transformation using

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Review Questions

1. *Write a note on transformation conventions.*
2. *Explain post-multiplication and pre-multiplication transformations.*

3

4.4 Two-dimensional Transformations

AU : Dec.-12, May-13

6

4.4.1 Translation

- Translation is a process of changing the position of an object in a straight-line path from one co-ordinate location to another.
- We can translate a two dimensional point by adding translation distances, t_x and t_y, to the original co-ordinate position (x, y) to move the point to a new position (x′, y′), as shown in the Fig. 4.4.1.

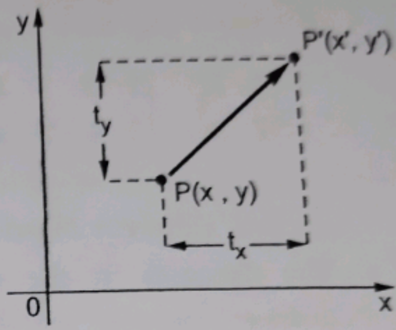


Fig. 4.4.1

$$x' = x + t_x \qquad \text{... (4.4.1)}$$

$$y' = y + t_y \qquad \text{... (4.4.2)}$$

- The translation distance pair (t_x, t_y) is called a **translation vector or shift vector.**
- It is possible to express the translation equations (4.4.1) and (4.4.2) as a single matrix equation by using column vectors to represent co-ordinate positions and the translation vector :

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \qquad P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad T = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

This allows us to write the two dimensional translation equations in the matrix form :

$$P' = P + T \qquad \dots (4.4.3)$$

Example 4.4.1 *Translate a polygon with co-ordinates A (2, 5), B (7, 10) and C (10, 2) by 3 units in x direction and 4 units in y direction.*

Solution : $A' = A + T = \begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$

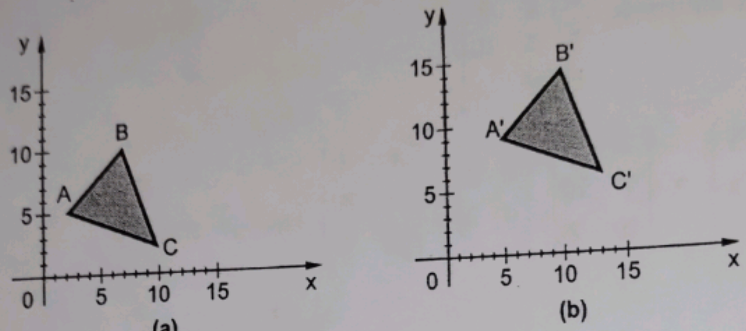


Fig. 4.4.2 Translation of polygon

$$B' = B + T = \begin{bmatrix} 7 \\ 10 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

$$C' = C + T = \begin{bmatrix} 10 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 13 \\ 6 \end{bmatrix}$$

4.4.2 Rotation

- A two dimensional rotation is applied to an object by repositioning it along a circular path in the xy plane.
- To generate a rotation, we specify a rotation angle θ and the position of the rotation point about which the object is to be rotated.
- Let us consider the rotation of the object about the origin, as shown in the Fig. 4.4.3.

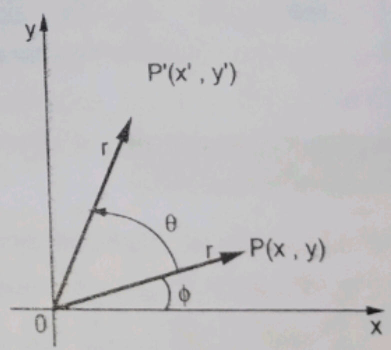


Fig. 4.4.3

- Here, r is the constant distance of the point from the origin, angle ϕ is the original angular position of the point from the horizontal, and θ is the rotation angle.
- Using standard trigonometric equations, we can express the transformed co-ordinates in terms of angles θ and ϕ as

$$\left.\begin{aligned} x' &= r\cos(\phi+\theta) = r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ y' &= r\sin(\phi+\theta) = r\cos\phi\sin\theta + r\sin\phi\cos\theta \end{aligned}\right\} \quad \text{... (4.4.4)}$$

- The original co-ordinates of the point in polar co-ordinates are given as

$$\left.\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}\right\} \quad \text{... (4.4.5)}$$

- Substituting equations (4.4.5) into (4.4.4), we get the transformation equations for rotating a point (x, y) through an angle θ about the origin as :

$$\left.\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned}\right\} \quad \text{... (4.4.6)}$$

- The above equations can be represented in the matrix form as given below

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$P' = P \cdot R \quad \text{... (4.4.7)}$$

where R is rotation matrix and it is given as

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \text{... (4.4.8)}$$

- It is important to note that positive values for the rotation angle define counterclockwise rotations about the rotation point and negative values rotate objects in the clockwise sense.
- For negative values of θ i.e. for clockwise rotation, the rotation matrix becomes

$$R = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \because \begin{array}{l} \cos(-\theta) = \cos\theta \text{ and} \\ \sin(-\theta) = -\sin\theta \end{array} \quad \text{... (4.4.9)}$$

Example 4.4.2 *A point (4, 3) is rotated counterclockwise by an angle of   Find the rotation matrix and the resultant point.*

Solution :

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\therefore \quad P^1 = \begin{bmatrix} 4 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 4/\sqrt{2} - 3/\sqrt{2} & 4/\sqrt{2} + 3/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 7/\sqrt{2} \end{bmatrix}$$

4.4.3 Scaling

- A scaling transformation changes the size of an object.
- This operation can be carried out for polygons by multiplying the co-ordinate values (x, y) of each vertex by scaling factors S_x and S_y to produce the transformed co-ordinates (x', y').

$$x' = x \cdot S_x \qquad \ldots (4.4.10)$$

and

$$y' = y \cdot S_y$$

- Scaling factor S_x scales object in the x direction and scaling factor S_y scales object in the y direction. The equation (4.4.10) can be written in the matrix form as given below :

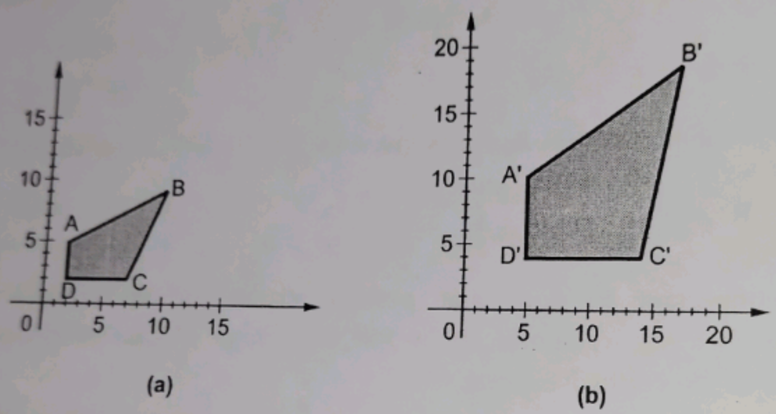


Fig. 4.4.4

$$\begin{aligned} [x' \;\; y'] &= [x \;\; y] \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \\ &= \begin{bmatrix} x \cdot S_x & y \cdot S_y \end{bmatrix} \\ &= P \cdot S \end{aligned} \qquad \ldots (4.4.11)$$

- Any positive numeric values are valid for scaling factors S_x and S_y. Values less than 1 reduce the size of the objects and values greater than 1 produce an enlarged object.
- For both S_x and S_y values equal to 1, the size of object does not change.
- To get uniform scaling it is necessary to assign same value for S_x and S_y. Unequal values for S_x and S_y result in a differential scaling.

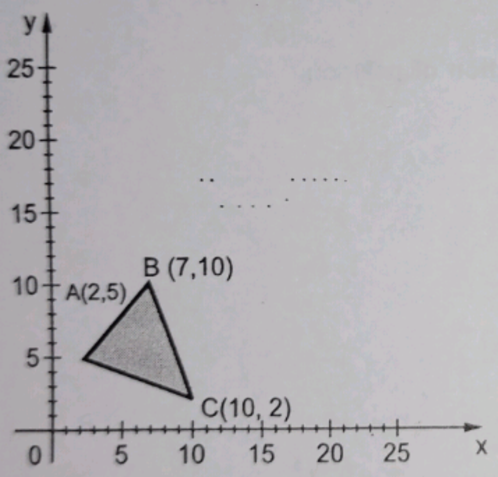
Example 4.4.3 *Scale the polygon with co-ordinates A (2, 5), B (7, 10) and C (10, 2) by two units in x direction and two units in y direction.*

Solution : Here $S_x = 2$ and $S_y = 2$. Therefore, transformation matrix is given as

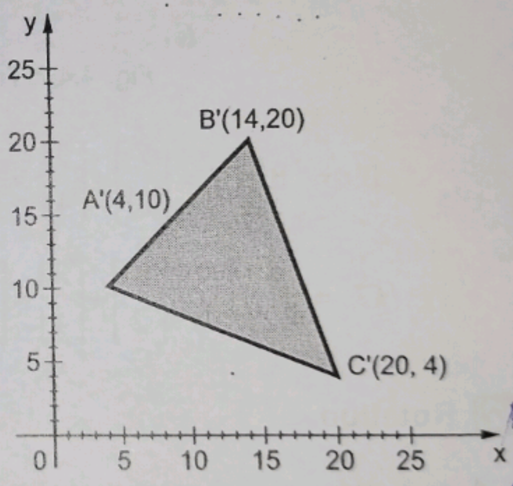
$$S = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The object matrix is :
$$\begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 2 & 5 \\ 7 & 10 \\ 10 & 2 \end{bmatrix} \end{array}$$

$$\therefore \begin{array}{c} A' \\ B' \\ C' \end{array} \begin{bmatrix} x'_1 & y'_1 \\ x'_2 & y'_2 \\ x'_3 & y'_3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 7 & 10 \\ 10 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 14 & 20 \\ 20 & 4 \end{bmatrix}$$



(a) Original object



(b) Scaled object

Fig. 4.4.5

Review Questions

1. *Describe w.r.t. 2D transformation : i) Scaling ii) Rotation iii) Translation*
2. *Explain two dimensional translation and scaling with an example.* **AU : May-13, Marks 8**
3. *With suitable example explain rotational transformation.* **AU : Dec.-12, Marks 8**

4.5 Homogeneous Co-ordinates

- In design and picture formation process, many times we may require to perform translation, rotations, and scaling to fit the picture components into their proper positions.

$$\begin{bmatrix} A' \\ B' \\ C' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2\cos 45 & 2\sin 45 & 0 \\ -2\sin 45 & 2\cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2.828 & 1 \\ 4.2426 & 9.898 & 1 \end{bmatrix}$$

Review Questions

1. *What is composite transformation ?*
2. *Obtain a transformation matrix for rotating an object about a specified pivot point.*

AU : May-13, Marks 8

3

4.7 Reflection and Shear Transformations

AU : Dec.-12, 13

- The three basic transformations of scaling, rotating, and translating are the most useful and most common.
- There are some other transformations which are useful in certain applications. Two such transformations are reflection and shear.

4.7.1 Reflection

- A reflection is a transformation that produces a mirror image of an object relative to an axis of reflection.
- We can choose an axis of reflection in the xy plane or perpendicular to the xy plane.
- Table 4.7.1 gives examples of some common reflections.

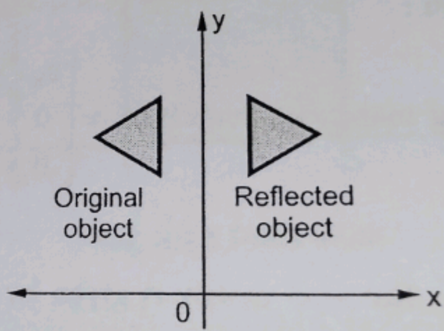
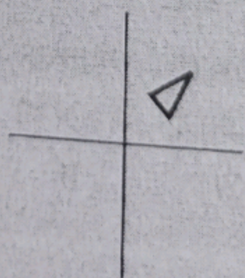
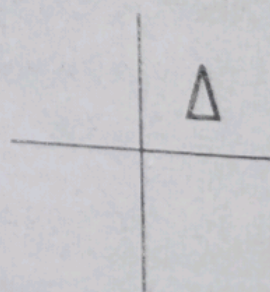


Fig. 4.7.1 Reflection about y axis

Reflection Transformation matrix Original image Reflected image

Reflection	Transformation matrix	Original image	Reflected image
Reflection about Y-axis	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		

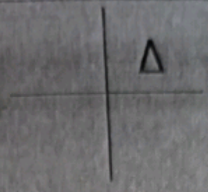
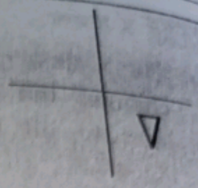
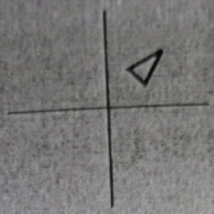
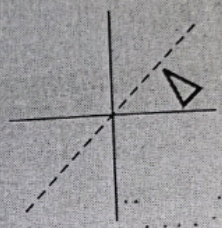
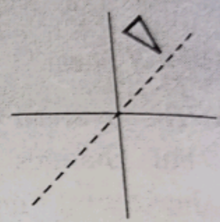
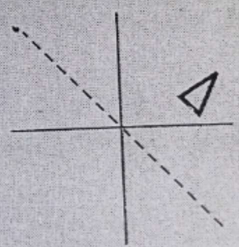
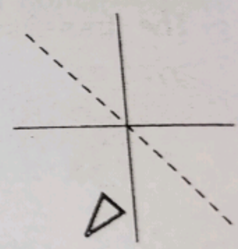
Reflection about X-axis	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		
Reflection about origin	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		
Reflection about line y = x	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		
Reflection about line y = − x	$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		

Table 4.7.1 Common reflections

4.7.2 Shear

- A transformation that slants the shape of an object is called the shear transformation.
- Two common shearing transformations are used. One shifts x co-ordinate values and other shifts y co-ordinate values. However, in both the cases only one co-ordinate (x or y) changes its co-ordinates and other preserves its values.

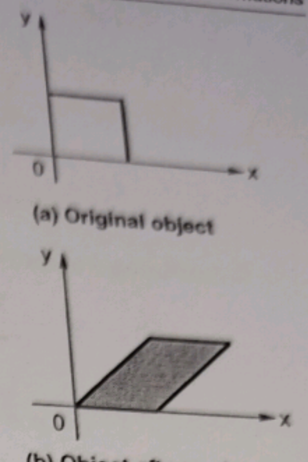
4.7.2.1 X Shear

- The x shear preserves the y co-ordinates, but changes the x values which causes vertical lines to tilt right or left as shown in the Fig. 4.7.2.
- The transformation matrix for x shear is given as

$$X\_sh = \begin{bmatrix} 1 & 0 & 0 \\ Sh_x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x' = x + Sh_x \cdot y \quad \text{and}$$

$$y' = y \qquad \ldots(4.7.1)$$



(a) Original object

(b) Object after x shear

Fig. 4.7.2

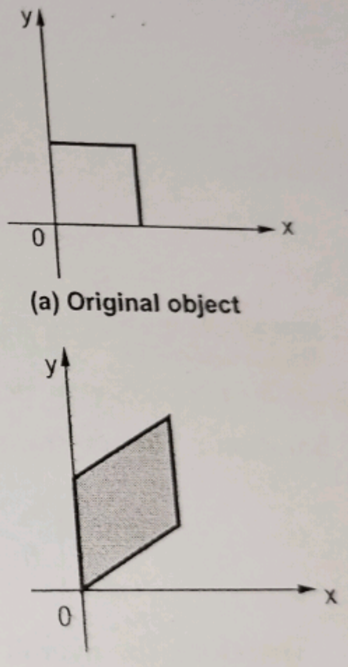
4.7.2.2 Y Shear

- The y shear preserves the x co-ordinates, but changes the y values which causes horizontal lines to transform into lines which slope up or down, as shown in the Fig. 4.7.3.
- The transformation matrix for y shear is given as

$$Y\_sh = \begin{bmatrix} 1 & Sh_y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x' = x \quad \text{and}$$

$$y' = y + Sh_y \cdot x$$



(a) Original object

(b) Object after y shear

Fig. 4.7.3

4.7.2.3 Shearing Relative to Other Reference Line

- We can apply x shear and y shear transformations relative to other reference lines. In x shear transformation we can use y reference line and in y shear we can use x reference line.
- The transformation matrices for both are given below :

x shear with y reference line : $\begin{bmatrix} 1 & 0 & 0 \\ Sh_x & 1 & 0 \\ -Sh_x \cdot y_{ref} & 0 & 1 \end{bmatrix}$