THE PENNSYLVANIA STATE UNIVERSITY
SCHREYER HONORS COLLEGE

HAROLD AND INGE MARCUS DEPARTMENT OF INDUSTRIAL ENGINEERING

Case study in Portfolio Optimization

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SPRING 2015

A thesis
submitted in partial fulfillment
of the requirements
for a baccalaureate degrees
in Industrial Engineering
with honors in Industrial Engineering

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Abstract

This thesis deals with investment portfolio selection problems. The goal is to determine an optimal investment policy. The securities chosen are 20 Vanguard mutual funds. Return values from the mutual funds over a 10-year period from 2004 to 2014 were collected, together with beta values of the funds using Morningstar.com. Standard deviation of return of the funds and the covariance between funds were calculated. The Sharpe's bi-criteria linear programming model and Markowitz's bi-criteria quadratic programming model are used to maximize return while minimizing risk for portfolios of a given risk or return level. Beta values were used to represent risk in the Sharpe's Model while variance-covariance matrix was used to represent risk in the Markowitz Model. For each model, maximum return and minimum risk portfolios were determined. Efficient frontier connecting the various efficient portfolios was drawn and 8 efficient portfolio options with different return and risk values were presented for an investor to choose from. It was found that as portfolio return increased, risk of investing in the portfolio increased as well. This is because return and risk are conflicting objectives. The efficient portfolios are different depending on the type of investor. A conservative investor intending to minimize risk invested in a portfolio with low risk and accepted a low return. An aggressive investor invested in a portfolio with high return without much consideration to the risk. Within portfolios, it was observed that funds with a lower beta risk and lower standard deviation values were often chosen over funds that had higher risk for the same return. This helps to maximize return while minimizing risk. As the risk of a portfolio increased, funds that provided greater returns were chosen. Covariance was an additional component of risk in the Markowitz Model. When 2 funds had lower covariance between each other, overall risk of the portfolio decreased. It was found that the standard deviation

of a portfolio in the Markowitz Model could be reduced to some extent by choosing a diverse group of funds that have low covariance between each other. The beta risk of a portfolio in the Sharpe's Model can be reduced by choosing funds with lower beta values.

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Acknowledgments

I would like to express my sincere gratitude to Professor Ravindran for helping me discover my interest in bi-criteria optimization and guiding me through every step of the journey very patiently. It has been a great learning experience.

I would like to specially thank Dr Griffin for supporting me throughout my three years at Penn State and never failing to have faith in my abilities.

I would like to thank my parents, brother, grandparents and family for believing in my dreams.

Chapter 1
Introduction

1.1. Background

Investment companies are often faced with the problem of determining optimal investment portfolios. Optimal investment portfolios provide the largest amount of return to the investor for a given amount of risk they are willing to take. This is a complex problem with a large economic significance (Ravindran et. al., 2006). In today's world, determining an optimal portfolio through intuition or experience will put an investment company out of business in no time. Modern portfolio theory provides a solution to this problem.

Modern Portfolio Theory or simply Portfolio Theory is a mathematical approach to quantifying the trade-off between risk and return of a portfolio. An economist named Harry Markowitz at the City University of New York's Baruch College first introduced it in 1952. For the first time, it allowed fund managers and investors to mathematically predict the future performance of a portfolio. It is regarded as an important advancement in the field of Finance and Markowitz won a Nobel Prize for the development of this theory (Egan).

Understanding Portfolio theory is crucial for any type of investor, inexperienced or experienced, small capital or large capital investor. It is assumed that a rational investor will not pick a portfolio that has a greater amount of risk between two portfolios with the same amount of return. The investor wants the best possible "value" for their investment in terms of return, based on the amount of risk they take.

The “best” portfolios for a given risk level are called efficient portfolios. Efficient portfolios provide the highest possible return for a given risk level or the lowest possible risk for a given return level. There is no better “solution” than an efficient portfolio (Markowitz, 1956). Efficient portfolios vary depending on the risk level an investor is willing to take. The Markowitz Model provides a mathematical model to find the efficient portfolios at different risk levels.

The Markowitz Model measures the risk of a portfolio based on the fluctuation of the returns of the securities in it and how they work in combination with each other. The idea is that prices of securities fluctuate in relation to the performance of the market as a whole and also in relation to one another (Ravindran et. al., 2006). Also, prices of certain securities move up and down together in a similar way at roughly the same time. It is suggested that these securities do not act independently but act with respect to one another. This has to be factored into the risk of a portfolio. Let us consider a simple small-scale portfolio with 2 securities. It might not seem obvious that picking two securities that experienced a raise in price at the same time might be a bad idea. However, what this means is that when the price of one of these securities drops, it is highly likely that the price of the other security would drop as well. This is called positive covariance. This means that the return of the portfolio would decrease even further during bad times, more so than if one security was to experience a price drop, while the price of the other security stayed the same. This translates to the portfolio having a higher risk associated with it.

However, two different securities might work differently from each other. When the price of one security increases, the price of the other security falls. This is called negative covariance. Using these two securities that work differently from each other is a smarter option to reduce the risk of a portfolio. It might also be that the returns from two securities might not be connected with each other. This means that the securities have very low covariance. This means that price drop or price rise of one security does not affect the other security and vice-versa. Picking two securities that have very little correlation between them is also a good idea (Markowitz, 1956). This is a key insight from the Markowitz Model.

It is important to choose several securities so that the risk of a portfolio can be substantially reduced without losing much in return. This is termed diversification. It can be achieved by purchasing a wide variety of securities such as stocks, bonds and mutual funds. Diversification can also be achieved within a category of securities by choosing securities that have a low covariance. This helps to reduce the risk of a portfolio further, allowing for returns to be maximized.

An alternative version of the Markowitz Model is the Sharpe's Model. William Sharpe of Stanford University developed it in 1963. The motivation for the simplified model came from the fact that the Markowitz Model took a long time to run given the processing power of computers at that time. This can be attributed to the complex risk function of the Markowitz Model, which is a quadratic in nature. One the other hand, the Sharpe's Model is a fully linear model and it takes much less power to run on the computer. It is also easier to consider a much larger group of securities using the Sharpe's Model due to the simplicity of its risk function compared to the Markowitz Model (Egan).

The risk of a portfolio is measured differently in the Sharpe's Model. It neither considers how the price or return of a security changes with regards to another security nor how the price fluctuates over time. Risk is measured by the fluctuation of the return of the security as compared to the fluctuation of prices or return in the entire stock market, usually S&P 500 index. This is a new way of measuring risk that greatly simplifies the Markowitz Model and still has a large amount of validity to it. The results of the Sharpe's Model are often close to the results of the Markowitz Model, with small differences in security choices in a portfolio. Sharpe was given the Nobel Prize (Egan) for building on the discovery of Markowitz by simplifying the original model and coming up with a new way to measure risk.

It has been stated so far that the Markowitz Model and Sharpe's Model guide an investor on exactly which securities to invest in. They go one step further and provide the critical information on exactly how much to invest in each security in a portfolio in order to maximize returns and minimize risk at the same time. The proportion of money invested in each security has a large bearing on the return. In order to maximize return, the proportion of money invested in each security has to be determined accurately with the help of these mathematical models.

1.2. Problem Statement

An investor has $250,000 to invest in a portfolio and wants to get the highest return possible for their risk appetite. The investor could be conservative, moderate or aggressive. A diverse group of 20 mutual funds from The Vanguard Group are chosen. The goal is to find the efficient portfolios that provide the highest possible return for different risk levels and the proportion of money required to be spent on each type of fund for a given efficient portfolio. The Markowitz Model and Sharpe's Model are used. Eight options of efficient portfolios are presented in each model for an investor to choose from. A comparison of results from both models is made in order to understand how similar the solutions are to each other.

1.3. Literature Review

1.3.1 Sharpe's Model

The Sharpe's Model is given by Equations (1.1) to (1.4) where x_J is the amount of money invested in security j and μ_j is the average return over a period of years (Sharpe, 1963). z_1 refers to the expected return function of the portfolio, which is made up of the sum of average return values of each security (μ_j) and the amount of money spend on that security x_J multiplied together, while z_2 referes to the portfolio risk function. β_j is the beta risk of the security j. Beta value is the indicator of risk used in the Sharpe's Model and it measures fluctuation of return of a security as compared to the fluctuation of return in the stock market. z_2 is made up of the beta values of each security β_j multiplied with the corresponding amount of money spend on that security x_J (Sharpe, 1963). C is the capital available for investment.

$$\text{MAX} \quad z_1 = \sum_{j=1}^{N} \mu_j x_j \qquad (1.1)$$

$$\text{MIN} \quad z_2 = \sum_{j=1}^{N} \beta_j x_j \qquad (1.2)$$

$$\text{Subject to} \sum_{j=1}^{N} x_j \leq \text{C} \qquad (1.3)$$

$$x_j \geq 0 \qquad (1.4)$$

1.3.2 Markowitz Model

The Markowitz Model is given by Equations (1.5) to (1.9) where x_J is the amount of money invested in security j and μ_J is the average return value of x_J over a period of years. z_1 refers to the portfolio return function, z_2 refers to the variance of the portfolio return. It is obtained by the matrix multiplication of the transpose of x with matrix Q, the variance-covariance matrix. This is then multiplied by a matrix of x to get the function for risk. q_{ij} is the covariance between security i and security j. It can be obtained by Equation 1.7 where T is the time period that is considered, t is a given time period, $r_i(t)$ and $r_j(t)$ are the total return per dollar invested in security i and security j respectively in year t (Ravindran et. al., 2006). When i and j are equal, q_{ij} represents the variance of return of security i. C is the capital available for investment.

$$\text{MAX } z_1 = \sum_{j=1}^{N} \mu_j x_j \quad (1.5)$$

$$\text{MIN } z_2 = x^T Q x = \sum_{i=1}^{N} \sum_{j=1}^{N} q_{ij} x_j x_j \quad (1.6)$$

$$Q = [q_{ij}] = [\sigma_{ij}^2] = \frac{1}{T} \sum_{t=1}^{T} [r_i(t) - \mu_i][r_j(t) - \mu_j] \quad (1.7)$$

$$\text{Subject to } \sum_{j=1}^{N} x_j \leq \text{C} \quad (1.8)$$

$$x_j \geq 0 \quad (1.9)$$

1.4. Organization of the Thesis

This thesis is an application with real data testing the Markowitz Model and Sharpe's Model and comparing their solutions. Chapter 2 provides the method that was used to choose mutual funds, collect the data and run the Sharpe's Model and the Markowitz Model. Chapter 3 discusses the results obtained from the Sharpe's Model and Markowitz Model in terms of ideal solution, efficient portfolios, 8 portfolio options, trend by fund category, most used funds, unused funds and confidence intervals. At the end of Chapter 3, a comparison is made between both models with regards to their linear and quadratic nature, the selection of funds, the portfolio options and the proportion of funds. Chapter 4 is the conclusion. It provides key findings that were reached and extensions that can be made to this thesis.

Chapter 2
Methodology

2.1. Choice of Mutual Funds

Mutual Funds are the securities chosen to test the Sharpe's Model and Markowitz Model. Buying mutual funds involves buying a basket of various stocks, bonds and money market funds. This means the buying a single mutual fund could involve buying small portions of various stocks and/or bonds that have been grouped together. In this way mutual funds naturally provide greater diversification and generally involve lower risk than individual stocks and better return than bonds. A stock mutual fund made up of different types of stocks grouped together will have lower risk than the individual stocks and yet offer a good value of return for a lower risk level (Investopedia, LLC, 2015b). Mutual funds were chosen due to the diversification they provide. They are typically safer investments than stocks and various types of investors, including conservative investors can benefit from putting their money in them. However, investing all the money in just one mutual fund might not be a good idea as it is still prone to the fluctuations of the market and finding one fund that meets all the investor's needs is very difficult. Choosing a diverse group of mutual funds is necessary (Markowitz, 1956). In this thesis, 20 mutual funds have been chosen so that there is a good number and variety of options to create portfolios of varying risk and return, at the same time without having two or three funds that provide a similar type of risk and return. Table 2.1 shows the 20 mutual funds that were selected.

Number	Name	Ticker Symbol	Morningstar® Category	MorningStar Rating™
1	Vanguard Small Capitalization Value Index Fund Inv	VISVX	Small Cap Value Index Fund	5
2	Vanguard International Explorer Inv	VINEX	Foreign Small/Mid Blend Mid Growth Stoc	3
3	Vanguard International Growth Inv	VWIGX	Large Growth Foreign Large Growth	3
4	Vanguard International Value Inv	VTRIX	Foreign Large Value Large Blend	4
5	Vanguard International Value Inv	VGTSX	Foreign Large Blend Large Blend	3
6	Vanguard Precious Metals and Mining Inv	VGPMX	Precious Metals and Mining	4
7	Vanguard Energy Inv	VGENX	Energy	4
8	Vanguard Inflation-Protected Secs Inv	VIPSX	Bond Inter-term Government	4
9	Vanguard Total Bond Market Index Adm	VBTLX	Bond Inter-term Investment	4
10	Vanguard Long-Term Treasury Inv	VUSTX	Bond Long-term Government	4
11	Vanguard Long-Term Investment-Grade Inv	VWESX	Bond Long-term Investment	4
12	Vanguard Short-Term Treasury Inv	VFISX	Bond Short-term Government	3
13	Vanguard Short-Term Investment-Grade Inv	VFSTX	Bond Short-term Investment	4
14	Vanguard Prime Money Market Inv	VMMXX	Money Market Fund	Not Rated
15	Vanguard Explorer Inv	VEXPX	Small Cap Growth Stock	4
16	Vanguard Mid Cap Growth Inv	VMGRX	Mid Cap Growth Stock	4
17	Vanguard Capital Value Inv	VCVLX	Mid Cap Value Stock	2
18	Vanguard Strategic Equity Inv	VSEQX	Mid Cap Blend Stock	4
19	Vanguard US Growth Inv	VWUSX	Large Cap Growth Stock	3
20	Vanguard US Value Inv	VUVLX	Large Cap Value Stock	4

Table 2.1: Mutual Funds, their Ticker Symbol, Morningstar Category and Morningstar Rating

Mutual Funds were chosen from the Morningstar website (www.morningstar.com). Morningstar is a reputable company that collects data on different types of securities including stocks, bonds and mutual funds and has valuable information on these securities that can guide an investor's decision making process as to which securities to invest in and how much to invest in them.

For this thesis, mutual funds from The Vanguard Group were solely used. It is the largest mutual fund company in the United States and offers more than 170 types of funds. When choosing mutual funds to be used, the category of the funds was considered so as to provide diversification (The Vanguard Group, Inc., 2015). Morningstar rating was used in the selection of funds as well. Morningstar funds are rated from 1 to 5 based on past performance in comparison to similar funds. Ratings are based on mathematical evaluation and are objective (Morningstar, Inc., 2015b). The 20 funds chosen had an average Morningstar rating of 3.7 out of 5, which means that are high quality funds. It was not so much a choice of which individual funds were used but rather which fund categories they represented and whether they contributed to the diversification of the portfolio. Diversification offers better returns for the same risk level and this is an important finding from the Markowitz Model. The funds also had to be in existence for 10 years so that they are somewhat established and also provide a good amount of data required for generating and running the models. Mutual funds had to have a minimum investment amount of $3000 and no more than that so that the cost of buying the fund is minimal.

The mutual funds were arranged in fund categories in as shown Table 2.2 by simplifying their initial Morningstar Category as shown in Table 2.1. There are 6 categories of mutual funds: index funds, international funds, specialty funds, government and investment bond funds, money market funds and stock funds. Index funds follow the fluctuations of market indices such as S&P 500 or the Barclays Bond Index, while Money Market funds invest in low risk securities (Investopedia, LLC, 2015b). A fund from each of these categories was used to contribute to the diversification of the portfolio. Two Specialty funds were used to provide options for investors to invest in metals and energy. The majority of the funds were international funds, bond funds and stock funds. The international funds used are typically international stock funds that contain a variety of stocks from outside the United States. Six different stock funds and 6 different bond funds were chosen to provide an investor with equal opportunities to make a safer portfolio by investing in bonds or a higher return portfolio by investing in stocks.

Funds
Index Funds
Small Cap Value Index Fund Inv
International Funds
International Explorer Inv
International Growth Inv
International Blend Value Inv
International Value Inv
Speciality Funds
Precious Metals and Mining Inv
Energy Inv
Government and Investment Bond Funds
Inflation-Protected Secs Inv
Total Bond Market Index Adm
Long-Term Treasury Inv
Long-Term Investment-Grade Inv
Short-Term Treasury Inv
Short-Term Investment-Grade Inv
Money Market Funds
Prime Money Market Inv
Stock Funds
Explorer Inv
Mid Cap Growth Inv
Capital Value Inv
Strategic Equity Inv
US Growth Inv
US Value Inv

Table 2.2: Categories of Mutual Funds

The investment amount chosen is $250,000. It is large enough to do a portfolio analysis and at the same time a realistic amount that quite a few actual investors could have accumulated.

2.2. Data Collection

For each of the mutual funds, average annual returns for 10 years from 2004 to 2014 were collected and simple average was calculated to get a 10-year average return for each fund. Beta risk for each fund was collected for use in the Sharpe's Model. With the values of annual return for the 10 years, standard deviations of the funds were calculated for the Markowitz Model. Variances of the funds and the covariances between funds were also tabulated for use in the risk objective function of the

Markowitz Model, which was discussed in Section 2.4. The data collected from the Morningstar website (Morningstar, Inc., 2014a) is summarized in Table 2.3.

Funds	10 yr return	Beta Risk	Std Dev
Index Funds			
Small Cap Value Index Fund Inv	11.6%	1.24	20.9%
International Funds			
International Explorer Inv	13.9%	1	27.8%
International Growth Inv	11.7%	1.02	24.1%
International Blend Value Inv	10.5%	0.99	22.5%
International Value Inv	10.1%	1.01	23.1%
Speciality Funds			
Precious Metals and Mining Inv	11.0%	1.44	41.5%
Energy Inv	16.6%	1.14	26.2%
Government and Investment Bond Funds			
Inflation-Protected Secs Inv	4.8%	1.49	7.0%
Total Bond Market Index Adm	4.6%	1.01	2.8%
Long-Term Treasury Inv	6.4%	2.73	13.1%
Long-Term Investment-Grade Inv	6.5%	2.65	6.3%
Short-Term Treasury Inv	2.8%	0.34	2.6%
Short-Term Investment-Grade Inv	3.7%	0.46	4.7%
Money Market Funds			
Prime Money Market Inv	1.8%	0	2.0%
Stock Funds			
Explorer Inv	11.8%	1.18	23.3%
Mid Cap Growth Inv	12.1%	1.06	21.4%
Capital Value Inv	13.6%	1.38	34.8%
Strategic Equity Inv	11.6%	1.22	22.8%
US Growth Inv	9.2%	1.02	20.6%
US Value Inv	8.4%	1.03	17.9%

Table 2.3: 20 mutual funds, their 10-year average return, beta risk and standard deviation

2.3. Mathematical Models

The Sharpe's Model and the Markowitz Model were used in this thesis. Both models have the same constraints as shown in Equations (2.1) to (2.9). There are 9 constraints, 20 variables and two conflicting objective functions for both the models.

X_i=Dollars invested in mutual fund i. i=1,2, ... , 20

$$X_i \leq 50{,}000 \text{ (No single mutual fund should be worth more than \$50,000)} \quad (2.1)$$

$$\sum X_i = 250{,}000 \text{ (Total sum of mutual funds is equal to \$250,000)} \quad (2.2)$$

$$50{,}000 \leq \sum_8^{13} X_i \leq 100{,}000 \text{ (U.S. Bonds 20-40\%)} \quad (2.3)$$

$$75{,}000 \leq \sum_{15}^{20} X_i \leq 125{,}000 \text{ (U.S. stocks 30-50\%)} \quad (2.4)$$

$$\sum_2^5 X_i + \sum_{15}^{20} X_i \leq 187{,}500 \text{ (Total stock less than 75\%)} \quad (2.5)$$

$$25{,}000 \leq \sum_2^5 X_i \leq 62{,}500 \text{(International funds between 10 and 25\%)} \quad (2.6)$$

$$X_{14} \geq 12{,}500 \text{ (Money market fund more than 5\%)} \quad (2.7)$$

$$X_1 + X_6 + X_7 \leq 50{,}000$$
$$\text{(Index Funds and Specialty funds less than or equal to 20\%)} \quad (2.8)$$

$$X_i \geq 0 \text{ (Money invested in each fund has to be a positive number)} \quad (2.9)$$

X_i is the amount of dollars invested in mutual fund i, where i takes values from 1 through 20. The constraints allow for no single mutual fund to be worth more than 20% of the total portfolio. It equates the sum of money invested in each of the funds to the investment amount of $250,000. The constraints provide a range to the proportion of money that should be invested in each category of mutual fund shown in Table 2.2. 20% to 40% of the money can be invested in U.S. Bond Funds, 30% to 50% of the money can be invested in U.S. Stock Funds, 10% to 25% of the money can be invested in International Funds, at least 5% of the money can be invested in Money Market funds and no more than 20% of the money can be invested in Index Funds and Specialty funds. Providing a range to the proportion of money that can be invested in each category allows for a variety of risk and return values to be generated with different combinations of investment in each fund. A conservative investor might invest 40% in US Bonds and 30% in US stocks while an aggressive investor might invest 50% in US Stocks and 20% in US Bonds given that bond funds have lower risk than stock funds.

Both the Sharpe's Model and the Markowitz Model have the same expression for the objective function for maximizing return from the investment. The objective function for return was calculated by first multiplying the 10-year average return for each fund with X_i, the amount invested in that fund and then adding up these values for all funds (Sharpe, 1963). Then these sum products were divided by the investment amount of \$250,000 to get final objective function for return shown in Equation (2.10).

$$\text{maximize} \quad \frac{\begin{array}{c}11.571X_1 + 13.923X_2 + 11.748X_3 + 10.485X_4 + 10.094X_5 + 11.039X_6 + \\ 16.592X_7 + 4.81X_8 + 4.562X_9 + 6.378X_{10} + 6.54X_{11} + 2.806X_{12} + 3.708X_{13} + \\ 1.76X_{14} + 11.839X_{15} + 12.074X_{16} + 13.587X_{17} + 11.559X_{18} + 9.2X_{19} + 8.408X_{20}\end{array}}{250{,}000} \tag{2.10}$$

The objective function for risk is different in the Sharpe's Model and in the Markowitz Model and is discussed in Sections 2.4. and 2.5.

2.4. Sharpe's Model

The Sharpe's Model is a bi-criteria linear programming model where the objective functions and constraints are linear. We have seen the constraints and objective function for return in the Sharpe's Model in Equations (2.1) to (2.10). While maximizing return is an objective, minimizing risk is also an objective. It is important to minimize risk while trying to maximize return. The objective function for risk is given by Equation (2.11).

$$\text{minimize} \quad \frac{\begin{array}{c}1.24X_1 + 1X_2 + 1.02X_3 + 0.99X_4 + 1.01X_5 + 1.44X_6 + \\ 1.14X_7 + 1.49X_8 + 1.01X_9 + 2.73X_{10} + 2.65X_{11} + 0.34X_{12} + 0.46X_{13} + \\ 0X_{14} + 1.18X_{15} + 1.06X_{16} + 1.38X_{17} + 1.22X_{18} + 1.02X_{19} + 1.03X_{20}\end{array}}{250{,}000} \tag{2.11}$$

Equation (2.11) was calculated by multiplying the Beta risk of each of the funds with X_i, the amount invested in that fund, adding up these values and dividing the total sum by the investment amount of $250,000. This represents the portfolio's risk (Sharpe, 1963). In this model, the risk increases (or decreases) linearly as the amount invested in a fund increases (or decreases). For example, when $100 is invested in fund X_1, the risk increases by $124, which is $100 multiplied by the beta risk of 1.24 of fund X_1.

2.5. Markowitz Model

Risk in the Markowitz Model is a quadratic function. It is obtained by the matrix multiplication of the transpose of X_i with matrix Q, the matrix of quadratic coefficients. This is then multiplied by a matrix of X_i to get the function for risk (Markowitz, 1956). This value is then square rooted and divided by the investment amount of $250,000 to get the standard deviation of a portfolio. Matrix Q consists of covariance between the funds and the variance of each fund. Covariance between fund X_1 and fund X_2 can be obtained from matrix Q by finding the row corresponding to X_1 and matching it with the column corresponding to X_2. Note that the covariance between a fund and itself is simply the variance of the fund. Covariance between fund X_2 and itself is 773.9492 as shown in Table 2.4, which is also the variance of X_2. The variance of each fund is shown on the diagonal of matrix Q in Table 2.4.

Q=	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	X11	X12	X13	X14	X15	X16	X17	X18	X19	X20
X1	435.9941	480.8279	391.9650	358.3932	356.8788	319.8521	291.9452	-10.1618	-22.9257	-195.9968	-9.7437	-37.8293	50.6649	-17.1592	413.4718	360.7158	584.1050	416.9103	332.6768	316.5500
X2	480.8279	773.9492	587.5099	545.3326	556.8804	703.6288	529.5505	9.0209	-22.8566	-275.2307	-17.5869	-38.6201	84.9260	-7.5853	515.2940	479.8216	779.0690	514.3816	436.3970	377.9740
X3	391.9650	587.5099	582.5884	483.1715	497.3537	649.7635	474.3016	25.0977	-14.0429	-232.6645	-9.5254	-27.8251	82.5330	-3.2722	443.5618	431.5058	667.3079	434.9041	395.5469	325.5692
X4	358.3932	545.3326	483.1715	507.7736	463.2787	565.5574	445.7012	13.9165	-17.8973	-217.1830	-15.8100	-27.1195	69.7492	-0.6510	394.6967	382.0083	593.8616	397.2508	356.2649	304.1131
X5	356.8788	556.8804	497.3537	463.2787	532.5794	645.8060	467.7392	30.5521	-10.7943	-209.8581	-4.6993	-24.5672	77.8594	0.0541	395.4402	391.5267	603.8128	394.1170	355.3923	296.0511
X6	319.8521	703.6288	649.7635	565.5574	645.8060	1722.9606	763.3155	123.7379	31.6676	-203.4881	43.2491	-2.9640	148.5717	18.5099	407.4681	475.6221	737.4446	356.7890	370.8497	204.2092
X7	291.9452	529.5505	474.3016	445.7012	467.7392	763.3155	684.9609	51.0414	-8.4189	-171.2246	-5.7563	-18.2473	71.2914	7.1913	366.8698	376.2936	495.5419	357.2425	344.9912	256.6661
X8	-10.1618	9.0209	25.0977	13.9165	30.5521	123.7379	51.0414	49.1019	14.6100	26.4916	33.0271	1.9434	15.9581	-0.7187	-1.3201	19.9013	12.2692	-0.6702	11.8665	-5.8291
X9	-22.9257	-22.8566	-14.0429	-17.8973	-10.7943	31.6676	-8.4189	14.6100	7.9864	19.7656	11.9734	3.3440	3.4267	0.7659	-23.7711	-13.9046	-25.0595	-25.0324	-19.5847	-21.7414
X10	-195.9968	-275.2307	-232.6645	-217.1830	-209.8581	-203.4881	-171.2246	26.4916	19.7656	171.7998	38.1312	14.5831	-31.4037	2.9076	-221.5451	-197.2171	-357.2086	-209.1026	-195.4703	-150.5224
X11	-9.7437	-17.5869	-9.5254	-15.8100	-4.6993	43.2491	-5.7563	33.0271	11.9734	38.1312	40.3158	-2.1331	7.1766	-4.3848	-15.8171	-5.6191	-12.4122	-7.9040	-10.0932	-7.6077
X12	-37.8293	-38.6201	-27.8251	-27.1195	-24.5672	-2.9640	-18.2473	1.9434	3.3440	14.5831	-2.1331	6.7921	-2.3306	3.5578	-36.9144	-28.4051	-52.3460	-41.2840	-30.8774	-32.0163
X13	50.6649	84.9260	82.5330	69.7492	77.8594	148.5717	71.2914	15.9581	3.4267	-31.4037	7.1766	-2.3306	22.5098	-0.6929	63.7610	69.9254	118.3512	57.8722	61.2642	37.4217
X14	-17.1592	-7.5853	-3.2722	-0.6510	0.0541	18.5099	7.1913	-0.7187	0.7659	2.9076	-4.3848	3.5578	-0.6929	4.1679	-18.1490	-12.1358	-24.6166	-18.9679	-14.3873	-13.4458
X15	413.4718	515.2940	443.5618	394.6967	395.4402	407.4681	366.8698	-1.3201	-23.7711	-221.5451	-15.8171	-36.9144	63.7610	-18.1490	542.4531	438.7407	656.8034	467.3024	413.9254	350.9285
X16	360.7158	479.8216	431.5058	382.0083	391.5267	475.6221	376.2936	19.9013	-13.9046	-197.2171	-5.6191	-28.4051	69.9254	-12.1358	438.7407	459.0879	605.8471	415.7416	384.2133	310.9409
X17	584.1050	779.0690	667.3079	593.8616	603.8128	737.4446	495.5419	12.2692	-25.0595	-357.2086	-12.4122	-52.3460	118.3512	-24.6166	656.8034	605.8471	1207.8690	635.2948	576.5566	446.3268
X18	416.9103	514.3816	434.9041	397.2508	394.1170	356.7890	357.2425	-0.6702	-25.0324	-209.1026	-7.9040	-41.2840	57.8722	-18.9679	467.3024	415.7416	635.2948	517.6739	395.5580	356.9544
X19	332.6768	436.3970	395.5469	356.2649	355.3923	370.8497	344.9912	11.8665	-19.5847	-195.4703	-10.0932	-30.8774	61.2642	-14.3873	413.9254	384.2133	576.5566	395.5580	424.2180	297.6870
X20	316.5500	377.9740	325.5692	304.1131	296.0511	204.2092	256.6661	-5.8291	-21.7414	-150.5224	-7.6077	-32.0163	37.4217	-13.4458	350.9285	310.9409	446.3268	356.9544	297.6870	320.7165

Table 2.4: Matrix Q depicting the covariance between funds and the variance of each fund on diagonal

The function for risk in the Markowitz Model is as given in Equations (2.12) and (2.13).

$$\text{Min}\,\frac{\sqrt{\mathrm{X}^T QX}}{250{,}000} \qquad (2.12)$$

$$\mathrm{X}^T QX = 382.5936X_1^2 + 382.2025X_2^2 + 381.0304X_3^2 + 358.3449X_4^2 + 369.0241X_5^2 +
1002.9889X_6^2 + 543.8224X_7^2 + 40.1956X_8^2 + 10.6929X_9^2 + 124.3225X_{10}^2 +
94.4784X_{11}^2 + 2.6244X_{12}^2 + 6.4009X_{13}^2 + 0.3364X_{14}^2 + 350.4384X_{15}^2 +
284.2596X_{16}^2 + 479.61X_{17}^2 + 356.4544X_{18}^2 + 250.5889X_{19}^2 + 237.4681X_{20}^2 +
507.8599X_1X_2 + 587.5099X_1X_3 + 483.1715X_1X_4 + 463.2787X_1X_5 +
645.8060X_1X_6 + 763.3155X_1X_7 + 51.0414X_1X_8 + 14.6099X_1X_9 + 19.7656X_1X_{10} +
38.1312X_1X_{11} - 2.1331X_1X_{12} - 2.3306X_1X_{13} - 0.6929X_1X_{14} - 18.1490X_1X_{15} +
438.7407X_1X_{16} + 605.8471X_1X_{17} + 635.2948X_1X_{18} + 395.5579X_1X_{19} +
297.6870X_1X_{20} + 424.8657X_2X_3 + 545.3326X_2X_4 + 497.3537X_2X_5 +
565.5574X_2X_6 + 467.7392X_2X_7 + 123.7379X_2X_8 - 8.4189X_2X_9 + 26.4916X_2X_{10} +
11.9734X_2X_{11} + 14.5831X_2X_{12} + 7.1766X_2X_{13} + 3.5578X_2X_{14} + 63.761X_2X_{15} -
12.1358X_2X_{16} + 656.8034X_2X_{17} + 415.7416X_2X_{18} + 576.5566X_2X_{19} +
356.9544X_2X_{20} + 384.4832X_3X_4 + 556.8804X_3X_5 + 649.7635X_3X_6 +
5445.7012X_3X_7 + 30.5521X_3X_8 + 31.6676X_3X_9 - 171.2246X_3X_{10} +
33.0271X_3X_{11} + 3.3440X_3X_{12} - 31.4037X_3X_{13} - 4.3848X_3X_{14} - 36.9144X_3X_{15} +
69.9254X_3X_{16} - 24.6166X_3X_{17} + 467.3024X_3X_{18} + 384.2133X_3X_{19} +
446.3268X_3X_{20}+380.6399X_4X_5+703.6288X_4X_6 +474.3016X_4X_7+13.9165X_4X_8-
10.7943X_4X_9 -203.4881X_4X_{10} -
5.7563X_4X_{11}+1.9434X_4X_{12} +3.4267X_4X_{13} +2.9076X_4X_{14}
-15.8171X_4X_{15}-28.4051X_4X_{16}+118.3512X_4X_{17}-18.9679X_4X_{18}+413.9254X_4X_{19}
+310.9409X_4X_{20}+359.7525X_5X_6+529.5505X_5X_7+25.0977X_5X_8-17.8973X_5X_9-
209.8581X_5X_{10}+43.2491X_5X_{11}-18.2473X_5X_{12} +15.9581X_5X_{13}+0.7659X_5X_{14}-
221.5451X_5X_{15}-5.6191X_5X_{16}-52.3460X_5X_{17}+57.8722X_5X_{18}-
14.3873X_5X_{19}+350.9285X_5X_{20}+294.7200X_6X_7+9.0209X_6X_8-14.0429X_6X_9
-217.1830X_6X_{10}-4.6993X_6X_{11}-2.9640X_6X_{12}+71.2914X_6X_{13}-0.7187X_6X_{14}-
23.7711X_6X_{15}-197.2171X_6X_{16}-12.4122X_6X_{17}-41.2840X_6X_{18} +61.2642X_6X_{19}-
13.4458X_6X_{20} -16.4079X_7X_8-22.8566X_7X_9-232.6645X_7X_{10}-15.8100X_7X_{11}-
24.5672X_7X_{12} +148.5717X_7X_{13}+7.1913X_7X_{14}-1.3201X_7X_{15}-13.9046X_7X_{16}-
357.2086X_7X_{17}-7.9040X_7X_{18}-30.8774X_7X_{19}+37.4217X_7X_{20} -25.1299X_8X_9-
275.2307X_8X_{10} -9.5254X_8X_{11}-
27.1195X_8X_{12}+77.8594X_8X_{13}+18.5099X_8X_{14}+366.8698X_8X_{15}+19.9013X_8X_{16}-
25.0595X_8X_{17}-209.1026X_8X_{18}-10.0932X_8X_{19}-32.0163X_8X_{20} -218.8716X_9X_{10}-
17.5869X_9X_{11}-
27.8251X_9X_{12}+69.7492X_9X_{13}+0.0541X_9X_{14}+407.4681X_9X_{15}+376.2936X_9X_{16}+12.26$$

92X_9X_{17}-25.0324X_9X_{18}-195.4703X_9X_{19}-7.6077X_9X_{20} -14.3756$X_{10}X_{11}$-38.6201$X_{10}X_{12}$+82.5330$X_{10}X_{13}$-0.6510$X_{10}X_{14}$+395.4402$X_{10}X_{15}$+475.6221$X_{10}X_{16}$+495.5419$X_{10}X_{17}$-0.6702$X_{10}X_{18}$-19.5847$X_{10}X_{19}$-150.5224$X_{10}X_{20}$ -39.4061$X_{11}X_{12}$+84.9260$X_{11}X_{13}$-3.2722$X_{11}X_{14}$+394.6967$X_{11}X_{15}$+391.5267$X_{11}X_{16}$+737.4446$X_{11}X_{17}$+357.2425$X_{11}X_{18}$+11.8665$X_{11}X_{19}$-21.7414$X_{11}X_{20}$+58.6576$X_{12}X_{13}$-7.5853$X_{12}X_{14}$+443.5618$X_{12}X_{15}$+382.0083$X_{12}X_{16}$+603.8128$X_{12}X_{17}$+356.7890$X_{12}X_{18}$+344.9912$X_{12}X_{19}$-5.8291$X_{12}X_{20}$-18.1045$X_{13}X_{14}$+515.2940$X_{13}X_{15}$+431.5058$X_{13}X_{16}$+593.8616$X_{13}X_{17}$+394.1170$X_{13}X_{18}$+370.8497$X_{13}X_{19}$+256.6661$X_{13}X_{20}$+456.5869$X_{14}X_{15}$+479.8216$X_{14}X_{16}$+667.3079$X_{14}X_{17}$+397.2508$X_{14}X_{18}$+355.3923$X_{14}X_{19}$+204.2092$X_{14}X_{20}$+401.8216$X_{15}X_{16}$+779.0690$X_{15}X_{17}$+434.9041$X_{15}X_{18}$+356.2649$X_{15}X_{19}$+296.0511$X_{15}X_{20}$+646.3539$X_{16}X_{17}$+514.3816$X_{16}X_{18}$+395.5469$X_{16}X_{19}$+304.1131$X_{16}X_{20}$+450.0257$X_{17}X_{18}$+436.3970$X_{17}X_{19}$+325.5692$X_{17}X_{20}$+372.8501$X_{18}X_{19}$+377.9740$X_{18}X_{20}$+343.8959$X_{19}X_{20}$ (2.13)

Chapter 3
Discussion of Results

3.1. Sharpe's Model

3.1.1. Ideal Solution

Ideal solution is the portfolio that has the highest return and lowest return at the same time (Ravindran et. al., 2006). In the Sharpe's Model, the ideal solution occurs at (12.0%, 0.57). Ideal solution is the best possible case in terms of return and risk. It represents the point where there is maximum return (12.0%) and minimum beta risk (0.57) for the given set of constraints. It is obtained in two steps. First, the maximization function given by Figure 2.2 is ignored and only the minimization function is run on Excel solver with all the constraints. This gives a portfolio with 0.57 beta risk and 5.38% return as shown on the bottom left corner of Figure 3.1. This represents the lowest possible beta risk that can be obtained from investing in the 20 mutual funds. Then, the minimization risk function given by Equation (2.12) is ignored and only the maximization return function is run on Excel solver with all the constraints. This gives us a portfolio with 12.0% return and 1.39 beta risk as shown on the top right corner of Figure 3.1. This solution represents the maximum possible return that can be obtained from investing in the 20 mutual funds. The results of maximum return portfolio and the minimum risk portfolio are given in Table 3.1. When the two extreme points are combined, choosing the highest return and the lowest risk value, the ideal solution is obtained. It can be seen on the top left corner of Figure 3.1. It represents the point that has both the highest return and the lowest beta risk. This point is not achievable. In other words, the ideal solution is not feasible as the two objectives conflict with each other (Ravindran et. al., 2006). Risk increases as return increases. Risk cannot be at the lowest value, when return is at the highest value as risk and return are conflicting objectives.

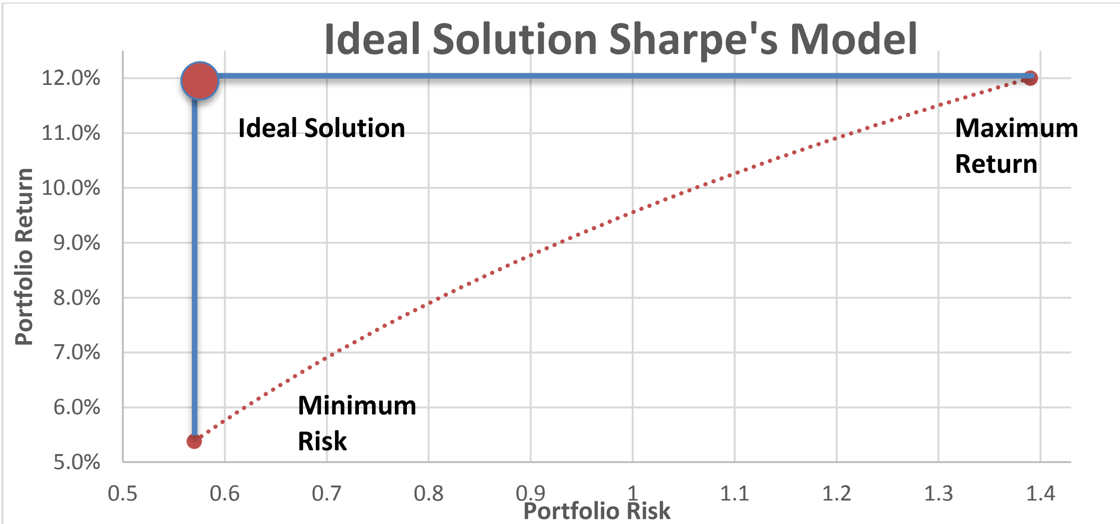

Figure 3.1: Ideal Solution Sharpe's Model

`	Portfolios	
	(5.38%, 0.57)	(12.0%,1.39)
Index Funds		
Small Cap Value Index Fund Inv	-	-
International Funds		
International Explorer Inv	-	20.0%
International Growth Inv	-	5.0%
International Blend Value Inv	10.0%	-
International Value Inv	-	-
Specialty Funds		
Precious Metals and Mining Inv	-	-
Energy Inv	-	20.0%
Government and Investment Bond Funds		
Inflation-Protected Secs Inv	-	-
Total Bond Market Index Adm	-	-
Long-Term Treasury Inv	-	-
Long-Term Investment-Grade Inv	-	20.0%
Short-Term Treasury Inv	20.0%	-
Short-Term Investment-Grade Inv	20.0%	-
Money Market Funds		
Prime Money Market Inv	20.0%	5.0%
Stock Funds		
Explorer Inv	-	-
Mid Cap Growth Inv	-	10.0%
Capital Value Inv	-	20.0%
Strategic Equity Inv	-	-
US Growth Inv	20.0%	-
US Value Inv	10%	-

Table 3.1: Maximum return portfolio and minimum risk portfolio

3.1.2. Efficient Portfolios

Efficient portfolios are portfolios such that no other portfolio can provide a greater return for the given risk level or no other portfolio can provide a lower risk for a given return level (Ravindran et. al., 2006). Efficient portfolios are the points shown in Figure 3.2. Efficient portfolios can be obtained by minimizing risk as an objective and adding return as a constraint, together with all the other constraints given in Equations (2.1) to (2.9). Another method to find efficient portfolios is by maximizing return as an objective and adding risk as a constraint together with the other constraints (Ravindran et. al., 2006). There are many combinations of portfolios with different risk and return values that can be obtained based on the risk an investor is willing to take. Not all the portfolios are worth investing in. There are portfolios that provide a lower return for a given risk level than the maximum possible return. For example, for the risk level of 0.79, a portfolio can provide a return of 8.25% shown by the 'x' symbol on Figure 3.2 based on how much the investor decides to invest in each of the funds. This is lower than the maximum possible return of 10.0% that can be achieved at that risk level. The proportion of money invested in various funds affects return for a given risk level. It is often difficult to decide what proportion of money is to be invested in each fund. Efficient portfolios resolve this problem by suggesting the correct amount of money to invest in each fund in order to maximize return for a given risk level. It is important to find these portfolios so that the investor can gain the best benefit from investing in the 20 mutual funds.

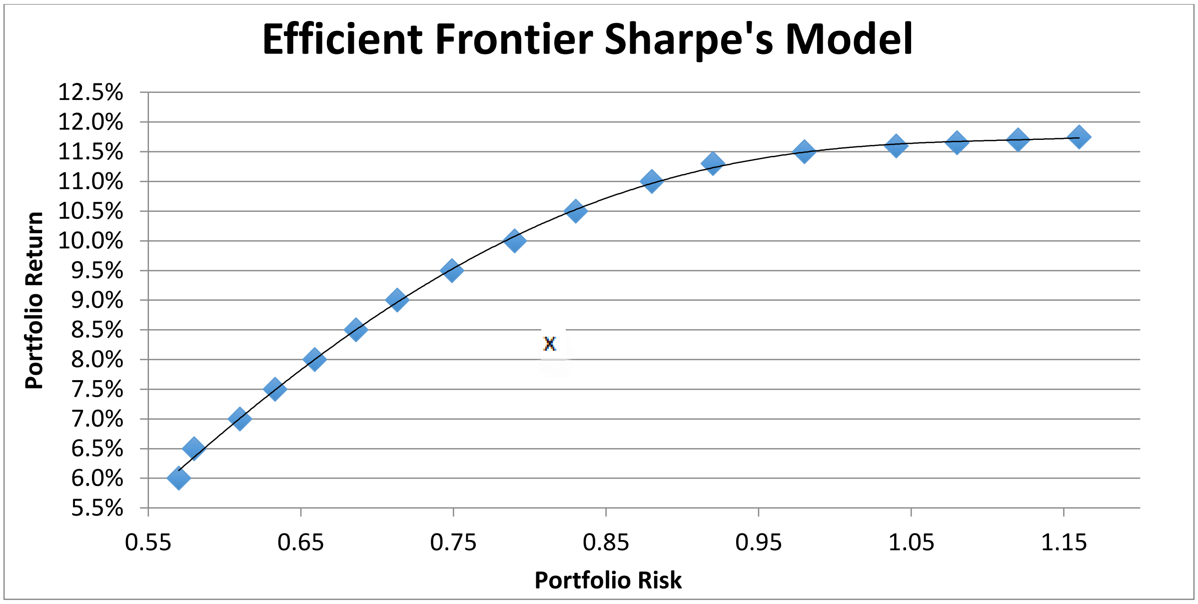

Figure 3.2: Efficient Portfolios Sharpe's Model

The set of efficient portfolios in Figure 3.2 were obtained by minimizing risk and using return as a constraint whose value was increased in increments of 0.5% -1.0% from 5.38% to 12.0%. An efficient frontier, shown by the trend line in Figure 3.2 is formed when efficient portfolios of varying risk and return values are joined together. Any portfolio on the efficient frontier gives the most amount of return for a given risk level or gives the least amount of beta risk for a given return level (Sharpe, 1963). The efficient portfolios are different alternatives that an investor can choose from to get various returns and incur different risk levels. Portfolios under the efficient frontier such as the point 'x' in Figure 3.2 satisfy the constraints given but provide the highest return for a given risk level as they are dominated by other portfolios that have both a greater return and a lower risk (Ravindran et. al., 2006). Note that there are more risk and return values used for Figure 3.2 than shown on Table 3.1 to develop a good shape for the efficient frontier.

3.1.3. Portfolio Options

In Table 3.2, eight different efficient portfolio options with different return and risk levels are presented for an investor to choose from. For each portfolio, the proportion of money to be invested in each fund is given. For example, for 9.0% return and 0.71 beta risk, 10.0% will be invested in the International Explorer Inv fund, 20.0% will be invested in the Energy Inv fund, 20.0% will be invested in the Short-Term Treasury Inv fund, 20.0% will be invested in the Prime Money Market Inv fund, 1.6% will be invested in the Explorer Inv fund, 20.0% will be invested in the Mid Cap Growth Inv fund, 8.4% will be invested in the US Growth Inv fund and no money will be invested in the remaining 13 funds. This allocation of money in the 20 funds allows for the greatest return of 9.0% for 0.71 beta risk. It is interesting to note that 70% of the money has been allocated for two funds with the highest returns of 16.6% (Energy Inv fund) and 13.9% (International Explorer Inv fund) and two funds with the lowest risk of 0 (Prime Money Market Inv fund) and 0.34 (Short-Term Treasury Inv fund). The funds with high return balance out the funds with low risk to create a diversified portfolio with lower risk than the highest return funds and higher returns than the low risk funds. The other three funds Explorer Inv fund, Mid Cap Growth Inv fund and US Growth Inv fund have moderate risk and return values. This is the benefit of having diverse funds with different risk and return values to choose from. The proportion of money allocated to each of the funds is determined by meeting the constraints and minimizing risk for a given return level.

Funds	Portfolios							
	(5.38%, 0.57)	(7.0%,0.61)	(8.0%,0.66)	(9.0%,0.71)	(10.0%,0.79)	(11.0%,0.88)	(11.75%,1.16)	(12.0%,1.39)
Index Funds								
Small Cap Value Index Fund Inv	-	-	-	-	-	-	-	-
International Funds								
International Explorer Inv	-	10.0%	10.0%	10.0%	16.4%	20.0%	20.0%	20.0%
International Growth Inv	-	-	-	-	-	5.0%	-	5.0%
International Blend Value Inv	10.0%	-	-	-	-	-	-	-
International Value Inv	-	-	-	-	-	-	-	-
Specialty Funds								
Precious Metals and Mining Inv	-	-	-	-	-	-	-	-
Energy Inv	-	4.8%	12.6%	20.0%	20.0%	20.0%	20.0%	20.0%
Government and Investment Bond Funds								
Inflation-Protected Secs Inv	-	-	-	-	-	-	-	-
Total Bond Market Index Adm	-	-	-	-	-	-	14.0%	-
Long-Term Treasury Inv	-	-	-	-	-	-	-	-
Long-Term Investment-Grade Inv	-	-	-	-	-	-	6.0%	20.0%
Short-Term Treasury Inv	20.0%	20.0%	20.0%	20.0%	20.0%	20.0%	-	-
Short-Term Investment-Grade Inv	20.0%	15.2%	7.4%	-	-	-	-	-
Money Market Funds								
Prime Money Market Inv	20.0%	20.0%	20.0%	20.0%	13.6%	5.0%	5.0%	5.0%
Stock Funds								
Explorer Inv	-	-	-	1.6%	10.0%	6.5%	-	-
Mid Cap Growth Inv	-	20.0%	20.0%	20.0%	20.0%	20.0%	15.0%	10.0%
Capital Value Inv	-	-	-	-	-	3.5%	20.0%	20.0%
Strategic Equity Inv	-	-	-	-	-	-	-	-
US Growth Inv	20.0%	10.0%	10.0%	8.4%	-	-	-	-
US Value Inv	10%	-	-	-	-	-	-	-

Table 3.2: Efficient portfolios using the Sharpe's Model and proportion spent on each fund

The return value of the portfolio options increases from 5.38% to 12.0% and beta risk value increases from 0.57 to 1.39 going from the left side to the right side in Table 3.2. The maximum return portfolio (12.0%, 1.39) and minimum risk portfolio (5.38%, 0.57) are also included. The other portfolios have been obtained by increasing the return value in increments of 1.0% from 7.0% up to 11.0%. An increment of 1.0% return was used so that a clear difference in the risk levels could be observed and also a clear difference in the proportion of money allocated to different funds can be seen. The portfolio with 11.75% return and 1.16 beta risk has been included as beta risk values increase significantly from 11.0% to the maximum return value of 12.0%. Beta risk value goes from 0.88 to 1.39. An investor has to experience a much greater increase in beta risk if they were to try to increase their return from 11.0% to 11.1%-12.0% than if they were to try to increase their return from 10.0% to 10.1%-11.0%. This is supported the decreasing gradient of the trend line from low portfolio risk to high portfolio risk in Figure 3.2. This means that between 11.0% and 12.0%, a greater amount of risk has to be incurred for the same amount of increase in return. This effect is more pronounced from 11.5% to 12.0% as seen by the sharper decrease in gradient in that region. This additional higher increase in beta risk may not be favorable to the investor and this is something they have to be cognizant of before deciding on how much portfolio return they want.

There are different types of investors and the choices made by an aggressive investor might be very different from the choices made by a conservative investor. An aggressive investor focused on getting higher returns would take on a higher degree of risk and probably choose the portfolio with 11.75% return and 1.16 beta risk. On the other hand, a conservative investor concerned with having a very ‘safe’ portfolio would probably go with 0.61 beta risk and 7.0% return. The proportion of funds allocated both the investors would be different. The conservative investor invests about 35% of the portfolio in Bond Funds that have lower beta risk while the aggressive investor invests 35% in Stock Funds that have a greater return. The aggressive investor invests the minimum required proportion of 5% in Money Market Funds while conservative investor invests 20% on it due to its 0 beta risk. Similarly, the aggressive investor invests greater proportions on the International Funds and Specialty funds than the conservative investor. An investor does not have to be very conservative or very aggressive; they can fall somewhere in the middle. A moderate investor might invest in the efficient portfolio with 9% or 10% return. The investor determines which portfolio is a better option for them. They have to decide if it is meaningful to get a 1% higher return of 10% while the beta risk increases by 0.08 or if it is sensible to have a 0.08 lower beta risk of 0.71 and get a lower return of 9.0%.

These 8 portfolios are not the only efficient portfolios that an investor can choose from. There are several other portfolios, which can be obtained by using different return values. When choosing between different portfolios, the investor has to decide on the range of risk they want to take and consider different alternatives in that range. They could also do trade-off analysis between risk and return, by considering whether a small additional amount of risk is worth the increase in return they receive.

3.1.4. Trend by Fund Category

In general, the proportion of International Funds and Specialty Funds used increases as the expected return and risk increase. On the other hand, the proportion of Government and Investment Bond Funds and Money Market Funds decreases as expected return and risk increase. This follows the general pattern that money market funds and bonds have the lower risk values than that of the

International Funds and Specialty Funds chosen. However, this depends on the selection of funds for each category. For the 20 mutual funds chosen, International Funds and Specialty Funds category do tend to have a higher risk. Hence, in order to increase return, lower risk fund categories that provide lower returns such as Government and Investment Bond Funds, are dropped and higher risk fund categories, such as Specialty Funds, are used in their place.

The proportion of Stock Funds used remained mostly the same at 30%, except for the portfolio with 11.75% return and 1.16 beta risk where 35% of the portfolio is invested in stock funds. This is interesting to note as the constraints allow for Stock Funds to be up to 75% of the portfolio. One would expect that the percentage of stock funds used would increase as portfolios with higher risk and return values are generated. Instead the percentage of International Funds used steadily increases together with Specialty Funds, in particular the Energy Inv fund. The International Funds that have been selected provide a moderate average risk of 1.01 and a high return of 11.6%. They are better than the Stock Funds chosen that provide a return of 11.1% and a risk of 1.15 on average.

3.1.5. Most Used Funds

Some funds were used more frequently than others. The Prime Money Market Inv fund was used on all the portfolios due to the constraint that the money market fund has to be at least 5% of the total investment. The International Explorer Inv fund, Energy Inv fund, Short-Term Treasury Inv fund and Mid Cap Growth Inv fund were used in 6 or 7 out of the 8 portfolios. The percentage of the funds as part of the total investment amount varied.

The International Explorer Inv fund increased from being 10.0% of the investment for a 0.61 beta value portfolio to 20% of the investment for a 0.88 beta value portfolio. The International Explorer Inv fund has the second highest return value. It could not be used in large proportions in the lower risk investments as it has a relatively high risk value that accompanies its high return value. However, it is one of the better funds to invest in as it provides a high return of 13.9% for an average beta value of 1.0. The proportions of funds invested in the Energy Inv fund increased from 4.8% to 20% as the risk of the portfolios increased. The Energy Inv fund offers the highest return of 16.6% at a moderately high beta value of 1.14. Funds with higher return values can be used in greater proportion as the risk appetite increases. This is the risk-return trade-off.

The proportion of Prime Money Market Inv fund decreased and Mid Cap Growth Inv fund decreased as the return value of the portfolios increased. These funds do not offer the highest returns and alternative funds that offer much higher returns for slightly greater risk are used as the amount of return expected increases. It has to be noted that the Prime Money Market Inv has a beta value of 0. This means that the risk of the fund is uncorrelated with the market. The performance of the market does not affect the returns gained from this mutual fund. This however does not mean that the Prime Money Market Inv fund is risk-free. It probably has low absolute risk as it is a money market fund. This is supported by standard deviation of the fund, which is 2.0%, the lowest among all the mutual funds.

3.1.6. Unused Funds

Certain funds were unused in any of the 8 portfolio options. These funds are the Small Cap Value Index Fund Inv, International Value Inv, Precious Metals and Mining Inv, Inflation-Protected Secs Inv, Long-Term Treasury Inv and Strategic Equity Inv. These funds were unused as they have a higher beta risk for the same or lower return value. Fox example, the Small Cap Value Index Fund Inv fund and Precious Metals and Mining Inv fund have 10 year return values of 11.6% and 11.0% and beta risk of 1.24 and 1.44 respectively. In contrast to these, the Explorer Inv fund, has a higher return value of 11.8% and with a lower beta risk value of 1.18. Thus, the Small Cap Value Index Fund Inv fund and Precious Metals and Mining Inv fund do not help to maximize return while minimizing risk as much as the Explorer Inv fund does and hence is not used. A similar comparison can be made for the other unused funds.

3.2. Markowitz Model

3.2.1. Ideal Solution

In the Markowitz Model, risk is measured by the standard deviation of the return value of the portfolio. The standard deviation of the return value was determined as shown in Equation (2.12). Standard deviation measures the actual risk of the portfolio as compared to beta risk, which measures the market risk. In the Markowitz Model, the ideal solution is (12.0%, 5.2%). The portfolio has the highest return value of 12.0% and the lowest risk/ standard deviation of 5.2%. However, ideal solution is not feasible. It is not possible to get the highest return and at the same time, have the lowest risk, as risk and return are conflicting objectives (Ravindran et. al., 2006). Return increases as risk increases. The ideal solution is obtained in two steps. First, the maximization of return objective is ignored and only the minimization of risk objective is used together with all the constraints. This gives us the portfolio (5.8 %, 5.2%) that has the lowest value of risk possible as shown on the bottom left corner of Figure 3.3. 5.2% is the lowest risk that can be achieved by investing a quarter million dollars in the 20 mutual funds. We get a return value of 5.8%. Then, the minimization objective is ignored and only the maximization objective together with all the constraints (Ravindran et. al., 2006). This means that there is no limit on the amount of risk that can be taken. Only the maximum amount of return that can be obtained is considered. This gives the portfolio (12.0%, 18.8%) as shown on the top right corner of Figure 3.3. The maximum possible return from investing in the 20 mutual funds is 12.0%. The maximum possible return and the minimum possible risk are combined to form the ideal solution (12.0%, 5.2%). It is shown on the top left corner of Figure 3.3. However, such a portfolio is not achievable and the ideal solution is infeasible.

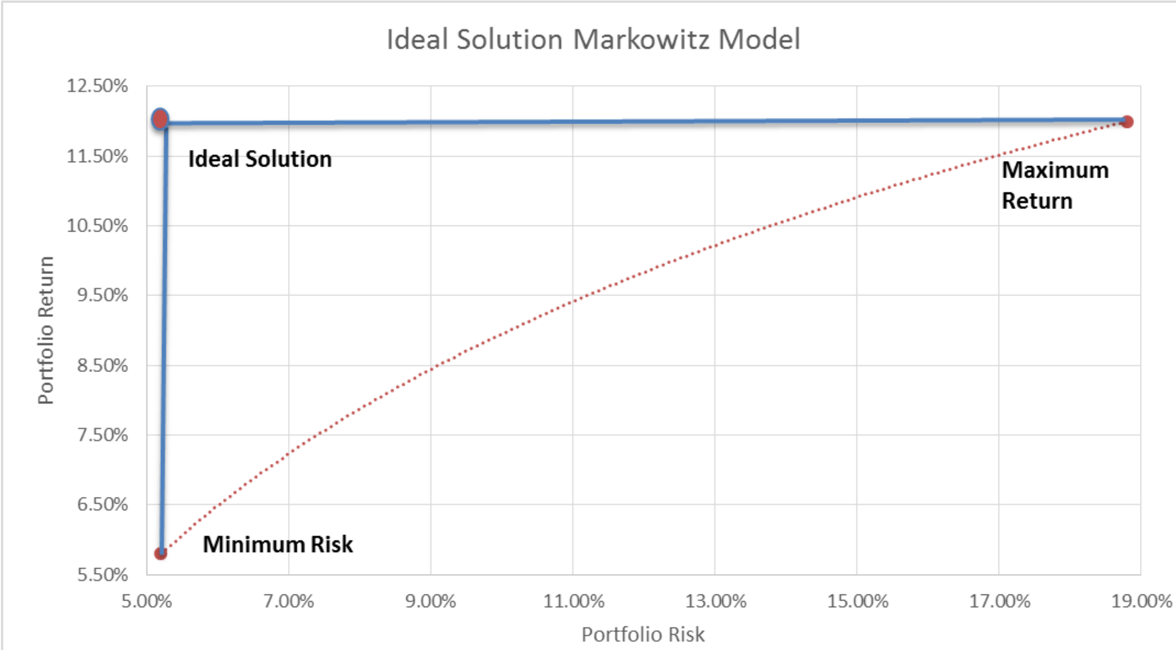

Figure 3.3: Ideal Solution Markowitz Model

3.2.2. Efficient Portfolios

Efficient portfolios are solutions where no higher return can be obtained for a given risk level and no lower risk can be achieved for a given return level. Efficient solutions were obtained by minimizing the risk function and using return as a constraint together with all the other constraints given in Equations (2.1) to (2.9). The value of return was increased in increments of about 1.0% from 5.8% to 12.0%. The minimum risk and maximum return portfolios provide the bounds for the values that return can take in different portfolios (Ravindran et. al., 2006). The minimum risk portfolio is (5.8%, 5.2%) and the maximum return portfolio is (12.0%, 18.8%). This is shown in Table 3.3. Minimum return is 8.3% and maximum return in 12.0%. All portfolios that can be obtained have a return value that falls within this range. Increments of about 1.0% were used to provide 8 different combinations of efficient portfolios that an investor can invest in as shown in

Table 3.4. These are alternatives that an investor can choose from to get various returns and incur different risk levels.

Funds	Portfolios	
	(5.8%, 5.2%)	(12.0%,18.8%)
Index Funds		
Small Cap Value Index Fund Inv	-	-
International Funds		
International Explorer Inv	-	20.0%
International Growth Inv	-	5.0%
International Blend Value Inv	6.5%	-
International Value Inv	3.5%	-
Specialty Funds		
Precious Metals and Mining Inv	-	-
Energy Inv	-	20.0%
Government and Investment Bond Funds		
Inflation-Protected Secs Inv	-	-
Total Bond Market Index Adm	-	-
Long-Term Treasury Inv	20.0%	-
Long-Term Investment-Grade Inv	-	20.0%
Short-Term Treasury Inv	20.0%	-
Short-Term Investment-Grade Inv	-	-
Money Market Funds		
Prime Money Market Inv	20.0%	5.0%
Stock Funds		
Explorer Inv	-	-
Mid Cap Growth Inv	-	10.0%
Capital Value Inv	-	20.0%
Strategic Equity Inv	-	-
US Growth Inv	10.0%	-
US Value Inv	20.0%	-

Table 3.3: Minimum risk and maximum return portfolios

Efficient solutions can be connected by the trend line shown in Figure 3.4 to form an efficient frontier. Every point on the efficient frontier represents a feasible solution that has the highest return for a given risk level or the lowest risk for a given return level (Markowitz, 1956). Portfolios other than those that are on the efficient frontier can be feasible. However, these portfolios consist of risk and return combinations that lie below the efficient frontier as shown by the 'x' symbol in Figure 3.4. For 6.0% risk, the portfolio only provides 6.0% return, which is lower than the maximum possible return of 7.0% at that risk level. Hence, it would not be sensible for anyone to invest in these portfolios. Portfolios under the efficient frontier are dominated by portfolios on the efficient frontier that have both a higher return and a lower risk.

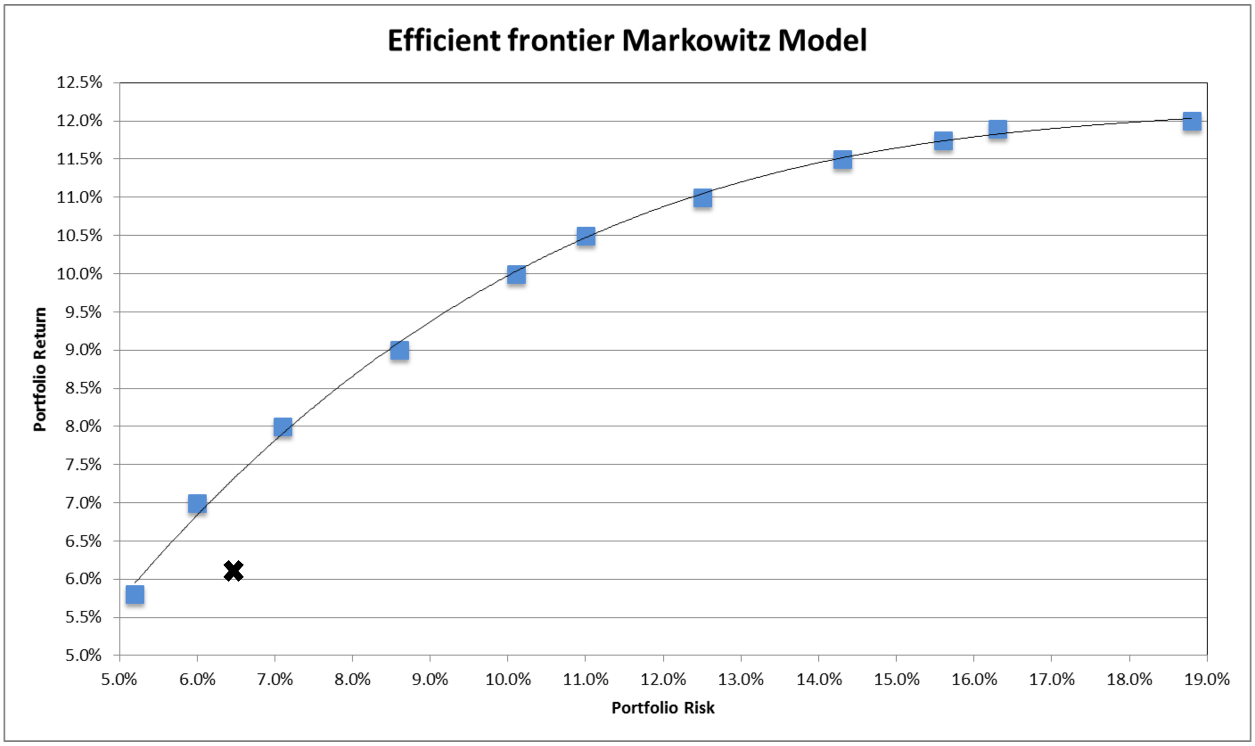

Figure 3.4: Portfolios on the efficient frontier of the Markowitz Model

3.2.3. Portfolio Options

Eight different efficient portfolio options have been presented for investors to choose from as shown in Table 3.4. These are different alternatives an investor can choose from to obtain different risk and return values. These are not the only efficient portfolios than an investor can invest in (Ravindran et. al., 2006). These portfolios have been chosen as they provide near a whole number value of return in 1.0% increments, which makes it is easy for investors to understand and make comparisons. The portfolio with 10.5% return is included as risk increases significantly between 10% return and 11.0% return from 10.1% to12.5%. This is a large increase as compared to the increase in risk from 8.6% to 10.1% for the increase in return from 9.0% to 10.0%. The return value of the portfolios increases from 8.3% to 12.0% going from the left to the right side of the table while risk increases from 5.0% to 17.6%. The proportion of money invested in each fund for each of the portfolios is also given. For example, for the portfolio with 10.5% return and 11.1% risk in the form of standard deviation, 10.0% of the money is invested in the International Explorer Inv fund, 18.5% of the money is invested in the Energy Inv Fund, 20.0% of the money is invested in the Long-Term Treasury Inv fund, 16.5% of the money is invested in the Long-Term Investment-Grade Inv fund, 5.0% of the money is invested in the Prime Money Market Inv fund, 7.3% of the money is invested in the Explorer Inv fund, 20.0% of the money is invested in the Mid Cap Growth Inv fund and 2.7% of the money is invested in the Strategic Equity Inv fund. No money has been invested in the 11 other funds.

About 40.0% of the money has been invested in the two funds that provide the highest returns, the International Explorer Inv fund and the Energy Inv fund. However, these funds do not have the two highest risk values, which make them the better funds to invest in when a moderate amount of return is expected and risk higher than the minimum amount can be taken without hesitation. Funds with the highest risk are the Previous Metals and Mining Inv fund and the Capital Value Inv fund. They have standard deviations of 41.5% and 34.8% respectively. Despite their very high risk values, the Precious Metals and Mining Inv fund and the Capital Value Inv fund only provide returns of 11.0% and 13.6%. The Energy Inv fund provides a return of 16.6% at a much lower risk value of 26.2%. Hence, these two funds were not used in the portfolio. 40% of the money has been invested in stock funds while about 37% has been invested in bond funds. Given that bond funds typically have lower risk and stock funds typically have higher returns, investing approximately 40% in each helps create a portfolio with moderate return and moderate risk. 10.5% return falls approximately in the middle between 8.3% minimum return and 12.0% maximum return.

Similarly, 11.0% risk falls approximately in the middle between 5.0% minimum risk and 17.6% maximum risk.

In the Markowitz model, it is sometimes not immediately obvious why some funds are chosen over other funds that might have slightly higher return at slightly lower risks. This can be seen in the portfolio with 10.5% return where 20.0% is invested in the Long-Term Treasury Inv fund which provides 6.4% return and 13.1% risk while only 16.5% is invested in the Long-term Investment-Grade Inv fund which has 6.5% return and only 6.3% risk. This choice of the proportion of money invested in each of these funds can be attributed to the covariance between funds. When risk is calculated in the Markowitz model as shown in Equation (2.13), covariance between a fund and each of the 19 other funds is considered for all 20 of the funds. Covariance is a measure of how the returns of two funds change with each other over the 10 year period. Positive covariance indicates that if the return for one fund increases, the return for the other fund in consideration increases too. Negative covariance means that when the return for one fund increases, the return for the other fund decreases. Zero covariance means that the performance of one fund does not affect the performance of the other fund. Another important idea is the absolute value of the covariance. Higher the absolute value of covariance between two funds, the greater the two funds affect each other. Either one fund increases greatly as the other fund increases (positive covariance) or it decreases greatly as the other fund increases (negative covariance).

When funds have high covariance between each other, there is not much diversification. The lack of diversification increases the risk of a portfolio. This is one of the findings made by Harry Markowitz when the Markowitz Model was originally created. On the other hand, diversification of funds is recommended as it reduces the risk of a portfolio. Hence, the proportion of funds was determined in such a way to provide maximum possible diversification for a given risk level so at to keep the portfolio risk to a minimum. Hence, although not intuitive, the proportion of each type of fund chosen using the Markowitz model is based on choosing funds that provide the best return and risk while meeting the constraints, as well as choosing the proportion of funds that keeps covariance to a minimum. Covariance is a component of risk in the Markowitz Model and covariance is minimized as risk is minimized (Markowitz, 1956).

From the 8 portfolios given in Table 3.4, different investors will choose different portfolios. A conservative investor might pick the portfolio with 7.0% return and 6.0% risk. Their goal is to have a safe investment and a risk of 6.0% is a low amount to incur. Whether a conservative investor chooses between the portfolio that offers a 7.0% return and 6.0% risk and the portfolio that offers 8.0% return and 7.1% risk depends on how much risk they want to take. It also depends on if they consider the additional risk of about 1.0% taken to be worth the increase in return by 1.0%. An aggressive investor on the other hand might pick the portfolio that provides 11.0% return and 12.5% risk. The aggressive investor's goal is to have returns and the amount of risk taken is not their primary concern. In other words, they are willing to take on the higher risk to get a better return from their investment. As mentioned in the beginning of this section, the 8 efficient portfolios presented in the table are not the only portfolios that an investor can invest in. There are several other efficient portfolios on the efficient frontier shown in Figure 3.4 that an investor can choose from and we can obtain these portfolios by using different return values. It is up to the investor to consider the trade-off between risk and return and decide which option is the most suitable for them. It must be appreciated that as return increases, risk hikes up at increasing rate (Ravindran et. al., 2006). The decreasing slope of the efficient frontier in Figure 3.4 supports this.

Funds	Portfolios							
	(5.8%, 5.2%)	(7.0%, 6.0%)	(8.0%,7.1%)	(9.0%,8.6%)	(10.0%,10.1%)	(10.5%, 11.0%)	(11.0%, 12.5%)	(12.0%,18.8%)
Index Funds								
Small Cap Value Index Fund Inv	-	-	-	-	-	-	-	-
International Funds								
International Explorer Inv	-	1.1%	5.9%	8.0%	10.0%	10.0%	14.7%	20.0%
International Growth Inv	-	0.2%	-	-	-	-	-	5.0%
International Blend Value Inv	6.5%	8.6%	4.1%	2.0%	-	-	-	-
International Value Inv	3.5%	-	-	-	-	-	-	-
Specialty Funds								
Precious Metals and Mining Inv	-	-	-	-	-	-	-	-
Energy Inv	-	-	2.8%	8.5%	15.0%	18.5%	20.0%	20.0%
Government and Investment Bond Funds								
Inflation-Protected Secs Inv	-	-	-	-	-	-	-	-
Total Bond Market Index Adm	-	4.6%	-	-	-	-	-	-
Long-Term Treasury Inv	20.0%	20.0%	20.0%	20.0%	20.0%	20.0%	20.0%	-
Long-Term Investment-Grade Inv	-	15.4%	20.0%	20.0%	20.0%	16.5%	10.3%	20.0%
Short-Term Treasury Inv	20.0%	-	-	-	-	-	-	-
Short-Term Investment-Grade Inv	-	-	-	-	-	-	-	-
Money Market Funds								
Prime Money Market Inv	20.0%	20.0%	17.2%	11.5%	5.0%	5.0%	5.0%	5.0%
Stock Funds								
Explorer Inv	-	2.1%	4.0%	4.4%	5.6%	7.3%	10.0%	-
Mid Cap Growth Inv	-	12.6%	18.9%	20.0%	20.0%	20.0%	20.0%	10.0%
Capital Value Inv	-	-	-	-	-	-	-	20.0%
Strategic Equity Inv	-	-	1.2%	2.1%	-	2.7%	-	-
US Growth Inv	10.0%	0.9%	-	-	-	-	-	-
US Value Inv	20.0%	14.3%	5.8%	3.4%	4.4%	-	-	-

Table 3.4: 8 portfolio options for the Markowitz Model

3.2.4. Trend by Fund Category

As expected return increases for the portfolios going from the left to the right of Table 3.4, the proportion of money invested in International Funds increases from the minimum required amount of 10% following from the constraints in Figure 2.1 to the maximum required amount of 25%. Specialty Funds, specifically the Energy Inv fund increases from 0.0% to the maximum allowable proportion of 20.0% as expected return increases. The proportion of money invested in Bond Funds decreases from the maximum amount of 40.0% to the minimum amount of 20.0%. Money Market Funds decreases from 20.0% to a minimum value of 5.0% as return increases. The proportion of Stock Funds remained constant at the minimum allowable value of 30.0% across all the 8 portfolios. The range of values for each type of fund is determined by the constraints listed in Figure 2.1. International Funds and Specialty Funds offer greater returns and are used in larger proportions as the risk appetite of the investor increases. On the other hand, Bond Funds and Money Market Funds are funds with a low risk value and are preferred if the risk of the portfolio is to be kept to a low value. These funds offer lower returns as well and hence are not used as much in portfolios that have high return values. Stock Funds chosen include low risk funds such as the US Value Inv fund and high returns funds such as the Capital Value Inv fund. It has a variety of funds but no fund is provides the highest return or the lowest risk. Moreover, some of the stock funds have higher risk and lower return that funds in other categories and hence are not used much. An example is the Capital Value Inv fund which has a return of 13.6% and a risk of 34.8%. The International Explorer Inv fund has a higher return of 13.9% while having a much lower risk of 27.8%. Stock Funds in general do not have funds that provide the best return and risk values as compared to the other categories. This explains why they are used minimally.

3.2.5. Most Used Funds

The funds that were used in most portfolios are the International Explorer Inv fund, Energy Inv fund, Long-Term Treasury Inv fund, Long-Term Investment-Grade Inv fund, Prime Money Market Inv Fund and Mid Cap Growth Inv fund. It is interesting to note that the Long-Term Treasury Inv fund has been used in the proportion of 20% in 7 out of 8 of the portfolios. It offers a return of 6.4% and a risk of 13.1% risk is worse deal than that of the Long-Term Investment-Grade Inv fund. However, it has a lower covariance with the other funds than the Investment Grade fund and it has a negative correlation with 14 out of the 20 funds. It can be thought that the Long-Term Treasury Inv fund acts more independently with respect to the other funds and acts to balance out the times when they are losses in the other funds due to its negative covariance with them. The proportion of low risk funds, such as the Prime Money Market Inv fund, decreased and the proportion of high risk funds, such as the International Explorer Inv fund and Energy Inv fund increased as the return value of the investment increased. The Long-Term Investment-Grade Inv fund, which has lower return and risk and Mid Cap Growth Inv fund, which has higher return and risk were used in roughly the same proportion in most of the portfolios. This could be attributed to the covariance that these funds have with other funds. The Investment-Grade Inv fund has negative covariance with majority of the funds while the Mid Cap Growth Inv fund has positive covariance with majority of the funds.

3.2.6. Unused Funds

Certain funds were unused in most of the 8 portfolios. These funds include the Small Cap Value Index Fund Inv fund, International Growth Inv fund, International Value Inv fund, Precious Metals and Mining Inv fund, Inflation-Protected Secs Inv fund, Total Bond Market Index Adm fund, Short-Term Investment-Grade Inv and Strategic Equity Inv fund. This could be attributed to their unfavorable risk and return values or their strong covariance with the other funds.

3.2.7. Confidence Intervals

Based on the portfolios generated by the Markowitz Model, confidence intervals can be found that describe the probability that the 10-year average return of these portfolios fall with a certain range of values. Let us take the portfolio with 10.0% return and 10.1% risk. Assuming that the portfolio return follows a Normal distribution with mean 10.0% and standard deviation 10.1%, we can say that there is a 68% probability that the actual portfolio return will fall within one standard deviation. In other words, we can say that we can be 68% confident that the actual yearly return of the portfolio would fall in the range (-0.1%, 20.1%). This means that 68% of the time, the return of this portfolio lies between -0.1% and 20.1%. Extending it further by using two standard deviations from the mean in both directions, we can say that 95% of the time, the yearly return of the portfolio would fall in the range given by (-10.2%, 30.2%). We can be 95% confident that the portfolio's actual return is between -10.2% and 30.2%. It can also be said that the actual return of the portfolio lies between (-20.3%, 40.3%) 99% of the time. It is important to realize that actual yearly returns from an efficient portfolio can vary based on how different funds react to the market conditions and a variety of other reasons. Selecting diversified funds and choosing a portfolio along the efficient frontier definitely improves the investor's chance at getting better returns for lower risks over the long run (Markowitz, 1956). However, the actual return from a portfolio cannot be determined with complete accuracy until the returns are calculated during the end of each time period. Given the uncertainty of investing in securities, specifically mutual funds, confidence intervals offer investors a gauge of how their portfolio might perform and the lower bound of the confidence interval offers them a word of caution as to how badly their portfolio might perform during troubled times in the economy and stock market.

Another observation is that standard deviation has larger values than returns for the portfolios on the right side of the table where higher return values are offered. This means there is greater chance that the return from these portfolios can be zero or take on negative values. These portfolios are riskier investments. The higher value of the standard deviation as compared to the average return becomes more significant moving to the right of the table. In the portfolio with 10.0% return, the standard deviation is 0.1% higher while in the portfolio with 12.0% return, the standard deviation is 6.8% more than the average return. Towards the right of the table, the portfolios are getting increasingly riskier reaching a peak value of 18.8% risk for a maximum possible return of 12.0%.

3.3 Comparison of Sharpe's Model and Markowitz Model

3.3.1. Linear Model versus Quadratic Model

The Sharpe's Model is a linear programming model, while the Markowitz Model is a quadratic programming model for maximizing returns from a portfolio for a given risk level that an investor is willing to undertake (Ravindran et. al., 2006). Although both models essentially maximize returns, there are a few differences between them.

The Sharpe's Model and the Markowitz Model have the same linear objective function for return given by the 10-year average return of each mutual fund multiplied by the amount invested in each fund. This means that portfolios from both the models have the same highest maximum possible return of 12.0%. A value of 12.0% is obtained by ignoring the risk function and only maximizing the return function with the constraints. The objective function for risk in the Markowitz Model is what makes the model quadratic. Risk is given by $X^T QX$. On the other hand, in the Sharpe's Model, risk consists of the beta risk values for each fund multiplied by the amount invested in each fund.

The risk measures in the Sharpe's Model and Markowitz Model differ from each other. Volatility is a measure of the variation of the return over time. Beta risk in the Sharpe's Model measures the volatility of a fund or portfolio relative to the market's volatility, generally S&P 500 index, while standard deviation in the Markowitz Model measures the volatility of return of the fund/portfolio (Investopedia, LLC, 2015a). As a result, standard deviation measures actual risk while beta risk measures risk relative to the market. The market has a risk value of 1. A beta value less than 1 means that the portfolio's return is less volatile than the market. It is safer to invest in the portfolio than it is to invest in the market on average. A beta risk of more than 1 means the portfolio's return are more volatile than the market. It is riskier to invest in the portfolio than to invest in the market on average (Investopedia, LLC, 2015a).

It is important to note that the risk of a portfolio can be reduced by diversification in the Markowitz Model, while the beta risk of a portfolio cannot be reduced by diversification alone in Sharpe's Model (Ravindran et. al., 2006). This is because beta risk measures the fluctuations of a portfolio with respect to the stock market. Choosing a diverse group funds cannot reduce the fluctuations. However, choosing funds with lower beta values can reduce the portfolio's risk against the market. Risk in the Markowitz model measures individual funds' fluctuations in returns and how they are correlated with similar funds. Choosing a diverse group of funds that have smaller correlation between each other will help to reduce Markowitz risk to some extent.

In the Markowitz Model, a confidence interval for the return value can be calculated with the help of standard deviation. One standard deviation represents 68.27% of the values where return might lie as seen in Section 3.2.7. However, this is not possible for the Sharpe's Model. With the help of confidence intervals, the calculated risk and return values of different portfolios can be understood and appreciated better by investors. It is more difficult to appreciate beta risk values and the corresponding return values in the Markowitz Model.

3.3.2. Selection of Funds

As the return value increased for portfolios in both models, the proportion of International funds and Specialty funds used increased and the proportion of Government and Investment Bond funds and Money Market funds used decreased. This is because International funds and Specialty funds provide greater returns together with greater risk than Bond funds and Money Market funds.

We can compare the most used funds for both models. Table 3.5 provides a summary of the funds that were used in most of the portfolios in the Sharpe's Model and Markowitz Model. Both models used the International Explorer Inv fund, Energy Inv fund, Prime Money Market Inv fund and Mid Cap Growth Inv fund widely in many of the portfolio options. However, they used different bond funds the most. The Sharpe's Model used the Short-Term Treasury Inv fund in many of its portfolios while the Markowitz Model used the Long-Term Treasury Inv fund and Long-Term Investment-Grade Inv fund more often.

Most Used Funds	Sharpe's Model	Markowitz Model
International Explorer Inv	✔	✔
Energy Inv	✔	✔
Long-Term Treasury Inv	-	✔
Long-Term Investment-Grade Inv	-	✔
Prime Money Market Inv	✔	✔
Mid Cap Growth Inv	✔	✔
Short-Term Treasury Inv	✔	-

Table 3.5: Most used funds in Sharpe's Model and Markowitz Model

Similar funds such as the Small Cap Value Index Fund Inv fund, International Value Inv fund, Precious Metals and Mining Inv fund, Inflation-Protected Secs Inv fund and Strategic Equity Inv fund were unused in both models as the return and risk values of the funds were dominated by other funds that provided greater returns for a lower level of risk.

3.3.3. Portfolios and Proportion of Funds

We can make a comparison between portfolios with the same return value in the Sharpe's Model and the Markowitz Model. For example, we can consider the portfolio that has a moderate return value of 10.0% as shown in Table 3.6. It provides a summary of the funds that were chosen and the proportion of money that was spent on them for both models. This can be thought of as a portfolio that a moderate risk taking investor would choose. If the investor was to make a decision using the Sharpe's Model, the investor might invest in different funds than if the investor was to use the Markowitz Model to make a decision. This could be a small difference or a significant difference. However, we cannot determine which model is better than the other in allocating proportions of money to different funds.

Funds	Portfolios	
	Sharpe	Markowitz
	(10.0%,0.79)	(10.0%,10.1%)
Index Funds		
Small Cap Value Index Fund Inv	-	-
International Funds		
International Explorer Inv	16.4%	10.0%
International Growth Inv	-	-
International Blend Value Inv	-	-
International Value Inv	-	-
Specialty Funds		
Precious Metals and Mining Inv	-	-
Energy Inv	20.0%	15.0%
Government and Investment Bond Funds		
Inflation-Protected Secs Inv	-	-
Total Bond Market Index Adm	-	-
Long-Term Treasury Inv	-	20.0%
Long-Term Investment-Grade Inv	-	20.0%
Short-Term Treasury Inv	20.0%	-
Short-Term Investment-Grade Inv	-	-
Money Market Funds		
Prime Money Market Inv	13.6%	5.0%
Stock Funds		
Explorer Inv	10.0%	5.6%
Mid Cap Growth Inv	20.0%	20.0%
Capital Value Inv	-	-
Strategic Equity Inv	-	-
US Growth Inv	-	-
US Value Inv	-	4.4%

Table 3.6: Portfolios in the Sharpe's Model and Markowitz Model with 10.0% return

The proportion of money spent on each category was somewhat different for the 10% return portfolio in the Sharpe' Model and in the Markowitz Model. In the Sharpe's Model, 16.4% was spent on International Funds, 20.0% was spent on Specialty Funds, 20.0% was spent on Bond Funds and 13.6% was spent on Money Market Funds while in the Markowitz Model, 10.0% was spent on International Funds, 15.0% on was spent on Specialty Funds, 40.0% was spent on Bond Funds and 5.0% was spent on Money Market Funds. This choice depends on the beta risk valve in Sharpe's Model while it depends on standard deviation of the fund and the covariance between them in the Markowitz Model. Stock Funds made up 30% of the money in both portfolios and Index Funds were unused in both cases.

Generally, the same funds were used in both portfolios as beta risk values and standard deviation values corresponded to each other to some extent for the same fund as seen in Table 2.3. The Mid Cap Growth Inv fund was used in the maximum proportion of 20% in both portfolios. However, a short-term bond fund was used in maximum proportion in the Sharpe's Model while two long-term bond funds were used in maximum proportion the Markowitz model. Additionally, the US Value Inv stock fund was used in the Markowitz Model, which is not seen in the Sharpe's Model.

A similar comparison can be made for the minimum risk portfolio and the maximum return portfolio of both models. The minimum risk portfolio from both models is given in Table. 3.7. For the Sharpe's Model, the minimum risk is 0.57 and for the Markowitz Model it is 5.2%. The same proportion of funds was used in each fund category. There was a minor difference in the choice of funds. An additional International Fund, International Value Inv fund was used in the Markowitz Model and two short-term bond funds were used in the Sharpe's Model while one long term bond fund and one short term bond fund were used in the Markowitz Model.

Funds	Portfolios	
	Sharpe	Markowitz
	(5.38%, 0.57)	(5.8%, 5.2%)
Index Funds		
Small Cap Value Index Fund Inv	-	-
International Funds		
International Explorer Inv	-	-
International Growth Inv	-	-
International Blend Value Inv	10.0%	6.5%
International Value Inv	-	3.5%
Specialty Funds		
Precious Metals and Mining Inv	-	-
Energy Inv	-	-
Government and Investment Bond Funds		
Inflation-Protected Secs Inv	-	-
Total Bond Market Index Adm	-	-
Long-Term Treasury Inv	-	20.0%
Long-Term Investment-Grade Inv	-	-
Short-Term Treasury Inv	20.0%	20.0%
Short-Term Investment-Grade Inv	20.0%	-
Money Market Funds		
Prime Money Market Inv	20.0%	20.0%
Stock Funds		
Explorer Inv	-	-
Mid Cap Growth Inv	-	-
Capital Value Inv	-	-
Strategic Equity Inv	-	-
US Growth Inv	20.0%	10.0%
US Value Inv	10%	20.0%

Table 3.7: Minimum Risk Portfolios in the Sharpe's Model and Markowitz Model

Maximum return portfolios can be compared as well. The maximum return portfolios from the Sharpe's Model and Markowitz Model are given in Table 3.8. It is very interesting to note that the same fund categories were used in equal proportions, the same funds were chosen and the same proportion of each fund was chosen. The portfolio generated by the Sharpe's Model is exactly the same as the Markowitz Model for the maximum risk value of 12.0%.

Funds	Portfolios	
	Sharpe	Markowitz
	(12.0%,18.8%)	(12.0%,1.39)
Index Funds		
Small Cap Value Index Fund Inv	-	-
International Funds		
International Explorer Inv	20.0%	20.0%
International Growth Inv	5.0%	5.0%
International Blend Value Inv	-	-
International Value Inv	-	-
Specialty Funds		
Precious Metals and Mining Inv	-	-
Energy Inv	20.0%	20.0%
Government and Investment Bond Funds		
Inflation-Protected Secs Inv	-	-
Total Bond Market Index Adm	-	-
Long-Term Treasury Inv	-	-
Long-Term Investment-Grade Inv	20.0%	20.0%
Short-Term Treasury Inv	-	-
Short-Term Investment-Grade Inv	-	-
Money Market Funds		
Prime Money Market Inv	5.0%	5.0%
Stock Funds		
Explorer Inv	-	-
Mid Cap Growth Inv	10.0%	10.0%
Capital Value Inv	20.0%	20.0%
Strategic Equity Inv	-	-
US Growth Inv	-	-
US Value Inv	-	-

Table 3.8. Maximum Return Portfolios in the Sharpe's Model and Markowitz Model

This means that at 12.0%, the Sharpe's Model best fits the Markowitz Model. It approximates the Markowitz Model the best at this value. It can be noted that the Sharpe's Model approximates the Markowitz Model better for the minimum risk and maximum return portfolios than for portfolios with other return values. There is also a tendency to use short-term bond funds in the Sharpe's Model while long-term bond funds are preferred in the Markowitz Model.

Chapter 4
Conclusion

4.1. Summary of Findings

For the portfolio selection problem, where there are 20 mutual funds to choose from and $250,000 to invest, the optimal investment policy was determined using the Sharpe's Model and Markowitz Model. Return values for the mutual funds over a 10-year period from 2004 to 2014 were collected, together with beta values of the funds using Morningstar.com. Standard deviation of return of the funds and the covariances between funds were calculated. The Sharpe's Model and Markowitz Model were developed with a variety of constraints with ranges that allowed different investors to invest the amount that they wanted in the different category of funds.

The Sharpe's Model and Markowitz Model were used to maximize return while minimizing risk for portfolios of a given risk or return level. Beta values were used to represent risk in the Sharpe's Model, while standard deviation and covariance were used to represent risk in the Markowitz Model. For each model, maximum return and minimum risk portfolios were determined, the efficient frontier connecting various efficient portfolios was drawn and 8 different efficient portfolio options with different return and risk values were presented for an investor to choose from.

It was found that as a portfolio's return increased, its risk increased as well. This is because maximizing return and minimizing risk are conflicting objectives. Different types of investors invested in different types of portfolios. A conservative investor intending to minimize risk invested in a portfolio with low risk and accepted the low return that came from the portfolio. An aggressive investor invests in a portfolio with high return without much consideration to the risk that was attached to it. Within portfolios, it was observed that funds with a lower beta risk or lower standard deviation values for a given return value were often chosen over funds that had higher risk for the same return value. This helps to maximize return while minimizing risk. As the risk of a portfolio increased, funds that provided greater returns were chosen.

Going from a lower return level to a higher return level for both models, the amount of additional risk that had to be taken for the same percentage increase in return increased. This means that more risk had to be taken to increase return by a particular percentage when return is already at a relatively high level than when it is at a lower value. This is supported by the decreasing slope of the efficient frontiers in both models as shown in Figure 3.3 and 3.4. A much larger amount of risk has to be taken for the same increase in return.

As the return value increased for portfolios in both models, the proportion of International funds and Specialty funds increased and the proportion of Government and Investment Bond funds and Money Market funds decreased.

Covariance was an additional component of risk in the Markowitz Model. When 2 funds had lower covariance between each other, overall risk of the portfolio decreased. This suggests that it is not so much of a choice of an individual fund that matters but a choice of funds that differ from each other that adds value. It was found that the standard deviation of a portfolio in the Markowitz

Model could be reduced to some extent by choosing a diverse group of funds that have low covariance between each other. However, the beta risk of a portfolio in the Sharpe's Model cannot be reduced in a similar manner.

When the Sharpe's Model and Markowitz Model were compared, it was found that both models suggested investing in similar funds with some variation. The proportion of money invested in each fund varied by a large amount for some of the portfolios. It can be noted that the Sharpe's Model approximates the Markowitz Model better for the minimum risk and maximum return portfolios than for portfolios with other return values. There is also a tendency to use short-term bond funds in the Sharpe's Model while long-term bond funds are preferred in the Markowitz Model.

4.2. Extensions to thesis

There are extensions to this thesis that can be performed so that a real-world scenario can be more closely represented and a better understanding of portfolio selection can be made.

A variety of securities can be used other than just mutual funds. It is more likely that an investor is going to invest in a variety of different securities to increase return and reduce risk through diversification instead of just mutual funds. A wider selection of securities can also be presented instead of just 20 of them. Choosing more securities that are different from each other also improves diversification to some extent.

The selection of US Stock Funds in this thesis can also be improved on. Many of the stock funds in the U.S. Stock Funds section provide a higher risk than other types of funds for similar return value. In that case, selecting and using a large proportion of stock funds will not benefit the investor. It was result in them having feasible, but non-efficient portfolios.

It can be also considered in greater detail in which cases the Sharpe's Model provides similar results as the Markowitz Model and in which cases it provides different results. The Markowitz was created first and the Sharpe's Model is developed based on it. In that sense, the Sharpe's Model is seen as an approximation of the Markowitz Model although it cannot proved that one model necessarily represents the actual returns in the future better than the other model. Portfolios where the Sharpe's Model and Markowitz Model provide similar results can probably be relied on with greater confidence.

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Education

B.S. Industrial Engineering, Schreyer Honors College, Penn State University *Spring 2015*

Work Experience

Management Consulting Intern

PwC, Capital Projects & Infrastructure (CP&I), San Francisco, CA *Summer 2014*

- Gained the confidence of PwC team to interact with and submit work to clients from day 1
- Created consistent error-free work that was presented to client's Steering Committee
- Built solid relationships with 20-member client team to understand services and work streams
- Commended by client team for successful assistance during high-pressure presentation
- Developed sustainable network of 18 contacts within CP&I and 7 contacts outside CP&I

Engineering Intern

Rockwell Automation Control Systems, Chelmsford, MA *Summer 2013*

- Ran 6 process capability studies to improve assembly efficiency of safety product
- Documented 30 product and process improvements to highlight business objectives they met
- Top 5 improvements were shared with firm's Senior Vice President
- Collaborated with hourly and salaried workforce to meet project's intended outcomes

Teaching Assistant

Penn State School of Engineering Design *Spring 2013*

- Guided students to complete SolidWorks tutorials for 2 hours every week

Awards

- Harold and Inge Marcus Endowment Scholarship, Industrial Engineering, '14
- Phillip Seville Outstanding Community Service Award, IIE Penn State Chapter, '14
- Dean's List, Fall '12, Spring '13, Fall '13, Spring '14
- SINDA Excellence Award for performance in IB diploma '12

Leadership

Consulting Team

Schreyer Consulting Group *2014*

- Performed peer analysis of 6 companies for Malini Foundation, a non-profit to understand the landscape and strategies for sale of its product
- Produced well-formatted deck to communicate research implications and key takeaways to founder, a former Deloitte consultant

Volunteer Chair

Biomedical Engineering Society *2012-2013*

- Planned and executed efforts to increase members' participation in volunteer events by 25%
- Strengthened relationships with chairpersons of other clubs to plan joint events

Facilitator

Students Engaging Students *2013-2014*

- Conducted lively leadership training workshops for students that enhanced teaming skills
- Led volunteer service activities and increased sense of community among students

Facilitator

30-Hour Famine, World Vision Singapore *2012*

- Supported 10-student team to raise awareness about world hunger while fasting for 30 hours

Skills

Access, R, Lean Six Sigma, C++, MATLAB, Visio, SolidWorks, Minitab, SharePoint