \times \frac{\pi}{180} = 3\pi \text{ cm}^2$$

Exercise:

1) Find the radian measures corresponding to the following degree measures:

i) 250^o

ii) 2400^o

iii) 5200^o

iv) $250^o.4'$

2) Find the degree measures corresponding to the following radian measures.

i) $\frac{\pi}{16}$

ii) $\frac{7\pi}{6}$

iii) $\frac{5\pi}{3}$

iv) 4π

3) A central angle of a circle of radius 30 cm intercepts an arc of $6\ cm$. Express the central angle θ in radians and in degrees.

4) A railroad curve is to be laid out on a circle. What radius should be used if the track is to change direction by $25°$ in a distance of $120\ m$?

5) Find the angle in radians through which a pendulum swings if its length is $75\ cm$ and the tip describes an arc of length $(i)\ 10\ cm$ $(ii)\ 15\ cm$ $(iii)\ 21\ cm$.

6) The radius of a circle is $30\ cm$. Find the length of an arc of this circle, if the length of the chord of the arc is $30\ cm$.

7) A railway train is travelling on a circular curve of $1500\ metres$ radius at the rate of $66\ km/hr$. Through what angle has it turned in 10 seconds?

8) Find the diameter of the sun in km supposing that it subtends an angle of $32'$ at the eye of an observer. Given that the distance of the sun is $91 \times 10^6 km$.

9) If the arcs of the same length in two circles subtend angles 65^0 and 100^0 at the centre, find the ratio of their radii.

10) Find the degree measure of the angle subtended at the centre of a circle of radius $100\ cm$ by an arc of length $22cm\ (Use\ \pi = 22/7\ )$

11) For a circle of radius 20 cm, the area of the sector intercepted by a central angle of 90° is

12) For a circle of radius r cm, the area of the sector intercepted by a central angle of 45^0 is $\frac{\pi}{2}$. Then radius r is equal to

13) For a circle radius $2\sqrt{2}$ cm, the area of the sector intercepted by a central angle θ is π. Then the θ is equal to?

4.2. Trigonometric Functions of Acute Angles

Learning objectives:

- To study the Trigonometric Functions for acute angles.
- To derive Fundamental Trigonometric identities.
- To determine the Limits of the values of Trigonometrical functions.
- To find the values of the Trigonometrical functions for angles $0°, 30°, 45°, 60°\ and\ 90°$.
  And
- To solve related problems.

Trigonometric Functions

An angle whose measure is greater than 0^o but less than 90^o is called an **acute angle**. Let a revolving line OP start from OA and revolve into the position OP, thus tracing out the angle AOP. In the revolving line take any point P and draw PM perpendicular to the initial line OA.

Let the angle AOP be denoted by θ. In the triangle, we call OM *adjacent side,* MP *opposite side,* and OP *hypotenuse*. We also denote the adjacent side, opposite side and hypotenuse by the symbols a, b, c respectively.

Trigonometry deals with the measurement of the sides and angles of a triangle.
The six basic Trigonometrical functions of an acute angle are defined in terms of the sides of a right triangle.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{b}{c}, \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{c}{b}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{a}{c}, \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{c}{a}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{b}{a}, \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{a}{b}$$

The abbreviations stand for $sine$,$cosine$, $tangent$, $cosecant$, $secant$ and $cotangent$ of the angle θ.

From the definitions, it is evident that the $cosecant$ is the reciprocal of the $sine$, $secant$ is the reciprocal of the $cosine$ and the $cotangent$ is the reciprocal of the $tangent$.

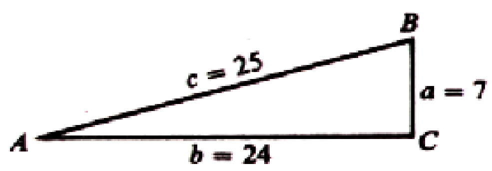
$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

We also note that $\tan\theta = \frac{b}{a} = \frac{b}{c} \times \frac{c}{a} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{\sin\theta}{\cos\theta}$

And similarly $\cot\theta = \frac{\cos\theta}{\sin\theta}$

It is a standard notation that in a triangle ABC, the sides opposite to the angles A, B, C are denoted by the letters by a, b, c respectively.

Example 1: Find the trigonometric functions of the acute angles of the right triangle ABC, given $b = 24$ and $c = 25$



Solution: $a = \sqrt{c^2 - b^2} = \sqrt{25^2 - 24^2} = 7$

$$\sin A = \frac{7}{25} \qquad \csc A = \frac{25}{7}$$

$$\cos A = \frac{24}{25} \qquad \sec A = \frac{25}{24}$$

$$\tan A = \frac{7}{24} \qquad \cot A = \frac{24}{7}$$

$$\sin B = \frac{24}{25} \qquad \csc B = \frac{25}{24}$$

$$\cos B = \frac{7}{25} \qquad \sec B = \frac{25}{7}$$

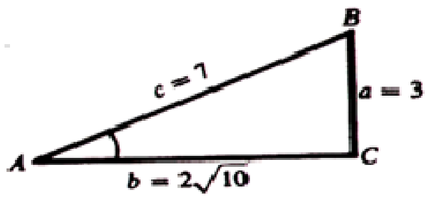
$$\tan B = \frac{24}{7} \qquad \cot B = \frac{7}{24}$$

Example 2:

Find the values of the trigonometric functions of the acute angle A, given $\sin A = \frac{3}{7}$.

Solution: Construct the right triangle ABC with

$a = 3,\ c = 7,$ and $b = \sqrt{7^2 - 3^2} = 2\sqrt{10}$.



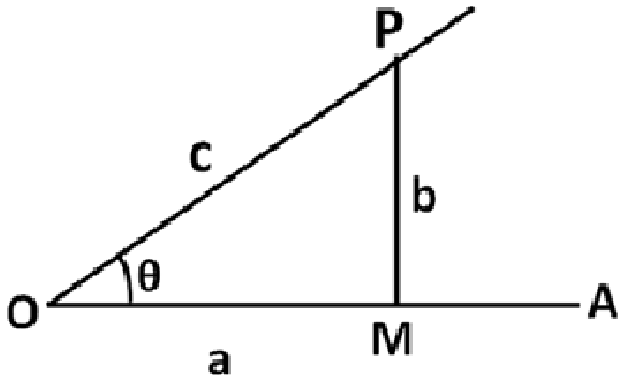
$$\sin A = \frac{3}{7} \qquad \csc A = \frac{7}{3}$$

$$\cos A = \frac{2\sqrt{10}}{7} \qquad \sec A = \frac{7}{2\sqrt{10}}$$

$$\tan A = \frac{3}{2\sqrt{10}} \qquad \cot A = \frac{2\sqrt{10}}{3}$$

Fundamental Relations

We shall find it that if one of the Trigonometrical ratios of an angle is known, then the numerical magnitudes of each of the others can be determined by using the fundamental relations.



In the right angled triangle MOP, $(OM = a, MP = b, OP = c, \text{ angle MOP} = \theta)$

we have $b^2 + a^2 = c^2$

Dividing by c^2, we have

$$\left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 = 1 \Rightarrow \boxed{\sin^2\theta + \cos^2\theta = 1}$$

If we divide by a^2, we obtain,

$$\left(\frac{b}{a}\right)^2 + 1 = \left(\frac{c}{a}\right)^2 \Rightarrow \boxed{1 + \tan^2\theta = \sec^2\theta}$$

and, on the other hand, if we divide by b^2, we get the relation

$$1 + \left(\frac{a}{b}\right)^2 = \left(\frac{c}{b}\right)^2 \Rightarrow \boxed{1 + \cot^2\theta = \csc^2\theta}$$

An equation is said to be an **identity** if it is satisfied for all values of the variable for which it is defined.

The three fundamental trigonometric identities are

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Example 3: Prove that $\sqrt{\frac{1-\cos A}{1+\cos A}} = \csc A - \cot A$

Solution:

$$\sqrt{\frac{1-\cos A}{1+\cos A}} = \sqrt{\frac{(1-\cos A)(1-\cos A)}{(1+\cos A)(1-\cos A)}}$$

$$= \sqrt{\frac{(1-\cos A)^2}{1-\cos^2 A}}$$

$$= \sqrt{\frac{(1-\cos A)^2}{\sin^2 A}}$$

$$= \frac{1-\cos A}{\sin A}$$

$$= \frac{1}{\sin A} - \frac{\cos A}{\sin A}$$

$$= \csc A - \cot A$$

Example 4: Prove that $\sqrt{\sec^2 A + \csc^2 A} = \tan A + \cot A$

Solution:

$$\sqrt{\sec^2 A + \csc^2 A} = \sqrt{\frac{1}{\cos^2 A} + \frac{1}{\sin^2 A}}$$

$$= \sqrt{\frac{\sin^2 A + \cos^2 A}{\sin^2 A \cos^2 A}}$$

$$= \frac{1}{\sin A \cos A}$$

$$= \frac{\cos^2 A + \sin^2 A}{\sin A \cos A}$$

$$= \frac{\cos A}{\sin A} + \frac{\sin A}{\cos A}$$

$$= \tan A + \cot A$$

Example 5: **Prove that** $(\csc A - \sin A)(\sec A - \cos A)(\tan A + \cot A) = 1$

$$(\csc A - \sin A)(\sec A - \cos A)(\tan A + \cot A)$$

$$= \left(\frac{1}{\sin A} - \sin A\right)\left(\frac{1}{\cos A} - \cos A\right)\left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right)$$

$$= \frac{1-\sin^2 A}{\sin A} \cdot \frac{1-\cos^2 A}{\cos A} \cdot \frac{\sin^2 A + \cos^2 A}{\sin A \cos A}$$

$$= \frac{\cos^2 A}{\sin A} \cdot \frac{\sin^2 A}{\cos A} \cdot \frac{1}{\sin A \cos A}$$

$$= 1$$

Limits of the values of Trigonometrical Functions

We have

$$\sin^2\theta + \cos^2\theta = 1$$

Both terms, being squares, are necessarily positive. Hence, neither of them can be greater than unity, since their sum is unity. Hence neither the sine nor the cosine can be numerically greater than unity.

Since $\sin\theta$ cannot be greater than unity, $\csc\theta, \left(\text{which is } \dfrac{1}{\sin\theta}\right)$

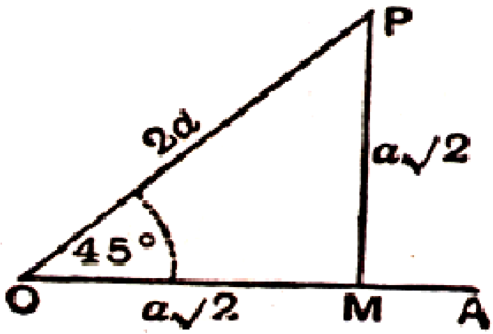
Cannot be numerically less than unity.

Similarly, $\sec\theta, \left(\text{which is } \dfrac{1}{\cos\theta}\right)$ cannot be numerically less than unity.

Values of the Trigonometrical Ratios

Angle of 45°

Let $OP = 2a$, angle $OPM = 45°$



Then, the angle OPM is equal to 45°. Therefore $OM = MP$. We have

$(OM)^2 + (MP)^2 = (OP)^2$

$$\Rightarrow 2(OM)^2 = (2a)^2 = 4a^2$$

$$\Rightarrow OM = MP = a\sqrt{2}$$

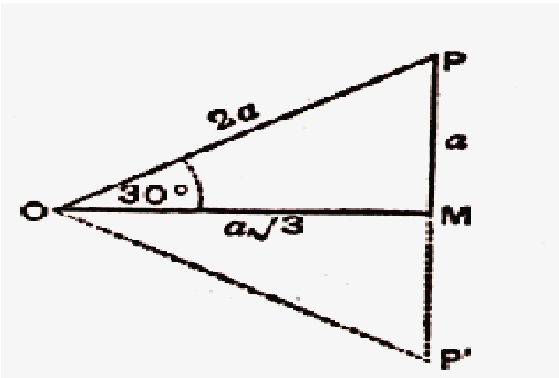
$$\sin 45^\circ = \frac{a\sqrt{2}}{2a} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{a\sqrt{2}}{2a} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{a\sqrt{2}}{a\sqrt{2}} = 1$$

Angle of 30° and 60°

Let $OP = 2a$ and angle $POM = 30°$



Produce PM to P', making MP' equal to MP. The two triangles OMP and OMP' are congruent. Therefore $OP' = OP$ and angle $OP'P = OPP' = 60°$ The triangle $P'OP$ is equilateral. Hence $OP = OP' = PP' = 2a$

$$\therefore\ MP = \frac{1}{2}OP = a$$

$$OM = \sqrt{4a^2 - a^2} = a\sqrt{3}$$

Therefore, In ΔOPM ,

$$\sin 30^\circ = \frac{a}{2a} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Angle APM is 60°. We obtain

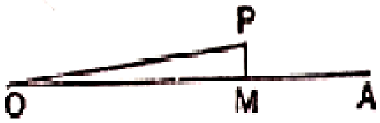
$$\sin 60^\circ = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{a}{2a} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{a\sqrt{3}}{} = \sqrt{3}$$

Angle of 0°

Let the revolving line OP turn through a very small angle AOP. When the angle AOP tends to zero, the lengths OM and OP are equal and MP is zero.



Hence

$$\sin 0^\circ = \frac{MP}{OP} = \frac{0}{OP} = 0$$

$$\cos 0^\circ = \frac{OM}{OP} = \frac{OP}{OP} = 1$$

$$\tan 0^\circ = \frac{0}{1} = 0$$

$$\cot 0^\circ = \frac{OM}{MP} = \frac{OM}{0} = \infty$$

$$\csc 0^\circ = \frac{OP}{MP} = \frac{OP}{0} = \infty$$

$$\sec 0^\circ = \frac{OP}{OM} = \frac{OM}{OM} = 1$$

Angle of 90°

Let the angle AOP be very nearly a right angle. When the angle AOP tends to be a right angle, the point M coincides with O. Then OM is zero and OP and MP are equal

Hence

$$\sin 90^\circ = \frac{MP}{OP} = \frac{OP}{OP} = 1$$

$$\cos 90^\circ = \frac{OM}{OP} = \frac{0}{OP} = 0$$

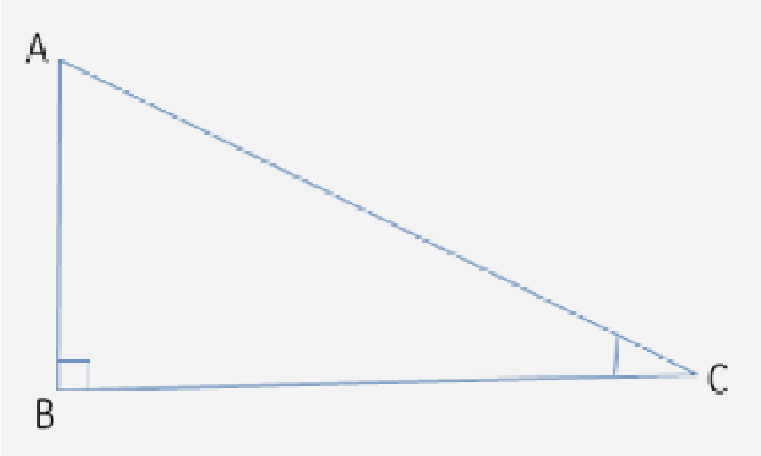
$$\tan 90^\circ = \frac{MP}{OM} = \frac{MP}{0} = \infty$$

$$\cot 90^\circ = \frac{OM}{MP} = \frac{0}{MP} = 0$$

$$\sec 90^\circ = \frac{OP}{OM} = \frac{OP}{0} = \infty$$

$$\csc 90^\circ = \frac{OP}{MP} = \frac{OP}{OP} = 1$$

IP1: In a Right angled triangle ABC



If $a = 12, b = 13$ then the value of $\left[\frac{\csc A+\sec C}{\cot A+\tan C}\right] - \left[\frac{\sin A+\cos C}{\tan A+\cot C}\right] =$

Solution:

Step1: By the hypothesis, we have $a = 12$ and $b = 13$

By Pythagoras theorem, we have

$$(13)^2 = (12)^2 + (AB)^2$$

$$\Longrightarrow \quad AB^2 = 169 - 144 = 25$$

$$\Longrightarrow \quad AB^2 = 5$$

Step 2: Now

$$\left[\frac{\csc A+\sec C}{\cot A+\tan C}\right] - \left[\frac{\sin A+\cos C}{\tan A+\cot C}\right]$$

$$= \left[\frac{\frac{13}{12}+\frac{13}{12}}{\frac{5}{12}+\frac{5}{12}}\right] - \left[\frac{\frac{12}{13}+\frac{12}{13}}{\frac{12}{5}+\frac{12}{5}}\right]$$

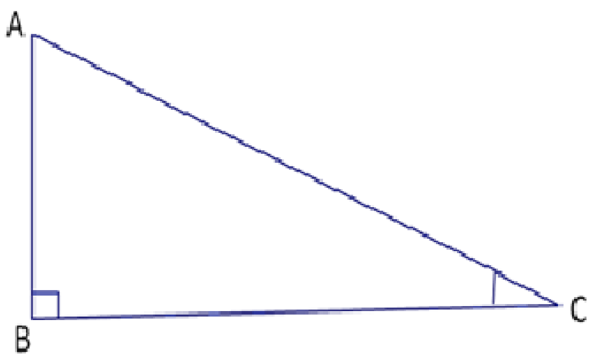
$$= \frac{\frac{13}{12}}{\frac{5}{12}} - \frac{\frac{12}{13}}{\frac{12}{5}}$$

$$= \frac{13}{5} - \frac{5}{13}$$

$$= \frac{169-25}{65} = \frac{144}{65}$$

Step 3: Hence $\left[\frac{\csc A+\sec C}{\cot A+\tan C}\right] - \left[\frac{\sin A+\cos C}{\tan A+\cot C}\right] = \frac{144}{65}$

P1: In a Right angled triangle ABC



If $a = 4, b = 5$ then the value of $10\left[\frac{\sin A+\cos C}{\tan A+\cot C}\right] + 18\left[\frac{\csc A+\sec C}{\cot A+\tan C}\right] =$

A. 30

B. 36
C. 25
D. 17

Answer: B

Solution:

By hypothesis, we have $a = 4, b = 5$

By Pythagoras theorem, we have

$$(5)^2 = (4)^2 + AB^2$$
$$\Longrightarrow 25 - 16 = AB^2$$
$$\Longrightarrow AB^2 = 9$$
$$\Longrightarrow AB = \sqrt{9} = 3 = c$$

Now,

$$10\left[\frac{\sin A+\cos C}{\tan A+\cot C}\right] + 18\left[\frac{\csc A+\sec C}{\cot A+\tan C}\right]$$
$$= 10\left[\frac{\frac{4}{5}+\frac{4}{5}}{\frac{4}{3}+\frac{4}{3}}\right] + 18\left[\frac{\frac{5}{4}+\frac{5}{4}}{\frac{3}{4}+\frac{3}{4}}\right]$$
$$= 10\left[\frac{\frac{4}{5}}{\frac{4}{3}}\right] + 18\left[\frac{\frac{5}{4}}{\frac{3}{4}}\right]$$
$$= 10\left[\frac{3}{5}\right] + 18\left[\frac{5}{3}\right]$$
$$= 6 + 30 = 36$$

Hence $10\left[\frac{\sin A+\cos C}{\tan A+\cot C}\right] + 18\left[\frac{\csc A+\sec C}{\cot A+\tan C}\right] = 36$

IP2: Prove that $(\boldsymbol{tan\theta} + \boldsymbol{cot\theta})^2 = \boldsymbol{sec^2\theta} + \boldsymbol{cosec^2\theta} = \boldsymbol{sec^2\theta.cosec^2\theta}$

Solution:

Step1:

$$(tan\theta + cot\theta)^2 = tan^2\theta + cot^2\theta + 2tan\theta cot\theta$$
$$= tan^2\theta + cot^2\theta + 2$$
$$= (1 + tan^2\theta) + (1 + cot^2\theta)$$
$$= sec^2\theta + cosec^2\theta \quad \text{.... (1)}$$

Step2:

$$sec^2\theta + cosec^2\theta = \frac{1}{cos^2\theta} + \frac{1}{sin^2\theta}$$
$$= \frac{sin^2\theta + cos^2\theta}{sin^2\theta.cos^2\theta}$$
$$= \frac{1}{sin^2\theta.cos^2\theta}$$
$$= sec^2\theta.cosec^2\theta \quad \text{............ (2)}$$

Step3:

From (1) and (2), we have

$$(tan\theta + cot\theta)^2 = sec^2\theta + cosec^2\theta = sec^2\theta.cosec^2\theta$$

P2: $\frac{\boldsymbol{tan\theta+sec\theta-1}}{\boldsymbol{tan\theta-sec\theta+1}} =$

A. $\frac{1+sin\theta}{cos\theta}$
B. $\frac{1-sin\theta}{cos\theta}$
C. $\frac{1+cos\theta}{sin\theta}$
D. $\frac{1-cos\theta}{cos\theta}$

Answer: **A**

Solution:

$$\frac{tan\theta+sec\theta-1}{tan\theta-sec\theta+1} = \frac{tan\theta+sec\theta-1}{tan\theta-sec\theta+1} \times \frac{tan\theta-sec\theta-1}{tan\theta-sec\theta-1}$$
$$= \frac{(tan\theta+sec\theta-1)\times(tan\theta-sec\theta-1)}{(tan\theta-sec\theta)^2-1}$$
$$= \frac{[(tan\theta-1)+sec\theta]\times[(tan\theta-1)-sec\theta]}{tan^2\theta+sec^2\theta-2tan\theta.sec\theta-1}$$
$$= \frac{(tan\theta-1)^2-sec^2\theta}{2tan^2\theta-2tan\theta.sec\theta}$$
$$= \frac{tan^2\theta+1-2tan\theta-sec^2\theta}{2tan\theta(tan\theta-sec\theta)}$$
$$= \frac{sec^2\theta-sec^2\theta-2tan\theta}{2tan\theta(tan\theta-sec\theta)} = \frac{2tan\theta}{2tan\theta(sec\theta-tan\theta)}$$
$$= \frac{1}{(sec\theta-tan\theta)} = sec\theta + tan\theta$$
$$= \frac{1}{cos\theta} + \frac{sin\theta}{cos\theta} = \frac{1+sin\theta}{cos\theta}$$

IP3: $\frac{\boldsymbol{\cos^2 30^0+\tan^2 30^0}}{\boldsymbol{\cot^2 30^0+\sec^2 30^0}} + \frac{\boldsymbol{\sin^2 45^0+\csc^2 60^0}}{\boldsymbol{\sec^2 0^0+\csc^2 90^0}} =$

Solution:

Step 1:

$$\frac{\cos^2 30^0+\tan^2 30^0}{\cot^2 30^0+\sec^2 30^0} + \frac{\sin^2 45^0+\csc^2 60^0}{\sec^2 0^0+\csc^2 90^0}$$
$$= \frac{\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2}{\left(\sqrt{3}\right)^2+\left(\frac{2}{\sqrt{3}}\right)^2} + \frac{\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{2}{\sqrt{3}}\right)^2}{(1)^2+(1)^2}$$
$$= \frac{\frac{3}{4}+\frac{1}{3}}{3+\frac{4}{3}} + \frac{\frac{1}{2}+\frac{4}{3}}{2}$$
$$= \frac{\frac{9+4}{12}}{\frac{9+4}{3}} + \frac{\frac{3+8}{6}}{2}$$
$$= \frac{3}{12} + \frac{11}{12} = \frac{14}{12} = \frac{7}{6}$$

Step 2:

$$\therefore \frac{\cos^2 30^0+\tan^2 30^0}{\cot^2 30^0+\sec^2 30^0} + \frac{\sin^2 45^0+\csc^2 60^0}{\sec^2 0^0+\csc^2 90^0} = \frac{7}{6}$$

P3: $\frac{\sin 30^0 - \cos 60^0}{\tan 0^0 + \cot 45^0} + \frac{\csc^2 60^0 + \sec^2 30^0}{\cot^2 90° - \sec^2 60^0} =$

A. $\frac{3}{2}$

B. $\frac{2}{3}$

C. $-\frac{2}{3}$

D. $\frac{8}{3}$

Answer: **C**

Solution:

$$\frac{\sin 30^0 - \cos 60^0}{\tan 0^0 + \cot 45^0} + \frac{\csc^2 60^0 + \sec^2 30^0}{\cot^2 90^\circ - \sec^2 60^0}$$

$$= \frac{\frac{1}{2} - \frac{1}{2}}{0+1} + \frac{\left(\frac{2}{\sqrt{3}}\right)^2 + \left(\frac{2}{\sqrt{3}}\right)^2}{(0)^2 - (2)^2}$$

$$= \frac{0}{1} + \frac{\frac{4}{3} + \frac{4}{3}}{-(2)^2}$$

$$= 0 + \frac{\frac{8}{3}}{-4} = -\frac{2}{3}$$

IP4: If $\sin\theta = \frac{15}{17}$ then find the value of $\frac{15\cot\theta + 17\sin\theta}{8\tan\theta + 16\sec\theta}$

Solution:

Step 1: Given that $\sin\theta = \frac{15}{17}$

We know that $\sin^2\theta + \cos^2\theta = 1$

$$\Longrightarrow \cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \left(\frac{15}{17}\right)^2} = \sqrt{1 - \frac{225}{289}} = \sqrt{\frac{64}{289}} = \frac{8}{17}$$

Step 2:

Since $\cos\theta = \frac{8}{17} \Longrightarrow \sec\theta = \frac{17}{8}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{5}{17}}{\frac{8}{17}} = \frac{15}{8}$$

$$\Longrightarrow \cot\theta = \frac{8}{15}$$

Step 3: Now,

$$\frac{15\cot\theta + 17\sin\theta}{8\tan\theta + 16\sec\theta} = \frac{15\left(\frac{8}{15}\right) + 17\left(\frac{15}{17}\right)}{8\left(\frac{15}{8}\right) + 16\left(\frac{17}{8}\right)} = \frac{8+15}{15+34} = \frac{23}{49}$$

P4: If $\sin\theta = \frac{3}{5}$ then find the value of $\frac{\tan\theta + 3\sec\theta}{5\csc\theta + 7\cot\theta}$

A. $\frac{9}{106}$

B. $\frac{27}{106}$

C. $\frac{9}{53}$

D. $\frac{3}{53}$

Answer: **B**

Solution: Given that $\sin\theta = \frac{3}{5} \implies \csc\theta = \frac{5}{3}$

We have $\sin^2\theta + \cos^2\theta = 1$

$$\implies \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\implies \sec\theta = \frac{5}{4}$$

Again, we have $\tan\theta = \frac{\sin\theta}{\cos\theta} \implies \tan\theta = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4} \implies \cot\theta = \frac{4}{3}$

Now,

$$\frac{\tan\theta+3\sec\theta}{5\csc\theta+7\cot\theta} = \frac{\frac{3}{4}+3\left(\frac{5}{4}\right)}{5\left(\frac{5}{3}\right)+7\left(\frac{4}{3}\right)} = \frac{\frac{3+15}{4}}{\frac{25+28}{3}} = \frac{18}{42}\times\frac{3}{53} = \frac{27}{106}$$

Exercise:

1. Prove the following identities
   a. $1-\sin^2\theta-\cos^2\theta=0$
   b. $\cos\theta\cdot\tan\theta=\sin\theta$
   c. $\sin^2\theta\left(1+\cot^2\theta\right)=1$
   d. $\cos^2\theta\left(1+\tan^2\theta\right)=1$
2. Prove that
   a. $\dfrac{\sin\theta-\cos\theta}{\sec\theta-\csc\theta}=\sin\theta\cdot\cos\theta$
   b. $\cos^4\theta-\sin^4\theta=\cos^2\theta-\sin^2\theta$
   c. $\dfrac{1}{\csc\theta-\cot\theta}=\csc\theta+\cot\theta$
   d. $\dfrac{\sqrt{\sec^2\theta-1}}{\sec\theta}=\sin\theta$
   e. $\sec^2\theta-\csc^2\theta=\tan^2\theta-\cot^2\theta$
3. Prove that
   a. $\left(\sin^2\theta-\cos^2\theta\right)^2=1-4\sin^2\theta\cos^2\theta$
   b. $\tan\theta+\cot\theta=\sec\theta\cdot\csc\theta$

c. $\tan\theta\csc\theta(\csc\theta-\sin\theta)=\cot\theta$

d. $\tan\theta+\sec\theta=\dfrac{1}{\sec\theta-\tan\theta}$

e. $\sin\theta=\dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}$

4. Prove that

a. $(\sin\theta+\cos\theta)(\cot\theta+\tan\theta)=\sec\theta+\csc\theta$

b. $(\sec\theta-\cos\theta)(\csc\theta-\sin\theta)=\dfrac{1}{\tan\theta+\cot\theta}$

c. $\tan^2 A-\tan^2 B=\dfrac{\cos^2 B-\cos^2 A}{\cos^2 A\cdot\cos^2 B}$

d. $\sin\theta\cdot\cos\theta=\dfrac{\sin\theta+\cos\theta}{\sec\theta+\csc\theta}$

e. $\cos^4\theta-\sin^4\theta=2\cos^2\theta-1$

5. Prove that

a. $\dfrac{1-\sin\theta}{\cos\theta}=\dfrac{\cos\theta}{1+\sin\theta}$

b. $\dfrac{1-\cos^2\theta}{\sqrt{\cos^2\theta}}=\tan\theta\cdot\sin\theta$

c. $\dfrac{\sin\theta}{1-\cos\theta}=\dfrac{1+\cos\theta}{\sin\theta}$

d. $\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{1+\cos\theta}{\sin\theta}=2\csc\theta$

6. Prove that

a. $\dfrac{\cos^2\theta+\tan^2\theta-1}{\sin^2\theta}=\tan^2\theta$

b. $\tan^4\theta+\tan^2\theta=\sec^4\theta-\sec^2\theta$

c. $\dfrac{\csc\theta}{\csc\theta-1}+\dfrac{\csc\theta}{\csc\theta+1}=2\sec^2\theta$

d. $\dfrac{\tan^2\theta}{\tan^2\theta-1}+\dfrac{\csc^2\theta}{\sec^2\theta-\csc^2\theta}=\dfrac{1}{\sin^2\theta-\cos^2\theta}$

7. Prove that $\sec\theta(1+\sin\theta)(\sec\theta-\tan\theta)=1$

8. If $\sin\theta+\cos\theta=x,\ \sin\theta-\cos\theta=y$, prove that $x^2+y^2=2$

9. If $x=a\cos\theta,\ y=b\cot\theta$ prove that $\frac{a^2}{x^2}-\frac{b^2}{y^2}=1$

10. If $x=a\sec\theta+b\tan\theta,\ y=a\tan\theta+b\sec\theta$ prove that $x^2-y^2=a^2-b^2$

11. Express all the trigonometric ratios in terms of sine.

12. Express all the trigonometric ratios in terms of the Cotangent.

13. If $\cos\theta$ equals $\frac{3}{5}$, find the value of the other ratios.

14. Supposing θ to be an angle whose sine is $\frac{1}{3}$, find the numerical magnitude of the other Trigonometrical ratios.

15. Find the values of the trigonometric functions of the acute angles of the right triangle ABC, given $a=2$ and $c=2\sqrt{5}$. Angle C is right angle and A and B are acute angles.

16. Find the values of the trigonometric functions of the acute angle B, given $\tan B=1.5$. ABC is right triangle with the right angle at C.

17. If $cos\theta + sin\theta = \sqrt{2}cos\theta$ then prove that

$$cos\theta - sin\theta = \sqrt{2}sin\theta$$

18. Show that $2(sin^6\theta + cos^6\theta) - 3(sin^4\theta + cos^4\theta) = -1$

19. Find the values of the following
    a. $sin30°.cos60° + cos30°.sin60°$
    b. $tan45°.sec30° - cot90°.cosec45°$
    c. $\frac{sec30°}{cosec30°} + \frac{cot30°}{tan60°} - \frac{cosec60°}{sin60°}$
    d. $\frac{sin30°}{cos45°} - \frac{cos60° - cosec60°}{sec60° - tan30°} + cot90°$
    e. $\frac{sin0°}{cos0°} + \frac{tan45°}{cot45°} - \frac{cos60°}{cot45°}$

4.3 Trigonometric Functions of General Angles

Learning Objectives:

- To define the trigonometric functions of any angle
- To derive three trigonometric identities in the general set up
- To find the values of trigonometric functions for any general angle

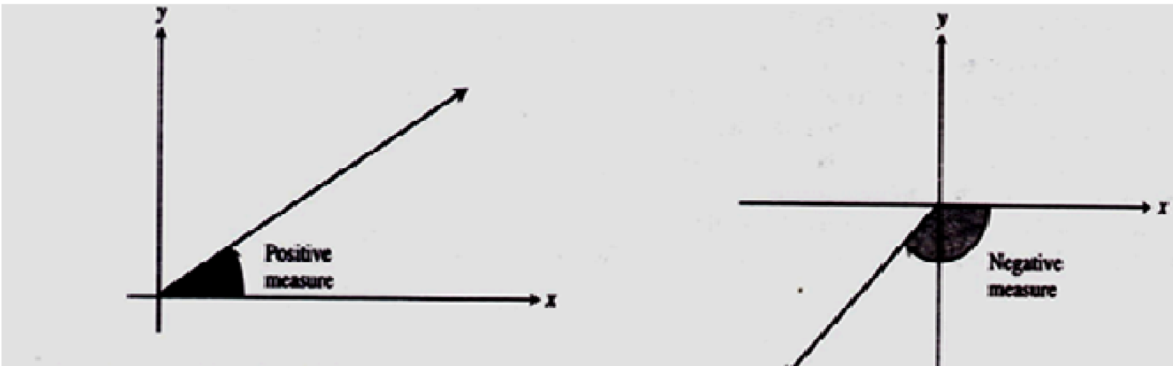
AND

- To solve related problems

Trigonometric Functions of General Angles

Angles of any Size and Sign

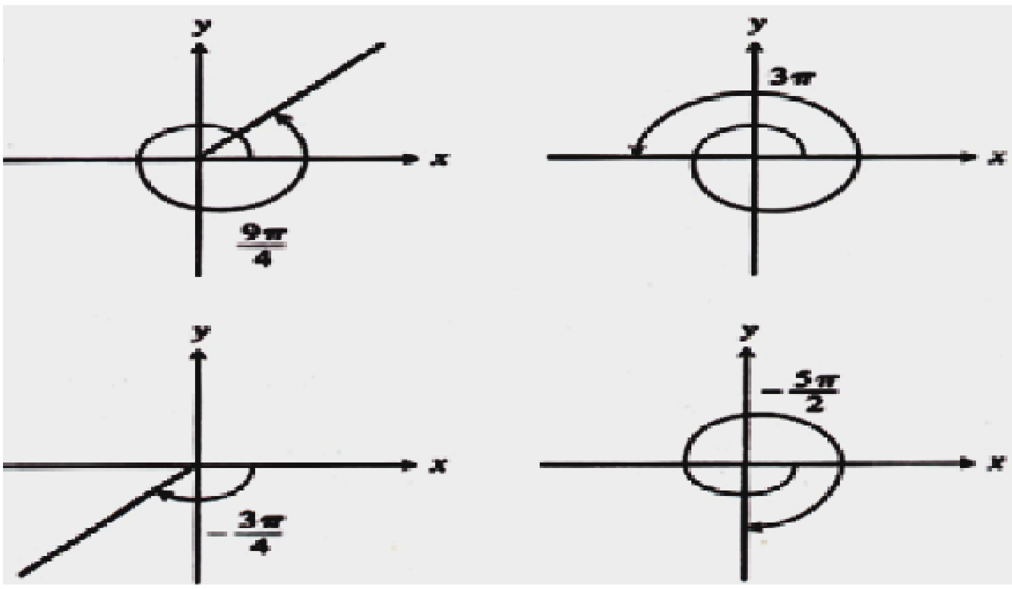
An angle in the xy-plane is said to be in **standard position** if its vertex lies at the origin and its initial ray lies along the positive x-axis.



Angles in standard position in the xy-plane.

Angles measured counterclockwise from the positive x −axis are assigned positive measures; angles measured clockwise are assigned negative measures.

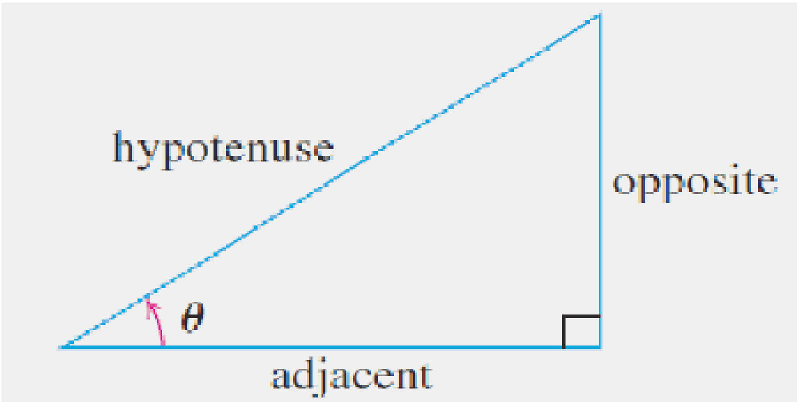
We now, allow for the possibility that the ray may make more than one complete revolution.



When the revolving line has made a complete revolution, so that it is again along the positive x-axis, the angle through which it has turned is 4 right angles or 2π radians or $360°$. If the line still continues to revolve, the angle through which it has turned will be more than 360°. Thus, when angles are used to describe counterclockwise rotations, our measurements can go arbitrarily far beyond 2π radians or 360°. Similarly, angles describing clockwise rotations can have measures of all sizes.

The Six Basic Trigonometric Functions

The trigonometric functions of an acute angle are defined in terms of the sides of a right triangle.

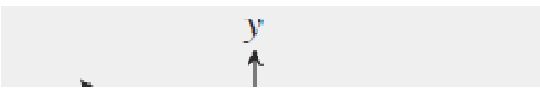


$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

This definition is extended to obtuse and negative angles by first placing the angle in standard position in a circle of radius r. **The trigonometric functions are then defined in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle.**



$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

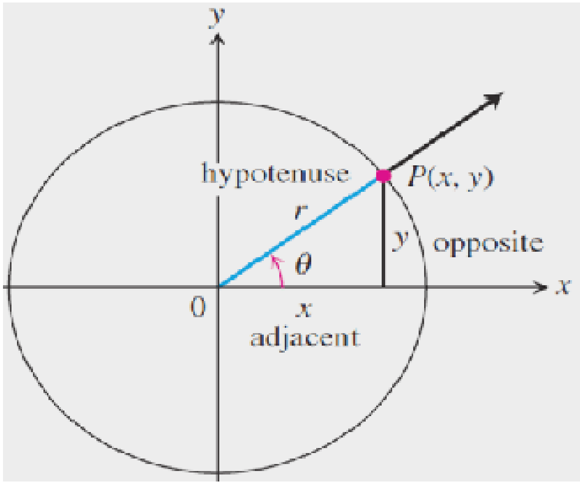
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

If $x = 0$, $\tan\theta$ *and* $\sec\theta$ *are not defined. This means they are not defined if* $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots$

Similarly, $\cot\theta$ *and* $\csc\theta$ *are not defined for values of* θ *for which* $y = 0$*, namely* $\theta = 0, \pm\pi, \pm 2\pi, \cdots$

These extended definitions agree with the right-triangle definitions when the angle is acute.

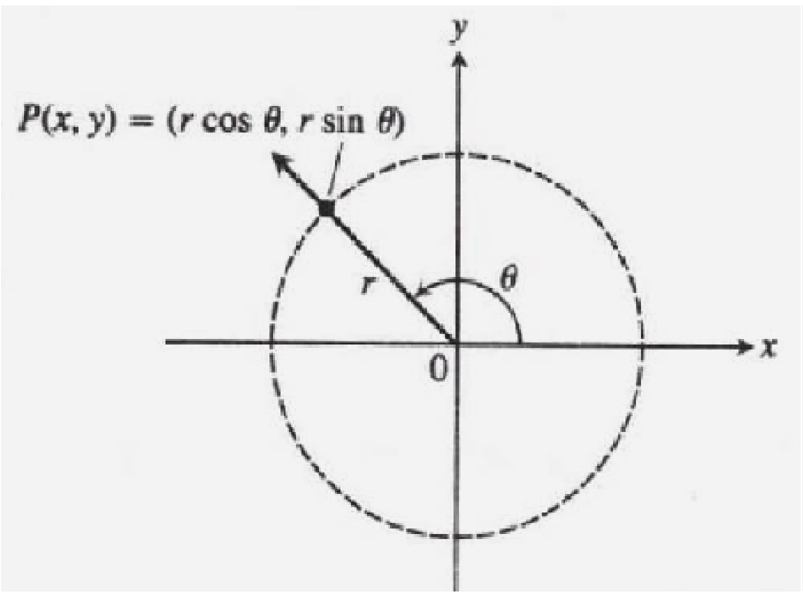


We note the following relations:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

The coordinates of any point $P(x, y)$ in the plane can now be expressed in terms of the point's distance from the origin and the angle that ray OP makes with the positive x −axis.



Since $\frac{x}{r} = \cos\theta$ *and* $\frac{y}{r} = \sin\theta$, we have

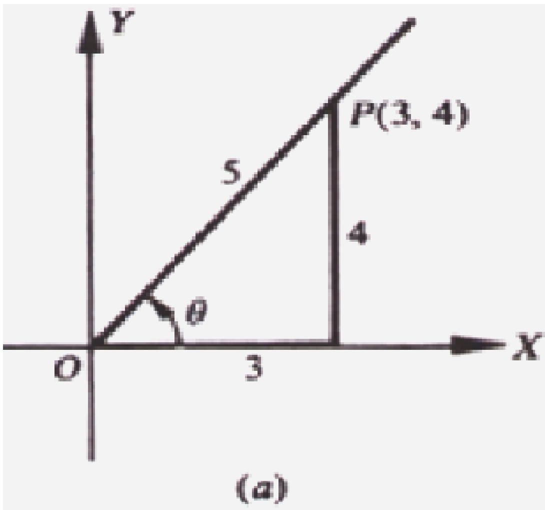
$$x = r\cos\theta \quad and \quad y = r\sin\theta$$

The signs of the trigonometric functions of an angle are determined by the quadrant in which the terminal ray of the angle falls. For example, if the terminal ray falls in the first quadrant, then x and y are positive, so all the trigonometric functions are positive. If the terminal ray falls in the second quadrant, then x is negative and y is positive, so sine and cosecant are positive, but all other trigonometric functions are negative. In the third quadrant, tangent and cotangent are positive, while in the fourth quadrant, cosine and secant are positive.

Example1 : Determine the values of the trigonometric functions of angle θ (smallest positive angle in standard position) if P is a point on the terminal ray of θ and the coordinates of P are (a) $P(3,4)$, (b) $P(-3,4)$, (c) $P(-1,-3)$.

Solution:

(a)



$$r = \sqrt{3^2 + 4^2} = 5$$

$$\sin\theta = \frac{y}{r} = \frac{4}{5}, \qquad \csc\theta = \frac{5}{4}$$

$$\cos\theta = \frac{x}{r} = \frac{3}{5}, \qquad \sec\theta = \frac{5}{3}$$

$$\tan\theta = \frac{x}{y} = \frac{4}{3}, \qquad \cot\theta = \frac{3}{4}$$

(b)

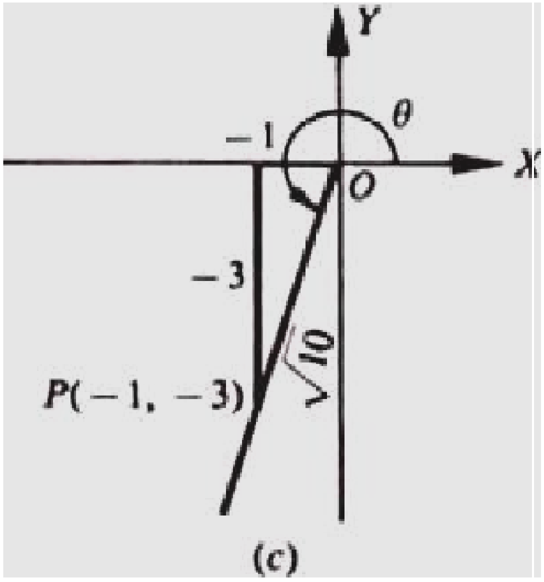
$$r = \sqrt{(-3)^2 + 4^2} = 5$$

$$\sin\theta = \frac{y}{r} = \frac{4}{5}, \qquad \csc\theta = \frac{5}{4}$$

$$\cos\theta = \frac{x}{r} = \frac{-3}{5}, \qquad \sec\theta = -\frac{5}{3}$$

$$\tan\theta = \frac{x}{y} = \frac{4}{-3}, \qquad \cot\theta = -\frac{3}{4}$$

(c)



$$r=\sqrt{(-1)^2+(-3)^2}=\sqrt{10}$$

$$\sin\theta=\frac{y}{r}=\frac{-3}{\sqrt{10}},\qquad \csc\theta=-\frac{\sqrt{10}}{3}$$

$$\cos\theta=\frac{x}{r}=\frac{-1}{\sqrt{10}},\qquad \sec\theta=-\sqrt{10}$$

$$\tan\theta=\frac{x}{y}=\frac{-3}{-1}=3,\qquad \cot\theta=\frac{1}{3}$$

Example 2: Find the values of

$\cos\,\theta$ and $\tan\,\theta$, given $\sin\,\theta=\dfrac{8}{17}$ and $\theta\, is$ in quadrant I.

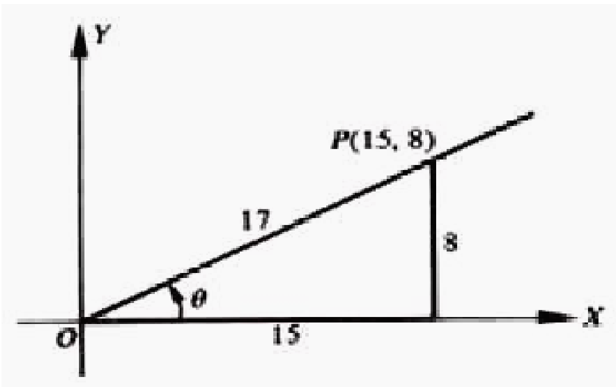
Solution:

Let P be a point on the terminal line of θ.

Since, $\sin\theta = y/r = 8/17$, we take $y = 8$ and $r = 17$.

Since θ is in quadrant I, x is positive; thus

$$x=\sqrt{r^2-y^2}=\sqrt{17^2-8^2}=15$$

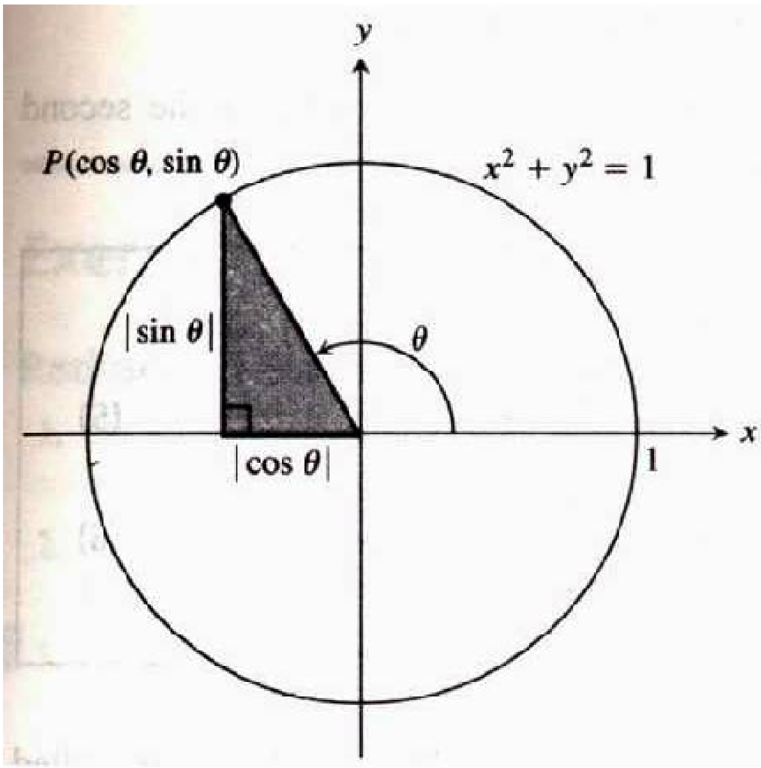


$$\cos\theta=\frac{x}{r}=\frac{15}{17}\qquad \tan\theta=\frac{y}{x}=\frac{8}{15}$$

Identities

A ***trigonometric identity*** is an equation involving trigonometric functions that is true for all angles for which both sides of the equation are defined.

The reference right triangle for a general angle θ is obtained by dropping a perpendicular from the point $P(\cos\theta\,,\sin\theta)$ on the unit circle to the x −axis.



Applying the Pythagorean Theorem to the reference right triangle, we obtain $cos^2\theta + sin^2\theta = 1$

This is the well-known identity in trigonometry. Dividing this equation in turn by $cos^2\theta$ and $sin^2\theta$ gives the identities

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Example 3: Prove that $\dfrac{\sin^2\theta + 2\cos^2\theta}{\sin\theta\cos\theta} = \tan\theta + 2\cot\theta$

Solution:

$$\frac{\sin^2\theta + 2\cos^2\theta}{\sin\theta\cos\theta} = \frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{2\cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta} + \frac{2\cos\theta}{\sin\theta} = \tan\theta + 2\cot\theta$$

Example 4: Prove the identity $\tan x + \cot x = \dfrac{\csc x}{\cos x}$

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$

Solution:

$$= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x} = \frac{\csc x}{\cos x}$$

Example 5: Prove the identity $\dfrac{\sec x}{\cot x + \tan x} = \sin x$

Solution: $$\frac{\sec x}{\cot x + \tan x} = \frac{1}{\cos x\left(\dfrac{\cos x}{\sin x} + \dfrac{\sin x}{\cos x}\right)} = \frac{\sin x}{\cos^2 x + \sin^2 x} = \sin x$$

Example 6: **Prove the identity** $\dfrac{\sin x}{1+\cos x} = \dfrac{1-\cos x}{\sin x}$

Solution:
$$\begin{aligned}\frac{\sin x}{1+\cos x} &= \frac{1-\cos x}{1-\cos x}\cdot\frac{\sin x}{1+\cos x}\\ &= \frac{(1-\cos x)\sin x}{1-\cos^2 x}\\ &= \frac{(1-\cos x)\sin x}{\sin^2 x} = \frac{1-\cos x}{\sin x}\end{aligned}$$

Example 7: Prove the identity $\sqrt{\dfrac{\sec x - \tan x}{\sec x + \tan x}} = \dfrac{1}{\sec x + \tan x}$

Solution:
$$\begin{aligned}\sqrt{\frac{\sec x - \tan x}{\sec x + \tan x}} &= \sqrt{\frac{\sec x - \tan x}{\sec x + \tan x}.\frac{\sec x + \tan x}{\sec x + \tan x}}\\ &= \frac{\sqrt{\sec^2 x - \tan^2 x}}{\sec x + \tan x} = \frac{1}{\sec x + \tan x}\end{aligned}$$

Values of Trigonometric Functions

The coordinates of a point P on the unit circle are $(\cos\theta, \sin\theta)$.

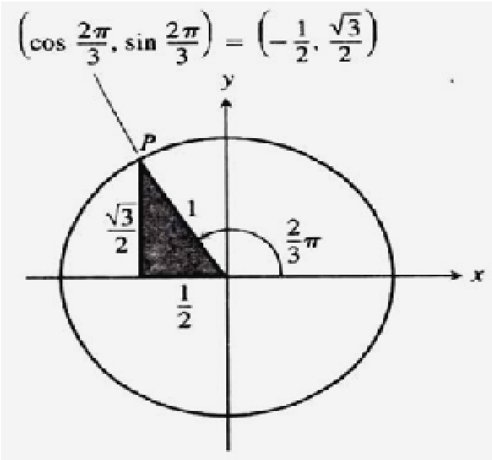
$$cos\,\theta = x, sin\,\theta = y$$

We can then calculate the values of the cosine and sine directly from the coordinates of P, if we happen to know them, or indirectly from the acute reference triangle made by dropping a perpendicular from P to the x-axis.

We read the magnitudes of x and y from the triangle's sides. The signs of x and y are determined by the quadrant in which the triangle lies.

Example 8: Find the sine and cosine of 2π/3 radians.

Solution: We draw the angle in standard position in the unit circle and write the lengths of the sides of the reference triangle.



We recall from geometry that the hypotenuse of a $30^\circ - 60^\circ - 90^\circ$ triangle is twice the shorter leg, where the shorter leg is opposite 30^o angle.

The coordinates of the point P, where the angle's terminal ray cuts the circle, are $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Therefore,

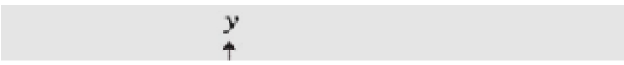
$$\cos\frac{2\pi}{3} = x\text{ - coordinate of P} = -\frac{1}{2}$$

$$\sin\frac{2\pi}{3} = y\text{ - coordinate of P} = \frac{\sqrt{3}}{2}$$

Example 9:Find the sine and cosine of $-\pi/4$ ***radians.***

Solution:

We draw the angle in standard position in the unit circle and write the lengths of sides of the reference triangle.



We recall from geometry that the two legs of a $45^\circ - 45^\circ - 90^\circ$ triangle are of equal size.

The coordinates of the point P, where the angle's terminal ray cuts the circle, are $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. Therefore,

$$\cos\left(-\frac{\pi}{4}\right) = x\text{-coordinate of P} = \frac{\sqrt{2}}{2}$$

$$\sin\left(-\frac{\pi}{4}\right) = y\text{-coordinate of P} = -\frac{\sqrt{2}}{2}$$

PROBLEM SET

IP1: $\frac{\tan^2\theta(\csc\theta-1)}{1+\cos\theta} + \frac{\csc^2\theta(\cos\theta-1)}{1+\csc\theta} =$

Solution:

Step1:

$$\frac{\tan^2\theta(\csc\theta-1)}{1+\cos\theta} + \frac{\csc^2\theta(\cos\theta-1)}{1+\csc\theta}$$

$$= \frac{\tan^2\theta(\csc\theta-1)(\csc\theta+1)+\csc^2\theta(\cos\theta-1)(\cos\theta+1)}{(1+\cos\theta)(1+\csc\theta)}$$

$$= \frac{\tan^2\theta(\csc^2\theta-1)+\csc^2\theta(\cos^2\theta-1)}{1+\csc\theta+\cos\theta+\cos\theta\csc\theta}$$

$$= \frac{\tan^2\theta\cot^2\theta+\csc^2\theta(1-\sin^2\theta-1)}{1+\csc\theta+\cos\theta+\cos\theta\csc\theta}$$

$$= \frac{\tan^2\theta\frac{1}{\tan^2\theta}-\csc^2\theta\sin^2\theta}{1+\csc\theta+\cos\theta+\cos\theta\csc\theta}$$

$$= \frac{1-\frac{1}{\sin^2\theta}\sin^2\theta}{1+\csc\theta+\cos\theta+\cos\theta\csc\theta}$$

$$= \frac{1-1}{1+\csc\theta+\cos\theta+\cos\theta\csc\theta}$$

$$= 0$$

Step2:

Therefore, $\frac{\tan^2\theta(\csc\theta-1)}{1+\cos\theta} + \frac{\csc^2\theta(\cos\theta-1)}{1+\csc\theta} = 0$

P1: $\left[\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}\right]^2 =$

A. $\frac{1+\cos\theta}{1-\cos\theta}$

B. $\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^2$

C. $\frac{1-\cos\theta}{1+\cos\theta}$

D. $\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^2$

Answer: C

Solution:

$$\left[\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}\right]^2 = \left[\frac{(1+\sin\theta)-\cos\theta}{(1+\sin\theta)+\cos\theta}\times\frac{(1+\sin\theta)-\cos\theta}{(1+\sin\theta)-\cos\theta}\right]^2$$

$$= \left[\frac{\{(1+\sin\theta)-\cos\theta\}^2}{(1+\sin\theta)^2-\cos^2\theta}\right]^2$$

$$= \left[\frac{(1+\sin\theta)^2+\cos^2\theta-2(1+\sin\theta)\cos\theta}{1+\sin^2\theta+2\sin\theta-1+\sin^2\theta}\right]^2$$

$$= \left[\frac{1+\sin^2\theta+2\sin\theta+\cos^2\theta-2\cos\theta(1+\sin\theta)}{2\sin^2\theta+2\sin\theta}\right]^2$$

$$= \left[\frac{1+(\sin^2\theta+\cos^2\theta)+2\sin\theta-2\cos\theta(1+\sin\theta)}{2\sin\theta(\sin\theta+1)}\right]^2$$

$$= \left[\frac{1+1+2\sin\theta-2\cos\theta(1+\sin\theta)}{2\sin\theta(1+\sin\theta)}\right]^2$$

$$= \left[\frac{2(1+\sin\theta)-2\cos\theta(1+\sin\theta)}{2\sin\theta(1+\sin\theta)}\right]^2$$

$$= \left[\frac{2(1+\sin\theta)(1-\cos\theta)}{2\sin\theta(1+\sin\theta)}\right]^2$$

$$= \left[\frac{(1-\cos\theta)}{\sin\theta}\right]^2$$

$$= \frac{(1-\cos\theta)^2}{\sin^2\theta}$$

$$= \frac{(1-\cos\theta)^2}{1-\cos^2\theta}$$

$$= \frac{(1-\cos\theta)^2}{(1-\cos\theta)(1+\cos\theta)}$$

$$= \frac{1-\cos\theta}{1+\cos\theta}$$

Therefore, $\left[\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}\right]^2 = \frac{1-\cos\theta}{1+\cos\theta}$

IP2: If $\boldsymbol{sec\theta + tan\theta = p}$, then find the value of $\boldsymbol{sin\theta}$ in terms of $\boldsymbol{p}$.

Solution:

Given: $sec\theta + tan\theta = p \quad \ldots(1)$

Step1:

Since, $sec^2\theta - tan^2\theta = 1$

$\Longrightarrow (sec\theta + tan\theta)\,(sec\theta - tan\theta) = 1$

$\Longrightarrow sec\theta - tan\theta = \frac{1}{p} \quad \ldots(2)$

Step2:

$(1) + (2) \Longrightarrow (sec\theta + tan\theta) + (sec\theta - tan\theta) = p + \frac{1}{p}$

$$\Longrightarrow 2sec\theta = \frac{p^2+1}{p}$$

$$\implies sec\theta = \frac{p^2+1}{2p}$$

$$\implies cos\theta = \frac{2p}{p^2+1}$$

Step3: Since, $sin^2\theta + cos^2\theta = 1$

$$\implies sin^2\theta = 1 - cos^2\theta$$

$$\implies sin^2\theta = 1 - \frac{4p^2}{(p^2+1)^2}$$

$$\implies sin^2\theta = \frac{(p^2+1)^2 - 4p^2}{(p^2+1)^2}$$

$$\implies sin^2\theta = \frac{(p^2-1)^2}{(p^2+1)^2}$$

$$\implies sin\theta = \pm \frac{p^2-1}{p^2+1}$$

P2: If $\boldsymbol{sec\theta + tan\theta = \frac{2}{3}}$, then determine the quadrant in which $\boldsymbol{\theta}$ lies.

A. I^{st} Quadrant
B. II^{nd} Quadrant
C. III^{rd} Quadrant
D. IV^{th} Quadrant

Answer: D

Solution:

Given: $sec\theta + tan\theta = \frac{2}{3}$ ... (1)

We know that $sec^2\theta - tan^2\theta = 1$

$$\implies (sec\theta + tan\theta)\,(sec\theta - tan\theta) = 1$$

$$\implies \frac{2}{3}\,(sec\theta - tan\theta) = 1$$

$$\implies sec\theta - tan\theta = \frac{3}{2} \quad ... (2)$$

$(1) + (2) \implies (sec\theta + tan\theta) + (sec\theta - tan\theta) = \frac{2}{3} + \frac{3}{2}$

$$\implies 2sec\theta = \frac{13}{6}$$

$$\implies sec\theta = \frac{13}{12}$$

And (1) - (2) $\implies (sec\theta + tan\theta) - (sec\theta - tan\theta) = \frac{2}{3} - \frac{3}{2}$

$$\implies 2tan\theta = \frac{-5}{6}$$

$$\implies tan\theta = \frac{-5}{12}$$

Since, $sec\theta$ is positive and $tan\theta$ is negative, then we say that θ lies in IV^{th} Quadrant.

IP3: Prove that

$$\boldsymbol{3(sin\theta - cos\theta)^4 + 6(sin\theta + cos\theta)^2 + 4(sin^6\theta + cos^6\theta) = 13}$$

Solution:

Step1:

Let, $3(sin\theta - cos\theta)^4 + 6(sin\theta + cos\theta)^2 + 4(sin^6\theta + cos^6\theta)$

$= 3\,\{(sin\theta - cos\theta)^2\}^2 + 6(sin\theta + cos\theta)^2$
$+4\{(sin^2\theta)^3 + (cos^2\theta)^3\}$

$= 3(sin^2\theta + cos^2\theta - 2\sin\theta\cos\theta)^2$
$+6(sin^2\theta + cos^2\theta - 2\sin\theta\cos\theta)$
$+4\{(sin^2\theta + cos^2\theta)^3 - 3sin^2\theta.cos^2\theta(sin^2\theta + cos^2\theta)\}$

$[\because\ (a-b)^2 = a^2 + b^2 - 2ab,\ (a+b)^2 = a^2 + b^2 + 2ab$
and $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$
$\Longrightarrow a^3 + b^3 = (a+b)^3 - 3ab(a+b)]$

$= 3(1 - 2\sin\theta\cos\theta)^2 + 6(1 + 2\sin\theta\cos\theta)$
$+4(1 - 3sin^2\theta.cos^2)\quad [\because\ \ sin^2\theta + cos^2\theta = 1]$

$= 3(1 + 4\,sin^2\theta.cos^2\theta - 4\sin\theta\cos\theta) + 6 + 12\sin\theta\cos\theta$
$+4 - 12\,sin^2\theta\,cos^2\theta$

$= 3 + 12\,sin^2\theta\,cos^2\theta - 12\sin\theta\cos\theta + 6 + 12\sin\theta\cos\theta$
$+4 - 12\,sin^2\theta\,cos^2\theta$

$= 13$

Step2:

Therefore,

$$3(sin\theta - cos\theta)^4 + 6(sin\theta + cos\theta)^2 + 4(sin^6\theta + cos^6\theta) = 13$$

P3: $(\boldsymbol{sin\theta + cosec\theta})^2 + (\boldsymbol{cos\theta + sec\theta})^2 - (\boldsymbol{tan^2\theta + cot^2\theta}) =$

A. 5
B. 7
C. 9
D. 11

Answer: B

Solution:

$$(sin\theta + cosec\theta)^2 + (cos\theta + sec\theta)^2 - (tan^2\theta + cot^2\theta)$$

$= sin^2\theta + cosec^2\theta + 2\,sin\theta.cosec\theta + cos^2\theta + sec^2\theta$
$+2\,cos\theta.sec\theta - tan^2\theta - cot^2\theta$

$= sin^2\theta + cosec^2\theta + 2\,sin\theta.\frac{1}{sin\theta} + cos^2\theta + sec^2\theta$
$+\,2\,cos\theta.\frac{1}{cos\theta} - tan^2\theta - cot^2\theta$

$= (sin^2\theta + cos^2\theta) + (cosec^2\theta - cot^2\theta) + 2$
$+(sec^2\theta - tan^2\theta) + 2$

$= 1 + 1 + 2 + 1 + 2$
$= 7$

IP4: If $\cos\boldsymbol{\theta} > 0$, $\tan\boldsymbol{\theta} + \sin\boldsymbol{\theta} = \boldsymbol{m}$ and $\tan\boldsymbol{\theta} - \sin\boldsymbol{\theta} = \boldsymbol{n}$, then show that

$$\boldsymbol{m^2 - n^2 = 4\sqrt{mn}.}$$

Solution:

Given, $m = \tan\theta + \sin\theta$ and $n = \tan\theta - \sin\theta$

Step 1:

$m + n = 2\tan\theta$, $m - n = 2\sin\theta$ and $mn = \tan^2\theta - \sin^2\theta$

Step 2:

$$
\begin{aligned}
m^2 - n^2 &= (m+n)(m-n) \\
&= (2\tan\theta).(2\sin\theta) \\
&= 4\tan\theta\sin\theta \\
&= 4\sqrt{\tan^2\theta\,.\sin^2\theta} \\
&= 4\sqrt{\tan^2\theta\,(1-\cos^2\theta)} \qquad [since, \cos\theta > 0] \\
&= 4\sqrt{\tan^2\theta - \sin^2\theta} \\
&4\sqrt{mn}
\end{aligned}
$$

Step 3: Therefore , $m^2 - n^2 = 4\sqrt{mn}$

P4: If $\boldsymbol{acos^3\theta + 3a\,cos\theta sin^2\theta = m}$ and $\boldsymbol{asin^3\theta + 3acos^2\theta sin\theta = n}$, then

$$\boldsymbol{(m+n)^{\frac{2}{3}} + (m-n)^{\frac{2}{3}} =}$$

A. $2\sqrt{a^3}$

B. $3\sqrt[3]{a^2}$

C. $3\sqrt{a^3}$

D. $2\sqrt[3]{a^2}$

Answer: D

Solution: We have, $acos^3\theta + 3a\,cos\theta sin^2\theta = m$

and $asin^3\theta + 3acos^2\theta sin\theta = n$

$$
\begin{aligned}
m + n &= acos^3\theta + 3a\,cos\theta sin^2\theta + asin^3\theta + 3acos^2\theta sin\theta \\
&= a(sin^3\theta + cos^3\theta + 3\sin\theta\,cos\,\theta(sin\,\theta + cos\,\theta)
\end{aligned}
$$

$\therefore m + n = a(\cos\theta + \sin\theta)^3$

And

$$
\begin{aligned}
m - n &= acos^3\theta + 3a\,cos\theta sin^2\theta - asin^3\theta - 3acos^2\theta sin\theta \\
&= a(cos^3\theta - sin^3\theta - 3\,cos\,\theta\,sin\,\theta(cos\,\theta - sin\,\theta)
\end{aligned}
$$

$$\therefore m - n = a(\cos\theta - \sin\theta)^3$$

$\therefore\ (m+n)^{\frac{2}{3}} + (m-n)^{\frac{2}{3}}$

$= [a(\cos\theta + \sin\theta)^3]^{\frac{2}{3}} + [a(\cos\theta - \sin\theta)^3]^{\frac{2}{3}}$

$= a^{\frac{2}{3}}(\cos\theta + \sin\theta)^2 + a^{\frac{2}{3}}(\cos\theta - \sin\theta)^2$

$= a^{\frac{2}{3}}(1 + 2\cos\theta\sin\theta + 1 - 2\cos\theta\sin\theta)$

$= 2a^{\frac{2}{3}}$

$= 2\sqrt[3]{a^2}$

Exercises:

1. In what quadrant θ will terminate if
   a. $\sin\theta$ and $\cos\theta$ are both negative?
   b. $\sin\theta$ and $\tan\theta$ are both positive?
   c. $\sin\theta$ is positive and $\sec\theta$ is negative?
   d. $\sec\theta$ is negative and $\tan\theta$ is negative?
2. In what quadrants θ will terminate if
   a. $\sin\theta$ is positive?
   b. $\cos\theta$ is negative?
   c. $\tan\theta$ is negative?
   d. $\sec\theta$ is positive?
3. Find the values of $\sin\theta$ and $\tan\theta$, given $\cos\theta = \dfrac{5}{6}$.
4. Find the values of $\sin\theta$ and $\cos\theta$, given $\tan\theta = -\dfrac{3}{4}$.
5. Find $\sin\theta$, given $\cos\theta = -\dfrac{4}{5}$ and that $\tan\theta$ is positive .
6. Find the values of the remaining functions of θ, given

   $$\sin\theta = \sqrt{3}/2 \text{ and } \cos\theta = -1/2$$
7. Prove the identities
   a. $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$
   b. $\dfrac{\sin x}{1+\cos x} + \dfrac{1+\cos x}{\sin x} = 2\csc x$
   c. $\dfrac{1-\sin x}{\cos x} = \dfrac{\cos x}{1+\sin x}$
   d. $\dfrac{\tan x - \sin x}{\sin^3 x} = \dfrac{\sec x}{1+\cos x}$
8. Evaluate the trigonometric functions of $\theta = 150°$.
9. Evaluate the trigonometric functions of $\theta = 180°$
10. Evaluate the trigonometric functions of $\theta = 270°$
11. Evaluate the trigonometric functions of $\theta = \frac{5\pi}{6}$.
12. Evaluate the trigonometric functions of $\theta = -\frac{\pi}{2}$.
13. Evaluate the trigonometric functions of $\theta = \frac{29\pi}{6}$.

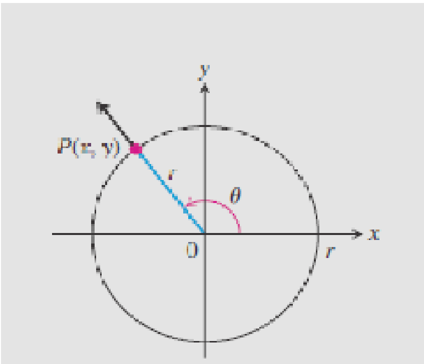
14. In a right angled triangle ΔABC, the right angle is at B and θ is an acute angle at C. If opposite side of the angle is 5cm and hypotenuse is $5\sqrt{5}cm$, then find all trigonometric values.
15. If the coordinates of the point on the circle which is plotted above with the radius r is $P(\sqrt{3}, 1)$, then find the angle that ray OP makes with the positive x-axis.
16. If the terminal side falls in the third quadrant and the point which has $\sqrt{2}$ units distance from the origin is $P(-1, -1)$, then find the angle.
17. In Quadrant IV, $\sec\theta = \frac{13}{12}$, then find all other trigonometric values.

4.4. Graphs of Trigonometric Functions

Learning Objectives:

- To study the variations of trigonometric functions and to view their graphs
- To understand the concept of periodicity and even and odd properties of trigonometric functions through their graphs

Variations of Trigonometric Functions

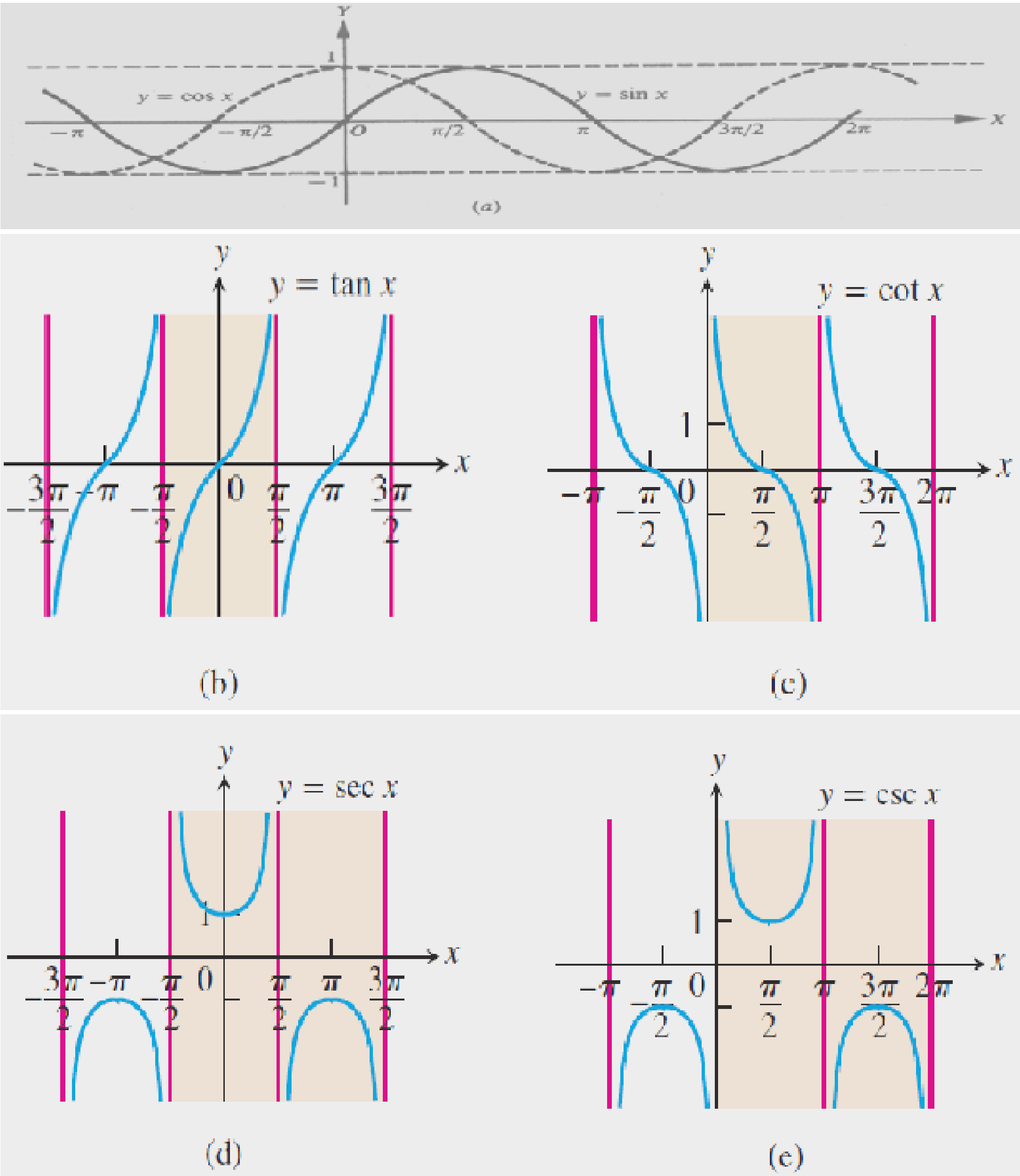


As the point P moves counter clockwise along the circle, the angle θ varies continuously from 0° to 360°. Correspondingly, the trigonometric functions vary as shown below.

θ	0° to 90°	90° to 180°	180° to 270°	270° to 360°
$\sin\theta$	Increases from 0 to 1	Decreases from 1 to 0	Decreases from 0 to -1	Increases from -1 to 0
$\cos\theta$	Decreases from 1 to 0	Decreases from 0 to –1	Increases from –1 to 0	Increases from 0 to 1
$\tan\theta$	0 to $+\infty$	$-\infty$ to 0	0 to $+\infty$	$-\infty$ to 0
$\csc\theta$	$+\infty$ to 1	1 to $+\infty$	$-\infty$ to -1	-1 to $-\infty$
$\sec\theta$	1 to $+\infty$	$-\infty$ to -1	-1 to $-\infty$	$+\infty$ to 1
$\cot\theta$	$+\infty$ to 0	0 to $-\infty$	$+\infty$ to 0	0 to $-\infty$

Graphs:

The values of the trigonometric functions corresponding to the several values of the angle x are given below. The graphs of trigonometric functions are shown below. When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by x instead of θ.



Periodicity

When an angle of measure x and an angle of measure $x + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric values. For example, $\cos(x+2\pi) = \cos x$.

Trigonometric functions whose values repeat at regular intervals are called periodic.

Periodic function and period:

Let $A \subseteq \boldsymbol{R}$. A function $f: A \longrightarrow \boldsymbol{R}$ is said to be a ***periodic function*** if there exists a positive number p such that $f(x+p) = f(x)$, for all $x \in A$. Then ***p is said to be a period of f.*** If p is the least positive real number such that

$f(x+p) = f(x)$, for all $x \in A$.Then p is called ***the period of f***.

If f is a periodic function with the period p. Then

i) The function cf is also a periodic function with the period p, where c is a real number.

ii) $f(x+np) = f(x), \forall\, x \in A$ and $n \in \boldsymbol{Z}$.

iii) The function $g: A \longrightarrow \boldsymbol{R}$ defined by $g(x) = f(ax+b) + c\,, x \in A$, where $a, b, c \in \boldsymbol{R}, a \neq 0$ is also a periodic function with the period $\frac{p}{|a|}$

Note:

Let $f: A \rightarrow \boldsymbol{R}$, $g: B \rightarrow \boldsymbol{R}$ be two periodic functions with the periods p and q respectively. Then the functions $(f \mp g)$, (fg) and $\left(\frac{f}{g}\right)$ are all periodic functions with *a period* $\boldsymbol{r} = l.c.m.\{p, q\}$.In these cases the period is either $\boldsymbol{r}$ or a sub multiple of $\boldsymbol{r}$.

As seen from their graphs, the tangent and cotangent functions have the period $p = \pi$. The other four functions have the period 2π.

$$\tan(x + \pi) = \tan x$$

$$\cot(x + \pi) = \cot x$$

$$\sin(x + 2\pi) = \sin x$$

$$\cos(x + 2\pi) = \cos x$$

$$\sec(x + 2\pi) = \sec x$$

$$\csc(x + 2\pi) = \csc x$$

The graphs of $y = \cos 2x$ *and* $y = \cos\left(\frac{x}{2}\right)$ plotted against the graph of $y = \cos x$ are shown below.

Multiplying x by a number greater than 1 speeds up a trigonometric function (increases the frequency) and shortens its period. Multiplying x by a positive number less than 1 slows a trigonometric function down and lengthens its period.

Even Versus Odd function:

The symmetries in the graphs of trigonometric functions reveal that the cosine and secant functions are even and the other four functions are odd.

Even functions	Odd functions
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

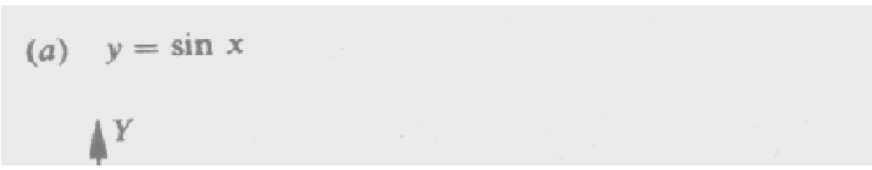
Shifting Graphs

The graph of a trigonometric function can be shifted vertically by adding a nonzero constant c to the function. If c is a positive number, then adding it to the right hand side of $y = f(x)$ results in the graph being shifted up by c units and subtracting it results in the graph being shifted down by c units.

The graph of a trigonometric function can be shifted horizontally by adding a nonzero constant d to the angle of the trigonometric function. For a positive number d, the graph of a trigonometric function is shifted left *d* units when *d* is added to the angle and shifted right *d* units when *d* is subtracted from the angle.

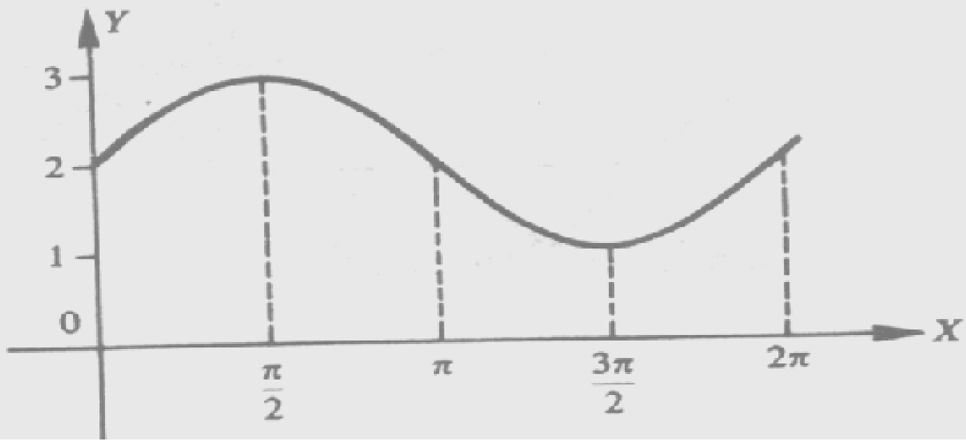
Example1: The graphs of

$y = \sin x, y = \sin x + 2, y = \sin x - 1, y = \sin x\left(x + \frac{\pi}{4}\right), y = \sin x\left(x - \frac{\pi}{3}\right)$ are

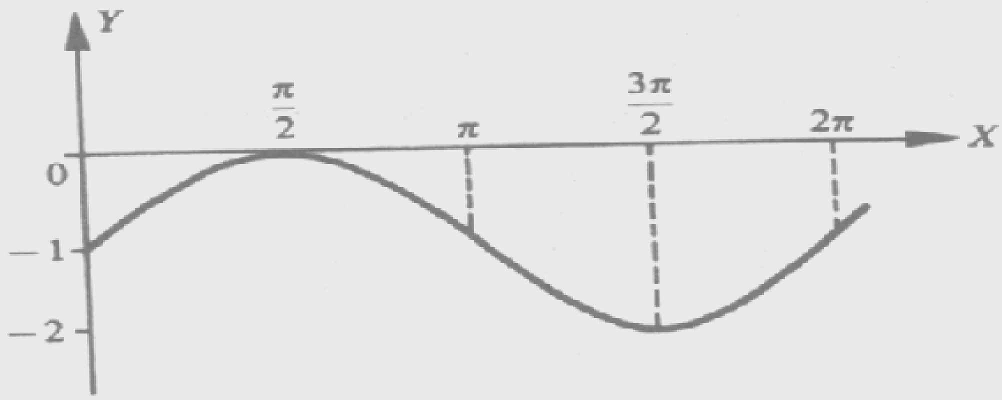


shown below.

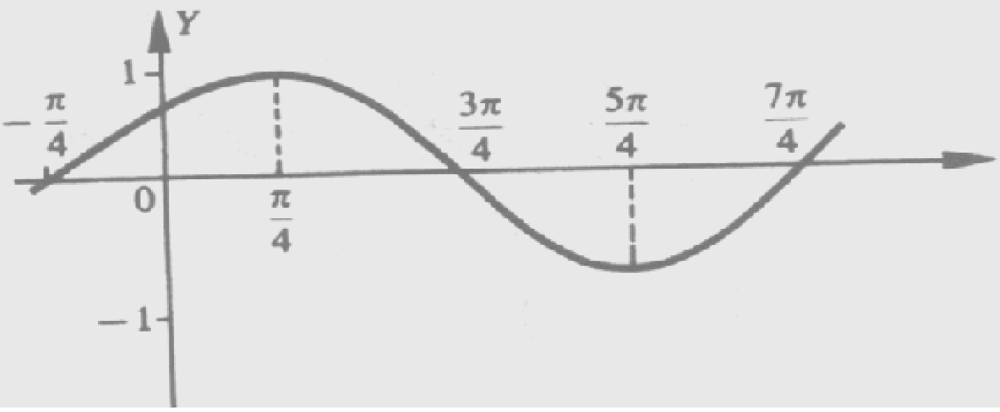
(b) $y = \sin x + 2$



(c) $y = \sin x - 1$



(d) $y = \sin (x + \pi/4)$



(e) $y = \sin (x - \pi/3)$

PROBLEM SET

IP1: Period of the function f defined by $f(x) = tan\left(\frac{3-5x}{2}\right)$ for all $x \in R$.

Solution:

We have, $f(x) = tan\left(\frac{3-5x}{2}\right)$

We know that the function $g(x) = tanx$ for all $x \in \boldsymbol{R}$, has the period π.

Let $f(x) = g(\frac{3-5x}{2})$. Therefore, the period of $f(x)$ is $\frac{\pi}{\left|\frac{-5}{2}\right|} = \frac{2\pi}{5}$

P1: What is the period of the function f defined by

$f(x) = \sin(5x+3)$ for all $x \in R$.

A. $\frac{\pi}{3}$

B. $\frac{2\pi}{3}$

C. $\frac{\pi}{5}$

D. $\frac{2\pi}{5}$

Answer: D

Solution: $f(x) = \sin(5x+3)$

We know that the function $g(x) = sinx$ for all $x \in \boldsymbol{R}$, has period2π.

Let $f(x) = g(5x+3)$. Therefore, the period of $f(x)$ is $\frac{2\pi}{5}$

IP2: Find the period of the function f defined by

$f(x) = \cos(x + 2^2x + 3^2x + \cdots + n^2x)$ for all $x \in R$.

Solution:

$f(x) = \cos(x + 2^2x + 3^2x + \cdots + n^2x)$

$\quad = \cos(\frac{n(n+1)(2n+1)}{6}x)$

We know that the function $g(x) = cosx$ for all $x \in \boldsymbol{R}$, has period 2π.

Then the period of $f(x)$ is $\frac{2\pi}{\frac{n(n+1)(2n+1)}{6}} = \frac{12\pi}{n(n+1)(2n+1)}$.

P2: What is the period of the function f defined by

$f(x) = \sin(x + 2x + 3x + \cdots + nx)$ for all $x \in R$.

E. 2π

F. π

G. $\frac{4\pi}{n(n+1)}$

H. $n(n+1)\pi$

Answer: C

Solution: $f(x) = \sin(x + 2x + 3x + \cdots + nx)$

$\quad = \sin(\frac{n(n+1)}{2}x)$

We know that the function $g(x) = sinx$ for all $x \in \boldsymbol{R}$, has period 2π.

Then the period of $f(x)$ is $\frac{2\pi}{\frac{n(n+1)}{2}} = \frac{4\pi}{n(n+1)}$

IP3: What is the period of $2sinx - 6cosx + 9$

Solution: Let $f(x) = 2sinx - 6cosx + 9$

Now $f(2\pi + x) = 2\sin(2\pi + x) - 6\cos(2\pi + x) + 9$

$$= 2sinx - 6cosx + 9 = f(x)$$

We know that period of $sinx$ is 2π and period of $cosx$ is 2π

The L.C.M. of their periods is 2π and it is the least positive value.

Hence the period of the given function is 2π

P3: What is the period of $2sinx - 3cosx + 4$

A. 2π

B. 4π

C. 0

D. π

Answer: A

Solution:

Let $f(x) = 2sinx - 3cosx + 4$

Now $f(2\pi + x) = 2\sin(2\pi + x) - 3\cos(2\pi + x) + 4$

$$= 2sinx - 3cosx + 4 = f(x)$$

We know that period of $sinx$ is 2π and period of $cosx$ is 2π

The $l.c.m.$ of their periods is 2π and it is the least positive value.

Hence the period of the given function is 2π

IP4: Find a tangent function whose period is 4

Solution:

Let the function be $tankx$ where k is to be determined.

The period of $tankx = \frac{\pi}{|k|} = 4$

$|k| = \frac{\pi}{4} \Longrightarrow k = \pm\frac{\pi}{4}$

Required function= $tan\left(\pm\frac{\pi}{4}\right)x$

P4: Find a sine function whose period is 3

A. $sin\left(\frac{2\pi}{3}\right)x$

B. $sin(6\pi)x$

C. $\pm sin\left(\frac{2\pi}{3}\right)x$

D. $\pm sin(6\pi)x$

Answer: C

Solution:

Let the function be $sinkx$ where k is to be determined.

The period of $sinkx = \frac{2\pi}{|k|} = 3$

$|k| = \frac{2\pi}{3} \implies k = \pm\frac{2\pi}{3}$

Required function= $sin\left(\pm\frac{2\pi}{3}\right)x$

Exercise:

1. Sketch the graphs of the following for one period

$$y = 4\sin x, y = \sin 3x,$$

$$y = 3\sin\frac{1}{2}x, y = 2\cos x,$$

$$y = 3\cos\frac{1}{2}x$$

2. Sketch the graph of each of the following.

$$y = \frac{1}{2}\tan x, y = 3\tan x, y = \tan 3x, y = \tan\frac{1}{4}x$$

3. Sketch the graph of each of the following

$$y = 3\sin x + 1, y = \sin x - 2, y = \cos x + 2, y = \frac{1}{2}\cos x - 1$$

4. Sketch a graph of each of the following

$$y = \sin\left(x - \frac{\pi}{6}\right), y = \sin\left(x + \frac{\pi}{6}\right)$$

$$y = \cos\left(x - \frac{\pi}{4}\right), y = \cos\left(x + \frac{\pi}{3}\right)$$

5. Find the period of 2 $sin2x$

6. Find the period of the $\tan(x + 4x + 9x + \cdots + n^2x)$

7. Find the period of $sin\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots to\ \infty\right) x$.

8. Find the period of $tan\ (x + 8x + 27x + .. + n^3x)$.

9. Find the cosine function whose period is 7.

10. Find the cosecant function whose period is 3.

4.5. Reductions to Functions of Positive Acute Angles

Learning objectives:

- The values of Trigonometrical ratios of the angles
  $(90° - \theta), (90° + \theta),\ (180° - \theta),\ (180° + \theta),$
  $(270° - \theta),\ (270° + \theta)\quad (360° - \theta),\ (360° + \theta).$

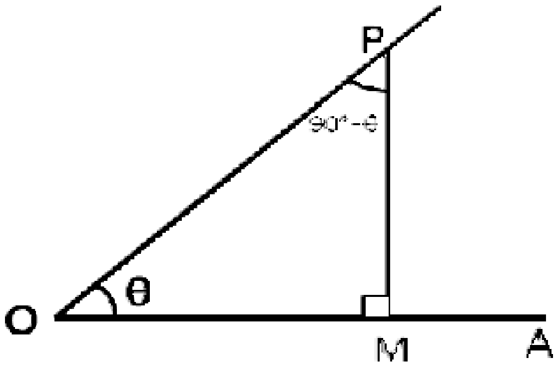
  And
- Co-terminal angles and Angles with a given trigonometric function value.

  And
- Solve the problems related to the above concepts.

Trigonometric Ratios of the Angle $(\mathbf{90°} - \boldsymbol{\theta})$

Two angles are said to be ***complementary*** when their sum is equal to a right angle. Thus any angle θ and the angle $(90^o - \theta)$ are complementary.

Let the revolving line, starting from OA, trace out an acute angle AOP, equal to θ. From any point P on it draw PM perpendicular to OA. Clearly, the angles MOP and OPM are complementary and angle OPM $= 90^o - \theta$.



$$\sin\left(90^\circ - \theta\right) = \frac{OM}{OP} = \cos\theta, \qquad \cos\left(90^\circ - \theta\right) = \frac{MP}{OP} = \sin\theta$$

$$\tan\left(90^\circ - \theta\right) = \frac{OM}{MP} = \cot\theta, \qquad \cot\left(90^\circ - \theta\right) = \frac{PM}{OM} = \tan\theta$$

$$\csc\left(90^\circ - \theta\right) = \frac{OP}{OM} = \sec\theta, \qquad \sec\left(90^\circ - \theta\right) = \frac{OP}{PM} = \csc\theta$$

Hence we observe that

- The Sine of any angle is equal to the Cosine of its complement,
- The Tangent of any angle is equal to the Cotangent of its complement, and
- The Secant of any angle is equal to the Cosecant of its complement.

This shows how the names **co**sine, **co**tangent, and **co**secant are derived from sine, tangent and secant.

Example1: Evaluate $\frac{\sin 23^\circ}{\cos 67^\circ}, \frac{\csc 29^\circ}{\sec 61^\circ}$

Solution:

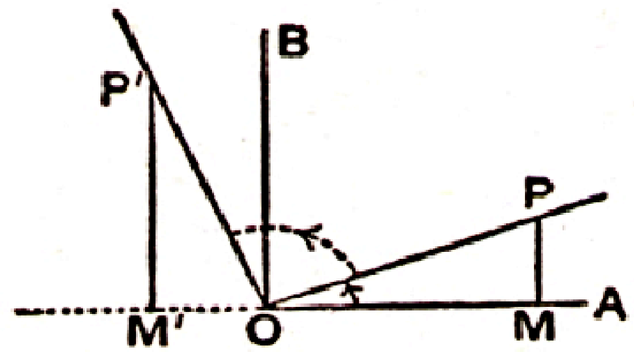
$$\frac{\sin 23^\circ}{\cos 67^\circ} = \frac{\sin(90-67)^\circ}{\cos 67^\circ} = \frac{\cos 67^\circ}{\cos 67^\circ} = 1$$

$$\frac{\csc 29^\circ}{\sec 61^\circ} = \frac{\csc(90-61)^\circ}{\sec 61^\circ} = \frac{\sec 61^\circ}{\sec 61^\circ} = 1$$

Trigonometric Ratios of the Angle (90° + θ)

The line OA, rotating through an angle θ, assumes the position OP

If it rotates further through an angle of 90°, it will take the position OP'



The triangles OMP and $OM'P'$ are equal.

The angle $M'OP'$= 90°– θ; $P'M' = OM$ and $OM' = PM$ therefore, if the coordinates of P are (x, y), then the coordinates of P' are $(-y, x)$. Let OP be r.

$$\sin(90^\circ + \theta) = \frac{x}{r} = \cos\theta, \quad \csc(90^\circ + \theta) = \sec\theta$$

$$\cos(90^\circ + \theta) = \frac{-y}{r} = -\sin\theta, \quad \sec(90^\circ + \theta) = -\csc\theta$$

$$\tan(90^\circ + \theta) = \frac{x}{-y} = -\cot\theta, \quad \cot(90^\circ + \theta) = -\tan\theta$$

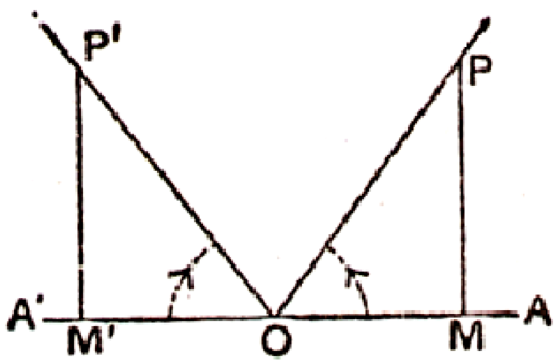
$$\sin 150^\circ = \sin(90^\circ + 60^\circ) = \cos 60^\circ = \frac{1}{2}$$

Example 2: $$\cos 135^\circ = \cos(90^\circ + 45^\circ) = -\sin 45^\circ = -\frac{1}{\sqrt{2}}$$

$$\tan 120^\circ = \tan(90^\circ + 30^\circ) = -\cot 30^\circ = -\sqrt{3}$$

Trigonometric Ratios of the Angle (180° $-\ \theta$)

Two angles are said to be **supplementary** when their sum is equal to two right angles. Thus, the supplementary angle of θ is $180° - \theta$.



Let the revolving line start from OA and describe angle AOP, θ. Also, let the revolving line assume the position OP' so that $A'OP'$ is θ. The angle AOP' is 180° − θ. Let $OP = OP' = r$

The angles MOP and $M'OP'$ are equal, and hence triangles MOP and $M'OP'$ are congruent. Hence $OM = OM'$ and also $MP = M'P'$. So, if the coordinates of P are (x, y), then the coordinates of P' are $(-x, y)$. Hence, we have

$$\sin(180^\circ - \theta) = \frac{y}{r} = \sin\theta \qquad \csc(180^\circ - \theta) = \csc\theta$$

$$\cos(180^\circ - \theta) = \frac{-x}{r} = -\cos\theta \qquad \sec(180^\circ - \theta) = -\sec\theta$$

$$\tan((180^\circ - \theta) = \frac{y}{-x} = -\tan\theta \qquad \cot(180^\circ - \theta) = -\cot\theta$$

Example 3:

$$\sin 120^\circ = \sin(180^\circ - 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\cos 135^\circ = \cos(180^\circ - 45^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}}$$

$$\tan 150^\circ = \tan(180^\circ - 30^\circ) = -\tan 30^\circ = -\frac{1}{\sqrt{3}}$$

Trigonometric Ratios of the Angle $(180° + \theta)$

These relations can be easily derived from the previous results. For example,

$$\sin(180^\circ + \theta) = \sin(90^\circ + 90^\circ + \theta)$$
$$= \cos(90^\circ + \theta) = -\sin\theta$$

Similarly, we can derive the other relations. Thus,

$$\sin(180^\circ + \theta) = -\sin\theta \qquad \csc(180^\circ + \theta) = -\csc\theta$$
$$\cos(180^\circ + \theta) = -\cos\theta \qquad \sec(180^\circ + \theta) = -\sec\theta$$
$$\tan(180^\circ + \theta) = \tan\theta \qquad \cot(180^\circ + \theta) = \cot\theta$$

Trigonometric Ratios of Angles $(270° - \theta)$

These relations can be easily derived from the previous results. For example,

$$\sin(270^\circ - \theta) = \sin(180^\circ + 90^\circ - \theta)$$
$$= -\sin(90^\circ - \theta) = -\cos\theta$$

Similarly, we can derive the other relations. Thus,

$$\sin(270^\circ - \theta) = -\cos\theta \qquad \csc(270^\circ - \theta) = -\sec\theta$$
$$\cos(270^\circ - \theta) = -\sin\theta \qquad \sec(270^\circ - \theta) = -\csc\theta$$
$$\tan(270^\circ - \theta) = \cot\theta \qquad \cot(270^\circ - \theta) = \tan\theta$$

Trigonometric Ratios of Angles $(270° + \theta)$

These relations can be easily derived from the previous results. For example,

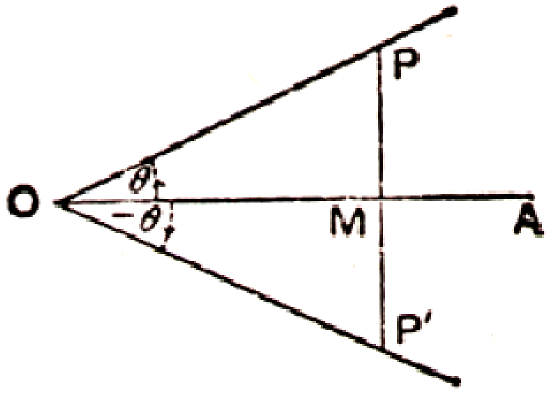
$$\sin(270^\circ + \theta) = \sin(180^\circ + 90^\circ + \theta)$$
$$= -\sin(90^\circ + \theta) = -\cos\theta$$

Similarly, we can derive the other relations. Thus,

$$\sin(270^\circ + \theta) = -\cos\theta \qquad \csc(270^\circ + \theta) = -\sec\theta$$
$$\cos(270^\circ + \theta) = \sin\theta \qquad \sec(270^\circ + \theta) = \csc\theta$$
$$\tan(270^\circ + \theta) = -\cot\theta \qquad \cot(270^\circ + \theta) = -\tan\theta$$

Trigonometrical Ratios of an angle $(-\theta)$

Let the line OP rotate through an angle θ. The coordinates of P are (x, y)



If the line OP revolves through an angle $-\theta$, it will take the position OP' and the coordinates of P' are $(x, -y)$.

Let OP be r then

$$\sin(-\theta) = \frac{-y}{r} = -\sin\theta, \quad \csc(-\theta) = -\csc\theta$$
$$\cos(-\theta) = \frac{x}{r} = \cos\theta, \quad \sec(-\theta) = \sec\theta$$
$$\tan(-\theta) = \frac{-y}{x} = -\tan\theta, \quad \cot(-\theta) = -\cot\theta$$

The angle $(-\theta)$ is the same as $360° - \theta$. Therefore, the above expressions are valid for the trigonometric ratios of an angle $360° - \theta$.

Example 4:

$$\sin(-30^\circ) = -\sin(30^\circ) = -\frac{1}{2}$$
$$\tan(-60^\circ) = -\tan(60^\circ) = -\sqrt{3}$$
$$\cos(-45^\circ) = \cos(45^\circ) = \frac{1}{\sqrt{2}}$$

Co-terminal Angles

The angle θ is the same as the angle $360° + \theta$. Therefore $\sin(360^\circ + \theta)$ is the same as the $\sin\theta$, and similarly the other ratios are. Let θ be any angle and let *n* be any integer. We have,

$$\sin(\theta + n\cdot 360^\circ) = \sin\theta \qquad \csc(\theta + n\cdot 360^\circ) = \csc\theta$$

$$\cos(\theta + n\cdot 360^\circ) = \cos\theta \qquad \sec(\theta + n\cdot 360^\circ) = \sec\theta$$

$$\tan(\theta + n\cdot 360^\circ) = \tan\theta \qquad \cot(\theta + n\cdot 360^\circ) = \cot\theta$$

The angles θ and $\theta + n.360°$ are co-terminal angles.

Example 5:

$$\sin 400^\circ = \sin(40^\circ + 360^\circ) = \sin 40^\circ$$

$$\cos 850^\circ = \sin(130^\circ + 2\cdot 360^\circ) = \cos 130^\circ$$

$$\tan(-1000)^\circ = \tan(80^\circ - 3\cdot 360^\circ) = \tan 80^\circ$$

If x is an angle in radian measure and n is any integer, then

$$\sin(x + 2n\pi) = \sin x \qquad \csc(x + 2n\pi) = \csc x$$

$$\cos(x + 2n\pi) = \cos x \qquad \sec(x + 2n\pi) = \sec x$$

$$\tan(x + 2n\pi) = \sin x \qquad \cot(x + 2n\pi) = \cot x$$

The trigonometrical ratios of any angle can be reduced to the determination of the trigonometrical ratios of an acute angle.

First, multiples of 360° are subtracted until the angle lies between 0° and 360°; if it be then greater than 180°, it is reduced by 180°.

Example 6: Reduce $\sin 1765^\circ$

Solution:

$$1765 = 4\times 360 + 325; \qquad 325 = 180 + 145; \qquad 145 = 180 - 35$$

$$\sin 1765^\circ = -\sin 145^\circ = -\sin(180^\circ - 35^\circ) = -\sin 35^\circ$$

Angles with a Given Function Value

Since co-terminal angles have the same value for a function, there are an unlimited number of angles that have the same value for a trigonometric function. Even when we restrict the angles to the interval of 0° to 360°, there are usually two angles that have the same function value. The quadrants for the angle are determined by the sign of the function value.

There are numerous situations in which it is necessary to find an unknown angle from a known value of one of its trigonometric functions. The following example illustrates a method for doing this.

Example 7: Find θ if $\sin\theta = \frac{1}{2}$

Solution: We begin by looking for positive angles that satisfy the equation. Because $\sin\theta$ is positive, the angle θ must terminate in the first or second quadrant.

If it terminates in the first quadrant, it must be

$$\theta = 30^\circ = \frac{\pi}{6} \text{ radians.}$$

If it terminates in the second quadrant, we have

$$180^\circ - \theta = 30^\circ$$

$$\theta = 180^\circ - 30^\circ = 150^\circ = \frac{5\pi}{6}$$

Now that we have found these two solutions, all other solutions are obtained by adding or subtracting multiples of 360° (2π radians) to or from them. Thus the entire set of solutions is given by the formulas (*n* is an integer)

$$\theta = 30^\circ \pm n \cdot 360^\circ \quad , \quad \theta = 150^\circ \pm n \cdot 360^\circ$$

or in radian measure

$$\theta = \frac{\pi}{6} \pm n \cdot 2\pi \quad , \theta = \frac{5\pi}{6} \pm n \cdot 2\pi$$

PROBLEM SET

IP1: Simplify $\dfrac{\sin\left(-\frac{11\pi}{3}\right).\tan\left(\frac{35\pi}{6}\right).\sec\left(-\frac{7\pi}{3}\right)}{\cos\left(\frac{5\pi}{4}\right).\csc\left(\frac{7\pi}{4}\right).\cos\left(\frac{17\pi}{6}\right)}$

Solution:

Step 1: Given

$$\frac{\sin\left(-\frac{11\pi}{3}\right).\tan\left(\frac{35\pi}{6}\right).\sec\left(-\frac{7\pi}{3}\right)}{\cos\left(\frac{5\pi}{4}\right).\csc\left(\frac{7\pi}{4}\right).\cos\left(\frac{17\pi}{6}\right)}$$

Step 2:

$$\sin\left(\frac{-11\pi}{3}\right) = -\sin(11 \times 60°)$$
$$= -\sin(660^0)$$
$$= -\sin(2.360^0 - 60^0)$$
$$= -(-\sin 60^0) = \sin 60^0 = \frac{\sqrt{3}}{2}$$

$$\tan\left(\frac{35\pi}{6}\right) = \tan 1050^0$$
$$= \tan(3.360^0 - 30^0)$$
$$= \tan(-30^0) = -\tan 30^0 = -\frac{1}{\sqrt{3}}$$
$$\sec\left(\frac{-7\pi}{3}\right) = \sec(-420^0)$$
$$= \sec 420^0$$
$$= \sec(1.360^0 + 60^0) = \sec 60^0 = 2$$
$$\text{Cos}\left(\frac{5\pi}{4}\right) = \cos 225^0$$
$$= \cos(270^0 - 45^0) = -\sin 45^0 = -\frac{1}{\sqrt{2}}$$
$$\text{Csc}\left(\frac{7\pi}{4}\right) = \csc 315^0$$
$$= \csc(360^0 - 45^0) = -\csc 45^0 = -\sqrt{2}$$
$$\cos\left(\frac{17\pi}{6}\right) = \cos 510^0 = \cos(1.360^0 + 150^0)$$
$$= \cos(150^0)$$
$$= \cos(90^0 + 60^0)$$
$$= -\sin 60^0 = -\frac{\sqrt{3}}{2}$$

Step 3:

$$\frac{\sin\left(-\frac{11\pi}{3}\right).\tan\left(\frac{35\pi}{6}\right).\sec\left(-\frac{7\pi}{3}\right)}{\cos\left(\frac{5\pi}{4}\right).\csc\left(\frac{7\pi}{4}\right).\cos\left(\frac{17\pi}{6}\right)} = \frac{\left(\frac{\sqrt{3}}{2}\right).\left(\frac{-1}{\sqrt{3}}\right).(2)}{\left(\frac{-1}{\sqrt{2}}\right).(-\sqrt{2}).\left(-\frac{\sqrt{3}}{2}\right)} = \frac{2}{\sqrt{3}}$$

P1: $\dfrac{\boldsymbol{\sin(-\theta)\ \tan(90^0+\theta)\ \sin(180^0+\theta)\ \sec(270^0+\theta)}}{\boldsymbol{\sin(360^0-\theta).\cos(270^0-\theta)cosec(180^0-\theta)\cot(360^0-\theta)}} =$

A. 3

B. 2

C. 1

D. 0

Answer: **C**

Solution:

Given that

$$\frac{\sin(-\theta)\tan(90^0+\theta)\sin(180^0+\theta)\sec(270^0+\theta)}{\sin(360^0-\theta).\cos(270^0-\theta)cosec(180^0-\theta)\cot(360^0-\theta)}$$
$$= \frac{(-\sin\theta)\ (-\cot\theta)\ (-\sin\theta)\ (cosec\theta)}{(-\sin\theta)\ (-\sin\theta)\ (cosec\theta)\ (-\cot\theta)}$$
$$= 1$$

IP2: If $\boldsymbol{\tan 20^0 = p}$, prove that $\dfrac{\boldsymbol{\tan 610^0+\tan 700^0}}{\boldsymbol{\tan 560^0-\tan 470^0}} = \dfrac{\boldsymbol{1-p^2}}{\boldsymbol{1+p^2}}$

Solution:

Step 1: Given that $\tan 20^0 = p$

Step 2: Now, $\dfrac{\tan 610^0+\tan 700^0}{\tan 560^0-\tan 470^0}$

$$= \frac{\tan(360^0+250^\circ)+\tan(2.360^0-20^0)}{\tan(360^0+200^0)-\tan(360^0+110^0)}$$

$$= \frac{\tan(250^0)-tan20^0}{\tan(200^0)-tan110^0}$$

$$= \frac{\tan(270^0-20^0)-\tan 20^0}{\tan(180^0+20^0)-\tan(90^0+20^0)}$$

$$= \frac{\cot 20^0-\tan 20^0}{\tan 20^0+\tan 20^0}$$

$$= \frac{\frac{1}{p}-p}{p+\frac{1}{p}} = \frac{1-p^2}{1+p^2}$$

Hence $\frac{\tan 610^0+\tan 700^0}{\tan 560^0-\tan 470^0} = \frac{1-p^2}{1+p^2}$

P2: If $\tan 20^0 = \lambda, then\ \frac{\tan 160^0-\tan 110^0}{1+\tan 160^0.\tan 110^0}$

A. $\frac{1+\lambda^2}{\lambda^2}$

B. $\frac{1-\lambda^2}{2\lambda}$

C. $\frac{1+\lambda^2}{2\lambda}$

D. $\frac{2\lambda}{1-\lambda^2}$

Answer: **B**

Solution:

Given $\tan 20^0 = \lambda$

Now,

$$\frac{\tan 160^0-\tan 110^0}{1+\tan 160^0.\tan 110^0} = \frac{\tan(180^0-20^0)-\tan(90^0+20^0)}{1+\tan(180^0-20^0).\tan(90^0+20^0)}$$

$$= \frac{-\tan 20^0+\cot 20^0}{1+(-\tan 20^0)(-\cot 20^0)}$$

$$= \frac{-\lambda+\frac{1}{\lambda}}{1+1} = \frac{1-\lambda^2}{2\lambda}$$

$$\therefore \frac{\tan 160^0-\tan 110^0}{1+\tan 160^0.\tan 110^0} = \frac{1-\lambda^2}{2\lambda}$$

IP3: Prove that $cot\frac{\pi}{20} . \cot\frac{3\pi}{20} . \cot\frac{5\pi}{20} . \cot\frac{7\pi}{20} . \cot\frac{9\pi}{20} = 1$

Solution:

Step1: $cot\frac{\pi}{20} . \cot\frac{3\pi}{20} . \cot\frac{5\pi}{20} . \cot\frac{7\pi}{20} . \cot\frac{9\pi}{20}$

$$= \left(\cot\frac{\pi}{20} . \cot\frac{9\pi}{20}\right) . \left(\cot\frac{3\pi}{20} . \cot\frac{7\pi}{20}\right) . \cot\frac{5\pi}{20}$$

$$= \left[\cot\frac{\pi}{20} . cot\left(\frac{\pi}{2}-\frac{\pi}{20}\right)\right] . \left[\cot\frac{3\pi}{20} . cot\left(\frac{\pi}{2}-\frac{3\pi}{20}\right)\right] . \cot\frac{\pi}{4}$$

$$= \left(\cot\frac{\pi}{20} . \tan\frac{\pi}{20}\right) . \left(\cot\frac{3\pi}{20} . \tan\frac{3\pi}{20}\right) (1)$$

$$= (1)\ (1)\ (1) = 1$$

Step2:

$$cot\frac{\pi}{20} . \cot\frac{3\pi}{20} . \cot\frac{5\pi}{20} . \cot\frac{7\pi}{20} . \cot\frac{9\pi}{20} = 1$$

P3: $\boldsymbol{cot\frac{\pi}{16}\,.\cot\frac{2\pi}{16}\,.\cot\frac{3\pi}{16}\,.\,\ldots\,\cot\frac{7\pi}{16}} =$

A. 1

B. 2

C. 3

D. 4

Answer: **A**

Solution:

Given that

$$cot\frac{\pi}{16}\,.\cot\frac{2\pi}{16}\,.\cot\frac{3\pi}{16}\,.\,\ldots\,\cot\frac{7\pi}{16}$$

$$= \left(cot\frac{\pi}{16}\ .\ \cot\frac{7\pi}{16}\right).\left[cot\left(\frac{2\pi}{16}\right)\ .\ \cot\frac{6\pi}{16}\right]$$
$$.\left[cot\left(\frac{3\pi}{16}\right)\ .\ \cot\frac{5\pi}{16}\right].\left(cot\frac{4\pi}{16}\right)$$

$$= \left[\cot\frac{\pi}{16}\ .\,cot\left(\frac{\pi}{2}-\frac{\pi}{16}\right)\right] = \left[\cot\frac{2\pi}{16}\ .\,cot\left(\frac{\pi}{2}-\frac{2\pi}{16}\right)\right]$$
$$.\left[\cot\frac{3\pi}{16}\ .\,cot\left(\frac{\pi}{2}-\frac{3\pi}{16}\right)\right].\left(cot\frac{\pi}{4}\right)$$

$$= \left(cot\frac{\pi}{16}\ .\ \tan\frac{\pi}{16}\right)\left(cot\frac{2\pi}{16}\ .\ \cot\frac{2\pi}{16}\right)\left(cot\frac{2\pi}{16}\ .\ \tan\frac{2\pi}{16}\right).(1)$$

$$= (1).(1).(1) = 1$$

$$\therefore\ cot\frac{\pi}{16}\ .\cot\frac{2\pi}{16}\ .\ \cot\frac{3\pi}{16}\,.\ldots\,\cot\frac{7\pi}{16} = 1$$

IP4: If $\boldsymbol{sin\theta = \frac{\sqrt{3}}{2}}$ *and* $\boldsymbol{cos\theta = -\frac{1}{2}}$ where the angle $\boldsymbol{\theta}$ terminates in the 2^{nd} Quadrant then the entire solution set is

Solution:

Step1: Given that $sin\theta = \frac{\sqrt{3}}{2}, cos\theta = -\frac{1}{2}$

Step2: $sin\theta$ is +ve and $cos\theta$ is –ve $\Longrightarrow \theta \in 2^{nd}$ Quadrant

$$\therefore 90° < \theta < 180°$$

Step3:

We have $sin120^0 = \sin\,(90^0 + 30^0)$

$$= \cos\ 30^0 = \frac{\sqrt{3}}{2}$$

$$cos120^0 = cos\,(90^0 + 30^0)$$

$$= -\sin 30^0 = -\frac{1}{2}$$

$$\Longrightarrow\ \theta = 120^0 = \frac{2\pi}{3}$$

$\therefore$ The entire solution set is $120^0 \pm n.360° \Longrightarrow \frac{2\pi}{3} \pm 2n\pi\ , n \in \boldsymbol{Z}$

Step4: Hence the entire solution set is $\frac{2\pi}{3} \pm 2n\pi\ , n \in \boldsymbol{Z}$

P4: If $\boldsymbol{tan\theta = \frac{1}{\sqrt{3}}\ and\ sin\theta = -\frac{1}{2}}$ then entire solution set is

A. $\frac{7\pi}{3} \pm 2n\pi$

B. $\frac{3\pi}{7} \pm 2n\pi$

C. $\frac{7\pi}{6} \pm 2n.\pi$

D. $\frac{6\pi}{7} \pm 2n\pi$

Answer: **C**

Solution:

Given that $tan\theta = \frac{1}{\sqrt{3}}\ and\ sin\theta = -\frac{1}{2}$

We know that $tan\theta$ is $+ve$ and $sin\theta$ is $-ve$, only in 3rd Quadrant.

$\Longrightarrow\ \theta \in$ 3rd Quadrant

$\Longrightarrow\ 180^0 < \theta < 270^0$

We have, $tan210^0 = \tan\,(180^0 + 30^0)$

$= \tan\ 30^0 = \frac{1}{\sqrt{3}}$

$sin210^0 = sin\,(180^0 + 30^0)$

$= -\sin 30^0 = -\frac{1}{2}$

$\Longrightarrow\ \theta = 210^0$

$\therefore$ The entire solution set is

$$210^0\ \pm n.360° \Longrightarrow \frac{7\pi}{6} \pm 2n.\pi\,,\ n \in \boldsymbol{Z}$$

Exercise

1. Find θ if $\sin\theta = \dfrac{\sqrt{3}}{2}$
2. Find θ if $\cos\theta = \dfrac{1}{2}$
3. Find θ if $\tan\theta = \dfrac{1}{\sqrt{3}}$
4. Evaluate

    a. $\csc^2 57^\circ - \tan^2 33^\circ$

    b. $\left(\dfrac{\sin 46^\circ}{\cos 44^\circ}\right)^2 + \left(\dfrac{\cos 44^\circ}{\sin 46^\circ}\right)^2$

    c. $\dfrac{\cos^2 10^\circ + \cos^2 80^\circ}{\sin^2 49^\circ + \sin^2 41^\circ}$

d. $\frac{\cos 50^\circ}{\sin 40^\circ}+\frac{\cos 49^\circ}{\sin 41^\circ}-8\sin^2 30^\circ$

e. $2\frac{\tan 43^\circ}{\cot 47^\circ}-\frac{\cot 55^\circ}{\tan 35^\circ}$

f. $\sec 40^\circ \sin 50^\circ+\cos 50^\circ \csc 40^\circ$

5. Evaluate $\sin\theta\cos\theta-\frac{\sin\theta\cos(90-\theta)\cos\theta}{\sec(90-\theta)}-\frac{\cos\theta\sin(90-\theta)\sin\theta}{\csc(90-\theta)}$

6. Show that $\tan 7^\circ.\tan 23^\circ.\tan 60^\circ.\tan 67^\circ.\tan 83^\circ=\sqrt{3}$

7. Prove that

a. $\cos^2 35^\circ+\cos^2 55^\circ=1$

b. $\sin 53^\circ\cos 37^\circ+\cos 53^\circ\sin 37^\circ=1$

c. $\sin 50^\circ\sin 40^\circ-\cos 40^\circ\csc 50^\circ=0$

8. $2\frac{\tan 43^\circ}{\cot 47^\circ}-\frac{\cot 70^\circ}{\tan 20^\circ}=1$

9.

a. $tan210° + cot240°$

b. $sin210° + cos240°$

c. $(sec300° + tan300°)(sec300° - tan300°)$

d. The value of $\frac{\sin(-\theta)+\cos(-\theta)}{cos\theta-sin\theta}+\frac{\tan(-\theta)+\cot(-\theta)}{cot\theta+tan\theta}$

e. $cot765° =$

10. If $tan\theta=\frac{1}{\sqrt{3}}$ and the angle θ terminates in third quadrant then the entire solution set is

11. Find the values of

a. $\tan(\theta-2520°)$

b. $\sin(540°-\theta)$

c. $\cos(\theta+1440°)$

d. $sec9720°-\theta)$

e. $cosec(1890°-\theta)$

f. $cot(900°+\theta)$

g. $\sin(-405°)$

h. $\cos\left(-\frac{7\pi}{2}\right)$

i. $\sec(2100°)$

j. $\cot(-315°)$

k. $\sin\left(\frac{20\pi}{3}\right)\cos\left(\frac{25\pi}{6}\right)$

l. $\tan\left(-\frac{2\pi}{3}\right).\cot\left(-\frac{3\pi}{4}\right)$

12. Evaluate the values of

a. $cos^2 45° + cos^2 135° + cos^2 225° + cos^2 315°$
b. $cos 5° + cos 24° + cos 175° + cos 204° + cos 300°$
c. $cos 225° - sin 225° + tan 495° - cot 495°$
d. $sin^2 \frac{2\pi}{3} + cos^2 \frac{5\pi}{6} - tan^2 \frac{3\pi}{4}$

13. Determine the value of

(i) $cos 1°. cos 2°. cos 3° \ldots\ldots\ldots\ldots\ldots.. cos 100°$
(ii) $tan 1°. tan 2°. tan 3° \ldots\ldots\ldots\ldots\ldots\ldots. tan 89°$
(iii) $log(tan 1°)\, log(tan 2°)\, log(tan 3°) \ldots\ldots log(tan 89°)$
(iv) $log(tan 17°) + log(tan 37°) + log(tan 53°) + log(tan 73°)$

14. If $\tan 20^0 = k, then\ \frac{\tan 250^0 - \tan 340^0}{\tan 200^0 - \tan 110^0} = \frac{1-k^2}{1+k^2}$

15. Prove the following

a. $\frac{cos(180° - A) cot(90° + A) cos(-A)}{tan(180° + A) tan(270° + A) sin(360° - A)} = cos A$
b. $\frac{cos(90° + \theta) sec(-\theta) tan(180° - \theta)}{sec(360° - \theta) sin(180° + \theta) cot(90° - \theta)} = -1$
c. $\frac{sin(3\pi - A) cos\left(A - \frac{\pi}{2}\right) tan\left(\frac{3\pi}{2} - A\right)}{cosec\left(\frac{13\pi}{2} + A\right) sec(3\pi + A) cot\left(A - \frac{\pi}{2}\right)} = cos^4 A$

5.1. Trigonometric Functions of Two Angles

Learning Objectives:

- To derive the angle sum and the angle difference formulae of trigonometric functions

  AND

- To solve problems related to above formulae

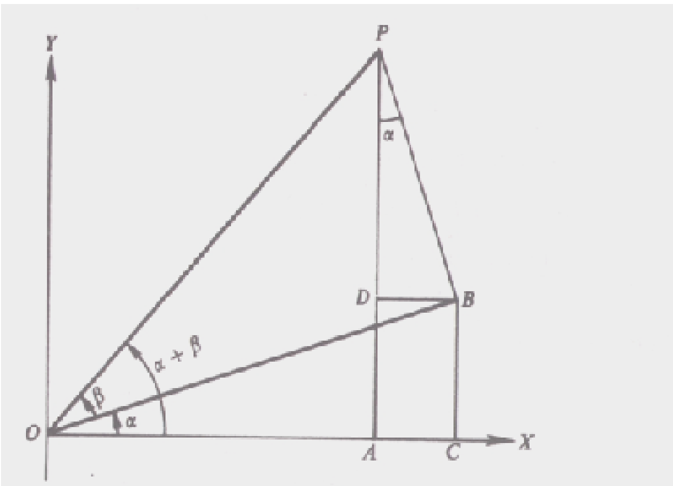
Angle Sum Formulas

We prove the following, the angle sum formulas.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$



Let OX and OB be the initial and terminal rays of the angle α in the standard position. Let OP be the terminal ray of the angle $\alpha+\beta$. Let PA and PB be perpendiculars on OX and OB respectively. Draw $BC \perp OX$ and $DB \parallel OX$.

Now, angle OBD $=\alpha$

The angle APB = $90^\circ - angle\ DBP = angle\ OBD = \alpha$

$$\sin(\alpha+\beta)=\frac{PA}{OP}=\frac{PD+DA}{OP}=\frac{PD+BC}{OP}=\frac{PD}{OP}+\frac{BC}{OP}$$

$$=\frac{CB}{OB}\cdot\frac{OB}{OP}+\frac{PD}{PB}\cdot\frac{PB}{OP}$$

$$=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\cos(\alpha+\beta)=\frac{OA}{OP}=\frac{OC-AC}{OP}=\frac{OC-DB}{OP}=\frac{OC}{OP}-\frac{DB}{OP}$$
$$=\frac{OC}{OB}\cdot\frac{OB}{OP}-\frac{DB}{PB}\cdot\frac{PB}{OP}$$
$$=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$
$$=\frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$
$$=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

Angle Difference Formulas

The angle difference formulas are obtained by replacing β by $-\beta$ in the angle sum formulas.

$$\sin(\alpha-\beta)=\sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)$$
$$=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\cos(\alpha-\beta)=\cos\alpha\cos(-\beta)-\sin\alpha\sin(-\beta)$$
$$=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$
$$\tan(\alpha-\beta)=\frac{\tan\alpha+\tan(-\beta)}{1-\tan\alpha\tan(-\beta)}$$
$$=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Some Useful Identities

The following identities are useful in solving the problems later. They can be easily proved by using the angle sum and difference formulas. (Prove!)

$$\sin(\alpha+\beta)\sin(\alpha-\beta)=\sin^2\alpha-\sin^2\beta=\cos^2\beta-\cos^2\alpha$$
$$\cos(\alpha+\beta)\cos(\alpha-\beta)=\cos^2\alpha-\sin^2\beta=\cos^2\beta-\sin^2\alpha$$
$$\cot(\alpha+\beta)=\frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$$
$$\cot(\alpha-\beta)=-\frac{\cot\alpha\cot\beta+1}{\cot\alpha-\cot\beta}$$

$$\sin(\alpha+\beta+\gamma)=\sum\sin\alpha\cos\beta\cos\gamma-\sin\alpha\sin\beta\sin\gamma$$

$$\cos(\alpha+\beta+\gamma)=\cos\alpha\cos\beta\cos\gamma-\sum\sin\alpha\sin\beta\cos\gamma$$

$$\tan(\alpha+\beta+\gamma)=\frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta-\tan\beta\tan\gamma-\tan\gamma\tan\alpha}$$

$$\tan(\frac{\pi}{4}+\alpha)=\frac{1+\tan\alpha}{1-\tan\alpha}=\frac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}$$

$$\tan(\frac{\pi}{4}-\alpha)=\frac{1-\tan\alpha}{1+\tan\alpha}=\frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}$$

Example 1: Find the values of sine, cosine, and tangent of 15°.

***Solution*:** $\sin 15^\circ=\sin(45^\circ-30^\circ)=\sin 45^\circ\cos 30^\circ-\cos 45^\circ\sin 30^\circ=\frac{\sqrt{6}-\sqrt{2}}{4}$

$$\therefore \boldsymbol{\sin 15^\circ=\frac{\sqrt{3}-1}{2\sqrt{2}}}$$

$$\cos 15^\circ=\cos(45^\circ-30^\circ)=\cos 45^\circ\cos 30^\circ+\sin 45^\circ\sin 30^\circ=\frac{\sqrt{6}+\sqrt{2}}{4}$$

$\therefore \boldsymbol{\cos 15^\circ=\frac{\sqrt{3}+1}{2\sqrt{2}}}$, $\tan 15^\circ=\tan(45^\circ-30^\circ)=\frac{\tan 45^\circ-\tan 30^\circ}{1+\tan 45^\circ\tan 30^\circ}=2-\sqrt{3}$

$$\therefore \boldsymbol{\tan 15^\circ=2-\sqrt{3}}$$

Example 2: Find the values of sine, cosine and tangent of 75^0.

Solution:

$$\sin 75^\circ=\sin(90^\circ-15^\circ)=\cos 15^\circ=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\cos 75^\circ=\cos(90^\circ-15^\circ)=\sin 15^\circ=\frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\tan 75^\circ=\tan(90^\circ-15^\circ)=\cot 15^\circ=\frac{1}{\tan 15^\circ}=\frac{1}{2-\sqrt{3}}=2+\sqrt{3}$$

Example 3: Rewrite $\sin 75^\circ\cos 28^\circ-\cos 75^\circ\sin 28^\circ$ **as a single function of an angle.**

***Solution*:**

$$\sin 75^\circ\cos 28^\circ-\cos 75^\circ\sin 28^\circ=\sin(75^\circ-28^\circ)=\sin 47^\circ$$

Example 4: Prove $\boldsymbol{\sin(45^o+\theta)-\sin(45^o-\theta)=\sqrt{2}\sin\theta}$

Solution:

$$\sin(45^\circ+\theta)-\sin(45^\circ-\theta)$$
$$=(\sin 45^\circ\cos\theta+\cos 45^\circ\sin\theta)-(\sin 45^\circ\cos\theta-\cos 45^\circ\sin\theta)$$
$$=2\cos 45^\circ\sin\theta=2\times\frac{1}{\sqrt{2}}\sin\theta=\sqrt{2}\sin\theta$$

PROBLEM SET:

IP1: Show that $\boldsymbol{\sin^2\left(52\frac{1}{2}\right)^o-\sin^2\left(22\frac{1}{2}\right)^o=\frac{\sqrt{3}+1}{4\sqrt{2}}}$

Solution:

Step1:

Let, L.H.S $=\sin^2\left(52\frac{1}{2}\right)^o-\sin^2\left(22\frac{1}{2}\right)^o$

$=\sin\left(\frac{105}{2}+\frac{45}{2}\right)^o\sin\left(\frac{105}{2}-\frac{45}{2}\right)^o$

$\because \sin^2 A-\sin^2 B=\sin(A+B)\sin(A-B)$

$=\sin 75^o\sin 30^o$

$=\sin(30+45)^o\sin 30^o$

$=(\sin 30^o\cos 45^o+\cos 30^o\sin 45^o)\sin 30^o$

$=\left(\frac{1}{2}.\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}}\right)\frac{1}{2}$

$=\frac{\sqrt{3}+1}{4\sqrt{2}}=$ R.H.S

Step2:

Therefore, $\sin^2\left(52\frac{1}{2}\right)^o-\sin^2\left(22\frac{1}{2}\right)^o=\frac{\sqrt{3}+1}{4\sqrt{2}}$

P1: Show that $\boldsymbol{\cos 42^0+\cos 78^0+\cos 162^0=0}$

A. $2\cos 18^0$

B. $\left(\sqrt{3}-1\right)\cos 18^0$

C. 0

D. 1

Answer: C

Solution:

Let, L.H.S $=\cos 42^0+\cos 78^0+\cos 162^0$

$=\cos(60^0-18^0)+\cos(60^0+18^0)+\cos(180^0-18^0)$

$=\cos 60^0\cos 18^0+\sin 60^0\sin 18^0+\cos 60^0\cos 18^0$

$-\sin 60^0\sin 18^0-\cos 18^0$

$=2\cos 60^0\cos 18^0-\cos 18^0$

$=2.\frac{1}{2}.\cos 18^0-\cos 18^0$

$=0=$ R.H.S

Therefore, $\cos 42^0 + \cos 78^0 + \cos 162^0 = 0$

IP2: If $\boldsymbol{A + B = \frac{\pi}{4}}$, prove that $\boldsymbol{(\cot A - 1)(\cot B - 1) = 2}$

Solution:

We have, $A + B = \frac{\pi}{4}$

Step1:

$$\Rightarrow \tan(A + B) = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$$

$$\Rightarrow \tan A + \tan B = 1 - \tan A \tan B$$

$$\Rightarrow \tan A + \tan B + \tan A \tan B = 1$$

Step2:

$$\Rightarrow \frac{\tan A + \tan B + \tan A \tan B}{\tan A \tan B} = \frac{1}{\tan A \tan B}$$

$$\Rightarrow \cot B + \cot A + 1 = \cot A \cot B$$

$$\Rightarrow \cot A \cot B - \cot A - \cot B = 1$$

$$\Rightarrow \cot A \cot B - \cot A - \cot B + 1 = 2$$

$$\Rightarrow \cot A(\cot B - 1) - 1(\cot B - 1) = 2$$

$$\Rightarrow (\cot B - 1)(\cot A - 1) = 2$$

Step3: Therefore, $(\cot B - 1)(\cot A - 1) = 2$

P2: If $\boldsymbol{A + B = 225^o}$, then $\boldsymbol{\frac{\cot A}{1 + \cot A} \cdot \frac{\cot B}{1 + \cot B} =}$

A. $-\frac{1}{2}$

B. $\frac{1}{2}$

C. -1

D. 1

Answer: B

Solution:

We have, $A + B = 225^o$

$$\Rightarrow \cot(A + B) = \cot 225^o$$

$$\Rightarrow \frac{\cot A \cot B - 1}{\cot A + \cot B} = \cot(180^o + 45^o)$$

$$\Rightarrow \frac{\cot A \cot B - 1}{\cot A + \cot B} = \cot 45^o$$

$$\Rightarrow \frac{\cot A \cot B - 1}{\cot A + \cot B} = 1$$

$$\Rightarrow \cot A \cot B = 1 + \cot A + \cot B \qquad \ldots.(1)$$

Now, $\frac{\cot A}{1 + \cot A} \cdot \frac{\cot B}{1 + \cot B} = \frac{\cot A \cot B}{(1 + \cot A + \cot B) + \cot A \cot B}$

$= \frac{\cot A \cot B}{\cot A \cot B + \cot A \cot B}$ [By (1)]

$$= \frac{\cot A \cot B}{2 \cot A \cot B} = \frac{1}{2}$$

Therefore, $\frac{\cot A}{1+\cot A} \cdot \frac{\cot B}{1+\cot B} = \frac{1}{2}$

IP3: Prove that $\frac{\tan(A+B)}{\cot(A-B)} = \frac{\sin^2 A - \sin^2 B}{\cos^2 A - \sin^2 B}$

(when the denominators on both sides are non-zero)

Solution:

Step1: Let, L.H.S $= \frac{\tan(A+B)}{\cot(A-B)}$

$$= \tan(A+B) . \tan(A-B)$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B} \cdot \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$= \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \ \tan^2 B}$$

$$= \frac{\frac{\sin^2 A}{\cos^2 A} - \frac{\sin^2 B}{\cos^2 B}}{1 - \frac{\sin^2 A}{\cos^2 A}\frac{\sin^2 B}{\cos^2 B}}$$

$$= \frac{\sin^2 A \cos^2 B - \cos^2 A \sin^2 B}{\cos^2 A \cos^2 B - \sin^2 A \sin^2 B}$$

$$= \frac{\sin^2 A(1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B}{\cos^2 A \cos^2 B - (1 - \cos^2 A)(\sin^2 B)}$$

$$= \frac{\sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 A \sin^2 B}{\cos^2 A \cos^2 B - \sin^2 B + \cos^2 A \sin^2 B}$$

$$= \frac{\sin^2 A - \sin^2 B}{\cos^2 A \cos^2 B - \sin^2 B + \cos^2 A(1 - \cos^2 B)}$$

$$= \frac{\sin^2 A - \sin^2 B}{\cos^2 A \cos^2 B - \sin^2 B + \cos^2 A - \cos^2 A \cos^2 B}$$

$$= \frac{\sin^2 A - \sin^2 B}{\cos^2 A - \sin^2 B} = \text{R.H.S}$$

Step2: Therefore, $\frac{\tan(A+B)}{\cot(A-B)} = \frac{\sin^2 A - \sin^2 B}{\cos^2 A - \sin^2 B}$

P3: If $\tan A - \tan B = x$ and $\cot B - \cot A = y$, then $\cot(A - B) =$

A. $\frac{1}{x} + \frac{1}{y}$
B. $\frac{1}{x} - \frac{1}{y}$
C. $-\frac{1}{x} + \frac{1}{y}$
D. $-\frac{1}{x} - \frac{1}{y}$

Answer: A

Solution: We have, $\tan A - \tan B = x$ and $\cot B - \cot A = y$

Now, $\cot B - \cot A = y$

$$\Rightarrow \frac{\tan A - \tan B}{\tan A \tan B} = y$$

$$\Rightarrow \frac{x}{\tan A \tan B} = y$$

$$\Rightarrow \tan A \tan B = \frac{x}{y}$$

$$\therefore \cot(A-B) = \frac{1}{\tan(A-B)} = \frac{1+\tan A \tan B}{\tan A - \tan B} = \frac{1+\frac{x}{y}}{x} = \frac{x+y}{xy} = \frac{1}{x} + \frac{1}{y}$$

IP4: If α *and* β are the solutions of $a\cos\theta + b\sin\theta = c$,

then show that $\cos(\alpha+\beta) = \frac{a^2-b^2}{a^2+b^2}$.

Solution: We have, $a\cos\theta + b\sin\theta = c$ ...(1)

Step1:

$\Rightarrow a\cos\theta = c - b\, s\,in\theta$

$\Rightarrow a^2\cos^2\theta = (c - b\sin\theta)^2$

$\Rightarrow a^2(1-\sin^2\theta) = c^2 - 2bc\sin\theta + b^2\sin^2\theta$

$\Rightarrow (a^2+b^2)\sin^2\theta - 2bc\sin\theta + (c^2-a^2) = 0$ ....(2)

Step2:

Since α , β are roots of equation (1).

Therefore, $\sin\alpha$ *and* $\sin\beta$ are roots of equation (2).

$\therefore \sin\alpha\sin\beta = \frac{c^2-a^2}{a^2+b^2}$ ...(3)

Step3:

Again, $a\cos\theta + b\sin\theta = c$

$\Rightarrow b\sin\theta = c - a\cos\theta$

$\Rightarrow b^2\sin^2\theta = (c - a\cos\theta)^2$

$\Rightarrow b^2(1-\cos^2\theta) = (c - a\cos\theta)^2$

$\Rightarrow (a^2+b^2)\cos^2\theta - 2ac\cos\theta + (c^2-b^2) = 0$ ....(4)

Step4:

It is given that α *and* β are the roots of equation (1).

So, $\cos\alpha$, $\cos\beta$ are the roots of equation (4).

$\therefore \cos\alpha\cos\beta = \frac{c^2-b^2}{a^2+b^2}$ ...(5)

Step5:

Now, $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\Rightarrow \cos(\alpha+\beta) = \frac{c^2-b^2}{a^2+b^2} - \frac{c^2-a^2}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2}$

Step6:

Therefore, $\cos(\alpha+\beta) = \frac{a^2-b^2}{a^2+b^2}$

P4: If α *and* β are the solutions of the equation

$$\boldsymbol{a\tan\theta + b\sec\theta = c}, \textbf{then } \boldsymbol{\tan(\alpha+\beta)} =$$

A. $\frac{2bc}{b^2-c^2}$
B. $\frac{2bc}{c^2-b^2}$
C. $\frac{2ac}{a^2-c^2}$
D. $\frac{2ac}{c^2-a^2}$

Answer: C

Solution:

We have, $a\tan\theta + b\sec\theta = c$ ... (1)

$\Longrightarrow c - a\tan\theta = b\sec\theta$

$\Longrightarrow (c - a\tan\theta)^2 = b^2\sec^2\theta$

$\Longrightarrow c^2 + a^2\tan^2\theta - 2ac\tan\theta = b^2(1+\tan^2\theta)$

$\Longrightarrow \tan^2\theta(a^2-b^2) - 2ac\tan\theta + (c^2-b^2) = 0$ ...(2)

It is given that α *and* β are the solutions of (1).

Therefore, $\tan\alpha$ and $\tan\beta$ are roots of equation (2).

$\therefore$ $\tan\alpha + \tan\beta = \frac{2ac}{a^2-b^2}$ and $\tan\alpha\tan\beta = \frac{c^2-b^2}{a^2-b^2}$

$\therefore \tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{\frac{2ac}{a^2-b^2}}{1-\frac{c^2-b^2}{a^2-b^2}} = \frac{2ac}{a^2-c^2}$

Exercises:

1. Find the values by using angle sum formulae or angle difference formulae
   a. $\sin 195^o$
   b. $\cos 165^o$
   c. $\tan 255^o$
   d. $\cot 345^o$
2. Prove that $\cos 340^o . \cos 40^o + \sin 200^o \sin 140^o = \frac{1}{2}$
3. Prove that $\cos(60+\theta) + \cos(60-\theta) = \cos\theta$
4. If $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$ and A, B are acute angles, find $A + B$.
5. If $\sin\alpha = -\frac{4}{5}$ and $\cos\beta = \frac{7}{25}$ and if none of α, β is in 4th quadrant, then find the values of
   a. $\sin(\alpha-\beta)$
   b. $\cos(\alpha+\beta)$
   c. $\tan(\alpha+\beta)$
6. If $0 < A < B < \frac{\pi}{4}$, $\sin(A+B) = \frac{24}{25}$, $\cos(A-B) = \frac{4}{5}$, then find the value of $\tan 2A$.
7. Express $a\cos\theta + b\sin\theta$ in the form

   $k\cos\varphi$ where $k = \sqrt{a^2+b^2}$, $\varphi = \theta - \alpha$

$k\sin\varphi$ where $k=\sqrt{a^2+b^2}$, $\varphi=\theta+\alpha$

8. Express $\sqrt{3}\sin\theta+\cos\theta$ as a sine of the angle
9. Find the value of sine, cosine, and tangent of $\frac{\pi}{12}$ radians.
10. Find the value of sine, cosine, and tangent of $\frac{5\pi}{12}$ radians.
11. If $\alpha+\beta+\gamma=180^0$, then prove that
    $\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma$
12. Find the values of sine, cosine and tangent of 105^0.
13. Rewrite $\cos 31^\circ\cos 48^\circ-\sin 31^\circ\sin 48^\circ$ as a single function of an angle.
14. Rewrite $\dfrac{\tan 37^\circ+\tan 68^\circ}{1-\tan 37^\circ\tan 68^\circ}$ as a single function of an angle.
15. Prove $\sin(30^\circ+\theta)+\cos(60^\circ+\theta)=\cos\theta$
16. Simplify
    a. $\dfrac{\tan(\alpha+\beta)-\tan\alpha}{1+\tan(\alpha+\beta)\tan\alpha}$
    b. $(\sin\alpha\cos\beta-\cos\alpha\sin\beta)^2+(\cos\alpha\cos\beta+\sin\alpha\sin\beta)^2$
17. Find $\sin(\alpha+\beta),\cos(\alpha+\beta),\sin(\alpha-\beta),\cos(\alpha-\beta)$ and determine the quadrants in which $(\alpha+\beta)$ and $(\alpha-\beta)$ terminate, given
    a. $sin\,\alpha=\dfrac{4}{5}, cos\,\beta=\dfrac{5}{13}$; α and β in quadrant I
    b. $sin\,\alpha=\dfrac{2}{3}, cos\,\beta=\dfrac{3}{4}$; α in quadrant II and β in quadrant IV

5.2. Trigonometric Functions of Multiple Angles

Learning Objectives

- To derive double - angle and triple - angle formulae
  And
- To solve problems related to them

Double-Angle Formulas

We propose to find the trigonometrical functions of an angle 2θ in terms of those of the angle θ .

In the angle sum formulas

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

we substitute θ for both α and β. This gives three more identities:

$$\cos 2\theta=\cos^2\theta-\sin^2\theta$$
$$\sin 2\theta=2\sin\theta\cos\theta$$
$$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$$

These identities are known as ***double-angle*** formulas.

Example 1: Write the expression $2\sin 75^\circ\cos 75^\circ$ as a single function of an angle.

Solution: $2\sin 75^\circ\cos 75^\circ=\sin(2\times 75^\circ)=\sin 150^\circ$

Additional Double-angle Formulas

Additional formulas come from combining the equations

$$\cos^2\theta+\sin^2\theta=1$$
$$\cos^2\theta-\sin^2\theta=\cos 2\theta$$

We add and subtract the equations in turn to get the following formulas:

$$\cos^2\theta=\frac{1+\cos 2\theta}{2}$$
$$\sin^2\theta=\frac{1-\cos 2\theta}{2}$$

These formulas can also be written as

$$\cos 2\theta=2\cos^2\theta-1$$
$$\cos 2\theta=1-2\sin^2\theta$$

Example 2: Write the expression $1-2\sin^2 37^\circ$ as a single function of an angle.

Solution: $1-2\sin^2 37^\circ=\cos(2\times 37^\circ)=\cos 74^\circ$

Some Useful Identities

The following identities are useful in solving the problems. They are easily derived using the above formulas.

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$$

Triple-Angle Formulas

We derive the formulas for sine, cosine, tangent and cotangent of the angle 3θ.

$$\begin{aligned}\sin 3\theta &= \sin(2\theta+\theta) = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta \\ &= 2\sin\theta\cos\theta\cos\theta + (1-2\sin^2\theta)\sin\theta \\ &= 2\sin\theta(1-\sin^2\theta) + (1-2\sin^2\theta)\sin\theta \\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

$\boldsymbol{sin3\theta = 3sin\theta - 4sin^3\theta}$

$$\begin{aligned}\cos 3\theta &= \cos(2\theta+\theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta \\ &= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta \\ &= (2\cos^2\theta - 1)\cos\theta - 2(1-\cos^2\theta)\cos\theta \\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

$$\boldsymbol{cos3\theta = 4cos^3\theta - 3cos\theta}$$

These two formulas can also be rearranged as

$$\sin^3\theta = \frac{1}{4}\left(3\sin\theta - \sin 3\theta\right)$$

$$\cos^3\theta = \frac{1}{4}\left(3\cos\theta + \cos 3\theta\right)$$

In the formula

$$\tan\left(\alpha+\beta+\gamma\right) = \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha}$$

we set $\alpha = \beta = \gamma = \theta$. We then obtain the identity

$\boldsymbol{tan3\theta = \frac{3tan\theta - tan^3\theta}{1-3tan^2\theta}}$ Similarly, $\boldsymbol{cot3\theta = \frac{cot^3\theta - 3cot\theta}{3cot^2\theta - 1}}$

By a process similar to that we used, we may express the trigonometric functions of any higher multiples of θ in terms of those of θ.

Example 3: Prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

$$\begin{aligned}
\cos 5\theta &= \cos(3\theta + 2\theta) \\
&= \cos 3\theta \cos 2\theta - \sin 3\theta \sin 2\theta \\
&= (4\cos^3\theta - 3\cos\theta)(2\cos^2\theta - 1) - (3\sin\theta - 4\sin^3\theta)\cdot 2\sin\theta\cos\theta \\
&= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) - 2\cos\theta\sin^2\theta(3 - 4\sin^2\theta) \\
&= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) - 2\cos\theta(1 - \cos^2\theta)(4\cos^2\theta - 1) \\
&= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) - 2\cos\theta(5\cos^2\theta - 4\cos^4\theta - 1) \\
&= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta
\end{aligned}$$

PROBLEM SET

IP1: Sin18$^\circ =$

Solution:

Step1: $Let\ A = 18^\circ \Rightarrow 5A = 90^\circ \Rightarrow 2A + 3A = 90^\circ$

$\Rightarrow 2A = 90^\circ - 3A$

Step2: $sin2A = \sin(90^\circ - 3A) = cos3A$

$\Rightarrow 2sinA\ cosA = 4cos^3A - 3cosA$

$\Rightarrow 4cos^3A - 2sinA\ cosA - 3\ cosA = 0$

$\Rightarrow cosA(4cos^2A - 2sinA - 3) = 0$

Step3:

$\Rightarrow Either\ cosA = 0\ or\ 4cos^2A - 2sinA - 3 = 0$

But $cos18^\circ \neq 0$

$Therefore,\ \ 4cos^2A - 2sinA - 3 = 0$

$\Rightarrow 4(1 - sin^2A) - 2sinA - 3 = 0$

$\Rightarrow 4 - 4sin^2A - 2sinA - 3 = 0$

$\Rightarrow 4sin^2A + 2sinA - 1 = 0$

Step4: $\therefore sinA = \frac{-2\pm\sqrt{4-4.4(-1)}}{2(4)} = \frac{-2\pm\sqrt{20}}{8} = \frac{-1\pm\sqrt{5}}{4}$

Since $sin18^\circ$ is positive, $sin18^\circ = \frac{\sqrt{5}-1}{4}$.

P1: $\boldsymbol{cos36^\circ} =$

A. $\frac{\sqrt{5}+1}{4}$

B. $\frac{\sqrt{5}-1}{4}$

C. $\frac{\sqrt{5}+1}{2}$

D. $\frac{\sqrt{5}-1}{2}$

Answer: A

Solution: We write

$cos36^{\circ} = 1 - 2sin^2 18^{\circ}$

We know that

$$Sin18^{\circ} = \frac{\sqrt{5}-1}{4}$$

$cos36^{\circ} = 1 - 2sin^2 18^{\circ} = 1 - 2\left(\frac{5-2\sqrt{5}+1}{16}\right) = 1 - \left(\frac{3-\sqrt{5}}{4}\right) = \frac{1+\sqrt{5}}{4}$

IP2: If $\boldsymbol{sin18^{\circ}} = \frac{\sqrt{5}-1}{4}$ then $\boldsymbol{Sin54^{\circ}} =$

Solution:

Step1: Given, $sin18^{\circ} = \frac{\sqrt{5}-1}{4}$

Step2: $Sin\ 3A\ = 3sinA - 4sin^3 A$

Step3: $sin54^{\circ} = Sin\ 3.18\ ^{\circ} = 3sin18\ ^{\circ} - 4sin^3 18\ ^{\circ}$

$$= 3\left(\frac{\sqrt{5}-1}{4}\right) - 4\left(\frac{\sqrt{5}-1}{4}\right)^3$$

After simplification we get

Step4: $sin54^{\circ} = \frac{\sqrt{5}+1}{4}$

P2: If $\boldsymbol{sin18^{\circ}} = \frac{\sqrt{5}-1}{4}$ then $\boldsymbol{cos72^{\circ}} =$

A. $\frac{\sqrt{5}+1}{4}$

B. $\frac{\sqrt{5}-1}{4}$

C. $\frac{\sqrt{5}+1}{2}$

D. $\frac{\sqrt{5}-1}{2}$

Answer: B

Solution: Given, $sin18^{\circ} = \frac{\sqrt{5}-1}{4}$

$$cos72^{\circ} = \cos\ (90^{\circ} - 18^{\circ})$$

$$= \sin\ 18^{\circ}$$

$$= \frac{\sqrt{5}-1}{4}$$

IP3: $cosA\ cos2A\ cos4A\ cos8A =$

A. $\frac{sin2^4 A}{2^4 sinA}$

B. $\frac{cos2^4A}{2^4cosA}$

C. $\frac{sin2^8A}{2^8sinA}$

D. $\frac{cos2^8A}{2^8cosA}$

Answer: A

Solution:

Step1: Given, $cosA\ cos2A\ cos4A\ cos8A =$

$$= \left\{\frac{1}{16sinA}\right\}[2sinA\ cosA](2\ cos2A)(2\ cos4A)(2cos8A)$$

Step2:

$$= \left\{\frac{1}{16sinA}\right\}[sin2A\ 2cos2A](2\ cos4A)(2cos8A)$$

$$= \left\{\frac{1}{16sinA}\right\}[sin4A\ 2cos4A](2cos8A)\ = \left\{\frac{1}{16sinA}\right\}sin8A(2cos8A)$$

Step3:

$$= \left\{\frac{1}{16sinA}\right\}sin16A$$

$$= \frac{1}{2^4\ \text{sinA}}.sin(2^4A)$$

P3: $\boldsymbol{cos20°cos40°cos60°cos80°} =$

A. 4

B. $\frac{1}{4}$

C. 16

D. $\frac{1}{16}$

Answer: D

Solution:

Given, $cos20°cos40°cos60°cos80° = \frac{1}{2}\ cos20^{\circ}\ cos40^{\circ}\ cos80^{\circ}\ \left(\because\ cos60^{\circ} = \frac{1}{2}\right)$

Since $cosA\ cos2A\ cos4A\ cos8A = \frac{sin2^4A}{2^4sinA}$

From this formula

$$cos20°cos40°cos60°cos80° = \frac{1}{2}\left[\frac{1}{2^3\sin 20°}\right]\sin(2^3 20°)$$

$$= \frac{1}{16\sin 20°}\sin 160°$$

$$= \frac{1}{16\sin 20°}\sin(180° - 20°)$$

$$= \left(\frac{1}{16}\right).\frac{(\sin 20°)}{(\sin 20°)} = \frac{1}{16}$$

IP4: $\boldsymbol{cotx + cot\left(x + \frac{\pi}{3}\right) + cot\left(x + \frac{2\pi}{3}\right) = 3}$ **then prove that** $\boldsymbol{cot3x = 1}$

Solution:

Step1: $cotx + cot\left(x + \frac{\pi}{3}\right) + cot\left(x + \frac{2\pi}{3}\right) = 3$

Step2: $cotx+\frac{cotx.\frac{1}{\sqrt{3}}}{cotx+\frac{1}{\sqrt{3}}}+\frac{cotx.\left(\frac{-1}{\sqrt{3}}\right)-1}{cotx-\frac{1}{\sqrt{3}}}=3$

Step3: $cotx+\frac{cotx-\sqrt{3}}{\sqrt{3}\,cotx+1}+\frac{-cotx-\sqrt{3}}{\sqrt{3}\,cotx-1}=3$

$$cotx+\frac{cotx-\sqrt{3}}{\sqrt{3}\,cotx+1}-\frac{cotx+\sqrt{3}}{\sqrt{3}\,cotx-1}=3$$

Step4: $\frac{cotx(3cot^2x-1)+(cotx-\sqrt{3})(\sqrt{3}\,cotx-1)-(cotx+\sqrt{3})(\sqrt{3}\,cotx+1)}{(3cot^2x-1)}=3$

Step5: $\frac{(3cot^3x-9cotx)}{3cot^2x-1}=3,\quad \frac{3(cot^3x-3cotx)}{3cot^2x-1}=3$

Step6: $3cot3x=3\Longrightarrow\ cot3x=1$

P4: If $\tan x+\tan(x+\frac{\pi}{3})+tan(x+\frac{2\pi}{3})=3$ then

(a) $\tan x=1$ $\qquad$ (b) $\tan 2x=1$

(c) $\tan 3x=1$ $\qquad$ (d) $3\tan 3x=1$

Answer. C

Solution:

The given equation can be written as

$$\tan x+\frac{\tan x+\tan\left(\frac{\pi}{3}\right)}{1-\tan x\tan\left(\frac{\pi}{3}\right)}+\frac{\tan x+\tan\left(\frac{2\pi}{3}\right)}{1-\tan x\tan\left(2\frac{\pi}{3}\right)}=3$$

$$\Longrightarrow\quad \tan x+\frac{\tan x+\sqrt{3}}{1-\sqrt{3}\tan x}+\frac{\tan x-\sqrt{3}}{1+\sqrt{3}\tan x}=3$$

$$\Longrightarrow\quad \tan x+\frac{(1-\sqrt{3}\tan x)(\tan x-\sqrt{3})}{1-3\tan^2 x}=3$$

$$\Longrightarrow\quad \tan x+\frac{8\tan x}{1-3\tan^2 x}=3$$

$$\Longrightarrow\quad \frac{3(3\tan x-\tan^3 x)}{1-3\tan^2 x}=3$$

$$\Longrightarrow\quad 3\tan 3x=3$$

$$\Longrightarrow\quad \tan 3x=1$$

Exercise:

1. Write the expression $\frac{2\tan 31^\circ}{1-\tan^2 31^\circ}$ as a single function of an angle.
2. Prove $cos\,2x=cos^4 x-sin^4 x$
3. Prove $2\,tan\,2x=\frac{cos\,x+sin\,x}{cos\,x-sin\,x}-\frac{cos\,x-sin\,x}{cos\,x+sin\,x}$

4. Find the value of
    a. $sin\ 18°$
    b. $cos\ 18°$
    c. $sin\ 36°$
    d. $cos\ 36°$
    e. $sin\ 54°$
    f. $cos\ 54°$
    g. $sin\ 72°$
    h. $cos\ 72°$
5. *Prove that*

i) $\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}} = \tan x$

ii) $\tan A - \cot A = -2\cot 2A$

6. Prove that $\dfrac{\cos A}{1-\sin A} = \tan\left(45^\circ + \dfrac{A}{2}\right)$
7. Prove that $\tan\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)+\tan\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)=2\sec\theta$
8. Prove that $sin^2\frac{\pi}{8} + sin^2\frac{3\pi}{8} = 1.$

9. Find the value of $tan100^0 + tan125^0 + tan100^0.tan125^0$.
10. Prove that $sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right) - sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right) = \frac{1}{\sqrt{2}} sinA$
11. Prove that

$$(cos\alpha - cos\beta)^2 + (sin\alpha - sin\,\beta)^2 = 4\,sin^2\left(\frac{\alpha-\beta}{2}\right)$$

12. If $2\ tanA + cotA = tanB$, then prove that

$$2\tan(B-A) = cotA\ .$$

13. If $\tan A = \frac{4}{3}$, find the values of . $sin2A$ $\quad ii. cos2A$ $\quad iii.\ tan2A$ $\quad iv.\ cot2A$.
14. Prove that

$sinA\ \sin(60^0 + A)\sin(60^0 - A) = \frac{1}{4}\ sin3A$

and hence deduce that $sin\frac{\pi}{9}\ sin\frac{2\pi}{9}\ \ sin\frac{3\pi}{9}\ \ sin\frac{4\pi}{9} = \frac{3}{16}$.
15. Show that $\sum\frac{\sin\,(B-C)}{sinB\ sinC} = 0.$
16. $\tan(1^0)\ .\tan(2^0) \ldots\ldots \tan(89^0) =$
17. $log\ tan\,1^0\ .log\ tan\,2^0 \ldots\ldots log\ tan\,89^0 =$

5.3. Trigonometric Functions of Sub-multiple Angles

Learning objectives:

- We derive Half-angle formulae and one-third angle formulae of the trigonometric functions.
  And
- Solve the problems related to the above concepts.

Half-Angle Formulas

When θ is replaced by $\frac{\theta}{2}$, the resulting formulas are called **Half-angle** formulas. All the previous formulas will apply.

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \qquad \ldots(1)$$

$$\left.\begin{aligned}\cos\theta &= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\\ &= 2\cos^2\frac{\theta}{2} - 1\\ &= 1 - 2\sin^2\frac{\theta}{2}\end{aligned}\right] \qquad \ldots(2)$$

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} \qquad \ldots(3)$$

From (1), we also have

$$\sin\theta = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}$$

We divide both numerator and denominator by $\cos^2\frac{\theta}{2}$. We obtain

$$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \qquad \text{...(4)}$$

From (2),
$$\cos\theta = \frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}}$$

Dividing both numerator and denominator by $\cos^2\frac{\theta}{2}$,

$$\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \qquad \text{...(5)}$$

Again from the formula (2), we can also obtain the formulas

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}, \quad \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \qquad \text{...(6)}$$

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}, \quad \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad \text{...(7)}$$

From this, we deduce that

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \qquad \text{...(8)}$$

Example 1:

Write the expression $\sqrt{\frac{1-\cos 84^\circ}{2}}$ as a single function of an angle.

Solution: $\sqrt{\dfrac{1-\cos 84^\circ}{2}} = \sin\dfrac{84^\circ}{2} = \sin 42^\circ$

Example 2: Find the value of $tan7\frac{1}{2}{}^\circ$, $cot7\frac{1}{2}{}^\circ$

Solution:

$$tan7\frac{1}{2}{}^\circ = \frac{sin7\frac{1}{2}{}^\circ}{cos7\frac{1}{2}{}^\circ} = \frac{2\,sin^2 7\frac{1}{2}{}^\circ}{2sin7\frac{1}{2}{}^\circ\, cos7\frac{1}{2}{}^\circ} = \frac{1-cos15^\circ}{sin15^\circ} = \frac{1-\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$$

$$= \frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1} = \frac{(2\sqrt{2}-\sqrt{3}-1)(\sqrt{3}+1)}{3-1}$$

$$= \sqrt{6}-\sqrt{3}+\sqrt{2}-2 = (\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$$

$$\therefore\ tan7\frac{1}{2}{}^\circ = (\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$$

$$\therefore\ cot7\frac{1}{2}{}^\circ = \frac{1}{tan7\frac{1}{2}{}^\circ} = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$$

One-third Angle Formulas:

All the required formulas are obtained by replacing θ by $\frac{\theta}{3}$ in the relevant formulas derived earlier. For example,

$$\sin\theta = 3\sin\frac{\theta}{3} - 4\sin^3\frac{\theta}{3}$$

$$\cos\theta = 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3}$$

$$\tan\theta = \frac{3\tan\frac{\theta}{3} - \tan^3\frac{\theta}{3}}{1-3\tan^2\frac{\theta}{3}}$$

Example 3:

$$\cos 81^\circ = 4\cos^3 27^\circ - 3\cos 27^\circ$$

$$\sin 6A = 3\sin 2A - 4\sin^3 2A$$

$$\tan 45^\circ = \frac{3\tan 15^\circ - \tan^3 15^\circ}{1-3\tan^2 15^\circ}$$

$$8\cos^3 20^\circ - 6\cos 20^\circ = 2\left(4\cos^3 20^\circ - 3\cos 20^\circ\right)$$

$$= 2\cos\left(3\times 20^\circ\right)$$

$$= 2\cos 60^\circ = 2\times\frac{1}{2} = 1$$

PROBLEM SET

IP1: The value of $\cos 7\frac{1}{2}°$

Solution:

Step1: We know that $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$ ....(1)

Step 2: We have $\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos A}{2}}$ ....(2)

Step3: By putting $A = 15°$ in (2), we get

$$\cos 7\frac{1}{2}° = \sqrt{\frac{1+\cos 15°}{2}} \qquad (since\ \ \cos 7\tfrac{1}{2}° \text{ is + ve})$$

$$= \sqrt{\frac{1+\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{2}} \qquad \text{(from (1))}$$

$$= \sqrt{\frac{2\sqrt{2}+\sqrt{3}+1}{4\sqrt{2}}}$$

$$= \sqrt{\frac{2\sqrt{2}+\sqrt{3}+1}{4\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}} = \sqrt{\frac{4+\sqrt{6}+\sqrt{2}}{8}} = \frac{\sqrt{4+\sqrt{6}+\sqrt{2}}}{2\sqrt{2}}$$

Step 4:

Hence $\cos 7\frac{1}{2}° = \frac{\sqrt{4+\sqrt{6}+\sqrt{2}}}{2\sqrt{2}}$

P1: The value of $\sin 7\frac{1}{2}°$

A. $\frac{\sqrt{4-\sqrt{6}-\sqrt{2}}}{2\sqrt{2}}$

B. $\frac{\sqrt{4+\sqrt{6}+\sqrt{2}}}{2\sqrt{2}}$

C. $\frac{\sqrt{4-\sqrt{6}+\sqrt{2}}}{2\sqrt{2}}$

D. $\frac{\sqrt{4+\sqrt{6}-\sqrt{2}}}{2\sqrt{2}}$

Answer: A

Solution: We know that $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$

Now, $\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{(1-\cos A)}{2}}$

Put $A = 15°$, we get

$$\sin 7\frac{1}{2}° = \sqrt{\frac{1-\cos 15^0}{2}} \qquad \text{(Since } \sin 7\tfrac{1}{2}° \text{ is + ve)}$$

$$= \sqrt{\frac{1-\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{2}} = \sqrt{\frac{2\sqrt{2}-\sqrt{3}-1}{4\sqrt{2}}}$$

$$= \sqrt{\frac{2\sqrt{2}-\sqrt{3}-1}{4\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}}$$

$$= \sqrt{\frac{4-\sqrt{6}-\sqrt{2}}{8}} = \frac{\sqrt{4-\sqrt{6}-\sqrt{2}}}{2\sqrt{2}}$$

IP2: If $\cos\theta = \frac{\cos\alpha\cos\beta}{1-\sin\alpha\sin\beta}$ then prove that $\tan\frac{\theta}{2} = \frac{\tan\frac{\alpha}{2}-\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}}$

Solution:

Step 1: Given that $\cos\theta = \frac{\cos\alpha\cos\beta}{1-\sin\alpha\sin\beta}$ ... (1)

Step 2: We know than $\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

$$\Rightarrow \tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta}$$

$$\Rightarrow \tan^2\frac{\theta}{2} = \frac{1-\frac{\cos\alpha\cos\beta}{1-\sin\alpha\sin\beta}}{1+\frac{\cos\alpha\cos\beta}{1-\sin\alpha\sin\beta}} \quad \text{(from (1))}$$

$$\Rightarrow \tan^2\frac{\theta}{2} = \frac{1-\sin\alpha\sin\beta-\cos\alpha\cos\beta}{1-\sin\alpha\sin\beta+\cos\alpha\cos\beta}$$

$$\Rightarrow \tan^2\frac{\theta}{2} = \frac{1-(\cos\alpha\cos\beta+\sin\alpha\sin\beta)}{1+(\cos\alpha\cos\beta-\sin\alpha\sin\beta)}$$

$$\Rightarrow \tan^2\frac{\theta}{2} = \frac{1-\cos(\alpha-\beta)}{1+\cos(\alpha+\beta)}$$

$$\Rightarrow \tan^2\frac{\theta}{2} = \frac{2\sin^2\left(\frac{\alpha-\beta}{2}\right)}{2\cos^2\left(\frac{\alpha+\beta}{2}\right)}$$

$$\Rightarrow \tan^2\frac{\theta}{2} = \frac{\sin^2\left(\frac{\alpha-\beta}{2}\right)}{\cos^2\left(\frac{\alpha+\beta}{2}\right)}$$

$$\Rightarrow \tan\frac{\theta}{2} = \pm\frac{\sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)} = \pm\frac{sin\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)}{cos\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)}$$

$$\Rightarrow \tan\frac{\theta}{2} = \pm\frac{\sin\frac{\alpha}{2}.\cos\frac{\beta}{2}-\cos\frac{\alpha}{2}.\sin\frac{\beta}{2}}{\cos\frac{\alpha}{2}.\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}.\sin\frac{\beta}{2}} \quad \text{... (2)}$$

Step 3:

Dividing numerator and denominator by $\left(\cos\frac{\alpha}{2}.\cos\frac{\beta}{2}\right)$,

we get

$$(2) \Rightarrow \tan\frac{\theta}{2} = \pm\frac{\tan\frac{\alpha}{2}-\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}}$$

Hence proved.

P2: If $\cos\theta = \frac{\cos\alpha-\cos\beta}{1-\cos\alpha.\cos\beta}$ then $\tan\frac{\theta}{2} =$

A. $\pm\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}$

B. $\pm\cot\frac{\alpha}{2}.\cot\frac{\beta}{2}$

C. $\pm\tan\frac{\alpha}{2}.\cot\frac{\beta}{2}$

D. $\pm\cot\frac{\alpha}{2}.\tan\frac{\beta}{2}$

Answer: **C**

Solution:

Given that $\cos\theta = \frac{\cos\alpha-\cos\beta}{1-\cos\alpha.\cos\beta}$

$$\Rightarrow \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{\cos\alpha-\cos\beta}{1-\cos\alpha.\cos\beta}$$

Applying componendo and dividendo, we get

$$\Rightarrow \frac{\left(1-\tan^2\frac{\theta}{2}\right)+\left(1+\tan^2\frac{\theta}{2}\right)}{\left(1-\tan^2\frac{\theta}{2}\right)-\left(1+\tan^2\frac{\theta}{2}\right)} = \frac{(\cos\alpha-\cos\beta)+(1-\cos\alpha.\cos\beta)}{(\cos\alpha-\cos\beta)-(1-\cos\alpha.\cos\beta)}$$

$$\Rightarrow \frac{2}{-2\tan^2\frac{\theta}{2}} = \frac{1+\cos\alpha-\cos\beta-\cos\alpha\cos\beta}{-\{1-\cos\alpha+\cos\beta-\cos\alpha\cos\beta\}}$$

$$\Rightarrow \frac{1}{\tan^2\frac{\theta}{2}} = \frac{1+\cos\alpha-\cos\beta-\cos\alpha\cos\beta}{\{1-\cos\alpha+\cos\beta-\cos\alpha\cos\beta\}} = \frac{(1+cos\alpha)(1-cos\beta)}{(1-cos\alpha)(1+cos\beta)}$$

$$\Rightarrow \frac{1}{\tan^2\frac{\theta}{2}} = \frac{2\cos^2\frac{\alpha}{2}\times 2\sin^2\frac{\beta}{2}}{2\sin^2\frac{\alpha}{2}\times 2\cos^2\frac{\beta}{2}}$$

$$\Rightarrow \frac{1}{\tan^2\frac{\theta}{2}} = \cot^2\frac{\alpha}{2}.\tan^2\frac{\beta}{2}$$

$$\Rightarrow \tan\frac{\theta}{2} = \pm\tan\frac{\alpha}{2}.\cot\frac{\beta}{2}$$

IP3: If $\boldsymbol{tan\frac{\theta}{2} = \sqrt{\frac{a-b}{a+b}}\ tan\ \frac{\emptyset}{2}}$ then prove that $\boldsymbol{cos\theta = \frac{a\cos\emptyset+b}{a+b\cos\emptyset}}$

Solution:

Step1: Given that $tan\frac{\theta}{2} = \sqrt{\frac{a-b}{a+b}}\ tan\ \frac{\emptyset}{2}$ ... (1)

Step2: We have $cos\theta = \frac{1-tan^2\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}$

$$\Rightarrow cos\theta = \frac{1-\frac{a-b}{a+b}\ .\ tan^2\frac{\emptyset}{2}}{1+\frac{a-b}{a+b}\ .tan^2\frac{\emptyset}{2}}$$

$$\Rightarrow cos\theta = \frac{(a+b)-(a-b)\ tan^2\frac{\emptyset}{2}}{(a+b)+(a-b)\ tan^2\frac{\emptyset}{2}}$$

$$\Rightarrow cos\theta = \frac{a\left[1-tan^2\frac{\emptyset}{2}\right]+b\left[1+tan^2\frac{\emptyset}{2}\right]}{a\left[1+tan^2\frac{\emptyset}{2}\right]+b\left[1-tan^2\frac{\emptyset}{2}\right]}$$

$$\Rightarrow cos\theta = \frac{\left[1+tan^2\frac{\emptyset}{2}\right]\left\{a\left[\frac{1-tan^2\frac{\emptyset}{2}}{1+tan^2\frac{\emptyset}{2}}\right]+b\right\}}{\left[1+tan^2\frac{\emptyset}{2}\right]\left\{a+b\left[\frac{1-tan^2\frac{\emptyset}{2}}{1+tan^2\frac{\emptyset}{2}}\right]\right\}}$$

$$\Longrightarrow\ cos\theta = \frac{a\left[\frac{1-tan^2\frac{\emptyset}{2}}{1+tan^2\frac{\emptyset}{2}}\right]+b}{a+b\left[\frac{1-tan^2\frac{\emptyset}{2}}{1+tan^2\frac{\emptyset}{2}}\right]} = \frac{a\ cos\emptyset+b}{a+b\ cos\emptyset}$$

Step3:

$\therefore$ If $tan\frac{\theta}{2} = \sqrt{\frac{a-b}{a+b}}\ tan\ \frac{\emptyset}{2}$ then $cos\theta = \frac{a\ cos\emptyset+b}{a+b\ cos\emptyset}$

Hence proved

P3: If $2\tan\frac{\alpha}{2} = \tan\frac{\beta}{2}$ then $\frac{3+5\cos\beta}{5+3\cos\beta}$

A. $\cos\frac{\beta}{2}$

B. $\cos^2\alpha$

C. $\cos\beta$

D. $\cos\alpha$

Answer: **D**

Solution:

Given that $2\tan\frac{\alpha}{2} = \tan\frac{\beta}{2}$

Squaring on both sides, we get

$$\Longrightarrow 4\tan^2\frac{\alpha}{2} = \tan^2\frac{\beta}{2}$$

$$\Longrightarrow 4\left(\frac{1-\cos\alpha}{1+\cos\alpha}\right) = \left(\frac{1-\cos\beta}{1+\cos\beta}\right)$$

$$\Longrightarrow 4(1-\cos\alpha)(1+\cos\beta) = (1+\cos\alpha)(1-\cos\beta)$$

$$\Longrightarrow 4[1+\cos\beta-\cos\alpha-\cos\alpha\cos\beta]$$

$$= [1-\cos\beta+\cos\alpha-\cos\alpha\cos\beta]$$

$$\Longrightarrow 4+4\cos\beta-4\cos\alpha-4\cos\alpha\cos\beta$$

$$= 1-\cos\beta+\cos\alpha-\cos\alpha\cos\beta$$

$$\Longrightarrow 3+5\cos\beta = 5\cos\alpha+3\cos\alpha\cos\beta$$

$$\Longrightarrow 3+5\cos\beta = \cos\alpha\,(5+3\cos\beta)$$

$$\Longrightarrow\ \cos\alpha = \frac{3+5\cos\beta}{5+3\cos\beta}$$

IP4: If $cos\left(\frac{\alpha}{3}\right)+cos\left(\frac{\beta}{3}\right)+cos\left(\frac{\gamma}{3}\right) = 0$ then show that $cos\alpha+cos\beta+cos\gamma = 12\ cos\left(\frac{\alpha}{3}\right).cos\left(\frac{\beta}{3}\right).cos\left(\frac{\gamma}{3}\right)$

Solution:

Step1: Given $cos\left(\frac{\alpha}{3}\right)+cos\left(\frac{\beta}{3}\right)+cos\left(\frac{\gamma}{3}\right) = 0$ ...(1)

Step2: Now,

$$cos\alpha+cos\beta+cos\gamma$$

$$= 4\ cos^3\left(\frac{\alpha}{3}\right)-3cos\frac{\alpha}{3}+4\ cos^3\left(\frac{\beta}{3}\right)-3cos\left(\frac{\beta}{3}\right)$$

$$+4\ cos^3\left(\frac{\gamma}{3}\right) - 3cos\left(\frac{\gamma}{3}\right)$$

$$= 4\ \left[cos^3\left(\frac{\alpha}{3}\right) + cos^3\left(\frac{\beta}{3}\right) + cos^3\left(\frac{\gamma}{3}\right)\right]$$

$$-3\ \left[cos\frac{\alpha}{3} + cos\frac{\beta}{3} + cos\frac{\gamma}{3}\right]$$

$$= 4\ \left[cos^3\left(\frac{\alpha}{3}\right) + cos^3\left(\frac{\beta}{3}\right) + cos^3\left(\frac{\gamma}{3}\right)\right]\ - 3(0)$$

(from (1))

$$= 4\ \left[cos^3\left(\frac{\alpha}{3}\right) + cos^3\left(\frac{\beta}{3}\right) + cos^3\left(\frac{\gamma}{3}\right)\right] \qquad \dots(2)$$

Step3:

We know that if $a + b + c\ =\ 0$ then $a^3 + b^3 + c^3 = 3abc$

$$\therefore\ cos\left(\frac{\alpha}{3}\right) + cos\left(\frac{\beta}{3}\right) + cos\left(\frac{\gamma}{3}\right) = 0$$

$$\Longrightarrow cos^3\left(\frac{\alpha}{3}\right) + cos^3\left(\frac{\beta}{3}\right) + cos^3\left(\frac{\gamma}{3}\right) = 3cos\left(\frac{\alpha}{3}\right)cos\left(\frac{\beta}{3}\right)cos\left(\frac{\gamma}{3}\right)$$

$$\therefore\ \ (2)\ \ \Longrightarrow cos\alpha + cos\beta + cos\gamma$$

$$= 4\ \left[3cos\frac{\alpha}{3}cos\frac{\beta}{3}cos\frac{\gamma}{3}\right]$$

$$= 12\ cos\left(\frac{\alpha}{3}\right)cos\left(\frac{\beta}{3}\right)cos\left(\frac{\gamma}{3}\right).$$

P4:

$\boldsymbol{sinA\,.sin^3\frac{A}{3} +\ cosA\,.\ cos^3\frac{A}{3} =}$

A. $\frac{1}{4}\left\{3\ cos\frac{A}{3} +\ cos2A\right\}$

B. $cos^3\left(\frac{2A}{3}\right)$

C. $cos\left(\frac{2A}{3}\right)$

D. $\frac{1}{4}\left[\ 3\ cos\left(\frac{2A}{3}\right) + cosA\ \right]$

Answer: **B**

Solution:

$$sinA\ .\ sin^3\frac{A}{3} +\ cosA\ .\ cos^3\frac{A}{3}$$

$$= sinA\,.\left[\frac{3\ sin\frac{A}{3} - sinA}{4}\right] + cosA\ \left[\frac{3\ cos\frac{A}{3} + cosA}{4}\right]$$

$$=\ \frac{1}{4}\left[3sinA.\,sin\frac{A}{3} - sin^2A + 3cosA\,.cos\frac{A}{3} + cos^2A\ \right]$$

$$= \frac{1}{4}\ \left[3\left(cosA.cos\frac{A}{3} + SinA.sin\frac{A}{3}\right) + (cos^2A - sin^2A)\right]$$

$$= \frac{1}{4}\ \left[3\ cos\left(A - \frac{A}{3}\right) + cos2A\right]$$

$$= \frac{1}{4}\ \left[3\ cos\left(\frac{2A}{3}\right) + \cos\ (2A)\right]$$

$$= \frac{1}{4}\ \left[3\ cos\left(\frac{2A}{3}\right) + 4\ cos^3\left(\frac{2A}{3}\right) - 3cos\left(\frac{2A}{3}\right)\right]$$

$$= \frac{1}{4}\left[4\, cos^3\left(\frac{2A}{3}\right)\right]$$
$$= cos^3\left(\frac{2A}{3}\right)$$
$$\therefore\ sinA\,.\ sin^3\left(\frac{A}{3}\right) + cosA\,.\ cos^3\left(\frac{A}{3}\right) = cos^3\left(\frac{2A}{3}\right)$$

Exercise:

1. Prove $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = 2$
2. Obtain the formulas for $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$ in terms of $\sin\theta$.
3. Write each expression as a single function of an angle.
   a. $\sqrt{\frac{1+\cos 160^\circ}{2}}$
   b. $\frac{\sin 142^\circ}{1+\cos 142^\circ}$
   c. $\frac{1-\cos 184^\circ}{\sin 184^\circ}$
4. Prove $1 - \frac{1}{2}\sin 2x = \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x}$
5. Prove $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$
6. Prove $\sin^4 A = \frac{3}{8} - \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A$
7. Find the values of
   a) $\sin 22\frac{1}{2}^0, \cos 22\frac{1}{2}^0, \tan 22\frac{1}{2}^0$
   b) Show that $\cos\frac{\pi}{8} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$, $\sin\frac{\pi}{8} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$
8.
   I. If $\tan\left(\frac{A}{2}\right) = t$ then show that $\sin A + \tan A = \frac{4t}{1-t^4}$

   and $\tan A + \sec A = \frac{1+t}{1-t}$

II. In a ΔABC, if $\tan\left(\frac{A}{2}\right)=\frac{5}{6}$, $\tan\left(\frac{B}{2}\right)=\frac{20}{37}$

then show that $\tan\left(\frac{C}{2}\right)=\frac{2}{5}$

9. Prove that

I. If $\theta<\frac{\pi}{8}$, show that $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 4\theta}}}=2\cos\left(\frac{\theta}{2}\right)$

II. If $\alpha+\beta+\gamma=2\pi$ then prove that

$$\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$$

10. For $\alpha,\beta\in R$, prove that

a. $(\cos\alpha+\cos\beta\,)^2+(\sin\alpha+\sin\beta)^2=4\cos^2\left(\frac{\alpha-\beta}{2}\right)$

b. $(\cos\alpha-\cos\beta\,)^2+(\sin\alpha-\sin\beta)^2=4\sin^2\left(\frac{\alpha-\beta}{2}\right)$

11. Show that

a) $\cot\frac{\pi}{24}=\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$

b) $\tan\frac{\pi}{16}=\sqrt{4+2\sqrt{2}}-(\sqrt{2}+1)$

c) $\tan 142\frac{1}{2}^{0}=2+\sqrt{2}+\sqrt{3}-\sqrt{6}$

12. If $0\le x\le 2\pi$ find $\sin\frac{x}{2}$, $\cos\frac{x}{2}\tan\frac{x}{2}$ when

A. $\tan x=-\frac{4}{3}$, x lies in 2nd quadrant

B. $\cos x=\frac{-1}{3}$, $x\in third\ quandrant$

C. $\sin x=\frac{-1}{2}$, $x\in fourth\ quandrant$

13. If $\cos\theta=\cos\alpha\,.\cos\beta$ then prove that

$$\tan\left(\frac{\theta+\alpha}{2}\right)\tan\left(\frac{\theta-\alpha}{2}\right)=\tan^2\frac{\beta}{2}$$

14. If $\sin\alpha+\sin\beta=a$, $\cos\alpha+\cos\beta=b$ then prove that

(a) $\cos(\alpha-\beta)=\frac{a^2+b^2-2}{2}$

(b) $\tan\left(\frac{\alpha-\beta}{2}\right)=\pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$

(c) $\sin(\alpha+\beta)=\frac{2ab}{a^2+b^2}$

5.4. Inverse Trigonometric Functions

Learning Objectives:

- To define Inverse Trigonometric Functions

- To derive some formulae of Inverse Trigonometric Functions
  AND
- To solve related problems

Inverse Trigonometric Relations

The equation

$$x = sin\, y$$

defines a unique value of x for each given angle y. But when x is given, the equation may have no solution or many solutions. For example: if $=\ 2$, there is no solution, since the sine of an angle never exceeds 1. If $x = \frac{1}{2}$ there are many solutions $y = 30^o, 150^o, \ldots$.

The symbol $sin^{-1} x$ is used to denote the smallest angle, whether positive or negative, that has x for its sine.

$$y = \sin^{-1} x$$

The symbol $\sin^{-1} x$ is read in words as "inverse sine of x". This shall not be confused with $\frac{1}{\sin x}$ which would be written in the form $(\sin x)^{-1}$.

Thus $\mathbf{sin^{-1}}\, \boldsymbol{x}$ **is an *angle*, and denotes the *smallest numerical* angle whose sine is $\boldsymbol{x}$.**

An alternative notation for the inverse relation is

$$y = arc\, sin\, x$$

This equation is to be interpreted as stating that "y is an angle whose sine is x."

So, $\cos^{-1} x$ or $arc \cos x$ means the smallest numerical angle whose cosine is x.

Similarly, the inverse of other trigonometric ratios are defined.

Inverse Trigonometric functions

If we propose to consider the inverse trigonometric relations as functions, then there must be one value of y corresponding to each admissible value of x. For this purpose, we select one out of the many angles corresponding to the given values of x. For example, when $x = \frac{1}{2}$, we select the value $y = \frac{\pi}{6}$, and when $x = -\frac{1}{2}$, we select the value $y = -\frac{\pi}{6}$. This selected value is called the ***principal value*** of $arc \sin x$.

When x is positive or zero and the inverse function exists, the principal value is defined as that value of y which lies between 0 and $\frac{\pi}{2}$ inclusive.

Example 1:

$$arcsin\ \frac{\sqrt{3}}{2} = \frac{\pi}{3} \qquad arccos\ \frac{\sqrt{3}}{2} = \frac{\pi}{6} \qquad arctan\ 1 = \frac{\pi}{4}$$

When x is negative and the inverse function exists, the principal value is defined as follows:

$$-\frac{\pi}{2} \le \sin^{-1} x < 0 \qquad -\frac{\pi}{2} \le \csc^{-1} x < 0$$

$$\frac{\pi}{2} < \cos^{-1} x \le \pi \qquad \frac{\pi}{2} < \sec^{-1} x \le \pi$$

$$-\frac{\pi}{2} < \tan^{-1} x < 0 \qquad -\frac{\pi}{2} < \cot^{-1} x < 0$$

The principal value of $\cot^{-1} x$ is in the interval $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. In this interval, $\cot y$ is not defined at all points (not defined at $y = 0$). In the interval $0 < y < \pi$, $\cot y$ is defined at all points. *It is a standard convention to take the interval* $0 < y < \pi$ *for the principal value of inverse cotangent function.*

Example 2:

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3} \qquad \csc^{-1}\left(-\sqrt{2}\right) = -\frac{\pi}{4}$$

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \qquad \sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) = \frac{5\pi}{6}$$

$$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6} \qquad \cot^{-1}(-1) = \frac{3\pi}{4}$$

The domain and the principal-value range of Inverse trigonometric functions are summarized below.

Inverse Function	Domain	Principal-value Range
$y = \sin^{-1} x$	$[-1,1]$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \cos^{-1} x$	$[-1,1]$	$0 \le y \le \pi$
$y = \tan^{-1} x$	$\boldsymbol{R}$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \csc^{-1} x$	$(-\infty,-1] \cup [1,\infty)$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$
$y = \sec^{-1} x$	$(-\infty,-1] \cup [1,\infty)$	$0 \le y \le \pi,\ y \ne \frac{\pi}{2}$
$y = \cot^{-1} x$	$\boldsymbol{R}$	$0 < y < \pi$

Some Formulae

(1) $\sin^{-1}x+\cos^{-1}x=\dfrac{\pi}{2}$

Proof:

We have $x \in [-1,1]$,

Let $\sin^{-1}x=y$ then $y \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $x = \sin y$

$$Now,\ x=\sin y=\cos\left(\frac{\pi}{2}-y\right)\ and\ \frac{\pi}{2}-y\in[0,\pi]$$

$$\Rightarrow \quad \frac{\pi}{2}-y=\cos^{-1}x$$

$$\Rightarrow y+\cos^{-1}x=\frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$$

Similarly,

(2) $\tan^{-1}x+\cot^{-1}x=\dfrac{\pi}{2}\quad \forall\, x\in R$

(3) $\sec^{-1}x+\csc^{-1}x=\dfrac{\pi}{2}\quad \forall\, x\in(-\infty,-1]\cup[1,\infty)$

The following are the properties of inverse trigonometric functions:

(4) For $x\in[-1,0)\cup(0,1]$, $\sin^{-1}x=\csc^{-1}\dfrac{1}{x}$

(5) For $x\in[-1,0)\cup(0,1]$, $\cos^{-1}x=\sec^{-1}\dfrac{1}{x}$

(6) For $x>0$, $\tan^{-1}x=\cot^{-1}\dfrac{1}{x}$

(7) For $x<0$, $\tan^{-1}x=\cot^{-1}\dfrac{1}{x}-\pi$

Proof of (7):

Now, let $x \in (-\infty, 0)$ and $\tan^{-1} x = \theta$.

Then $\theta \in \left(-\dfrac{\pi}{2}, 0\right)$ and $\tan\theta = x$. That is,

$$\theta + \pi \in \left(\frac{\pi}{2}, \pi\right) \text{ and } \tan(\pi + \theta) = \tan\theta = x$$

Therefore, $\theta + \pi \in \left(\dfrac{\pi}{2}, \pi\right)$ and $\cot(\pi + \theta) = \dfrac{1}{x}$

$$\cot^{-1}\frac{1}{x} = \theta + \pi = \pi + \tan^{-1}\frac{1}{x} \text{ or } \tan^{-1}\frac{1}{x} = \cot^{-1}\frac{1}{x} - \pi$$

To convert any inverse function into another, we draw a triangle and mark the ratios of the sides. From this, we can get the desired conversion. For example:

Let $\cot^{-1} x = y$, then $\cot y = x$

$$\sin y = \frac{1}{\sqrt{1+x^2}} \Rightarrow y = \sin^{-1}\frac{1}{\sqrt{1+x^2}} \Rightarrow \cot^{-1} x = \sin^{-1}\frac{1}{\sqrt{1+x^2}}$$

Example 3: Prove that $\sin^{-1}\dfrac{3}{5} - \cos^{-1}\dfrac{12}{13} = \sin^{-1}\dfrac{16}{65}$

Let $\sin^{-1}\dfrac{3}{5} = \alpha$, so that $\sin\alpha = \dfrac{3}{5}$

And therefore $\cos\alpha = \dfrac{4}{5}$

Let $\cos^{-1}\dfrac{12}{13} = \beta$, so that $\cos\beta = \dfrac{12}{13}$

And therefore $\sin\beta = \dfrac{5}{13}$

Let $\sin^{-1}\dfrac{16}{65} = \gamma$, so that $\sin\gamma = \dfrac{16}{65}$

We have then to prove that $\alpha - \beta = \gamma$

$i.e.$ to show that $\sin(\alpha - \beta) = \sin\gamma$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \frac{3}{5}\cdot\frac{12}{13} - \frac{4}{5}\cdot\frac{5}{13} = \frac{16}{65} = \sin\gamma$$

Hence the relation is proved.

Example 4: **Solve the equation** :

$$\tan^{-1}\frac{x+1}{x-1}+\tan^{-1}\frac{x-1}{x}=\tan^{-1}(-7)$$

Taking the tangents of both sides of the equation, we have

$$\frac{\tan\left[\tan^{-1}\frac{x+1}{x-1}\right]+\tan\left[\tan^{-1}\frac{x-1}{x}\right]}{1-\tan\left[\tan^{-1}\frac{x+1}{x-1}\right]\tan\left[\tan^{-1}\frac{x-1}{x}\right]}=-7$$

$$\Rightarrow \frac{\frac{x+1}{x-1}+\frac{x-1}{x}}{1-\frac{x+1}{x-1}\cdot\frac{x-1}{x}}=-7$$

$$\Rightarrow \frac{2x^2-x+1}{1-x}=-7$$

$$\Rightarrow x=2$$

PROBLEM SET

IP1: Theorem:

If $\mathbf{0 \le x \le 1, 0 \le y \le 1}$ then

$$\mathbf{Sin^{-1}\,x+Sin^{-1}\,y=Sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)\ \text{for}\ x^2+y^2\le 1}$$

$$\mathbf{=\pi-Sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)\ \text{for}\ x^2+y^2>1}$$

Proof: Let $\mathrm{Sin}^{-1}x=\alpha, \mathrm{Sin}^{-1}y=\beta$

$\Rightarrow x=\sin\alpha\,, y=\sin\beta$

$0\le x\le 1\Rightarrow 0\le\alpha\le\frac{\pi}{2}$

$0\le y\le 1\Rightarrow 0\le\beta\le\frac{\pi}{2}$

$0\le\alpha\le\frac{\pi}{2}, 0\le\beta\le\frac{\pi}{2}\Rightarrow 0\le\alpha+\beta\le\pi$

$$\begin{aligned}\sin(\alpha+\beta)&=\sin\alpha\cos\beta+\cos\alpha\sin\beta\\&=\sin\alpha\sqrt{1-\sin^2\beta}+\sqrt{1-\sin^2\alpha}\sin\beta\\&=x\sqrt{1-y^2}+y\sqrt{1-x^2}\end{aligned}$$

$0\le x\le 1, 0\le y\le 1\Rightarrow x\sqrt{1-y^2}+y\sqrt{1-x^2}\ge 0$

Case (i): Suppose $x^2+y^2\le 1$

$$\begin{aligned}&\Rightarrow x^2\le 1-y^2\\&\Rightarrow \sin^2\alpha\le 1-\sin^2\beta\\&\Rightarrow \sin^2\alpha\le\cos^2\beta\\&\Rightarrow \sin^2\alpha\le\sin^2\left(\frac{\pi}{2}-\beta\right)\end{aligned}$$

$$\Rightarrow \sin\alpha \le \sin\left(\frac{\pi}{2}-\beta\right)$$

$$\Rightarrow \alpha \le \frac{\pi}{2}-\beta$$

$$\Rightarrow \alpha+\beta \le \frac{\pi}{2}$$

$0 \le \alpha+\beta \le \frac{\pi}{2} \Rightarrow \mathrm{Sin}(\alpha+\beta) = \sin(\alpha+\beta)$

$$= x\sqrt{1-y^2}+y\sqrt{1-x^2}$$

$$\Rightarrow \alpha+\beta = \mathrm{Sin}^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$$

$$\Rightarrow \mathrm{Sin}^{-1}x+\mathrm{Sin}^{-1}y = \mathrm{Sin}^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$$

Case (ii): Suppose $x^2+y^2>1$

$$\Rightarrow x^2 > 1-y^2$$

$$\Rightarrow \sin^2\alpha > 1-\sin^2\beta$$

$$\Rightarrow \sin^2\alpha > \cos^2\beta$$

$$\Rightarrow \sin^2\alpha > \sin^2\left(\frac{\pi}{2}-\beta\right)$$

$$\Rightarrow \sin\alpha > \sin\left(\frac{\pi}{2}-\beta\right)$$

$$\Rightarrow \alpha > \frac{\pi}{2}-\beta$$

$$\Rightarrow \alpha+\beta > \frac{\pi}{2}$$

$\frac{\pi}{2} < \alpha+\beta \le \pi \Rightarrow -\frac{\pi}{2} > -(\alpha+\beta) \ge -\pi$

By adding π throughout, we get

$$\Rightarrow \frac{\pi}{2} > \pi-(\alpha+\beta) \ge 0$$

$\sin[\pi-(\alpha+\beta)] = \sin(\alpha+\beta)$

$$= x\sqrt{1-y^2}+y\sqrt{1-x^2}$$

$$\Rightarrow \pi-(\alpha+\beta) = \sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$$

$$\Rightarrow \alpha+\beta = \pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$$

$$\Rightarrow \sin^{-1}x+\sin^{-1}y = \pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$$

P1: If $0 \le x \le 1, 0 \le y \le 1$ and $x^2+y^2 \ge 1$ then $\mathrm{Cos}^{-1}x+\mathrm{Cos}^{-1}y =$

A. $\pi-\mathrm{Cos}^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)$
B. $\pi-\mathrm{Cos}^{-1}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right)$
C. $\mathrm{Cos}^{-1}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right)$
D. $\mathrm{Cos}^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)$

Answer: D

Proof:

Let $\mathrm{Cos}^{-1}x=\alpha, \mathrm{Cos}^{-1}y=\beta$. Then $x=\cos\alpha, y=\cos\beta$.

$0 \leq x \leq 1 \Rightarrow 0 \leq \alpha \leq \frac{\pi}{2}$

$0 \leq y \leq 1 \Rightarrow 0 \leq \beta \leq \frac{\pi}{2}$

$0 \leq \alpha \leq \frac{\pi}{2}, 0 \leq \beta \leq \frac{\pi}{2} \Rightarrow 0 \leq \alpha + \beta \leq \pi$

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \cos\alpha\cos\beta - \sqrt{1-\cos^2\alpha}\sqrt{1-\cos^2\beta} \\ &= xy - \sqrt{1-x^2}\sqrt{1-y^2}\end{aligned}$$

Case (i): Suppose $x^2 + y^2 \geq 1$

$$\begin{aligned}x^2 + y^2 \geq 1 &\Rightarrow x^2 \geq 1 - y^2 \\ &\Rightarrow \cos^2\alpha \geq 1 - \cos^2\beta \\ &\Rightarrow \cos^2\alpha \geq \sin^2\beta \\ &\Rightarrow \cos^2\alpha \geq \cos^2\left(\frac{\pi}{2} - \beta\right) \\ &\Rightarrow \cos\alpha \geq \cos\left(\frac{\pi}{2} - \beta\right) \\ &\Rightarrow \alpha \leq \frac{\pi}{2} - \beta \\ &\Rightarrow \alpha + \beta \leq \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}0 \leq \alpha + \beta \leq \frac{\pi}{2} \Rightarrow \mathrm{Cos}(\alpha+\beta) &= \cos(\alpha+\beta) \\ &= xy - \sqrt{1-x^2}\sqrt{1-y^2}\end{aligned}$$

$$\Rightarrow \alpha + \beta = \mathrm{Cos}^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$$

$$\Rightarrow \mathrm{Cos}^{-1}x + \mathrm{Cos}^{-1}y = \mathrm{Cos}^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$$

Note:

Suppose $x^2 + y^2 < 1$

$$\begin{aligned}&\Rightarrow x^2 < 1 - y^2 \\ &\Rightarrow \cos^2\alpha < 1 - \cos^2\beta \\ &\Rightarrow \cos^2\alpha < \sin^2\beta \\ &\Rightarrow \cos^2\alpha < \cos^2\left(\frac{\pi}{2} - \beta\right) \\ &\Rightarrow \cos\alpha < \cos\left(\frac{\pi}{2} - \beta\right) \\ &\Rightarrow \alpha > \frac{\pi}{2} - \beta \\ &\Rightarrow \alpha + \beta > \frac{\pi}{2}\end{aligned}$$

$\frac{\pi}{2} < \alpha + \beta < \pi \Rightarrow \frac{-\pi}{2} > -(\alpha+\beta) > -\pi$

Add π throughout, then we get

$$\Rightarrow \frac{\pi}{2} > \pi - (\alpha+\beta) > 0$$

Now $\mathrm{Cos}[\pi - (\alpha+\beta)] = \cos[\pi - (\alpha+\beta)]$

$$= -\cos(\alpha+\beta)$$
$$= \sqrt{1-x^2}\sqrt{1-y^2} - xy$$
$$\Rightarrow \pi - (\alpha+\beta) = \mathrm{Cos}^{-1}\left(\sqrt{1-x^2}\sqrt{1-y^2} - xy\right)$$
$$\Rightarrow \alpha+\beta = \pi - \mathrm{Cos}^{-1}\left(\sqrt{1-x^2}\sqrt{1-y^2} - xy\right)$$
$$\Rightarrow \mathrm{Cos}^{-1} x + \mathrm{Cos}^{-1} y = \pi - \cos^{-1}\left(\sqrt{1-x^2}\sqrt{1-y^2} - xy\right)$$

IP2: $\boldsymbol{Tan^{-1}\, x + Tan^{-1}\, y = Tan^{-1}\left(\frac{x+y}{1-xy}\right)}$ ***for*** $\boldsymbol{x} > 0, y > 0, xy < 1$

$$= \boldsymbol{\pi + Tan^{-1}\left(\frac{x+y}{1-xy}\right)}\ \textit{\textbf{for}}\ \boldsymbol{x} > 0, y > 0, xy > 1.$$

Proof:

Case (i): Suppose $x > 0, y > 0, xy < 1$

$x > 0 \Rightarrow 0 < \mathrm{Tan}^{-1} x < \frac{\pi}{2}$

$y > 0 \Rightarrow 0 < \mathrm{Tan}^{-1} y < \frac{\pi}{2}$

Let $\mathrm{Tan}^{-1} x = \alpha, \mathrm{Tan}^{-1} y = \beta$. Then $x = Tan\ \alpha, y = Tan\ \beta$

$0 < \mathrm{Tan}^{-1} x < \frac{\pi}{2} \Rightarrow 0 < \alpha < \frac{\pi}{2}$

$0 < \mathrm{Tan}^{-1} y < \frac{\pi}{2} \Rightarrow 0 < \beta < \frac{\pi}{2}$

$0 < \alpha < \frac{\pi}{2}, 0 < \beta < \frac{\pi}{2} \Rightarrow 0 < \alpha+\beta < \pi$

$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{Tan\ \alpha+Tan\ \beta}{1-Tan\ \alpha\ Tan\ \beta} = \frac{x+y}{1-xy} > 0$

$0 < \alpha+\beta < \pi, \tan(\alpha+\beta) > 0 \Rightarrow 0 < \alpha+\beta < \frac{\pi}{2}$

$\therefore Tan\ (\alpha+\beta) = \tan(\alpha+\beta) = \frac{x+y}{1-xy} \Rightarrow \alpha+\beta = \mathrm{Tan}^{-1}\frac{x+y}{1-xy}$

$\Rightarrow \mathrm{Tan}^{-1} x + \mathrm{Tan}^{-1} y = \mathrm{Tan}^{-1}\frac{x+y}{1-xy}$

Case (ii): Suppose $x > 0, y > 0, xy > 1$

$x > 0 \Rightarrow 0 < \mathrm{Tan}^{-1} x < \frac{\pi}{2}$

$y > 0 \Rightarrow 0 < \mathrm{Tan}^{-1} y < \frac{\pi}{2}$

Let $\mathrm{Tan}^{-1} x = \alpha, \mathrm{Tan}^{-1} y = \beta$. Then $x = Tan\ \alpha, y = Tan\ \beta$

$0 < \mathrm{Tan}^{-1} x < \frac{\pi}{2} \Rightarrow 0 < \alpha < \frac{\pi}{2}$

$0 < \mathrm{Tan}^{-1} y < \frac{\pi}{2} \Rightarrow 0 < \beta < \frac{\pi}{2}$

$0 < \alpha < \frac{\pi}{2}, 0 < \beta < \frac{\pi}{2} \Rightarrow 0 < \alpha+\beta < \pi$

$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{Tan\ \alpha+Tan\ \beta}{1-Tan\ \alpha\ Tan\ \beta} = \frac{x+y}{1-xy} < 0$

$0 < \alpha + \beta < \pi, \tan(\alpha + \beta) < 0 \Rightarrow \frac{\pi}{2} < \alpha + \beta < \pi$

$$\Rightarrow -\frac{\pi}{2} < \alpha + \beta - \pi < 0$$

$\therefore Tan\ (\alpha + \beta - \pi) = \tan(\alpha + \beta - \pi) = -\tan[\pi - (\alpha + \beta)]$

$$= \tan(\alpha + \beta) = \frac{x+y}{1-xy}$$

$$\Rightarrow \alpha + \beta - \pi = \mathrm{Tan}^{-1} \frac{x+y}{1-xy}$$

$$\Rightarrow \alpha + \beta = \pi + \mathrm{Tan}^{-1} \frac{x+y}{1-xy}$$

$\Rightarrow \mathrm{Tan}^{-1} x + \mathrm{Tan}^{-1} y = \pi + \mathrm{Tan}^{-1} \frac{x+y}{1-xy}$

P2: For $\boldsymbol{x < 0, y < 0}$ and $\boldsymbol{xy > 1}$, $\mathbf{Tan^{-1}}\, \boldsymbol{x} + \mathbf{Tan^{-1}}\, \boldsymbol{y} =$

A. $\boldsymbol{-\pi + Tan^{-1} \frac{x+y}{1-xy}}$

B. $\boldsymbol{-\pi - Tan^{-1} \frac{x+y}{1-xy}}$

C. $\boldsymbol{\pi - Tan^{-1} \frac{x+y}{1-xy}}$

D. $\boldsymbol{\pi + Tan^{-1} \frac{x+y}{1-xy}}$

Answer: A

Proof:

Given, $x < 0, y < 0, xy > 1$

Let $x = -a, y = -b$

$\therefore a > 0, b > 0, ab > 1$

Thus $Tan^{-1}a + Tan^{-1}\, b = \pi + Tan^{-1} \frac{a+b}{1-ab}$ [From IP2]

$\Rightarrow Tan^{-1}(-x) + Tan^{-1}\,(-y) = \pi + Tan^{-1} \frac{-x-y}{1-xy}$

$\Rightarrow -Tan^{-1}x - Tan^{-1}\, y = \pi - Tan^{-1} \frac{x+y}{1-xy}$

$\Rightarrow Tan^{-1}x + Tan^{-1}\, y = -\pi + Tan^{-1} \frac{x+y}{1-xy}$

Note:

For $\boldsymbol{x < 0, y < 0}$ and $\boldsymbol{xy > 1}$,

$\mathbf{Tan^{-1}}\, \boldsymbol{x} + \mathbf{Tan^{-1}}\, \boldsymbol{y} = \boldsymbol{Tan^{-1} \frac{x+y}{1-xy}}$

Proof:

Suppose $x < 0, y < 0, xy < 1$

Let $x = -a, y = -b$

$\therefore a > 0, b > 0, ab < 1$

Thus $Tan^{-1}a + Tan^{-1}\, b = Tan^{-1} \frac{a+b}{1-ab}$

$\Rightarrow Tan^{-1}(-x) + Tan^{-1}\,(-y) = Tan^{-1} \left(\frac{-x-y}{1-xy}\right)$

$$\Rightarrow -Tan^{-1}x - Tan^{-1}\,y = -Tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$Tan^{-1}x + Tan^{-1}\,y = Tan^{-1}\frac{x+y}{1-xy}$$

IP3: If $\cos^{-1}\frac{p}{a} + \cos^{-1}\frac{q}{b} = \alpha$, then prove that

$$\frac{p^2}{a^2} - \frac{2pq}{ab}.\cos\alpha + \frac{q^2}{b^2} = \sin^2\alpha$$

Solution: Let $\cos^{-1}\frac{p}{a} = A$ and $\cos^{-1}\frac{q}{b} = B$.

Then $\cos A = \frac{p}{a}$, $\cos B = \frac{q}{b}$ and $A + B = \alpha$ (given)

Now, $\cos\alpha = \cos(A+B) = \cos A\cos B - \sin A\sin B$

$$= \frac{p}{a}.\frac{q}{b} - \sqrt{1-\frac{p^2}{a^2}}\,.\sqrt{1-\frac{q^2}{b^2}}$$

$$\Longrightarrow \sqrt{1-\frac{p^2}{a^2}}\,.\sqrt{1-\frac{q^2}{b^2}} = \frac{pq}{ab} - \cos\alpha$$

On squaring both sides, we get

$$\left(1-\frac{p^2}{a^2}\right).\left(1-\frac{q^2}{b^2}\right) = \left(\frac{pq}{ab} - \cos\alpha\right)^2$$

$$\Longrightarrow 1 - \frac{p^2}{a^2} - \frac{q^2}{b^2} + \frac{p^2q^2}{a^2b^2} = \frac{p^2q^2}{a^2b^2} - \frac{2pq}{ab}\cos\alpha + \cos^2\alpha$$

$$\Longrightarrow \frac{p^2}{a^2} - \frac{2pq}{ab}\cos\alpha + \frac{q^2}{b^2} = 1 - \cos^2\alpha$$

$$\Longrightarrow \frac{p^2}{a^2} - \frac{2pq}{ab}\cos\alpha + \frac{q^2}{b^2} = \sin^2\alpha$$

P3: If $\mathrm{Sin}^{-1}\left(\frac{2p}{1+p^2}\right) - \mathrm{Cos}^{-1}\left(\frac{1-q^2}{1+q^2}\right) = \mathrm{Tan}^{-1}\left(\frac{2x}{1-x^2}\right)$ then $x =$

A. $\frac{p-q}{1-pq}$

B. $\frac{p+q}{1+pq}$

C. $\frac{p-q}{1+pq}$

D. $\frac{p+q}{1-pq}$

Answer: C

Solution: Let $p = \tan\alpha, q = \tan\beta$ *and* $x = \tan\theta$

$$\frac{2p}{1+p^2} = \frac{2\tan\alpha}{1+\tan^2\alpha} = \sin 2\alpha,$$

$$\frac{1-q^2}{1+q^2} = \frac{1-\tan^2\beta}{1+\tan^2\beta} = \cos 2\beta,$$

$$\frac{2x}{1-x^2} = \frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta$$

$$\mathrm{Tan}^{-1}\left(\frac{2x}{1+x^2}\right) = \mathrm{Sin}^{-1}\left(\frac{2p}{1+p^2}\right) - \mathrm{Cos}^{-1}\left(\frac{1-q^2}{1+q^2}\right)$$

$$\Rightarrow \mathrm{Tan}^{-1}(\tan 2\theta) = \mathrm{Sin}^{-1}(\sin 2\alpha) - \mathrm{Cos}^{-1}(\cos 2\beta)$$

$$\Rightarrow 2\theta = 2\alpha - 2\beta \Rightarrow \theta = \alpha - \beta$$

$\Rightarrow \tan\theta = \tan(\alpha-\beta) \Rightarrow x = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{p-q}{1+pq}$

IP4: Find x when $sin[2\,Cos^{-1}\{cot(2\,Tan^{-1}x)\}] = 0$

Solution: Given, $sin[2\,Cos^{-1}\{cot(2\,Tan^{-1}x)\}] = 0$

$\Rightarrow sin\left[2\,Cos^{-1}\left\{cot\left(Tan^{-1}\left(\frac{2x}{1-x^2}\right)\right)\right\}\right] = 0$

$\Rightarrow sin\left[2\,Cos^{-1}\left\{cot\left(Cot^{-1}\left(\frac{1-x^2}{2x}\right)\right)\right\}\right] = 0$

$\Rightarrow sin\left[2\,Cos^{-1}\left\{\frac{1-x^2}{2x}\right\}\right] = 0$

$\Rightarrow sin\left[Sin^{-1}\left\{2\left(\frac{1-x^2}{2x}\right)\sqrt{1-\left(\frac{1-x^2}{2x}\right)^2}\right\}\right] = 0$

$\Rightarrow sin\left[Sin^{-1}\left\{\left(\frac{1-x^2}{x}\right)\sqrt{\frac{4x^2-(1-x^2)^2}{4x^2}}\right\}\right] = 0$

$\Rightarrow \left(\frac{1-x^2}{2x^2}\right)\sqrt{4x^2-1-x^4+2x^2} = 0$

$\Rightarrow 1-x^2 = 0 \; or \; \sqrt{-x^4+6x^2-1} = 0$

$\Rightarrow 1-x^2 = 0 \; or \; x^4-6x^2+1 = 0$

If $1-x^2 = 0 \Rightarrow x = \pm 1$

If $x^4-6x^2+1 = 0$

$\Rightarrow x^2 = \frac{6\pm\sqrt{36-4}}{2} = \frac{6\pm\sqrt{32}}{2} = \frac{6\pm 2\sqrt{8}}{2} = 3 \pm 2\sqrt{2}$

$\Rightarrow x = \pm\sqrt{3\pm 2\sqrt{2}} = 1\pm\sqrt{2}, -1\pm\sqrt{2}$

$\therefore\; x = \pm 1, 1\pm\sqrt{2}, -1\pm\sqrt{2}$

P4: $\mathbf{cos[Tan^{-1}\{sin(Cot^{-1}x)\}]} =$

A. $\sqrt{\frac{x^2+2}{x^2+1}}$

B. $\sqrt{\frac{x^2+1}{x^2+2}}$

C. $\sqrt{\frac{1-x^2}{x^2+2}}$

D. $\sqrt{\frac{2-x^2}{x^2+1}}$ **Answer:** B

Solution: Let $\mathrm{Cot}^{-1}x = \theta$. Then $\cot\theta = x$ and $0 < \theta < \pi$.

$\sin(\mathrm{Cot}^{-1}x) = \sin\theta = \frac{1}{\csc\theta} = \frac{1}{\sqrt{1+\cot^2\theta}} = \frac{1}{\sqrt{1+x^2}}$ (Since $0 < \theta < \pi$

Now $\mathrm{Tan}^{-1}(\sin(\mathrm{Cot}^{-1}x)) = \mathrm{Tan}^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \alpha$ (say)

Then $\tan\alpha = \frac{1}{\sqrt{1+x^2}}$ and $0 < \alpha < \frac{\pi}{2}$

$\cos[\mathrm{Tan}^{-1}\{\sin(\mathrm{Cot}^{-1} x)\}] = \cos\alpha = \frac{1}{\sec\alpha} = \frac{1}{\sqrt{1+\tan^2\alpha}}$ (Since $0 < \alpha < \frac{\pi}{2}$)

$$= \frac{1}{\sqrt{1+\frac{1}{1+x^2}}} = \sqrt{\frac{x^2+1}{x^2+2}}$$

Exercise:

Prove that

1. $\sin^{-1}(-x) = -\sin^{-1} x$
2. $\cos^{-1}(-x) = \pi - \cos^{-1} x$
3. $\tan^{-1}(-x) = -\tan^{-1} x$
4. $\cot^{-1}(-x) = \pi - \cot^{-1} x$
5. $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$
6. $\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$
7. $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$
8. $\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right),\ x > 0$

   $\cot^{-1} x = \pi + \tan^{-1}\left(\frac{1}{x}\right),\ x < 0$
9. $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$ for $0 \le x \le 1$

   $= -\cos^{-1}\sqrt{1-x^2}$ for $-1 \le x < 0$
10. $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy},\quad x > 0, y > 0$
11. $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\frac{x+y+z-xyz}{1-xy-yz-zx}$
12. $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$
13. $2\tan^{-1} x = \sin^{-1}\frac{2x}{1+x^2},\quad |x| \le 1$

14. $2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}, \quad x \geq 0$

15. $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}, \quad |x| < 1$

16. $2\sin^{-1}x = \sin^{-1}\left[2x\sqrt{1-x^2}\right]$

17. $2\cos^{-1}x = \cos^{-1}\left[2x^2-1\right]$

18. $3\sin^{-1}x = \sin^{-1}\left(3x-4x^3\right)$

19. $3\cos^{-1}x = \cos^{-1}\left(4x^3-3x\right)$

20. $3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

21. $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

22. $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4}$

23. $\cos^{-1}\frac{63}{65} + 2\tan^{-1}\frac{1}{5} = \sin^{-1}\frac{3}{5}$

24. Solve the equation $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$

6.1. Cartesian Coordinates

Learning objectives:

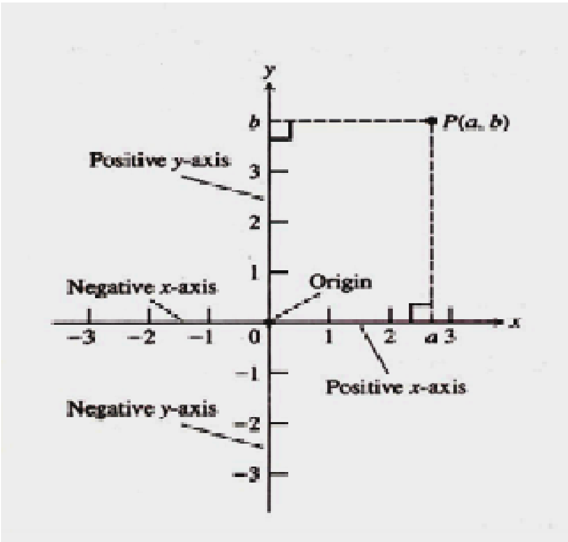
- To learn the concept of Cartesian Coordinate System and the co-ordinates of a point

- To derive a formula for the distance between two points and section formula

- To solve related problems

Cartesian Coordinates

A ***rectangular coordinate system*** (also called a ***Cartesian coordinate system***) consists of two perpendicular lines (Real lines), called ***coordinate axes***, which intersect at their origins. The intersection of the axes is called the ***origin*** of the coordinate system. It is common to call the horizontal axis the x-axis and the vertical axis the y-axis, and the plane and the axes together are referred to as the xy-plane. The position of all points in the plane can be measured with respect to these two axes. On the horizontal x-axis, (Real) numbers are denoted by x and increase to the right. On the vertical y-axis, numbers are denoted by y and increase upwards. The point where x and y are both 0 is the origin of the coordinate system, denoted by the letter O.



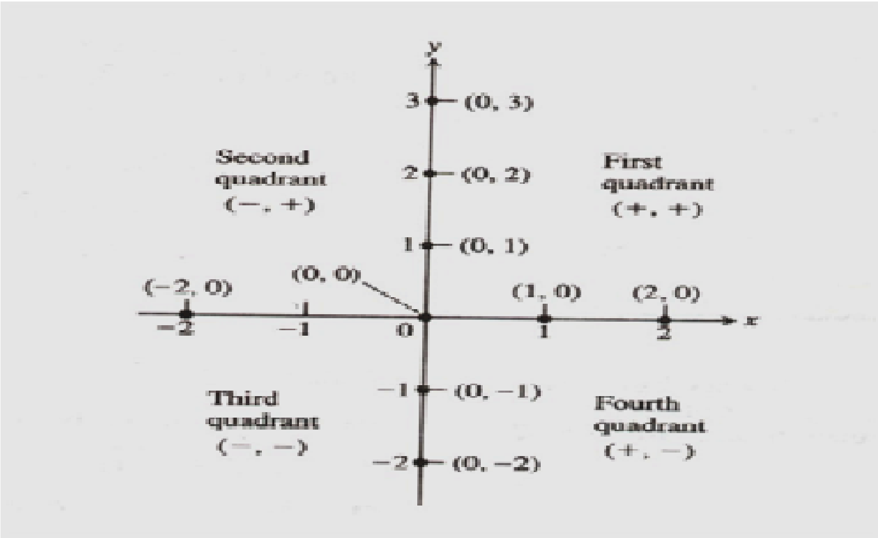
Just as points on a coordinate line can be associated with real numbers, so points in a plane can be associated with pairs of real numbers using the rectangular coordinate system.

If P is any point in the plane, we can draw lines through P perpendicular to the two coordinate axes. If the lines meet the x-axis at a and the y-axis at b, then a is the ***x*-coordinate** (also called ***abscissa***) of P, and b is the ***y*-coordinate (**also called $\boldsymbol{ordinate}$). The *ordered pair* (a, b) is the point's **coordinate pair** or **the coordinates of the point P**.

The x-coordinate of every point on the y-axis is 0. The y-coordinate of every point on the x-axis is 0. The origin is the point $(0,0)$.

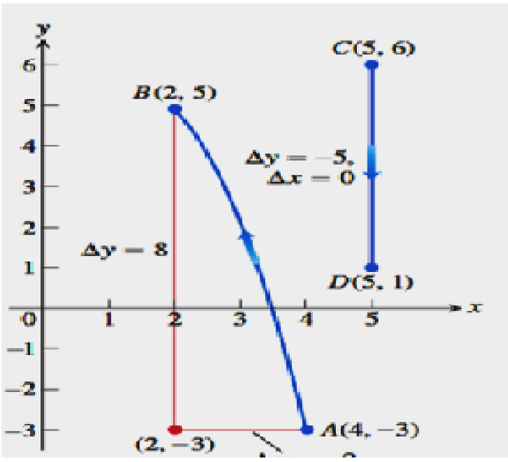
The origin divides the x-axis into the **positive x-axis** to the right and the **negative x-axis** to the left. It divides the y-axis into the **positive** and **negative y-axis** above and below.

The axes divide the plane into four regions called **quadrants**, numbered counter clockwise.



When a particle moves from one point in the plane to another, the net changes in its coordinates are called ***increments***. They are calculated by subtracting the coordinates of the starting point from the coordinates of the ending point.

An increment in a variable is a net change in that variable. If x changes from x_1 to x_2, the increment in x is $\Delta x = x_2 - x_1$



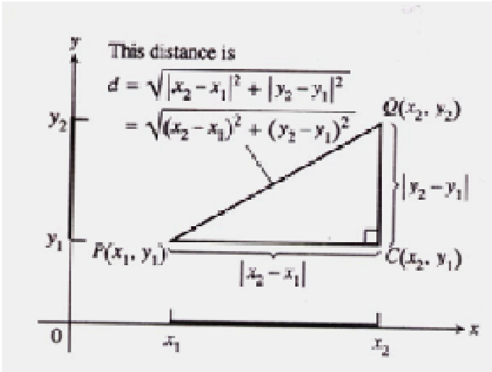
In going from the point $A\ (4,-3)$ to the point $B\ (2,5)$, the increments in the x and y coordinates are

$$\Delta x = 2-4 = -2\ ,\ \Delta y = 5-(-3) = 8$$

From $C\ (5,6)$ to $D\ (5,1)$, the coordinate increments are
$$\Delta x = 5-5 = 0$$
$$\Delta y = 1-6 = -5$$

Distance between Two Points in the Plane

Suppose that we are interested in finding the distance *d* between two points $P\ (x_1, y_1)$, and $Q\ (x_2, y_2\ )$ in the xy −plane. If we form a right triangle with P and Q as vertices, then it follows that the sides of that triangle have lengths $|x_2 - x_1|$ and $|y_2 - y_1|$. We apply the Pythagorean Theorem to triangle PCQ.



The distance between $P\ (x_1, y_1)$, and $Q\ (x_2, y_2)$ is

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 1:

a) The distance between $P\ (-1, 2)$ and $Q\ (3, 4)$ is

$$\sqrt{(3-(-1))^2 + (4-2)^2} = \sqrt{(4)^2 + (2)^2} = \sqrt{20} = 2\sqrt{5}$$

b) The distance between the origin to $P\ (x, y)$ is

$$\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

Example 2: Find the point on the $x - axis$ equidistant from (6,3) and (5,4).

On the $x - axis$, the y −coordinate is zero.

$$(x-6)^2 + (0-3)^2 = (x-5)^2 + (0-4)^2 \Rightarrow 2x = 4 \Rightarrow x = 2$$

Hence the required point is (2,0).

Section Formula

A point C divides the straight line joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m_1 : m_2$. We derive a formula to find the coordinates of C.

From the similar triangles ACP and CBQ, we have

$$\frac{x - x_1}{x_2 - x} = \frac{m_1}{m_2}$$

$$\Rightarrow \quad m_2(x - x_1) = m_1(x_2 - x)$$

$$\Rightarrow \quad (m_1 + m_2)x = m_1x_2 + m_2x_1$$

$$\Rightarrow \quad x = \frac{m_1x_2 + m_2x_1}{(m_1 + m_2)}$$

Again from similar triangles ACP and CBQ, we have

$$\frac{y - y_1}{y_2 - y} = \frac{m_1}{m_2}$$

$$\Rightarrow \quad m_2(y - y_1) = m_1(y_2 - y)$$

$$\Rightarrow \quad (m_1 + m_2)y = m_1y_2 + m_2y_1$$

$$\Rightarrow \quad y = \frac{m_1y_2 + m_2y_1}{(m_1 + m_2)}$$

Hence *the coordinates of point C which divides the straight line joining two points* (x_1, y_1) *and* (x_2, y_2) *internally in the ratio* $m_1 : m_2$ *are*

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$

From this formula, we can easily deduce the coordinates of the midpoint of a line segment joining two points in the plane. If C is the midpoint of AB, then it will divide AB in the ratio $1:1$ ($m_1 = m_2 = 1$), then the coordinates of C are

$$x = \frac{x_1 + x_2}{2} \quad , y = \frac{y_1 + y_2}{2}$$

The coordinates of a point which divides the straight line joining two points (x_1, y_1) *and* (x_2, y_2) *externally in the ratio* $m_1 : m_2$ *are* $\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right)$

Example 3:

Find the midpoint of the line segment joining $(3, -4)$ and $(7, 2)$.

Solution

Midpoint = $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{3 + 7}{2}, \frac{-4 + 2}{2}\right) = (5, -1)$

Example 4:

Find the coordinates of the point which divides the join of $(-1,7)$ and $(4,-3)$ in the ratio $2:3$.

The coordinates of the required point are

$$x=\frac{2\times4+3\times(-1)}{2+3}=1\ ,\ y=\frac{2\times(-3)+3\times7}{2+3}=3$$

Example 5:

Find the coordinates of the points which divide

i. **internally and**
ii. **externally**

the line joining $A(-1,2)$ to $B(4,-5)$ in the ratio $2:3$.

For the internal division

$$x=\frac{2\times4+3\times(-1)}{2+3}=1\ \ ,\ y=\frac{2\times(-5)+3\times2}{2+3}=-\frac{4}{5}$$

For external division

$$x=\frac{2\times4-3\times(-1)}{2-3}=-11\ ,\ y=\frac{2\times(-5)-3\times2}{2-3}=16$$

The coordinates of point C which divides the straight line joining two points (x_1,y_1) and (x_2,y_2) internally in the ratio $m_1:m_2$ are $\left(\frac{m_1x_2+m_2x_1}{m_1+m_2},\frac{m_1y_2+m_2y_1}{m_1+m_2}\right)$

We divide the numerator and the denominator by m_2 and set $\frac{m_1}{m_2}=\lambda$. The

coordinates of point C are then given by $\left(\frac{\lambda x_2+x_1}{\lambda+1},\frac{\lambda y_2+y_1}{\lambda+1}\right)$

This form of the coordinates is called the **parametric representation** of any point P on the line segment AB.

Example 6:

Find the ratio in which $(2,7)$ divides the join of the points $(3,5)$ and $(1,9)$.

Let $\lambda:1$ be the required ratio. The coordinates of the point of division are

$$\frac{\lambda\times1+3}{\lambda+1}=2,\quad \frac{\lambda\times9+5}{\lambda+1}=7$$

$$\lambda+3=2\lambda+2\Rightarrow\lambda=1$$

The required ratio is 1: 1.

PROBLEM SET

IP1: Prove that $(12,8)$, $B(-2,6)$and $C(6,0)$ are the vertices of a right-angled triangle.

Solution:

Step 1: $A(12,8), B(-2,6)$ and $C(6,0)$

$$AB = \sqrt{(-2-12)^2 + (6-8)^2}$$
$$= \sqrt{(-14)^2 + (-2)^2}$$
$$= \sqrt{196+4} = \sqrt{200} = 10\sqrt{2}$$

Step 2:

$$BC = \sqrt{(6-(-2))^2 + (0-6)^2}$$
$$= \sqrt{64+36} = \sqrt{100} = 10$$

Step 3:

$$CA = \sqrt{(6-12)^2 + (0-8)^2}$$
$$= \sqrt{36+64}$$
$$= \sqrt{100} = 10$$

Step 4:

Notice that, $AB^2 = BC^2 + CA^2$

$$200 = 100 + 100$$

Therefore ABC is a right angled triangle, right angle at C.

P1: $(5,-2)$, $B(6,4)$and $C(7,-2)$ are the vertices of

A. A right-angled triangle
B. An isosceles triangle
C. An equilateral triangle
D. A scalene triangle

Answer: B

Solution: Given the points $(5,-2)$, $B(6,4)$and $C(7,-2)$

Distance between $AB = \sqrt{(6-5)^2 + (4-(-2))^2}$

$$= \sqrt{(6-5)^2 + (4+2)^2}$$
$$= \sqrt{1+36} = \sqrt{37}$$

Distance between $BC = \sqrt{(7-6)^2 + (-2-4)^2}$

$$= \sqrt{1+36} = \sqrt{37}$$

Distance between $CA = \sqrt{(7-5)^2 + (-2-(-2))^2}$

$$= \sqrt{(7-5)^2 + 0}$$
$$= \sqrt{4} = 2$$

Here two sides AB, BC are equal. Then it is an isosceles triangle.

IP2: Find a point on the y- axis which is equidistant from the points $(6,5)$ and $(-4,3)$.

Solution:

Step1: Let A(6, 5) and B(−4, 3), a point on the y- axis is $(0, y)$

Step2: A and B are equidistant from the point $(0, y)$

Distance from A to $(0, y) =$ distance from B to $(0, y)$

Step3:

$\sqrt{(0-6)^2+(y-5)^2} = \sqrt{(0-(-4))^2+(y-3)^2}$

$\sqrt{36+y^2-10y+25} = \sqrt{16+y^2-6y+9}$

$\Rightarrow 51+y^2-10y = 25+y^2-6y$

$\Rightarrow -10y+6y = 25-51$

$\Rightarrow -4y = -36 \Rightarrow y = 9$

Therefore the required point on y- axis is $(0,9)$

P2: Find the relation between x and y such that the point $(x, y)$is equidistant from the points $(7, 1)$ and $(3, 5)$

A. $\boldsymbol{x+y=2}$

B. $\boldsymbol{x+y=-2}$

C. $\boldsymbol{x-y=2}$

D. $\boldsymbol{x-y=-2}$ **Answer: C**

Solution: Let $A\ (7, 1)$ and $B\ (3, 5)$.

Let the point equidistant from the points A and B is(x, y).

Therefore,

The distance from A to $(x, y) =$ the distance from B to (x, y)

$\Rightarrow \sqrt{(x-7)^2+(y-1)^2} = \sqrt{(x-3)^2+(y-5)^2}$

$\Rightarrow (x-7)^2+(y-1)^2 = (x-3)^2+(y-5)^2$

$\Rightarrow x^2-14x+49+y^2-2y+1 = x^2-6x+9+y^2-10y+25$

$\Rightarrow -14x+49-2y+1 = -6x+9-10y+25$

$\Rightarrow -14x-2y+50 = -6x-10y+34$

$\Rightarrow -14x+6x-2y+10y = -50+34$

$\Rightarrow -8x+8y = -16$

$\Rightarrow x-y = 2$

IP3: Find the coordinates of the point which divides the line segment joining the points $(1, -3)$ and $(-3, 9)$ in the ratio $1 : 3$ externally.

Solution:

Step1: Let a point (x, y) divides the joining points $(1, -3)$ and $(-3, 9)$ in the ratio $1: 3$ externally.

Step2: *The coordinates of the point which divides the straight line joining two points (x_1, y_1) and (x_2, y_2) externally in the ratio $m_1 : m_2$ are*

$$\left(\frac{m_1x_2-m_2x_1}{m_1-m_2},\frac{m_1y_2-m_2y_1}{m_1-m_2}\right)$$

Step3: $(x,y)=\left(\frac{1\times(-3)-3\times(1)}{1-3},\frac{1\times 9-3\times(-3)}{1-3}\right)$

Step4: $(x,y)=(3,-9)$

P3: Find the coordinates of the point which divides the line segment joining the points $(4,-3)$ and $(8,5)$ in the ratio 3:1 internally.

Solution: Let a point (x,y) divides the line joining the points $(4,-3)$ and $(8,5)$ in the ratio $3:1$ internally

Then

The coordinates of the point which divides the straight line joining two points (x_1,y_1) and (x_2,y_2) internally in the ratio $m_1:m_2$ are

$$\left(\frac{m_1x_2+m_2x_1}{m_1+m_2},\frac{m_1y_2+m_2y_1}{m_1+m_2}\right)$$

$$(x,y)=\left(\frac{3\times 8+1\times 4}{4},\frac{3\times 5+1\times(-3)}{4}\right)$$

$$(x,y)=\left(\frac{28}{4},\frac{15-3}{4}\right)$$

$(x,y)=(7,3)$

IP4: If $(1,2),(4,y),(x,6)$ and $(3,5)$ are the vertices of a parallelogram taken in order, find $x+y$.

Solution:

STEP1: Given $A(1,2), B\ (4,y), C(x,6)$ and $D(3,5)$ are the vertices of a parallelogram taken in order.

STEP2: Diagonal of a parallelogram bisect each other, then

The midpoint of AC $=$ the midpoint of BD

$$\Rightarrow\left(\frac{1+x}{2},\frac{6+2}{2}\right)=\left(\frac{4+3}{2},\frac{y+5}{2}\right)$$

STEP3: $1+x=7\ \ ,\ y+5=8$

$\Rightarrow x=6\ \ ,y=3$

STEP4: Therefore , $\quad x+y=9$

P4: If the points $A(6,1)$, $B(8,2)$, $C(9,4)$ and $D(p,3)$ are the vertices of a parallelogram, taken in order, then value of p.

Solution: Given $A\ (6,1), B\ (8,2), C\ (9,4)$ and $D\ (p,3)$ are the vertices of a parallelogram taken in order.

We know that the diagonals of a parallelogram bisect each other. Therefore, The coordinates of the mid-point of AC = The coordinates of the mid-point of BD

$$\left(\frac{6+9}{2},\frac{1+4}{2}\right)=\left(\frac{8+p}{2},\frac{2+3}{2}\right)$$

$$\Rightarrow 8+p=15$$

$$\Rightarrow p=7$$

Exercise:

1. Find the distance between the points $P(7,5)$ and $Q(2,5)$.
2. Find the length of the line AB formed by two points $A\ (4,10)$ and $B\ (7,-6)$.
3. Find a point on the x-axis which is equidistant from the points $(5,4)$ and $(-2,3)$.
4. If the distance of $P(x,y)$ from $A\ (5,1)$ and $B\ (-1,5)$ are equal, prove that $3x=2y$.
5. In what ratio does the point $(-4,6)$ divide the line segment joining the points $A(-6,10)\ and\ B(3,-8)$.
6. Find the values of y for which the distance between the points $P\ (2,-3)$ and $Q\ (10,y)$ is 10 units.
7. Show that the points $(12,8),(-2,6)$ and $(6,0)$ are the vertices of a right angled triangle.
8. Find the vertices of a triangle, the mid points of whose sides are $(3,1),(5,6)$ and $(-3,2)$.
9. Find the lengths of the medians of a triangle whose vertices are $(3,5),(5,3)$ and $(7,7)$.
10. Find the coordinates of a point A, where AB is the diameter of a circle whose centre is $(2,-3)$ and B is $(1,4)$
11. Find the point (x,y) such that $(2,4)$ is the midpoint of the line segment connecting (x,y) and $(1,5)$.
12. If A and B are $(-2,-2)\ and\ (2,-4)$, respectively, find the Co ordinates of P such that $AP=\frac{3}{7}\ AB$ and P lies on the line segment AB.
13. Find the area of a rhombus if its vertices are $(3,0),(4,5),(-1,4)\ and\ (-2,-1)$ taken in order. [Hint: Area of a rhombus = $\frac{1}{2}$ (product of its diagonals)]
14. By calculating the lengths of its sides, show that the triangle with vertices at the points $A(1,2),B\ (5,5)$, and $C(4,-2)$ is isosceles but not equilateral.
15. If the points $A(2,-1)$,$B\ (1,3)$, $C(-3,2)$ and $D(x,y)$are vertices of a square, and find the fourth vertex D.

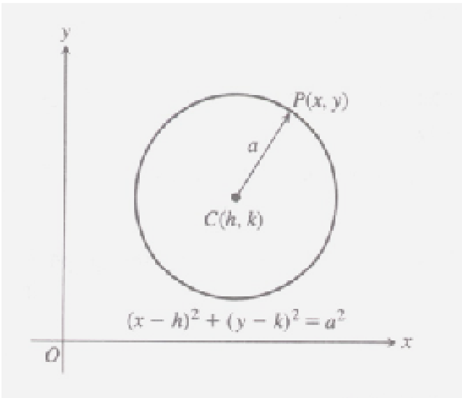
6.2. Locus

Learning Objectives:

1. To define locus of a point subject to a given set of geometric conditions and to find the equation of the locus.
2. To discuss the standard and general equations of a circle.
   And
3. To practice the related problems.

Locus of a point is the curve along which the particle moves fulfilling a given set of geometric conditions.

For example C is a given point in the plane of the paper and that a point moves so that its distance from C is constant and equal to *a*. The locus of the point P is the circle with center at C and radius *a*.



The radius *a* of the circle is equal to the distance from $C(h,k)$ to the point $P(x,y)$ on the circle which is given by $\sqrt{(x-h)^2+(y-k)^2}$.

Thus, $\sqrt{(x-h)^2+(y-k)^2} = a$. Squaring both sides,

$$(\boldsymbol{x}-\boldsymbol{h})^2+(\boldsymbol{y}-\boldsymbol{k})^2=\boldsymbol{a}^2$$

This is the algebraic statement of the given geometrical condition. When any point P satisfies the given geometrical condition, its coordinates will satisfy the algebraic relation. The algebraic relation which is satisfied for all positions of the moving point is called the ***equation of locus*** described by the moving point.

Equation of the Locus

Locus, in Latin, means *location*. The plural of locus is *loci*. **The locus of a point, is the *set of points*, and only those points, that satisfy the specified conditions**.

To determine the locus:

1. *State the condition to be satisfied.*
2. *Find several points satisfying the condition.*
3. *Sketch the locus by connecting the several points.*

Definition: The equation of the locus is a relation between x and y which is satisfied by the coordinates of all points of the locus and by no others.

The *locus of a point* is simply the *graph of the equation* that expresses geometrical condition algebraically.

Degenerate Loci

The locus of an equation $E(x,y)=0$ is called ***degenerate*** if $E(x,y)$ is the product of two or more factors $G(x,y), H(x,y)$, …. The locus of $E(x,y)=0$ then consists of the loci of $G(x,y)=0, H(x,y)=0$, ….

Example:

a) The locus $xy+x^2=0$ consists of the lines $x=0 \text{ and } x+y=0.$

b) The locus $y^2+xy^2-xy-x^2=0$ consists of the line $x+y=0$ and the parabola $y^2-x=0.$

c) The locus $y^4+y^2-x^2-x=0$ consists of the parabolas $y^2-x=0 \text{ and } y^2+x+1=0.$

Procedure to find the Equation of the Locus

The point P when denoted by (x,y) is a general point and covers all its positions on the locus. Some times it is convenient to use some symbols such as (h,k) to denote any point on the locus and replace them by (x,y) in the end.

The procedure to determine the equation of a locus may be specified as follows:

1. *Take any point (h,k) or (x,y) or (α,β) on the locus.*
2. *Use the given condition to write a relation between h and k.*
3. *Simplify the relation.*
4. *Change (h,k) into the general coordinates (x,y).*

The equation so obtained is the required equation of the locus.

Example: Find the locus of a point which moves so that its distance from the x-axis is twice the distance from the y-axis.

Solution: Let (h,k) be any point on the locus. Then by the given condition,

$$|k|=2|h|$$
$$k=\pm 2h$$

The locus is $y=\pm 2x$

Example: Find the equation of the set of points equidistant from A(−1, −1) and B(4,2).

Solution:

Let P(x,y) be a general point on the locus. By the given condition, we have $PA=PB$

$$\sqrt{(x+1)^2+(y+1)^2}=\sqrt{(x-4)^2+(y-2)^2}$$
$$(x+1)^2+(y+1)^2=(x-4)^2+(y-2)^2$$
$$x^2+2x+1+y^2+2y+1=x^2-8x+16+y^2-4y+4$$

The required equation is $5x+3y-9=0$.

Example:

The sum of the squares of the distances of a moving point from the two fixed points $A(a,0)$ and $B(-a,0)$ is equal to a constant quantity $2c^2$. Find the equation to its locus.

Solution: Let $P(x,y)$ be any position of the moving point.

The given condition yields $PA^2 + PB^2 = 2c^2$

$$\{(x-a)^2 + (y-0)^2\} + \{(x+a)^2 + (y-0)^2\} = 2c^2$$

$$x^2 - 2ax + a^2 + y^2 + x^2 + 2ax + a^2 + y^2 = 2c^2$$

$$2x^2 + 2y^2 + 2a^2 = 2c^2$$

$$x^2 + y^2 = c^2 - a^2$$

This is the equation to the required locus.

This equation tells us that the square of the distance of the point (x,y) from the origin is constant and equal to $c^2 - a^2$, and therefore the locus of the point is a circle whose center is at the origin.

Example:

A point moves so that its distance from the point (−1,0) is always three times its distance from the point (0,2). Find the equation to its locus.

Solution: Let (x,y) be any point satisfying the given condition. We then have

$$The\ distance\ of\ (x,y)\ from\ (-1,0) = 3(The\ distance\ of\ (x,y)\ from\ (0,2))$$

$$\sqrt{(x+1)^2 + (y-0)^2} = 3\sqrt{(x-0)^2 + (y-2)^2}$$

On squaring both sides , we get

$$(x+1)^2 + (y-0)^2 = 9((x-0)^2 + (y-2)^2)$$

$$x^2 + 2x + 1 + y^2 = 9(x^2 + y^2 - 4y + 4)$$

$$x^2 + 2\,x + 1 + y^2 = 9x^2 + 9y^2 - 36y + 36$$

$$9x^2 + 9y^2 - 36y + 36 - x^2 - 2x - 1 - y^2 = 0$$

$$8x^2 + 8y^2 - 2x - 36y + 35 = 0$$

This is the required equation. This equation represents a circle.

Standard Form of the Equation of a Circle

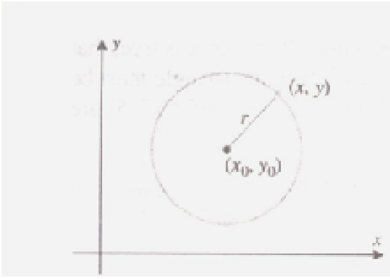
The circle of radius *a* centered at the origin (0, 0) is the set of all points in the plane whose distance from the origin is *a*. The radius *a* of the circle is equal to the distance from the origin to any point on the circle which is given by $\sqrt{x^2 + y^2}$ Thus, $\sqrt{x^2 + y^2} = a$

$$\boldsymbol{x^2 + y^2 = a^2}$$

The circle is the graph of the equation $x^2 + y^2 = a^2$

Points (x, y) whose coordinates satisfy the inequality $x^2 + y^2 \leq a^2$ all have distance less than or equal to *a* form the origin. The graph of the inequality is therefore the circle of radius *a* centered at the origin together with its interior.

If (x_0, y_0) is a fixed point in the plane, then the circle of radius *r* centered at (x_0, y_0) is the set of all points in the plane whose distance from (x_0, y_0) is *r*.



Thus a point (x, y) will lie on the circle if and only if

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = r$$

$$\boldsymbol{(x - x_0)^2 + (y - y_0)^2 = r^2}$$

This is called the ***standard form of the equation of a circle***.

Example: Find an equation for the circle of radius 4 centered at (-5, 3).

Solution: $$\boldsymbol{(x - (-5))^2 + (y - 3)^2 = 4^2}$$

$$(x + 5)^2 + (y - 3)^2 = 16$$

If this equation is written in an expanded form by squaring the terms and then simplifying, we obtain

$$x^2 + y^2 + 10x - 6y + 18 = 0$$

Example: Find an equation for the circle with center (1, -2) that passes through (4, 2).

Solution :

$$\boldsymbol{a = \sqrt{(4 - 1)^2 + (2 - (-2))^2} = \sqrt{9 + 16} = \sqrt{25} = 5}$$

The equation of the required circle is

$$(x - 1)^2 + (y + 2)^2 = 25$$

When expanded, the equation becomes

$$x^2 + y^2 - 2x + 4y - 20 = 0$$

The circle $x^2 + y^2 = 1$, which is centered at the origin and has radius 1, is of special importance; it is called the ***unit circle***.

General Form of the Equation of a Circle

As seen from the examples above, we see that the equation of circle has the general form $x^2 + y^2 + 2gx + 2fy + c = 0$

where *g, f, c* are constants. If the equation of a circle is given in the general form, then the center and radius can be found by first writing the equation in standard form, then reading off the center and radius from that equation. The following example shows how to do this using the technique of *completing the square*.

Example: Find the center and radius of the circle with the equation

(a) $x^2 + y^2 - 8x + 2y + 8 = 0$

(b) $2x^2 + 2y^2 + 24x - 81 = 0$

Solution

(a)

$(x^2 - 8x) + (y^2 + 2y) = -8$

$(x^2 - 8x + 16) + (y^2 + 2y + 1) = -8 + 16 + 1$

$(x - 4)^2 + (y + 1)^2 = 9$

Center: $(4, -1)$, radius $= 3$

(b)

$$x^2 + y^2 + 12x - \frac{81}{2} = 0$$

$$(x^2 + 12x) + y^2 = \frac{81}{2}$$

$$(x^2 + 12x + 36) + y^2 = \frac{81}{2} + 36$$

$$(x + 6)^2 + y^2 = \frac{153}{2}$$

The circle has center $(-6, 0)$ and radius $\sqrt{\frac{153}{2}}$

Degenerate Cases of a Circle

The equation in the general form may not represent a circle in all the cases. Suppose that the given general form is changed into the standard form

$(x - x_0)^2 + (y - y_0)^2 = k$

Depending on the value of *k*, the following situations occur:

$\boldsymbol{k} > 0$**:**

The graph is a circle with center(x_0, y_0) and radius $\sqrt{k}$.

$\boldsymbol{k = 0}$**:**

The only solution of the equation is $x = x_0, y = y_0$, so the graph is the single point (x_0, y_0).

$\boldsymbol{k} < 0$**:**

The equation has no real solutions and consequently no graph.

PROBLEM SET

IP1. $\boldsymbol{A(2,0)}$ **and** $\boldsymbol{B(4,0)}$ **are two given points. A point** $\boldsymbol{P}$ **moves so that**
$\boldsymbol{PA^2 + PB^2 = 10}$**. Find the locus of** $\boldsymbol{P}$**.**

Solution: Let P (x , y) be a given point on the locus.

Given, $(2,0)$, $B(4,0)$ and $PA^2 + PB^2 = 10$.

$i.e., \left(\sqrt{(x-2)^2+y^2}\right)^2 + \left(\sqrt{(x-4)^2+y^2}\right)^2 = 10$

$\Rightarrow (x-2)^2 + y^2 + (x-4)^2 + y^2 = 10$

$\Rightarrow x^2 - 4x + 4 + y^2 + x^2 - 8x + 16 + y^2 = 10$

$\Rightarrow 2x^2 + 2y^2 - 12x + 20 - 10 = 0$

$\Rightarrow 2x^2 + 2y^2 - 12x + 10 = 0$

$i.e., \quad x^2 + y^2 - 6x + 5 = 0$

The required locus is $x^2 + y^2 - 6x + 5 = 0$ and it is a circle.

P1. Find the locus of the point equidistant from (−1, 4) and (3,2).

Solution:

Let P(x , y)be a general point on the locus.

Given that P is equidistant from $A(-1,4)$ and $B(3,2)$.

By the given condition, $PA^2 = PB^2$

$\Rightarrow (x+1)^2 + (y-4)^2 = (x-3)^2 + (y-2)^2$

$\Rightarrow x^2 + 2x + 1 + y^2 - 8y + 16 = x^2 - 6x + 9 + y^2 - 4y + 4$

$\Rightarrow 2x + 1 - 8y + 16 = -6x + 9 - 4y + 4$

$\Rightarrow 2x + 6x - 8y + 4y = 9 + 4 - 1 - 16$

$\Rightarrow 8\mathrm{x} - 4y = -4 \Rightarrow 2\mathrm{x} - y = -1 \Rightarrow 2\mathrm{x} - y + 1 = 0$

The required locus is $8x - 4y + 4 = 0$ $i.e., 2x - y + 1 = 0$ and it is a straight line.

IP2. A point moves so that the sum of its distances from $\boldsymbol{(4,0)}$ **and** $\boldsymbol{(-4,0)}$ **is** $\boldsymbol{4}$**. Prove that the equation to its locus is** $\boldsymbol{\frac{x^2}{4} - \frac{y^2}{12} = 1}$**,**

Solution: Given, $A(4,0)$ and $B(-4,0)$

Let $P(x,y)$be any point on the locus. Given $PA + PB = 4$

$\sqrt{(x-4)^2+y^2} + \sqrt{(x+4)^2+y^2} = 4$

$\Rightarrow \sqrt{(x-4)^2+y^2} = 4 - \sqrt{(x+4)^2+y^2}$

Squaring both sides

$\Rightarrow (x-4)^2 + y^2 = 16 + (x+4)^2 + y^2 - 8\sqrt{(x+4)^2+y^2}$

$\Rightarrow (x-4)^2 - (x+4)^2 - 16 = -8\sqrt{(x+4)^2+y^2}$

$\Rightarrow 2x + 2 = \sqrt{(x+4)^2+y^2}$

$\Rightarrow 4x^2 + 4 + 8x = x^2 + 16 + 8x + y^2$

$\Rightarrow 3x^2 - y^2 = 12 \Rightarrow \frac{x^2}{4} - \frac{y^2}{12} = 1$; The required locus is $\frac{x^2}{4} - \frac{y^2}{12} = 1$.

P2. A point moves so that the sum of its distances from $(1,0)$ and $(-1,0)$ is 4. Prove that the equation to its locus is $\frac{x^2}{4}+\frac{y^2}{3}=1$.

Solution: Given, $A(1,0)$ and $B(-1,0)$

Let $P(x,y)$be any point on the locus. Given $PA+PB=4$

$\sqrt{(x-1)^2+y^2}+\sqrt{(x+1)^2+y^2}=4$

$\Rightarrow \sqrt{(x-1)^2+y^2}=4-\sqrt{(x+1)^2+y^2}$

Squaring both sides

$\Rightarrow \left(\sqrt{(x-1)^2+y^2}\right)^2=\left(4-\sqrt{(x+1)^2+y^2}\right)^2$

$\Rightarrow (x-1)^2+y^2=16+(x+1)^2+y^2-8\sqrt{(x+1)^2+y^2}$

$\Rightarrow (x-1)^2-(x+1)^2-16=-8\sqrt{(x+1)^2+y^2}$

$\Rightarrow x^2-2x+1-x^2-2x-1-16=-8\sqrt{(x+1)^2+y^2}$

$\Rightarrow -4x-16=-8\sqrt{(x+1)^2+y^2}$

$\Rightarrow 4x+16=8\sqrt{(x+1)^2+y^2}$ $\Rightarrow 4(x+4)=8\sqrt{(x+1)^2+y^2}$

$\Rightarrow x+4=2\sqrt{(x+1)^2+y^2}$

Squaring on bothside

$\Rightarrow (x+4)^2=2^2\left(\sqrt{(x+1)^2+y^2}\right)^2$

$\Rightarrow x^2+16+8x=4((x+1)^2+y^2)=4(x^2+2x+1+y^2)$

$\Rightarrow x^2+16+8x=4x^2+4+8x+4y^2$

$\Rightarrow 4x^2+4y^2-x^2=16-4$

$\Rightarrow 3x^2+4y^2=12 \Rightarrow \frac{3x^2}{12}+\frac{4y^2}{12}=\frac{12}{12}$

$\Rightarrow \frac{x^2}{4}+\frac{y^2}{3}=1$,The required locus is $\frac{x^2}{4}+\frac{y^2}{3}=1$.

IP3. Describe the graphs of

a) $\boldsymbol{x^2+y^2-4x-6y+12=0}$
b) $\boldsymbol{x^2+y^2-4x-6y+13=0}$
c) $\boldsymbol{x^2+y^2-4x-6y+14=0}$

Solution:

a) Given, $x^2+y^2-4x-6y+12=0$

$\Rightarrow (x-2)^2+(y-3)^2-4-9+12=0$

$\Rightarrow (x-2)^2+(y-3)^2=1$

It is a circle with centre $(2,3)$ and radius is 1.

b) Given, $x^2+y^2-4x-6y+13=0$

$\Rightarrow (x-2)^2+(y-3)^2-4-9+13=0$

$\Rightarrow (x-2)^2+(y-3)^2=0$

The graph is the single point $(2,3)$.

c) Given, $x^2 + y^2 - 4x - 6y + 14 = 0$

$\Longrightarrow (x-2)^2 + (y-3)^2 - 4 - 9 + 14 = 0$

$\Longrightarrow (x-2)^2 + (y-3)^2 = -1$

The equation has no real solutions so there is no graph.

The equation of the required circle is $x^2 + y^2 - 4x - 6y = 0$

P3. Describe the graphs of

d) $\boldsymbol{x^2 + y^2 + 8x - 6y + 21 = 0}$

e) $\boldsymbol{x^2 + y^2 + 8x - 6y + 25 = 0}$

f) $\boldsymbol{x^2 + y^2 + 8x - 6y + 26 = 0}$

Solution:

d) Given, $x^2 + y^2 + 8x - 6y + 21 = 0$

$\Longrightarrow (x+4)^2 + (y-3)^2 - 16 - 9 + 21 = 0$

$\Longrightarrow (x+4)^2 + (y-3)^2 = 2^2$

It is a circle with centre $(-4,3)$ and radius is 2.

e) Given, $x^2 + y^2 + 8x - 6y + 25 = 0$

$\Longrightarrow (x+4)^2 + (y-3)^2 - 16 - 9 + 25 = 0$

$\Longrightarrow (x+4)^2 + (y-3)^2 = 0$

The graph is the single point $(-4,3)$.

f) Given, $x^2 + y^2 + 8x - 6y + 26 = 0$

$\Longrightarrow (x+4)^2 + (y-3)^2 - 16 - 9 + 26 = 0$

$\Longrightarrow (x+4)^2 + (y-3)^2 = -1$

The equation has no real solutions so there is no graph.

IP4.

Find the standard equation of the circle having centre $(\mathbf{3}, \mathbf{2})$ and one of the end points of a diameter is $(\mathbf{5}, \mathbf{1})$.

Solution:

Given centre of the circle is $(3,2)$ and one of the end points of the diameter is $(5,1)$.

Let (p, q) is another end point of the diameter.

Centre of the circle = midpoint of the diameter.

$(3,2) = \left(\frac{p+5}{2}, \frac{q+1}{2}\right) \Longrightarrow p = 1 \text{ and } q = 3.$

Diameter of the circle $= \sqrt{4^2 + 2^2} = 2\sqrt{5}$

Radius of the circle $= \frac{\text{diameter}}{2} = \sqrt{5}$

Standard equation of a circle with centre $(3,2)$ and radius $\sqrt{5}$ is

$(x-3)^2 + (y-2)^2 = 5$

P4. Find the standard equation of the circle having as a diameter the segment joining $(2,-3)$ and $(6,5)$.

Solution: Given, $(2,-3)$ and $(6,5)$ are joining end points of the segment as a diameter.

Diameter = $\sqrt{(6-2)^2+(5+3)^2} = \sqrt{16+64} = \sqrt{80} = 4\sqrt{5}$

Then the radius of the circle $= \frac{\text{diameter}}{2} = 2\sqrt{5}$.

Center of the circle= midpoint of the line segment joining $(2,-3)$and $(6,5)$

$\Rightarrow \left(\frac{2+6}{2},\frac{5-3}{2}\right) = (4,1)$

Standard form of the equation of the required circle is

$(x-4)^2+(y-1)^2 = 20$

Exercises

1. Find the locus of a point which is equidistant from the points $(1,0)$ and $(-1,0)$.
2. Find the locus of a point which moves so that the sum of the squares of its distances from the points $(2,4)$ and $(-3,-1)$ is 30.
3. Two points A and B with coordinates $(5,3),(3,-2)$ are given. A point P moves so that the area of triangle PAB is constant and equal to 9 square units. Find the equation to the locus of the point P.
4. Find the center and radius of the following circles:
   a. $x^2+y^2+14x+2y+34=0$
   b. $x^2+y^2=25$
   c. $x^2+y^2-9x+16=0$
   d. $x^2+y^2-4x-10y+20=0$
5. Find the standard equation of the circle passing through the points $P(3,8), Q(9,6)$, and $R(13,-2)$.
6. Describe the graphs of
   a. $2x^2+2y^2-4x+y+1=0$
   b. $x^2+y^2-4y+7=0$
   c. $x^2+y^2-6x-2y+10=0$
7. For what values of *k* does the circle $(x+2k)^2+(y-3k)^2=10$ Passes through the Point $(1,0)$?
8. Find the standard equation of every circle that passes through the origin, has radius 5, and is such that the y-coordinate of its center is -4.

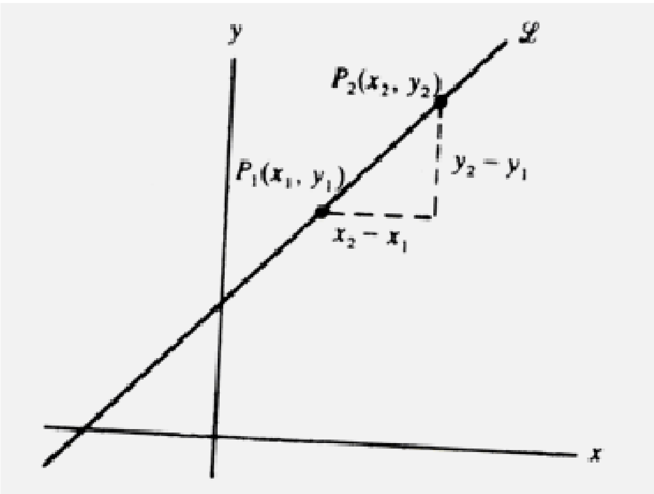
6.3. Slope of a Line

Learning objectives:

- To define slope of a line
- To derive the relationship between the slopes of parallel and perpendicular lines
- To derive a formula for the angle between two given lines in terms of their slopes And
- To solve related problems.

Slope of a Line

Given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the plane, we call the increments $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ the ***run*** and the ***rise***, respectively, between P_1 and P_2. Two such points always determine a unique straight line passing through them. We call the line P_1P_2.

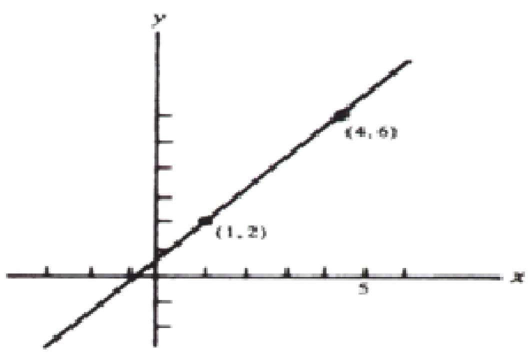


The constant

$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

is the **slope** of the *non-vertical line P_1P_2*.

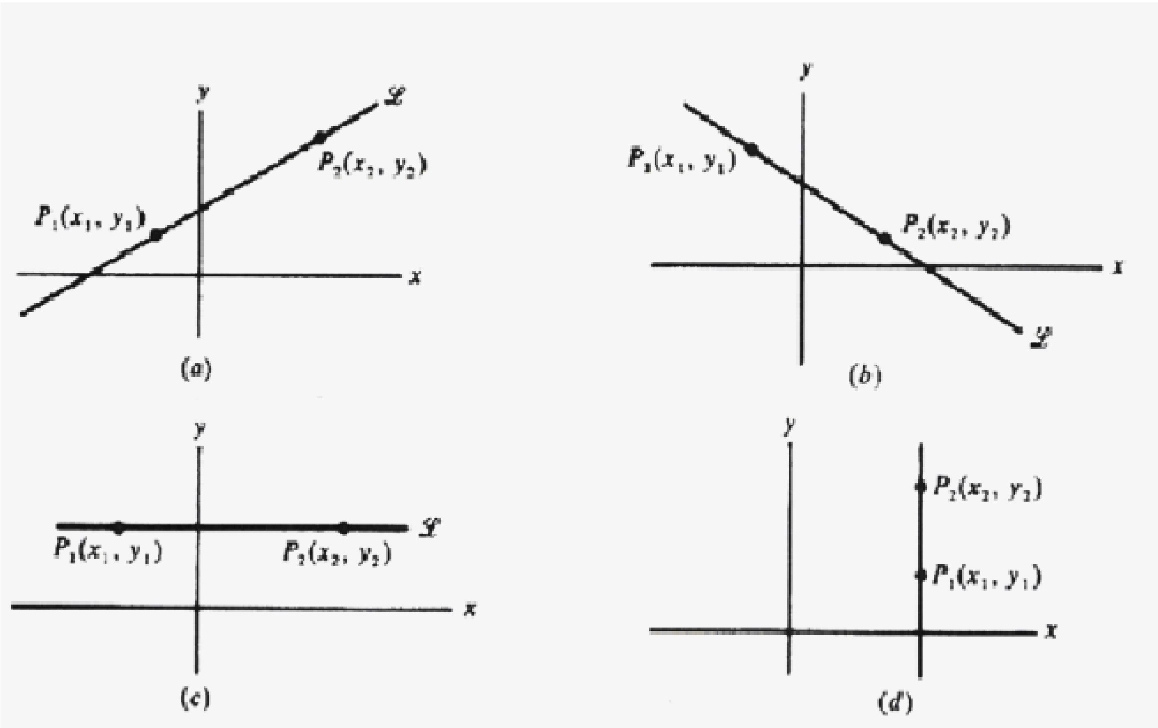
Example 1: The slope of the line joining the points $(\mathbf{1}, \mathbf{2})$ and $(\mathbf{4}, \mathbf{6})$ is $m = \frac{6-2}{4-1} = \frac{4}{3}$



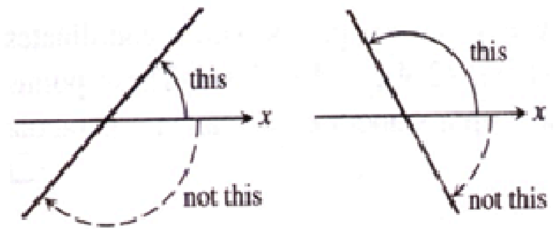
The slope tells us the direction (uphill, downhill) and steepness of a line. A line with positive slope rises uphill to the right and the line with negative slope falls downhill to the right. The greater the absolute value of the slope, the more rapid the rise or fall.

The slope of a horizontal line is zero.

The slope of a vertical line is undefined. Since the run Δx is zero for a vertical line, we cannot form the ratio m.



The direction and steepness of a line can also be measured with an angle. *The angle of inclination of a line that crosses the x-axis is the smallest counterclockwise angle from the x-axis to the line.*



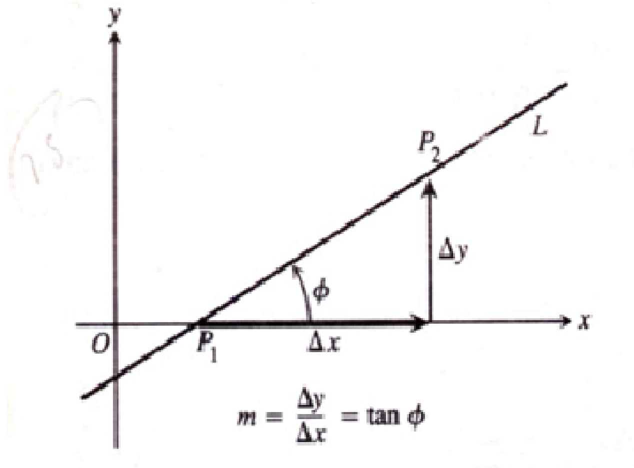
The inclination of a horizontal line is 0°.

The inclination of a vertical line is 90°.

If φ is the inclination of a line, then $0 \leq \varphi < 180^{\circ}$.

The relationship between the slope m of a non-vertical line and the line's inclination is

$m = tan\,\varphi$



Example 2:

The slope of the line joining the points $(1, 0)$ and $(5, 4)$ is given by

$$m = \frac{4-0}{5-1} = \frac{4}{4} = 1$$

The angle of inclination of the line is given by

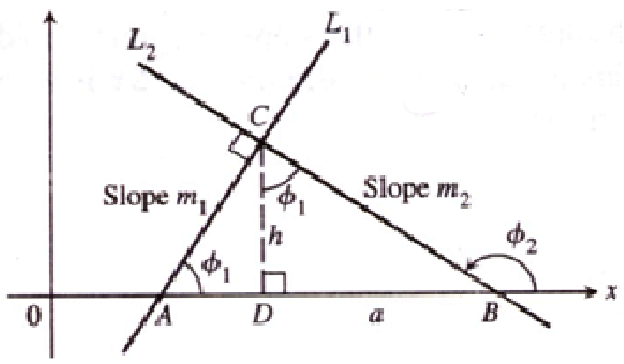
$$\varphi = \arctan m = \tan^{-1} 1 = \frac{\pi}{4}$$

Parallel and Perpendicular Lines

Lines that are parallel have equal angles of inclination, and therefore they have the same slope.

If two non-vertical lines L_1 and L_2 with slopes m_1 and m_2 are perpendicular, then $m_1 m_2 = -1$.

This is shown as follows:



$$m_2 = \tan\phi_2 = \tan(90^\circ + \phi_1) = -\cot\phi_1 = -\frac{1}{\tan\phi_1} = -\frac{1}{m_1}$$

$$m_1 = -\frac{1}{m_2}$$

So each slope is the negative reciprocal of the other and

$$m_1 m_2 = -1.$$

Example 3: Show that A$(-5, 4)$, B$(-1, -2)$,C$(5, 2)$ are the vertices of a right angle triangle.

Slope of AB $= m_1 = \dfrac{-2-4}{-1-(-5)} = -\dfrac{3}{2}$

Slope of BC $= m_2 = \dfrac{2-(-2)}{5-(-1)} = \dfrac{2}{3}$

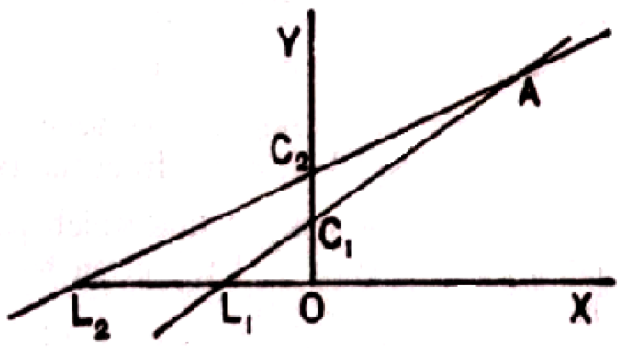
Slope of AC $= m_3 = \dfrac{2-4}{5-(-5)} = -\dfrac{2}{10} = -\dfrac{1}{5}$

Notice that , $$m_1 m_2 = -1$$

Thus, AB is perpendicular to BC. Therefore ABC is a right angled triangle with right angle at B.

Angle between two Lines

We wish to find the angle between two lines, whose inclinations to the x-axis are $\alpha_1$and α_2. That is, angle $L_1L_2A = \alpha_1$ and angle $OL_1A = \alpha_2$. Let θ be the angle between the given lines. That is, angle $L_1AL_2 = \theta$.



Then from the figure,

$\alpha_2 = \alpha_1 + \theta$ (exterior angle is equal to the sum of the angle
of the interior opposite angles in a triangle)

$\Longrightarrow\ \theta = \alpha_2 - \alpha_1 \Longrightarrow \tan\theta = \tan(\alpha_2 - \alpha_1)$

$\therefore\ \tan\theta = \frac{\tan\alpha_2 - \tan\alpha_1}{1+\tan\alpha_1\tan\alpha_2} = \frac{m_2 - m_1}{1+m_1m_2}$

If ϕ is the exterior angle between the lines then

$\tan\phi = \tan(\pi - \theta) = -\tan\theta = -\frac{m_2 - m_1}{1+m_1m_2}$

Hence the angle formula is $\tan\theta = \pm\frac{m_2 - m_1}{1+m_1m_2} = \pm\frac{\text{Difference of the slopes}}{1+\text{product of the slopes}}$

If two lines are parallel, then $\theta = 0$, which implies $m_1 = m_2$.

So, two lines are parallel if their slopes are equal.

If two lines are perpendicular,$i.e.\ \theta = 90°$, then $m_1m_2 = -1$. This is the condition of perpendicularity of two lines.

Example 4: Find the angle between the lines whose slopes are $m_1 = \sqrt{3}$ and $m_2 = \frac{1}{\sqrt{3}}$.

Solution: Here, $m_1 = \sqrt{3}$ and $m_2 = \frac{1}{\sqrt{3}}$.

The angle θ between the two lines is given by

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\sqrt{3} - \frac{1}{\sqrt{3}}}{1 + \sqrt{3}.\frac{1}{\sqrt{3}}} = \frac{3-1}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

The angle $\theta = 30°$, and the supplementary angle $= 180 - 30 = 150°$.

Example 5: Two lines passing through the point $(2,3)$ make an angle of $45°$. If the slope of one of the lines is 2,then find the slope of the other.

Solution: Slope of the given line $m_1 = 2$; let the slope of other line be m.

Angle between these two lines $= 45°$.

$$\tan 45° = \left|\frac{2 - m}{1 + 2m}\right|$$

$$\Longrightarrow \quad \frac{2-m}{1+2m} = \pm 1$$

Now , $1 + 2m = 2 - m \implies 3m = 1 \implies m = \frac{1}{3}$

$$\frac{2-m}{1+2m} = -1 \implies -1 - 2m = 2 - m \implies m = -3$$

Hence the slope of the second line is $\frac{1}{3}$ $or - 3$.

PROBLEM SET

IP1: If the Line through the points $(3, -2)$ and $(-1, 4)$ is perpendicular to the line through the points $(3, 4)$and $(-5, y)$, then find the value of y.

Solution:

Step1: Let L_1be the line passing through $(3, -2)$ and $(-1,4)$

We know that the slope(m) of a line passing through $(x_1, y_1), (x_2, y_2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope of $L_1 = m_1 = \frac{4+2}{-1-3} = \frac{-3}{2}$

Let L_2 be the line passing through $(3,4)$ and $(-5, y)$

Slope of $L_2 = m_2 = \frac{y-4}{-5-3} = \frac{y-4}{-8}$

Step2: By the hypothesis, $L_1 \perp L_2$

$\Longrightarrow \quad m_1 m_2 = -1$

$\Longrightarrow \quad \frac{-3}{2}\left(\frac{y-4}{-8}\right) = -1 \implies 3(y-4) = -16 \implies y = -\frac{4}{3}$

P1: If the Line through the points $(-2, 6)$ and $(4, 8)$ is perpendicular to the line through the points $(8, 12)$ and $(x, 24)$, then find the value of x.

A. 1

B. 2

C. 3

D. 4 Answer: D

Solution: Let $\boldsymbol{L_1}$ be the line passing through $(-\mathbf{2}, \mathbf{6})$ and $(\mathbf{4}, \mathbf{8})$

The slope$(\boldsymbol{m})$ of a line passing through $(\boldsymbol{x_1}, \boldsymbol{y_1}), (\boldsymbol{x_2}, \boldsymbol{y_2})$ is

$$\boldsymbol{m} = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope of $\boldsymbol{L_1} = \boldsymbol{m_1} = \frac{\mathbf{8-6}}{\mathbf{4+2}} = \frac{\mathbf{1}}{\mathbf{3}}$

Let $\boldsymbol{L_2}$ be the line passing through $(\mathbf{8}, \mathbf{12})$ and $(\boldsymbol{x}, \mathbf{24})$

Slope of $\boldsymbol{L_2} = \boldsymbol{m_2} = \frac{\mathbf{24-12}}{\boldsymbol{x}\mathbf{-8}} = \frac{\mathbf{12}}{\boldsymbol{x}\mathbf{-8}}$

By the hypothesis, $\boldsymbol{L_1} \perp \boldsymbol{L_2}$

$\Rightarrow \quad m_1 m_2 = -1$

$\Rightarrow \quad \frac{1}{3}\left(\frac{12}{x-8}\right) = -1 \quad \Rightarrow 12 = -(3x - 24) \quad \Rightarrow x = 4$

IP2: If the Line through the points $(4, -7)$ and $(1, -5)$ is parallal to the line through the points $(2, 4)$and $(x, 9)$, then find the value of x.

Solution:

Step1: Let $\boldsymbol{L_1}$ be the line passing through $(\mathbf{4}, -\mathbf{7})$ and $(\mathbf{1}, -\mathbf{5})$

We know that the slope$(\boldsymbol{m})$ of a line passing through $(\boldsymbol{x_1}, \boldsymbol{y_1}), (\boldsymbol{x_2}, \boldsymbol{y_2})$ is

$$\boldsymbol{m} = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope of $\boldsymbol{L_1} = \boldsymbol{m_1} = \frac{\mathbf{-5+7}}{\mathbf{1-4}} = \frac{\mathbf{-2}}{\mathbf{3}}$

Let $\boldsymbol{L_2}$be the line passing through $(\mathbf{2}, \mathbf{4})$and $(\boldsymbol{x}, \mathbf{9})$

Slope of $\boldsymbol{L_2} = \boldsymbol{m_2} = \frac{\mathbf{9-4}}{\boldsymbol{x}\mathbf{-2}} = \frac{\mathbf{5}}{\boldsymbol{x}\mathbf{-2}}$

Step2: By the hypothesis, $\boldsymbol{L_1} \parallel \boldsymbol{L_2}$

$\Rightarrow \quad \boldsymbol{m_1} = \boldsymbol{m_2}$

$\Rightarrow \quad \frac{-2}{3} = \frac{5}{x-2} \quad \Rightarrow \quad -2x + 4 = 15 \quad \Rightarrow \quad x = -\frac{11}{2}$

P2: If the Line through the points $(5, 6)$ and $(10, 2)$ is parallal to the line through the points $(2, 5)$and $(8, y)$, then find the value of y.

E. $-\mathbf{5}$

F. $-\frac{\mathbf{1}}{\mathbf{5}}$

G. $\frac{\mathbf{1}}{\mathbf{5}}$

H. $\mathbf{5}$ Answer: C

Solution: Let $\boldsymbol{L_1}$ be the line passing through $(\mathbf{5}, \mathbf{6})$ and $(\mathbf{10}, \mathbf{2})$

We know that the slope$(\boldsymbol{m})$ of a line passing through

$(\boldsymbol{x_1}, \boldsymbol{y_1}), (\boldsymbol{x_2}, \boldsymbol{y_2})$ is

$$\boldsymbol{m} = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope of $L_1 = m_1 = \frac{2-6}{10-5} = \frac{-4}{5}$

Let L_2 be the line passing through $(2, 5)$ and $(8, y)$

Slope of $L_2 = m_2 = \frac{y-5}{8-2} = \frac{y-5}{6}$

By the hypothesis, $L_1 \parallel L_2$

$\Rightarrow \quad m_1 = m_2$

$\Rightarrow \quad \frac{-4}{5} = \frac{y-5}{6} \quad \Rightarrow \quad 5(y-5) = -24 \quad \Rightarrow \quad y = \frac{1}{5}$

IP3: If the angle between two lines is $\frac{\pi}{4}$ and the slope of one line is $\frac{3}{4}$, then find the slope of the other line.

Solution:

Step1: We know that the acute angle θ between two lines with Slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Let $m_1 = \frac{3}{4}$, $m_2 = m$ and $\theta = \frac{\pi}{4}$

Step2: Now putting these values in above formula then we get

$$\tan\frac{\pi}{4} = \left|\frac{m - \frac{3}{4}}{1 + \frac{3}{4}m}\right|$$

$$1 = \left|\frac{m - \frac{3}{4}}{1 + \frac{3}{4}m}\right|$$

Step3: $\Rightarrow \quad \frac{m - \frac{3}{4}}{1 + \frac{3}{4}m} = 1$ or $\frac{m - \frac{3}{4}}{1 + \frac{3}{4}m} = -1$

$\Rightarrow \quad m - \frac{3}{4} = 1 + \frac{3}{4}m$ or $m - \frac{3}{4} = -\left(1 + \frac{3}{4}m\right)$

$\Rightarrow \quad m = 7 \ or \ -\frac{1}{7}$

P3: If the angle between two lines is $\frac{\pi}{4}$ and the slope of one line is $\frac{1}{2}$, then find the slope of the other line.

A. $-3, \frac{1}{3}$

B. $3, -\frac{1}{3}$

C. $3, -3$

D. $\frac{1}{3}, -\frac{1}{3}$ Answer: B

Solution:

We know that the acute angle θ between two lines with slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Let $m_1 = \frac{1}{2}$, $m_2 = m$ and $\theta = \frac{\pi}{4}$

Now putting these values, we get

$$\tan\frac{\pi}{4} = \left|\frac{m-\frac{1}{2}}{1+\frac{1}{2}m}\right|$$

$$1 = \left|\frac{m-\frac{1}{2}}{1+\frac{1}{2}m}\right|$$

$$\Rightarrow \quad \frac{m-\frac{1}{2}}{1+\frac{1}{2}m} = 1 \text{ or } \frac{m-\frac{1}{2}}{1+\frac{1}{2}m} = -1$$

$$\Rightarrow \quad m-\frac{1}{2} = 1+\frac{1}{2}m \text{ or } m-\frac{1}{2} = -\left(1+\frac{1}{2}m\right)$$

$$\Rightarrow \quad m = 3 \ or -\frac{1}{3}$$

IP4: Find the angle between the diagonals of the quadrilateral formed by the points $\mathbf{A(7,8), B(1,6), C(1,0), D(5,2)}$.

Solution:

Step1: Let the slope of the diagonal through the points $A(7,8), C(1,0)$ be m_1 and

$$m_1 = \frac{0-8}{-1-7} = \frac{-8}{-8} = 1$$

Let the slope of the diagonal through the points $B(1,6), D(5,2)$ be m_2 and

$$m_2 = \frac{2-6}{5-1} = \frac{-4}{4} = -1$$

Step2: Notice that $m_1 m_2 = -1$

$\therefore$ AC $\perp$ BD

The angle between the diagonals of the quadrilateral formed by the points A, B, C and D is 90°

P4: Find the angle between the diagonals of the quadrilateral formed by the points $\mathbf{A(-2,-3), B(5,1), C(6,9), D(-1,5)}$.

Solution:

Let the slope of the diagonal through the points $A(-2,-3), C(6,9)$ be m_1 and

$$m_1 = \frac{9+3}{6+2} = \frac{3}{2}$$

Let the slope of the diagonal through the points $B(5,1), D(-1,5)$ be m_2 and

$$m_2 = \frac{5-1}{-1-5} = -\frac{2}{3}$$

Notice that, $m_1 m_2 = \frac{3}{2} \times -\frac{2}{3} = -1$

$\therefore$ AC $\perp$ BD

The angle between the diagonals of the quadrilateral formed by the points A, B, C and D is 90°

Exercise:

1. In each part find the slope of the line through:
   a. The points $(6,2)$ and $(9,8)$
   b. The points $(2,9)$ and $(4,3)$
   c. The points $(-2,7)$ and $(5,7)$
2. Use slopes to show that the points $A(1,3)$, $B(3,7)$, and $C(7,5)$ are vertices of a right triangle
3. Show that the points $(4,4),(3,5) and (-1,-1)$ are the vertices of a right angle triangle
4. Find the slope of a line whose inclination is a) $30°$ (b) $45°$ (c) $60°$ (d) $15°$ (e) $135°$.
5. Using slopes determine which of the following sets of three points are collinear?
   a. $(5,-2),(7,6).(0,-2)$
   b. $(-2,3),(8,-5),(5,4)$
   c. $(6,-1),(5,0),(2,3)$
   d. $(1-5),(3,1),(5,7)$
6. Find the angle between the lines whose slopes are (a) 2 and -1; (b) 2 and $\frac{3}{4}$.
7. A line pass through the $(x,y)$and (h,k) If slope of the line is m, show that $(k-y)=m(h-x)$.
8. If $(\boldsymbol{h},\mathbf{0}),(\boldsymbol{a},\boldsymbol{b})\ \boldsymbol{and}\ (\mathbf{0},\boldsymbol{k})$ lie on a line show that $\frac{\boldsymbol{a}}{\boldsymbol{h}}+\frac{\boldsymbol{b}}{\boldsymbol{k}}=1$.
9. Find the angles between $\boldsymbol{x-axis}$ and the line joining the points $(\mathbf{3},-\mathbf{1}),(\mathbf{4},-\mathbf{2})$.
10. Find the slope of a line which makes an angle of $45°$ with a line of slope $-\frac{6}{5}$.
11. Find y if the slope of line joining $(-8,11),(2,y)$ is $-\frac{4}{3}$.
12. Find the slope of a line parallel to the line which passes through the pair of points $(-1,3)$ and $(4,7)$.
13. Find the slope of a line perpendicular to the line which passes through the pair of points $(0,8)$ and $(-5,2)$.
14. The line joining $(-5,7)$ and $(0,-2)$ is perpendicular to the line joining $(1,-3)$ and $(4,x)$. Find x.
15. The slope of a line is double of the slope of a another line. If tangent of the angle between them is $\frac{\mathbf{1}}{\mathbf{3}}$ find the slope of the lines.
16. Find the angle of the triangle formed by
    a) $(-\mathbf{2},\mathbf{3})(\mathbf{3},\mathbf{8})(\mathbf{4},\mathbf{1})$
    b) $(\mathbf{3},\mathbf{2})(\mathbf{11},\mathbf{8})(\mathbf{8},\mathbf{12})$
17. Find the angle between the diagonals of the quadrilateral formed by the following points $(\mathbf{9},\mathbf{2}),(\mathbf{17},\mathbf{11}),(\mathbf{5},-\mathbf{3})\boldsymbol{and}(-\mathbf{3},-\mathbf{2})$.

6.4. Various Forms of Equation of a Line

Learning objectives:

- To derive the equation of a line in various forms
- To derive the equation of a line in
  Point-slope form
  Slope-intercept form
  Two point form
  Intercept form
  And Normal form
- To derive the parametric equations of a line
  And
- To solve related problems.

Equation of a Straight Line

Suppose r is the radius of the circle and C its circumference. Both r and C are variables. The Circumference and the radius of a circle are related by the equation $C = 2\pi r$. For any given radius, there is one and only one circumference. So we say r is an independent variable and C is a dependent variable. We can also view that when C changes, r changes according to the relation $r = \frac{C}{2\pi}$. So, here we can say that C is the independent variable and r is the dependent variable.

Thus, if there is a relation between two variables x and y, then if one is regarded as the independent variable, then other will be the dependent variable.

The equation of a line is an expression showing a relation between the x-coordinate and the y-coordinate of all the points that lie on the straight line. The relation is called a function with x as independent variable and y as dependent variable.

Lines determined by Point and Slope

We can write an equation for a non-vertical straight line L if we know its slope m and the coordinates of one point $P_1(x_1, y_1)$. *If* $P(x, y)$ is any other point on L, then

$$\frac{y - y_1}{x - x_1} = m$$

$$y - y_1 = m(x - x_1)$$

The equation $y = y_1 + m(x - x_1)$

is the **point-slope equation** of the line that passes through the point (x_1, y_1) and has slope m.

Example 1: Write an equation for the line through the point $(2,3)$ with slope $-\frac{3}{2}$.

Solution:

$$y = 3 - \frac{3}{2}(x - 2)$$

$$y = -\frac{3}{2}x + 6$$

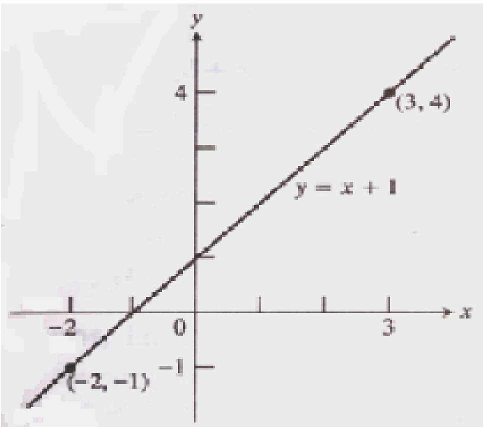
Example 2: Write an equation for the line through $(-2,-1)$ and $(3,4)$.

Solution:

$$m = \frac{-1-4}{-2-3} = \frac{-5}{-5} = 1$$

We can use this slope with either of the two given points. With the point $(x_1, y_1) = (-2,-1)$,

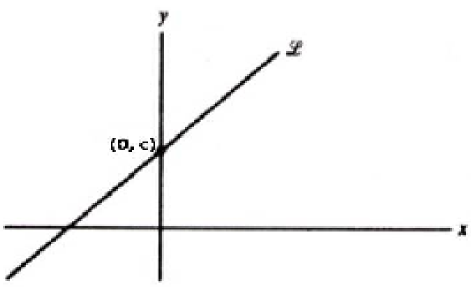
$$y = -1 + 1.(x - (-2))$$
$$y = -1 + x + 2$$
$$y = x + 1$$



From the preceding description of the line we see that one point on the straight line and the direction of the straight line will determine it. Thus the fixation of a straight line requires the specification of two quantities. For example, we may specify slope and intercept or two points lying on the straight line and likewise. This will lead to various forms of equations which are known as the ***standard forms*** for the equation of straight line.

Lines determined by Slope and y-intercept

A line with slope m and y −intercept c passes through $(0, c)$.



So, it has the equation $y = c + m(x - 0)$

The equation $y = mx + c$ is called the **slope-intercept equation** of the line with the slope m and y −intercept c.

Example 3: The line $y = 2x - 5$ has slope 2 and y −intercept -5.

Example 4: Find the equation of a line which passes through $(2,2)$ and inclined to x −axis at $45°$.

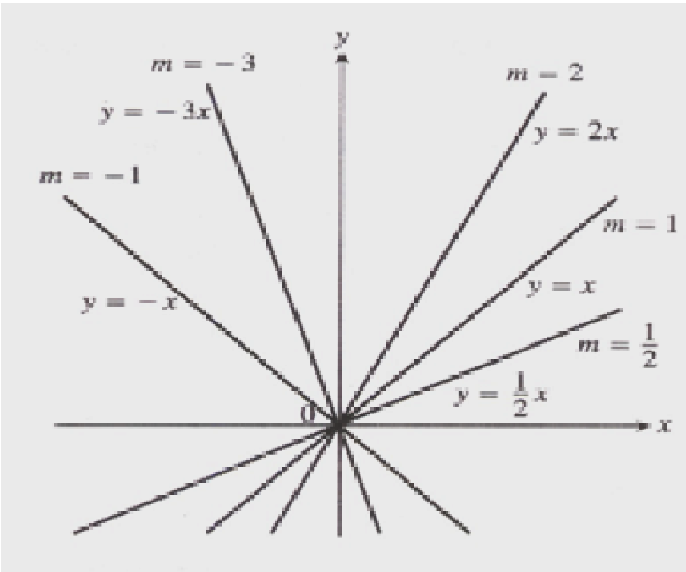
Here $m = \tan 45^\circ = 1$

$$\therefore y - 2 = 1.(x - 2)$$

The equation of the line is $x - y = 0$

Example 5:

Lines with equation of the form $y = mx$ have y −intercept 0 and so pass through the origin. Several examples are shown below in the figure.



The Two Point Form of a Line

The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

This equation is called the **two point form** of a line

Example 6:

The relation between Fahrenheit temperature (F) and Celsius temperature (C) is a linear relation and its graph is a straight line. The freezing state of water, $F = 32°$ or $C = 0°$, is a point on the line and the boiling state, $F = 212°$ or $C = 100°$, is also a point on the line. Determine the equation of line.

Solution:

The two points on the line are $(0,32)$ and $(100,212)$. Therefore

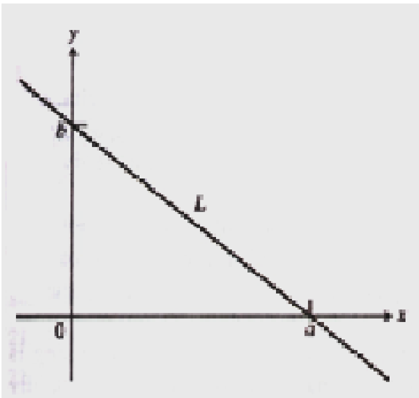
$$F-32=\frac{212-32}{100-0}(C-0)=\frac{180}{100}C=\frac{9}{5}C$$

The equation of the line is given by $F=\frac{9}{5}C+32$.

Two-Intercept Form of the Equation of Line

The y −coordinate of the point where a non-vertical line intersects the y −axis is called the y −intercept of the line. Similarly, the x −intercept of a non-horizontal line is the x −coordinate of the point where it crosses the x −axis.

Let a and b be x − and y −intercepts respectively. Then the line passes through the points $(a,0)$ and $(0,b)$.



$$m=\frac{b-0}{0-a}=-\frac{b}{a}$$

Using the point-slope form,

$$(y-b)=-\frac{b}{a}(x-0)$$

$$\Rightarrow ay-ab=-bx$$

$$\Rightarrow bx+ay=ab$$

Dividing through out by ab

$$\frac{x}{a}+\frac{y}{b}=1$$

This form of equation is called ***two-intercept form***.

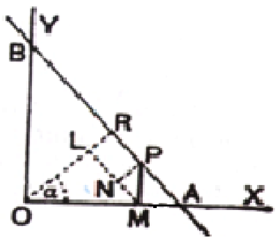
Example 7: Find the equation of the line which cuts off an intercept 5 on the negative direction of the x −axis and an intercept of 4 on positive direction of y −axis.

Solution: Here $a=-5, b=4$. So, $\frac{x}{-5}+\frac{y}{4}=1$

The equation of the line is $4x-5y+20=0$

The Normal Form of a Line

Consider a line AB and let p denote the length of the perpendicular from the origin to the line. Let α denote the angle of inclination of the perpendicular



Let $P(x, y)$be any point on the line. We note that the angle $NMP = \alpha$

$$p = OR = OL + LR = OL + NP = x\cos\alpha + y\sin\alpha$$

Therefore, the normal form of the equation of the straight line is

$$x\cos\alpha + y\sin\alpha = p$$

Where p is the length of the perpendicular from the origin to the line and α is the angle of inclination of the perpendicular. This form is also called **Normal form or perpendicular form** of equation of a line.

Example 8: Find the equation of a line which is at a distance $\frac{1}{2}$from the origin and pass through the point $(0,1)$.

Solution: The equation of any line which is at a distance $\frac{1}{2}$ from the origin is

$$x\cos\alpha + y\sin\alpha = \frac{1}{2}$$

It passes through the point $(0,1)$. So,

$$\sin\alpha = \frac{1}{2} \implies \cos\alpha = \pm\sqrt{1 - sin^2\alpha} = \pm\frac{\sqrt{3}}{2}$$

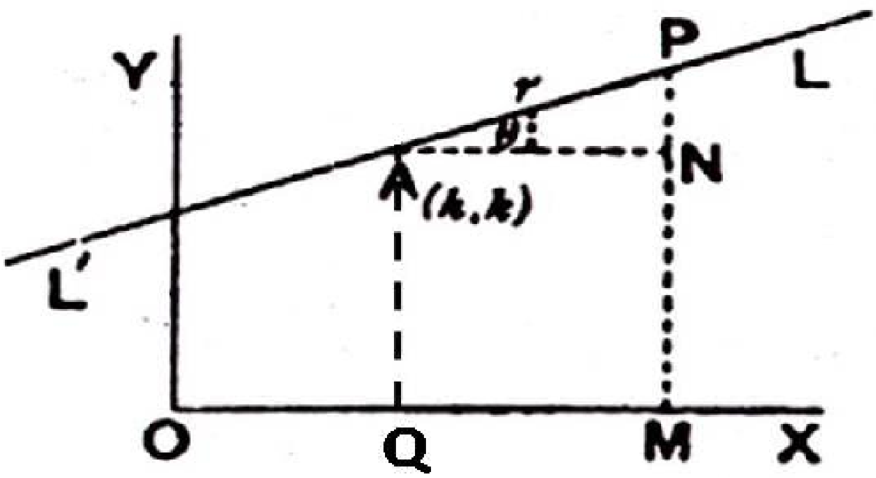
Using these values,

$$\pm\sqrt{3}x + y - 1 = 0$$

There are two lines, $\sqrt{3}x + y - 1 = 0$ and $-\sqrt{3}x + y - 1 = 0$, which fulfill the given requirement.

Parametric equations of a line

Suppose a line passes through a point $A(h, k)$ and makes an angle θ with the axis of x.



Let $P(x, y)$ be any point on the line and let AP be r. From the figure,

$$x = OM = h + r\cos\theta$$

$$y = MP = k + r\sin\theta$$

These equations give the coordinates of any point on the given line. They are known as parametric equations of the line. They are also written in the alternative form

$$\frac{x-h}{\cos\theta}=\frac{y-k}{\sin\theta}=r$$

Example 9: Find the equation of the line through $(-2,1)$ using the parametric form when the angle made by the line with positive direction of x-axis is 45°.

Solution: The parametric form is

$$\frac{x+2}{\cos 45^\circ}=\frac{y-1}{\sin 45^\circ}$$

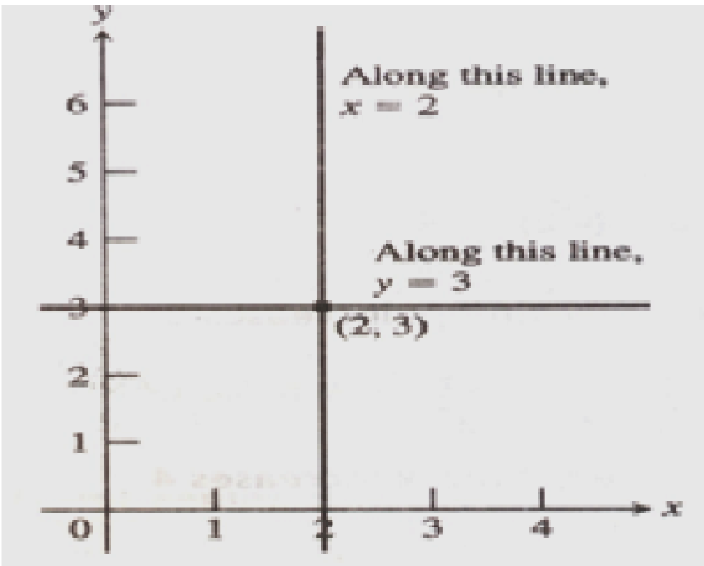
The equation of the line is $x-y+3=0$

Lines Parallel to the Coordinate Axes

All points on the vertical line through the point $(a,0)$ on the $x-axis$ have x —coordinate equal to a. Thus, $x = a$ is an equation for the vertical line. Similarly, $y = b$ is an equation for the horizontal line meeting the y-axis at $(0,b)$.

Example 10:

The vertical and horizontal lines through the point $(2,3)$ have equations $x = 2$ *and* $y = 3$, respectively.

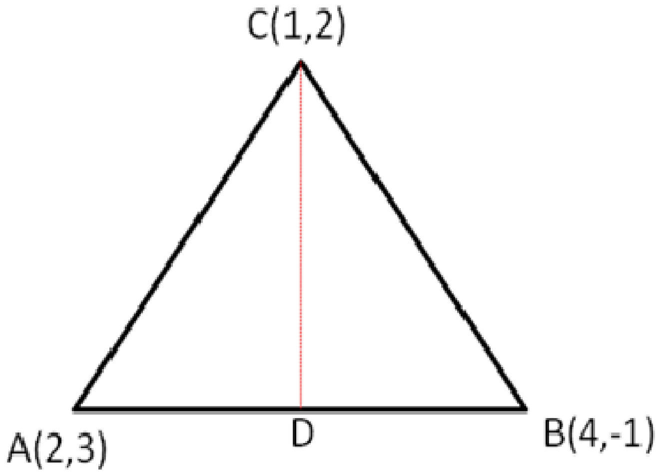


PROBLEM SET

IP1: The vertices of the ΔABC are $A(2,3), B(4,-1)$ and $C(1,2)$. Find the equation of the altitude through the vertex C.

Solution:

Given that the vertices of a $\Delta \boldsymbol{ABC}$ are $\boldsymbol{A(2,3), B(4,-1)}$ and $\boldsymbol{C(1,2)}$



Step1:

Now we have to find the slope of line $\boldsymbol{AB}$.

Slope of $\boldsymbol{AB} = \frac{-1-3}{4-2} = -2$

Let $\boldsymbol{CD}$ be the altitude through $\boldsymbol{C}$

$\therefore \boldsymbol{CD} \perp \boldsymbol{AB}$

If two lines are perpendicular then the slope one line is negative reciprocal of the other

$\therefore$The slope of $\boldsymbol{CD} = \frac{1}{2}$

Step2:

The equation of the line passing through the point $(\boldsymbol{x_1}, \boldsymbol{y_1})$ and having slope $\boldsymbol{m}$ is

$$\boldsymbol{y - y_1 = m(x - x_1)}$$

The Equation of CD is the equation of the straight line passing through the point $\boldsymbol{C(1,2)}$ with slope $\frac{1}{2}$

$\therefore$ The equation of $\boldsymbol{CD}$ is

$$\boldsymbol{y - 2 = \frac{1}{2}(x - 1)}$$

$$\Rightarrow \boldsymbol{2(y-2) = x - 1}$$

$$\Rightarrow \boldsymbol{x - 2y + 3 = 0}$$

$\therefore$The Equation of the altitude through the vertex $\boldsymbol{C}$ is

$$\boldsymbol{x - 2y + 3 = 0}$$

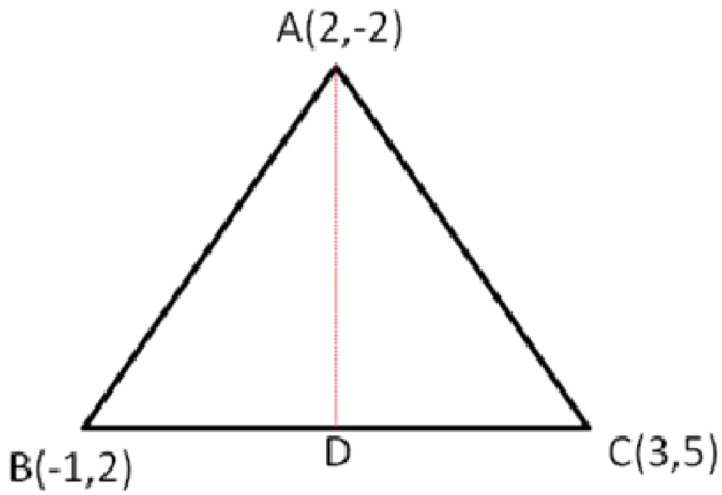
P1: The vertices of the ΔABC are $A(2,-2), B(-1,2)$ and $C(3,5)$. Find the equation of the altitude through the vertex A.

A. $4x+3y+14=0$
B. $4x+3y-14=0$
C. $4x+3y-2=0$
D. $4x+3y+2=0$

Answer: C

Solution:

Given that the vertices of the ΔABC are $A(2,-2), B(-1,2)$ and $C(3,5)$.



Slope of $BC = \frac{5-2}{3+1} = \frac{3}{4}$

Let AD be the altitude through A

$$\therefore AD \perp BC$$

If two lines are perpendicular then the slope of one is the negative reciprocal of the other

The slope of $AD = -\frac{4}{3}$

Then the equation of the line passing through the point (x_1, y_1) and having slope m is

$$y - y_1 = m(x - x_1)$$

The Equation of AD is the equation of the straight line passing through the point $A(2,-2)$ with slope $-\frac{4}{3}$

$\therefore$ The equation of AD is

$$y+2 = -\frac{4}{3}(x-2)$$

$$\Rightarrow 3y+6 = -4x+8$$

$$\Rightarrow 4x+3y-2=0$$

$\therefore$ The Equation of the altitude through the vertex A is

$$4x+3y-2=0$$

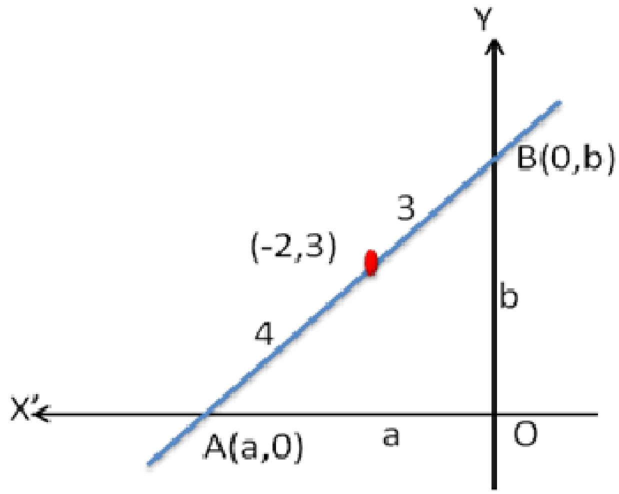
IP2: Find the equation of the line $\overleftrightarrow{AB}$ whose segment between x and y axes is divided by the point $(-2,3)$ in the ratio $4:3$

Solution:

Step1:

Suppose the line $\overleftrightarrow{AB}$ meet the axes in $\boldsymbol{A(a,0)}$ and $\boldsymbol{B(0,b)}$

Given that the point $(\mathbf{-2,3})$ divides $\overleftrightarrow{AB}$ in the ratio 4:3



The coordinates of a point $\boldsymbol{P}$ that divides the line Joining the points $(\boldsymbol{x_1},\boldsymbol{y_1})$ and $(\boldsymbol{x_2},\boldsymbol{y_2})$ in the ratio $\boldsymbol{m}:\boldsymbol{n}$ given by

$$= \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$

Here $(\boldsymbol{x_1},\boldsymbol{y_1}) = (\boldsymbol{a},\mathbf{0})$ and $(\boldsymbol{x_2},\boldsymbol{y_2}) = (\mathbf{0},\boldsymbol{b})$ and ratio $\boldsymbol{m}:\boldsymbol{n} = \mathbf{4:3}$

Then the point $(-2,3) = \left(\frac{(4\times0)+(3\times a)}{4+3}, \frac{(4\times b)+(3\times 0)}{4+3}\right)$

$$\Longrightarrow \quad (-2,3) = \left(\frac{3a}{7}, \frac{4b}{7}\right)$$

$$\Longrightarrow \quad \frac{3a}{7} = -2\ , \ \frac{4b}{7} = 3$$

$$\Longrightarrow \quad a = \frac{-14}{7}, b = \frac{21}{7}$$

Step2:

The equation of the line with $\boldsymbol{x}$ −intercept $\boldsymbol{a}$ and $\boldsymbol{y}$ −intercept $\boldsymbol{b}$ is

$$\frac{x}{a} + \frac{y}{b} = 1$$

The equation of the line $\overleftrightarrow{AB}$ with intercepts $\boldsymbol{a} = -\frac{14}{7}$, $\boldsymbol{b} = \frac{21}{7}$ is

$$\frac{x}{\frac{-14}{7}} + \frac{y}{\frac{21}{7}} = 1$$

$$\Longrightarrow \quad -9x + 8y = 42$$

$$\Longrightarrow \quad 9x - 8y + 42 = 0$$

P2: The portion of a line segment $\overleftrightarrow{AB}$ intercepted between the axes is divided by the point $(2,-1)$ in the ratio $3:2$. Find the equation of the line.

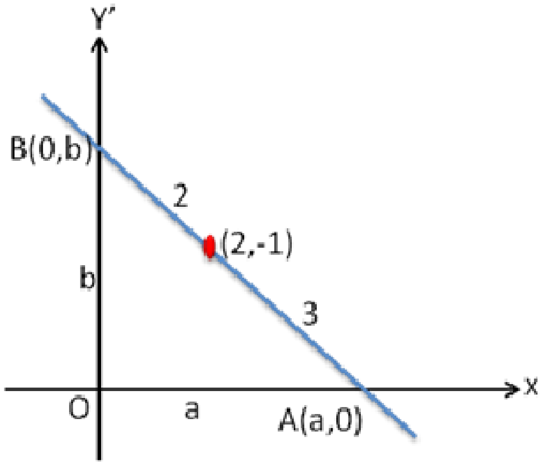
A. $\boldsymbol{x+3y-5=0}$
B. $\boldsymbol{x-3y+5=0}$
C. $\boldsymbol{x+3y+5=0}$
D. $\boldsymbol{x-3y-5=0}$

Answer: D

Solution:

Suppose the line $\overleftrightarrow{AB}$ meet the axes in $A(a,0)$ and $B(0,b)$

Given that the point $(2,-1)$ divides $\overleftrightarrow{AB}$ in the ratio $3:2$



The coordinates of a point P that divides the line Joining the points $(x_1,y_1)$and (x_2,y_2) in the ratio $m:n$ is given by

$$\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right)$$

Here $(x_1,y_1)=(a,0)$ and $(x_2,y_2)=(0,b)$ and ratio $m:n=3:2$

Then the point $(2,-1)=\left(\frac{(3\times 0)+(2\times a)}{2+3},\frac{(3\times b)+(2\times 0)}{2+3}\right)$

$$\Rightarrow \quad (2,-1)=\left(\frac{2a}{5},\frac{3b}{5}\right)$$

$$\Rightarrow \quad \frac{2a}{5}=2\,,\ \frac{3b}{5}=-1$$

$$\Rightarrow \quad a=5\,,b=\frac{-5}{3}$$

The equation of the line with x −intercept a and y −intercept b is $\frac{x}{a}+\frac{y}{b}=1$

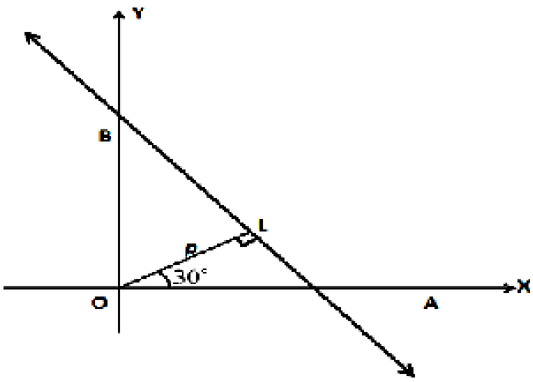
The equation of the line $\overleftrightarrow{AB}$ with intercepts $a=5\,,b=-\frac{5}{3}$ is

$$\frac{x}{5}+\frac{y}{\frac{-5}{3}}=1$$

$$\Rightarrow \quad x-3y-5=0$$

IP3: A line forms a triangle of area $96\sqrt{3}$ square units with the axes. Find the equation of the line if the perpendicular drawn from the origin to the line makes an angle of $30°$ with x –axis.

Solution:



Step1: Let AB be the required line and $OL = p$ be the perpendicular drawn from the origin on the line. It is given that the perpendicular OL makes $30°$ angle with OX.

$\therefore \quad \alpha = 30°$

Equation of the line AB is $x\cos\alpha + y\sin\alpha = p$ or $x\cos 30° + y\sin 30° = p$

or $\frac{\sqrt{3}x}{2} + \frac{y}{2} = p$ or $\sqrt{3}x + y = 2p$ ... (1)

Step2: write (1) in the intercept form, we get

$$\frac{x}{\left(\frac{2p}{\sqrt{3}}\right)} + \frac{y}{2p} = 1$$

x –intercept $= OA = \left(\frac{2p}{\sqrt{3}}\right)$; y –intercept $= OB = 2p$

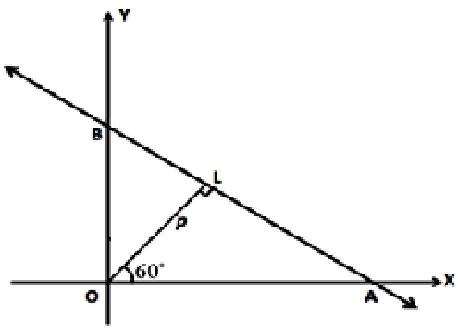
The area of the $\Delta OAB = \frac{1}{2} \times OA \times OB = \frac{1}{2} \times \frac{2p}{\sqrt{3}} \times 2p = \frac{2p^2}{\sqrt{3}}$

Now $\frac{2p^2}{\sqrt{3}} = 96\sqrt{3}$ (given) $\Longrightarrow p^2 = 144 \Longrightarrow p = 12$

The required equation is $\sqrt{3}x + y - 24 = 0$

P3: A line forms a triangle of area $54\sqrt{3}$ square units with the axes. Find the equation of the line if the perpendicular drawn from the origin to the line makes an angle of $60°$ with x –axis.

Solution:



Let AB be the required line and $OL = p$ be the perpendicular drawn from the origin on the line. It is given that the perpendicular OL makes $60°$ angle with OX.

$\therefore \quad \alpha = 60°$

Equation of the line AB is $x\cos\alpha + y\sin\alpha = p$ or $x\cos 60° + y\sin 60° = p$

or $\frac{x}{2} + \frac{\sqrt{3}y}{2} = p$ or $x + \sqrt{3}y = 2p$ ... (1)

write (1) in the intercept form, we get

$$\frac{x}{2p} + \frac{y}{\left(\frac{2p}{\sqrt{3}}\right)} = 1$$

x –intercept $= OA = 2p$; y –intercept $= OB = \frac{2p}{\sqrt{3}}$

The area of the $\Delta OAB = \frac{1}{2} \times OA \times OB = \frac{1}{2} \times 2p \times \frac{2p}{\sqrt{3}} = \frac{2p^2}{\sqrt{3}}$

Now $\frac{2p^2}{\sqrt{3}} = 54\sqrt{3}$ (given) $\Longrightarrow p^2 = 81 \Longrightarrow p = 9$

The required equation is $x + \sqrt{3}y - 18 = 0$

IP4: A straight line through $Q(\sqrt{3}, 2)$ makes an angle $\frac{\pi}{6}$ with $x - axis$ in positive direction. If the line intersect $\sqrt{3}x + 4y + 8 = 0$ at P. then find PQ.

Solution:

Step1: The Equation of the line in parametric form passing through the point (h, k) and having inclination angle θ is $\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r$

Now, the Equation of the line in parametric form passing Through the point $Q(\sqrt{3}, 2)$ and having inclination angle $\theta = \frac{\pi}{6}$ is

$$\frac{x-\sqrt{3}}{\cos\frac{\pi}{6}} = \frac{y-2}{\sin\frac{\pi}{6}} = r$$

$$\Longrightarrow \quad \frac{x-\sqrt{3}}{\frac{\sqrt{3}}{2}} = \frac{y-2}{\frac{1}{2}} = r$$

$$\Longrightarrow \quad x = \sqrt{3} + \frac{\sqrt{3}r}{2}, y = 2 + \frac{r}{2}$$

Any point on the line $= (\sqrt{3} + \frac{\sqrt{3}r}{2}, 2 + \frac{r}{2})$

Let this point be P. Then $P = (\sqrt{3} + \frac{\sqrt{3}r}{2}, 2 + \frac{r}{2})$.

Step2:

By the hypothesis point P lies on the line $\sqrt{3}x + 4y + 8 = 0$

Then P satisfies line equation

$$\sqrt{3}\left(\sqrt{3} + \frac{\sqrt{3}r}{2}\right) + 4\left(2 + \frac{r}{2}\right) + 8 = 0$$

$$\Longrightarrow \quad 6 - r = 0 \quad \Longrightarrow \quad r = 6$$

Substituting the value of r in point P then we get

$$P = (\sqrt{3} + \frac{6\sqrt{3}}{2}, 2 + \frac{6}{2})$$

$$P = (4\sqrt{3}, 5)$$

$\therefore$Length of $PQ = \sqrt{(\sqrt{3} - 4\sqrt{3})^2 + (2-5)^2} = \sqrt{36} = 6$

P4: A straight line through $\boldsymbol{Q(-3,5)}$ makes an angle $\boldsymbol{\frac{\pi}{4}}$ with $\boldsymbol{x - axis}$ in positive direction. If the line intersects $\boldsymbol{x + y - 6 = 0}$ at $\boldsymbol{P}$. Then find $\boldsymbol{PQ}$.

Solution: The Equation of the line in parametric form passing through the point $(\boldsymbol{h}, \boldsymbol{k})$ and having inclination angle $\boldsymbol{\theta}$ is

$$\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r$$

Now, the Equation of the line in parametric form passing through the point $\boldsymbol{Q(-3,5)}$ and having inclination angle $\boldsymbol{\theta = \frac{\pi}{4}}$ is

$$\frac{x+3}{\cos\frac{\pi}{4}} = \frac{y-5}{\sin\frac{\pi}{4}} = r$$

$$\Longrightarrow \quad \frac{x+3}{\frac{1}{\sqrt{2}}} = \frac{y-5}{\frac{1}{\sqrt{2}}} = r$$

$$\Longrightarrow \quad x = -3 + \frac{r}{\sqrt{2}},\ y = 5 + \frac{r}{\sqrt{2}}$$

Any point on the line $= (-3 + \frac{r}{\sqrt{2}}, 5 + \frac{r}{\sqrt{2}})$

Let this point be $\boldsymbol{P}$. Then $\boldsymbol{P} = (-3 + \frac{r}{\sqrt{2}}, 5 + \frac{r}{\sqrt{2}})$.

By the hypothesis P lies on the line $\boldsymbol{x + y - 6 = 0}$.

$\therefore$ P satisfies line equation

$$-3 + \frac{r}{\sqrt{2}} + 5 + \frac{r}{\sqrt{2}} - 6 = 0$$

$$\Longrightarrow \quad \sqrt{2}r - 4 = 0 \quad \Longrightarrow \quad r = 2\sqrt{2}$$

Substituting the value of $\boldsymbol{r}$ in point $\boldsymbol{P}$ then we get

$$P = (-3 + \frac{2\sqrt{2}}{\sqrt{2}}, 5 + \frac{2\sqrt{2}}{\sqrt{2}})$$

$$P = (-1, 7)$$

$\therefore$Length of $\boldsymbol{PQ} = \sqrt{(-3+1)^2 + (7-5)^2} = \sqrt{8} = 2\sqrt{2}$

Exercises:

18. Find the point-slope form of the line through $(4, -3)$ with slope 5.

19. Graph the equation $3x - 4y + 12 = 0$.

20. Find the slope of the line $3x - 4y + 12 = 0$.

21. Find the equation of the line passing through $(3,7)$ whose slope is $-\frac{3}{2}$.

22. Find the angle of inclination of the line $y = x$

23. Find the slope-intercept form of the equation of the line that satisfies the stated conditions:

a. Slope is -9; crosses the $y - axis$ at $(0, -4)$

b. Slope is 1; passes through the origin

c. Passes through $(5,-1)$; perpendicular to $y = 3x + 4$

d. Passes through $(3,4)$ and $(2,-5)$

24. Find the equation of the straight line cutting of an intercept of -5 on the axis of y and inclined to the axis of x at an angle θ such that $\tan\theta = \frac{3}{5}$.

25. Find the equation of the straight line passing through the points $(3,7)$ and $(5,4)$.

26. Find the equation of a line which passes through $(p\cos\alpha, pin\alpha)$ and making an angle of $(90^\circ + \alpha)$ with the positive direction of $x - axis$.

27. The vertices of triangle PQR are $P(2,1), Q(-2,3)$ and $R(4,5)$. Find equation of the median through the vertex R.

28. Find the ratio in which the line joining $(2,3)$ and $(4,1)$ divides the line joining $(1,2)$ and $(4,3)$.

29. Find the equation of the straight line passing through $(2,3)$ and cutting off equal intercepts along the positive direction of both the axes.

30. A straight line passes through the point $(2,3)$ and the portion of the line intercepted between the axes is bisected at this point. Find the equation of the line.

14. Find the equation of the straight line which passes through $(3,4)$ and the sum of whose intercepts on the coordinate axes is 14.

15. Find the equation of straight lines which pass through the origin and trisect the intercept of line $3x+4y=12$ between the axes.

16. If the straight line drawn through the point $P(\sqrt{3},2)$ and making an angle $\frac{\pi}{6}$ with the $x - axis$ meets the line $\sqrt{3}x-4y+8=0$ at Q, find the length of PQ.

17. Find the equation of the line through the point $A(2,3)$ and making an angle of $45°$ with the $x - axis$. Also, determine the length of intercept on it between A and the line $x+y+1=0$

18. The line joining two points $A(2,0), B(3,1)$ is rotated about A in anticlockwise direction through an angle of $15°$. Find the equation of the line in the new position. If B goes to C in the new position, what will be the coordinates of C?

6.5. Parallel and Perpendicular Lines

Learning objectives:

- To derive conditions for two lines $\boldsymbol{a_1x+b_1y+c_1=0}$ and $\boldsymbol{a_2x+b_2y+c_2=0}$ to be
  $(\boldsymbol{i})$ parallel and $(\boldsymbol{ii})$perpendicular.

  And
- To practice problems on parallel and perpendicular lines.

Parallel Lines

Lines that are parallel have equal angles of inclination, and therefore they have the same slope .

Two straight lines whose equations are given in terms of their slopes are parallel when their slopes are same, or, in other words, if their equations differ only in the constant term.

Therefore, the straight line $ax+by+c'=0$ is parallel to the straight line $ax+by+c=0$.

Again the equation $a(x-x')+b(y-y')=0$ clearly represents a straight line which passes through the point(x',y') and is parallel to the line $ax+by+c=0$.

The condition for the two lines $\boldsymbol{a_1x+b_1y+c_1=0}$ and $\boldsymbol{a_2x+b_2y+c_2=0}$ to be parallel

The condition is obtained by equating their slopes

$$-\frac{a_1}{b_1}=-\frac{a_2}{b_2}$$

That is, $a_1b_2-a_2b_1=0$ is the condition for the given lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ to be parallel

If two equations $\boldsymbol{ax+by+c=0}$ and $\boldsymbol{a'x+b'y+c'=0}$ represent the same straight line, then

$$\boldsymbol{\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}}$$

This can be proved as follows:

As the two equations represent the same straight line, they will have the same slope and the same y intercept.

Thus $-\frac{a}{b}=-\frac{a'}{b'}$ and $-\frac{c}{b}=-\frac{c'}{b'}$; Therefore $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$

Example 1:

Find the equation to the straight line, which passes through the point $(4,-5)$, and which is parallel to the straight line $3x+4y+5=0$

Solution: Any straight line which is parallel to the given line is of the form

$$3x + 4y + k = 0$$

The straight line passes through $(4, -5)$. Therefore,

$$3 \times 4 + 4 \times (-5) + k = 0$$

$$k = 8$$

Thus, the equation of the required line is

$$3x + 4y + 8 = 0$$

Perpendicular Lines

If two non-vertical lines L_1 and L_2 are perpendicular, their slopes m_1 and m_2 satisfy

$m_1 m_2 = -1$

The straight line $y = m_2 x + c_2$ is therefore perpendicular to $y = m_1 x + c_1$, if

$m_1 = -\frac{1}{m_2}$

Example 2:

Let L be the line $3x + 2y = 5$.

a. Find an equation of the line that is parallel to L and passes through $P(4,7)$.
b. Find an equation of the line that is perpendicular to L and passes through $P(4,7)$.

Solution:

$3x + 2y = 5 \quad \Longrightarrow \quad y = -\frac{3}{2}x + \frac{5}{2} \quad \Longrightarrow \quad m = -\frac{3}{2}$

a. $(y - 7) = -\frac{3}{2}(x - 4) \Longrightarrow 2y - 14 = -3x + 12$
$\Longrightarrow 3x + 2y - 26 = 0$
b. $m = \frac{2}{3} \Longrightarrow (y - 7) = \frac{2}{3}(x - 4) \Longrightarrow 3y - 21 = 2x - 8$
$\Longrightarrow 2x - 3y + 13 = 0$

The condition for the straight lines $\boldsymbol{a_1 x + b_1 y + c_1 = 0}$ and $\boldsymbol{a_2 x + b_2 y + c_2 = 0}$ to be perpendicular is

$\boldsymbol{a_1 a_2 + b_1 b_2 = 0}$

The slopes of the given lines are $m_1 = -\frac{a_1}{b_1}, m_2 = -\frac{a_2}{b_2}$.

The lines are at right angles if $m_1 m_2 = -1$

$$i.e., \qquad \left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right) = -1$$

That is, $a_1 a_2 + b_1 b_2 = 0$ is the condition of perpendicularity of the lines.

It now follows that the two straight lines $a_1 x + b_1 y + c_1 = 0$ and $b_1 x - a_1 y + c_2 = 0$ are at right angles since the product of their slopes is -1. Also, the second equation is derived from the first by interchanging the coefficients of x and y, changing the sign of one of them, and changing the constant into another constant.

Example 2: Find the equation to the straight line which passes through the point $(4, -5)$ and is perpendicular to the straight line $3x + 4y + 5 = 0$.

Solution: Any straight line perpendicular to the given line is

$$4x - 3y + k = 0$$

Since the straight line passes through $(4, -5)$, we have

$4 \times 4 - 3 \times (-5) + k = 0 \implies k = -31$

Therefore, the equation of the required line is

$$4x - 3y = 31$$

PROBLEM SET

IP1: Find the equation of the line passing through $(-2, 3)$ and parallal to the line through the points $(-3, 4)$ and $(2, -5)$.

Solution:

Step1: First we find the slope of the line passing through the points $(-3, 4)$ and $(2, -5)$.

$$m = \frac{-5-4}{2+3} = \frac{-9}{5}$$

We know that parallel lines have equal slopes.

Step2: Then the equation of the line parallel to the line through $(-3, 4)$ and $(2, -5)$ is

$$y = -\frac{9}{5}x + c$$

Since it passes through the point $(-2, 3)$,

$$3 = -\frac{9}{5}(-2) + c \quad \implies \quad c = -\frac{3}{5}$$

The required line equation is

$$y = -\frac{9}{5}x - \frac{3}{5} \quad \implies \quad 9x + 5y + 3 = 0$$

P1: Find the equation of the line passing through $(3, -4)$ and parallal to the line through the points $(4, 3)$ and $(6, 4)$.

Solution: First we find the slope of the line passing through the points $(4, 3)$ and $(6, 4)$.

$$m = \frac{4-3}{6-4} = \frac{1}{2}$$

We know that parallel lines have equal slopes.

Then the equation of the line parallel to the line through $(4, 3)$ and $(6, 4)$ is

$$y = \frac{1}{2}x + c$$

Since it passes through the point $(3, -4)$,

$$-4 = \frac{3}{2} + c \quad \implies \quad c = -\frac{11}{2}$$

The required line equation is

$$y = \frac{1}{2}x - \frac{11}{2} \quad \implies \quad x - 2y - 11 = 0$$

IP2: Find the value of k such that the line $4kx + (5k + 18)y + 7 = 0$ is parallel to the line $2x + 7y + 11 = 0$.

Solution:

Step1: The condition for the two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ to be parallel is

$$a_1b_2 - a_2b_1 = 0$$

Given that the line $4kx + (5k+18)y + 7 = 0$ is parallel to the line $2x + 7y + 11 = 0$.

Here $a_1 = 4k$, $b_1 = 5k + 18$

$a_2 = 2$, $b_2 = 7$

Step2: Substituting the values in the condition then we get k

$$7(4k) - 2(5k+18) = 0$$

$$\Rightarrow \quad 18k - 36 = 0 \quad \Rightarrow \quad k = 2$$

P2: The two lines $(3k-9)x + 5(k+1)y + 8 = 0$ and $3x - 5y - 11 = 0$ are parallel then find the value of k.

Solution: The Given lines are

$(3k-9)x + 5(k+1)y + 8 = 0$ and $3x - 5y - 11 = 0$

The condition for the two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ to be parallel is

$$a_1b_2 - a_2b_1 = 0$$

Here $a_1 = (3k-9)$, $b_1 = 5(k+1)$

$a_2 = 3$, $b_2 = -5$

Substituting the values in the condition then we get k

$$-5(3k-9) - 3(5)(k+1) = 0$$

$$\Rightarrow \quad -15k + 45 - 15k - 15 = 0$$

$$\Rightarrow \quad k = 1$$

IP3: Find the equation to the perpendicular bisector of the line segment joining the points $(2,-5)$ and $(0,-3)$

Solution:

Step1: Let $A = (2,-5)$ and $B = (0,-3)$

Let C midpoint of the line AB

$$C = \left(\frac{2+0}{2}, \frac{-5-3}{2}\right) = (1,-4)$$

Step2: The perpendicular bisector of the line segment joining the points A and B is the line perpendicular to AB passing through the mid point C

The slope of $AB = \frac{-3+5}{0-2} = -1$

$\therefore$ The slope of the perpendicular to $AB = 1$ $\quad (\because m_1m_2 = -1)$

The equation of the perpendicular bisector of AB is

$$(y+4) = (1)(x-1)$$

$$\Rightarrow \quad x - y - 5 = 0$$

P3: Find the equation to the perpendicular bisector of the line segment joining the points $(7,1)$ and $(3,-3)$

Solution: Let $A = (7,1)$ and $B = (3,-3)$

Let $\boldsymbol{C}$ be the midpoint of the line $\boldsymbol{AB}$

$$\therefore\ \boldsymbol{C} = \left(\frac{7+3}{2}, \frac{1-3}{2}\right) = (5, -1)$$

The perpendicular bisector of the line segment joining the points $\boldsymbol{A}$ and $\boldsymbol{B}$ is the line perpendicular to $\boldsymbol{AB}$ passing through the mid point $\boldsymbol{C}$

The slope of $\boldsymbol{AB} = \frac{-3-1}{3-7} = 1$

$\therefore$ The slope of the line perpendicular to $\boldsymbol{AB} = -1 (\because m_1 m_2 = -1)$

The equation of the perpendicular bisector of $\boldsymbol{AB}$ is

$$(y+1) = (-1)(x-5)$$

$$\Rightarrow\ x + y - 4 = 0$$

IP4: Find the equation of the line perpendicular to the line $3x - y - 2 = 0$ and having y-intercept – 5.

Solution:

Step1: Given that the required line is perpendicular to the line

$$3x - y - 2 = 0$$

We know that the equation of the line perpendicular to

$ax + by + c = 0$ is $bx - ay + k = 0$

The equation of the line perpendicular to $3x - y - 2 = 0$ is

$$-x - 3y + k = 0 \Rightarrow x + 3y - k = 0 \quad \ldots.(1)$$

Step2: Writing (1) in intercept form

$$x + 3y = k$$

$$\frac{x}{k} + \frac{y}{k/3} = 1$$

y –intercept of the line is $\frac{k}{3}$.

Given that the y –intercept -5

$$\therefore\quad \frac{k}{3} = -5 \quad \Rightarrow \quad k = -15$$

The required line equation of the line is

$$x + 3y - 15 = 0$$

P4: Find the equation of the line perpendicular to the line $x - 7y + 5 = 0$ and having x –intercept 3.

Solution: Given that the required line is perpendicular to the line

$$x - 7y + 5 = 0$$

We know that the equation of the line perpendicular to

$ax + by + c = 0$ is $bx - ay + k = 0$

The equation of the line perpendicular to $x - 7y + 5 = 0$ is

$$-7x - y + k = 0 \ \Rightarrow \ 7x + y - k = 0$$

Writing in intercept form

$$7x + y = k$$

$$\frac{x}{k/7}+\frac{y}{k}=1$$

x −intercept of the line is $\frac{k}{7}$.

Given that the x −intercept is 3.

$$\therefore\ \frac{k}{7}=3 \quad \Longrightarrow \quad k=21$$

The required line equation is

$$7x+y-21=0$$

Exercises:

31. Find the equation of the straight line which passes through the point $(\mathbf{5},\mathbf{6})$ and is perpendicular to the straight line $\mathbf{4x + 7y = 12}$.

32. Find the equation of the perpendicular bisector of the line segment joining the points $(\mathbf{3},\mathbf{4})$ and $(-\mathbf{1},\mathbf{2})$.

33. Find the coordinates of the foot of the perpendicular from the point $(-\mathbf{1},\mathbf{3})$ to the line $\mathbf{3x-4y-16=0}$.

34. The perpendicular from the origin to the line $\boldsymbol{y=mx+c}$ meets it at the point $(-\mathbf{1},\mathbf{2})$. Find the values of $\boldsymbol{m}$ and $\boldsymbol{c}$.

35. The perpendicular from the origin to a line meets it at the point $(-\mathbf{2},\mathbf{9})$. Find the equation of the line.

36. In a triangle $\boldsymbol{ABC}$ with vertices $\boldsymbol{A}(\mathbf{2},\mathbf{3}),\boldsymbol{B}(\mathbf{4},-\mathbf{1}),\boldsymbol{C}(\mathbf{1},\mathbf{2})$, find the equation of the altitude from the vertex $\boldsymbol{A}$.

37. Find the equation of a line passing through $(-\mathbf{3},\mathbf{5})$ and perpendicular to the line through the points $(\mathbf{2},\mathbf{5})$ and $(-\mathbf{3},\mathbf{6})$.

38. Show that the perpendicular drawn from the point $(\mathbf{4},\mathbf{1})$ on the line joining $(\mathbf{6},\mathbf{5})$ and $(\mathbf{2},-\mathbf{1})$ divide it in the ratio $\mathbf{8}:\mathbf{5}$.

39. Find the distance of the point $(\mathbf{2},\mathbf{5})$ from the line $\mathbf{3x+y+4=0}$. measured parallel to a line having slope $\frac{\mathbf{3}}{\mathbf{4}}$.

40. Find the value of k such that the line $\boldsymbol{(k-2)x+(k+3)y-5=0}$ is parallel to the line $\mathbf{2x-y+7=0}$.

41. Prove that the lines $\boldsymbol{px+qy-r=0}$ and $\boldsymbol{-4px-4qy+5s=0}$ are parallel.

42. Find the equation of the straight line which passes through $(\mathbf{4},-\mathbf{5})$ and is parallel to the straight line $\mathbf{3x-4y+9\ =0}$.

6.6. General Equation of a Line

Learning objectives:

- To study the general equation of a line.
- To study the reductions of the general equation of line to standard forms.
- To introduce the concept of positive and negative sides of a line and to derive a condition for a given two points to lie on the same side or opposite side of a line.
- To derive the equation of line passing through a given point and making a given angle with a given line.
  And
- To practice related problems.

The General equation of a Line

We have seen that all the forms of the equations of straight lines are only of the first degree in x and y. An equation that is expressible in the form

$$Ax + By + C = 0$$

where $A, B,$ and C are constants and at least one of A, B is not zero, is called a *first-degree equation* in x and y. It is called the *general linear equation* in x and y since its graph always represents a line.

Every first degree equation of the form $Ax + By + C = 0$,
where at least one of A, B is not zero, is called the ***general equation*** of a straight line.
Such a general equation can be reduced to the standard forms of the equations of lines.
We demonstrate the reduction of the general linear equation to the standard forms in the following.

Lines Parallel to Coordinate axes

Suppose $B = 0$, then A is not zero. Then the equation becomes $x = -\frac{C}{A}$,which is the equation of a straight line parallel to the $y - axis$, at $-\frac{C}{A}$ units from it.

On the other hand, if $A = 0$; then B is not zero and the equation becomes $y = -\frac{C}{B}$,which is the equation of a straight line parallel to the $x - axis$, at $-\frac{C}{B}$ units from it.

Slope Intercept Form

If $B \neq 0$, then we divide by B and solve for y, getting $y = -\frac{A}{B}x - \frac{C}{B}$

This is the ***slope-intercept form*** of the equation of the line, with slope $-\frac{A}{B}$ and y −intercept $-\frac{C}{B}$.

Example 1: Find the slope and y −intercept of the line $8x + 5y = 20$.

Solution: $5y = -8x + 20 \implies y = -\frac{8}{5}x + 4$

The slope $m = -\frac{8}{5}$ and the y −intercept is $c = 4$

Intercept Form

Consider the general equation of the line $Ax + By + C = 0$

If $C \neq 0$ then dividing by $-C$, and transposing, we obtain

$$\frac{A}{-C}x + \frac{B}{-C}y = 1 \Longrightarrow \frac{x}{\left(-\frac{C}{A}\right)} + \frac{y}{\left(-\frac{C}{B}\right)} = 1$$

This is the ***intercept form*** of the equation of the line with x and y intercepts as $-\frac{C}{A}$ and $-\frac{C}{B}$ respectively.

Example 2: Reduce the equation $3x+4y-5=0$ into the intercept form.

Solution: Dividing both sides by 5, we get ,

$$\frac{3}{5}x + \frac{4}{5}y = 1$$

The required form is $\frac{x}{\left(\frac{5}{3}\right)} + \frac{y}{\left(\frac{5}{4}\right)} = 1$

The intercept on the $x - axis$ is $\frac{5}{3}$ and the intercept on the $y - axis$ is $\frac{5}{4}$.

Normal Form

We take the general linear equation as $ax + by = c$,

where the right-hand side is assumed to be positive. If the right-hand side is not positive, we make it positive by multiplying the whole equation by -1.

We divide both sides of the equation by $\sqrt{a^2 + b^2}$. This gives

$$\left(\frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y\right) = \frac{c}{\sqrt{a^2+b^2}}$$

We choose α such that $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$

Let $p = \frac{c}{\sqrt{a^2+b^2}}$. Then, we obtain the required reduction

$$x\cos\alpha + y\sin\alpha = p$$

Example 3: Reduce the equation $x + \sqrt{3}y + 4 = 0$ into the perpendicular form and hence find the length of the perpendicular from the origin on the straight line.

Solution: On rearrangement, $-x - \sqrt{3}y = 4$

Dividing through out by $\sqrt{(-1)^2 + (-\sqrt{3})^2} = 2$, we get

$$-\frac{1}{2}x - \frac{\sqrt{3}}{2}y = 2$$

Choose α such that $\cos\alpha = -\frac{1}{2}$, $\sin\alpha = -\frac{\sqrt{3}}{2}$

$\Longrightarrow$ α is in 3rd quadrant. Further $\alpha = 240° = \frac{4\pi}{3}$

The required form is, $x\cos\frac{4\pi}{3} + y\sin\frac{4\pi}{3} = 2$ and

the length of the perpendicular from the origin is 2.

Positive and Negative Sides of a Line

If the origin makes $ax + by + c$ positive, then this side is known as the positives side of the line $ax + by + c$; if negative, then this side is known as the negative side.

The point(x', y') and the origin are on the same side of the given line if $ax' + by' + c$ and $a \times 0 + b \times 0 + c$ have the same signs, i.e., if $ax' + by' + c$ has the same sign as c. If these two quantities have opposite signs, then the origin and the point (x', y') are on opposite sides of the given line.

This principle is applicable for any two points (x_1, y_1) and (x_2, y_2). If two points (x_1, y_1) and (x_2, y_2) make $ax + by + c$ of the same sign, then both the points lie on the same side; if opposite signs, then they lie on the opposite sides of the line.

Example 4: Show that the points $(1,1)$ and $(2, -1)$ lie on the same side of the line $2x + 3y + 4 = 0$.

Solution:

Let $f(x) = 2x + 3y + 4 = 0$

For the point $(1,1), f_1 = 2 \times 1 + 3 \times 1 + 4 = +9$

For the point $(2, -1), f_2 = 2 \times 2 + 3 \times -1 + 4 = +5$

Hence, both points lie on the same side of the given line.

Example 5: Find the angle between the lines $y - \sqrt{3}x - 5 = 0$ and $\sqrt{3}y - x + 6 = 0$.

Solution: Here, $y = \sqrt{3}x + 5, m_1 = \sqrt{3}$

$$y = \frac{1}{\sqrt{3}}x - 2\sqrt{3}\,, m_2 = \frac{1}{\sqrt{3}}$$

The angle θ between the two lines is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{\frac{1}{\sqrt{3}} - \sqrt{3}}{1 + \sqrt{3}.\frac{1}{\sqrt{3}}}\right| = \left|\frac{1-3}{2\sqrt{3}}\right| = \frac{1}{\sqrt{3}}$$

The angle $\theta = 30^o$, and the supplemental angle is $180° - 30° = 150°$.

Type equation here.

Note: The formula $\tan\theta = \pm\frac{m_2 - m_1}{1 + m_1 m_2}$ fails when one of the lines is parallel to the $y - axis$. In such a case we can use the alternative formula derived below.

The angle between vertical line and non vertical line:

Let the lines be $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$

Then their slopes are given by

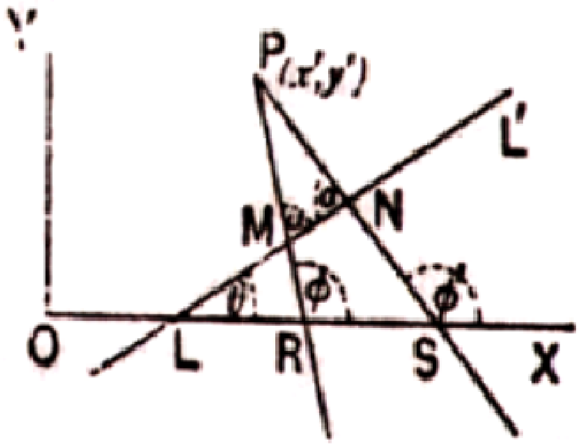
$$m_1 = -\frac{a_1}{b_1}\,, m_2 = -\frac{a_2}{b_2}$$

The formula for the angle between the two lines is given by

$$\tan\theta = \pm\frac{m_2 - m_1}{1 + m_1 m_2} = \pm\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}$$

Equations of Straight Lines passing through a given point and making a given angle with a given line

Let $P(x', y')$ be the given point and let the given straight line be LMN, making an angle θ with the positive direction of the axis of x. Let $m = \tan\theta$



In general there are two straight lines PMR and PNS making an angle α with the given line.

Let these lines meet the axis of x in R and S and let them make angles ∅ and ∅′ with the positive direction of the axis of x.

The equations to the two required straight lines are

$$y - y' = \tan\emptyset\,(x - x')$$
$$y - y' = \tan\emptyset'\,(x - x')$$

From the figure $\emptyset = \alpha + \theta$

$$\emptyset' = (180° - \alpha) + \theta$$

Hence $\tan\emptyset = \tan(\alpha+\theta) = \frac{\tan\alpha + \tan\theta}{1 - \tan\alpha\tan\theta} = \frac{m + \tan\alpha}{1 - m\tan\alpha}$

$\tan\emptyset' = \tan(180° + \theta - \alpha) = \tan(\theta - \alpha) = \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha} = \frac{m - \tan\alpha}{1 + m\tan\alpha}$

Therefore the required equations are

$$y - y' = \frac{m + \tan\alpha}{1 - m\tan\alpha}(x - x')$$

And

$$y - y' = \frac{m - \tan\alpha}{1 + m\tan\alpha}(x - x')$$

Example 6: Find the equations of the two lines passing through the point $(1, -1)$ and inclined at an angle of 45° with the line $2x - 5y + 7 = 0$.

Solution: Here $m = \frac{2}{5}$

So, the equations of the required lines are

$y - (-1) = \frac{\frac{2}{5} + \tan 45°}{1 - \frac{2}{5}\tan 45°}(x - 1)$ and $y - (-1) = \frac{\frac{2}{5} + \tan 45°}{1 - \frac{2}{5}\tan 45°}(x - 1)$

$7x - 3y - 10 = 0$ and $3x + 7y + 4 = 0$

PROBLEM SET

IP1: Reduce the equation $4x-3y+12=0$ into slope intercept form , Two intercept form and normal form.

Solution:

Step1: The equation of the given line $\mathbf{4x-3y+12=0}$

$$\Rightarrow 3y=4x+12 \quad \Rightarrow y=\frac{4}{3}x+4$$

This is the slope intercept form, Slope $\boldsymbol{m}=\frac{4}{3}$ and $\boldsymbol{y-intercept}$ is 4

Step2: Now reduce the equation $\mathbf{4x-3y+12=0}$ in to intercept form

$$4x-3y=-12 \Rightarrow -\frac{4}{12}x+\frac{3}{12}y=1$$

$$\Rightarrow \frac{x}{-3}+\frac{y}{4}=1$$

This is the intercept form with $\boldsymbol{x}$ and $\boldsymbol{y}$ intercepts $\mathbf{-3}$ and 4 respectively

Step3: Now we reduce the equation $\mathbf{4x-3y+12=0}$ into normal form

$$4x-3y+12=0 \Rightarrow -4x+3y=12$$

$$\Rightarrow \frac{-4}{\sqrt{(-4)^2+3^2}}x+\frac{3}{\sqrt{(-4)^2+3^2}}y=\frac{12}{\sqrt{(-4)^2+3^2}}$$

$$\Rightarrow \frac{-4}{5}x+\frac{3}{5}y=\frac{12}{5}$$

This is the normal form of the equation $\mathbf{4x-3y+12=0}$

P1: Match the following

1. **Slope of the line $3x-2y+8=0$** — a. $\frac{2}{3}$
2. **x-intercept of the line $3x+6y-2=0$** — b. $-\frac{2}{3}$
3. **Length of the perpendicular from the origin to the line $3x+4y+5=0$** — c. $\frac{3}{2}$

d. 1

e. -1

A. $\mathbf{1\longrightarrow a, 2\longrightarrow c, 3\longrightarrow e}$

B. $\mathbf{1\longrightarrow b, 2\longrightarrow c, 3\longrightarrow d}$

C. $\mathbf{1\longrightarrow c, 2\longrightarrow a, 3\longrightarrow d}$

D. $\mathbf{1\longrightarrow c, 2\longrightarrow b, 3\longrightarrow e}$ **Answer:** C

Solution:

1. Slope of the line $3x-2y+8=0$

$$3x+2y-8=0 \quad \Rightarrow \quad 2y=3x+8$$

$$\Rightarrow \quad y=\frac{3}{2}x+4$$

Comparing the equation with $y=mx+c$

Then slope $m=\frac{3}{2}$

2. x-intercept of the line $3x + 6y - 2 = 0$

   We convert the equation into intercept form

$$3x + 6y - 2 = 0 \quad \Longrightarrow \quad 3x + 6y = 2$$

$$\Longrightarrow \quad \frac{3x}{2} + \frac{6y}{2} = 1 \qquad \Longrightarrow \quad \frac{x}{2/3} + \frac{y}{1/3} = 1$$

   x-intercept of the line $3x + 6y - 2 = 0$ is $\frac{2}{3}$

3. Converting the line equation $3x + 4y + 5 = 0$ in normal form

$$3x + 4y + 5 = 0 \quad \Longrightarrow \quad -3x - 4y = 5$$

   Diving both sides with $\sqrt{3^2 + 4^2}$

$$\frac{-3}{\sqrt{3^2+4^2}}x - \frac{4}{\sqrt{3^2+4^2}}y = \frac{5}{\sqrt{3^2+4^2}}$$

$$\Longrightarrow \quad \frac{-3}{5}x - \frac{4}{5}y = 1$$

   Comparing the equation with $x\cos\alpha + y\sin\alpha = p$

   Then $\cos\alpha = \frac{-3}{5}$, $\sin\alpha = \frac{-4}{5}$ and $p = 1$

IP2: Let $A(3,-2), B(1,2)$ and $C(1,1)$ be three points. Then which pair of points lies on different sides of the line $2x + 3y = 5$.

Solution:

Let $f = 2x + 3y - 5 = 0$

For the point $A(3,-2)$, $f_A = 2(3) + 3(-2) - 5 = -5 < 0$

For the point $B(3,-2)$, $f_B = 2(1) + 3(2) - 5 = 0 > 0$

For the point $(1,1)$, $f_C = 2(1) + 3(1) - 5 = 0$

Hence A, B are on different sides of the line $2x + 3y - 5 = 0$

P2: For what values of a the points $(1,2)$ and $(3,4)$lie on the same side of the line $3x - 5y + a = 0$

A. $7 < a < 11$

B. $a = 7$

C. $a = 11$

D. $a < 7$ **or** $a > 11$

Answer: D

Solution:

If the points $(1,2)$ and $(3,4)$ lies on same side of line

$3x - 5y + a = 0$, then $3 - 10 + a$ and $9 - 20 + a$ must be same sign

$3 - 10 + a > 0$ and $9 - 20 + a > 0$

Or $3 - 10 + a < 0$ and $9 - 20 + a < 0$

$\therefore \quad (3 - 10 + a)(9 - 20 + a) > 0$

$(a - 7)(a - 11) > 0$

$a < 7$ or $a > 11$

IP3: Find the value of k if the acute angle between the lines $4x - y + 7 = 0$ and $kx + 5y = 9$ is $\frac{\pi}{4}$.

Solution:

Step1: Given that the acute angle between the lines

$$4x - y + 7 = 0 \text{ and } kx + 5y = 9 \text{ is } \frac{\pi}{4}.$$

If θ is the angle between the lines

$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ then

$$\tan\theta = \pm\frac{a_1b_2 - a_2b_1}{a_1a_2 + b_1b_2}$$

Here $\theta = \frac{\pi}{4}$ and

$$a_1 = 4, \quad b_1 = -1$$

$$a_2 = k, \quad b_2 = 5$$

Step2: Substitute these values in the above formula

$$\tan\frac{\pi}{4} = \pm\frac{20+k}{4k-5}$$

$$1 = \pm\frac{20+k}{4k-5}$$

$$1 = \frac{20+k}{4k-5} \implies 4k - 5 = 20 + k \implies k = \frac{25}{3}$$

$$1 = -\frac{20+k}{4k-5} \implies 4k - 5 = -(20 + k) \implies k = -3$$

Hence $k = \frac{25}{3}$ and -3

P3: The angle between the lines $3x - y + 1 = 0$ and $2x + ky + 5 = 0$ is 45° then find the value of k.

A. 1 and 4
B. $-\mathbf{1}$ and 4
C. $\mathbf{1}$ **and** $-\mathbf{4}$
D. $-\mathbf{1}$ and $-\mathbf{4}$

Answer: C

Solution: If $\boldsymbol{\theta}$ is the angle between the line $\boldsymbol{a_1x + b_1y + c_1 = 0}$ and $\boldsymbol{a_2x + b_2y + c_2 = 0}$ then

$$\boldsymbol{\tan\theta = \pm\frac{a_1b_2 - a_2b_1}{a_1a_2 + b_1b_2}}$$

Given that the angle between the lines $\boldsymbol{3x - y + 1 = 0}$ and $\boldsymbol{2x + ky + 5 = 0}$ is 45°

Here $\boldsymbol{\theta = 45°}$ and

$$\boldsymbol{a_1 = 3, \quad b_1 = -1}$$

$$\boldsymbol{a_2 = 2, \quad b_2 = k}$$

Substitute these values in the above formula

$$\boldsymbol{\tan 45° = \pm\frac{3k+2}{6-k}}$$

$$1 = \pm \frac{3k+2}{6-k}$$

$$1 = \frac{3k+2}{6-k} \quad \Rightarrow \quad 6 - k = 3k + 2 \quad \Rightarrow \quad k = 1$$

$$1 = -\frac{3k+2}{6-k} \quad \Rightarrow \quad 6 - k = -(3k + 2) \quad \Rightarrow \quad k = -4$$

$k = 1$ and -4

IP4: Find the equations to the straight lines passing through the point $(-5, -3)$ and inclined 60° to the straight line $\sqrt{3}x - y + 8 = 0$

Solution:

Step1: We know that the equations of straight lines passing through (x_1, y_1) and making angle α with the given line with slope m is

$$y - y_1 = \frac{m - \tan\alpha}{1 + m\tan\alpha}(x - x_1)$$

and

$$y - y_1 = \frac{m + \tan\alpha}{1 - m\tan\alpha}(x - x_1)$$

Step2:

Given that $(x_1, y_1) = (-5, -3)$ and $\alpha = 60°$

Slope of the given line $\sqrt{3}x - y + 8 = 0$ is $m = \sqrt{3}$

Substituting these values in above line equations then we get

$$y + 3 = \frac{\sqrt{3} - \tan 60°}{1 + \sqrt{3}\tan 60°}(x + 5) \text{ and } \quad y + 3 = \frac{\sqrt{3} + \tan 60°}{1 - \sqrt{3}\tan 60°}(x + 5)$$

$$i.e., \ y + 3 = \frac{\sqrt{3} - \sqrt{3}}{1 + \sqrt{3}(\sqrt{3})}(x + 5) \quad \text{and} \quad y + 3 = \frac{\sqrt{3} + \sqrt{3}}{1 - \sqrt{3}(\sqrt{3})}(x + 5)$$

$$i.e., \ y + 3 = 0(x + 5) \qquad \text{and} \qquad y + 3 = \frac{2\sqrt{3}}{-2}(x + 5)$$

$$i.e., \ y + 3 = 0 \qquad \text{and} \qquad \sqrt{3}x + y + (5\sqrt{3} + 3) = 0$$

Step3:

The required equations of the lines are

$$y + 3 = 0 \ \text{ and } \ \sqrt{3}x + y + (5\sqrt{3} + 3) = 0$$

P4: Find the equations of the straight lines which pass through the point $(-1, 6)$ and inclined at an angle 45° to the straight line $4x + 5y - 6 = 0$

A. $x + 9y - 55 = 0, 9x + y + 3 = 0$
B. $x - 9y + 55 = 0, 9x + y + 3 = 0$
C. $x - 9y + 3 = 0, 9x - y - 55 = 0$
D. $x + 9y + 55 = 0, 9x - y + 3 = 0$

Answer: B

Solution:

We know that the equations of straight lines passing through (x_1, y_1) and making angle α with the given line with slope m is

$$y - y_1 = \frac{m - \tan\alpha}{1 + m\tan\alpha}(x - x_1) \quad \text{and} \qquad y - y_1 = \frac{m + \tan\alpha}{1 - m\tan\alpha}(x - x_1)$$

Given that $(x_1, y_1) = (-1,6)$ and $\alpha = 45°$

Slope of the given line $4x + 5y - 6 = 0$ is $m = \frac{-4}{5}$

Substituting these values in above line equations then

$$y - 6 = \frac{\frac{-4}{5} - \tan 45°}{1 - \frac{4}{5}\tan 45°}(x+1) \quad \text{and} \quad y - 6 = \frac{\frac{-4}{5} + \tan 45°}{1 + \frac{4}{5}\tan 45°}(x+1)$$

$i.e.,\ y - 6 = \frac{\frac{-4}{5} - 1}{1 - \frac{4}{5}}(x+1)$ and $y - 6 = \frac{\frac{-4}{5} + 1}{1 + \frac{4}{5}}(x+1)$

$i.e.,\ y - 6 = -9(x+1)$ and $y - 6 = \frac{1}{9}(x+1)$

$i.e.,\ 9x + y + 3 = 0$ and $x - 9y + 55 = 0$

The required equations of the lines are

$9x + y + 3 = 0$ and $x - 9y + 55 = 0$

Exercise:

1. Write the slope-intercept form of the equation $2x + 3y + 1 = 0$.
2. Reduce the equation $7x - 5y + 8 = 0$ into the intercept form.
3. Reduce the following equations to the normal form and find the values of p and α.
   a. $\sqrt{3}x - y + 2 = 0$
   b. $3x + 4y + 10 = 0$
   c. $12y = 5x + 65$
   d. $x + y + \sqrt{2} = 0$
4. Transform the equation of the line $2\sqrt{2}x + y - 3 = 0$ into
   a. slope intercept form
   b. intercept form
   c. normal form
5. Determine that the points $(2,3)$ and $(1,3)$ are on the same side or on opposite sides of the line $x - 2y + 3 = 0$.
6. Show that $(2,-1)$ and $(1,1)$ are on opposite sides of $3x + 4y - 6 = 0$.
7. The sides of a triangle are given by the equations $3x + 4y = 10$, $4x - 3y = 5$ and $7x + y + 10 = 0$ show that the origin lies within the triangle.
8. Show that the points (1,4) and (0,-3) lie on the opposite sides of the line $x + 3y + 7 = 0$.

9. Determine the angle between the lines whose equations are

   a. $3x + y - 7 = 0$ and $x + 2y + 9 = 0$

   b. $2x - 5y + 3 = 0$ and $x + y - 2 = 0$

10. Find the acute angle between the lines $x + 2y = 0$ and $\frac{x}{1} + \frac{y}{2} = 2$

11. Find the tangent of the angle between the lines whose Intercepts on the axes are respectively $a, -b$ and $b, -a$.

12. Through the point $(3, 4)$ are drawn two straight lines each Inclined at 45° to the straight line $x - y = 2$. Find their equations.

13. Find the equations to the straight lines which pass through the origin and are inclined at 75° to the straight line $x + y + \sqrt{3}(y - x) = a$.

14. If the points $(4,7)$ and $(\cos\theta, \sin\theta)$ where $0 < \theta < \pi$, lie on the same side of the line $x + y - 1 = 0$, then prove that θ lies in the first quadrant.

6.7. Distance of a Point from a Line

Learning objectives:

- To derive a formula for the distance of a point from a line.
- To find the distance between two parallel lines.

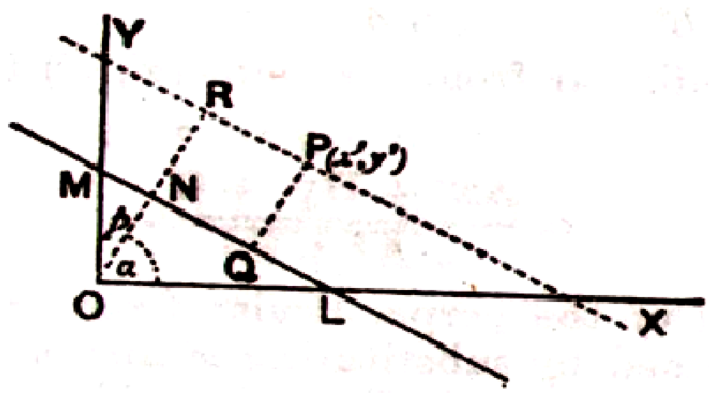
  And
- To practice the related problems.

Distance of a Point from a Line

Let the equation of the given straight line LM be

$$x\cos\alpha + y\sin\alpha - p = 0$$

So, in the figure, $ON = p$ and angle $XON = \alpha$



Let the given point P be (x', y'). Through P draw PR parallel to the given line. PQ is the required distance.

If OR is p', the equation to PR is

$$x \cos \alpha + y \sin \alpha - p' = 0.$$

Since this passes through the point (x', y'), we have

$$x' \cos \alpha + y' \sin \alpha - p' = 0.$$

So, $\quad p' = x' \cos \alpha + y' \sin \alpha$

$\therefore$ The Required distance $= p' - p = x' \cos \alpha + y' \sin \alpha - p$

The length of the required perpendicular is therefore obtained by substituting x' and y' for x and y in the given equation.

Since the normal form of an equation is equivalent to the general form, the following result follows.

The length of the perpendicular from (x', y') on $ax + by + c = 0$, where c *is negative*, is obtained by substituting x' and y' for x and y in the given equation and dividing the result so obtained by the square root of the sum of the squares of the coefficients of x and y. Required distance $= \frac{ax'+by'+c}{\sqrt{a^2+b^2}}$

It follows that the length of the perpendicular from the origin is $\frac{c}{\sqrt{a^2+b^2}}$.

In practice we take, the length of the perpendicular from a point $P(x', y')$ to the straight line $ax + by + c = 0$ ------(1) as

$$\frac{|ax'+by'+c|}{\sqrt{a^2+b^2}}$$

and from this it follows that, the length of perpendicular from the origin to the line (1) is $\frac{|c|}{\sqrt{a^2+b^2}}$

Example 1: Find the distance of the point $(-3,4)$ from the line $3x + 4y - 5 = 0$.

Solution:

Distance $= \frac{3(-3)+4(-4)-5}{\sqrt{a^2+b^2}} = \frac{|3(-3) + 4(4) - 5|}{\sqrt{3^2 + 4^2}} = \frac{2}{5}$

Example 2: Find the points on the y-axis whose perpendicular distance from the line $4x - 3y - 12 = 0$ is 3.

Solution: Let $(0, k)$ is the required point.

$$\frac{|4(0)-3k-12|}{\sqrt{4^2+(-3)^2}} = 3$$

$$\Longrightarrow \quad \frac{-3k-12}{5} = \pm 3$$

$$\therefore \quad -3k - 12 = 15 \quad \text{and} \quad -3k - 12 = -15$$

$$\Longrightarrow \quad k = -9 \quad \text{and} \quad k = 1$$

The required points are $(0,1)$ and $(0, -9)$

Distance between Two Parallel Lines

The distance between two parallel lines is the same throughout. Therefore, to find the distance between them, choose an arbitrary point on one of them and find the length of the perpendicular on the other.

The distance between two parallel straight lines

$\boldsymbol{ax+by+c_1=0}$ **and** $\boldsymbol{ax+by+c_2=0}$ **is** $\boldsymbol{\frac{|c_1-c_2|}{\sqrt{a^2+b^2}}}$

Proof: Let $P(x',y')$ be a point on the line $L_1 \equiv ax+by+c_1=0$

$$\therefore \; ax'+by'+c_1=0$$

Let $L_2 \equiv ax+by+c_2=0$ be the other line, Now

The distance between L_1 and L_2 = The perpendicular distance of P from L_2

$$=\frac{|ax'+by'+c_2|}{\sqrt{a^2+b^2}}=\frac{|ax+by+c_1+(c_2-c_1)|}{\sqrt{a^2+b^2}}=\frac{|c_1-c_2|}{\sqrt{a^2+b^2}}$$

Example 3: Find the perpendicular distance between the lines

$3x+4y-5=0$ and $6x+8y-45=0$.

Solution: Putting $x=0$ in the first equation, we get $y=\frac{5}{4}$

Consider the point$\left(0,\frac{5}{4}\right)$ on the first line. Perpendicular distance of this point from the second line is $\frac{\left|6\times0+8\times\frac{5}{4}-45\right|}{\sqrt{6^2+8^2}}=\frac{|-35|}{10}=\frac{7}{2}$; Hence, the required distance is $\frac{7}{2}$.

PROBLEM SET

IP1: If the length of the perpendicular from $(-1,2)$ to the line $3x+ky-4=0$ is 3 units then $k=$

Solution:

Step1: We know that the length of the perpendicular from (x',y') to the line

$ax+by+c=0$ is $\left|\frac{ax'+by'+c}{\sqrt{a^2+b^2}}\right|$

Given that the length of the perpendicular from $(-1,2)$ to the line $3x+ky-4=0$ is 3 units then

Step2: $3=\left|\frac{3(-1)+k(2)-4}{\sqrt{3^2+k^2}}\right| \Longrightarrow \quad 3=\left|\frac{2k-7}{\sqrt{3^2+k^2}}\right|$

$$\Longrightarrow \quad 2k-7=\pm3\sqrt{3^2+k^2}$$

Squaring on both sides then

$$(2k-7)^2=9(9+k^2)$$

$$\Longrightarrow \quad 5k^2+28k+32=0 \Longrightarrow \quad (5k+8)(k+4)=0$$

$$\Longrightarrow \quad k=-4 \text{ and } -\frac{8}{5}$$

P1: If the perpendicular distance from $(1,1)$ to the straightline $3x+4y-a=0$ is 5 units then $a=$

A. $\mathbf{32, 18}$
B. $\mathbf{32, -18}$
C. $\mathbf{-32, 18}$
D. $\mathbf{-32, -18}$

Answer: B

Solution: Length of the perpendicular from (x', y') to the line

$ax+by+c=0$ is $=\left|\frac{ax'+by'+c}{\sqrt{a^2+b^2}}\right|$

Given that the perpendicular distance from $(1,1)$ to the line

$3x+4y-a=0$ is 5 units then

$$5=\left|\frac{3(1)+4(1)-a}{\sqrt{3^2+4^2}}\right| \Longrightarrow 5=\left|\frac{7-a}{5}\right| \Longrightarrow |7-a|=25$$

$$\Longrightarrow 7-a=\pm 25$$

$\therefore$ $7-a=25$ and $7-a=-25 \Longrightarrow a=-18$ and $a=32$

IP2: Show that the distance of the point $(6,-2)$ from the line $4x+3y=12$ is half the distance of the point $(3,4)$ from the line $4x-3y=12$

Solution:

Step1: Distance from the point (x', y') to the line $ax+by+c=0$ is

$$=\left|\frac{ax'+by'+c}{\sqrt{a^2+b^2}}\right|$$

Let the distance from the point $(6,-2)$ to the line

$4x+3y-12=0$ be d_1

$$\therefore d_1=\left|\frac{4(6)+3(-2)-12}{\sqrt{4^2+3^2}}\right|=\frac{6}{5}$$

Let the distance from the point $(3,4)$ to the line

$4x-3y-12=0$ be d_2

$$\therefore d_2=\left|\frac{4(3)-3(4)-12}{\sqrt{4^2+3^2}}\right|=\frac{12}{5}$$

Step2: Notice that $d_2=2d_1$ $\quad \therefore d_1=\frac{1}{2}d_2$

Thus the distance of the point $(6,-2)$ from the line

$4x+3y=12$ is half the distance of the point $(3,4)$ from the

line $4x-3y=12$

P2: Find the value of a if the distances of the points $(2,3)$ and $(-4,a)$ from the straight line $3x+4y=8$ are equal.

A. $-\frac{15}{2}, \frac{5}{2}$
B. $-\frac{15}{2}, -\frac{5}{2}$
C. $\frac{15}{2}, \frac{5}{2}$

D. $\frac{15}{2}, -\frac{5}{2}$; **Answer:** C

Solution:

Distance from the point (x', y') to the line $ax + by + c = 0$ is

$$= \left|\frac{ax' + by' + c}{\sqrt{a^2 + b^2}}\right|$$

Distance from the point $(2,3)$ to the line $3x + 4y - 8 = 0$ is

$$= \left|\frac{3(2) + 4(3) - 8}{\sqrt{3^2 + 4^2}}\right| = \frac{10}{5} = 2$$

Distance from the point $(-4, a)$ to the line $3x + 4y - 8 = 0$ is

$$= \left|\frac{3(-4) + 4(a) - 8}{\sqrt{3^2 + 4^2}}\right| = \left|\frac{4a - 20}{5}\right|$$

Given that the distances of the points $(2,3)$ and $(-4, a)$ from the straight line $3x + 4y = 8$ are equal then

$$\left|\frac{4a - 20}{5}\right| = 2 \implies 4a - 20 = \pm 10$$

$$\therefore \quad 4a - 20 = 10 \quad \text{and} \quad 4a - 20 = -10$$

$$\implies a = \frac{15}{2} \qquad \text{and} \qquad a = \frac{5}{2}$$

IP3: Find the distance between the parallel lines

$\mathbf{5x - 3y - 4 = 0\,,\, 10x - 6y + 9 = 0}$

Solution:

Step1: Given parallel lines are

$$5x - 3y - 4 = 0\,,\, 10x - 6y + 9 = 0$$

Putting $x = 0$ in the first equation, we get $y = -\frac{4}{3}$

Step2: Therefore, the point $\left(0, -\frac{4}{3}\right)$ is on the first line. Now the perpendicular distance of this point from the second line is

$$= \left|\frac{10(0) - 6\left(\frac{-4}{3}\right) + 9}{\sqrt{10^2 + (-6)^2}}\right| = \left|\frac{17}{\sqrt{136}}\right| = \frac{1}{2}\sqrt{\frac{17}{2}}$$

The distance between given parallel lines is $\frac{1}{2}\sqrt{\frac{17}{2}}$ units

Remark:

The given parallel lines are

$$5x - 3y - 4 = 0 \qquad \ldots(1)$$

$$10x - 6y + 9 = 0 \qquad \ldots(2)$$

Write (1) as $10x - 6y - 8 = 0$ ... (3), Notice that (3) is same as (1).

Now distance between $(1)\&(2)$ = The distance between $(2)\&(3)$

$$= \frac{|9 - (-8)|}{\sqrt{(10)^2 + (-6)^2}} = \frac{17}{\sqrt{136}} = \frac{1}{2}\sqrt{\frac{17}{2}}$$

P3: Find the distance between the parallel lines $x+4y-3=0$, $6x-8y-1=0$

Solution: The given parallel lines are

$$3x+4y-3=0\,,\,6x-8y-1=0$$

Putting $x=0$ in the first equation, we get $y=\frac{3}{4}$

Therefore, the point $\left(0,\frac{3}{4}\right)$ is on the first line. Now, the perpendicular distance of this point from the second line is

$$=\left|\frac{6(0)+8\left(\frac{3}{4}\right)-1}{\sqrt{6^2+(-8)^2}}\right|=\left|\frac{6-1}{10}\right|=\frac{1}{2}$$

The distance between the given parallel lines is $\frac{1}{2}$ units

Remark: The given parallel lines are

$$3x+4y-3=0 \qquad \dots(1)$$

$$6x-8y-1=0 \qquad \dots(2)$$

Write (1) as $6x+8y-6=0$ -------(3), Notice that (3) is same as (1).

Now, distance between $(1)\&(2)=$ The distance between $(2)\&(3)$

$$=\frac{|-6-(-1)|}{\sqrt{6^2+8^2}}=\frac{5}{10}=\frac{1}{2}$$

IP4: Find the equations of the lines which are parallel to $8x-6y+5=0$ and at a distance of 1 unit from $(1,3)$.

Solution:

Step1: A line parallel to $8x-6y+5=0$ is

$$8x-6y+k=0$$

Step2: If the distance from $(1{,}3)$ to the line $8x-6y+k=0$ is 1 unit Then

$$\left|\frac{8(1)-6(3)+k}{\sqrt{8^2+(-6)^2}}\right|=1\implies\left|\frac{k-10}{10}\right|=1\implies k-10=\pm10$$

$$\therefore\; k-10=10 \text{ and } k-10=-10$$

$$\implies\quad k=20 \quad\text{and}\quad k=0$$

The required lines are

$$8x-6y=0 \text{ and } 8x-6y+20=0$$

or $\quad 4x-3y=0$ and $4x-3y+10=0$

P4: Find the equations of the lines which are parallel to $5x-12y+20=0$ and at a distance of 2 units from $(3,2)$.

Solution: A line parallel to $5x-12y+20=0$ is

$$5x-12y+k=0$$

If the distance from $(3{,}2)$ to the line $5x-12y+k=0$ is 2 units then

$$\left|\frac{5(3)-12(2)+k}{\sqrt{5^2+12^2}}\right|=2\implies\left|\frac{k-9}{13}\right|=2\implies k-9=\pm26$$

$$\therefore\; k-9=26 \text{ and } k-9=-26$$

$\Longrightarrow k = 35$ and $k = -17$

$\therefore$ The required lines are

$5x - 12y - 17 = 0$ and $5x - 12y + 35 = 0$

Exercise:

43. Find the perpendicular distance from origin to the line $x - 3y - 4 = 0$

44. Find the perpendicular distance from the point

- **i.** $(-2, -3)$ to the line $5x - 2y + 4 = 0$
- **ii.** $(-3,4)$ to the line $5x - 12y = 2$
- **iii.** $(3,4)$ to the line $3x - 4y + 10 = 0$
- **iv.** $(-1,1)$ to the line $12(x + 6) = 5(y - 2)$

45. If p and q are the lengths of perpendiculars from the origin to the lines $x \cos\theta - y \sin\theta = k \cos 2\theta$ and $x \sec\theta + y \csc\theta = k$ respectively, prove that $p^2 + 4q^2 = k^2$.

46. Which of the lines $2x - y + 3 = 0$ and $x - 4y - 7 = 0$ is farther from the origin.

47. In a triangle with vertices $A(2,3), B(4,-1)$ and $C(-1,2)$, find the length of the altitude from the vertex A.

48. Find the distance between parallel lines

- **a.** $15x + 8y - 34 = 0$ and $15x + 8y + 31 = 0$.
- **b.** $l(x + y) + p = 0$ and $lx + ly - r = 0$.

49. Find the equation of the straight lines parallel to the line $3x - 4y = 1$ and at a distance of 4 units from (3,4).

50. Show that the product of the perpendiculars drawn from the two points $(\pm\sqrt{a^2 - b^2}, 0)$ to the line $\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1$ is b^2.

51. Find the points on $y - axis$ which is at a distance of b units to the line $\frac{x}{a} + \frac{y}{b} = 1$.

52. If p is the length of perpendicular from the origin to the line $\frac{x}{a} + \frac{y}{b} = 1$, then prove that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

53. Find the points on $x + y = 4$ that lie at a unit distance from the line $4x + 3y - 10 = 0$

54. If the length of the perpendicular from the point $(1,1)$ to the line $ax - by + c = 0$ be unity, show that

$\frac{1}{c} + \frac{1}{a} - \frac{1}{b} = \frac{c}{2ab}$.

6.8. Family of Lines

Learning objectives:

- To derive a formula for the coordinates of the point of intersection of two lines.
- To derive the condition for the concurrency of three lines.
- To derive the equation for the family of lines through the intersection of two given lines.
  And
- To practice related problems.

Intersection of Straight Lines

Consider two lines whose equations are

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

The coordinates of the point of intersection satisfy both the equations. Solving these equations,

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \quad , \quad y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

Therefore, the lines ***intersect*** at the point with the above coordinates provided $a_1b_2 - a_2b_1 \neq 0$. This means that $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$. In case $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then the lines are ***parallel*** and ***there is no point of intersection***. On the other hand if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the lines are ***coincident*** and every point on them is a point of intersection.

Example 1: Find the point of intersection of the line $\frac{x}{3} - \frac{y}{4} = 0, \frac{x}{2} + \frac{y}{3} = 1$.

Solution: The given lines are $4x - 3y = 0$, $3x + 2y - 6 = 0$

Solving these equations, we get

$$x = \frac{(-3)\times(-6) - (2)\times 0}{4\times 2 - 3\times(-3)} = \frac{18}{17} \quad , \quad y = \frac{0\times 3 - (-6)\times 4}{4\times 2 - 3\times(-3)} = \frac{24}{17}$$

Concurrent Lines

Three or more lines are said to be ***concurrent*** *if they intersect at a point*.

It means that any one of these three lines shall pass through the point of intersection of the other two lines.

Consider the lines whose equations are

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$
$$a_3x + b_3y + c_3 = 0$$

The coordinates of the point of intersection of the first two lines are

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \quad y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

These coordinates should satisfy the third equation since the third line passes through the point of intersection of the first two lines. Therefore,

$$a_3\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}\right)+b_3\left(\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)+c_3=0$$

$$a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1)=0$$

This is the condition of the concurrency of three lines.

Note: The above condition is equivalent to

$$a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)=0$$

$$i.e., \quad \sum a_1(b_2c_3-b_3c_2)=0$$

Example 2: Prove that the lines $5x-3y=1;\ 2x+3y=23;$ $42x+21y=257$ are concurrent.

Solution: Given lines are

$$5x-3y-1=0$$
$$2x+3y-23=0$$
$$42x+21y-257=0$$

The condition for three lines

$$a_1x+b_1y+c_1=0$$
$$a_2x+b_2y+c_2=0$$
$$a_3x+b_3y+c_3=0$$

to be concurrent is

$$a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)=0$$

Now,

$$\sum a_1(b_2c_3-b_3c_2)$$
$$=5[-771+483]+2[-21-771]+42[69+3]$$
$$=5(-288)+2(-792)+42(72)$$
$$=-1440-1584+3024=0$$

∴ The given lines are concurrent

Family of Lines through the Intersection of Two Lines

The equation of the family of lines through the intersection of the two lines

$$a_1x+b_1y+c_1=0$$
$$a_2x+b_2y+c_2=0$$

is given by the equation

$$a_1x+b_1y+c_1+\lambda(a_2x+b_2y+c_2)=0$$

As this is a first-degree equation and evidently passes through the intersection of the two given lines. The symbol λ is a parameter and a specific value of λ yields a particular member of the family.

Example 3: Find the equation of the line passing through the point of intersection of $x + 2y = 5, x - 3y = 7$ and passing through the point $(2, -3)$.

Solution:

The equation of any line through the intersection of the given lines is

$$(x + 2y - 5) + \lambda(x - 3y - 7) = 0$$

If the point $(2, -3)$ lies on this equation, then

$$2 + 2(-3) - 5 + \lambda(2 - 3(-3) - 7) = 0 \Longrightarrow \lambda = \frac{9}{4}$$

Using this value of λ, we get

$$(x + 2y - 5) + \frac{9}{4}(x - 3y - 7) = 0$$

$$\Longrightarrow \quad 13x - 19y - 83 = 0$$

PROBLEM SET

IP1: Find the equation of the line parallel to $5x - 2y = 7$ and passing through the point of intersection of the lines $2x + 3y = 1$ and $3x + 4y = 6$.

Solution:

Step1: Equation of the line parallel to $5x - 2y - 7 = 0$ is

$$5x - 2y + k = 0 \qquad \ldots(1)$$

Step2:

By hypothesis (1) passes through the point of intersection of lines $2x + 3y - 1 = 0$ and $3x + 4y - 6 = 0$.

We know that the point of intersection of two lines

$a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0$ is

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \ y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

We have,

$$a_1 = 2, b_1 = 3, c_1 = -1$$

$$a_2 = 3, b_2 = 4, \ c_2 = -6$$

$$\therefore \quad x = \frac{(3\times-6)-(4\times-1)}{(2\times4)-(3\times3)} \ , \ \ y = \frac{(-1\times3)-(-6\times2)}{(2\times4)-(3\times3)}$$

$$\Longrightarrow \quad x = \frac{-18+4}{8-9} = 14 \ , \quad y = \frac{-3+12}{8-9} = -9$$

$$\therefore (x, y) = (14, -9)$$

Step3:

Since the (1) passes through the point $(14, -9)$

$$5(14) - 2(-9) + k = 0 \Longrightarrow \ 88 + k = 0 \Longrightarrow k = -88$$

By substituting the value of k in (1), we get the require line

$$5x - 2y - 88 = 0$$

P1: Find the equation of the straight line parallel to $3x + 4y = 7$ and passing through the point of intersection of the lines $x - 2y - 3 = 0$ and $x + 3y - 6 = 0$.

A. $\mathbf{3x + 4y - 25 = 0}$
B. $\mathbf{3x + 4y + 25 = 0}$
C. $\mathbf{3x + 4y - 15 = 0}$
D. $\mathbf{3x + 4y + 15 = 0}$

Answer: C

Solution:

Equation of the line parallel to $3x + 4y - 7 = 0$ is

$$3x + 4y + k = 0 \qquad \ldots (1)$$

By hypothesis (1) passes through the point of intersection of lines $x - 2y - 3 = 0$ and $x + 3y - 6 = 0$

We know that the point of intersection of two lines

$a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0$ is

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \; y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

We have,

$$a_1 = 1, b_1 = -2, c_1 = -3$$
$$a_2 = 1, b_2 = 3, \; c_2 = -6$$

$$\therefore \; x = \frac{(-2\times-6)-(3\times-3)}{(1\times3)-(1\times-2)} \;, \; y = \frac{(-3\times1)-(-6\times1)}{(1\times3)-(1\times-2)}$$

$$\Rightarrow x = \frac{12+9}{3+2} = \frac{21}{5} \qquad , \quad y = \frac{-3+6}{3+2} = \frac{3}{5}$$

$$\therefore (x, y) = \left(\frac{21}{5}, \frac{3}{5}\right)$$

Since the line (1) passes through the point $(\frac{21}{5}, \frac{3}{5})$

$$3\left(\frac{21}{5}\right) + 4\left(\frac{3}{5}\right) + k = 0 \; \Rightarrow \; \frac{75}{5} + k = 0 \Rightarrow \; k = -15$$

By substituting the value of k in (1), we get the required line

$$3x + 4y - 15 = 0$$

IP2: Show that the lines $2x + y - 3 = 0, 3x + 2y - 2 = 0$ and $2x - 3y - 23 = 0$ are concurrent.

Solution:

Step1: The given lines are

$2x + y - 3 = 0, 3x + 2y - 2 = 0$ and $2x - 3y - 23 = 0$

Step2: We know that the condition for the three lines

$a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$

to be concurrent is

$$a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = 0$$

We have,

$$a_1 = 2, \; b_1 = 1, \; c_1 = -3$$

$a_2 = 3,\ b_2 = 2,\quad c_2 = -2$
$a_3 = 2,\ b_3 = -3, c_3 = -23$

Step3:

Substituting these values in the condition, we get

$$\sum a_1(b_2c_3 - b_3c_2) = 2[-46 - 6] + 3[9 + 23] + 2[-2 + 6]$$
$$= 2(-52) + 3(32) + 2(4)$$
$$= -104 + 96 + 8 = 0$$

∴ The given lines are concurrent

P2: If the lines $3x + y + 2 = 0, 2x - y + 3 = 0$ and $2x + ay - 6 = 0$ are concurrent. Then find the value of a.

A. **2**
B. **4**
C. **6**
D. **8**

Answer: D

Solution:

The given lines are

$3x + y + 2 = 0\ , 2x - y + 3 = 0$ and $2x + ay - 6 = 0$

We know that the condition for the three lines

$a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$

to be concurrent is

$$a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = 0$$

We have,

$a_1 = 3,\ b_1 = 1,\quad c_1 = 2$
$a_2 = 2,\ b_2 = -1,\ c_2 = 3$
$a_3 = 2,\ b_3 = a,\quad c_3 = -6$

Substituting these values in the condition, we get

$$3(6 - 3a) + 2(2a + 6) + 2(3 + 2) = 0$$
$$\Rightarrow 18 - 9a + 4a + 12 + 10 = 0$$
$$\Rightarrow\ -5a + 40 = 0 \ \Rightarrow\ a = 8$$

IP3: Find the equation of the line through the intersection of the lines $2x + 3y = 1, 8x + 3y + 5 = 0$ and perpendicular to the line $2x + 3y + 8 = 0$.

Solution:

Step1: The equation of the line through the intersection of the given lines

$2x + 3y - 1 = 0, 8x + 3y + 5 = 0$ is

$$(2x + 3y - 1) + \lambda(8x + 3y + 5) = 0$$

$i.e.,\ (8\lambda + 2)x + (3\lambda + 3)y + (5\lambda - 1) = 0 \qquad \ldots(1)$

Step2: By hypothesis (1) is perpendicular to the line $\quad 2x + 3y + 8 = 0$

We know that if two lines $a_1x + b_1y + c_1 = 0$,
$a_2x + b_2y + c_2 = 0$ are perpendicular $\Leftrightarrow$ $a_1a_2 + b_1b_2 = 0$

$$\therefore \quad 2(8\lambda + 2) + 3(3\lambda + 3) = 0$$
$$\Rightarrow 16\lambda + 4 + 9\lambda + 9 = 0$$
$$\Rightarrow 25\lambda + 13 = 0 \Rightarrow \lambda = \frac{-13}{25}$$

Step3: Substituting value of λ in (1), we get the required line

$$\left(8\left(\frac{-13}{25}\right) + 2\right)x + \left(3\left(\frac{-13}{25}\right) + 3\right)y + \left(5\left(\frac{-13}{25}\right) - 1\right) = 0$$
$$\Rightarrow \frac{-54}{25}x + \frac{36}{25}y - \frac{90}{25} = 0 \Rightarrow -54x + 36y - 90 = 0$$

$i.e.,\ 3x - 2y + 5 = 0$

P3: Find the equation of the line through the intersection of the lines $x + 3y - 1 = 0, x - 2y + 4 = 0$ and perpendicular to the line $2x + 3y = 0$.

A. $\mathbf{3x - 2y - 8 = 0}$
B. $\mathbf{3x + 2y - 8 = 0}$
C. $\mathbf{3x - 2y + 8 = 0}$
D. $\mathbf{3x - 2y - 10 = 0}$ **Answer:** C

Solution: The equation of the line through the intersection of the lines
$x + 3y - 1 = 0, x - 2y + 4 = 0$ is

$$(x + 3y - 1) + \lambda(x - 2y + 4) = 0$$

$i.e.,\ (\lambda + 1)x + (3 - 2\lambda)y + (4\lambda - 1) = 0 \qquad \ldots(1)$

By hypothesis (1) is perpendicular to $2x + 3y = 0$

We know that if two lines $a_1x + b_1y + c_1 = 0$,
$a_2x + b_2y + c_2 = 0$ are perpendicular $\Leftrightarrow$ $a_1a_2 + b_1b_2 = 0$

$$\therefore \quad 2(\lambda + 1) + 3(3 - 2\lambda) = 0$$
$$\Rightarrow 2\lambda + 2 + 9 - 6\lambda = 0$$
$$\Rightarrow 11 - 4\lambda = 0 \Rightarrow \lambda = \frac{11}{4}$$

Substituting the λ value in (1), we get the required line

$$\left(\frac{11}{4} + 1\right)x + \left(3 - 2\left(\frac{11}{4}\right)\right)y + \left(4\left(\frac{11}{4}\right) - 1\right) = 0$$
$$\Rightarrow \frac{15}{4}x - \frac{10}{4}y + 10 = 0 \Rightarrow 15x - 10y + 40 = 0$$

$i.e.,\ 3x - 2y + 8 = 0$

IP4: Find the equation of the line passing through the intersection of the lines $x + 2y - 5 = 0, 2x - y + 4 = 0$ and that has equal intercepts on the axes.

Solution:

Step1:

The equation of the family of lines passing through the intersection of lines

$$x + 2y - 5 = 0, 2x - y + 4 = 0 \text{ is}$$
$$(x + 2y - 5) + \lambda(2x - y + 4) = 0$$

$i.e.,\ (2\lambda+1)x+(2-\lambda)y+(4\lambda-5)=0 \qquad \dots(1)$

Step2:

By hypothesis (1) has equal intercepts on the axes

Now reduce (1) into intercept form

$$(2\lambda+1)x+(2-\lambda)y=-(4\lambda-5)$$

Dividing both sides by $-(4\lambda-5)$ then

$$\frac{(2\lambda+1)x}{-(4\lambda-5)}+\frac{(2-\lambda)y}{-(4\lambda-5)}=1$$

$$\Rightarrow \frac{x}{\frac{-(4\lambda-5)}{2\lambda+1}}+\frac{y}{\frac{-(4\lambda-5)}{2-\lambda}}=1$$

x-intercept is $\frac{-(4\lambda-5)}{2\lambda+1}$ and y-intercept is $\frac{-(4\lambda-5)}{2-\lambda}$

$\because$ (1) has equal intercepts

$$\frac{-(4\lambda-5)}{2\lambda+1}=\frac{-(4\lambda-5)}{2-\lambda} \Rightarrow 2-\lambda=2\lambda+1 \Rightarrow \lambda=\frac{1}{3}$$

Step3:

Substituting value of λ in (1), we get the require line

$$\left(2\left(\frac{1}{3}\right)+1\right)x+\left(2-\frac{1}{3}\right)y+\left(4\left(\frac{1}{3}\right)-5\right)=0$$

$$\Rightarrow \frac{5}{3}x+\frac{5}{3}y-\frac{11}{3}=0\ ,\ i.e.,\ 5x+5y-11=0$$

P4: Find the equation of the line passing through the intersection of the lines $\boldsymbol{2x-5y+1=0, 3x+2y=8}$ and that has equal intercepts on the axes.

A. $\boldsymbol{x+y+1=0}$

B. $\boldsymbol{x+y+3=0}$

C. $\boldsymbol{x+y-3=0}$

D. $\boldsymbol{x+y-1=0}$ **Answer:** C

Solution: The equation of the family of the lines through the intersection of the lines $2x-5y+1=0, 3x+2y-8=0$ is

$$(2x-5y+1)+\lambda(3x+2y-8)=0$$

$i.e.,\ (3\lambda+2)x+(2\lambda-5)y+(1-8\lambda)=0 \qquad \dots(1)$

By hypothesis (1) has equal intercepts on the axes

Now reduce (1) into intercept form

$$(3\lambda+2)x+(2\lambda-5)y=8\lambda-1$$

Dividing both sides by $8\lambda-1$ then

$$\frac{(3\lambda+2)x}{8\lambda-1}+\frac{(2\lambda-5)y}{8\lambda-1}=1$$

$$\Rightarrow \frac{x}{\frac{8\lambda-1}{3\lambda+2}}+\frac{y}{\frac{8\lambda-1}{2\lambda-5}}=1$$

x-intercept is $\frac{8\lambda-1}{3\lambda+2}$ and y-intercept is $\frac{8\lambda-1}{2\lambda-5}$

$\because$ (1) has equal intercepts

$$\frac{8\lambda-1}{3\lambda+2}=\frac{8\lambda-1}{2\lambda-5} \Rightarrow 2\lambda-5=3\lambda+2 \Rightarrow \lambda=-7$$

Substituting value of λ in (1), we get the required line

$$(3(-7)+2)x+(2(-7)-5)y+(1-8(-7))=0$$

$$\Longrightarrow\ -19x-19y+57=0\ \ , i.e.,\ \ x+y-3=0$$

Exercise:

1.

a. Find the equation of the straight line joining the origin to the point of intersection of $y-x+7=0$ and $y+2x-2=0$.

b. Determine k if the line $3x+3y+k=0$ passes through the point of intersection of $3x+4y+6=0$ and $6x+5y-9=0$.

2.

a. Find the equation of the line parallel to $3x-2y+5=0$ and passing through the point of the intersection of lines $5x-2y=12$ and $4x-7y-15=0$.

b. Find the equation of the line through the intersection of lines $x+2y-3=0, 4x-y+7=0$ and which is parallel to $5x+4y-20=0$.

c. Find the equation of the line through the intersection of the lines $5x-3y=1, 2x+3y-23=0$ and perpendicular to the line whose equation is $5x-3y-1=0$.

d. Find the equation of the line passing through intersection of lines $x-3y+1=0\ , 2x+5y-9=0$ and

$(\boldsymbol{i})$Parallel to $(\boldsymbol{ii})$ Perpendicular to the line $2x+y+5=0$

e. Find the equation of the line passing through the intersection of the lines $4x+7y-3=0$, $2x-3y+1=0$ that has equal intercepts on the axes.

3.

a. Prove that the three lines $2x-y-5=0$, $3x-y-6=0\ , 4x-y-7=0$ meet in a point.

b. Show that the lines $(a-b)x+(b-c)y=c-a$, $(b-c)x+(c-a)y=a-b$ and $(a-b)x+(c-a)y=b-c$ are concurrent. Also find the point of concurrence.

c. If the straight lines $4x-3y-7=0, 2x+\lambda y+2=0$ and $6x+5y-1=0$ are concurrent. Find the value of λ. Also find the point of concurrence.

d. For what values of a the three straight lines $3x+y+2=0, x-y+3=0,\ x+2ay+5=0$, are concurrent.

e. For what values of *k* are the three lines $4x+7y-9=0, 5x+ky+15=0, 9x-y+6=0$ are concurrent?